Power Method
Eigenvalue Computation

Section 8.3

CS 3010

Numerical Methods

Dr. Amar Raheja

Problem with finding Eigenvalues for large *n*

- Remember that the eigenvalues of an *n* x *n* matrix *A* are obtained by solving its characteristic equation
$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_0 = 0$$
- For large values of *n*, polynomial equations like this one are difficult and time-consuming to solve.
- Moreover, numerical techniques for approximating roots of polynomial equations of high degree are sensitive to rounding errors.
- Power method is an alternative method for approximating eigenvalues.
- This method can be used only to find the eigenvalue of *A* that is largest in absolute value—this eigenvalue is called the **dominant eigenvalue** of *A*.
- Although this restriction may seem severe, dominant eigenval- ues are of primary interest in many physical applications.

Definition and Note on Dominant Eigenvalue

Definition of Dominant Eigenvalue and Dominant Eigenvector

Let $\lambda_1, \lambda_2, \ldots,$ and λ_n be the eigenvalues of an $n \times n$ matrix A. λ_1 is called the **dominant eigenvalue** of A if

$$|\lambda_1| > |\lambda_i|, \quad i = 2, \ldots, n.$$

The eigenvectors corresponding to λ_1 are called **dominant eigenvectors** of A.

- Not every matrix has a dominant eigenvalue $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Matrix A above with eigenvalues of λ_1=1 and λ_2=-1 has no dominant eigenvalue.
- Similarly, the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- with eigenvalues of λ_1=2 and λ_2=2, and λ_2=1 has no dominant eigenvalue.

Find the dominant eigenvalue

- Find the dominant eigenvalue of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$
- The characteristic polynomial of A is $\lambda^2 + 3\lambda + 2$
- So the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$ which the dominant one is $\lambda_2 = -2$
- From the same example you know that the dominant eigenvectors of A is those corresponding to $\lambda_2 = -2$ is of the form

$$x = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ where } t \neq 0$$

Power Method for Approximating Eigenvalue

- A procedure called the **power method** can be employed to compute highest eigenvalue.
- It is an example of an *iterative* process that, under the right circumstances, will produce a sequence converging to an eigenvalue of a given matrix.
- Suppose that ***A*** is an $n \times n$ matrix, and that its eigenvalues (which we do not know) have the following property:

$$|\lambda_1| > |\lambda_2| \geqq |\lambda_3| \geqq \cdots \geqq |\lambda_n|$$

- This strict inequality is simply ordering the eigenvalues according to decreasing absolute value.
- Notice in equality that λ_1 is a dominant eigenvalue

Power Method for Approximating Eigenvalue

- Each eigenvalue has a nonzero eigenvector $\boldsymbol{u}^{(i)}$ and

$$\boldsymbol{A}\boldsymbol{u}^{(i)} = \lambda_i \boldsymbol{u}^{(i)} \qquad (i = 1, 2, \ldots, n) \qquad (1)$$

- We assume that there is a linearly independent set of n eigenvectors

$$\{\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \ldots, \boldsymbol{u}^{(n)}\}, \text{ which is a basis for } \mathbb{C}^n$$

- We want to compute the *single* eigenvalue of maximum modulus (the *dominant* eigen- value) and an associated eigenvector.
- We select an arbitrary starting vector, $\boldsymbol{x}^{(0)} \in \mathbb{C}^n$ and express it as a linear combination of $\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \ldots, \boldsymbol{u}^{(n)}$

$$\boldsymbol{x}^{(0)} = c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)} + \cdots + c_n\boldsymbol{u}^{(n)}$$

- In this equation, we must assume that $c_1 \neq 0$. Since the coefficients can be absorbed into the vectors $\boldsymbol{u}^{(i)}$, there is no loss of generality in assuming that

$$\boldsymbol{x}^{(0)} = \boldsymbol{u}^{(1)} + \boldsymbol{u}^{(2)} + \cdots + \boldsymbol{u}^{(n)} \qquad (2)$$

Power Method for Approximating Eigenvalue

- Choose an initial approximation $\boldsymbol{x}^{(0)}$ of one of the dominant eigen*vectors* of A.
- This initial approximation must be a *nonzero* vector in C^n. Finally, form the sequence given by repeatedly carrying out matrix-vector multiplication, using the matrix $\boldsymbol{A}$ to produce a sequence of vectors.

$$
\begin{cases}
\boldsymbol{x}^{(1)} = \boldsymbol{A}\boldsymbol{x}^{(0)} \\
\boldsymbol{x}^{(2)} = \boldsymbol{A}\boldsymbol{x}^{(1)} = \boldsymbol{A}^2\boldsymbol{x}^{(0)} \\
\boldsymbol{x}^{(3)} = \boldsymbol{A}\boldsymbol{x}^{(2)} = \boldsymbol{A}^3\boldsymbol{x}^{(0)} \\
\qquad \vdots \\
\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} = \boldsymbol{A}^k\boldsymbol{x}^{(0)} \\
\qquad \vdots
\end{cases}
$$

- In general, we have $\boldsymbol{x}^{(k)} = \boldsymbol{A}^k \boldsymbol{x}^{(0)}$ $\qquad (k = 1, 2, 3, \ldots)$

Power Method for Approximating Eigenvalue

- Substituting $\boldsymbol{x}^{(0)}$ in Equation (2), we obtain

$$\begin{aligned}\boldsymbol{x}^{(k)} &= \boldsymbol{A}^k \boldsymbol{x}^{(0)} \\ &= \boldsymbol{A}^k \boldsymbol{u}^{(1)} + \boldsymbol{A}^k \boldsymbol{u}^{(2)} + \boldsymbol{A}^k \boldsymbol{u}^{(3)} + \cdots + \boldsymbol{A}^k \boldsymbol{u}^{(n)} \\ &= \lambda_1^k \boldsymbol{u}^{(1)} + \lambda_2^k \boldsymbol{u}^{(2)} + \lambda_3^k \boldsymbol{u}^{(3)} + \cdots + \lambda_n^k \boldsymbol{u}^{(n)}\end{aligned}$$

- by using Equation (1). This can be written in the form

$$\boldsymbol{x}^{(k)} = \lambda_1^k \left[\boldsymbol{u}^{(1)} + \left(\frac{\lambda_2}{\lambda_1}\right)^k \boldsymbol{u}^{(2)} + \left(\frac{\lambda_3}{\lambda_1}\right)^k \boldsymbol{u}^{(3)} + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k \boldsymbol{u}^{(n)} \right]$$

Since $|\lambda_1| > |\lambda_j|$ for $j > 1$, we have $|\lambda_j/\lambda_1| < 1$ and $\left(\lambda_j/\lambda_1\right)^k \to 0$ as $k \to \infty$. To simplify the notation, we write the above equation in the form

$$\boldsymbol{x}^{(k)} = \lambda_1^k \left[\boldsymbol{u}^{(1)} + \boldsymbol{\varepsilon}^{(k)} \right] \tag{3}$$

where $\varepsilon^{(k)} \to 0$ as $k \to \infty$.

Power Method Example

- Find the dominant Eigenvalue and vector for $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$
- Begin with an initial nonzero approximation of

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Iteration — *"Scaled" Approximation*

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} \rightarrow -4\begin{bmatrix} 2.50 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix} \rightarrow 10\begin{bmatrix} 2.80 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix} \rightarrow -22\begin{bmatrix} 2.91 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -64 \\ -22 \end{bmatrix} = \begin{bmatrix} 136 \\ 46 \end{bmatrix} \rightarrow 46\begin{bmatrix} 2.96 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x}_5 = A\mathbf{x}_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 136 \\ 46 \end{bmatrix} = \begin{bmatrix} -280 \\ -94 \end{bmatrix} \rightarrow -94\begin{bmatrix} 2.98 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -280 \\ -94 \end{bmatrix} = \begin{bmatrix} 568 \\ 190 \end{bmatrix} \rightarrow 190\begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}$$

- Note that the approximations in this Example appear to be approaching scalar multiples of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Rayleigh Quotient

Determining an Eigenvalue from an Eigenvector

If $\mathbf{x}$ is an eigenvector of a matrix A, then its corresponding eigenvalue is given by

$$\lambda = \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}.$$

This quotient is called the **Rayleigh quotient.**

- Because $\mathbf{x}$ is an eigenvector of A, you know that $A\mathbf{x} = \lambda\mathbf{x}$ and can write

$$\frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\lambda\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\lambda(\mathbf{x} \cdot \mathbf{x})}{\mathbf{x} \cdot \mathbf{x}} = \lambda.$$

- In cases for which the power method generates a good approximation of a dominant eigenvector, the Rayleigh quotient provides a correspondingly good approximation of the dominant eigenvalue.

Example: Determining Eigenvalue from Eigenvector

- After the sixth iteration of the power method $\mathbf{x}_6 = \begin{bmatrix} 568 \\ 190 \end{bmatrix} \approx 190 \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}$
- With **x =** (2.99, 1) as the approximation of a dominant eigenvector of *A*, use the Rayleigh quotient to obtain an approximation of the dominant eigenvalue of *A*. First compute the product *A***x**.

$$A\mathbf{x} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = \begin{bmatrix} -6.02 \\ -2.01 \end{bmatrix}$$

- Then, because

$$A\boldsymbol{x} \cdot \boldsymbol{x} = (-6.02)(2.99) + (-2.01)(1) \approx -20.0$$

and $\boldsymbol{x} \cdot \boldsymbol{x} = (2.99)(2.99) + (1)(1) \approx 9.94$

$$\lambda = \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} \approx \frac{-20.0}{9.94} \approx -2.01$$

- which is a good approximation of the dominant eigenvalue λ = -2.

Class Exercise

- Calculate seven iterations of the power method with *scaling* to approximate a dominant eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

- Use $\mathbf{x}_0$ (1, 1, 1) as the initial approximation.
- One iteration of the power method produces and by scaling one obtains the approximation

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \qquad \mathbf{x}_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}$$

Class Exercise

- A second iteration yields

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.00 \\ 2.20 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{1}{2.20}\begin{bmatrix} 1.00 \\ 1.00 \\ 2.20 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}$$

- Continuing this process, you obtain the sequence of approximations shown below

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$
$\begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.48 \\ 0.55 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.51 \\ 0.51 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.49 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$

- Using the Rayleigh quotient, you can approximate the dominant eigenvalue of A to be $\lambda = 3$.

Remark on convergence of Power Method Eigenvalue

- REMARK: Note that the *scaling factors* used to obtain the vectors in Table are approaching the dominant eigenvalue λ = 3

$$
\begin{array}{ccccccc}
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 & \mathbf{x}_6 & \mathbf{x}_7 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
5.00 & 2.20 & 2.82 & 3.13 & 3.02 & 2.99 & 3.00,
\end{array}
$$