Polynomial Interpolation

CS3010

Numerical Methods

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Section 4.1

Determining coefficients

- Systematically determine coefficients by setting x equal to $x_0, x_1, \ldots, x_n$ in the previous equation

$$
\begin{aligned}
f(x_0) &= a_0 \\
f(x_1) &= a_0 + a_1(x_1 - x_0) \\
f(x_2) &= a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
&\ldots\ldots
\end{aligned}
$$

$$f(x_k) = \sum_{i=0}^{k} a_i \prod_{j=0}^{i-1} (x_k - x_j) \qquad (0 \le k \le n)$$

- As seen from the above equation, a_k depends on the values of f at nodes $x_0, x_1, \ldots, x_k$ and the notation used is

$$a_k = f[x_0, x_1, \ldots., x_k]$$

Newton's Form of interpolating polynomial

- Set $f(x_0) = f(x_0)$ and compute the rest for k = 1, 2,...., n

$$p_n(x) = a_0 + \sum_{i=1}^{n} f[x_0, x_1, ..., x_i] \left[\prod_{j=0}^{j=i-1} (x - x_j) \right]$$

$$f(x_k) = a_k \prod_{j=0}^{j=k-1} (x_k - x_j) + \sum_{i=1}^{k-1} a_i \left[\prod_{j=0}^{j=i-1} (x_k - x_j) \right]$$

and

$$a_k = \frac{f(x_k) - \sum_{i=1}^{k-1} a_i \left[\prod_{j=0}^{j=i-1} (x_k - x_j) \right]}{\prod_{j=0}^{j=k-1} (x_k - x_j)}$$

Substituding for a_k from the previous equation

$$f[x_0, x_1, ..., x_k] = \frac{f(x_k) - \sum_{i=1}^{k-1} f[x_0, x_1, ..., x_i] \left[\prod_{j=0}^{j=i-1} (x_k - x_j) \right]}{\prod_{j=0}^{j=k-1} (x_k - x_j)}$$

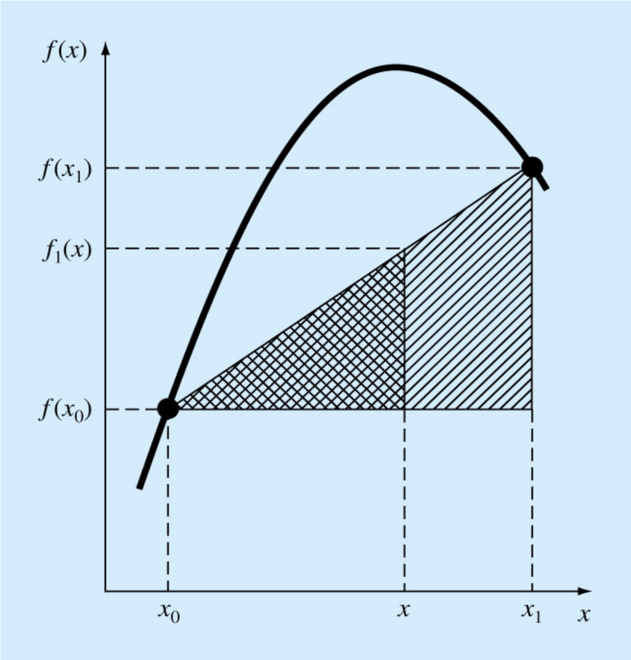
Newton's Divided-Difference Interpolating Polynomials

- Linear Interpolation
  - Is the simplest form of interpolation, connecting two data points with a straight line.

Slope and a finite divided difference approximation to 1st derivative

$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\boxed{f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)}$$



- $p(x)$ designates that this is a first-order interpolating polynomial.
- It is clearly seen that $p(x_0) = y_0$ and $p(x_1) = y_1$

Quadratic Interpolation

- If three data points are available, the estimate is improved by introducing some curvature into the line connecting the points.

$$f_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

- A simple procedure can be used to determine the values of the coefficients.

$$x = x_0 \qquad a_0 = f(x_0)$$

$$x = x_1 \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$x = x_2 \qquad a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Recursive Property of Divided Differences

$$f_n(x) = f(x_0) + (x - x_0)f[x_1,x_0] + (x - x_0)(x - x_1)f[x_2,x_1,x_0]$$
$$+ \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_n,x_{n-1},\cdots,x_0]$$

$$a_0 = f(x_0)$$

$$a_1 = f[x_1,x_0]$$

$$a_2 = f[x_2,x_1,x_0]$$

$$\vdots$$

$$a_n = f[x_n,x_{n-1},\cdots,x_1,x_0]$$

$$f[x_i,x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i,x_j,x_k] = \frac{f[x_i,x_j] - f[x_j,x_k]}{x_i - x_k}$$

Bracketed function evaluations are finite divided differences

$$\vdots$$

$$\boxed{f[x_n,x_{n-1},\ldots,x_1,x_0] = \frac{f[x_n,x_{n-1},\ldots,x_1] - f[x_{n-1},x_{n-2},\ldots,x_0]}{x_n - x_0}}$$

Invariance Theorem

- The divided difference $f[x_0,x_1,\ldots.,x_k]$ is invariant under all permutations of arguments $x_0,x_1,\ldots.x_k$.
- This is true because $f[]$'s are the coefficients of the interpolating polynomial
- $f[x_0,x_1,\ldots.,x_k]$ doesn't change in value if the nodes $x_0,x_1,\ldots.x_k$ are permuted, hence $f[x_0,x_1,x_2] = f[x_2,x_1,x_0] = f[x_1,x_2,x_0]$

i	x_i	$f(x_i)$	**First**	**Second**	**Third**
0	x_0	$f(x_0)$	$f[x_1, x_0]$	$f[x_2, x_1, x_0]$	$f[x_3, x_2, x_1, x_0]$
1	x_1	$f(x_1)$	$f[x_2, x_1]$	$f[x_3, x_2, x_1]$	
2	x_2	$f(x_2)$	$f[x_3, x_2]$		
3	x_3	$f(x_3)$			

Divided-Difference Table

- First 3 divided differences are

$$f_n(x) = f(x_0) + (x - x_0) f[x_1, x_0] + (x - x_0)(x - x_1) f[x_2, x_1, x_0]$$
$$+ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_n, x_{n-1}, \cdots, x_0]$$

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_1, x_0]}{x_2 - x_0}$$

x	f[]	f[,]	f[, ,]	f[, , ,]
x_0	$f[x_0]$			
		$f[x_0,x_1]$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2]$	
		$f[x_1,x_2]$		$f[x_0,x_1,x_2,x_3]$
x_2	$f[x_2]$		$f[x_1,x_2,x_3]$	
		$f[x_1,x_2]$		
x_3	$f[x_3]$			

- Coefficients along the top diagonal are the ones needed for the IP

Examples

- Using Newton's form, find the interpolating polynomial of least order for the following table

x	1	-4	0
$f(x)$	3	13	-23

- Determine $f[x_0], f[x_0, x_1]$ and $f[x_0, x_1, x_2]$
- $f[x_0] = 3$
- $f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = -2$
- $f[x_0, x_1, x_2] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = 7$

Example

x	1	3/2	0	2
f(x)	3	13/4	3	5/3

- Complete table of divided-difference is:

x	f[]	f[,]	f[, ,]	f[, , ,]
1	3			
		1/2		
3/2	13/4		1/3	
		1/6		-2
0	3		-5/3	
		-2/3		
2	5/3			

$p_3(x) = 3 + 1/2(x-1) + 1/3(x-1)(x-3/2) - 2(x-1)(x-3/2)x$

$p_3(x) = -2x^3+16/3x^2-10/3x+3$