MMAE 411

Spacecraft Dynamics

Lesson 10

Dr. Seebany Datta-Barua

Outline

- Center of Mass
- Translational EOM for Rigid Body
- Inertia Tensor
- Angular Momentum

Rigid body dynamics

Until now we have considered the objects we've dealt with to be a single point. In reality they are made of many point particles. When the distances between all the particles remains fixed, the object is a *rigid body*.

The state of a system and its motion are parameterized in terms of coordinates and velocities. The motion of a system is influenced by the forces acting on it and by the geometry of the system's mass. In other words, the manner in which mass is *distributed* throughout a system will affect the resulting motion.

Center of Mass

The center of mass $\vec{r}_c \equiv \vec{r}^{C/O}$ of a discrete system of N particles P_i with respect to some reference origin O is defined as

$$\boxed{\vec{r}_c = \frac{\sum_{i=1}^{N} m_i \vec{r}^{P_i/O}}{\sum_{i=1}^{N} m_i}}$$

At the center of mass point C, the weighted sum of the position vectors from this point to every one of the discrete points P_i is equal to zero:

$$\sum_{i=1}^{N} m_i \vec{r}^{P_i/C} = 0 \tag{1}$$

where m_i is the mass of the ith particle and $\vec{r}^{P_i/C}$ is the position vector from the point C to each particle P_i.

Center of Mass Intuition

To see that this is true, notice that we can write for the i^{th} point

$$\vec{r}_c = \vec{r}^{P_i/O} - \vec{r}^{P_i/C} \tag{2}$$

Multiplying each side by the mass of the i^{th} particle and summing over all particles we obtain

$$\sum_{i=1}^{N} m_i \vec{r}_c = \sum_{i=1}^{N} m_i \vec{r}^{P_i/O} - \sum_{i=1}^{N} m_i \vec{r}^{P_i/C} \tag{3}$$

Substituting in the equation for the center of mass,

$$\begin{aligned} \sum_{i=1}^{N} m_i \frac{\sum_{i=1}^{N} m_i \vec{r}^{P_i/O}}{\sum_{i=1}^{N} m_i} &= \sum_{i=1}^{N} m_i \vec{r}^{P_i/O} - \sum_{i=1}^{N} m_i \vec{r}^{P_i/C} && (4) \\ 0 &= \sum_{i=1}^{N} m_i \vec{r}^{P_i/C} && (5) \end{aligned}$$

It is the weighted average location of the particles in the system.

Continuous Systems

Generally, mass is not distributed in a discrete fashion, but rather is continuously distributed throughout a body. In this case we replace the summations by integrals. Instead of using the mass of individual particles we use the *density* $\rho = \rho(\vec{r})$ of the body to express the mass of a volume element dV and integrate over the volume V of the body

$$\boxed{\vec{r}_c = \frac{\int_V \rho\, \vec{r}\, dV}{\int_V \rho\, dV}}$$

Again $\vec{r}$ refers to the position vector of the volume element dV and $\vec{r}_c$ is the position vector of the mass center C. In the special case of a surface-shaped body (e.g., a body defined as a surface, like a plate) we sum over its area rather than its volume, using an area density. In the case of a line-shaped body (e.g., a wire) we sum over its length using a length density.

Degrees of freedom

Because the relative positions of all the particles in a rigid body stay fixed, it has only six degrees of freedom. To specify the system completely we only need six quantities, six equations of motion. Let's look at Newton's laws for the rigid body. Consider the i^{th} particle's EOM:

$$\begin{aligned}
\vec{F}_i &= \vec{F}_{ext,i} + \vec{F}_{int} && (6)\\
&= \vec{F}_{ext,i} + \sum_{j \neq i}^{N} \vec{f}_{ij} && (7)\\
&= m_i \ddot{\vec{r}}_i^{\,I} && (8)
\end{aligned}$$

3 translational EOM

If we add up the total force on all the i particles, we get:

$$\sum_{i=1}^{N} \vec{F}_{ext,i} + \sum_{i=1}^{N} \sum_{j \neq i}^{N} \vec{f}_{ij} = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i^{\,I} \tag{9}$$

$$\sum_{i=1}^{N} \vec{F}_{ext,i} = M \ddot{\vec{r}}_c^{\,I} \tag{10}$$

where M is the total mass of the system. The ij^{th} internal force is equal and opposite the ji^{th} internal force, so the internal forces all cancel in pairs. The sum of all the external forces acting on the rigid body cause its center of mass to accelerate.

Eq. (10) is a set of 3 equations, to allow us to solve for 3 of the 6 unknowns.

Inertia Tensor

Introduction

Given a *rigid* body B consisting of N particles $P_1, ...P_N$ of mass $m_1, ..., m_N$. Once the center of mass has been defined, computing the *linear momentum* ${}^I\vec{p}^B$ of B with respect to an inertial reference frame $\mathcal{I}$ is simple:

$$ {}^I\vec{p}^B = \sum_{i=1}^{N} m_i {}^I\vec{v}^{P_i} = M {}^I\vec{v}^C \tag{11} $$

where M is the total mass of B (i.e., $M = \sum_{i=1}^{N} m_i$) and ${}^I\vec{v}^C$ represents the velocity of the center of mass of B in the I frame.

The linear momentum of the system is equivalent to that of a single particle of mass M and velocity ${}^I\vec{v}^C$.

Angular momentum about origin $\mathcal{O}$

In a similar way, the *angular momentum* ${}^I\vec{H}^{B/P}$ of body B about a given point P is obtained by summing the angular momentum of each of its constituents:

$$ {}^I\vec{H}^{B/P} = \sum_{i=1}^{N} \vec{r}^{P_i/P} \times m_i {}^I\vec{v}^{P_i/P} \tag{12} $$

When the point P is chosen as the center of mass B_{cm}, the previous definition becomes

$$ {}^I\vec{H}^{B/B_{cm}} = \sum_{i=1}^{N} \vec{r}^{P_i/B_{cm}} \times m_i {}^I\vec{v}^{P_i} \tag{13} $$

From angular momentum to inertia

Let's consider the angular momentum of a rigid system $\mathcal{B}$ of N particles of mass $m_i\,(i = 1, ..., N)$ about the origin:

$$ {}^I\vec{H}^{B/B_{cm}} = \sum_{i=1}^{N} m_i(\vec{r}^{P_i} \times {}^I\vec{v}^{P_i}) \tag{14} $$

By noticing that the velocity of particle i can be expressed using the Golden Rule for vector differentiation:

$$ \begin{aligned} {}^I\vec{v}^{P_i} &= \frac{{}^I d\vec{r}^{P_i}}{dt} && (15) \\ &= \frac{{}^B d\vec{r}^{P_i}}{dt} + {}^I\omega^B \times \vec{r}^{P_i} && (16) \\ &= {}^I\omega^B \times \vec{r}^{P_i} && (17) \end{aligned} $$

we obtain the following expression for angular momentum ${}^I\vec{H}^{B/B_{cm}}$:

$$ {}^I\vec{H}^{B/B_{cm}} = \sum_{i=1}^{N} m_i(\vec{r}^{P_i} \times ({}^I\vec{\omega}^B \times \vec{r}^{P_i})) \tag{18} $$

Rigid body angular momentum

Attaching a reference frame and coordinate system to $\mathcal{B}$, we express the angular velocity of B in frame $\mathcal{I}$ and the position vector $\vec{r}^{P_i/B_{cm}}$:

$$
\begin{aligned}
{}^I\vec{\omega}^B &= \omega_1\hat{b}_1 + \omega_2\hat{b}_2 + \omega_3\hat{b}_3 && (19)\\
\vec{r}^{P_i/B_{cm}} &= x\hat{b}_1 + y\hat{b}_2 + z\hat{b}_3 && (20)
\end{aligned}
$$

Turning the summation into an integral for a continuous distribution of particles,

$$
\begin{aligned}
{}^I\vec{H}^{B/B_{cm}} &= \sum_{i=1}^{N} m_i(\vec{r}^{P_i} \times ({}^I\vec{\omega}^B \times \vec{r}^{P_i})) && (21)\\
&= \int \vec{r}^{P_i} \times ({}^I\vec{\omega}^B \times \vec{r}^{P_i})dm && (22)
\end{aligned}
$$

$$\vec{H} = \int \begin{bmatrix} (y^2+z^2)\omega_1 + (-xy)\omega_2 + (-xz)\omega_3 \\ -xy\omega_1 + (x^2+z^2)\omega_2 + (-yz)\omega_3 \\ -xz\omega_1 + (-yz)\omega_2 + (x^2+y^2)\omega_3 \end{bmatrix} dm \tag{23}$$

$$= \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} dm \tag{24}$$

$$= \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} \int(y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int(x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int(x^2+y^2)dm \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{26}$$

Inertia matrix

The 3x3 matrix of integrals can be written as:

$$ {}^{I}\vec{H}^{B/B_{cm}} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{27} $$

The diagonal elements are the moments of inertia and the off-diagonal elements are products of inertia.