Locating Roots of Equations Secant Method

CS3010

Numerical Methods

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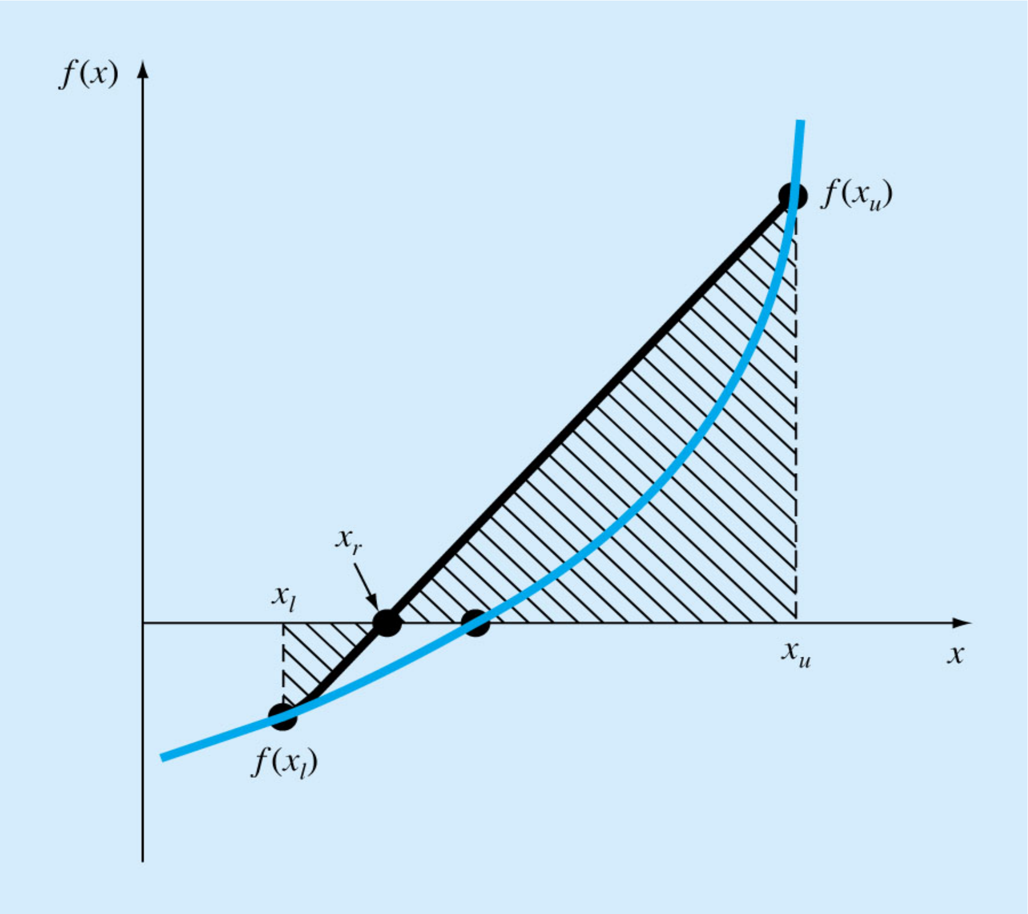
Section 3.3

Lecture 6

What's a Secant?

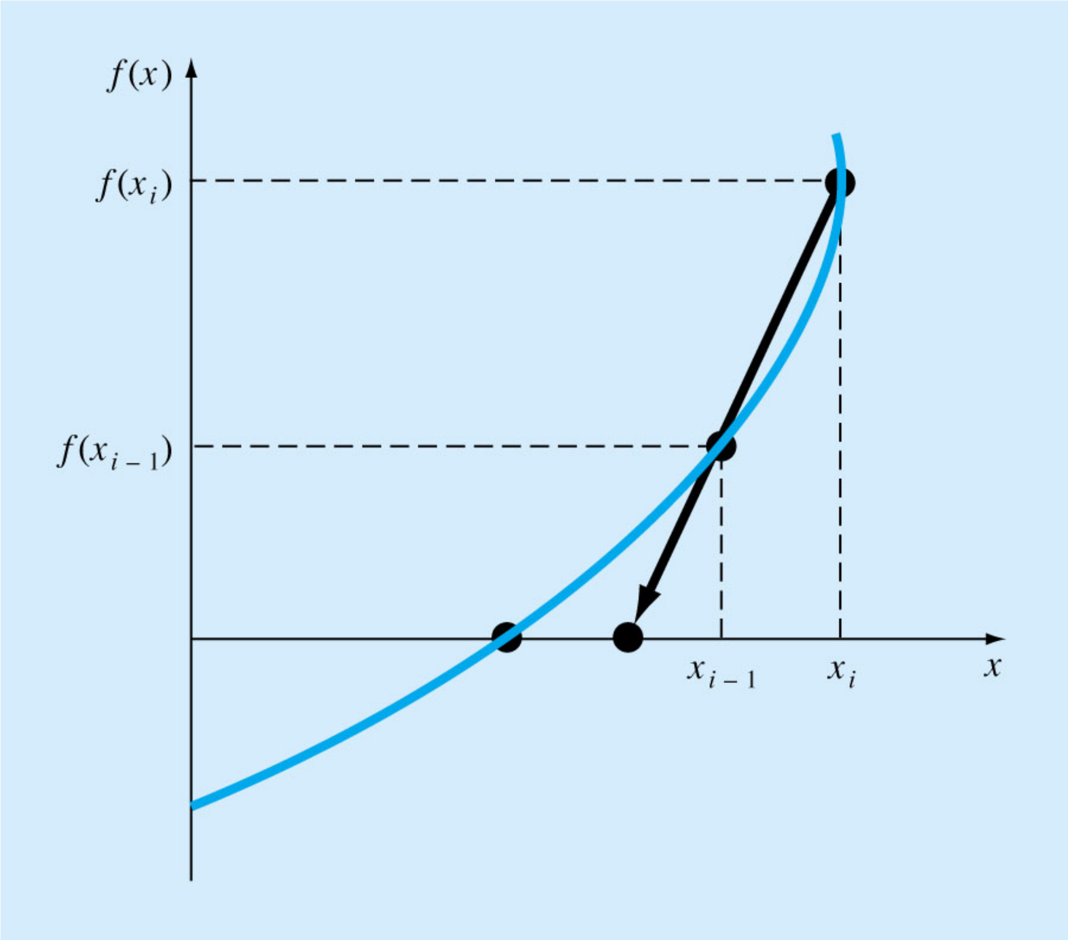
- Secant is a line that crosses the function at two points.
- False-Position is a method that talked about secant
- Secant method is the same as False-Position with the difference being that the two points to be picked don't have to bracket the root.
- Formulation is exactly the same



$$c = b - f(b)\left[\frac{a-b}{f(a)-f(b)}\right] = a - f(a)\left[\frac{b-a}{f(b)-f(a)}\right] = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

The Secant Method

- Requires two initial estimates of x , e.g, x_o, x_1. However, because $f(x)$ is not required to change signs between estimates, it is not classified as a "bracketing" method.
- The secant method has the same properties as Newton's method. Convergence is not guaranteed for all $x_o, x_1, f(x)$.



The Secant Method (alternative formulation)

- A slight variation of Newton's method for functions whose derivatives are difficult to evaluate.
- For these cases, the derivative can be approximated by a backward finite divided difference.
- This approximation can be got from the Taylor series expansion using only first two terms of the series.

$$f(x_i + h) = f(x_i) + (x_{i-1} - x_i)f'(x_i)$$

$$f'(x_i) = \frac{f(x_i + h) - f(x_i)}{(x_{i-1} - x_i)} = \frac{f(x_{i-1}) - f(x_i)}{(x_{i-1} - x_i)}$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}$$

Using this in the Newton-Raphson iterative formulation, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$$\boxed{x_{i+1} = x_i - \left(\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}\right) f(x_i)}$$

Convergence

- While programming the secant method, calculate and test the quantity $f(xi) - f(x_{i-1})$
- Method is succeeding as x_i is approaching a root i.e. $f(xi)$converges to zero.
- f$f(x_{i-1})$
- will also be converging to zero, and hence $f(xi) - f(x_{i-1})$ will approach zero.
- Iterations can be halted when for some specified tolerance δ

$$\left|f(x_i) - f(x_{i-1})\right| \leq \delta \left|f(x_i)\right|$$

- Best method to halt iterations: use relative approximate error

Algorithm

```
procedure Secant( f, a, b, nmax, ε) integer n,nmax; real a,b,fa,fb,ε,d
external function f
fa← f(a)
fb ← f(b)
if |fa| > |fb| then
            a ⟷ b
            fa ⟷ fb
end if
output 0, a, fa output 1, b, fb
for n = 2 to nmax do
            if |fa| > |fb| then
                        a ⟷ b
                        fa ⟷ fb
            end if
            d ← (b −a)/(fb −fa) b←a
            fb ← fa
            d ← d · fa
            if |d| < ε then
                        output "convergence"
            return
            end if
            a ← a −d fa ← f(a)
            output n,a, fa
end for
end procedure Secant
```

Here ⟷ means interchange values. The endpoints $[a, b]$ are interchanged, if necessary, to keep $|f(a)| \leqq |f(b)|$. Consequently, the absolute values of the function are nonincreasing; thus, we have $|f(x_n)| \leqq |f(x_{n+1})|$ for $n \leqq 1$.

Secant Method example

$f(x) = \cos(x) + 2\sin(x) + x^2$

use

$x_0 = 0$ and $x_1 = -0.1$ as initial approximations

n	x_{n-1}	x_n	x_{n+1}	$\lvert f(x_{n+1}) \rvert$	$\lvert x_{n+1} - x_n \rvert$
1	0.0	-0.1	-0.5136	0.1522	0.4136
2	-0.1	-0.5136	-0.6100	0.0457	0.0964
3	-0.5136	-0.6100	-0.6514	0.0065	0.0414
4	-0.6100	-0.6514	-0.6582	0.0013	0.0068
5	-0.6514	-0.6582	-0.6598	0.0006	0.0016
6	-0.6582	-0.6598	-0.6595	0.0002	0.0003

Modified Secant Method

- Instead of using two arbitrary starting values, use one starting value a fractional perturbation.

$$f'(x_i) \cong \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

$$x_{i+1} = x_i - f(x_i) \frac{\delta x_i}{f(x_i + \delta x_i) - f(x_i)} \qquad i = 1,2,3,\ldots$$

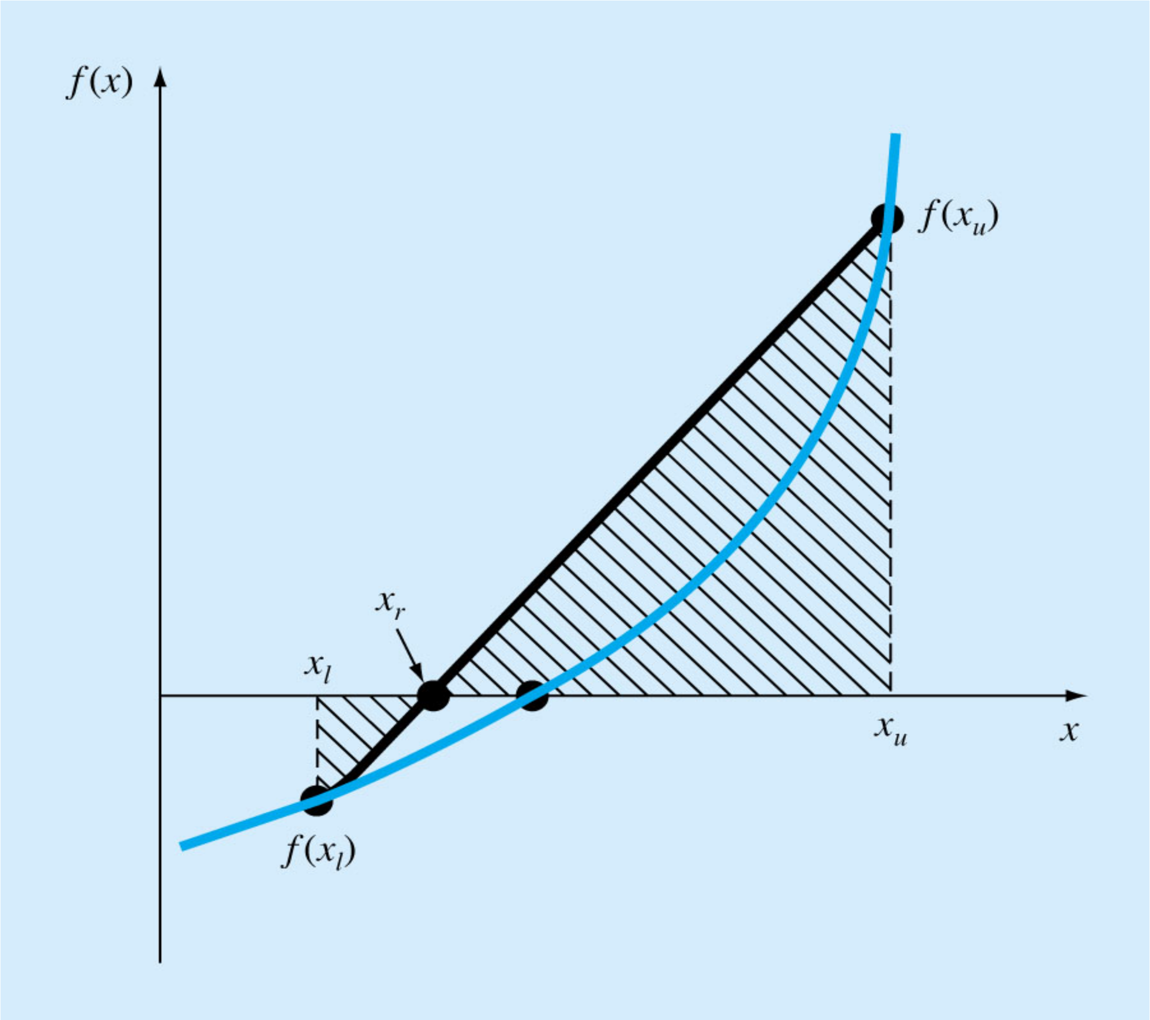
- Choice of a value for δ is not automatic
  - If it is too small, you could be swamped by round off error caused by subtractive cancellation in the denominator
  - If too big, it can become inefficient and even divergent
  - If chosen properly, it is a nice alternative for cases where evaluating the derivative is difficult and developing two initial guesses is inconvenient.

Modified Secant Example

- Using Modified Secant method, estimate the root of f(x) = $e^{-x} - x$ using δ = 0.01 and start with x_0 = 1.0. True root is 0.56714329. Calculate for 3 iterations with % relative error.

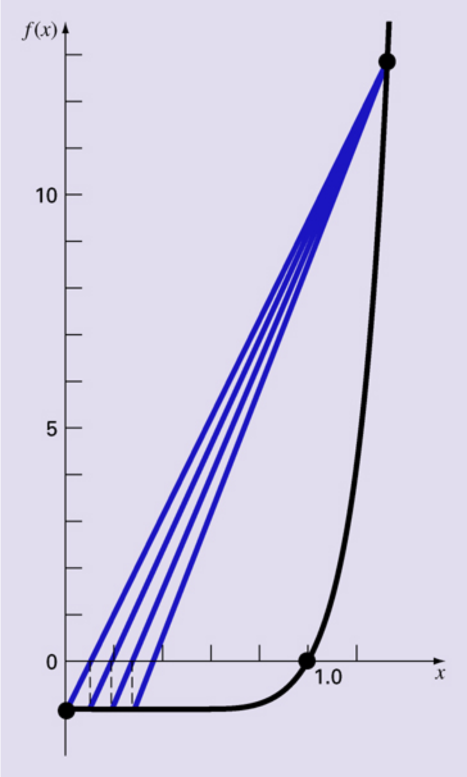
The False-Position Method (Regula-Falsi)

- If a real root is bounded by x_l and x_u of $f(x)=0$, then we can approximate the solution by doing a linear interpolation between the points $[x_l, f(x_l)]$ and $[x_u, f(x_u)]$ to find the x_r value such that $l(x_r)=0$, $l(x)$ is the linear approximation of $f(x)$.



Pros and Cons of False-Position Method

- Pros:
  - Faster
  - Always converges for a single root.
- Cons
  - One sided in that one bracketing point will tend to stay fixed
  - Leads to poor convergence, especially for functions with significant curvature
  - Plot of $f(x) = x^{10} - 1$, illustrating slow convergence of the false-position method.
- Modified False-Position method
  - Modify the stuck bound by halving it each time
- Note: Always check by substituting estimated root in the original equation to determine whether $f(x_r) \approx 0$.



Comparison of False-Position and Secant

- (a) & (b) First iteratoins for both methods are identical
- (c) & (d) Points differ for second iterations, as a result the secant method diverges.