Class exercise section 2.2 and section 8.4 | Christopher Koepke

1. Solve the following coefficient matrix system using Gaussian elimination with scaled partial pivoting: show intermediate matrices, vector b not provided, so just show the 3 intermediate matricies.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix}$$

scale vector $= [\overset{1}{3}, \overset{2}{3}, \overset{3}{6}, \overset{4}{6}]$

index vector $= [1, 3]$

scale ratio $= [\frac{3}{1}, \frac{6}{3}] = [3, 2]$ * 1st pivot $I = 1$

multipliers $= [\frac{3}{1}] = [3]$ * $\frac{3rd}{1st}$

$$\begin{array}{rl} \text{3rd} & 3 - 3 + 0 + 6 \\ \text{1st} & -(3)(1 + 0 + 3 + 0) \\ \hline & 0 - 3 - 9 + 6 \end{array}$$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & -3 & -9 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix}$$

$l = [1, 3, 2, 4]$

ratio $= [\overset{3}{\frac{3}{3}}, \overset{2}{\frac{3}{1}}, \overset{4}{\frac{6}{2}}] = [1, 3, 3]$ * 2nd pivot $I = 2$

multiplier $= [\frac{-3}{1}, \frac{2}{1}] = [\overset{3}{-3}, \overset{4}{2}]$

$$\begin{array}{rl} \text{3rd} & -3 - 9 + 6 \\ \text{2nd} & -(-3)(1 + 3 - 1) \\ \hline & 0 + 0 + 3 \end{array} \qquad \begin{array}{rl} \text{4th} & 2 + 4 - 6 \\ \text{2nd} & -(2)(1 + 3 - 1) \\ \hline & 0 - 2 - 4 \end{array}$$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

$l = [1, 2, 3, 4]$

* No need to show 3rd intermediate matrix as the 3rd row only has one non-zero coefficient.

2. Using the Jacobi, Gauss-Seidel, and S.O.R ($\omega = 1.1$) interative methods, solve the following linear system to four decimal places (rounded) of accuracy. Calculate 3 interations using each method starting with $[0\ 0\ 0\ 0]^T$. Check that the exact solution is $\bar{X} = [1\ -1\ 1\ -1]^T$.

$$\begin{bmatrix} 7 & 1 & -1 & 2 \\ 1 & 8 & 0 & -2 \\ -1 & 0 & 4 & -1 \\ 2 & -2 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \\ -3 \end{bmatrix}$$

- Jacobi

$$x_1^{(k)} = -(1x_2^{(k-1)} - 1x_3^{(k-1)} + 2x_4^{(k-1)} - 3)/7 = (-\tfrac{1}{7})x_2^{(k-1)} + (\tfrac{1}{7})x_3^{(k-1)} - (\tfrac{2}{7})x_4^{(k-1)} + \tfrac{3}{7}$$
$$x_2^{(k)} = -(1x_1^{(k-1)} + 0x_3^{(k-1)} - 2x_4^{(k-1)} + 5)/8 = (-\tfrac{1}{8})x_1^{(k-1)} + (\tfrac{1}{4})x_4^{(k-1)} - \tfrac{5}{8}$$
$$x_3^{(k)} = -(-1x_1^{(k-1)} + 0x_2^{(k-1)} - 1x_4^{(k-1)} - 4)/4 = (\tfrac{1}{4})x_1^{(k-1)} + (\tfrac{1}{4})x_4^{(k-1)} + 1$$
$$x_4^{(k)} = -(2x_1^{(k-1)} - 2x_2^{(k-1)} - 1x_3^{(k-1)} + 3)/6 = (-\tfrac{1}{3})x_1^{(k-1)} + (\tfrac{1}{3})x_2^{(k-1)} + (\tfrac{1}{6})x_3^{(k-1)} - \tfrac{1}{2}$$

starting solution $\bar{x} = [0\ 0\ 0\ 0]^T$

$x_1^{(1)} = \frac{3}{7}$ $\quad x_2^{(1)} = -\frac{5}{8}$ $\quad x_3^{(1)} = 1$ $\quad x_4^{(1)} = -\frac{1}{2}$

$$x_1^{(2)} = (-\tfrac{1}{7})(-\tfrac{5}{8}) + (\tfrac{1}{7})(1) - (\tfrac{2}{7})(-\tfrac{1}{2}) + \tfrac{3}{7} \Rightarrow x_1^{(2)} = \tfrac{45}{56}$$
$$x_2^{(2)} = (-\tfrac{1}{8})(\tfrac{3}{7}) + (\tfrac{1}{4})(-\tfrac{1}{2}) - \tfrac{5}{8} \Rightarrow x_2^{(2)} = -\tfrac{45}{56}$$
$$x_3^{(2)} = (\tfrac{1}{4})(\tfrac{3}{7}) + (\tfrac{1}{4})(\tfrac{1}{2}) + 1 \Rightarrow x_3^{(2)} = \tfrac{55}{56}$$
$$x_4^{(2)} = (-\tfrac{1}{3})(\tfrac{3}{7}) + (\tfrac{1}{3})(-\tfrac{5}{8}) + (\tfrac{1}{6})(1) - \tfrac{1}{2} \Rightarrow x_4^{(2)} = -\tfrac{115}{168}$$

$$x_1^{(3)} = -\tfrac{1}{7}(-\tfrac{45}{56}) + (\tfrac{1}{7})(\tfrac{55}{56}) - (\tfrac{2}{7})(-\tfrac{115}{168}) + \tfrac{3}{7} \Rightarrow x_1^{(3)} = \tfrac{517}{588}$$
$$x_2^{(3)} = -\tfrac{1}{8}(\tfrac{45}{56}) + (\tfrac{1}{4})(-\tfrac{115}{168}) - \tfrac{5}{8} \Rightarrow x_2^{(3)} = -\tfrac{1205}{1344}$$
$$x_3^{(3)} = \tfrac{1}{4}(\tfrac{45}{56}) + \tfrac{1}{4}(-\tfrac{115}{168}) + 1 \Rightarrow x_3^{(3)} = \tfrac{173}{168}$$
$$x_4^{(3)} = -\tfrac{1}{3}(\tfrac{45}{56}) + \tfrac{1}{3}(-\tfrac{45}{56}) + \tfrac{1}{6}(\tfrac{55}{56}) - \tfrac{1}{2} \Rightarrow x_4^{(3)} = -\tfrac{293}{336}$$

$$\boxed{x^{(3)} = [0.8979, -0.8966, 1.0298, -0.8720]}$$

• Gauss-seidel

starting solution
$\bar{x} = [0\ 0\ 0\ 0]^T$

$$X_1^{(k)} = (-\tfrac{1}{7})x_2^{(k-1)} + (\tfrac{1}{7})x_3^{(k-1)} - (\tfrac{2}{7})x_4^{(k-1)} + \tfrac{3}{7}$$
$$X_2^{(k)} = (-\tfrac{1}{8})x_1^{(k)} + (\tfrac{1}{4})x_4^{(k-1)} - \tfrac{5}{8}$$
$$X_3^{(k)} = (\tfrac{1}{4})x_1^{(k)} + (\tfrac{1}{4})x_4^{(k-1)} + 1$$
$$X_4^{(k)} = (-\tfrac{1}{3})x_1^{(k)} + (\tfrac{1}{3})x_2^{(k)} + (\tfrac{1}{6})x_3^{(k)} - \tfrac{1}{2}$$

$$X_1^{(1)} = (-\tfrac{1}{7})(0) + (\tfrac{1}{7})(0) - \tfrac{2}{7}(0) + \tfrac{3}{7} \Rightarrow X_1^{(1)} = \tfrac{3}{7}$$
$$X_2^{(1)} = (-\tfrac{1}{8})(\tfrac{3}{7}) + (\tfrac{1}{4})(0) - \tfrac{5}{8} \Rightarrow X_2^{(1)} = -\tfrac{19}{28}$$
$$X_3^{(1)} = (\tfrac{1}{4})(\tfrac{3}{7}) + (\tfrac{1}{4})(0) + 1 \Rightarrow X_3^{(1)} = \tfrac{31}{28}$$
$$X_4^{(1)} = (-\tfrac{1}{3})(\tfrac{3}{7}) + (\tfrac{1}{3})(-\tfrac{19}{28}) + (\tfrac{1}{6})(\tfrac{31}{28}) - \tfrac{1}{2} \Rightarrow X_4^{(1)} = -\tfrac{115}{168}$$

$$X_1^{(2)} = -\tfrac{1}{7}(-\tfrac{19}{28}) + \tfrac{1}{7}(\tfrac{31}{28}) - \tfrac{2}{7}(-\tfrac{115}{168}) + \tfrac{3}{7} \Rightarrow X_1^{(2)} = \tfrac{517}{588}$$
$$X_2^{(2)} = -\tfrac{1}{8}(\tfrac{517}{588}) + \tfrac{1}{4}(-\tfrac{115}{168}) - \tfrac{5}{8} \Rightarrow X_2^{(2)} = -\tfrac{2131}{2352}$$
$$X_3^{(2)} = \tfrac{1}{4}(\tfrac{517}{588}) + \tfrac{1}{4}(-\tfrac{115}{168}) + 1 \Rightarrow X_3^{(2)} = \tfrac{4933}{4704}$$
$$X_4^{(2)} = -\tfrac{1}{3}(\tfrac{517}{588}) + \tfrac{1}{3}(-\tfrac{2131}{2352}) + \tfrac{1}{6}(\tfrac{4933}{4704}) - \tfrac{1}{2} \Rightarrow X_4^{(2)} = -0.9203160431$$

$$X_1^{(3)} = -\tfrac{1}{7}(-\tfrac{2131}{2352}) + \tfrac{1}{7}(\tfrac{4933}{4704}) - \tfrac{2}{7}(-0.9203160431) + \tfrac{3}{7} \Rightarrow X_1^{(3)} = 0.9707644963$$
$$X_2^{(3)} = -\tfrac{1}{8}(0.9707644963) + \tfrac{1}{4}(-0.9203160431) - \tfrac{5}{8} \Rightarrow X_2^{(3)} = -0.9764245728$$
$$X_3^{(3)} = (\tfrac{1}{4})(0.9707644963) + (\tfrac{1}{4})(-0.9203160431) + 1 \Rightarrow X_3^{(3)} = 1.012612113$$
$$X_4^{(3)} = -\tfrac{1}{3}(0.9707644963) + \tfrac{1}{3}(-0.9764245728) + \tfrac{1}{6}(1.012612113) - \tfrac{1}{2}$$
$$\Rightarrow X_4^{(3)} = -0.9802943375$$

$$\bar{X} = [0.9708, -0.9764, 1.0126, -0.9803]$$

• Gauss-seidel w/ S.O.R $\quad * \; \omega = 1.1$

$$x_1^{(k)} = (1.1)\left[\left(-\tfrac{1}{7}\right)x_2^{(k-1)} + \left(\tfrac{1}{7}\right)x_3^{(k-1)} - \left(\tfrac{2}{7}\right)x_4^{(k-1)} + \tfrac{3}{7}\right] + (-0.1)\,x_1^{(k-1)}$$

$$x_2^{(k)} = (1.1)\left[\left(-\tfrac{1}{8}\right)x_1^{(k)} + \left(\tfrac{1}{4}\right)x_4^{(k-1)} - \tfrac{5}{8}\right] + (-0.1)\,x_2^{(k-1)}$$

starting solution
$\bar{x} = [0\ 0\ 0\ 0]^T$

$$x_3^{(k)} = (1.1)\left[\left(\tfrac{1}{4}\right)x_1^{(k)} + \left(\tfrac{1}{4}\right)x_4^{(k-1)} + 1\right] + (-0.1)\,x_3^{(k-1)}$$

$$x_4^{(k)} = (1.1)\left[\left(-\tfrac{1}{3}\right)x_1^{(k)} + \left(\tfrac{1}{3}\right)x_2^{(k)} + \left(\tfrac{1}{6}\right)x_3^{(k)} - \tfrac{1}{2}\right] + (-0.1)\,x_4^{(k-1)}$$

$$x_1^{(1)} = (1.1)\left[\left(-\tfrac{1}{7}\right)(0) + \left(\tfrac{1}{7}\right)(0) - \left(\tfrac{2}{7}\right)(0) + \tfrac{3}{7}\right] - 0.1(0) \Rightarrow x_1^{(1)} = 0.4714285714$$

$$x_2^{(1)} = (1.1)\left[\left(-\tfrac{1}{8}\right)x_1^{(1)} + \left(\tfrac{1}{4}\right)(0) - \tfrac{5}{8}\right] - 0.1(0) \Rightarrow x_2^{(1)} = -0.7523214285$$

$$x_3^{(1)} = (1.1)\left[\left(\tfrac{1}{4}\right)x_1^{(1)} + \left(\tfrac{1}{4}\right)(0) + 1\right] - 0.1(0) \Rightarrow x_3^{(1)} = 1.229642857$$

$$x_4^{(1)} = (1.1)\left[\left(-\tfrac{1}{3}\right)x_1^{(1)} + \left(\tfrac{1}{3}\right)x_2^{(1)} + \left(\tfrac{1}{6}\right)x_3^{(1)} - \tfrac{1}{2}\right] - 0.1(0) \Rightarrow x_4^{(1)} = -0.4049047633$$

$$x_1^{(2)} = (1.1)\left[\left(-\tfrac{1}{7}\right)x_2^{(1)} + \left(\tfrac{1}{7}\right)x_3^{(1)} - \left(\tfrac{2}{7}\right)x_4^{(1)} + \tfrac{3}{7}\right] - (0.1)\,x_1^{(1)} \Rightarrow x_1^{(2)} = 0.8629930276$$

$$x_2^{(2)} = (1.1)\left[\left(-\tfrac{1}{8}\right)x_1^{(2)} + \left(\tfrac{1}{4}\right)x_4^{(1)} - \tfrac{5}{8}\right] - (0.1)\,x_2^{(1)} \Rightarrow x_2^{(2)} = -0.8422782083$$

$$x_3^{(2)} = (1.1)\left[\left(\tfrac{1}{4}\right)x_1^{(2)} + \left(\tfrac{1}{4}\right)x_4^{(1)} + 1\right] - (0.1)\,x_3^{(1)} \Rightarrow x_3^{(2)} = 1.103009987$$

$$x_4^{(2)} = (1.1)\left[\left(-\tfrac{1}{3}\right)x_1^{(2)} + \left(\tfrac{1}{3}\right)x_2^{(2)} + \left(\tfrac{1}{6}\right)x_3^{(2)} - \tfrac{1}{2}\right] - (0.1)\,x_4^{(1)} \Rightarrow x_4^{(2)} = -0.932557145 7$$

$$x_1^{(3)} = (1.1)\left[\left(-\tfrac{1}{7}\right)x_2^{(2)} + \left(\tfrac{1}{7}\right)x_3^{(2)} - \left(\tfrac{2}{7}\right)x_4^{(2)} + \tfrac{3}{7}\right] - (0.1)\,x_1^{(2)} \Rightarrow x_1^{(3)} = 0.9839130775$$

$$x_2^{(3)} = (1.1)\left[\left(-\tfrac{1}{8}\right)x_1^{(3)} + \left(\tfrac{1}{4}\right)x_4^{(2)} - \tfrac{5}{8}\right] - (0.1)\,x_2^{(2)} \Rightarrow x_2^{(3)} = -0.9950134423$$

$$x_3^{(3)} = (1.1)\left[\left(\tfrac{1}{4}\right)x_1^{(3)} + \left(\tfrac{1}{4}\right)x_4^{(2)} + 1\right] - (0.1)\,x_3^{(2)} \Rightarrow x_3^{(3)} = 1.003821883$$

$$x_4^{(3)} = (1.1)\left[\left(-\tfrac{1}{3}\right)x_1^{(3)} + \left(\tfrac{1}{3}\right)x_2^{(3)} + \left(\tfrac{1}{6}\right)x_3^{(3)} - \tfrac{1}{2}\right] - (0.1)\,x_4^{(2)} \Rightarrow x_4^{(3)} = -0.9983166641$$

$$x^{(3)} = [0.9839,\ -0.9950,\ 1.0038,\ -0.9983]$$

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exact solution $x = [1\ \ -1\ \ 1\ \ -1]$

- $7(1) + 1(-1) - 1(1) + 2(-1) = 3 \Rightarrow 3 = 3$ ✓
- $1(1) + 8(-1) + 0(1) - 2(-1) = -5 \Rightarrow -5 = -5$ ✓
- $-1(1) + 0(-1) + 4(1) - 1(-1) = 4 \Rightarrow 4 = 4$ ✓
- $2(1) - 2(-1) - 1(1) + 6(-1) = -3 \Rightarrow -3 = -3$ ✓