CS 456: Quiz 8

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Problem

Using the pumping lemma for context-free languages, prove the following language is not context-free.

$$L = \{n_a(w) = n_b(w) < n_c(w) : w \in \{a, b, c\}^*\}$$

Solution, Proof by Contradiction

Assumption (Pumping Lemma for Context-free languages)

If L is a context-free language, for all strings s in L with length greater than or equal to P, s can be broken down into $uvxyz$ where the length of $vxy \geq P$ and the length of $vy \geq 1$. L is a context free language if the string $s_i = uv^ixy^Iz$ must also be in the language for all $i \geq 0$.

Counter example

If L is a context-free language, then all strings must adhere to the pumping lemma. Therefore if any string that does not adhere to the pumping lemma is found, L is not a context-free language. One string that can be used to show L is not context-free is $a^Pb^Pc^{P+1}$. This string is valid since its length is greater than or equal to P. s is in the language since $n_a(w) = n_b(w) < n_c(w)$

Cases

For the string to be impossible to "pump" it must be shown that no partitioning of s can be pumped up or down and stay in the language. There are 5 possible cases where this may be true for the test string.

Case 1: vy contains only as

$$\begin{aligned} vy &= a^k; \quad 1 \leq k \leq P \\ not(vy) &= a^{P-k}b^Pc^{P+1} \\ s_0 &= a^{P-k}b^Pc^{P+1} \end{aligned}$$

In s_0, the number of as is less than the number of bs. Therefore, s_0 is not in the language and this partitioning of s is invalid.

Case 2: vy contains only bs

$$
\begin{aligned}
vy &= b^k; \quad 1 \leq k \leq P \\
not(vy) &= a^P b^{P-k} c^{P+1} \\
s_0 &= a^P b^{P-k} c^{P+1}
\end{aligned}
$$

In s_0, the number of as is greater than the number of bs. Therefore, s_0 is not in the language and this partitioning of s is invalid.

Case 3: vy contains only cs

$$
\begin{aligned}
vy &= c^k; \quad 1 \leq k \leq P \\
not(vy) &= a^P b^P c^{P+1-k} \\
s_0 &= a^P b^P c^{P+1-k}
\end{aligned}
$$

In s_0, the number of cs is not greater than the number of bs since $k \geq 1$. Therefore, s_0 is not in the language and this partitioning of s is invalid.

Case 4: vy contains only as and bs

$$
\begin{aligned}
vy &= a^i b^j; i+j \geq 1; 0 \leq i \leq P; 0 \leq j \leq P \\
not(vy) &= a^{P-i} b^{P-j} c^{P+1} \\
s_3 &= a^{P+2i} b^{P+2j} c^{P+1}
\end{aligned}
$$

In s_3, the number of cs is not greater than the number of as and the number of bs since $i + j \geq 1$. This means that either i or j must be 1 or greater. If this is the case, then the number of as or the number of bs is guaranteed to be greater than the number of cs. Therefore, s_0 is not in the language and this partitioning of s is invalid.

Case 5: vy contains only bs and cs

$$
\begin{aligned}
vy &= b^i c^j; i+j \geq 1; 0 \leq i \leq P; 0 \leq j \leq P \\
not(vy) &= a^P b^{P-i} c^{P+1-j} \\
s_0 &= a^P b^{P-i} c^{P+1-j}
\end{aligned}
$$

In s_0, either the number of as is not equal to the number of bs, or the number of cs is not greater than the number of as since $i + j \geq 1$. This means that either i or j must be 1 or greater. If $i \geq 1$, then $n_a(s_0) \neq n_b(s_0)$ if $j \geq 1$, then $n_c(s_0) leP + 1 - 1$ and $n_c(s_0) \leq n_a(s_0)$. Therefore, s_0 is not in the language and this partitioning of s is invalid.

Conclusion

By proof by contradiction, the language L is not a regular language because it does not adhere to the pumping lemma. For a language to be regular, it must be possible to "pump" any string in the language. A counter example string s was found that for all partitioning of s into $uvxyz$, pumping s either up or down resulted in a string that was not contained in the language.