Mathematical Preliminaries
Taylor Series

CS3010

Numerical Methods

Dr. Amar Raheja

Section 1.2

Lecture 2

What is Taylor Series?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (|x| < \infty) \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad (|x| < \infty) \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad (|x| < \infty) \tag{3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \qquad (|x| < 1) \tag{4}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \qquad (-1 < x \leqq 1) \tag{5}$$

- For each case, the series represents the given function and converges in the interval specified.
- Series above are Taylor series expanded about $c = 0$.

Taylor Series Convergence

- Use five terms in Series (5) to approximate ln(1.1).
  Taking x = 0.1 in the first five terms of the series for ln(1 + x) gives us

$$\ln(1.1) \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} = 0.09531\,03333\ldots$$

- This value is correct to six decimal places of accuracy.
- Compute e^8 by using Series:

$$e^8 = 1 + \frac{8}{1} + \frac{64}{2} + \frac{512}{6} + \frac{4096}{24} + \frac{32768}{120} + \cdots \approx 570.06666\,5$$

By repeated squaring, we find e^2 =7.389056,e^4 =54.5981500,and e^8 =2980.957987

A Taylor series converges rapidly near the point of expansion and slowly (or not at all) at more remote points.

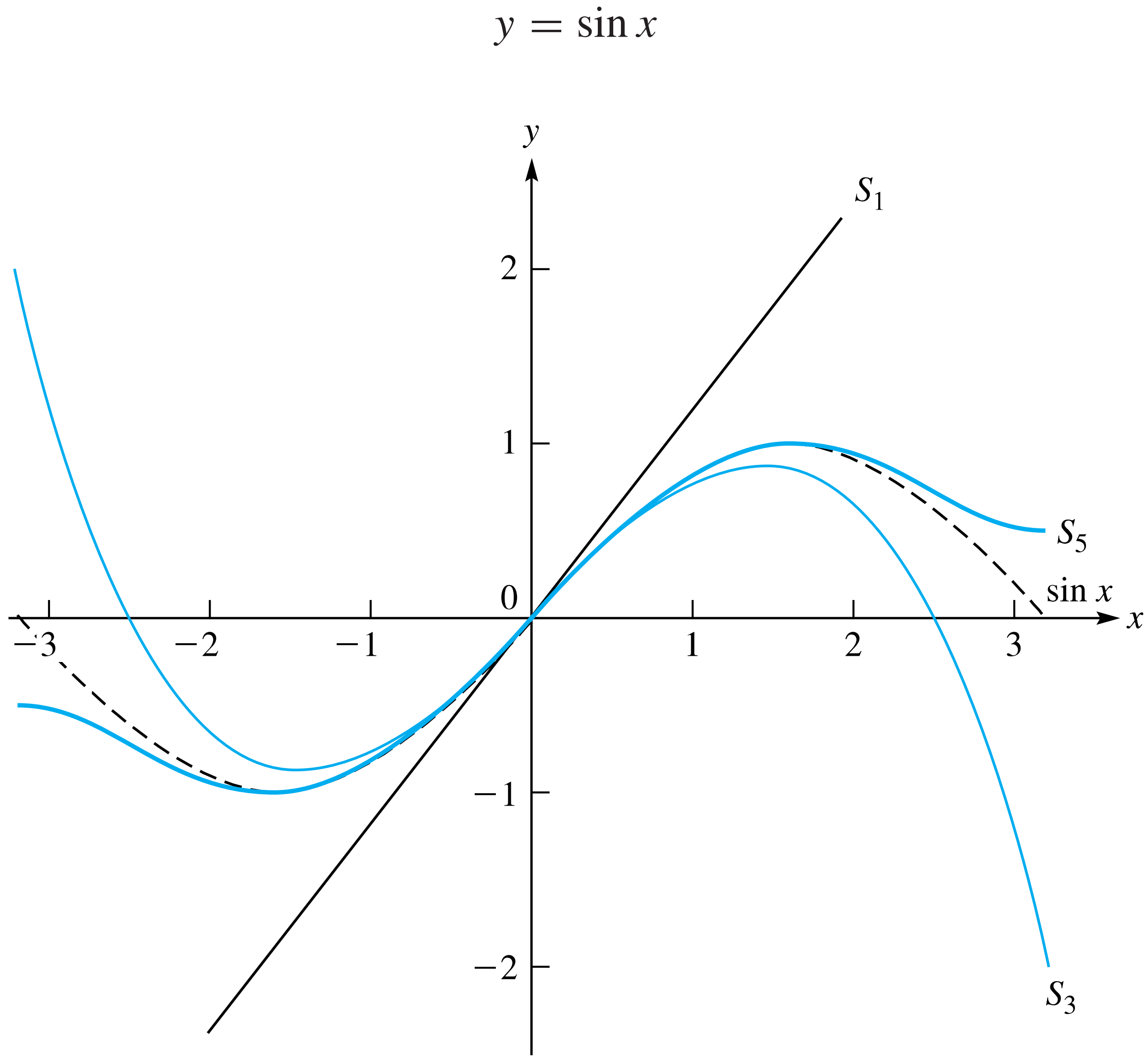
Example: Convergence of sin$x$

- Partial Sums:

$$S_1 = x$$

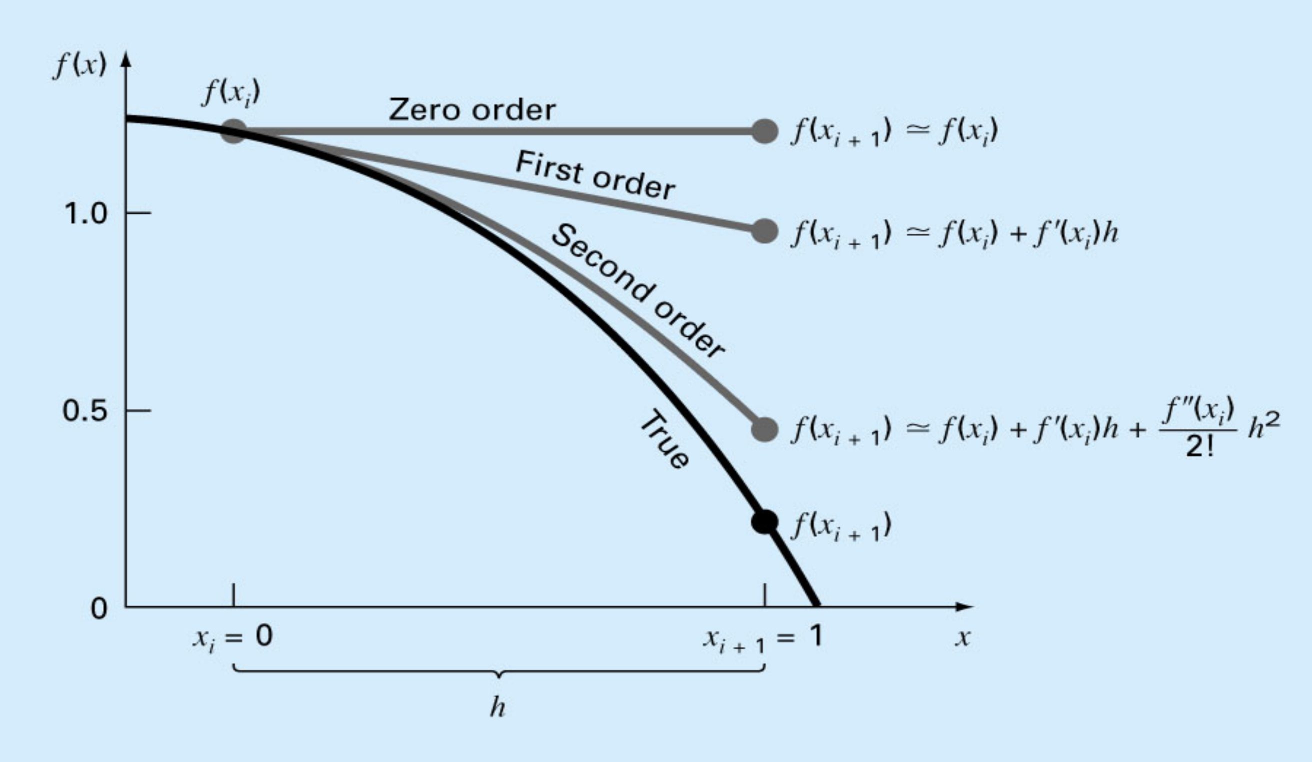
$$S_2 = x - \frac{x^3}{6}$$

$$S_3 = x - \frac{x^3}{6} + \frac{x^5}{120}$$



Taylor Series Conceptual Idea

- Any smooth function can be approximated as a polynomial.
- Taylor series provides a means to predict the value of a function at one point in terms of the function value and its derivatives at another point.



Theorem 1: Formal Taylor series for *f* about *c*

$$f(x) \sim f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \tag{6}$$

- In the special case $c = 0$, Series (6) is also called a **Maclaurin series**:

$$f(x) \sim f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k \tag{7}$$

- The first term is $f(0)$ when $k = 0$

Example using Theorem 1

Find Taylor Series about c=2 for the function

$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$

$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5 \Longrightarrow f(2) = 207$

$f'(x) = 15x^4 - 8x^3 + 45x^2 + 26x - 12 \Longrightarrow f'(2) = 396$

$f''(x) = 60x^3 - 24x^2 + 90x + 26 \Longrightarrow f''(2) = 590$

$f'''(x) = 180x^2 - 48x + 90 \Longrightarrow f'''(2) = 714$

$f^{(4)}(x) = 360x - 48 \Longrightarrow f^{(4)}(2) = 672$

$f^{(5)}(x) = 360 \Longrightarrow f^{(5)}(2) = 360$

$f^{(k)}(x) = 0$ for $k \geq 6$

Therefore, we have

$$f(x) \sim 207 + 396(x - 2) + 295(x-2)^2 + 119(x-2)^3 + 28(x-2)^4 + 3(x-2)^5$$

Complete Horner's Algorithm

- Using the complete Horner's algorithm, find the Taylor expansion of the polynomial $p(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$ about the point r = 3.

$$
\begin{array}{r|rrrrr}
 & 1 & -4 & 7 & -5 & 2 \\
3) & & 3 & -3 & 12 & 21 \\
\hline
 & 1 & -1 & 4 & 7 & 23 \\
 & & 3 & 6 & 30 & \\
\hline
 & 1 & 2 & 10 & 37 & \\
 & & 3 & 15 & & \\
\hline
 & 1 & 5 & 25 & & \\
 & & 3 & & & \\
\hline
 & 1 & 8 & & &
\end{array}
$$

- The calculation shows that

$$p(x) = (x-3)^4 + 8(x-3)^3 + 25(x-3)^2 + 37(x-3) + 23$$

Theorem 2: Taylor's theorem in terms of (x-c)

If the function f possesses continuous derivatives of orders 0, 1, 2, . . . , (n + 1) in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{(k)!}(x-c)^k + E_{n+1}$$

where the error term E_{n+1} can be given in the form

$$E_{n+1} = \sum_{k=0}^{\infty} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Here ξ is a point that lies between c and x and depends on both.

Example: Using Taylor's Theorem (1 of 2)

- Derive the Taylor series for e^x at *c* = 0, and prove that it converges to e^x by using Taylor's Theorem

If $f(x) = ex$, then $f^{(k)}(0) = e^0$ for $k \geq 0$. Therefore, $f^{(k)}(c) = f^{(k)}(0) = e^0 = 1$ for all *k*.

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\xi}}{(n+1)!} x^{n+1}$$

Now let us consider all the values of *x* in some symmetric interval around the origin, for example, $-s \ \leq \ x \ \leq s$. Then $|x| \leq s$, $|\xi| \leq s$, and $e^{\xi} \leq e^s$. Hence, the remainder term satisfies this inequality

$$\lim_{n \to \infty} \left| \frac{e^{\xi}}{(n+1)!} x^{n+1} \right| \leqq \lim_{n \to \infty} \frac{e^s}{(n+1)!} s^{n+1} = 0$$

Example: Using Taylor's Theorem (2 of 2)

- Thus, if we take the limit as $n \to \infty$ on both sides of the previous Equation, we obtain

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

- This example illustrates how we can establish, in specific cases, that a formal Taylor Series of e^x actually represents the function.
- Next example examines how the formal series can *fail* to represent the function.

Second Example: Using Taylor's Theorem (1 of 2)

- Derive the formal Taylor series for $f(x) = \ln(1 + x)$ at $c = 0$, and determine the range of positive x for which the series represents the function.
- Calculate $f^{(k)}(x)$ and $f^{(k)}(0)$ for $k \geq 1$. Here is the work:

$$
\begin{aligned}
f(x) &= \ln(1+x) & f(0) &= 0 \\
f'(x) &= (1+x)^{-1} & f'(0) &= 1 = 0! \\
f''(x) &= -(1+x)^{-2} & f''(0) &= -1 = -1! \\
f'''(x) &= 2(1+x)^{-3} & f'''(0) &= 2 = 2! \\
f^{(4)}(x) &= -6(1+x)^{-4} & f^{(4)}(0) &= -6 = -3!
\end{aligned}
$$

$$f^{(k)}(x) = (-1)^{k-1}\,(k-1)!\,(1+x)^{-k} \qquad f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

Second Example: Using Taylor's Theorem (2 of 2)

- Hence by Taylor's Theorem, we obtain

$$\ln(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}\,(k\ -\ 1)!\ x^k}{k!} + \frac{(-1)^n\,(n)!\,(1+\xi)^{-n-1}}{(n+1)!}x^{n+1}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}\,x^k}{k} + \frac{(-1)^n\,(1+\xi)^{-n-1}}{n+1}x^{n+1}$$

For the *infinite* series to represent ln(1 + *x*), it is necessary and sufficient that the error term converge to zero as $n \to \infty$.

Assume that $0 \leq x\ \leq 1$. Then $0 \leq \xi \leq x$ (because zero is the point of expansion); thus, $0\ \leq\ x\,/(1\ +\ \xi\,)\ \leq\ 1$. Hence, the error term converges to zero in this case.

If *x* > 1, the terms in the series do not approach zero, and the series does not converge. Hence, the series represents ln(1 + *x*) if $0 \leq x\ \leq 1$ but *not* if *x* > 1. (The series also represents $\ln(1+x)$ for $-1 < x < 0$ but not if $x\ \leq -1$.)

Derive Taylor Series for e^x *at c=0*

- $f(x) = e^x \Rightarrow f^{(k)}(x) = e^x$ *for* $k \geq 0$
- $f^{(k)}(c) = f^{(k)}(0) = e^0 = 1$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\xi}}{(n+1)!} x^{n+1}$$

- Consider $|x| \leq s$, $|\xi| \leq s$ and $e^{\xi} \leq e^s$, Hence the remainder term satisfies the property

$$e^x = \lim_{n \to \infty} \left| \frac{e^{\xi}}{(n+1)!} x^{n+1} \right| \leq \lim_{n \to \infty} \left| \frac{e^{s}}{(n+1)!} s^{n+1} \right| = 0$$

- Thus, when $n \to \infty$, one gets

$$e^x = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Class Exercise 3

- Determine how many terms are needed to compute *e* correctly to 15 decimal places (rounded) using Series (1) for e^x .
- Solution: Notice that we want to compute *e*, which means x=1

For this function, If $f(x) = e^x$, then $f^{(k)}(0) = e^0 = 1$ for $k \geq 0$

e^x= 1 + x + x^2/2! + x^3/3! +….

e=1 + 1 + 1/2! + 1/3! + ….

By Taylor's Theorem, E_{n+1}=$\frac{e^{\xi}}{(n+1)!}x^{n+1}$

So, the (n+1)th of the series above is for $k = n$, i.e. $\frac{e^{\xi}}{(n)!}$

$\frac{e^{\xi}}{(n)!} < 0.5 \times 10^{-15} \Longrightarrow loge - logn! < \log 0.5 - 15 \Longrightarrow n = 18$ terms

Theorem 3: Mean Value Theorem

If f is a continuous function on the closed interval $[a, b]$ and possesses a derivative at each point of the open interval (a, b), then

$$f(b) = f(a) + (b - a) f'(\xi)$$

for some ξ in (a, b).

- Hence, the ratio $[f(b) - f(a)]/(b - a)$ is equal to the derivative of f at some point ξ between a and b; that is, for some $\xi \in (a, b)$,

$$f'(\xi) = \frac{f(b) - f(a)}{(b - a)}$$

- The right-hand side could be used as an *approximation* for $f'(x)$ at any x within the interval (a, b).

Taylor's Theorem in Terms of *h*

If the function f possesses continuous derivatives of order $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any x in I,

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1} \tag{11}$$

where h is any value such that $x + h$ is in I and where

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

for some ξ between x and $x + h$.

- Note: *h* can be positive or negative, hence $x < \xi < x+h$ if $h > 0$ or $x+h < \xi < x$ if $h < 0$

COROLLARY: Taylor's Theorem in Terms of h

- The **error term** E_{n+1} depends on h in two ways: First, h^{n+1} is explicitly present; second, the point ξ generally depends on h. As h converges to zero, E_{n+1} converges to zero with essentially the same rapidity with which h^{n+1} converges to zero. For large n, this is quite rapid. To express this qualitative fact, we write

$$E_{n+1} = \mathcal{O}(h^{n+1}) \qquad \text{as } h \to 0.$$

This is called **big O notation**, and it is shorthand for the inequality

$$|\, E_{n+1} \,| \leqq C|h|^{n+1}$$

- where C is a constant. In the present circumstances, this constant could be any number for which $|f^{(n+1)}(t)|/(n+1)! \leqq C$, for all t in the initially given interval, I. Roughly speaking, $E_{n+1} = \mathcal{O}(h^{n+1})$ means that the behavior of E_{n+1} is similar to the much simpler expression h^{n+1}.

Example: Calculating $f(x+h)$

- What is the fifth term in the Taylor series of $(1 - 2h)^{1/2}$?
  Solution Hint: First try and look at given function above to figure out functional form in terms of $f(x)$. What is $f(x)$ looking at this?
- So, $f(x)= x^{1/2}$ and $f(x+h)= (x+h)^{1/2}$ and $x = 1$ and h = $-2h$

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k$$

- Fifth term means $k = 4$, so we have to find $f^{(4)}(x)$ which is $-\frac{15}{16}x^{-7/2}$

Hence, fifth term is $-\frac{15}{16}x^{-7/2}\frac{h^4}{4!}$ and putting $x = 1$ and h = $-2h$, we get

$$-\frac{5}{8}h^4$$

Another Example

- Determine the first three terms in the Taylor series in terms of h for e^{x-h}. Using three terms, one obtains $e^{0.999} \approx Ce$, where C is a constant. Determine C.
- Solution: First calculate first 3 terms of e^{x-h} and clearly $h = 0.001$ and x=1

Remember, $f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k$ and $f(x) = e^x$ and h = -h

Since $f^{(k)}(x) = e^x$, Hence, $f(x-h) = e^x\left(1 - h + \frac{h^2}{2!} - \cdots\right)$

$$e^{0.999} = e^{(1-0.001)}$$

$$e^{0.999} = e^x\left(1 - 0.001 + \frac{(0.001)^2}{2!}\right) = Ce \text{ so } C = 0.9990005$$

Alternating Series Theorem

If $a_1 \geq a_2 \geq .... \geq a_n$ .... 0 for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series

$$a_1 - a_2 + a_3 - a_4 + ...$$

converges; that is

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k = \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}a_k = \lim_{n\to\infty}S_n = S$$

where S is its sum and S_n is the nth partial sum. Moreover, for all n,

$$\left|S - S_n\right| \leq a_{n+1}$$

Example 1 for Alternating Series

- If the sine series is to be used in computing sin1 with an error less than 1/2 x 10^{-6}, how many terms are needed?

$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^k \frac{x^{(2k+1)}}{(2k+1)!}$ starting at n=0

$sin1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$+… (alternating series)

according to Alternating series theorem, $|S - S_n| < a_{n+1}$

So, for (n+1)th term, $k=n$

$$\frac{1}{(2n+1)!} < \frac{1}{2} \times 10^{-6}$$

Solving it, log(2n + 1)! > log 2 + 6 = 6.3

- Hence, if $\mathrm{n} \geq 5$, the error will be acceptable.