Locating Roots of Equations
Newton's Method

CS3010

Numerical Methods

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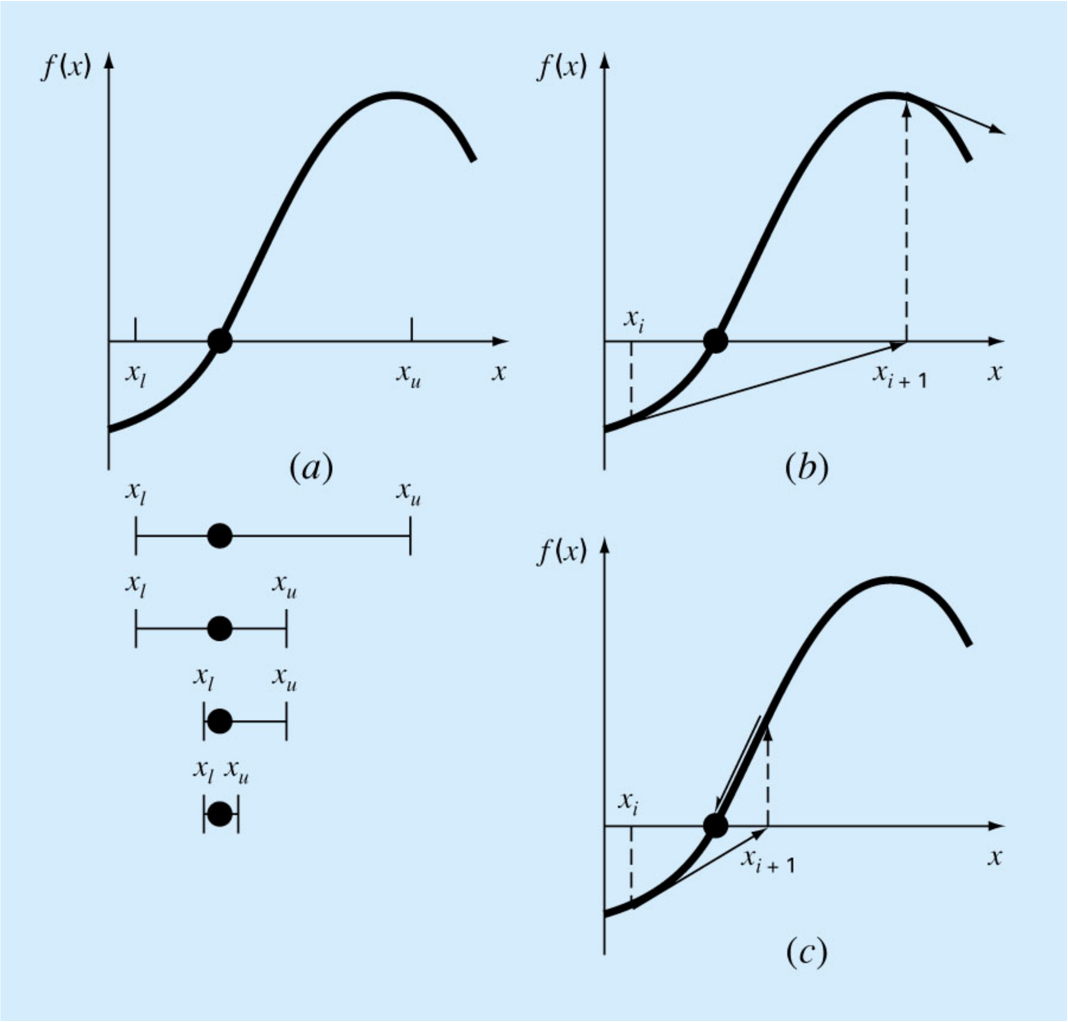
Section 3.2

Lecture 5

Open Methods

- Open methods are based on formulas that require only a single starting value of *x* (guess to the root) or two starting values that do not necessarily bracket the root.



Newton-Raphson Method

- Most widely used method.
- Based on Taylor series expansion of $f(x+h)$:

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k$$

$$f(x_i + h) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \cdots$$

Take only first two terms of Taylor Series to make it simple.

if $x_{i+1} = x_i + h$, then assuming x_{i+1} is the root, $f(x_i + h) = 0$

$$f(x_i + h) = f(x_i) + hf'(x_i) = 0$$

$$f(x_i + h) = f(x_i) + (x_{i+1} - x_i) f'(x_i) = 0$$

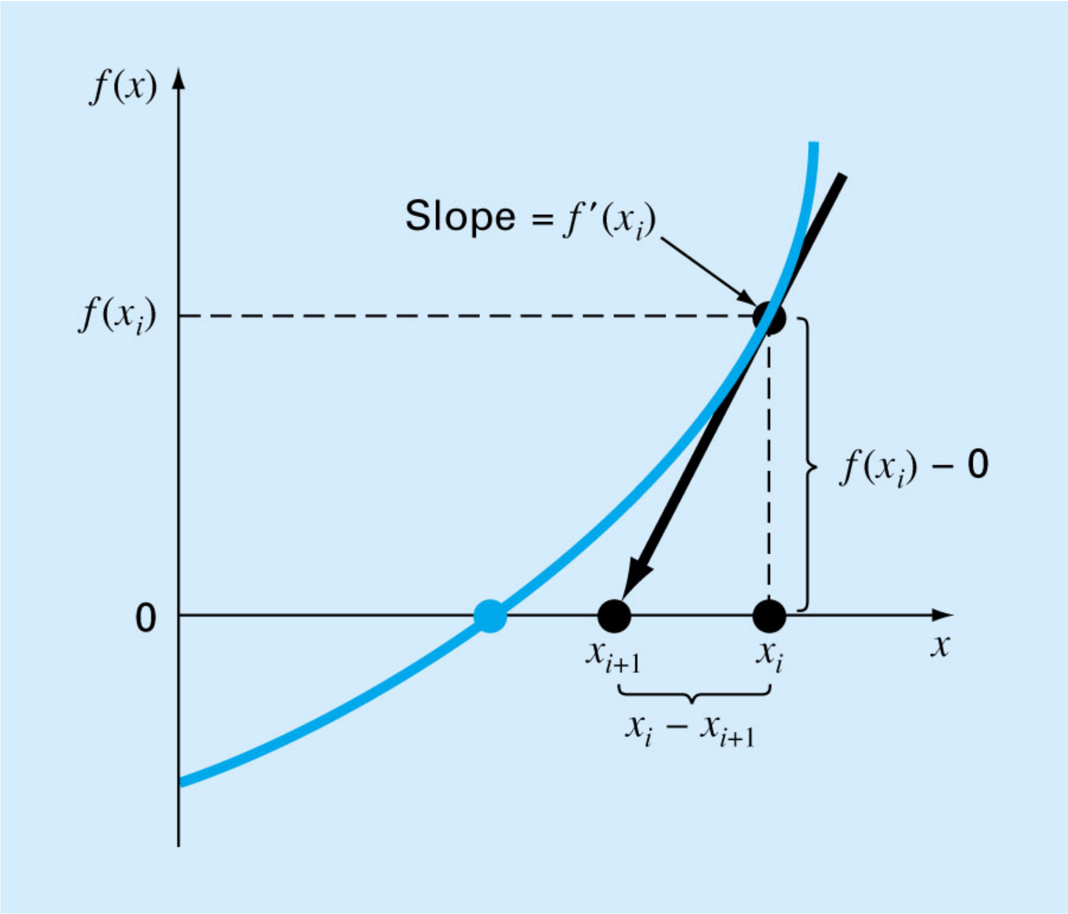
$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Interpretation of Newton-Raphson Method

- A convenient method for functions whose derivatives can be evaluated analytically.
- It may not be convenient for functions whose derivatives cannot be evaluated analytically.
- Equation of the line $l(x)$ give by

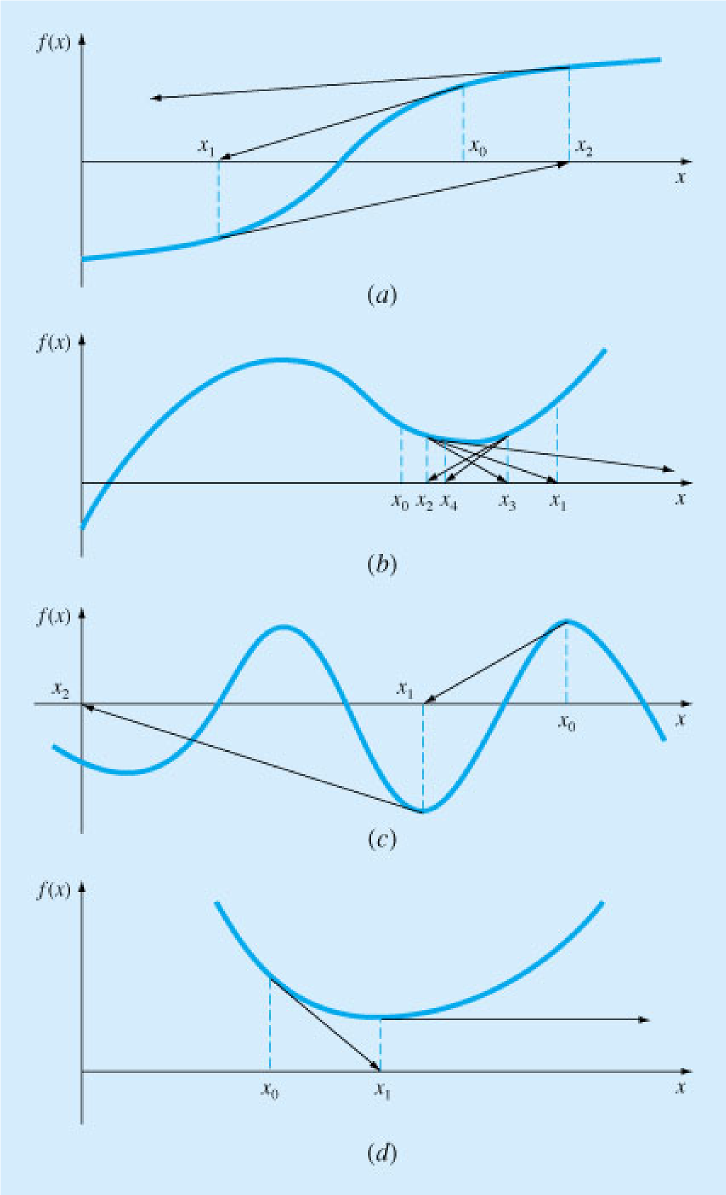
  $l(x) = f'(x_i)(x - x_i) + f(x_i)$
- Zero (root) is found at $x = x_{i+1}$, hence $l(x_{i+1}) = 0$

  $$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$
- This is the iterative formula for Newton's method
- In Newton's method, it is assumed at once that the function *f* is differentiable



Drawbacks of Newton-Raphson Method

- (a) Roots starting at x_0 diverge when inflection point ($f''(x) = 0$) in vicinity of root).
- (b) Oscillate around a local minimum or maximum. In this case around zero slope where as the root is far from this area of interest.
- (c) Initial guess close to one root can jump to a root several roots away. This is because near-zero slopes are encountered . Of course, a zero slope is disaster since it causes division by 0 is the NR formula
- (d) at zero slope, the solution shoots off horizontally and never hits the x axis.



Newton-Raphson Drawbacks

- No general convergence criterion for NR method
- Convergence depends on the accuracy of initial guess.
    - Only remedy is to have a guess close to the root. For some functions, no guess is good enough
- Keep an eye on the functional value at each approximation with each iteration to make sure functional value is approaching zero
- Good guess usually predicted on knowledge of the physical application being solved or graphs that provide some idea to the solution of the problem
- Lack of general convergence criterion implies clever software to recognize slow convergence or divergence.

Smart Newton-Raphson Program

- Final root estimate should be substituted into the original function to compute whether the result is close to zero. This check partially guards against those cases where slow or oscillating convergence may lead to a small value of ε_a while the solution is still far from a root.
- Program should always include upper limit on iterations to guard against oscillations, slow convergent or divergent solutions that could persist interminably.
- Program should alert user and take into account of the possibility that $f'(x)$ might equal zero at any time during the computation.