MMAE 411

Spacecraft Dynamics

Kepler's Equation

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Outline

- Circular orbits
- Parabolic orbits
- Hyperbolic orbits
- Kepler's equation

Circular orbits

$\boxed{\mathcal{E} < 0, e = 0.0, a < \infty}$

From our equation of a conic section $r = \frac{p}{1+e\cos\theta} = p$.

In a circular orbit $r = a$, so the speed is the same everywhere. It is:

$$\begin{aligned} \mathcal{E} &= \frac{v_{circ}^2}{2} - \frac{\mu}{r} && (1) \\ &= \frac{v^2}{2} - \frac{\mu}{a} && (2) \\ &= \frac{\mu}{2a} && (3) \\ \frac{v_{circ}^2}{2} &= \frac{2\mu - \mu}{2a} && (4) \end{aligned}$$

$$\boxed{v_{circ} = \sqrt{\frac{\mu}{a}}}$$

Typical speeds for circular orbits

The larger the radius of the orbit, the less speed is needed to stay in this orbit. Some typical orbits:

- $v_{LEO} \approx$ 25000 fps (7.6 km/s)

- $v_{☾} \approx$ 3000 fps

- In its heliocentric orbit, $v_{\oplus} = \sqrt{\frac{\mu_{\odot}}{1AU}} = 1 \approx$ 29.8 km/s

Low Earth Orbits (LEO) are frequently used as “parking” orbits. They are an efficient place to add Δv.

Parabolic orbits

$\boxed{\mathcal{E} = 0, e = 1.0, a = \infty}$

From our equation of a conic section $r = \frac{p}{1+e\cos\theta}$, we can see that at periapsis $r_\pi = \frac{p}{2}$.

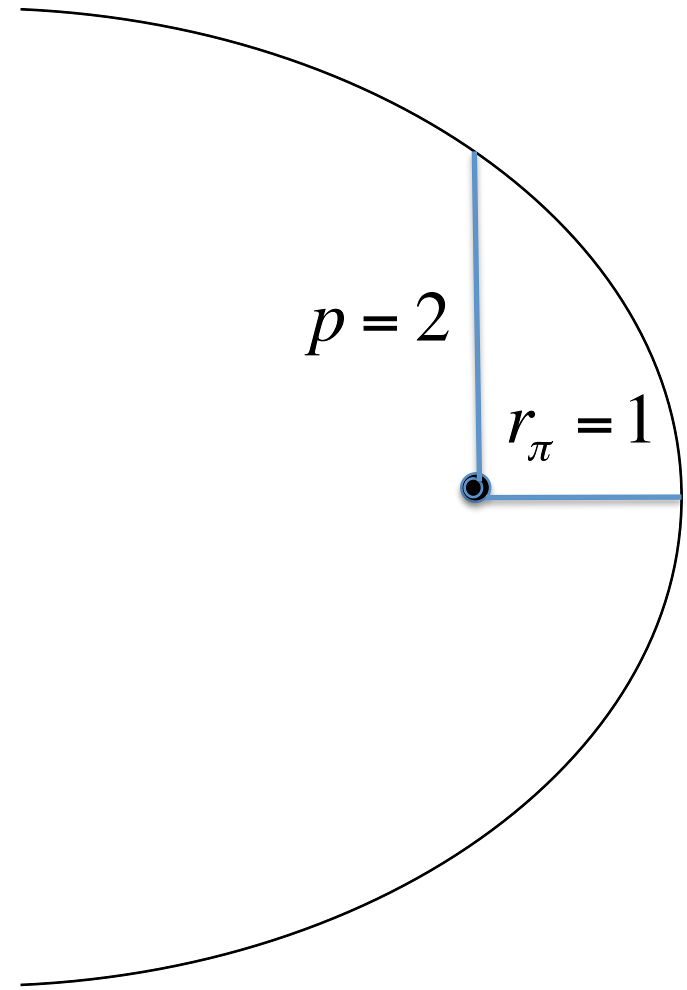
For a parabolic orbit there is now a meaning to the concept of "escape speed." What velocity will get the sv to ∞ with no kinetic energy left? In other words, $KE_\infty = 0, PE_\infty = 0$.

$$\mathcal{E} = \frac{v_{esc}^2}{2} - \frac{\mu}{r} = 0 \quad (5)$$

$$\Rightarrow v_{esc} = \sqrt{\frac{2\mu}{r}} \quad (6)$$

$$= \sqrt{2}v_{circ} \quad (7)$$

At 1 DU, $v_{esc} = \sqrt{2}$ DU/TU = (1.41)(7.9 km/s) = 11.17 km/s.



$p = 2$
$r_\pi = 1$

Hyperbolic orbits

$\boxed{\mathcal{E} > 0, e > 1.0, a < 0}$

From our equation of a conic section $r = \frac{p}{1+e\cos\theta}$, we can see that at periapsis $r_\pi = \frac{p}{1+e}$.

The asymptote of the hyperbola is $||$ to the point at which the denominator $= 0$.

$$\cos\theta_A = -\frac{1}{e} \tag{8}$$

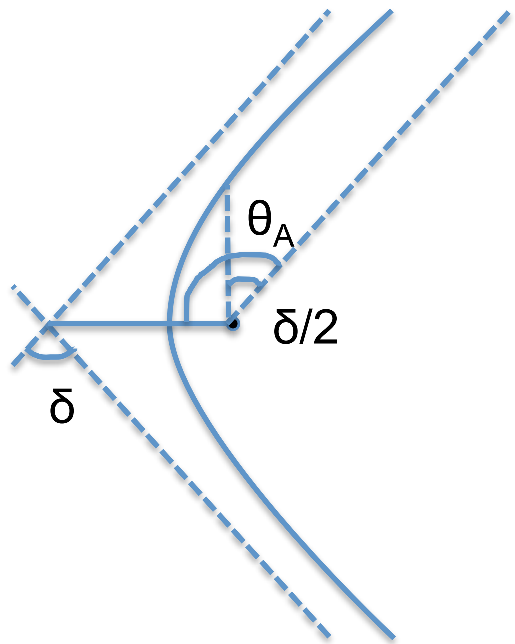
From the geometry of the figure, $\theta_A = \frac{\pi}{2} + \frac{\delta}{2}$. The use the sum of cosines rule to get:

$$\cos\left(\frac{\pi}{2} + \frac{\delta}{2}\right) = \cos\frac{\pi}{2}\cos\frac{\delta}{2} - \sin\frac{\pi}{2}\sin\frac{\delta}{2} \tag{9}$$

$$= -\frac{1}{e} \tag{10}$$

So,

$$\boxed{\sin\frac{\delta}{2} = \frac{1}{e}}$$



Hyperbolic orbits h and $\mathcal{E}$

Even though $e > 1$, p is still positive:

$$p = a(1 - e^2) = \frac{h^2}{\mu} \tag{11}$$

since $a < 0$.

Also $a < 0$ means that energy $\mathcal{E} = -\frac{\mu}{2a} > 0$. So a body with positive energy *must* escape.

Kepler's equation

From Kepler's 2nd law (equal areas in equal times), he developed a way to express the position of the satellite as a function of time.

Let's define:

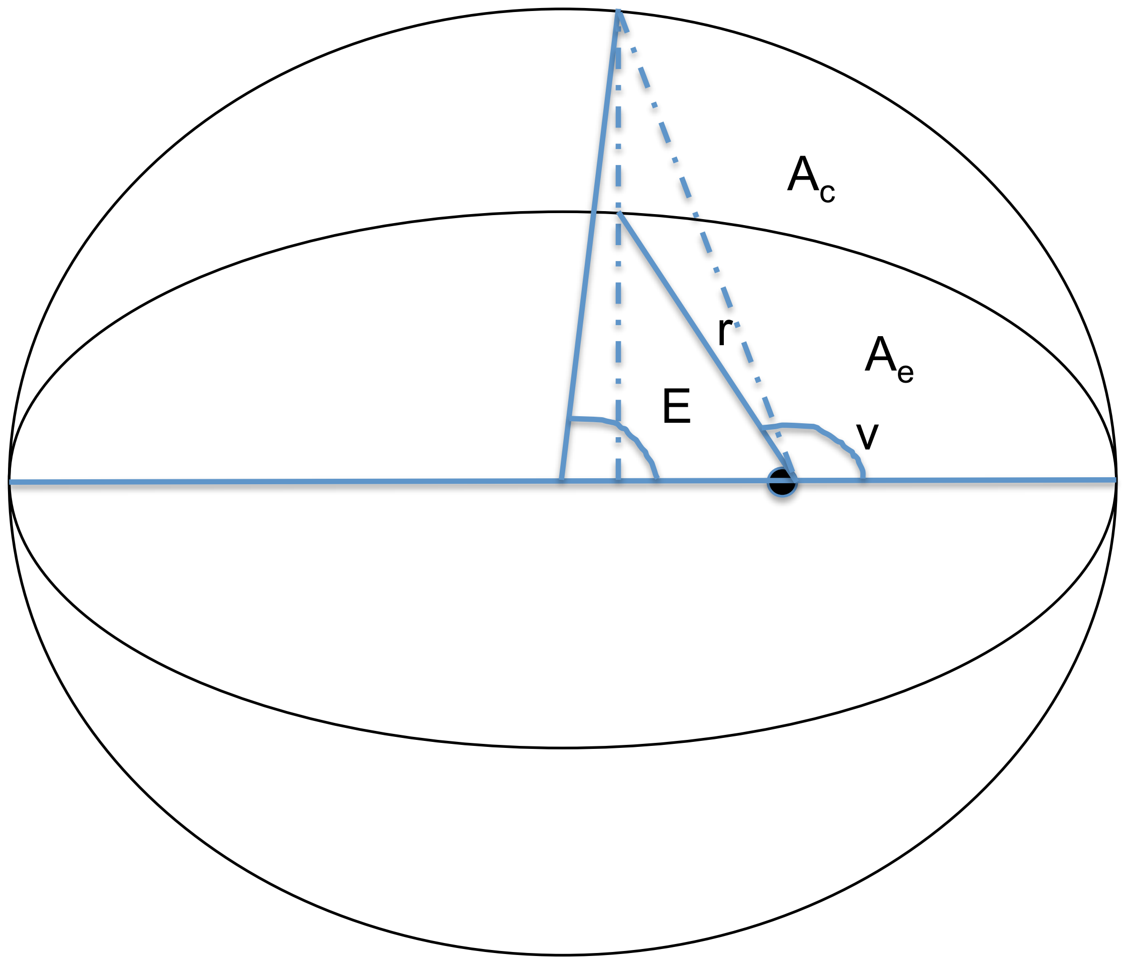
- ν: True anomaly – the "polar angle" measured from periapsis.

- M: Mean anomaly – the average orbital rate x time since periapsis (radians).

- n: Mean motion – the rate of mean anomaly [rad/TU], aka the average orbital rate.

- E: Eccentric anomaly – an intermediate geometric variable that allows transformation of position to time and vice versa.

Ellipse and auxiliary circle



Derivation of Kepler's Equation, part 1

The auxiliary circle allows us to compute the area of a section of the ellipse.

$$
\begin{aligned}
A_c &= \frac{a^2 E}{2} - \frac{a^2 E \sin E}{2} && (12)\\
A_e &= \frac{b}{a} A_c && (13)\\
&= \frac{ab}{2}(E - e \sin E) && (14)
\end{aligned}
$$

We know the total area of the ellipse will be covered during the period:

$$
T = 2\pi \sqrt{\frac{a^3}{\mu}} \qquad (15)
$$

Derivation of Kepler's Equation, part 2

From Kepler's 2nd law:

$$\frac{dA}{dt} = \frac{\pi ab}{T} \tag{16}$$

$$= \frac{ab}{2}\sqrt{\frac{\mu}{a^3}} \tag{17}$$

The area computed in (19) has this same ratio:

$$\frac{dA}{dt} = \frac{ab}{2(t - T_0)}(E - e \sin E) \tag{18}$$

where T_0 is the time the body was at periapsis.

Derivation of Kepler's Equation, part 3 of 3

Equating (22) and (23) and defining the mean motion n:

$$E - e\sin E \;=\; \underbrace{\sqrt{\frac{\mu}{a^3}}}_{n}(t - T_0) \tag{19}$$

Calling the righthand side of (24) the mean anomaly M, we have Kepler's Equation:

$$\boxed{M = E - e\sin E}$$

This form holds for circles and ellipses (see Wiesel for parabolic and hyperbolic forms).

To be useful in calculating position, though, we need a relationship to the true anomaly ν.

Mean anomaly related to true anomaly

The distance from center of the ellipse to the focus is ae. It is also the sum of the bases of the two triangles:

$$a \cos E - r \cos \nu \quad = \quad ae \tag{20}$$

Sub in the expression $r = p/(1 + e \cos \nu)$ and $p = a(1 - e^2)$ to get:

$$\cos E = \frac{e + \cos \nu}{1 + e \cos \nu} \tag{21}$$

From the geometry, E and ν are always in the same quadrant, so given one of them, you can determine the other using (26) without any quadrant ambiguity.

Where is this all going?

We will see in the next class how to completely describe the shape of the orbit and its orientation in space. And we will use these to compute the position of the satellite at any given time and vice versa.