1) Recurrence

Solve using repeated substitution

step 1 $T(n) = 3T(\frac{n}{4}) + 4n$

Substitute $\underline{T(\frac{n}{4})} = 3T(\frac{n}{16}) + 4(\frac{n}{4})$

Substitute back to the original equation $T(n) = 3[3T(\frac{n}{16}) + 4(\frac{n}{4})] + 4n$

Simplify $T(n) = 9T(\frac{n}{16}) + 3 \cdot 4(\frac{n}{4}) + 4n$

$T(n) = 9T(\frac{n}{16}) + 3n + 4n$

$T(n) = 9T(\frac{n}{16}) + 7n$

step 2 continue expanding

Substitute $T(\frac{n}{16})$ $T(\frac{n}{16}) = 3T(\frac{n}{64}) + 4(\frac{n}{16})$

Substitute back to the original equation

$T(n) = 9[3T(\frac{n}{64}) + 4(\frac{n}{16})] + 7n$

Simplify

$T(n) = 27T(\frac{n}{64}) + 9 \cdot 4(\frac{n}{16}) + 7n$

$T(n) = 27T(\frac{n}{64}) + \frac{9n}{4} + 7n$

$T(n) = 27T(\frac{n}{64}) + \frac{37n}{4}$

step 3 identify the pattern after k substitution

$T(n) = 3^k T(\frac{n}{4^k}) + 4n\left(1 + \frac{3}{4} + \frac{9}{16} + \dots + (\frac{3}{4})^{k-1}\right)$

geometric series with ratio $r = \frac{3}{4}$

$$\sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i = \frac{1 - (\frac{3}{4})^k}{1 - \frac{3}{4}} = 4\left(1 - \left(\frac{3}{4}\right)^k\right)$$

$T(n) = 3^k T(\frac{n}{4^k}) + 4n \cdot 4\left(1 - (\frac{3}{4})^k\right)$

$T(n) = 3^k T(\frac{n}{4^k}) + 16n\left(1 - (\frac{3}{4})^k\right)$

Step 4

Assume $T(1) = C$ a constant when $\frac{n}{4^k} = 1$ $k = \log_4 n$

Substitute $k = \log_4 n$

$$T(n) = 3^{\log_4 n} T(1) + 16n\left(1 - \left(\frac{3}{4}\right)^{\log_4 n}\right)$$

Simplify $3^{\log_4 n}$

$$3^{\log_4 n} = n^{\log_4 3}$$

$$T(n) = C \cdot n^{\log_4 3} + 16n\left(1 - n^{\log_4 \frac{3}{4}}\right)$$

Since $n^{\log_4 \frac{3}{4}}$ becomes negligible as n grows the dominant term is $16n$

Final solution

$$T(n) = \Theta(n)$$

2 Master Method

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

For $T(n) = 3T\left(\frac{n}{4}\right) + 4n$

- $a = 3$
- $b = 4$
- $f(n) = 4n$

Compute $\log_b a$

$$\log_b a = \log_4 3 \approx 0.792$$

Compare $f(n)$ with $n^{\log_b a}$

$$f(n) = 4n$$

$$n^{\log_b a} = n^{\log_4 3} \approx n^{0.792}$$

$4n$ grows faster than $n^{0.792}$

Substitute $f(n) = 4n$: $af\left(\frac{n}{b}\right) = 3 \cdot 4\left(\frac{n}{4}\right) = 3n$

$3n \leq k \cdot 4n$ which simplifies to $k \geq \frac{3}{4}$ $k = \frac{3}{4}$

regularity condition holds, the solution holds

$$T(n) = \Theta(f(n)) = \Theta(n)$$

Final

$T(n) = 3T\left(\frac{n}{4}\right) + 4n$ is $T(n) = \Theta(n)$

2) Master Theorem

A) $T(N) = 3T(\frac{n}{5}) + n^2$

$a=3$, $b=5$, $f(n)=n^2$

$\log_b a = \log_5 3 \approx 0.6826$

$f(n) = n^2 \quad n^{\log_b a} = n^{0.6826}$

n^2 grow faster than $n^{0.6826}$

Condition $f(\frac{n}{b}) = 3(\frac{n}{5})^2 = \frac{3n^2}{25} \leq kn^2$ for $k = \frac{3}{25} < 1$

Solution $T(n) = \Theta(n^2)$

B) $T(n) = 4T(\frac{n}{3}) + 7n$

$a=4$, $b=3$, $f(n)=7n$

$\log_b a = \log_3 4 \approx 1.2619$

Compare $f(n) = 7n \quad n^{\log_b a} = n^{1.2619}$

$7n$ grows slower than $n^{1.2619}$

Solution $T(n) = \Theta(n^{\log_3 4})$

C) $T(n) = 5T(\frac{n}{4}) + 10$

$a=5$, $b=4$, $f(n)=10$

$a = \log_4 5 \approx 1.1609$

$f(n) = 10 \quad n^{\log_b a} = n^{1.1609}$

10 is a constant, grows slower than $n^{1.1609}$

Solution $T(n) = \Theta(n^{\log_4 5})$

D) $T(n) = 9T(\frac{n}{3}) + n^4$

$a=9$, $b=3$ $f(n) = n^4$

Compute $\log_b a = \log_3 9 = 2$

Compare $f(n)\ n^4 \quad n^{\log_b a} = n^2$

n^4 grows faster than n^2

Check condition $af(\frac{n}{b}) = 9(\frac{n}{3})^4 = \frac{9n^4}{81} = \frac{n^4}{9} \leq kn^4$ for $k\ \frac{1}{9} < 1$

Solution $T(n) = \Theta(n^4)$

e) $T(n) = 6T(\frac{n}{8}) + n^3$

$a=6$, $b=8$, $f(n) = n^3$

Compute $\log_b a = \log_8 6 \approx 0.8617$

Compare $f(n) = n^3 \quad n^{\log_b a} = n^{0.8617}$

n^3 grows faster than $n^{0.8617}$

Check condition $af(\frac{n}{b}) = 6(\frac{n}{8})^3 = \frac{6n^3}{512} \leq kn^3$ for $k = \frac{6}{512} < 1$

Solution $T(n) = \Theta(n^3)$

3) Radix Sort = process strings character by character

Given

1) CAP COL USD SUN JPY VEE ROW JOB COX LOL RAT WOW WOW
DOD CAR FIG PIG VIS LOW LOX VEA CAD
DOG TSL 1st is to sort least significant character

CAD COL USD SUN JPY VEE ROW JOB COX LOL RAT WOW
DOD CAR FIG PIG VIS LOW LOX VEA CAP
DOG TSL RESULT Now for 2nd step

2) CAD CAP CAR COL COX DOG DOD FIG JOB JPY
LOL LOW LOX PIG RAT ROW SUN TSL VEA
VEE VIS WOW USD 2nd Sort by 2nd character

3) CAD CAP CAR COL COX DOG DOD FIG JOB JPY
LOL LOW LOX PIG RAT ROW SUN TSL USD
VEA VEE VIS WOW

Sort by the 1st Character

Final results

CAD CAP CAR COL COX DOG DOD FIG JOB JPY
LOL LOW LOX PIG RAT ROW SUN TSL USD
VEA VEE VIS WOW

4) Double Hashing M=13 slots

Hash function: $h_1(key) = \left(\frac{(key+19)(key+11)}{15} + key\right) \% 13$

Second hash function: Reverse (key)

key	0	1	2	3	4	5	6	7	8	9	10	11
value	-	-	-	-	-	-	-	-	-	-	-	-

Double hashing if a collision occurs

Keys in order [25, 14, 9, 7, 5, 3, 0, 21, 6, 33, 25, 42, 24, 107]

- Key 25 = $h_1(25) = \left(\frac{(25+19)(25+11)}{15} + 25\right) \% 13 = \left(\frac{44.36}{15} + 25\right) \% 13 = (105.6+25) \% 13 =$ $130.6 \% 13 = 0$

  collision none

  probe sequence [0]

  add to hash table

  $\begin{bmatrix} 0 & \cdots & 12 \\ 25 & & \end{bmatrix}$

- Key 14 = $h_1(14) = \left(\frac{(14+19)(14+11)}{15} + 14\right) \% 13 = \left(\frac{33.25}{15} + 14\right) \% 13 = (55+14) \% 13 =$ $69 \% 13 = 4$

  Collision none slot 4

  Probe sequence (4)

  add to hash table

  $\begin{bmatrix} 0 & \cdots & 4 & \cdots & 12 \\ 25 & & 14 & & \end{bmatrix}$

- Key 9 = $h_1(9) = \left(\frac{(9+19)(9+11)}{15} + 9\right) \% 13 = \left(\frac{28.20}{15} + 9\right) \% 13 = (37.33+9) \% 13 =$ $46.33 \% 13 = 7$

  Collision none slot 7

  Probe sequence (7)

  add to hash table

  $\begin{bmatrix} 0 & \cdots & 4 & \cdots & 7 & \cdots & 12 \\ 25 & & 14 & & 9 & & \end{bmatrix}$

- Key 7 = $h_1(7) = \left(\frac{(7+19)(7+11)}{15} + 7\right) \% 13 = \left(\frac{26.18}{15} + 7\right) \% 13 = (31.2+7) \% 13 =$ $38.2 \% 13 = 12$

  Collision none slot 12

  Probe sequence [12]

  add to hash table

  $\begin{bmatrix} 0 & \cdots & 4 & \cdots & 7 & \cdots & 12 \\ 25 & & 14 & & 9 & & 7 \end{bmatrix}$

- Key 5 $= h_1(5) = \left(\frac{(5+19)(5+11)}{15} + 5\right) \% 13 = \left(\frac{24 \cdot 16}{15} + 5\right) \% 13 = (25.6+5) \% 13 =$

  $30.6 \% 13 = 4$

  Collision slot 4 is occupied by 14

  Reverse(5)=5

  Prob sequence $h(5,1) = (4+1\cdot 5)\% 13\% = 9$

  Collision slot 9 is empty

  Probe sequence [4,9]

$$\begin{bmatrix} 0 & \cdots & 4 & \cdots & 7 & 9 & \cdots & 12 \\ 25 & & 14 & & 9 & 5 & & 7 \end{bmatrix}$$

- Key 3 $= h_1(3) = \left(\frac{(3+19)(3+11)}{15} + 3\right) \% 13 = \left(\frac{22 \cdot 14}{15} + 3\right) \% 13 = (20.53+3) \% 13 =$

  $23.53 \% 13 = 10$

  Collision none slot 10

  Probe sequence [10]

  add to hash table

$$\begin{bmatrix} 0 & \cdots & 4 & \cdots & 7 & 9 & 10 & \cdots & 12 \\ 25 & & 14 & & 9 & 5 & 3 & & 7 \end{bmatrix}$$

Key $= 0 = h_1(0) = \left(\frac{(0+19)(0+11)}{15} + 0\right) \% 13 = \left(\frac{19 \cdot 11}{15} + 0\right) \% 13 = (13.93+0) \% 13 =$

$13.93 \% 13 = 10$

Collision slot 0 is occupied by 25 use double hashing

Reverse (0) = 0

Probe sequence $h(0,1) = (0+1\cdot 0) \% 13 = 0$ still occupied

Prob sequence $h(0,2) = (0+2\cdot 0) \% 13 = 0$

Infinite loop. Resize and rehash

Double size of hash table to M=26

Rehash all existing keys into new table

Re insert the current key (0)

- Key 25 $= h_1(25) = \left(\frac{(25+19)(25+11)}{15} + 25\right) \% 26 = \left(\frac{44.36}{15} + 25\right) \% 26 = (105.6 + 25) \% 26$

  $130.6 \% 26 = 0$ ↳

- Key 14 $= h_1(14) = \left(\frac{(14+19)(14+11)}{15} + 14\right) \% 26 = \left(\frac{33.25}{15} + 14\right) \% 26 = (55 + 14) \% 26$

  $69 \% 26 = 17$

- Key 9 $= h_1(9) = \left(\frac{(25+19)(25+11)}{15} + 9\right) \% 26 = \left(\frac{28.20}{15} + 9\right) \% 26 = (37.33 + 9) \% 26$

  $46.33 \% 26 = 20$

- Key 7 $= h_1(7) = \left(\frac{(7+19)(7+11)}{15} + 7\right) \% 26 = \left(\frac{26.18}{15} + 7\right) \% 26 = (31.2 + 7) \% 26$

  $38.2 \% 26 = 12$

- Key 5 $= h_1(5) = \left(\frac{(5+19)(5+11)}{15} + 5\right) \% 26 = \left(\frac{24.16}{15} + 5\right) \% 26 = (25.6 + 5) \% 26$

  $30.6 \% 26 = 4$

- Key 3 $= h_1(3) = \left(\frac{(3+19)(3+11)}{15} + 3\right) \% 26 = \left(\frac{22.14}{15} + 3\right) \% 26 = (20.53 + 3) \% 26$

  $23.53 \% 26 = 23$

- Key 0 $= h_1(0) = \left(\frac{(0+19)(0+11)}{15} + 0\right) \% 26 = \left(\frac{19.11}{15} + 0\right) \% 26 = (13.93 + 0) \% 26$

  $13.93 \% 26 = 13$

7) Algorithm Analysis

- Problem 4 - Involves inserting keys using double hashing. If collision occurs the algorithms probes to the next slot until an empty slot is found

Time Complexity: O(N) When in the worst case all keys hash in the same slot. Best case O(1) when there are no collisions

Space Complexity: O(M) where M is the size of the hash table. Space is used to store the keys in the hash table

- Problem 5: Sorts an array of strings using Radix Sort. Processing each character position from the least significant character (rightmost) to the most significant and using counting sort to group strings

Time Complexity: O(K·n) where K is the maximum length of strings, and n is the number of strings

Space Complexity: O(n) where n is the number of strings

Problem 6: Algorithms checks if a string follows a given pattern p by mapping each character in the pattern to a word. It uses two HashMaps to store the mappings.

Time complexity: O(n) Where n is the length of the pattern or the number of words in S.

Space complexity O(n) where n is the number or unique characters in the pattern or unique words in S.