Class exercise 1.2

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- First two non-zero terms of series expansion about zero for
  a: $e^{\cos(x)}$ and b: $\sin(\cos(x))$

a: $f(x) = e^{\cos(x)}$, $f''(x) = -\sin(x)e^{\cos(x)}$, $f'''(x) = -\cos(x)e^{\cos(x)} + \sin^2(x)e^{\cos(x)}$

$f(0) = e^1$ $\quad f''(0) = 0$ $\quad f'''(x) = -e^1 + 0$

$P(x) = p(x) + \frac{p'(x)}{1!}(x-a) + \frac{p''(x)}{2!}(x-a)^2 + \frac{p'''(x)}{3!}(x-a)^3$

- $\boxed{f(0) = e^1 - \frac{e^1}{3!}x^3}$

b: $\sin(\cos x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\cos x)^{2n+1} = \cos(x) - \left(\frac{1}{3!}\right)\cos^3(x) + \left(\frac{1}{5!}\right)\cos^5(x)$

$f(x) = \sin(\cos x) =$

- $\boxed{f(0) = 1 - \frac{1}{3!}(1)^3}$

- What is the series for $\ln(1-x)$ and $\ln\left[\frac{(1+x)}{(1-x)}\right]$ when $c=0$?

$f(x) = \ln(1-x)$ $\quad f'(x) = \frac{-1}{1-x}$ $\quad f''(x) = \frac{-1}{(1-x)^2}$ $\quad f'''(x) = \frac{-2}{(1-x)^3}$

$f(0) = \ln(1-0) = 0$ $\quad f'(0) = -1$ $\quad f''(0) = -1$ $\quad f'''(0) = -2$

$f^4(x) = \frac{-6}{(1-x)^4}$ $\quad f^5(x) = \frac{-24}{(1-x)^5}$ $\quad f^6(x) = \frac{-120}{(1-x)^6}$

$f^4(0) = -6$ $\quad f^5(0) = -24$ $\quad f^6(0) = -120$

mclaurin series: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^4(0) + \cdots$

$\ln(1-x) = 0 + x(-1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(-24) + \frac{x^6}{6!}(-120)$

$$\boxed{= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \cdots - \frac{x^n}{n}}$$

$$\ln(1-x) = \sum_{n=0}^{\infty}\left(\frac{-(x)^{n+1}}{n+1}\right)$$

• series expansion for $\ln\left[\frac{1+x}{1-x}\right]$ when $c=0$

$$f(x) = \ln\left[\frac{1+x}{1-x}\right] \qquad f'(x) = \frac{1}{1+x} - \frac{-1}{1-x} = \frac{1-x+1+x}{(1+x)(1-x)} = \frac{\cancel{1}-\cancel{x}+\cancel{1}+\cancel{x}}{1-\cancel{x}+\cancel{x}-x^2}$$

$$f(0) = \ln\left[\frac{1+0}{1-0}\right] = 0 \qquad f'(x) = \frac{2}{1-x^2} = 2(1-x^2)^{-1} \qquad f'(0) = 2$$

$$f''(x) = 2\frac{d}{dx}(1-x^2)^{-1} = 2\left(\frac{-1}{(1-x^2)^2}(-2x)\right) = \frac{4x}{(1-x^2)^2} \qquad f''(0) = 0$$

$$f'''(x) = 4\frac{d}{dx}\left[x\cdot(1-x^2)^{-2}\right] = 4\left[(1-x^2)^{-2} + \frac{-2(-2x)}{(1-x^2)^3}\right] = \frac{4}{(1-x^2)^2} + \frac{16x}{(1-x^2)^3}$$

$$f'''(0) = 4$$

$$f^4(x) = \frac{48(x^2+1)}{(1-x^2)^4} \qquad f^4(0) = 0$$

$$f^5(x) = \frac{48(5x^4+10x^2+1)}{(1-x^2)^5} \qquad f^5(0) = 48$$

$$f^6(x) = \frac{480x(3x^4+10x^2+3)}{(1-x^2)^6}$$

$$f^6(0) = 0$$

$$f^7(x) = \frac{1440(7x^6+35x^4+21x^2+1)}{(1-x^2)^7}$$

$$f^7(0) = 1440$$

Mclaurin series of $\ln\left(\frac{1+x}{1-x}\right)$:

$$\ln\left(\frac{1+x}{1-x}\right) = 0 + \frac{x}{1!}(2) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(4) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(48) + \frac{x^6}{6!}(0) + \ldots$$

$$= 2x + \frac{4}{6}x^3 + \frac{48}{120}x^5 + \frac{1440}{5040}x^7 + \ldots$$

$$= 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \ldots\right)$$

$$\boxed{\ln\left(\frac{1+x}{1-x}\right) = (2)\sum_{n=0}^{\infty}\left(\frac{x^{2n+1}}{2n+1}\right)}$$

• Taylor Series for $\sin(x-3h)$

$$f(x-h) = f(x) + f'(x)(x-h-x) + \frac{f''(x)}{2!}(x-h-x)^2 + \frac{f'''(x)}{3!}(x-h-x)^3 + \cdots$$

$$= f(x) + f'(x)(-h) + \frac{f''(x)}{2!}(-h)^2 + \frac{f'''(x)}{3!}(-h)^3 + \cdots$$

$f(x) = \sin(x)$ $\quad f''(x) = \cos(x)$ $\quad f'''(x) = -\sin(x)$ $\quad f^4(x) = -\cos(x)$

$$\sin(x-3h) = \sin(x) + \cos(x)(-3h) + \frac{-\sin(x)}{2!}(-3h)^2 + \frac{-\cos(x)}{3}(-3h)^3 + \cdots$$

$$\boxed{\sin(x-3h) = \sum_{k=0}^{\infty} \frac{f^k(x)}{k!}(-3h)^k}$$

• How many terms required to have an error less than $(\frac{1}{2})\times 10^{-6}$ when computing $\ln(2)$ of the series of $\ln(x)$?

$$F(x) = \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} = (x-1) - (\tfrac{1}{2})(x-1)^2 + (\tfrac{1}{3})(x-1)^3 - (\tfrac{1}{4})(x-1)^4 +$$

$$\ln(2) = (2-1) - (\tfrac{1}{2})(2-1)^2 + (\tfrac{1}{3})(2-1)^3 - (\tfrac{1}{4})(2-1)^4 + (\tfrac{1}{5})(2-1)^5 - \cdots$$

$$= 1 - (\tfrac{1}{2}) + (\tfrac{1}{3}) - (\tfrac{1}{4}) + (\tfrac{1}{5}) - \cdots \frac{(-1)^n}{n+1}$$

$$\frac{(-1)^n}{n+1} < (\tfrac{1}{2})\times 10^{-6} \implies \frac{(-1)^n}{n+1} < 0.0000005$$

$$\log_{10}(\overset{0}{\cancel{-1^n}}) - \log_{10}(n+1) < \log_{10}(\overset{0}{\cancel{1}}) - [\log_{10}(2) + \log_{10}(10^{-6})]$$

$$-\log_{10}(n+1) < -\log_{10}(2) + (6)\log_{10}(\overset{1}{\cancel{10}})$$

$$\log_{10}(n+1) > \log_{10}(2) + 6$$

$$\log_{10}(n+1) > 6.3010299957$$

$1999998 < n < 1999999$ terms for an error value less than $(\frac{1}{2})\times 10^{-6}$