Class Exercise Section 2.2

1.) Solve using Gaussian Elimination w/ scaled partial pivoting

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix} \end{array} \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

① $\ell = [1,2,3,4]$ $\quad$ $*S = [3,3,6,6]$

1st pivotal eq.? Ratio: $\left\{ \frac{1}{3}, \frac{0}{3}, \frac{3}{6}, \frac{0}{6} \right\}$

→ eq. 3 will be 1st pivotal eq.

→ $\ell = [3,2,1,4]$

Solve:

* eq. 2 & 4 stay same for $k=1$ b/c x_1 is already 0

→ eq. 3 & 1:

$$\begin{array}{rrrr} & \cancel{-1} & 1 & 0 & -2 \\ + & \cancel{1} & 0 & 3 & 0 \\ \hline & & 1 & 3 & -2 \end{array}$$

After $k=1$: new matrix:

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

② $\ell = [3,2,1,4]$ $\quad$ 2nd pivotal eq.?
$S = [6,3,3,6]$

→ Ratio: $\left\{ \frac{1}{3}, \frac{1}{3}, \frac{2}{6} \right\}$ → all are $=$ → use first occurence

→ eq. 1 will be 2nd pivot eq.

→ $\ell = [3,1,2,4]$
$S = [6,3,3,6]$

Solve: $k=2$

eq. (1) & (2)

$$\begin{array}{rrrr} & \cancel{-1} & \cancel{-3} & 2 \\ + & \cancel{1} & \cancel{3} & -1 \\ \hline & 0 & 0 & 1 \end{array}$$

eq. (1) & (4)

$$\begin{array}{rrrr} & \cancel{-2} & -6 & 4 \\ + & \cancel{2} & 4 & -6 \\ \hline & & -2 & -2 \end{array}$$

After $k=2$: new matrix

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 3 & -3 & 0 & 6 \\ 0 & 0 & -2 & -2 \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

③ 3rd pivotal eq.? → Ratio: $\left\{ \begin{array}{cc} (2) & (4) \\ \frac{0}{3}, & \frac{27}{6} \end{array} \right\}$ → eq. 4 will be 3rd pivotal eq.

→ new $\ell = [3,1,4,2]$ → Solve: → eq 2's x_3 is already 0 → stay same

final matrix:

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 3 & -3 & 0 & 6 \\ 0 & 0 & -2 & -2 \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

2.) Use Jacobi, Gauss-Seidel, and SOR ($\omega = 1.1$) to solve to 4 dec. places

$$A = \begin{bmatrix} 7 & 1 & -1 & 2 \\ 1 & 8 & 0 & -2 \\ -1 & 0 & 4 & -1 \\ 2 & -2 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \\ -3 \end{bmatrix}$$

① Jacobi Method

1.) Equation form:

(1) $7x_1 + 1x_2 - x_3 + 2x_4 = 3$ → $x_1^k = \frac{3}{7} - \frac{x_2^{(k-1)}}{7} + \frac{x_3^{(k-1)}}{7} - \frac{2x_4^{(k-1)}}{7}$

(2) $x_1 + 8x_2 + 0 - 2x_4 = -5$ → $x_2^k = \frac{-5}{8} - \frac{x_1^{(k-1)}}{8} + \frac{x_4^{(k-1)}}{4}$

(3) $-x_1 + 0 + 4x_3 - x_4 = 4$ → $x_3^k = 1 + \frac{x_1^{(k-1)}}{4} + \frac{x_4^{(k-1)}}{4}$

(4) $2x_1 - 2x_2 - x_3 + 6x_4 = -3$ → $x_4^k = \frac{-1}{2} - \frac{x_1^{(k-1)}}{3} + \frac{x_2^{(k-1)}}{3} + \frac{x_3^{(k-1)}}{6}$

2.) our guess: $x^0 = [0\ 0\ 0\ 0]^T$

① k=1:

- $x_1^1 = \frac{3}{7} - 0 + 0 - 0 = 0.4286$
- $x_2^1 = \frac{-5}{8} - 0 + 0 = -0.625$
- $x_3^1 = 1 + 0 + 0 = 1$
- $x_4 = -\frac{1}{2} - 0 + 0 + 0 = -0.5$

$x^1 = [0.4286, -0.625, 1, -0.5]$

② $\underline{k=2}$, $x^1 = [\overset{x_1}{0.4286}, \overset{x_2}{-0.625}, \overset{x_3}{1}, \overset{x_4}{-0.5}]$

- $x_1^2 = \frac{3}{7} - \frac{(-0.625)}{7} + \frac{1}{7} - \frac{2(-0.5)}{7} = 0.8036$

- $x_2^2 = \frac{-5 - 0.4286 + 2(-0.5)}{8} = -0.8036$

- $x_3^2 = \frac{4 + 0.4286 + (-0.5)}{4} = 0.9822$

- $x_4^2 = \frac{-3 - (0.4286)2 + 2(-0.625) + 1}{6} = -0.6845$

$\rightarrow x^2 = [0.8036, -0.8036, 0.9822, -0.6845]$

③ $\underline{k=3:}$

- $x_1^3 = \frac{3 - (-0.8036) + 0.9822 - 2(-0.6845)}{7} = 0.8793$
- $x_2^3 = \frac{-5 - 0.8036 + 2(-0.6845)}{8} = -0.8966$
- $x_3^3 = \frac{4 + 0.8036 + (-0.6845)}{4} = 1.0298$
- $x_4^4 = \frac{-3 - 2(0.8036) + 2(-0.8036) + 0.9822}{6} = -0.8720$

$\boxed{\text{After 3 iterations: } x^3 = [0.8793, -0.8966, 1.0298, -0.8720]}$

Jacobi method answer

② Gauss-Seidel method:

*** New equations:**

- $x_1^k = \dfrac{3 - x_2^{(k-1)} + x_3^{(k-1)} - 2x_4^{(k-1)}}{7}$

- $x_2^k = \dfrac{-5 - x_1^k + 2x_4^{(k-1)}}{8}$

- $x_3^k = \dfrac{4 + x_1^k + x_4^{(k-1)}}{4}$

- $x_4^k = \dfrac{-3 - 2x_1^k + 2x_2^k + x_3^k}{6}$

1.) Using $x^0 = [0\ 0\ 0\ 0]^T$

① K=1:

$x_1^1 = \dfrac{3-0+0-0}{7} = 0.4286$

$x_2^1 = \dfrac{-5-0.4286+0}{8} = -0.6786$

$x_3^1 = \dfrac{4+0.4286+0}{4} = 1.1072$

$x_4^1 = \dfrac{-3-2(0.4286)+2(-0.6786)+1.1072}{6} = -0.6845$

$x^1 = [0.4286, -0.6786, 1.1072, -0.6845]$

② K=2:

- $x_1^2 = \dfrac{3-(-0.6786)+1.1072-2(-0.6845)}{7} = 0.8793$
- $x_2^2 = \dfrac{-5-0.8793+2(-0.6845)}{8} = -0.9060$
- $x_3^2 = \dfrac{4+0.8793+(-0.6845)}{4} = 1.0487$
- $x_4^2 = \dfrac{-3-2(0.8793)+2(-0.9060)+1.0487}{6} = -0.9203$

$\rightarrow x^2 = [0.8793, -0.9060, 1.0487, -0.9203]$

③ $k=3$:

$$x_1^3 = \frac{3-(-0.9060)+1.0487-2(-0.9203)}{7} = 0.9708$$

$$x_2^3 = \frac{-5-0.9708+2(-0.9203)}{8} = -0.9764$$

$$x_3^3 = \frac{4+0.9708+(-0.9203)}{4} = 1.0126$$

$$x_4^3 = \frac{-3-2(0.9708)+2(-0.9764)+1.0126}{6} = -0.9803$$

Gauss-Seidal method answer:

$$\boxed{\text{After 3 iterations: } x^3 = [0.9708, -0.9764, 1.0126, -0.9803]}$$

③ SOR method:

* New equations:

• $x_1^k = 1.1 \cdot \left[\frac{3-x_2^{(k-1)}+x_3^{(k-1)}-2x_4^{(k-1)}}{7}\right] + (1-1.1)\, x_1^{(k-1)}$

• $x_2^k = 1.1 \cdot \left[\frac{-5-x_1^k+2x_4^{(k-1)}}{8}\right] + (1-1.1)\, x_2^{(k-1)}$

• $x_3^k = 1.1 \cdot \left[\frac{4+x_1^k+x_4^{(k-1)}}{4}\right] + (1-1.1)\, x_3^{(k-1)}$

• $x_4^k = 1.1 \cdot \left[\frac{-3-2x_1^k+2x_2^k+x_3^k}{6}\right] + (1-1.1)\, x_4^{(k-1)}$

1.) Guess w/ $x^\circ = [0\ 0\ 0\ 0]^T$

① $k=1$:

$$x_1' = 1.1 \cdot \left[\frac{3-0+0-0}{7}\right] + (-0.1)(0) = 0.4714$$

$$x_2' = 1.1 \cdot \left[\frac{-5-0.4714+0}{8}\right] + (-0.1)(0) = -0.7523$$

$$x_3' = 1.1 \cdot \left[\frac{4+0.4714+0}{4}\right] + (-0.1)(0) = 1.230$$

$$x_4' = 1.1 \cdot \left[\frac{-3-2(0.4714)+2(-0.7523)+1.23}{6}\right] + 0 = -0.7732$$

$x' = [0.4714, -0.7523, 1.230, -0.7732]$

② k=2:

$$x_1^2 = 1.1\left[\frac{3-(-0.7523)+1.23-2(-0.7732)}{7}\right] + (-0.1)(0.4714)$$
$$= 0.9788$$

$$x_2^2 = 1.1\left[\frac{-5-0.9788+2(-0.7732)}{8}\right] + (-0.1 \cdot -0.7523)$$
$$= -0.9595$$

$$x_3^2 = 1.1\left[\frac{4+0.9788+(-0.7732)}{4}\right] + (-0.1 \cdot 1.23)$$
$$= 1.0335$$

$$x_4^2 = 1.1\left[\frac{-3-2(0.9788)+2(-0.9595)+1.0335}{6}\right] + (-0.1 \cdot -0.7732)$$
$$= -0.9939$$

$$x^2 = [0.9788, -0.9595, 1.0335, -0.9939]$$

③ k=3:

$$x_1^3 = 1.1\left[\frac{3-(-0.9595)+1.0335-2(-0.9939)}{7}\right] + (-0.1 \cdot 0.9788)$$
$$= 0.9991$$

$$x_2^3 = 1.1\left[\frac{-5-0.9991+2(-0.9939)}{8}\right] + (-0.1 \cdot -0.9595)$$
$$= -1.0022$$

$$x_3^3 = 1.1\left[\frac{4+0.991+(-0.9939)}{4}\right] + (-0.1 \cdot 1.0335)$$
$$= 0.9959$$

$$x_4^3 = 1.1\left[\frac{-3-2(0.9991)+2(-1.0022)+0.9959}{6}\right] + (-0.1 \cdot -0.9939)$$
$$= -1.0018$$

$$\boxed{x^3 = [0.9991, -1.0022, 0.9959, -1.0018]}$$

SOR answer