MMAE 411
Spacecraft Dynamics

Torque-free motion

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Recap

Euler's Equations describe the angular equation of motion of a body about its center of mass, as seen in an inertial frame, but using body-fixed coordinates:

$$\vec{M} = \frac{^B d}{dt}\vec{H} + {}^I\vec{\omega}^B \times \vec{H} \tag{1}$$

$$= \vec{\vec{I}}^I \vec{\alpha}^B + {}^I\vec{\omega}^B \times \vec{\vec{I}}^I \vec{\omega}^B \tag{2}$$

We also saw that we can express the angular velocity components in terms of Euler angles ϕ, θ, ψ and their derivatives. The exact form of the equations depends on which Euler angle sequence is used.

Torque-free motion

If there are no external torques acting on the body (space is almost torque-free), then Euler's equation becomes (dropping the superscripts):

$$\overset{\rightrightarrows}{I}\vec{\alpha} \;=\; -\vec{\omega} \times \overset{\rightrightarrows}{I}\vec{\omega} \tag{3}$$

In general $\overset{\rightrightarrows}{I}\vec{\omega}$ is not aligned with $\vec{\omega}$. This means that the cross product will generally be nonzero. So $\vec{\alpha} = \frac{{}^{I}d}{dt}\vec{\omega}$ will be some nonzero value.

Torque-free motion

In general for torque-free motion, the spin axis moves about in the β frame and wobbles about the inertially fixed $\vec{H}$ direction, as seen by an inertial observer.

Since there are no torques, $\vec{H}$ is constant. We can write this in principal axis coordinates. We now also use the vectors as column arrays so we can do matrix operations:

$$
\begin{aligned}
[H] &= [I][\omega] && (4)\\
H^2 &= [H]^T[H] = [\omega]^T[I]^T[I][\omega] && (5)\\
&= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 && (6)
\end{aligned}
$$

Torque-free Euler's Equations

In principal axes, the general rotational EOM (Euler's Equation) is

$$\begin{aligned} M_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) && (7) \\ M_2 &= I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) && (8) \\ M_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) && (9) \end{aligned}$$

In principal axes, torque-free motion becomes:

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) && (10) \\ I_2\dot{\omega}_2 &= \omega_1\omega_3(I_3 - I_1) && (11) \\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) && (12) \end{aligned}$$

Equilibrium of the motion

When analyzing a dynamical system, whether linear or nonlinear, after writing the equations of motion (EOM) one of the first things we can do is find the equilibrium points.

For a dynamical system described by a system of differential equations, an equilibrium is one where all the time derivatives vanish. When the time derivative is zero, the values stay constant over time.

In our case, equilibrium angular velocities are those where the satellite is spinning solely about one principal axis. If $\omega_j \neq 0$ but $\omega_{i \neq j} = 0$, then all $\dot{\omega} = 0$. We can see by inspection of the last three equations on the previous slide that this will be true.

Stability of the equilibria

After finding the equilibrium points, the next useful thing to do is to ask, "Is each equilibrium stable?"

Consider $\omega_3 \neq 0$, and $\omega_1 = \epsilon_1, \omega_2 = \epsilon_2$ where the epsilons are both "small." This represents a small perturbation, a slight deviation from equilibrium (not due to energy dissipation). $\dot{\omega}_3 \approx 0$.

Then

$$\dot{\epsilon}_1 = \left(\frac{I_2 - I_3}{I_1}\right) \epsilon_2 \omega_3 \tag{13}$$

$$\dot{\epsilon}_2 = \left(\frac{I_3 - I_1}{I_2}\right) \epsilon_1 \omega_3 \tag{14}$$

To find the stability, we differentiate (14) and plug (13) into it.

Solve the differential equation via Laplace Transform

Solution of

$$\ddot{\epsilon}_1 - \left(\frac{I_2 - I_3}{I_1}\right)\left(\frac{I_3 - I_1}{I_2}\right)\epsilon_1\omega_3^2 \;=\; 0 \tag{15}$$

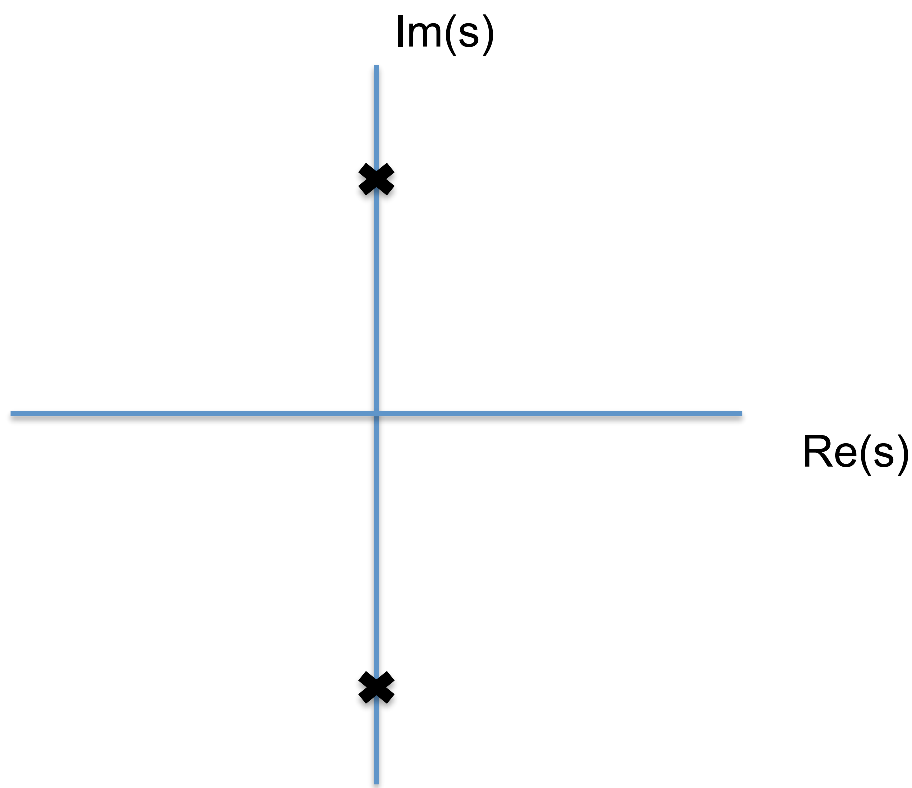
can be found with the Laplace transform (the technique of classical controls). The Laplace transform of this equation, and its solutions are

$$\epsilon_1 \;=\; \pm\sqrt{\left(\frac{I_2 - I_3}{I_1}\right)\left(\frac{I_3 - I_1}{I_2}\right)\omega_3^2} \tag{16}$$

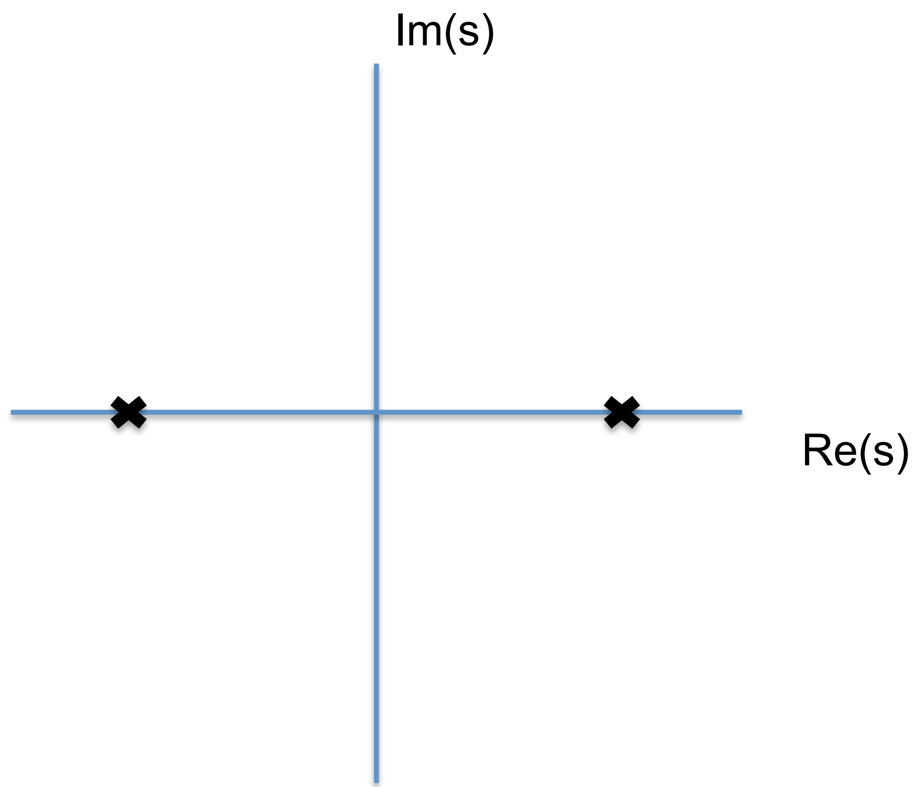
Roots of the diff. eq. in the complex plane

From classical controls, we know we can judge the stability of the system by plotting the roots in the complex plane. In the complex plane, any roots in the right half plane (RHP) mean the system will be unstable.

If I_3 is the largest or smallest, the square root is imaginary $\Rightarrow$ sinusoidal motion $\Rightarrow$ stable.


Im(s)
Re(s)

If I_3 is the intermediate value the square root is real. One solution is proportional to $e^{at}, a = \omega_3\sqrt{\frac{(I_2-I_3)(I_3-I_1)}{I_1 I_2}} > 0$. In this case, the system is *unstable*.



Phase plane

We can also plot the motion in the ϵ_1, ϵ_2 plane. This is a *phase portrait*.