Eigenvalues and Eigenvectors

Section 8.2

CS 3010

Numerical Methods

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Eigenvalues and Eigenvectors

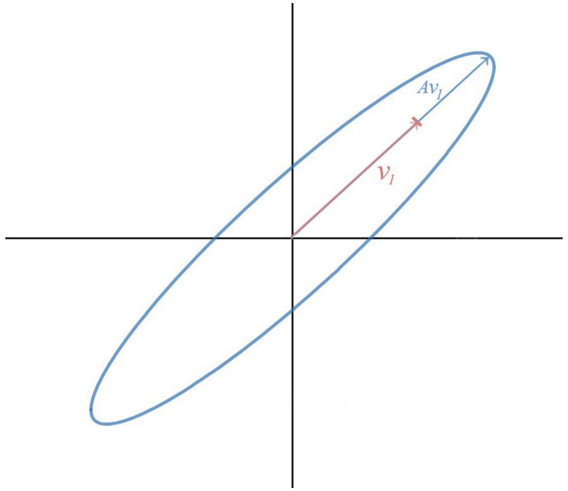
- Let $\boldsymbol{A}$ be an $n \times n$ matrix. We ask the following natural question about $\boldsymbol{A}$: *Are there nonzero vectors* v *for which* $\boldsymbol{A}$v *is a scalar multiple of* v*?*
- There are many situations in scientific computation in which this question arises. We must be willing to consider complex scalars, as well as vectors with complex components.
- Practical Applications:
  - Principal Component Analysis (PCA): reduction of dimensionality of large data (needed in many applications including machine learning)
  - Spectral Clustering: find K clusters using the **eigenvectors** of a matrix
  - Interest Point Detection in Computer Vision: detecting the corner points (one of the many features) in an image. Harris corner detector using **eigenvalues** and **eigenvectors**.
  - Google PageRank Algorithm uses calculation of eigenvectors

https://towardsdatascience.com/the-essence-of-eigenvalues-and-eigenvectors-in-machine-learning-f28c4727f56f

Eigenvalues and Eigenvectors

- By definition, scalar λ and vector v are the eigenvalue and eigenvector of A if

$$\boldsymbol{Av} = \lambda \boldsymbol{v}$$



- Visually, Av lies along the same line as the eigenvector v
- Av does not usually equal to λv.
- Only some exceptional vectors satisfy the condition.
- Here are some eigenvector examples

$$A = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A\begin{bmatrix} 2 \\ -7 \end{bmatrix} = \begin{bmatrix} -8 \\ 28 \end{bmatrix} = -4\begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = (2+i)\begin{bmatrix} 1 \\ 1+i \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = (2-i)\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}}_{A}\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = \underbrace{4}_{\text{eigenvalue}}\underbrace{\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}}_{\text{eigenvector}}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

https://medium.com/@jonathan_hui/machine-learning-linear-algebra-eigenvalue-and-eigenvector-f8d0493564c9

Eigenvalues and Eigenvectors

- Mathematically, eigenvalues and eigenvectors provide a way to identify the principal components of information stored in a matrix $\boldsymbol{A}$
- Eigenvectors identify the components and eigenvalues quantify its significance.
- Notice that an equation $\boldsymbol{A0} = \lambda\mathbf{0}$ and an equation $\boldsymbol{A0} = \mathbf{0}\boldsymbol{x}$ say nothing useful about eigenvalues and eigenvectors of $\boldsymbol{A}$
- Notice that if $\boldsymbol{Ax} = \lambda\boldsymbol{x}$ and $\boldsymbol{x} \neq 0$, then every nonzero multiple of $\boldsymbol{x}$ is an eigenvector (with the same eigenvalue).
- If λ is an eigenvalue of an $n \times n$ matrix $\boldsymbol{A}$, then the set $\{\boldsymbol{x}: \boldsymbol{Ax} = \lambda\boldsymbol{x}\}$ is a subspace of R^n called an **eigenspace**. It is necessarily of dimension at least 1

Calculating Eigenvalues and Eigenvectors

- The equation $\boldsymbol{Ax} = \lambda\boldsymbol{x}$ is equivalent to $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$
- Since we are interested in nonzero solutions to this equation, the matrix $\boldsymbol{A} - \lambda\boldsymbol{I}$ must be singular (noninvertible), and therefore

$$\mathrm{Det}(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$$

- Form the function p by the definition $p(\lambda) = \mathrm{Det}(\boldsymbol{A} - \lambda\boldsymbol{I})$, and find the zeros of p.
- It turns out that p is a polynomial of degree n and must have n zeros, provided that we allow complex zeros and count each zero a number of times equal to its multiplicity.
- Even if the matrix $\boldsymbol{A}$ is real, we must be prepared for complex eigenvalues.

Calculating Eigenvalues and Eigenvectors

- Examples of calculating characteristic polynomials and eigenvalues and eigenvectors

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- The eigenvalues are:

$$det\,|A - \lambda I| = det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 3 - 4\lambda + \lambda^2 = 0$$

characteristic polynomial

$$\lambda = 1 \text{ or } 3$$

- Apply $Av = \lambda v$, we solve:

$$v_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Calculating Eigenvalues and Eigenvectors

- Find Eigenvalues and Eigenvectors for $$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}$$
- The characteristic polynomial is

$$p(\lambda) = \operatorname{Det}(\boldsymbol{A} - \lambda \boldsymbol{I}) = \operatorname{Det} \begin{bmatrix} 3-\lambda & 2 \\ 7 & -2-\lambda \end{bmatrix} = (3-\lambda)(-2-\lambda) - 14$$

$$= \lambda^2 - \lambda - 20 = (\lambda - 5)(\lambda + 4)$$

- The eigenvalues are 5 and −4
- Once an eigenvalue λ has been determined for a matrix ***A***, an eigenvector can be computed by solving the system (***A*** − λ***I***)***x*** = **0**.
- Hence, solve (***A*** − 5***I***)***x*** = **0**, and (***A*** − (−4)***I***)***x*** = **0**

$$\begin{bmatrix} -2 & 2 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 7 & 2 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Eigenvector for this is $[1, 1]^T$. The other eigenvalue is treated in the same way, leading to an eigenvector $[2, -7]^T$.

Calculating Eigenvalues and Eigenvectors for 3x3 Matrix

- Steps with a more complicated example

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

- To find the eigenvalue λ,

$$det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} = 0$$

$$16 + 12\lambda - \lambda^3 = 0$$

$$\lambda^3 - 12\lambda - 16 = 0$$

when $\lambda = 4$, $\lambda^3 - 12\lambda - 16 = 0$

So $\lambda^3 - 12\lambda - 16 = (\lambda - 4)(\lambda^2 + 4\lambda + 4) = 0$

By solving the root, the eigenvalues are 4, -2.

Calculating Eigenvalues and Eigenvectors for 3x3 Matrix

- Calculate the eigenvector for eigenvalue *λ = 4* through **row reduction**

$$A - 4I = \begin{vmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{vmatrix}$$

Doing row reduction to solve the linear equation $(A - \lambda I)v = 0$

$$\begin{vmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{vmatrix}$$

Appending 0

Perform $R_1 = -\frac{1}{3}R_1$

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{vmatrix}$$

Perform row subtraction/multiplicatior

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{vmatrix}$$

After many more reductions:

$$\begin{vmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$x_1 - \frac{1}{2}x_3 = 0$$

$$x_2 - \frac{1}{2}x_3 = 0$$

Calculating Eigenvalues and Eigenvectors for 3x3 Matrix

- We have three variables with 2 equations. We set x_3 arbitrary to 1 and compute the other two variables. So for *λ=4*, the eigenvector is:

$$\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

- We repeat the calculation for *λ=-2* and get $x_1 - x_2 + x_3 = 0$
- With 3 variables and 1 equation, we have 2 degrees of freedom in our solution.
- Let's set the value to one in one of the degrees of freedom while other(s) to 0. i.e. setting $x_2=1, x_3=0,$ and $x_2=0, x_3=1$ separately, the calculated eigenvectors will be:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Calculating Eigenvalues and Eigenvectors for 3x3 Matrix

- Note that the solution set for eigenvalues and eigenvectors are not unique. One can rescale the eigenvectors.
- One can also set different values for x_2, x_3 above. Hence, it is possible and desirable to **choose** our eigenvectors to meet certain conditions.
- For example, for a symmetric matrix, it is always possible to choose the eigenvectors to have unit length and orthogonal to each other.
- In this example, we have a repeated eigenvalue "-2". It generates two different eigenvectors.
- However, this is not always the case — there are cases where repeated eigenvalues do not have more than one eigenvector.

Mathematical Software to Calculate EVs

- Matlab might be to compute the eigenvalues and eigenvalues of a matrix with a command such as [V,D] = *eig*(A) for the matrix
- Matlab responds instantly with the eigenvectors in the array V and the eigenvalues in the diagonal array D.
- Maple and Mathematica can be used to compute the eigenvalues and eigenvectors
- The best advice for anyone who is confronted with challenging eigenvalue problems is to use the software in the package LAPACK (Matlab uses some programs in LAPACK)

Python to calculate Eigenvalue and Eigenvector

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

```
>>> import numpy as np
>>> from numpy import linalg as LA
A = np.array([[1,2,3],[3,2,1],[1,0,-1]])
w, v = LA.eig(A)
>>> print(w)
[ 4.31662479e+00 -2.31662479e+00 3.43699053e-17]
>>> print(v)
[[ 0.58428153 0.73595785 0.40824829]
[ 0.80407569 -0.38198836 -0.81649658]
[ 0.10989708 -0.55897311 0.40824829]]
```

numpy.linalg.eig function returns a tuple consisting of a vector and an array

vector (here **w**) contains the eigenvalues

array (here **v**) contains the corresponding eigenvectors, one eigenvector per column. The eigenvectors are normalized so their Euclidean norms are 1

Python to calculate Eigenvalue and Eigenvector

- let's check the eigenvector/eigenvalue condition for the second eigenvalue and eigenvector given $Au = \lambda u$
- So we multiply the eigenvector **v[:,1]** by **A** and check that it is the same as multiplying the same eigenvector by its eigenvalue **w[1]**.

```
>>> u = v[:,1]
>>> print(u)
[ 0.73595785 -0.38198836 -0.55897311]
>>> lam = w[1]
>>> print(lam)
-2.31662479036
>>> print(np.dot(A,u))
[-1.7049382 0.88492371 1.29493096]
>>> print(lam*u)
[-1.7049382 0.88492371 1.29493096]
```

The eigenvalue **w[1]** goes with column **1**, etc.
To extract the ***i***th column vector

Diagonalizable Matrix A

- Let's assume a matrix *A* has two eigenvalues and eigenvectors

$$Av_1 = \lambda_1 v_1 \quad \text{and} \quad Av_2 = \lambda_2 v_2$$

- We can concatenate them together and rewrite the equations in the matrix form

$$A\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- We can generalize it into any number of eigenvectors as $\mathrm{A\,V} = \mathrm{V\,\Lambda}$

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{bmatrix} = AV$$

where *V* concatenates all the eigenvectors and Λ (the capital letter for λ) is the diagonal matrix containing the eigenvalues

Diagonalizable Matrix A

- *V* concatenates all the eigenvectors and Λ (the capital letter for λ) is the diagonal matrix containing the eigenvalues

$$\underbrace{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{\mathbf{V}} \underbrace{\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}}_{\text{Diagonal matrix } \Lambda}$$

- A square matrix ***A*** is **diagonalizable** if we can convert it into a diagonal matrix, like

$$\mathrm{V}^{-1}\mathrm{A}\,\mathrm{V} = \Lambda = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

- An n × n square matrix is diagonalizable if it has *n* linearly independent eigenvectors.

Diagonalizable Matrix A

- An $n \times n$ square matrix is diagonalizable if it has n linearly independent eigenvectors.
- If a matrix is symmetric, it is diagonalizable.
- If a matrix does not have repeated eigenvalue, it always generates enough linearly independent eigenvectors to diagonalize a vector.
- If it has repeated eigenvalues, there is no guarantee we have enough eigenvectors.
- Some will not be diagonalizable.

Why is eigendecomposition useful?

$$V^{-1}AV = \Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

This implies $A = V \Lambda V^{-1}$

- Eigendecomposition decomposes a matrix A into a multiplication of a matrix of eigenvectors V and a diagonal matrix of eigenvalues Λ.
- This can only be done if a matrix is **diagonalizable**. In fact, the definition of a diagonalizable matrix $A \in \mathbb{R}^{n \times n}$ is that it can be eigendecomposed into n eigenvectors, so that $V^{-1}AV = \Lambda$.

Matrix inverse with eigendecomposition

If $\boldsymbol{A}$ is a square matrix with N linearly independent eigenvectors (v_1, v_2, … & v_n and corresponding eigenvalues λ_1, λ_2, … & λ_n), we can rearrange to get $A = V \Lambda V^{-1}$

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}^{-1}$$

square matrix

the inverse exists only if eigenvectors are linearly independent

$A = V \Lambda V^{-1} \Rrightarrow V^{-1}A = V^{-1}V \Lambda V^{-1} \Rrightarrow V^{-1}A = I \Lambda V^{-1} \Rrightarrow V^{-1}AV = \Lambda V^{-1}V$

$\Rrightarrow V^{-1}AV = \Lambda I \Rrightarrow V^{-1}AV = \Lambda$

Matrix Inverse and Power with eigendecomposition

$$A^{-1} = V\Lambda^{-1}V^{-1}$$

- The inverse of Λ is just the inverse of each diagonal element (the eigenvalues).

$A^2 = V\Lambda V^{-1}V\Lambda V^{-1} = V\Lambda \mathrm{I}\, \Lambda V^{-1} = V\Lambda^2 V^{-1}$

Similarly, $A^n = V\Lambda^n V^{-1}$

- The power of Λ is just the power of each diagonal element. This becomes much simpler than *n* multiplications of A.

Properties of Eigendecomposition

- $det(A) = \prod_{i=1}^{n} \lambda_i$ (the determinant of A is equal to the product of its eigenvalues)
- $r(A) = \sum_{i=1}^{n} \lambda_i$ (the trace of A is equal to the sum of its eigenvalues)
- The eigenvalues of A^{-1} are λ_i^{-1}
- The eigenvalues of A^n are λ_i^n
- In general, the eigenvalues of $p(A)$ are $p(\lambda_i)$, for any polynomial p
- The eigenvectors of A^{-1} are the same as the eigenvectors of A
- If A is real and symmetric*, then its eigenvalues are real
- A is invertible if all its eigenvalues are different from zero and vice-versa.

*Recall that a matrix $\boldsymbol{A}$ is **symmetric** if $\boldsymbol{A} = \boldsymbol{A}^T$, where $\boldsymbol{A}^T = (a_{ji})$ is the **transpose** of $\boldsymbol{A} = (a_{ij})$.

Properties of Eigendecomposition

- if A is Hermitian and full-rank (all rows or columns are linearly independent), then the eigenvectors are mutually orthogonal (the dot-product between any two eigenvectors is zero) and the eigenvalues are real.
- If A is complex and Hermitian[&], then its eigenvalues are real
- If A is Hermitian[&] and positive definite[⌘], then its eigenvalues are positive
- If P is nonsingular, then A and PAP^{-1} have the same characteristic polynomial (and the same eigenvalues).

[&]a complex matrix $\boldsymbol{A}$ is **Hermitian** if $\boldsymbol{A} = \boldsymbol{A}^*$, where $\boldsymbol{A}^* = \boldsymbol{A}^T = (\bar{a}_{ji})$. Here $\boldsymbol{A}^*$ is the conjugate transpose of the matrix $\underline{\boldsymbol{A}.}$ Using the syntax of programming, we can write $\boldsymbol{A}^T(i,j) = \boldsymbol{A}(j,i)$ and $\boldsymbol{A}^*(i,j) = \bar{\boldsymbol{A}}\ (j,i)$.

[⌘] Recall also that $\boldsymbol{A}$ is **positive definite** if $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} > 0$ for all nonzero vectors $\boldsymbol{x}$.

Eigenvalues of Similar Matrices

- Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are **similar** to each other if there exists a nonsingular matrix $\boldsymbol{P}$ such that $\boldsymbol{B} = \boldsymbol{PAP}^{-1}$.
- Similar matrices have the same characteristic polynomial

$$\begin{aligned}\mathrm{Det}(\boldsymbol{B} - \lambda \boldsymbol{I}) &= \mathrm{Det}(\boldsymbol{PAP}^{-1} - \lambda \boldsymbol{I}) = \mathrm{Det}(\boldsymbol{P}(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{P}^{-1}) \\ &= \mathrm{Det}(\boldsymbol{P}) \cdot \mathrm{Det}(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \mathrm{Det}(\boldsymbol{P}^{-1}) = \mathrm{Det}(\boldsymbol{A} - \lambda \boldsymbol{I})\end{aligned}$$

- **Theorem: Similar matrices have the same eigenvalues**
- This theorem suggests a strategy for finding eigenvalues of $\boldsymbol{A}$. Transform the matrix $\boldsymbol{A}$ to a matrix $\boldsymbol{B}$ using a similarity transformation $\boldsymbol{B} = \boldsymbol{P}\,\boldsymbol{A}\,\boldsymbol{P}^{-1}$ in which $\boldsymbol{B}$ has a special structure, and then find the eigenvalues of matrix $\boldsymbol{B}$.
- Specifically, if $\boldsymbol{B}$ is *triangular* or *diagonal*, the eigenvalues of $\boldsymbol{B}$ (and those of $\boldsymbol{A}$) are simply the diagonal elements of $\boldsymbol{B}$.

Unitarily Similar Matrices

- Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be **unitarily similar** to each other if $\boldsymbol{B} = \boldsymbol{U}^*\boldsymbol{A}\boldsymbol{U}$ for some unitary matrix $\boldsymbol{U}$
- Recall that a matrix $\boldsymbol{U}$ is **unitary** if $\boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{I}$
- Important theorem and two corollaries that result from this:

Schur's Theorem: Every square matrix is unitarily similar to a triangular matrix

- In this theorem, an arbitrary complex $n \times n$ matrix $\boldsymbol{A}$ is given, and the assertion made is that a unitary matrix $\boldsymbol{U}$ exists such that:

$$\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}* = \boldsymbol{T}$$

where $\boldsymbol{U}\boldsymbol{U}* = \boldsymbol{I}$ and $\boldsymbol{T}$ is a triangular matrix

Proof beyond scope of this class

Corollaries related to Eigenvalues of Similar Matrices

COROLLARY 1 : Every square real matrix is similar to a triangular matrix.

- Thus the factorization $\boldsymbol{PAP}^{-1} = \boldsymbol{T}$ is possible, where $\boldsymbol{T}$ is triangular, $\boldsymbol{P}$ is invertible, and $\boldsymbol{A}$ is real.

COROLLARY 2: Every square Hermitian matrix is unitarily similar to a diagonal matrix.

- A Hermitian matrix, $\boldsymbol{A}$, is factored as $\boldsymbol{A} = \boldsymbol{U}^*\boldsymbol{DU}$

  where $\boldsymbol{D}$ is diagonal and $\boldsymbol{U}$ is unitary.

- Furthermore, $\boldsymbol{U}^*\boldsymbol{AU} = \boldsymbol{T}$ and $\boldsymbol{U}^*\boldsymbol{A}^*\boldsymbol{U} = \boldsymbol{T}^*$ and $\boldsymbol{A} = \boldsymbol{A}^*$, so $\boldsymbol{T} = \boldsymbol{T}^*$, which must be a diagonal matrix.
- Most numerical methods for finding eigenvalues of an $n \times n$ matrix $\boldsymbol{A}$ proceed by determining such similarity transformations.
- Then one eigenvalue at a time, say, λ, is computed, and a **deflation process** is used to produce an $(n - 1) \times (n - 1)$ matrix $\boldsymbol{A}$ whose eigenvalues are the same as those of $\boldsymbol{A}$, except for λ