MMAE 411
Spacecraft Dynamics

Useful frames/coordinates; orbital elements

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Outline

- Some commonly used coordinate systems

- Canonical units

- Classical orbital elements

Commonly used coordinate systems

In order to usefully describe an orbit we need an appropriate inertial reference frame and coordinate system. The following are typical coordinate systems and examples:

- Heliocentric-ecliptic: for describing the orbits of solar system objects about the sun.

- Geocentric-equatorial: satellites of Earth.

- Right ascension-declination: for computing star position.

- Perifocal: for coordinates aligned naturally with a satellite's orbit.

Heliocentric-ecliptic system

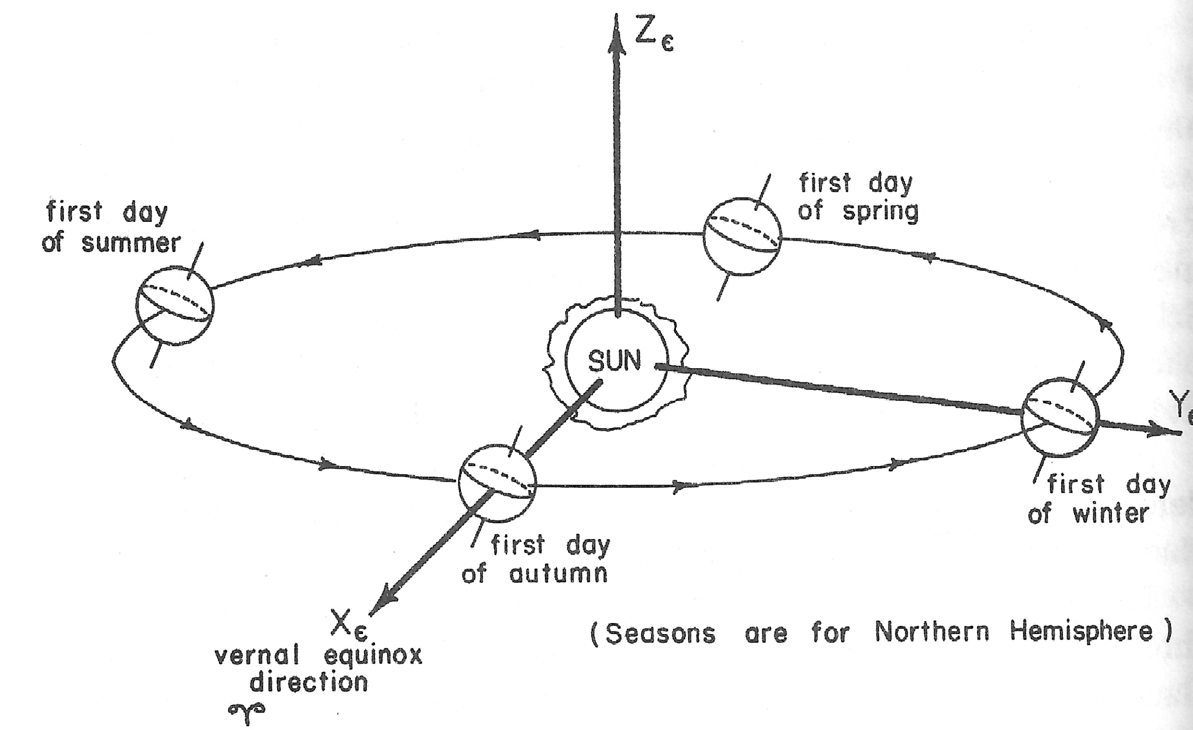

Figure 2.2-1 Heliocentric–ecliptic coordinate system

Geocentric-equatorial system

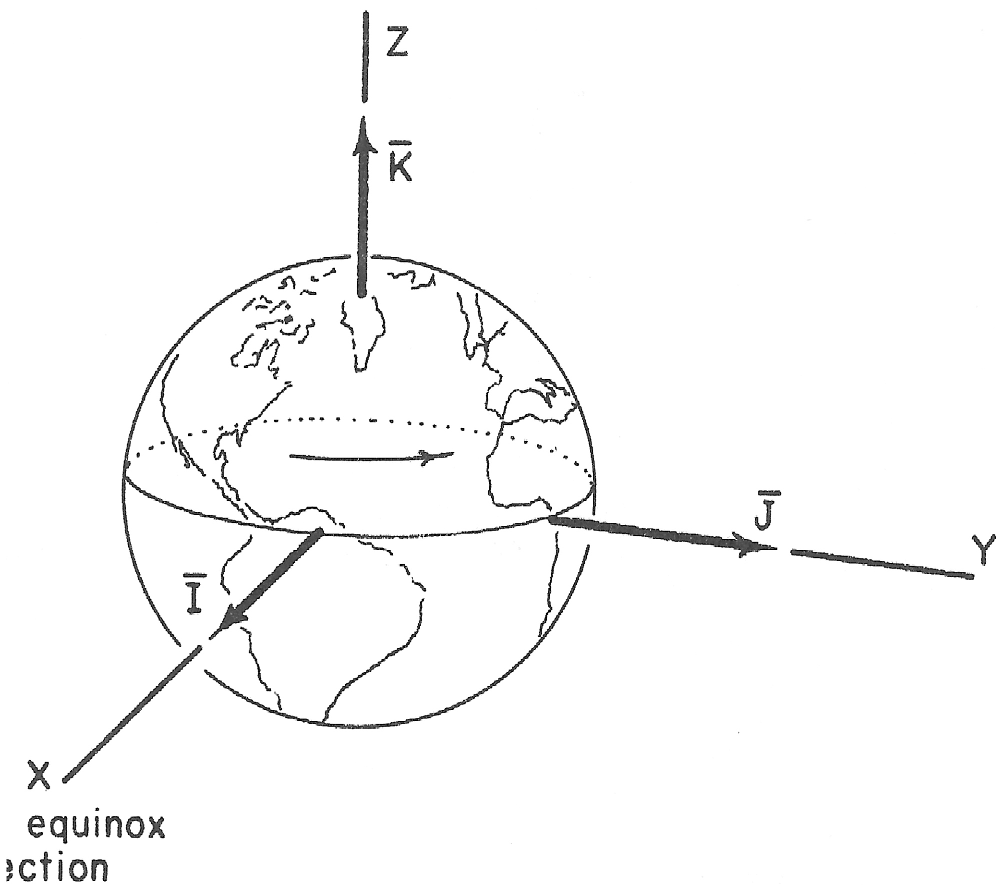

Figure 2.2-2 Geocentric-equatorial coordinate system

Right ascension-declination

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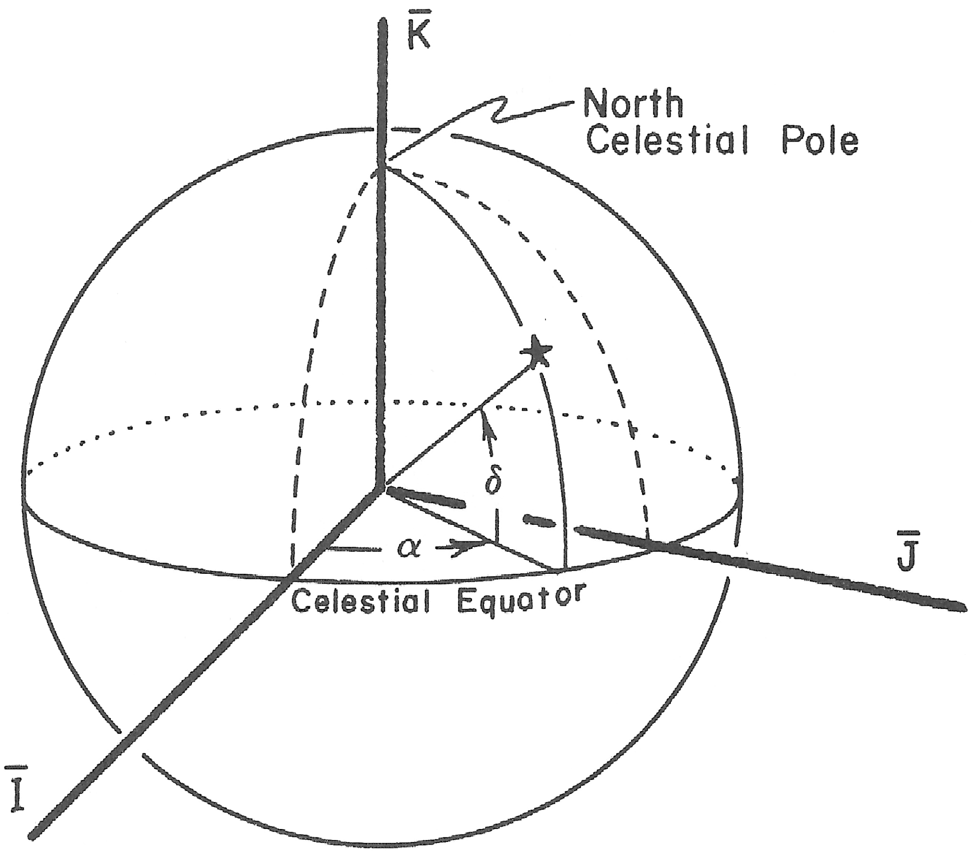


Figure 2.2-3 Right ascension—declination coordinate system

Perifocal

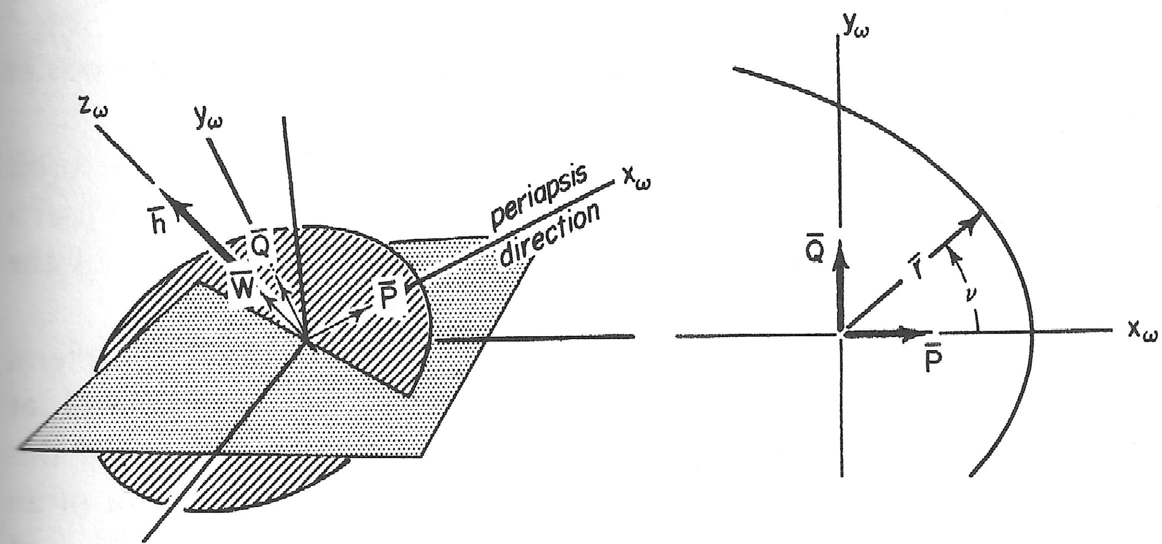


Figure 2.2-4 Perifocal coordinate system

Summary of astrodynamical coordinate systems

1. Heliocentric Ecliptic
   - X_E, Y_E in the plane of Earth's orbit.
   - X_E pointed at vernal equinox (known as the first point of Aries, the constellation ram).
2. Geocentric Equatorial
   - X_G, Y_G in the equatorial plane.
   - X_G pointed at vernal equinox (known as the first point of Aries, the constellation ram).
3. Right ascension, Declination
   - angles only (for objects a long distance away)
   - α is right ascension RA (celestial longitude).
   - δ is declination (celestial latitude).
4. Perifocal
   - Specified by $r = \frac{P}{1+e\cos\theta}$, so the coordinate system is unique to each satellite.
   - $\theta = \nu$.

Canonical Units concept

In orbit analysis, sometimes instead of SI or English units, we use different reference lengths and times called "canonical units". Using distance units (DU) and time units (TU), is helpful because:

- For certain gravitating bodies, we may not know the mean distances or mass well enough to specify μ_{body}.
- Numbers are often of order 1 rather than requiring exponential notation.
- They make the calculations simpler.

When we work in *canonical units*, all values are normalized so that instead of distance in km, which can be very large numbers, we can work with distance in distance units DU. Similarly time is not measured in seconds, but in time units TU.

Defining canonical units

With canonical units, the system of units for mass, length, and time is based on a hypothetical circular reference orbit.

- When the two-body system includes the sun, the reference orbit has a radius of 1 AU (1 astronomical unit = $1.4959965 \cdot 10^8$ km). For sun orbits, the canonical unit of distance is 1 AU.

- When the two-body system includes any other massive body as the central body, the reference orbit is the circular orbit that just grazes the surface (i.e., the radius of the body). So for Earth 1 DU = 1 $R_\oplus = 6378.145$ km.

Then instead of time in s, we work with time units TU. The time unit is chosen so that in the reference orbit, $v = 1 DU/TU$. Then by the energy equation, we find that $\mu = 1 DU^3/TU^2$.

Classical Orbital Elements

Five pieces of independent information are needed to specify the size, shape and orientation of an orbit. A sixth provides the position of the body at a given point in time. The classical (most commonly used) orbital elements are:

1. $a =$ semi-major axis
2. $e =$ eccentricity
3. $i =$ inclination
4. $\Omega =$ longitude of the ascending node
5. $\omega =$ argument of periapsis
6. $T =$ time of periapsis passage

The orbital elements are defined in the geocentric-equatorial (GE) coordinate system.

Computing the orbital elements

The orbital elements can be derived from a specification of position and velocity at a given instant. We have already seen two of them. First computing energy and angular momentum:

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} \tag{1}$$

$$\vec{h} = \vec{r} \times \vec{v} \tag{2}$$

The energy gives us the semi-major axis:

$$a = -\frac{\mu}{2\mathcal{E}} = \frac{p}{1-e^2} \tag{3}$$

We have seen how to compute the eccentricity vector as well:

$$\vec{e} = \frac{1}{\mu}\left(\vec{v} \times \vec{h} - \frac{\mu\vec{r}}{r}\right) \tag{4}$$

$$= \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r}\cdot\vec{v})\vec{v}\right] \tag{5}$$

Its magnitude is the orbital eccentricity.

Line of nodes and longitude of the ascending node Ω

The line of nodes is the line where the orbit slices through the GE system's x-y plane. This vector can be computed from the fact that it is perpendicular to the angular momentum (i.e., it lies in the orbit plane) and is perpendicular to the GE system z-axis.

The unit vector in the direction of the line of nodes is computed as:

$$\vec{n} = \frac{\vec{k} \times \vec{h}}{|\vec{k} \times \vec{h}|} \tag{6}$$

Since $\vec{n}$ lies in the equatorial plane (being perp to $\hat{k}$) and is unit length, it must have the form:

$$\vec{n} = \cos\Omega\hat{i} + \sin\Omega\hat{j} \tag{7}$$

So,

$$\Omega = \text{arccos}\left(\frac{n_{x,/GE}}{|n|}\right) \tag{8}$$

To determine which quadrant, notice that if $n_{y,GE} > 0, \Omega < 180^\circ$.

Inclination

The inclination is the angle at which the plane of the orbit is tilted with respect to the equatorial plane.

$$i = \text{arccos}\left(\frac{h_z}{h}\right) \tag{9}$$

There's no ambiguity because $i < 180^\circ$ always.

Argument of perigee ω

The argument of perigee is the angle between the node vector $\vec{n}$ and the eccentricity vector $\vec{e}$:

$$\omega = \text{arccos}\left(\frac{\vec{n} \cdot \vec{e}}{ne}\right) \qquad (10)$$

If $e_{z,/GE} < 0$ then it is in the southern hemisphere, so $\omega > 180^\circ$.

Time of periapsis passage T

The last element T comes from Kepler's equation:

$$M = \sqrt{\frac{\mu}{a^3}}(t_0 - T) \quad (11)$$

$$= E - e \sin E \quad (12)$$

To solve this for T, we need to know the eccentric anomaly E, which we found could be computed from:

$$\cos E = \frac{e + \cos\nu}{1 + e\cos\nu} \quad (13)$$

with E in the same quadrant as ν. The value ν is the true anomaly, the angle between $\vec{e}$ and $\vec{r}$ (which we had called θ before).

$$\nu = \arccos\frac{\vec{e}\cdot\vec{r}}{er} \quad (14)$$

Euler angles

Three of the orbital elements are angles: Ω, i, ω. Starting from the GE system, we can use these three to rotate into the perifocal system. A sequence of three simple rotations about different axes are often referred to as "Euler angles."*

The classical set of astrodynamic rotations is about the axes (3,1,3), although in other aspects of aerospace a (3,2,1) sequence of Euler angles is also common.

*Euler was the first to use these to describe the orientation of a rigid body. In general any orientation be achieved by a sequence of three simple rotations.

Rotation 1: around axis 3, goes from GE to AN

Longitude of Ascending Node Ω. Rotating about the z-axis of the GE system, measure Ω from the first point of Aries (x-axis in the GE system).

$$ {}^{AN}Q^{GE} = \begin{bmatrix} c\Omega & s\Omega & 0 \\ -s\Omega & c\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15} $$

This transforms from the GE frame to the “ascending node” (AN) frame, an intermediate frame.

Rotation 2: around axis 1, goes from AN to OP

Inclination angle i. Next tilt about the new x-axis by an amount i.

$$ {}^{OP}Q^{AN} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & ci & si \\ 0 & -si & ci \end{bmatrix} \tag{16} $$

This transforms from the AN frame to the "orbital plane" (OP) intermediate frame.

Rotation 3: around axis 3, goes from OP to PF

Argument of perigee ω. Rotate about the new z-axis by an amount ω to end up in the perifocal system.

$$
{}^{PF}Q^{OP} = \begin{bmatrix} c\omega & s\omega & 0 \\ -s\omega & c\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}
$$

This transforms from the OP frame to the final perifocal (PF) frame.

Transform to Perifocal system

Can transform $\vec{e}, \vec{r}, \vec{v}$ to the perifocal coordinates using:

$$ {}^{PF}Q^{GE} = [{}^{PF}Q^{OP}(\omega)][{}^{OP}Q^{AN}(i)][{}^{AN}Q^{GE}(\Omega)] \qquad (18) $$

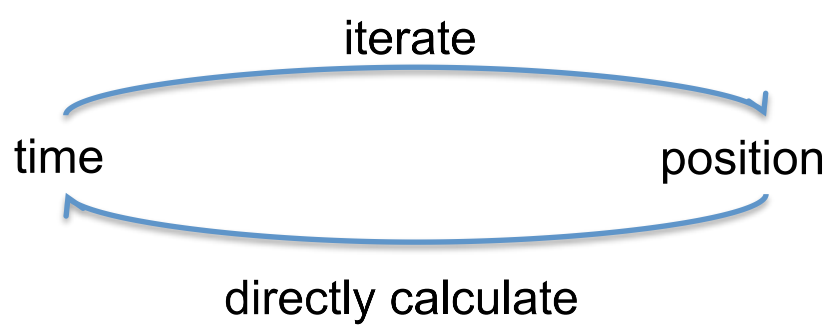
To transform from the geocentric equatorial system to the perifocal system, each angle is a positive rotation.

To transform from the perifocal system back to the geocentric equatorial system would require negative rotations in the reverse order (ω, i, Ω).

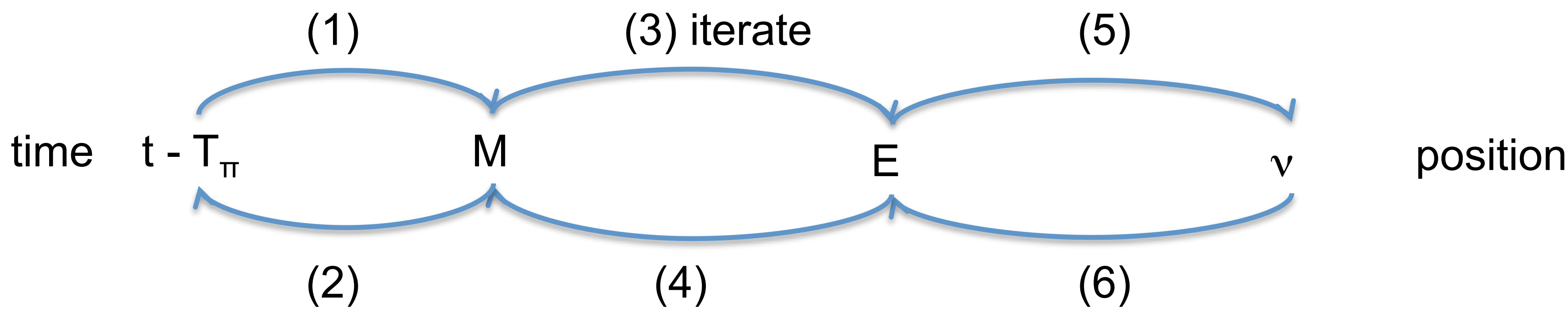
Computing position from orbital elements

We've shown how to compute orbital elements from $\vec{r}, \vec{v}$. To do it we use Kepler's equation to compute time of periapsis passage T directly.

Unfortunately, to compute $\vec{r}, \vec{v}$ from the orbital elements requires iteration. Kepler's equation cannot be solved directly.



The big idea



$$(1) M = (t - T_\pi) n \quad (3) E_{j+1} = E_j + \left. \frac{M - M_j}{\frac{dM}{dE}} \right|_{E=E_j} \quad (5) \cos\nu = \frac{e - \cos E}{e \cos E - 1} \tag{19}$$

$$(2) t - T_\pi = \frac{M}{n} \qquad (4) M = E - e \sin E \qquad (6) \cos E = \frac{e + \cos\nu}{1 + e\cos\nu}$$

where $n = \sqrt{\frac{\mu}{a^3}}$.

Computing $\vec{r}, \vec{v}$ from orbital elements

Once the true anomaly ν has been computed, the position and velocity are straightforward in the perifocal frame:

$$\vec{r} = r\cos\nu\widehat{p} + r\sin\nu\widehat{q} \tag{20}$$

Differentiate $\vec{r}$ to get the velocity. The perifocal system is inertial.

$$\vec{v} = (\dot{r}\cos\nu - r\dot{\nu}\sin\nu)\widehat{p} + (\dot{r}\sin\nu + r\dot{\nu}\cos\nu)\widehat{q} \tag{21}$$