MMAE 411
Spacecraft Dynamics

Rigid Body Inertia Matrices

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Outline

- Angular Momentum
- Kinetic Energy
- Principal Body Axes
- Parallel Axis Theorem

Overview

In space, satellites rotate due to

- natural motion $\Rightarrow$ Euler's equations.
- internal forces
  - outgassing
  - thrusters
  - control moment gyros and other momentum-storing devices
- external forces
  - magnetic torquing
  - asymmetrical radiation
  - gravity gradient

Motivation

Most satellites require attitude control and stabilization for

- sensor orientation
- communication
  - transmit
  - receive
- solar array orientation
- payload health

To do any of this we first have to understand the EOM. They tell us about the natural motion.

Angular momentum, summarized

To recap, angular momentum of a body B is defined as

$$\vec{H} \equiv \int_{Vol} \vec{r} \times ({}^{I}\vec{\omega}^{B} \times \vec{r})\rho dV \tag{1}$$

Coordinatizing in the body frame in terms of a $\hat{b}_1, \hat{b}_2, \hat{b}_3$, and multiplying out

$$[\vec{H}]_{\hat{b}} = \int_{Vol} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \rho dV \vec{\omega} \tag{2}$$

$$= [I]_{\hat{b}}[\omega]_{\hat{b}} \tag{3}$$

where the subscript $\hat{b}$ means each matrix is written in components along $\hat{b}_1, \hat{b}_2, \hat{b}_3$.

Angular momentum

We've defined the inertia dyadic $\vec{\vec{I}}$ which is closely related to the inertia matrix $[I]$, a 3x3 matrix that contains the moments of inertia on the diagonal. The off-diagonal elements are "products of inertia."

With the inertia matrix $[I]$, writing the angular momentum $\vec{H}$ becomes simply:

$$[H]_{\hat{b}} \;=\; [I]_{\hat{b}}[{}^{I}\omega^{B}]_{\hat{b}} \tag{4}$$

In general $\vec{H}$ and $\vec{\omega}$ do not have to be aligned.

The inertia tensor represented as a matrix is symmetric and positive definite ($\vec{a}\vec{\vec{I}}\vec{a} > 0$ for any $\vec{a}$). This property will help us to do something useful shortly.

Rotational kinetic energy

We can also use $\vec{\vec{I}}$ to compute kinetic energy:

$$
\begin{aligned}
KE_{rot} &\equiv \frac{1}{2}\int_B (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) \rho dV &(5)\\
&= \int_B (\vec{\omega} \cdot \vec{\omega} \times \vec{r}) \times (\vec{r} \cdot \vec{\omega} \times \vec{r}) \rho dV &(6)
\end{aligned}
$$

With some algebraic rearrangement we can write this as:

$$
\begin{aligned}
KE_{rot} &= \frac{1}{2}\int_B \vec{\omega} \cdot \vec{r} \times (\vec{\omega} \times \vec{r}) \rho dV &(7)\\
&= \frac{1}{2}\vec{\omega} \cdot \int_B \vec{r} \times (\vec{\omega} \times \vec{r}) \rho dV &(8)\\
&= \frac{1}{2}\vec{\omega} \cdot \vec{H} &(9)\\
&= \frac{1}{2}\vec{\omega} \cdot \vec{\vec{I}} \cdot \vec{\omega} &(10)
\end{aligned}
$$

In matrix form this is computed as $KE_{rot} = \frac{1}{2}[\omega]^T[I][\omega]$.

Eigenvectors and eigenvalues of inertia tensor

We looked at the inertia tensor and how to compute it. It can be expressed as a 3x3 matrix.

In general, all the elements will be nonzero. This means the inertia matrix $[I]$ of a body B with respect to a point generally changes both the *orientation* and *magnitude* of the vectors on which it is applied, i.e., given a vector $\vec{v} \in \mathbb{R}^3$ and $\vec{u}$ defined as

$$\vec{u} = \vec{\vec{I}}\vec{v} \tag{11}$$

then typically $\vec{u}$ is not parallel to $\vec{v}$. There are, however, some vectors $\vec{v}_e$ whose directions are *unchanged* under the action of the matrix $[I]$, i.e.,

$$[I][v_e] = \lambda[v_e] \tag{12}$$

These vectors are called the *eigenvectors* of $[I]$ and the corresponding λ are their *eigenvalues*.

Angular momentum in principal axis coordinates

If you express the inertia tensor in a basis made of the eigenvector, the input vector and the output vector have the same direction. This means the $[I]$ will be diagonal. You won't have to worry about I_{xy}, I_{xz}, I_{yz} terms.

Let's suppose the angular velocity in the eigenvector basis is:

$$\vec{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T \tag{13}$$

Then the angular momentum is:

$$\begin{aligned} \vec{H} &= \vec{\vec{I}}\vec{\omega} & (14) \\ &= \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} & (15) \end{aligned}$$

It really simplifies writing the EOM to work in the principal-body-axis coordinate system.

Principal Moments λ_j: eigenvalues of $\overset{\rightrightarrows}{I}$

Since $[I]$ is real and symmetric, it can be diagonalized. When you do this, the diagonal elements λ are eigenvalues. The eigenvalues are the *principal moments of inertia*.

To find the eigenvalues, we must solve:

$$|[I] - \lambda[\mathbf{1}]| = 0 \tag{16}$$

$[\mathbf{1}]$ is a 3x3 identity matrix. There are three λ_j and they will be real-valued. Usually these are listed from largest to smallest i.e.,

$$\lambda_1 = I_{max}, \lambda_2 = I_{med}, \lambda_3 = I_{min} \tag{17}$$

Principal Axes of a Body: eigenvectors of $[I]$

The directions associated with the eigenvalues are the eigenvectors. These three eigenvectors are the *principal axes*. They are mutually perpendicular. They represent the symmetry of the body.

Each λ_j that makes $|[I]-\lambda[\mathbf{1}]| = 0$ can be plugged back in to

$$[I - \lambda_j \mathbf{1}][v_j] = 0 \tag{18}$$

to solve for the vector $[v_j]$'s components. You can normalize each vector to get a unit vector by dividing by its length.

$$\widehat{e}_j = \frac{\vec{v}_j}{|\vec{v}_j|} \tag{19}$$

In Matlab, use "[v,d] = eig(I);" to solve for the eigenvalues and eigenvectors.

Transforming to the principal body axes frame

Suppose we started off in body coordinates B $= \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$ that did not make the inertial matrix diagonal. To find the transformation matrix ${}^{B}T^{\beta}$ that provides major axis orientation with axis set $\beta = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ we simply write the three orthonormal eigenvectors as columns of a matrix:

$$[{}^{B}T^{\beta}] = [\hat{e}_1|\hat{e}_2|\hat{e}_3] \quad (20)$$

Warning: make sure when arranging them col 1 $\times$ col 2 = col 3, or ${}^{B}T^{\beta}$ won't be a right-handed coordinate system!

Then we can write any vector $[d]_{/B}$ in the original frame as:

$$[d]_{\hat{b}} = [{}^{B}T^{\beta}][d]_{\hat{e}} \quad (21)$$

But we want the inverse: ${}^{\beta}T^{B}$, so we can do:

$$[d]_{\hat{e}} = [{}^{\beta}T^{B}][d]_{\hat{b}} \quad (22)$$

We're in luck because $[{}^{\beta}T^{B}]$ is just the transpose of $[{}^{B}T^{\beta}]$.

Transforming angular momentum to principal axes frame

In the original arbitrary $\widehat{b}$ coordinates we had

$$[H]_{\widehat{b}} = [I]_{\widehat{b}}[\omega]_{\widehat{b}} \tag{23}$$

This is difficult to analyze if it's not in the principal axis frame. So let's change it.

$$\underbrace{[{}^{\beta}T^{B}][H]_{\widehat{b}}}_{[H]_{\widehat{e}}} = [{}^{\beta}\underbrace{T^{B}][I]_{\widehat{b}}[{}^{B}T^{\beta}]}_{[I]_{\widehat{e}}}\,\underbrace{[{}^{\beta}T^{B}][\omega]_{\widehat{b}}}_{[\omega]_{\widehat{e}}} \tag{24}$$

The point: $[I]_{\widehat{e}}$ is diagonal. It's very helpful and we can always do it!

From now on we will assume principal axis body coordinates to allow a clearer understanding of the behavior of rotating bodies.

Parallel Axis Theorem

We showed how to compute the moment of inertia matrix about the center of mass C. Now suppose we change the origin of the body frame to some point P. We don't have to recompute all nine elements of the inertia matrix. Instead, we take advantage of the *parallel axis theorem*.

For the moments of inertia, this means:

$$\vec{\vec{I}}_{ii}^{B/P} = \vec{\vec{I}}_{ii}^{B/C} + m||R||^2 \tag{25}$$

where R is the perpendicular distance between the axes passing through C and those passing through P.