MMAE 411
Spacecraft Dynamics
Coordinate transformations/Rotation matrices

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One vector, two coordinate systems

Suppose we have two coordinate systems (right-handed orthogonal unit vector bases). We can use either system to describe a vector.

In system A consisting of $\widehat{a}_x, \widehat{a}_y, \widehat{a}_z$:

$$\vec{r}_{/A} = r_{ax}\widehat{a}_x + r_{ay}\widehat{a}_y + r_{az}\widehat{a}_z \quad (1)$$

In system B, with $\widehat{b}_x, \widehat{b}_y, \widehat{b}_z$:

$$\vec{r}_{/B} = r_{bx}\widehat{b}_x + r_{by}\widehat{b}_y + r_{bz}\widehat{b}_z \quad (2)$$

Note: Wiesel uses a superscript letter after the vector to write this $\mathbf{r}^{\mathbf{A}}$, e.g. in Eq. 1.20.

Coordinate transform

Dot each vector $\vec{r}$ with $\widehat{a}_x$:

$$\begin{aligned}
\vec{r} \cdot \widehat{a}_x &= r_{ax}(\widehat{a}_x \cdot \widehat{a}_x) + r_{ay}\cancelto{0}{(\widehat{a}_y \cdot \widehat{a}_x)} + r_{az}\cancelto{0}{(\widehat{a}_z \cdot \widehat{a}_x)} && (3) \\
&= r_{ax} && (4) \\
\vec{r} \cdot \widehat{a}_x &= r_{bx}(\widehat{b}_x \cdot \widehat{a}_x) + r_{by}(\widehat{b}_y \cdot \widehat{a}_x) + r_{bz}(\widehat{b}_z \cdot \widehat{a}_x) && (5) \\
r_{ax} &= r_{bx}(\widehat{b}_x \cdot \widehat{a}_x) + r_{by}(\widehat{b}_y \cdot \widehat{a}_x) + r_{bz}(\widehat{b}_z \cdot \widehat{a}_x) && (6)
\end{aligned}$$

Same for the other two components:

$$\begin{aligned}
r_{ay} &= r_{bx}(\widehat{b}_x \cdot \widehat{a}_y) + r_{by}(\widehat{b}_y \cdot \widehat{a}_y) + r_{bz}(\widehat{b}_z \cdot \widehat{a}_y) && (7) \\
r_{az} &= r_{bx}(\widehat{b}_x \cdot \widehat{a}_z) + r_{by}(\widehat{b}_y \cdot \widehat{a}_z) + r_{bz}(\widehat{b}_z \cdot \widehat{a}_z) && (8)
\end{aligned}$$

$\vec{r}$ transformed

Now express in matrix form $\Rightarrow$

$$\vec{r}_{/A} = \begin{bmatrix} r_{ax} \\ r_{ay} \\ r_{az} \end{bmatrix} = \begin{bmatrix} r_{bx}\widehat{a}_x \cdot \widehat{b}_x + r_{by}\widehat{a}_x \cdot \widehat{b}_y + r_{bz}\widehat{a}_x \cdot \widehat{b}_z \\ r_{bx}\widehat{a}_y \cdot \widehat{b}_x + r_{by}\widehat{a}_y \cdot \widehat{b}_y + r_{bz}\widehat{a}_y \cdot \widehat{b}_z \\ r_{bx}\widehat{a}_z \cdot \widehat{b}_x + r_{by}\widehat{a}_z \cdot \widehat{b}_y + r_{bz}\widehat{a}_z \cdot \widehat{b}_z \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} \widehat{a}_x \cdot \widehat{b}_x & \widehat{a}_x \cdot \widehat{b}_y & \widehat{a}_x \cdot \widehat{b}_z \\ \widehat{a}_y \cdot \widehat{b}_x & \widehat{a}_y \cdot \widehat{b}_y & \widehat{a}_y \cdot \widehat{b}_z \\ \widehat{a}_z \cdot \widehat{b}_x & \widehat{a}_z \cdot \widehat{b}_y & \widehat{a}_z \cdot \widehat{b}_z \end{bmatrix} \begin{bmatrix} r_{bx} \\ r_{by} \\ r_{bz} \end{bmatrix} \tag{10}$$

$$\mathbf{r}_{/A} = {}^{A}\mathbf{R}^{B}\mathbf{r}_{/B} \tag{11}$$

Note: Wiesel writes this with the superscript order flipped so beware.

$$\mathbf{r}^{A} = \mathbf{R}^{BA}\mathbf{r}^{B} \tag{12}$$

Direction cosine

Each element in the 3x3 matrix is a "direction cosine", the dot product

$$\hat{a}_i \cdot \hat{b}_j = \cancelto{1}{||\hat{a}_i||}\,\cancelto{1}{||\hat{b}_j||} \cos \angle(\hat{a}_i, \hat{b}_j) \quad (13)$$

Since they are all unit vectors, the elements end up being simply a cos (or a sin using simplifying trig). The whole 3x3 matrix is called a *rotation matrix* or *coordinate transformation matrix*.

Inverse transformation matrix

If we had dot multiplied by $\widehat{b}_x, \widehat{b}_y, \widehat{b}_z$ instead, we would instead have:

$$\begin{bmatrix} r_{bx} \\ r_{by} \\ r_{bz} \end{bmatrix} = \left({}^{A}\mathbf{R}^{B}\right)^{T} \begin{bmatrix} r_{ax} \\ r_{ay} \\ r_{az} \end{bmatrix} \tag{14}$$

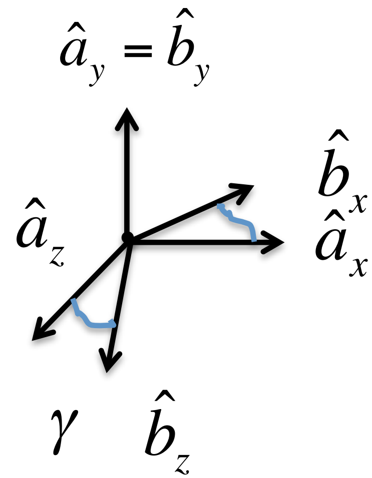
(Check this on your own.)

$\therefore \left({}^{A}\mathbf{R}^{B}\right)^{T} = \left({}^{A}\mathbf{R}^{B}\right)^{-1} = {}^{B}\mathbf{R}^{A}$

Matrices whose transpose is the inverse are called *orthogonal matrices*.

A matrix is orthogonal if component column (or row) vectors are $\perp$ to each other.

Simple rotations



A *simple rotation* is one in which one axis is shared (or stays fixed). In this example the y-axis is shared. So to build ${}^B\mathbf{R}_y^A$, compute the direction cosines

$$\hat{a}_x \cdot \hat{b}_x = \cancelto{1}{||\hat{a}_x||}\cancelto{1}{||\hat{b}_x||} \cos\gamma \tag{15}$$

$$\hat{a}_x \cdot \hat{b}_y = 0 \tag{16}$$

$$\hat{a}_x \cdot \hat{b}_z = \cos(90^\circ + \gamma) = +\sin\gamma \tag{17}$$

Also,

$$\begin{array}{rclrcl} \widehat{a}_y \cdot \widehat{b}_x & = & 0, & \widehat{a}_z \cdot \widehat{b}_x & = & -\sin\gamma \\ \widehat{a}_y \cdot \widehat{b}_y & = & 1, & \widehat{a}_z \cdot \widehat{b}_y & = & 0 \\ \widehat{a}_y \cdot \widehat{b}_z & = & 0, & \widehat{a}_z \cdot \widehat{b}_z & = & \cos\gamma \end{array} \tag{18}$$

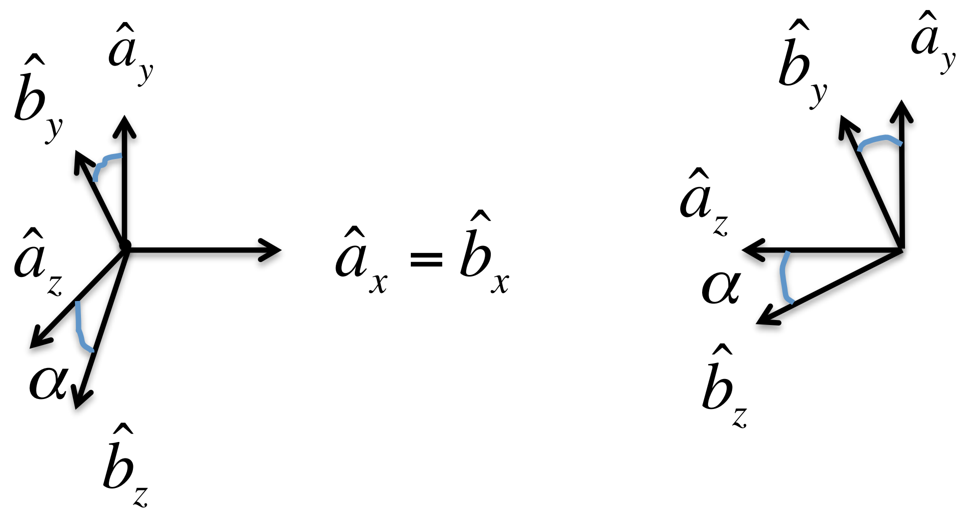
Simple rotation and its inverse

After computing the direction cosines, to build ${}^B\mathbf{R}_y^A$, list one $\widehat{b}_i$ for each row, and one $\widehat{a}_j$ for each column to determine which direction cosine goes in which element:

$$
{}^B\mathbf{R}^A \;=\; \begin{array}{c c} & \begin{array}{ccc} \widehat{a}_x & \widehat{a}_y & \widehat{a}_z \end{array} \\ \begin{array}{c} \widehat{b}_x \\ \widehat{b}_y \\ \widehat{b}_z \end{array} & \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \end{array} \tag{19}
$$

Other simple rotation matrices

When the x-axis is the common axis, we have:

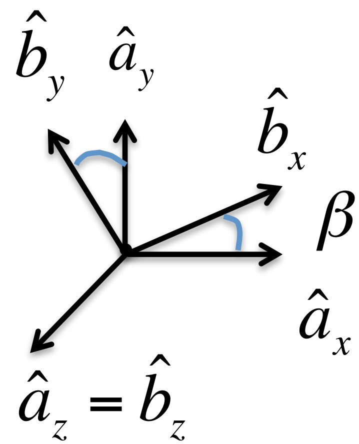


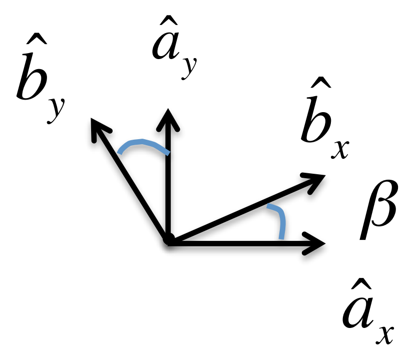
where the figure on the left shows a 3-D view, and the figure on the right shows a view looking along the $\widehat{a}_x = \widehat{b}_x$ axis, which is out of the page.

To build ${}^B\mathbf{R}^A_x$ here, again list one $\widehat{b}_i$ for each row, and one $\widehat{a}_j$ for each column to determine which direction cosine goes in which element:

$$
{}^{B}\mathbf{R}_x^{A} = \begin{array}{c} \\ \widehat{b}_x \\ \widehat{b}_y \\ \widehat{b}_z \end{array} \begin{array}{c} \begin{array}{ccc} \widehat{a}_x & \widehat{a}_y & \widehat{a}_z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \end{array} \tag{20}
$$

For a shared z-axis, we get:

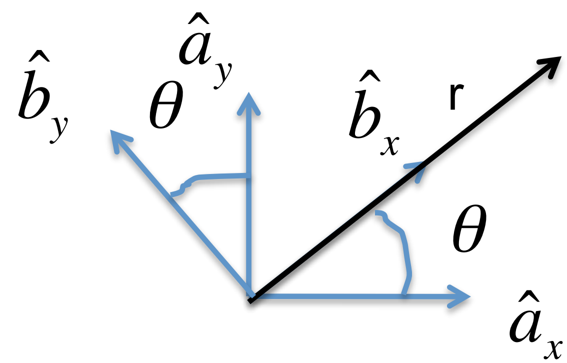




$$
{}^{B}\mathbf{R}_{z}^{A} \;=\; \begin{array}{c} \\ \widehat{b}_x \\ \widehat{b}_y \\ \widehat{b}_z \end{array} \begin{array}{c} \begin{array}{ccc} \;\;\widehat{a}_x\;\; & \;\;\widehat{a}_y\;\; & \;\;\widehat{a}_z \end{array} \\ \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{21}
$$

Interpretation #1: coordinate transform

When we use a rotation matrix to express a vector in different coordinate systems,

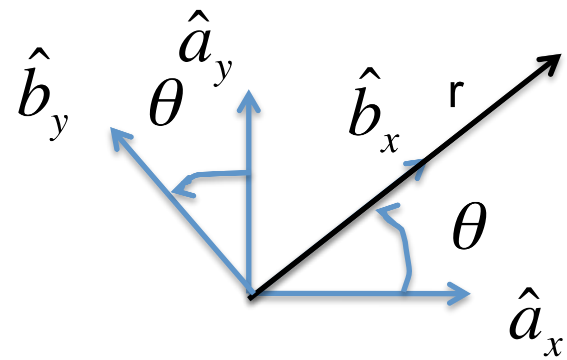


the rotation is "passive". (We're not physically rotating anything). To form the matrix, we project one basis onto the other using the direction cosines, as shown in previous slides. The passive interpretation is useful for thinking about different reference frames viewing the same vector. (Homework 1 will have you practice this.)

In this figure's example,

$$
\begin{aligned}
r\hat{b}_x &= r\cos\theta\hat{a}_x + r\sin\theta\hat{a}_y && (22)\\
\mathbf{r}_{/B} &= {}^{B}\mathbf{R}_z^{A}\mathbf{r}_{/A} && (23)\\
\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{bmatrix} && (24)
\end{aligned}
$$

Interpretation #2: Rotation of the coordinate system



Another point of view of this transformation is that the basis vectors $\widehat{a}_{xyz}$ are being rotated through an angle θ to *become* the $\widehat{b}_{xyz}$. The matrix describing this process is also ${}^B\mathbf{R}_z^A$. This interpretation, as a rotation of the coordinate system, is one that we will use later when talking about Euler angles, a sequence of rotations that describe the attitude of a rigid body.

Summary

Rotation matrices are useful in different ways in this class. The interpretations of what they do are consistent:

- as a coordinate transform, the matrix ${}^{B}\mathbf{R}^{A}$ multiples a vector given in A coordinates to express it in B coordinates.

- a rotation ${}^{B}\mathbf{R}^{A}$ of the *coordinates* through angle θ converts the A coordinates into the B coordinates.

Because of the way matrices operate, the superscripts in these lecture notes use the convention ${}^{end}\mathbf{R}^{start}$.