# MMAE 411

# Spacecraft Dynamics

## Euler's Equations and Euler Angles

Dr. Seebany Datta-Barua

# Outline

- Natural motions: Euler's Equations

- Euler angles

- Angular velocity in terms of Euler angles

# Recap

In general the inertia tensor $\vec{\vec{I}}$ of a body $B$ with respect to a point changes both the *orientation* and *magnitude* of the vectors on which it is applied, i.e., given a vector $\vec{\omega} \in \mathbb{R}^3$ and $\vec{H}$ defined as

$$\vec{H} = \vec{\vec{I}}\vec{\omega} \tag{1}$$

then typically $\vec{H}$ is not parallel to $\vec{\omega}$. However, we can always diagonalize $\vec{\vec{I}}$ by finding its eigenvalues and eigenvectors. These eigenvectors define the *principal axes* of the body.

From now on, we assume that we are working in the principal body axes coordinate system, so we can analyze the dynamics today.

# 3 Angular Eqs. of Motion

We know that external torques $i$ cause a change in the angular momentum of a body, as seen in an inertial reference frame, $I$:

$$\begin{aligned}
\vec{M} &= \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i && (2) \\
&= \frac{d^I}{dt}\vec{H} && (3) \\
&= \frac{d^I}{dt}(\vec{\vec{I}}\vec{\omega}) && (4) \\
&= {}^I\dot{\vec{\vec{I}}}{}^B\vec{\omega} + \vec{\vec{I}}{}^I\dot{\vec{\omega}}{}^B && (5)
\end{aligned}$$

Unfortunately, expressing $\vec{\vec{I}}$ in an inertial coordinate system would generally mean nonzero-off-diagonal terms. And then to get ${}^I\dot{\vec{\vec{I}}}{}^B$ would mean taking time derivatives of 9 elements, so it would be messy.

# Angular EOM expressed in body coordinates

Let's try to express the angular EOM again:

$$
\begin{aligned}
\vec{M} &= \frac{d^I}{dt}\vec{H} && (6)\\
&= \frac{d^B}{dt}\vec{H} + {}^I\vec{\omega}^B \times \vec{H} && (7)\\
&= \left(\frac{d^B}{dt}\vec{\vec{I}}\right)\vec{\omega} + \vec{\vec{I}}\left(\frac{d^B}{dt}\vec{\omega}\right) + {}^I\vec{\omega}^B \times \vec{\vec{I}}\,{}^I\vec{\omega}^B && (8)
\end{aligned}
$$

For a rigid body, $\frac{d^B}{dt}\vec{\vec{I}} = 0$ because the coordinates of $B$ are fixed to the body. Meanwhile, the time rate of change of angular velocity $\vec{\omega}$ (dropping the superscript "I" and "B" now) is the same no matter which frame you use:

$$
\frac{d^I}{dt}\vec{\omega} = \frac{d^B}{dt}\vec{\omega} + \cancelto{0}{\vec{\omega} \times \vec{\omega}} = \vec{\alpha} \qquad (9)
$$

# Euler's Equations

So, the angular EOM become

$$\boxed{\vec{M} = \vec{\vec{I}}\vec{\alpha} + \vec{\omega} \times \vec{\vec{I}}\vec{\omega}}$$

This set of three differential equations is known as *Euler's Equations*. They are the angular equations of motion for a rigid body expressed in the body frame.

Used together with the 3 translational EOM of the rigid body's center of mass (COM):

$$\boxed{\sum_{i=1}^{N} \vec{F}_{ext,i} = M\ddot{\vec{r}}_c^{\,I}}$$

we have six equations. Solving these allows us to completely specify the body's state.

In short, rigid body dynamics can be described as **translation of the COM plus rotation of the body about its COM.**

# Euler's Equations in Principal Axis Coordinates

If the body coordinates used in Euler's Eqs. are the body's principal axes, then they look like:

$$\vec{M} = \vec{\vec{I}}\vec{\alpha} + \vec{\omega} \times \vec{\vec{I}}\vec{\omega} \tag{10}$$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix} \tag{11}$$

$$= \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) \\ I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) \end{bmatrix} \tag{13}$$

Since the $i^{th}$ equation has components of $\omega$ that are not the $i^{th}$ component, these equations are *coupled*.

Since components of $\omega$ multiplied together, these equations are *nonlinear*.

# Classical Euler angles

The six equations of motion allow us to solve for six unknown coordinates. The rotational coordinates typically used are known as *Euler angles*. We've encountered Euler angles before, with $\Omega, i, \omega$ for an orbit.

1. Rotate by $\Omega$ about the initial third (i.e., z) axis.

2. Rotate by $i$ about the new first (i.e., x') axis.

3. Rotate by $\omega$ about the new new third (i.e., z") axis.

The transformation matrix for this is the product of three simple rotation matrices that take us from the geocentric equatorial (GE) inertial frame to the perifocal (PF) "body" frame:

$$ {}^{PF}Q^{GE} = \begin{bmatrix} c\Omega c\omega - s\Omega c i s\omega & s\Omega c\omega + c\Omega c i s\omega & s i s\omega \\ -c\Omega s\omega - s\Omega c i c\omega & -s\Omega s\omega + c\Omega c i c\omega & s i c\omega \\ s i s\Omega & -c\Omega s i & c i \end{bmatrix} \tag{14} $$

# NASA convention Euler angles

Euler angles refer to any general sequence of three rotations around three unique axes. It doesn't have to be axes (3-1-3). A commonly used set in aerospace are the NASA convention of Euler angles $\psi$=yaw, $\theta$=pitch, $\phi$=roll. Used in a (3-2-1) sequence:

1. Rotate by $\psi$ about the third (i.e., z) axis of inertial system I.

2. Rotate by $\theta$ about the new second (i.e., y') axis of intermediate system A1.

3. Rotate by $\phi$ about the new new first (i.e., x'') axis of intermediate system A2.

they will transform from a locally horizontal-vertical inertial system (I) to the body system (B).

# NASA convention Euler angle rotations

$$ {}^{A1}Q^{I} = Q_{\psi} \quad = \quad \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15} $$

$$ {}^{A2}Q^{A1} = Q_{\theta} \quad = \quad \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{16} $$

$$ {}^{B}Q^{A2} = Q_{\phi} \quad = \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{17} $$

$$ {}^{B}Q^{I} = Q_{\phi}Q_{\theta}Q_{\psi} \quad = \quad \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\ s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{bmatrix} \tag{18} $$

# Alternative Euler angles, and alternatives to Euler angles

One reason there are different sets of Euler angles is because any set will turn singular for some value of angle.

${}^{PF}Q^{GE}$ for the (3-1-3) rotation is singular when the second angle $i = 0$. Two of the rows of the matrix become identical. This corresponds to an equatorial orbit. The line of nodes is undefined, and there is no way to distinguish $\Omega$ and $\omega$ as two separate angles.

For the aerospace set (3-2-1), the matrix product $Q_\phi Q_\theta Q_\psi$ is singular for $\theta = 90^\circ$. For an aircraft this would mean the nose is pointing vertically, which doesn't often happen, and not for an extended period of time.

Another way of specifying orientation of a body that doesn't have a singularity issue is to use Euler parameters, aka quaternions. We will not discuss those here.

# Angular velocity in terms of Euler angles

We found the Euler Equations, which involves angular velocity $\vec{\omega}$, but we need to write its components in terms of $\psi, \theta, \phi$ and their time derivatives.

To relate each of the rotations to angular velocity $\vec{\omega} = [\omega_1, \omega_2, \omega_3]^T$, we have to keep in mind that they are in intermediate $I, A1, A2$ frames that are different from the body $B$ frame:

$$\vec{\omega}_B = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}_{/I} + \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}_{/A1} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}_{/A2} \tag{19}$$

So to add them together, we would need to apply the transformation matrices to get them all in the same coordinate system.

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = Q_\phi Q_\theta Q_\psi \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + Q_\phi Q_\theta \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + Q_\phi \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

## Angular velocity in terms of Euler angles, cont'd 2

Applying the transformation matrices, we have:

$$
\begin{aligned}
\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} &= Q_\phi Q_\theta Q_\psi \begin{bmatrix}0\\0\\ \dot\psi\end{bmatrix} + Q_\phi Q_\theta \begin{bmatrix}0\\ \dot\theta\\ 0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & c\phi & s\phi\\ 0 & -s\phi & c\phi\end{bmatrix}\begin{bmatrix}\dot\phi\\0\\0\end{bmatrix} && (21)\\
&= Q_\phi Q_\theta Q_\psi \begin{bmatrix}0\\0\\ \dot\psi\end{bmatrix} + Q_\phi \begin{bmatrix}c\theta & 0 & -s\theta\\ 0 & 1 & 0\\ s\theta & 0 & c\theta\end{bmatrix}\begin{bmatrix}0\\ \dot\theta\\ 0\end{bmatrix} + \begin{bmatrix}\dot\phi\\0\\0\end{bmatrix} && (22)\\
&= Q_\phi Q_\theta Q_\psi \begin{bmatrix}0\\0\\ \dot\psi\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & c\phi & s\phi\\ 0 & -s\phi & c\phi\end{bmatrix}\begin{bmatrix}0\\ \dot\theta\\ 0\end{bmatrix} + \begin{bmatrix}\dot\phi\\0\\0\end{bmatrix} && (23)\\
&= Q_\phi Q_\theta \begin{bmatrix}c\psi & s\psi & 0\\ -s\psi & c\psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\ \dot\psi\end{bmatrix} + \begin{bmatrix}0\\ \dot\theta c\phi\\ -\dot\theta s\phi\end{bmatrix} + \begin{bmatrix}\dot\phi\\0\\0\end{bmatrix} && (24)
\end{aligned}
$$

## Angular velocity in terms of Euler angles, cont'd 3

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = Q_\phi \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ \dot{\theta} c\phi \\ -\dot{\theta} s\phi \end{bmatrix} \tag{25}$$

$$= Q_\phi \begin{bmatrix} -\dot{\psi} s\theta \\ 0 \\ \dot{\psi} c\theta \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ \dot{\theta} c\phi \\ -\dot{\theta} s\phi \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} -\dot{\psi} s\theta \\ 0 \\ \dot{\psi} c\theta \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ \dot{\theta} c\phi \\ -\dot{\theta} s\phi \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} -\dot{\psi} s\theta \\ \dot{\psi} s\phi c\theta \\ \dot{\psi} c\phi c\theta \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ \dot{\theta} c\phi \\ -\dot{\theta} s\phi \end{bmatrix} \tag{28}$$

$$= \begin{bmatrix} \dot{\phi} - \dot{\psi} s\theta \\ \dot{\psi} s\phi c\theta + \dot{\theta} c\phi \\ \dot{\psi} c\phi c\theta - \dot{\theta} s\phi \end{bmatrix} \tag{29}$$

# Transform Euler angles rates to angular velocity

We can rearrange (29) to show the relationship between angular velocity and the Euler angle rates as:

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{30}$$

For $\theta \neq \pi/2$, you can express the Euler angle rates in terms of the body-frame angular velocity by inverting the matrix:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{c\theta} \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\theta c\phi & -s\phi c\theta \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{31}$$