MMAE 411

Spacecraft Dynamics

Conic sections

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Outline

- Orbit equation
- Eccentricity
- Conic sections – mainly the ellipse
- Energy
- Summary of formulas

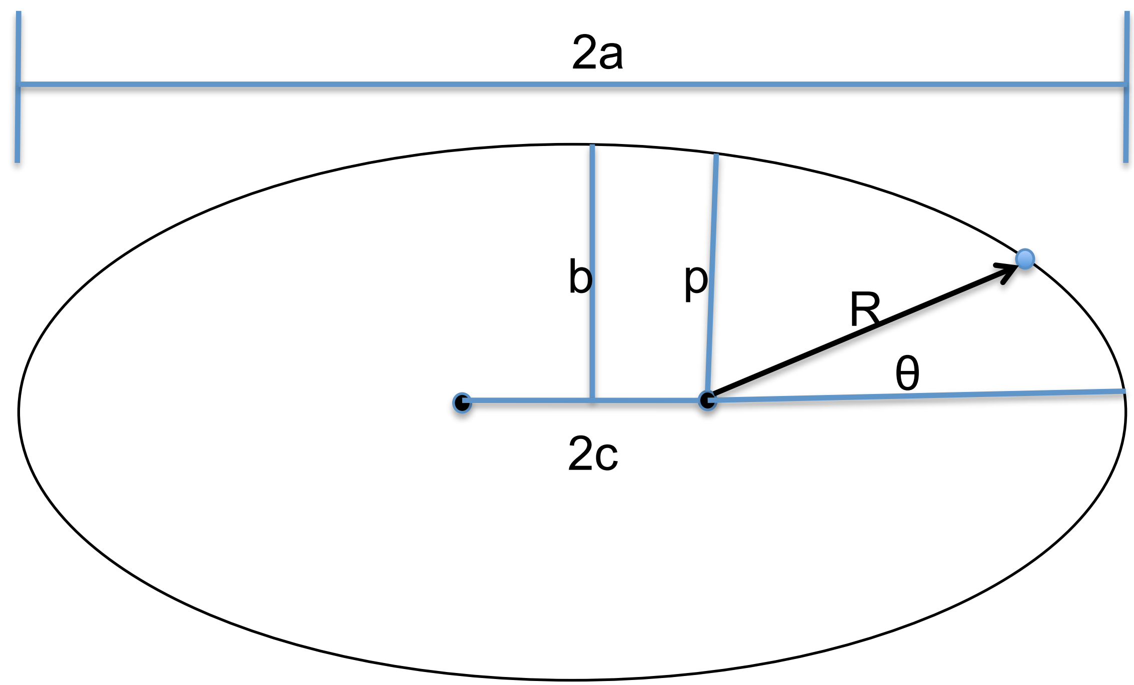
Orbit Equation

From the equation of motion for the two-body problem, $\ddot{\vec{r}} = \frac{\mu \vec{r}}{r^3}$, we derived the "orbit equation":

$$\boxed{r = \frac{\frac{h^2}{\mu}}{1 + e\cos\theta}}$$

This is the equation of an ellipse. It is in polar form, so (r, θ) are the coordinates. The origin is at one focus.

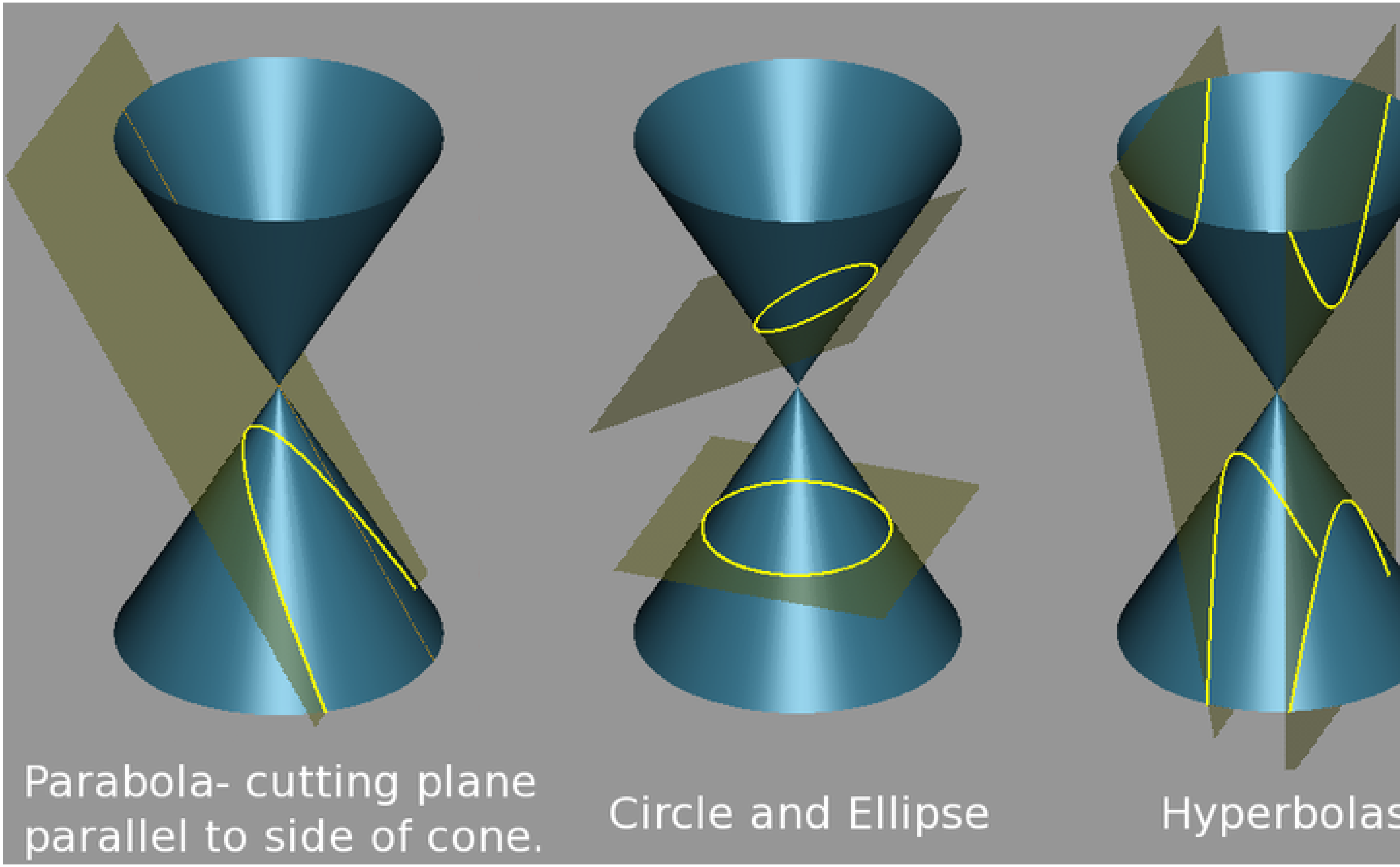
Ellipse



- a is the semi-major axis
- b is the semi-minor axis $b = \sqrt{ap}$
- c is half the distance between the foci $c = ae$
- p is the semi-latus rectum $p = \frac{h^2}{\mu}$

Conic sections

More generally, this is the equation of a conic section. A conic section is a form that can be created by slicing a plane through a circular cone.



(Courtesy: Wikimedia Commons)

Eccentricity

$\vec{e}$, the vector constant of integration, is the eccentricity vector. Its magnitude is the eccentricity e:

e	figure	a
0	circle	0
$0 < e < 1$	ellipse	$0 < a < \infty$
1	parabola	∞
$e > 1$	hyperbola	$a < 0$

The vector $\vec{e}$ points at the periapsis and has magnitude equal to eccentricity. Recall:

$$\dot{\vec{r}} \times \vec{h} \;=\; \mu\frac{\vec{r}}{r} + \mu\vec{e} \tag{1}$$

or

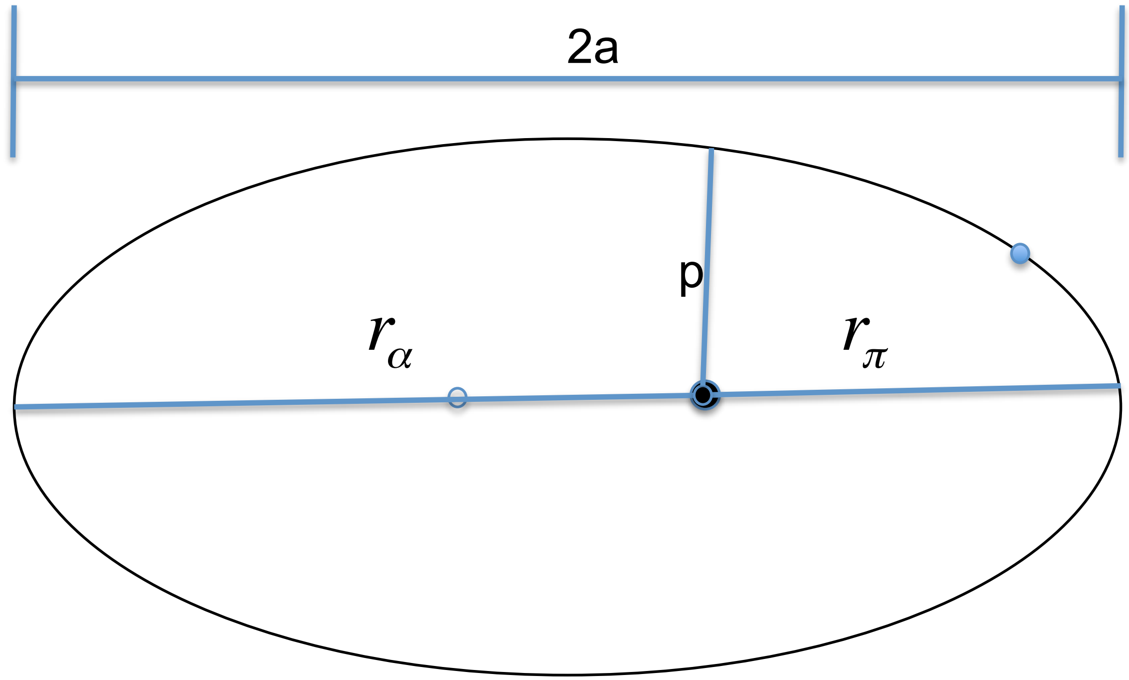
$$\boxed{\vec{e} = \frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r}}$$

Peri- and ap-

The closest point between the orbiting bodies occurs when $\cos\theta = 1, \theta = 0$. This is peri- (apsis/gee/lune) at distance r_π. The farthest distance between them is apo- (apsis/gee/lune), r_α.

figure	a	p
circle	$= r$	$= r$
ellipse	$r_\pi < a < r_\alpha$	$r_\pi < p < r_\alpha$
parabola	∞	$2r_\pi$
hyperbola	< 0	$> 2r_\pi$

Periapsis and apoapsis



From the equation $r = \frac{p}{1+e\cos\theta}$ we see that:

$$r_\pi = \frac{p}{1+e} \quad (2)$$

$$r_\alpha = \frac{p}{1-e} \quad (3)$$

where $p = h^2/\mu$. This means that p is solely determined by the angular momentum.

Computing p and e

$$
\begin{aligned}
2a &= r_\pi + r_\alpha && (4)\\
&= \frac{p}{1+e} + \frac{p}{1-e} && (5)\\
2a(1-e^2) &= 2p && (6)
\end{aligned}
$$

$$\boxed{p = a(1-e^2)}$$

Given that $c = ae$,

$$
\begin{aligned}
c &= ae && (7)\\
&= \frac{1}{2}(r_\alpha + r_\pi)e && (8)\\
&= r_\alpha - r_\pi && (9)
\end{aligned}
$$

$$\boxed{e = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi}}$$

Energy of the orbiting body, part 1

When talking about time derivatives in dynamics, we're talking about the rate of change in the inertial frame, so I will drop the superscript ^{,I}. I'll make the distinction when we're using multiple coordinate systems.

Beginning with the equation of motion, dot it with $\dot{\vec{r}}$:

$$\begin{aligned} \ddot{\vec{r}} &= -\frac{\mu \vec{r}}{r^3} && (10) \\ \dot{\vec{r}} \cdot \ddot{\vec{r}} &= -\frac{\mu \vec{r} \cdot \dot{\vec{r}}}{r^3} && (11) \end{aligned}$$

On the LHS, we use the trick of $\vec{r} \cdot \dot{\vec{r}} = r\dot{r}$ to get:

$$\begin{aligned} \dot{\vec{r}} \cdot \ddot{\vec{r}} &= \dot{r}\ddot{r} && (12) \\ \dot{\vec{r}} \cdot \ddot{\vec{r}} &= \frac{1}{2}\frac{d}{dt}(\dot{r}^2 + c_1) && (13) \end{aligned}$$

Energy of the orbiting body, part 2

Meanwhile, the RHS is, again using $\vec{r} \cdot \dot{\vec{r}} = r\dot{r}$:

$$
\begin{aligned}
-\frac{\mu \vec{r} \cdot \dot{\vec{r}}}{r^3} &= -\frac{\mu \dot{r}}{r^2} && (14) \\
&= \frac{d}{dt}\left(\frac{\mu}{r} + c_2\right) && (15)
\end{aligned}
$$

Putting the LHS and RHS together, we have:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\dot{r}^2 + c_1) &= \frac{d}{dt}\left(\frac{\mu}{r} + c_2\right) && (16) \\
\frac{d}{dt}\left(\frac{\dot{r}^2}{2} - \frac{\mu}{r} + c)\right) &= 0 && (17)
\end{aligned}
$$

Notice the "()" contains energy per unit mass.

$$
\begin{aligned}
\mathcal{E} &= \underbrace{\frac{\dot{r}^2}{2}}_{KE/unitmass} - \underbrace{\frac{\mu}{r}}_{PE/unitmass} + c && (18) \\
&= \text{constant} && (19)
\end{aligned}
$$

Potential energy zero reference level

When $\dot{r} = v = 0$, then $KE = 0$. In this situation $PE = -\frac{\mu}{r} + c$. The zero reference is defined by whatever constant c is.

We could use $c = \frac{\mu_\oplus}{R_\oplus} \Rightarrow$ PE is zero at Earth's surface. Instead we use $c = \frac{\mu_\oplus}{r=\infty} = 0$.

Finally the expression for energy per unit mass is:

$$\boxed{\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant.}}$$

Escaping gravity

Energy is constant; kinetic energy just gets traded for potential. So the speed at any point is just:

$$v = \sqrt{2\mathcal{E} + \frac{2\mu}{r}} \tag{20}$$

What minimum speed would the body need to have to just make it to $r = \infty$ but without any additional energy so $v(r = \infty) = 0$. The total energy would be:

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} \tag{21}$$
$$= 0 \tag{22}$$

This would be the same amount of energy as back when it is closer to the body, so:

$$\boxed{v_{esc} = \sqrt{\frac{2\mu}{r}}}$$

An important fact about energy

Consider the energy at periapsis. Use the fact that $h = r_\pi v_\pi$:

$$\mathcal{E} = \frac{v_\pi^2}{2} - \frac{\mu}{r_\pi} \qquad (23)$$

$$= \frac{h^2}{2r_\pi^2} - \frac{\mu}{r_\pi} \qquad (24)$$

For the conic section, $p = a(1-e^2)$ and $r_\pi = a(1-e)$. So:

$$\mathcal{E} = \frac{p\mu}{2r_\pi^2} - \frac{\mu}{r_\pi} \qquad (25)$$

$$= \frac{a(1-e^2)\mu}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)} \qquad (26)$$

$$= \frac{\mu}{2a}\left(\frac{1+e}{1-e} - \frac{2}{1-e}\right) = \frac{\mu}{2a}\left(\frac{e-1}{1-e}\right) \qquad (27)$$

$$\boxed{\mathcal{E} = -\frac{\mu}{2a}}$$

In other words, the energy is **only** a function of the semi-major axis!

Velocity in orbit

We can use the expression for energy to compute the velocity:

$$\begin{aligned} \mathcal{E} &= \frac{v^2}{2} - \frac{\mu}{r} && (28) \\ &= -\frac{\mu}{2a} && (29) \end{aligned}$$

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Using the above equation, you can compute the speed at any point on the orbit. If the orbit is circular, $r = a$ so $v = \sqrt{\frac{\mu}{a}}$.

Orbital Motion summary concepts

- The family of conics are the only possible paths.

- Focus is located at the central body

- Mechanical energy is constant $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$, and the semi-major axis a is determined solely by the energy.

- Orbital plane and conic orientation are fixed in inertial space.

- $\vec{h}$ and $\vec{e}$ are constant, and $\vec{h} \cdot \vec{e} = 0$ always.

- Angular momentum is a constant and directly determines p (semi-latus rectum).

- Period depends only on the energy (which also determines a).

Summary of Useful Formulas

$$\begin{aligned}
\vec{h} &= \vec{r} \times \vec{v} && (30)\\
&= rv\cos\gamma, \text{ where } \gamma \text{ is flight path angle} && (31)\\
p &= \frac{h^2}{\mu} && (32)\\
\frac{dA}{dt} &= \frac{|h|}{2}, \text{ Kepler's 2nd} && (33)\\
\vec{e} &= \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r} && (34)\\
\mathcal{P} &= \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} && (35)\\
\mathcal{E} &= \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} && (36)\\
v^2 &= \mu\left(\frac{2}{r} - \frac{1}{a}\right) && (37)\\
a &= \frac{\mu r}{2\mu - v^2 r} && (38)\\
e &= \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu^2}} && (39)
\end{aligned}$$

Useful formulas for an ellipse

$$e = \frac{c}{a} = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi} = 1 - \frac{r_\pi}{a} = \frac{r_\alpha}{a} - 1 = \frac{r_2 - r_1}{r_1 \cos\theta_1 - r_2 \cos\theta_2} \tag{40}$$

$$p = a(1 - e^2) = r_\pi(1 + e) \tag{41}$$

Useful formulas for velocity

- Escape velocity $v_{esc} = \sqrt{\frac{2\mu}{r}}$ at any point r

- Circular velocity $v_C = \sqrt{\frac{\mu}{r}}$