MMAE 411
Spacecraft Dynamics

Newton's Laws of Motion +

Newton's Law of Gravitation =

Kepler's Laws

Dr. Seebany Datta-Barua

Outline

- Single particle energy and momentum
- Newton's Laws of Motion
- Newton's Law of Gravitation
  - N-body problem
  - Two-body problem
- Kepler's Laws

Recap

In Dynamics when we time differentiate vectors such as position and velocity, we must specify which reference frame. We'll denote the time derivative of a vector $\vec{r}$ in a reference frame $\mathcal{A}$ by

$$\frac{d^A\vec{r}}{dt} = {}^A\dot{\vec{r}} \tag{1}$$

Read the righthand side of that expression as "one time derivative, as seen from the A frame, of $\vec{r}$." We'll start using the dot and double-dot notation in these notes to make equations more compact.

Newton's Laws of Motion

1. A body P continues at rest or in uniform motion unless compelled to change by forces acting on it.

2. The rate of change of momentum is proportional to the force.

    - when observed in an inertial frame

    - If m is not changing with time (usually the case) then $\vec{F} = m^I\vec{a}^P$. If the position of m is $\vec{r}$, we can write this in the new notation as $\vec{F} = m^I\ddot{\vec{r}}$.

3. To every action there is an equal and opposite reaction. If particle 1 exerts a force on particle 2 of $\vec{F}_{12}$, then particle 2 acts on 1 with a force $\vec{F}_{21} = -\vec{F}_{12}$.

Inertial frame

Very important: when applying Newton's Second Law, the acceleration ${}^{I}\vec{a}^{P}$ *must* be computed with respect to an *inertial* reference frame.

Inertial frame: in theory, any frame that does not rotate or accelerate with respect to a frame at "absolute rest." Of course "absolute rest" doesn't really exist. So, in practice, the choice of reference frame is based on what constitutes a good approximation of "at rest" for the problem of interest.

The "inertial" frames that we use are not truly inertial, but rather "inertial enough" for the problem.

Inertial frame examples

- Frame fixed to Earth rotating with Earth: good for terrestrial machines, cars, usually aircraft

- Frame fixed to Earth but with axes fixed to the stars: good for Earth-orbiting spacecraft, aircraft, inertial navigation instruments (i.e., gyroscopes)

- Frame centered at the sun with axes fixed to the stars: good for interplanetary spacecraft.

Single particle momentum

In the inertial frame, we are often interested in computing a particle's momentum and energy. For a particle located at a position vector $\vec{r}$ from an origin fixed in the inertial frame, angular momentum is defined as:

$$\vec{H} \;=\; m\vec{r} \times {}^{I}\dot{\vec{r}} \tag{2}$$

as in, the velocity $\dot{\vec{r}}$ is the time derivative of position, as seen in the inertial frame. Per unit mass we write specific angular momentum:

$$\vec{h} \;=\; \vec{r} \times {}^{I}\dot{\vec{r}} \tag{3}$$

Single particle energy

In the inertial frame, we are often interested in computing a particle's kinetic energy. For a particle located at a position vector $\vec{r}$ from an origin fixed in the inertial frame, its kinetic energy T is a scalar quantity defined as:

$$T = \frac{1}{2} m {}^I\dot{\vec{r}} \cdot {}^I\dot{\vec{r}} \tag{4}$$

where the velocity $\dot{\vec{r}}$ is the time derivative of position, as seen in the inertial frame. Per unit mass we write specific kinetic energy as:

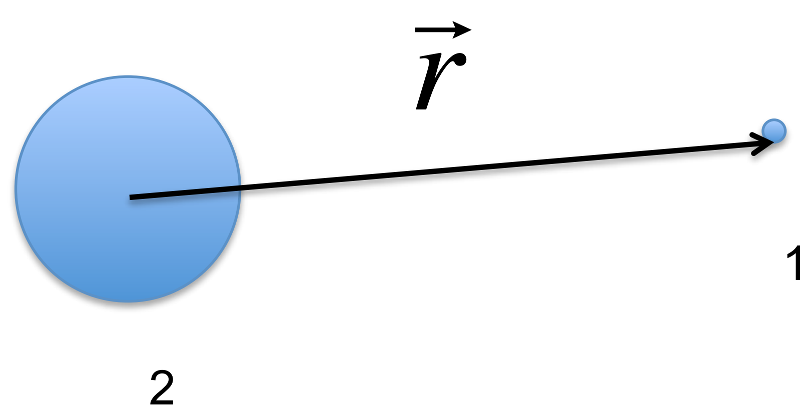
$$\mathcal{T} = \frac{1}{2} {}^I\dot{\vec{r}} \cdot {}^I\dot{\vec{r}} \tag{5}$$

Newton's Law of Gravitation

Until now, everything we've covered has been general Dynamics. Now bring in the other law Newton introduced in his book *Principia*.

In a system consisting of two bodies of mass m_1 and m_2, the force acting on m_1 due to the gravity of body 2 is:

$$\vec{F}_1 = -\frac{m_1 m_2 G}{r^2}\frac{\vec{r}}{r} \tag{6}$$



1. The bodies are spherically symmetrical.

2. The only force is gravity acting along the line of centers (gravity is a central force).

Gravity as a conservative force

Gravity is a conservative force, meaning that any work done moving from one point to another is undone when you get back to where you started. Mathematically,

$$\oint \vec{F} \cdot \vec{dr} = 0 \tag{7}$$

With a conservative force, the amount of work done to go from one point to another only depends on the start point and the end point, not the route used.

Another feature is that the total energy, potential V plus kinetic T, is conserved when the only external force is gravity.

$$V_1 + T_1 = V_2 + T_2 \tag{8}$$

Potential energy

Conservative forces can be described by a scalar function $V(\vec{r})$, which in a cartesian coordinate system looks like:

$$\begin{aligned}\vec{F} &= -\vec{\nabla} V && (9)\\ &= -\frac{\partial V}{\partial x}\widehat{x} - \frac{\partial V}{\partial y}\widehat{y} - \frac{\partial V}{\partial z}\widehat{z} && (10)\end{aligned}$$

For gravitation, the potential function is:

$$V = -\frac{Gm_1m_2}{r} + C \qquad (11)$$

where C is an arbitrary constant. Its negative gradient gives the gravitational force.

Low-altitude approximation

For the Earth-aircraft system, $r = R_E + h$ is the Earth's radius plus the altitude of the aircraft. Since $R_E >> h$, we can Taylor expand the potential to get a simple and familiar approximation.

The Taylor expansion formula for a function $f(x)$ for points $x = x_0 + \Delta x$ near x_0 is:

$$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x_0} \Delta x + \frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_{x_0} (\Delta x)^2 + ... \qquad (12)$$

Approximate potential energy

So for points near the Earth's radius R_E, we approximate V as:

$$V = -\frac{GMm}{r} \approx -\frac{GMm}{R_E} + \frac{GMm}{R_E^2}h + ... \quad (13)$$

$$\approx const + \underbrace{\frac{GM}{R_E^2}}_{g} mh + ... \quad (14)$$

For systems where m is located near Earth's surface, we often use $V = mgh$. But it is not valid as the second body gets further from Earth, e.g., with satellites.

N-body problem

For a system of N particles P_i with masses m_i with gravitation as the only force among them, the force on m_1 due to $N-1$ other particles is:

$$\vec{F}_1 \;=\; -Gm_1 \sum_{i=2}^{N} \frac{m_i \vec{r}_{1i}}{||\vec{r}_{1i}||^3} \tag{15}$$

or for the jth particle generally,

$$\vec{F}_j \;=\; -Gm_j \sum_{i=1, i\neq j}^{N} \frac{m_i \vec{r}_{ji}}{||\vec{r}_{ji}||^3} \tag{16}$$

where $\vec{r}_{ji} = \vec{r}^{P_j/P_i}$, meaning the position of P_j from P_i (i.e., the head of the arrow is at P_j).

If we combine this with Newton's 2nd Law $\vec{F}_j = m_j {}^I\ddot{\vec{r}}_j$ we get the equation of motion (EOM) for the jth particle:

$$m_j {}^I\ddot{\vec{r}}_j \;=\; -\sum_{i=1, i\neq j}^{N} \frac{Gm_j m_i \vec{r}_{ji}}{||\vec{r}_{ji}||^3},\; j = 1, 2, ..., N \tag{17}$$

The m_j cancels from both sides. This system of 3N equations is the "N-body problem." For $N \geqslant 3$, all 3 (or more) masses have gravitational fields that are interacting. To date, there is no known closed-form solution.

Two-body problem

Fortunately, for many applications, we can approximate a system as really involving only two objects, for example:

- Earth-Sun
- Earth-Moon
- Earth-satellite
- binary stars!

This scenario comes up so often that it is known as "the two-body problem." For the two-body problem there is a known solution to the motion of the objects.

It may seem like the 2-body problem is very limited and restricting, but for many application this is simple and usable. For example it explains a set of celestial observations from the early 1600s...

Kepler's Laws

Johannes Kepler discovered the following in the years 1609-1619:

1. The orbit of planets is a planar ellipse with one focus at the sun.

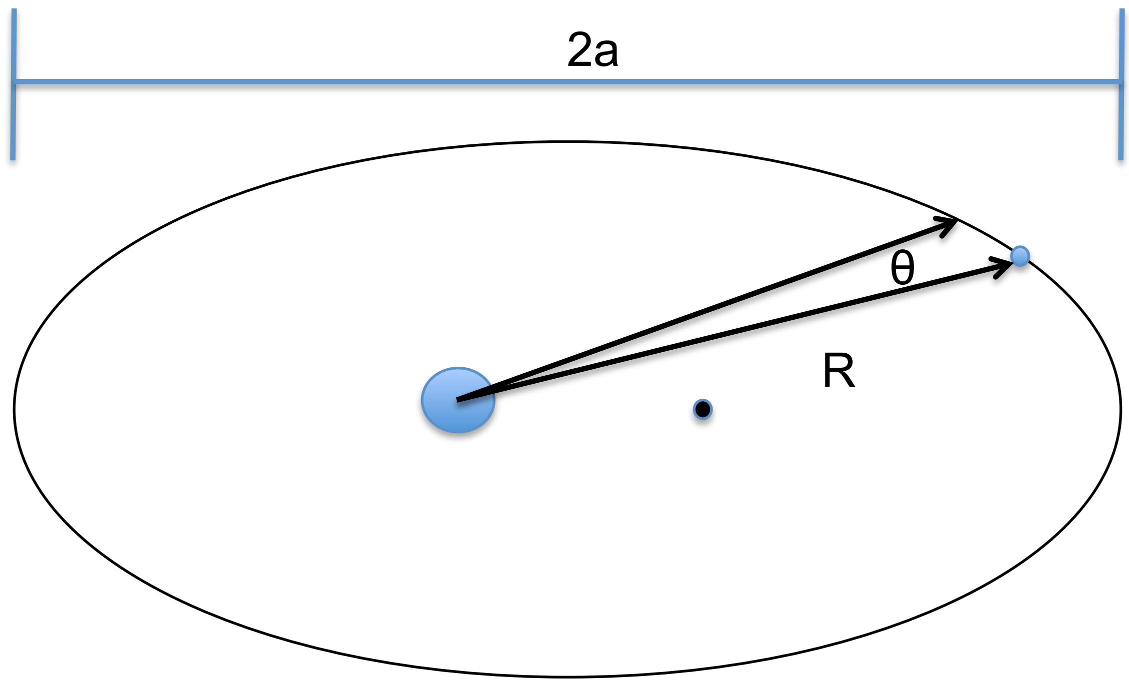
2. The line joining the body to the sun sweeps out equal areas in equal time.

$$R(t)\frac{d\theta}{dt} \;=\; \text{constant} \tag{18}$$

3. The square of the period is proportional to the cube of the mean distance.

$$\mathcal{P} \;\propto\; a^{\frac{3}{2}} \tag{19}$$

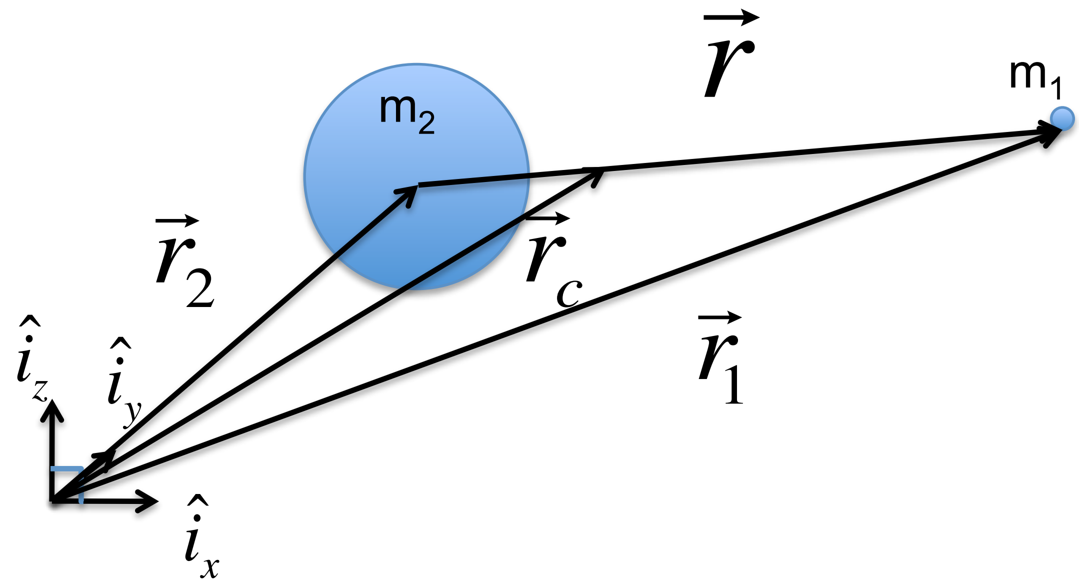
Kepler's Law's, illustrated



Kepler came up with these empirically (by observation). Newton "discovered" gravity and the laws of dynamics and was able to "prove" Kepler's laws in 1665. We'll go through and prove them today.

Two-body system in an inertial frame

Assume two point masses, spherically symmetric, that are subject only to each other's gravitational forces.



In an inertial frame,

$$\vec{F}_1 = -\frac{m_1 m_2 G}{r^2}\hat{r} = m_1 {}^I\ddot{\vec{r}}_1 \tag{20}$$

$$\vec{F}_2 = \frac{m_1 m_2 G}{r^2}\hat{r} = m_2 {}^I\ddot{\vec{r}}_2 \tag{21}$$

Fundamental Orbital Differential Equation

The center of mass (COM) $\vec{r}_c$ undergoes no acceleration in the inertial frame (add Eqs. (20) and (21) together):

$$\begin{aligned}
\vec{r}_c &\equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} && (22)\\
{}^I\ddot{\vec{r}}_c &= \frac{m_1{}^I\ddot{\vec{r}}_1 + m_2{}^I\ddot{\vec{r}}_2}{m_1 + m_2} && (23)\\
&= 0 && (24)\\
\rightarrow \vec{r}_c(t) &= \vec{r}_c(0) + \vec{v}_c(0)t && (25)
\end{aligned}$$

Re-expressing (20) and (21) in terms of $\vec{r}_c$ allows us to write the equations of motion (EOM) using only the relative position between them:

$$\begin{aligned}
{}^I\ddot{\vec{r}} &= -\frac{G(m_1 + m_2)\vec{r}}{r^3} && (26)\\
&= -\frac{\mu\vec{r}}{r^3} && (27)
\end{aligned}$$

Gravitational parameter

The fundamental orbital differential equation describes the motion of body 1 relative to body 2.

The *gravitational parameter* $\mu = G(m_1 + m_2)$ is often $\mu \approx Gm_2$ for common pairs (exception: Earth-Moon). It has units of $\left[\frac{l^3}{t^2}\right]$.

Commonly used values:

$$\mu_\oplus = 3.986e5\frac{km^3}{s^2} \tag{28}$$

$$\mu_\odot = 1.327e11\frac{km^3}{s^2} \tag{29}$$

Angular momentum

Cross product of $\vec{r}$ with (27) gives:

$$\begin{aligned} \vec{r} \times {}^I\ddot{\vec{r}} &= 0 && (30) \\ &= \frac{d^I}{dt}(\vec{r} \times \dot{\vec{r}}) && (31) \end{aligned}$$

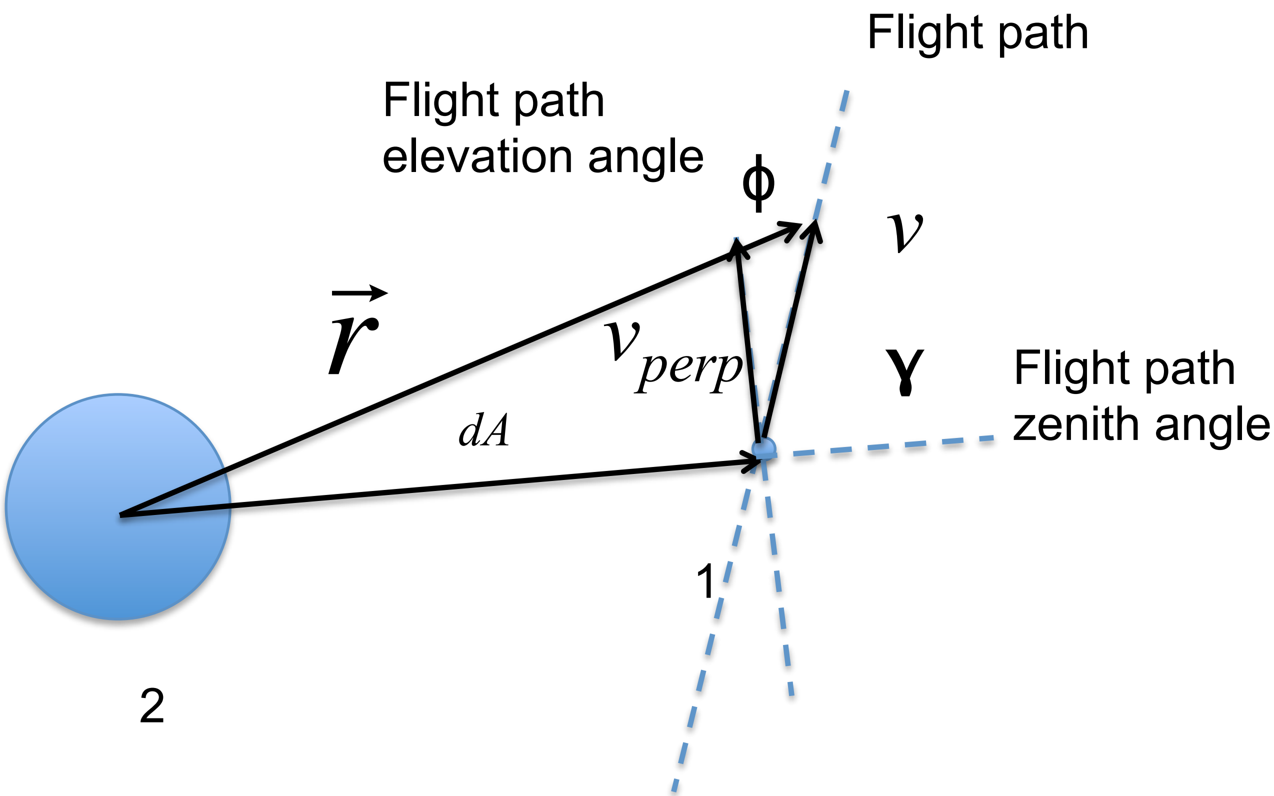
So the angular momentum per unit mass $\vec{h} = \vec{r} \times {}^I\dot{\vec{r}}$ is constant.

1. $\vec{h}$ is normal to $\vec{r}$ and $\vec{v}$, the inertial velocity.

2. Since $\vec{h}$ is fixed, the motion is planar.

Kepler's 2nd: area swept by the motion

Using $|h| = rv_{perp}$, the area of the triangle swept out by the body as it travels is dA in an amount of time dt.



$$\frac{dA}{dt} = \frac{1}{2} r v_{perp} = \frac{h}{2} = \text{constant} \tag{32}$$

Other neat tricks with the FODE

Cross (27) with $\vec{h}$:

$$ {}^I\ddot{\vec{r}} \times \vec{h} = \frac{\mu}{r^3}(\vec{h} \times \vec{r}) \tag{33} $$

Lefthand side (LHS) is $\frac{d^I}{dt}({}^I\dot{\vec{r}} \times \vec{h})$. The righthand side (RHS) takes a little more rearrangement:

$$ \begin{aligned} \frac{\mu}{r^3}(\vec{h} \times \vec{r}) &= \frac{\mu}{r^3}(\vec{r} \times {}^I\dot{\vec{r}}) \times \vec{r} && (34) \\ &= \frac{\mu}{r^3}\left({}^I\dot{\vec{r}}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot {}^I\dot{\vec{r}})\right) && (35) \end{aligned} $$

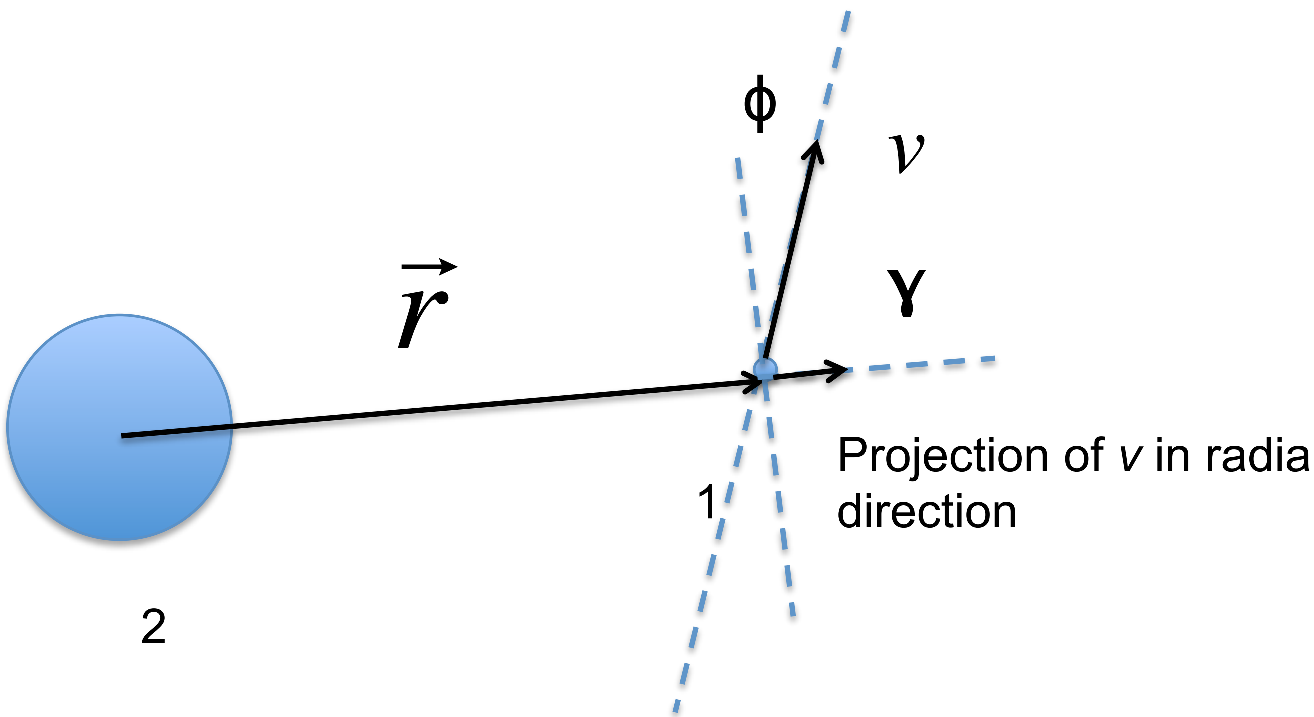
This last line uses the vector triple product identity.

Proof of $\vec{r} \cdot {}^I\dot{\vec{r}}$

Now we prove another useful identity, that $\vec{r} \cdot {}^I\dot{\vec{r}} = r^I\dot{r}$:

$$\vec{r} \cdot {}^I\dot{\vec{r}} = |r|\hat{r} \cdot {}^I\dot{\vec{r}} \tag{36}$$

The dot product is the projection of the velocity in the radial direction. But this is just the rate of change of the magnitude of the distance, $|\dot{r}|$. Thus, $\vec{r} \cdot {}^I\dot{\vec{r}} = |r||^I\dot{r}|$.



FODE constants of motion demonstration, cont'd

Returning to the RHS of (35):

$$\frac{\mu}{r^3}(\vec{h} \times \vec{r}) = \frac{\mu}{r^3}\left({}^I\dot{\vec{r}}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot {}^I\dot{\vec{r}})\right) \quad (37)$$

$$= \frac{\mu {}^I\dot{\vec{r}}}{r} - \frac{\mu {}^I\dot{r}\vec{r}}{r^2} \quad (38)$$

Realizing that this is a time derivative,

$$\mu \frac{d^I}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\mu}{r} {}^I\dot{\vec{r}} - \frac{\mu {}^I\dot{r}}{r^2}\vec{r} \quad (39)$$

we now have a differential expression for the RHS as well. The LHS and the RHS together become:

$$\frac{d^I}{dt}({}^I\dot{\vec{r}} \times \vec{h}) = \mu \frac{d^I}{dt}\left(\frac{\vec{r}}{r}\right) \quad (40)$$

The solution is:

$${}^I\dot{\vec{r}} \times \vec{h} = \mu\left(\frac{\vec{r}}{r}\right) + \mu\vec{e} \quad (41)$$

The last term is a vector constant of integration.

Toward a geometrical interpretation

What does the solution mean? Let's dot (41) with $\vec{r}$:

$$\vec{r} \cdot {}^I\dot{\vec{r}} \times \vec{h} = \mu \left(\vec{r} \cdot \frac{\vec{r}}{r} \right) + \mu \vec{r} \cdot \vec{e} \tag{42}$$

We use a property that the dot and the cross in a vector triple product can be swapped:

$$\begin{aligned} \vec{r} \cdot {}^I\dot{\vec{r}} \times \vec{h} &= \vec{r} \times {}^I\dot{\vec{r}} \cdot \vec{h} && (43) \\ &= \vec{h} \cdot \vec{h} && (44) \\ &= \mu \left(\vec{r} \cdot \frac{\vec{r}}{r} \right) + \mu \vec{r} \cdot \vec{e} && (45) \end{aligned}$$

So now this is essentially a scalar equation:

$$h^2 = \mu r + \mu r e \cos\theta \tag{46}$$

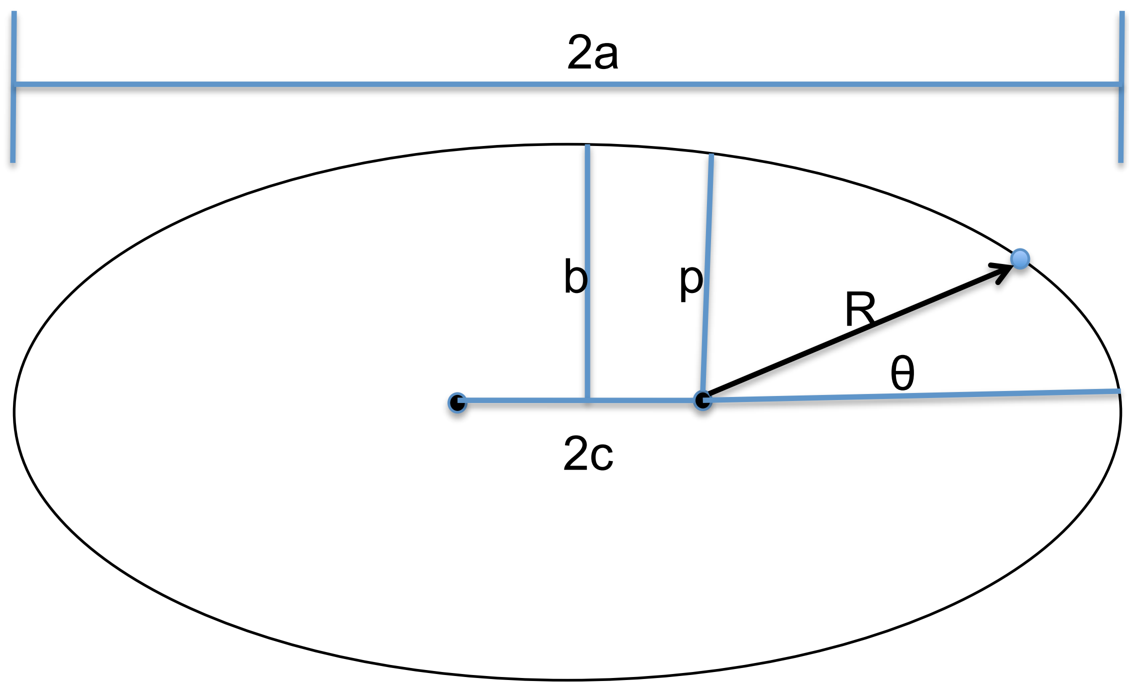
where θ is the angle between $\vec{r}$ and $\vec{e}$.

Kepler's Laws, illustrated

Solve for r:

$$\boxed{r = \frac{\frac{h^2}{\mu}}{1+e\cos\theta}}$$

This is the equation of an ellipse (in polar form with the origin at one focus).



Ellipse

- a is the semi-major axis
- b is the semi-minor axis $b = \sqrt{ap}$
- c is half the distance between the foci $c = ae$
- p is the semi-latus rectum $p = \frac{h^2}{\mu}$