MMAE 411

Spacecraft Dynamics

Vector time derivatives

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Outline

- Time derivatives of vectors
- Angular velocity
- Computation, interpretation of angular velocity
- Angular acceleration
- Time derivatives in moving frames

Reference frame and coordinate system

Each reference frame is an observer with their own set of rulers, known as a coordinate system. For example, let a person standing at a playground be frame $\mathcal{A}$ with coordinates $\widehat{a}_x, \widehat{a}_y, \widehat{a}_z$.

Meanwhile, a child on a swing at the playground can be represented by another frame $\mathcal{B}$ with coordinates $\widehat{b}_x, \widehat{b}_y, \widehat{b}_z$.

Observer point of view

Vectors can change in magnitude, they can change in direction, both, or neither.

Each frame's coordinate system consists of vectors. Those vectors are *fixed* (not changing in magnitude or direction) in that frame. For frame $\mathcal{A}$, if one unit vector $\widehat{a}_z$ points upward (from the ground to the sky). In $\mathcal{A}$'s perspective, it remains pointing up from the ground to the sky, not changing direction. It remains a unit vector, not changing in magnitude.

From frame $\mathcal{B}$, if one unit vector $\widehat{b}_z$ points "up" from the seat up the chain to the crossbar, then even as the child is swinging, it always seems to be pointing "up" from the seat to the crossbar. In $\mathcal{B}$'s perspective, it remains pointing "up," not changing direction (or magnitude).

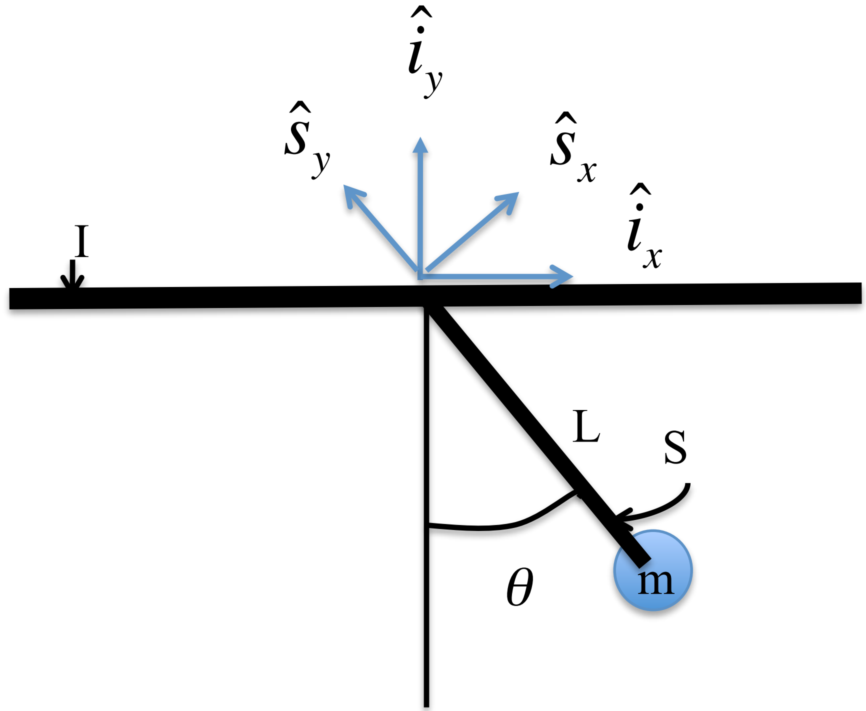
Vector time derivatives

The choice of reference frame affects the description of a given vector $\vec{r}$. So when computing the derivative of the vector, it is important to specify which reference frame the derivative is being computed *with respect to*. Let's denote the time derivative of a vector $\vec{r}$ in a reference frame $\mathcal{A}$ by

$$\frac{d^A\vec{r}}{dt} = {}^A\dot{\vec{r}} \tag{1}$$

This *implicitly* means that we choose $\mathcal{A}$ as a reference frame. As long as we use the associated *fixed* coordinate system $\hat{a}_{x,y,z}$ to express $\vec{r}$, we need not take further derivatives of the basis vectors.

Example: Pendulum



$$\vec{r}_{/I} = x\hat{i}_x + y\hat{i}_y \quad (2)$$

$$^{I}\dot{\vec{r}}_{/I} = \dot{x}\hat{i}_x + \dot{y}\hat{i}_y \quad (3)$$

But it's perfectly valid to write the position vector of the pendulum bob in S

coordinates instead:

$$\vec{r}_{/S} = -L\widehat{s}_y \tag{4}$$

$$\tag{5}$$

Now things get interesting...

However, when the basis vectors used to express $\vec{r}$ are *moving* in I, the expressions for the time derivatives of $\vec{r}$ contain additional terms accounting for the changing direction of the basis vectors.

Example: the pendulum $\vec{r}$ is convenient in coordinate system S. We can still take the time derivative of $\vec{r}$ with respect to the I frame, but must make sure to account for the time-varying basis vectors. Think of it as using the chain rule to account for changes in magnitude *and* changes in direction:

$$\vec{r}_{/S} = -L\widehat{s}_y \tag{6}$$

$${}^I\dot{\vec{r}}_{/S} = = -\cancelto{0}{\dot{L}}\widehat{s}_y + L\frac{d^I\widehat{s}_y}{dt} \tag{7}$$

The first term is the time derivative of the vector as it appears in the frame S whose coordinates it is written in. The second term is the time derivative of the basis vectors of S WRT frame I.

The idea behind angular velocity

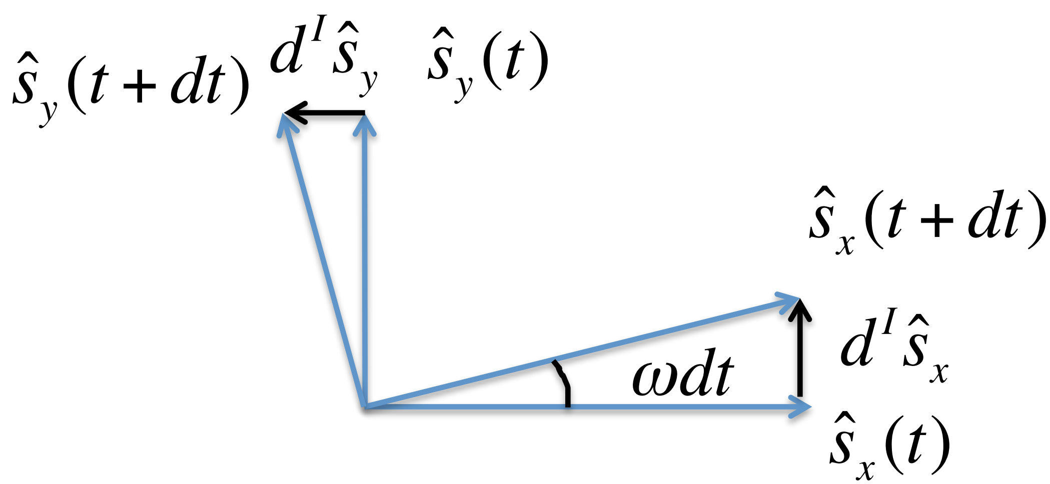
The pendulum S is spinning with respect to the inertial frame I. In a short time the base vectors $\hat{s}_x, \hat{s}_y$ have changed direction by a small amount:

$$d\hat{s}_x = \omega dt \hat{s}_y \tag{8}$$

$$d\hat{s}_y = -\omega dt \hat{s}_x \tag{9}$$

$$\tag{10}$$

The basis is still orthonormal.



Definition of angular velocity

The 3 components represent S-basis *components* of a vector ${}^{I}\vec{\omega}^{S}$: the *angular velocity of S in frame* I:

$$ {}^{I}\vec{\omega}^{S} = \omega_x \widehat{s}_x + \omega_y \widehat{s}_y + \omega_z \widehat{s}_z \tag{11} $$

By using this vector, very simple expressions can be obtained for the time derivative in the I frame of the basis vectors $\widehat{s}_j$ of S.

$$ \frac{d^I}{dt} \widehat{s}_j = {}^{I}\vec{\omega}^{S} \times \widehat{s}_j \tag{12} $$

Why the angular velocity is useful

The whole point of this is: angular velocity simplifies the differentiation of vectors. If you're given a vector $\vec{r}$ that is *fixed* in S,

$$\vec{r}_{/S} = r_{s_x}\widehat{s}_x + r_{s_y}\widehat{s}_y + r_{s_z}\widehat{s}_z \tag{13}$$

then

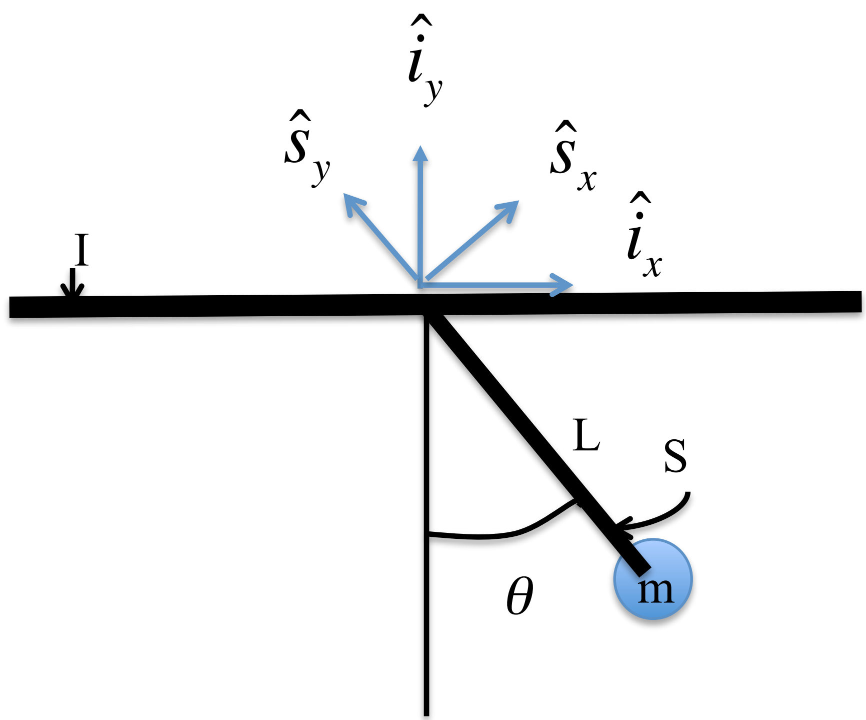
$$\begin{aligned} \frac{d^I}{dt}\vec{r} &= r_{s_x}\frac{d^I}{dt}\widehat{s}_x + r_{s_y}\frac{d^I}{dt}\widehat{s}_y + r_{s_z}\frac{d^I}{dt}\widehat{s}_z && (14)\\ &= {}^I\vec{\omega}^S \times \vec{r}_{/S} && (15) \end{aligned}$$

- No complicated derivatives of the transformation matrix
- Merely apply a cross product between two vectors
- You get to continue to express the derivative in the local basis of the moving body

Interpretation

The angular velocity is a vector whose direction is specified by the unit vector defining the *instantaneous axis of rotation of* S in frame I. The norm ω of ${}^I\vec{\omega}^S$ is given by an angular rate of rotation [rad/s].

For the pendulum, ${}^I\vec{\omega}^S{}_{/S} = [0, 0, \dot{\theta}]^T = \dot{\theta}\hat{s}_z$.



Generally the instantaneous axis of rotation may vary in time. The axis is *not necessarily fixed* in S or in I.

Addition theorem for angular velocities

Given three frames A, B, C such that B moves in A with angular velocity ${}^A\vec{\omega}^B$, C moves in B with angular velocity ${}^B\vec{\omega}^C$, and C moves in A with angular velocity ${}^A\vec{\omega}^C$.

The angular velocity of C in A can be expressed as:

$$\boxed{{}^A\vec{\omega}^C = {}^A\vec{\omega}^B + {}^B\vec{\omega}^C}$$

One practical use of this relation is the decomposition of a complicated angular velocity into the sum of simpler ones about locally fixed axes.

We will explore this concept more in a later lesson, but for now we can use this as needed.

Time derivative of vectors

The time derivative of a vector fixed in S, which is rotating with respect to I is

$$\frac{d^I}{dt}\vec{r}_{/S} \;=\; {}^I\vec{\omega}^S \times \vec{r} \tag{16}$$

But in general, our vector $\vec{r}$ won't be fixed in S, even when written in S's coordinates. In this case, the time derivative in I of a vector $\vec{r}$ not necessarily fixed in a frame S is given by

$$\frac{d^I\vec{r}}{dt} = = \boxed{\frac{d^S}{dt}\vec{r} + {}^I\vec{\omega}^S \times \vec{r}} \tag{17}$$

The time derivative of $\vec{r}$ WRT frame I is the time derivative of the components in the frame they're written in (the S frame) *plus* the time derivative of the basis vectors $\hat{s}_{x,y,z}$ WRT frame I.

Angular acceleration

The *angular acceleration* ${}^{A}\vec{\alpha}^{B}$ of a rigid body B with respect to a reference frame $\mathcal{A}$ is defined by

$$\boxed{{}^{A}\vec{\alpha}^{B} = \frac{d^{A}}{dt}{}^{A}\vec{\omega}^{B}}$$

i.e., it expresses the time rate of change of the angular velocity of B with respect to $\mathcal{A}$.

Velocity and acceleration of a point

The velocity ${}^A\vec{v}^P$ of a point $\mathcal{P}$ located by the position vector $\vec{r}^P$ in a reference frame A is defined by

$$\boxed{{}^A\vec{v}^P = \frac{d^A\vec{r}^P}{dt}}$$

Similarly the acceleration of $\mathcal{P}$ in A is

$$\boxed{{}^A\vec{a}^P = \frac{d^A\vec{v}^P}{dt}}$$

We can apply the "Golden Rule"

$$\frac{d^A}{dt}\vec{\blacksquare} = \frac{d^B}{dt}\vec{\blacksquare} + {}^A\vec{\omega}^B \times \vec{\blacksquare} \tag{18}$$

repeatedly to compute time derivatives of vectors.

Time Derivatives in Moving Frames

In the previous sections, we saw how to compute the time derivative of arbitrary vectors. In this section we will concentrate more specifically on the time derivative of position and velocity vectors to express their velocity and acceleration. These are routinely needed when forming the equations of motion of a dynamical system.

We will consider two cases when computing the velocity of a point associated with a rigid body:

- the point is *fixed* with respect to a frame attached to the rigid body

- the point *moves* with respect to a frame attached to the rigid body

Velocity and acceleration of a point

The velocity ${}^{A}\vec{v}^{P}$ of a point $\mathcal{P}$ located by the position vector $\vec{r}^{P}$ in a reference frame $\mathcal{A}$ (relative to the origin of A if no reference point is specified) is defined by

$$\boxed{{}^{A}\vec{v}^{P} = \frac{d\,{}^{A}\vec{r}^{P}}{dt}}$$

Similarly the acceleration of $\mathcal{P}$ in $\mathcal{R}_A$ is

$$\boxed{{}^{A}\vec{a}^{P} = \frac{d\,{}^{A}\vec{v}^{P}}{dt}}$$

Fixed points in a rigid body

Given two points $\mathcal{P}, \mathcal{Q}$ *fixed* in a rigid body B which is moving in A, the velocity and acceleration of $\mathcal{Q}$ in A frame can by derived from the position vector of $\mathcal{Q}$ in A as

$$\vec{r}^{Q} = \vec{r}^{Q/P} + \vec{r}^{P/O} \tag{19}$$

Hence,

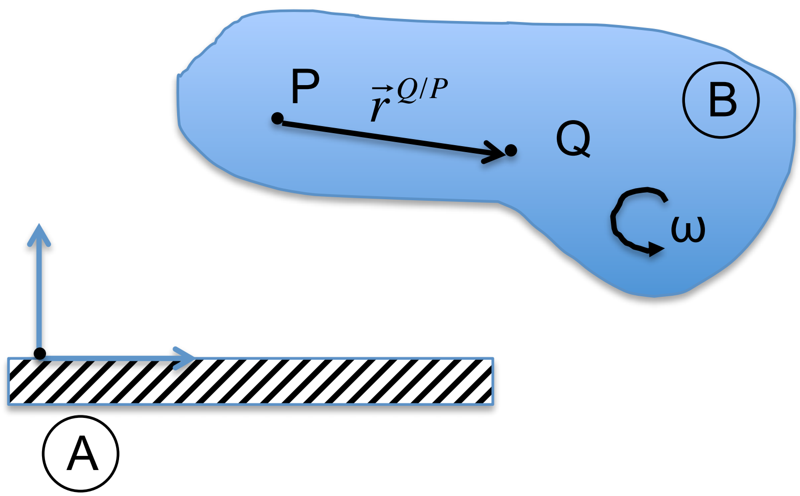
$$^{A}\vec{v}^{Q} = \frac{d^{A}\vec{r}^{Q}}{dt} = \frac{d^{A}\vec{r}^{P}}{dt} + \frac{d^{A}\vec{r}^{Q/P}}{dt} \tag{20}$$

which can also be written as

$$\boxed{\,^{A}\vec{v}^{Q} = {}^{A}\vec{v}^{P} + {}^{A}\vec{\omega}^{B} \times \vec{r}^{Q/P}}$$

(remember that $\vec{r}^{Q/P}$ is fixed in B.)

Illustration



$$ {}^{A}\vec{v}^{Q} = {}^{A}\vec{v}^{P} + {}^{A}\vec{\omega}^{B} \times \vec{r}^{Q/P} $$

Remark: These formulas are especially handy when P is any reference point in B whose velocity and acceleration are known, e.g., the center of mass.

Acceleration of fixed point in a rigid body

Similarly for acceleration we have:

$$
\begin{aligned}
{}^A\vec{a}^Q &= \frac{d^A({}^A\vec{v}^Q)}{dt} && (21)\\
&= \frac{d^A}{dt}\left({}^A\vec{v}^P + {}^A\vec{\omega}^B \times \vec{r}^{Q/P}\right) && (22)\\
&= \frac{d^A({}^A\vec{v}^P)}{dt} + \frac{d^A({}^A\vec{\omega}^B)}{dt} \times \vec{r}^{Q/P} + {}^A\vec{\omega}^B \times \left({}^A\vec{\omega}^B \times \vec{r}^{Q/P}\right) && (23)
\end{aligned}
$$

Thus

$$\boxed{{}^A\vec{a}^Q = {}^A\vec{a}^P + {}^A\vec{\alpha}^B \times \vec{r}^{Q/P} + {}^A\vec{\omega}^B \times \left({}^A\vec{\omega}^B \times \vec{r}^{Q/P}\right)}$$

where ${}^A\vec{\alpha}^B \equiv \frac{{}^Ad}{dt}({}^A\vec{\omega}^B)$ is the *angular acceleration* of B in A. The $\vec{\omega} \times \vec{\omega} \times \vec{r}$ term is known as the “centripetal acceleration.”

Moving point in a rigid body

Now consider a point $\mathcal{Q}$ that is *moving* in frame B. We will express its velocity and acceleration with respect to some reference frame A using a similar approach to the one introduced above (i.e., by using a reference point with known velocity).

The reference point $\mathcal{P}$ is assumed to be fixed in B. The position vector $\vec{r}^P$ of $\mathcal{P}$ in A is defined as

$$\vec{r}^Q = \vec{r}^{P/O} + \vec{r}^{Q/P} \tag{24}$$

where O is a reference point fixed in A.

The velocity of $\mathcal{Q}$ in A is then given by

$$^A\vec{v}^Q \;=\; {}^A\vec{v}^{P/O} + {}^B\vec{v}^{Q/P} + {}^A\vec{\omega}^B \times \vec{r}^{Q/P} \tag{25}$$

Acceleration of moving point in a rigid body

Similarly for acceleration we have:

$$
\begin{aligned}
{}^A\vec{a}^Q &= \frac{d^A}{dt}({}^A\vec{v}^{P/O}) + \frac{d^A}{dt}({}^B\vec{v}^{Q/P}) + \frac{d^A}{dt}\left({}^A\vec{\omega}^B \times \vec{r}^{Q/P}\right) && (26)\\
&= {}^A\vec{a}^P + {}^B\vec{a}^Q + {}^A\vec{\omega}^B \times {}^B\vec{v}^{Q/P} + {}^A\vec{\alpha}^B \times \vec{r}^{Q/P} + \\
&\quad {}^A\vec{\omega}^B \times \left({}^B\vec{v}^{Q/P} + {}^A\vec{\omega}^B \times \vec{r}^{Q/P}\right) && (27)\\
&= {}^A\vec{a}^P + {}^A\vec{\alpha}^B \times \vec{r}^{Q/P} + {}^A\vec{\omega}^B \times \left({}^A\vec{\omega}^B \times \vec{r}^{Q/P}\right) \\
&\quad + {}^B\vec{a}^Q + 2\left({}^A\vec{\omega}^B \times {}^B\vec{v}^{Q/P}\right) && (28)
\end{aligned}
$$

The term "$2({}^A\vec{\omega}^B \times {}^B\vec{v}^P)$" is known as the *Coriolis acceleration*.