MMAE 411

Spacecraft Dynamics

Non-spinning spacecraft in orbit

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Outline

- Non-spinning orbiting spacecraft
- Gravity gradient stabilization
- Pitch libration
- Roll-yaw libration

Recap

Euler's Equations describe the angular equation of motion of a body about its center of mass, as seen in an inertial frame, but using body-fixed coordinates:

$$\vec{M} \;=\; \vec{\vec{I}} \cdot {}^{I}\vec{\alpha}^{B} + {}^{I}\vec{\omega}^{B} \times \vec{\vec{I}}\,{}^{I}\vec{\omega}^{B} \tag{1}$$

We continue our analysis of Euler's Equations, for non-spinning orbiting spacecraft today.

Local Horizontal Frame

When a rigid body spacecraft orbits Earth, even if it is not spinning, the act of orbiting gives it angular velocity. We can see this by setting up a local horizontal frame A, in which $\widehat{a}_x$ points in-track (the direction of the orbit), $\widehat{a}_y$ points cross-track, and $\widehat{a}_z$ points down (always toward Earth).

This frame rotates with respect to inertial at a rate

$$ {}^{I}\vec{\omega}^{A} = -n\widehat{i}_y \tag{2} $$

The angular velocity vector in the body axes $\widehat{b}_{xyz}$ is then related to the Euler angle rates as before, but we separate this rotation as a dominant constant, compared to the Euler angle rates.

$$ {}^{I}\vec{\omega}^{B} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{3} $$

$$ = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + R_\phi R_\theta R_\psi \begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix} \tag{4} $$

EOM if LHA and body axes are not the same, cont'd

We can compute the last term as:

$$
\begin{aligned}
R_\phi R_\theta R_\psi \begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix} &= R_\phi R_\theta \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix} && (5) \\
&= R_\phi \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} -n\sin\psi \\ -n\cos\psi \\ 0 \end{bmatrix} && (6) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} -n\cos\theta\sin\psi \\ -n\cos\psi \\ -n\sin\theta\sin\psi \end{bmatrix} && (7) \\
&= -n \begin{bmatrix} \cos\theta\sin\psi \\ \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi \\ -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \end{bmatrix} && (8)
\end{aligned}
$$

Linearize EOM when A and B axes are not aligned

If $\phi, \theta, \psi << 1$ rad, then we can linearize the body angular velocity $[\omega_x, \omega_y, \omega_z]^T$:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} - n \begin{bmatrix} \cos\theta\sin\psi \\ \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi \\ -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \end{bmatrix}$$

$$\approx \begin{bmatrix} \dot\phi - \dot\psi\theta - n\psi \\ \dot\theta - \dot\psi\phi - n \\ -\dot\theta\phi + \dot\psi + n\phi \end{bmatrix}$$

If we consider that the average value of the nonlinear terms $\dot\psi\theta, \dot\psi\phi, \dot\theta\phi$ is small compared to the linear terms, or that $\dot\psi, \dot\theta << n$ then

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \approx \begin{bmatrix} \dot\phi - n\psi \\ \dot\theta - n \\ \dot\psi + n\phi \end{bmatrix} \tag{10}$$

EOM for an orbiting spacecraft

Use these angular velocity expressions in Euler's Equations:

$$\vec{M} = \frac{^I d}{dt}\vec{H} \tag{11}$$

$$= \begin{bmatrix} I_x\dot{\omega}_x \\ I_y\dot{\omega}_y \\ I_z\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} (I_z - I_y)\omega_y\omega_z \\ (I_x - I_z)\omega_x\omega_z \\ (I_y - I_x)\omega_x\omega_y \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} I_x\dot{\omega}_x \\ I_y\dot{\omega}_y \\ I_z\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} (I_z - I_y)(\dot{\theta} - n)(\dot{\psi} + n\phi) \\ (I_x - I_z)(\dot{\phi} - n\psi)(\dot{\psi} + n\phi) \\ (I_y - I_x)(\dot{\phi} - n\psi)(\dot{\theta} - n) \end{bmatrix} \tag{13}$$

EOM for an orbiting spacecraft, cont'd 1

In Eq. (13) there are products of the angular rates. Consider the first of these (ignoring the moments of inertia briefly):

$$(\dot{\theta} - n)(\dot{\psi} + n\phi) \;=\; (\dot{\theta}\dot{\psi} - n\dot{\psi} + n\dot{\theta}\phi - n^2\phi) \tag{14}$$

Any time we multiply two ϕ, θ, ψ or their derivatives together, the quantity is second order. In this linear approximation, we assume that the second order terms are small enough to be negligible. So this product becomes:

$$\begin{aligned}(\dot{\theta} - n)(\dot{\psi} + n\phi) &\approx -n\dot{\psi} - n^2\phi \qquad (15)\\ &= -n(\dot{\psi} + n\phi) \qquad (16)\\ &= -n\omega_z \qquad (17)\end{aligned}$$

With this same kind of approximation for all three directions, we have:

$$\vec{M} \;=\; \begin{bmatrix} I_x\dot{\omega}_x \\ I_y\dot{\omega}_y \\ I_z\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} (I_y - I_z)n\omega_z \\ 0 \\ -(I_y - I_x)n\omega_x \end{bmatrix} \tag{18}$$

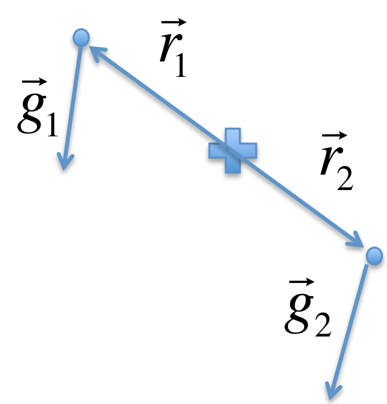
The dominant direction of spin remains a constant, and it couples the motion of the other two directions (just like a spinning spacecraft in deep space).

Gravity gradient moment

Now let's look at the external torques $\vec{M}$. In a configuration that is low altitude and low eccentricity, you can use the gradient (or difference) in gravity across a rigid body to stabilize its attitude. Example: a system of two masses at $\vec{r}_1$ and $\vec{r}_2 = -\vec{r}_1$ from the center of mass.

$$\vec{M}_g = \vec{r}_1 \times \vec{g}_1 + \vec{r}_2 \times \vec{g}_2 \tag{19}$$

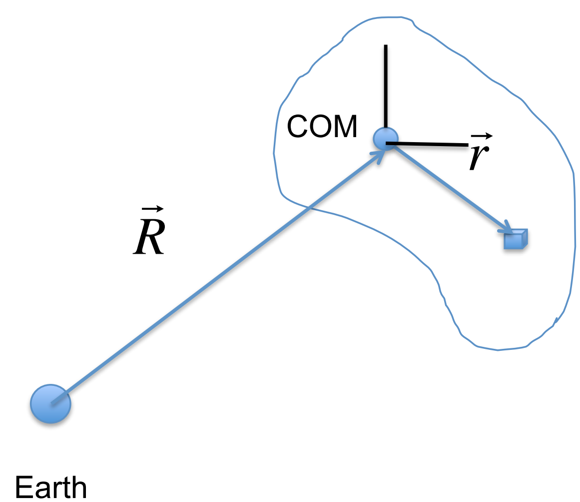
$$= \vec{r}_1 \times \underbrace{(\vec{g}_1 - \vec{g}_2)}_{\Delta \vec{g}} \tag{20}$$



General gravity gradient

Let's find a general expression for the gravity gradient torque. For a general rigid body:

$$\vec{M}_g = \int_{body} \vec{r} \times \left[\frac{-\mu_{\oplus}(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3} \right] dm \tag{21}$$



Gravity gradient expression

Let's simplify this by doing a binomial (aka Taylor) expansion of the denominator:

$$\frac{1}{(\vec{R}+\vec{r})^3} = \left[(\vec{R}+\vec{r})\cdot(\vec{R}+\vec{r})\right]^{-\frac{3}{2}} \tag{22}$$

$$= \frac{1}{R^3}\left[1+\frac{2\vec{R}\cdot\vec{r}}{R^2}+\frac{r^2}{R^2}\right]^{-\frac{3}{2}} \tag{23}$$

$$\approx \frac{1}{R^3}\left[1-\frac{3}{2}\cdot\frac{2\vec{R}\cdot\vec{r}}{R^2}+HOT\right] \tag{24}$$

Gravity gradient expression, cont'd

Plugging this simplified form of the denominator back into $\vec{M}_g$ we get:

$$\vec{M}_g \quad = \quad \frac{\mu_\oplus}{R^3}\int_{body} -\vec{r}\times(\vec{R}+\vec{r})\left[1-3\frac{\vec{r}\cdot\vec{R}}{R^2}\right]dm \tag{25}$$

Since $\vec{r}\times\vec{r}=0$ and $\int \vec{r}dm = 0$ because the origin is at the COM, this simplifies to

$$\vec{M}_g \quad = \quad -\frac{3\mu_\oplus}{R^5}\int_{body}(\vec{R}\times\vec{r})(\vec{r}\cdot\vec{R})dm \tag{26}$$

Now we do a trick of adding and subtracting $\vec{r}\cdot\vec{r}$ inside the integral. This will let us sub in the inertia dyadic:

$$\vec{M}_g \quad = \quad \frac{3\mu_\oplus}{R^5}[\vec{R}\times]\int_{body}(\underbrace{\vec{r}\cdot\vec{r}-\underbrace{\vec{r}\vec{r}}_{dyadic}}_{definition\,of\,\vec{\vec{I}}}-\vec{r}\cdot\vec{r})dm\cdot\vec{R} \tag{27}$$

The subtracted $\vec{r}\cdot\vec{r}$ is a scalar that is multiplied by $\vec{R}\times\vec{R}=0$ so finally

$$\boxed{\vec{M}_g = \frac{3\mu_\oplus}{R^5}\vec{R}\times\vec{\vec{I}}\cdot\vec{R}}$$

Coordinatizing gravity gradient torque

Using $n^2 = \frac{\mu_\oplus}{R^3}$, and $\widehat{r} \equiv \frac{\vec{R}}{R}$, torque $\vec{M}_g$ due to gravity:

$$\vec{M}_g = 3n^2 \widehat{r} \times (\vec{\vec{I}} \cdot \widehat{r}) \tag{28}$$

In body coordinates, for small angular perturbations from nominal,

$$\widehat{r} \approx \theta \widehat{b}_x - \phi \widehat{b}_y - \widehat{b}_z \tag{29}$$

$$\vec{\vec{I}} \cdot \widehat{r} = I_x \theta \widehat{b}_x - I_y \phi \widehat{b}_y - I_z \widehat{b}_z \tag{30}$$

$$\widehat{r} \times \vec{\vec{I}} \cdot \widehat{r} = \begin{vmatrix} \widehat{b}_x & \widehat{b}_y & \widehat{b}_z \\ \theta & -\phi & -1 \\ I_x \theta & -I_y \phi & -I_z \end{vmatrix} \tag{31}$$

$$\approx \begin{bmatrix} (I_z - I_y)\phi & (I_z - I_x)\theta & 0 \end{bmatrix}^T \tag{32}$$

$$\vec{M}_g = 3n^2 \begin{bmatrix} (I_z - I_y)\phi \\ (I_z - I_x)\theta \\ 0 \end{bmatrix} \tag{33}$$

Gravity gradient EOM for an orbiting spacecraft

Returning to our EOM with these expressions for the gravity gradient torque we have:

$$\vec{M} = \begin{bmatrix} 3n^2(I_z - I_y)\phi \\ 3n^2(I_z - I_x)\theta \\ 0 \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega}_x + n(I_y - I_z)\omega_z \\ I_y\dot{\omega}_y \\ I_z\dot{\omega}_z - n(I_y - I_x)\omega_x \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} - n\psi \\ \dot{\theta} - n \\ \dot{\psi} + n\phi \end{bmatrix} \tag{35}$$

The equations for the $\hat{\jmath}$ direction only involve the θ coordinate and ω_y, so these are uncoupled from the others.

EOM for an Earth-pointing spacecraft

The uncoupled equations are the *pitch set* of EOM:

$$
\begin{aligned}
I_y\dot{\omega}_y &\approx 3n^2(I_z - I_x)\theta + M_y && (36)\\
\omega_y &\approx \dot{\theta} - n && (37)
\end{aligned}
$$

The remaining 4 equations (two EOM plus two definitions of angular velocity that let us split our second-order ODE into pairs of first-order ODEs) are the *roll-yaw set* of EOM:

$$
\begin{aligned}
I_x\dot{\omega}_x + n(I_y - I_z)\omega_z &\approx M_x + 3n^2(I_z - I_y)\phi && (38)\\
I_z\dot{\omega}_z - n(I_y - I_x)\omega_x &\approx M_z && (39)\\
\omega_x &\approx \dot{\phi} - n\psi && (40)\\
\omega_z &\approx \dot{\psi} + n\phi && (41)
\end{aligned}
$$

Pitch Librations

By substituting in for $\omega_x, \omega_y, \omega_z$ we can turn the pitch EOM back into a second-order ODE and look at its characteristic equation (telling us where the poles of the transfer function are).

$$\begin{aligned}
\omega_y &= \dot{\theta} - n & (42)\\
\dot{\omega}_y &= \ddot{\theta} & (43)\\
I_y\ddot{\theta} &= 3n^2(I_z - I_x)\theta & (44)\\
\mathcal{L}\{I_y\ddot{\theta}\} &= \mathcal{L}\{3n^2(I_z - I_x)\theta\} & (45)\\
I_y s^2\Theta &= 3n^2(I_z - I_x)\Theta & (46)\\
(s^2 + 3n^2\frac{(I_x - I_z)}{I_y})\Theta &= 0 & (47)\\
\Rightarrow s^2 + 3n^2\frac{(I_x - I_z)}{I_y} &= 0 & (48)
\end{aligned}$$

Pitch Librations, cont'd

This is in the form of $s^2 + \omega_n^2 = 0$.

If $I_x > I_z$ then the system has undamped oscillations (*librations*) at the pitch libration frequency:

$$\omega_n = n\sqrt{\frac{3(I_x - I_z)}{I_y}} \tag{49}$$

Note that ω_n is not part of the angular velocity of the body, but the natural frequency at which the pitch angle will "nod" up and down.

If $I_x < I_z$ the system is unstable in pitch.

Roll-yaw Librations

We can eliminate ω_x, ω_z from the roll-yaw equations and take the Laplace transform to find its characteristic equation:

$$\begin{aligned} I_x\dot{\omega}_x + n(I_y - I_z)\omega_z &= 3n^2(I_z - I_y)\phi & (50) \\ I_z\dot{\omega}_z - n(I_y - I_x)\omega_x &= 0 & (51) \end{aligned}$$

To substitute, we use:

$$\begin{aligned} \omega_x &= \dot{\phi} - n\psi & (52) \\ \omega_z &= \dot{\psi} + n\phi & (53) \\ \dot{\omega}_x &= \ddot{\phi} - n\dot{\psi} & (54) \\ \dot{\omega}_z &= \ddot{\psi} + n\dot{\phi} & (55) \end{aligned}$$

Finally we have two coupled second-order ODEs:

$$\begin{aligned} I_x(\ddot{\phi} - n\dot{\psi}) + n(I_y - I_z)(\dot{\psi} + n\phi) &= 0 + 3n^2(I_z - I_y)\phi & (56) \\ I_z(\ddot{\psi} + n\dot{\phi}) - n(I_y - I_x)(\dot{\phi} - n\psi) &= 0 & (57) \end{aligned}$$

Laplace transform of roll-yaw EOM

Take the Laplace transform of the first EOM:

$$\begin{aligned}\mathcal{L}\{I_x(\ddot{\phi} - n\dot{\psi}) + n(I_y - I_z)(\dot{\psi} + n\phi) &= 3n^2(I_z - I_y)\phi\} \\ I_x(s^2\Phi - ns\Psi) + n(I_y - I_z)(s\Psi + n\Phi) &= 3n^2(I_z - I_y)\Phi\end{aligned}$$

Gather like terms in $\Psi(s)$ and $\Phi(s)$ together:

$$(I_x s^2 + 4n^2(I_y - I_z))\Phi + ((I_y - I_z) - I_x)ns\Psi = 0$$

Define a constant $a = (I_y - I_z)/I_x$. Then we have:

$$\begin{aligned}(s^2 + 4an^2)\Phi + (a - 1)ns\Psi &= 0 \\ \mathcal{L}\{I_z(\ddot{\psi} + n\dot{\phi}) &= 0\}\end{aligned} \tag{58}$$

By a similar process for the second EOM, we can define $b = (I_y - I_x)/I_z$ and get:

$$(s^2 + bn^2)\Psi - (b - 1)ns\Phi = 0 \tag{59}$$

Characteristic equation of roll-yaw EOM

We can write the coupled roll-yaw Laplace-transformed EOM as a matrix:

$$\begin{bmatrix} s^2 + 4an^2 & (a-1)ns \\ -(b-1)ns & s^2 + bn^2 \end{bmatrix} \begin{bmatrix} \Phi(s) \\ \Psi(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{60}$$

The characteristic equation of the system is the determinant of the matrix above:

$$\Delta(s) = \left(\frac{s}{n}\right)^4 + (3a + ab + 1)\left(\frac{s}{n}\right)^2 + 4ab = 0 \tag{61}$$

Based on the definitions of moments of inertia I_x, I_y, I_z, it turns out that $a, b \leqslant 1$.

Quadratic equation

The characteristic equation is a quadratic equation if we recognize x as $(s/n)^2$:

$$\begin{aligned} x^2 + Bx + C &= 0 & (62) \\ B &= 3a + ab + 1 & (63) \\ C &= 4ab & (64) \end{aligned}$$

The roots are:

$$\begin{aligned} x &= -\frac{B}{2} \pm \frac{\sqrt{B^2 - 4C}}{2} & (65) \\ &= -\frac{3a + ab + 1}{2} \pm \frac{\sqrt{(3a + ab + 1)^2 - 4(4ab)}}{2} & (66) \\ &= \left(\frac{s}{n}\right)^2 & (67) \end{aligned}$$

Requirements on a, b for imaginary roots (stability)

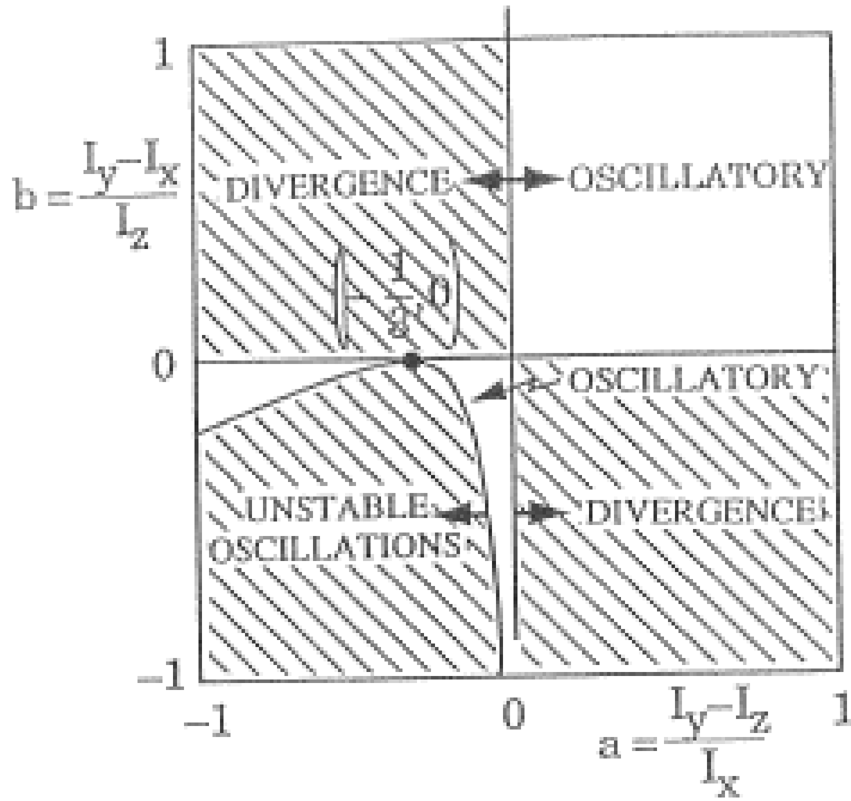
To get (s/n) to be imaginary, $(s/n)^2$ must be negative. So x must be negative and real. To get x to be negative and real, we need the discriminant to be:

$$
\begin{aligned}
B^2 - 4C &> 0 && (68)\\
3a + ab + 1 &> 4\sqrt{ab} && (69)
\end{aligned}
$$

To get the whole quantity to then be negative, we need the first fraction to be larger than the second, i.e.,

$$
\begin{aligned}
B^2 &> B^2 - 4C && (70)\\
C &> 0 && (71)\\
ab &> 0 && (72)
\end{aligned}
$$

Roll-yaw stability chart



For $a, b > 0$ and a small region where $a, b < 0$, the system is oscillatory. Otherwise, it is unstable in roll-yaw.

Summary conditions for stability of non-spinning orbiting spacecraft

Pitch librations of frequency $\omega_n = n\sqrt{3(I_x - I_z)/I_y}$ if

$$I_x > I_z \tag{73}$$

otherwise unstable.

Roll-yaw librations if either of the following is true:

$$3a + ab + 1 > 4\sqrt{ab} \tag{74}$$

$$\text{OR}$$

$$ab > 0 \tag{75}$$