Polynomial Interpolation

CS3010

Numerical Methods

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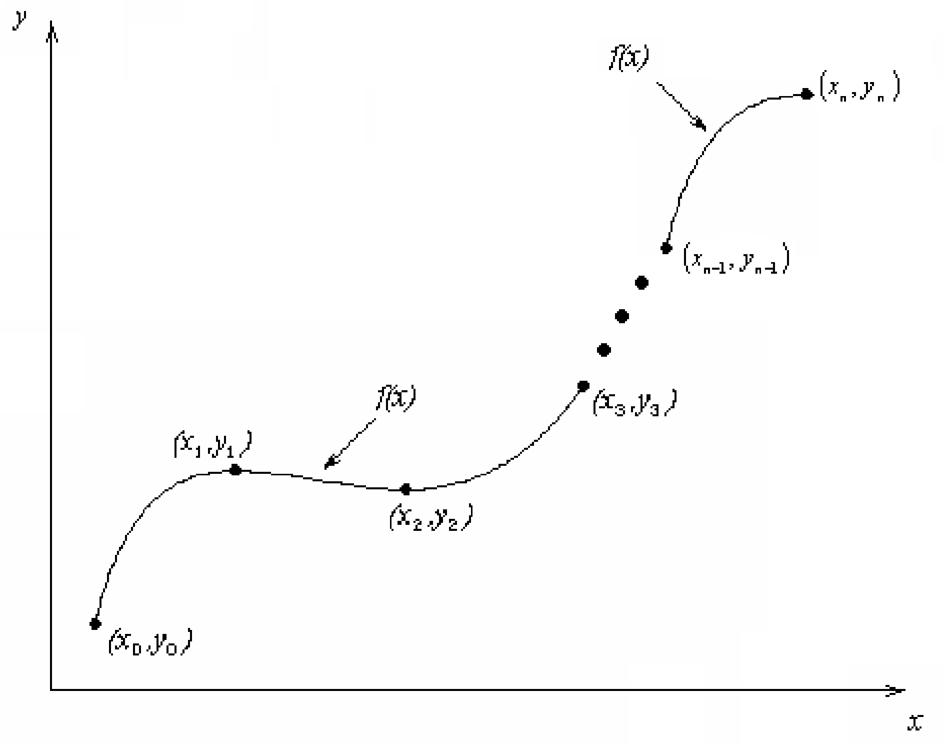
Section 4.1

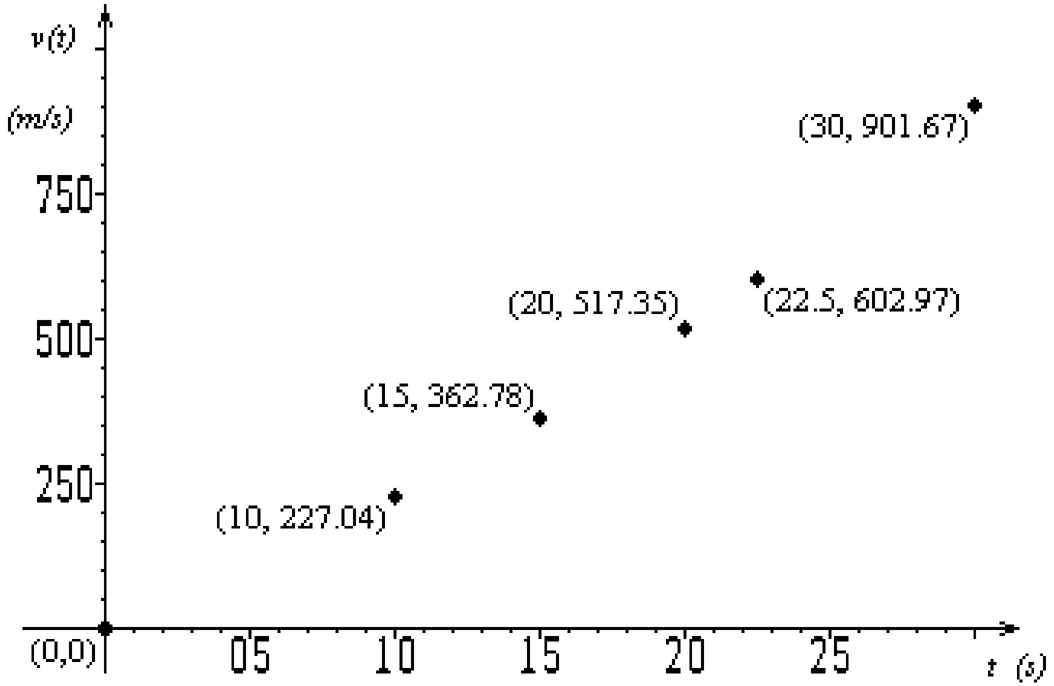
What is Interpolation ?

- Given $(x_0, y_0), (x_1, y_1), \ldots \ldots \ (x_n, yn)$, find the value of '$y$' at a value of 'x' that is not given.



Interpolants

- Polynomials are the most common choice of interpolants because they are easy to:
  - Evaluate
  - Differentiate, and
  - Integrate



Direct Method

- Given 'n+1' data points $(x_0, y_0), (x_1, y_1), \dots \dots \dots \dots . . (x_n, yn)$, pass a polynomial of order 'n' through the data as given below:

  $$y = \; a_0 + a_1x + \cdots + a_nx^n$$

  where $a_0, a_1, \dots \dots \dots \dots \dots, a_n$ are real constants.
- Set up '*n*+1' equations to find '*n*+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Polynomial Interpolation Theorem

- Existence of polynomial interpolation theorem:
- If points $x_0, x_1, \ldots . x_n$ are distinct, then for arbitrary real values $y_0, y_1 \ \ldots . y_n$, there is a unique polynomial p of degree at most n such that $p(xi) = y_i \, for \; 0 ? i \leq n$.

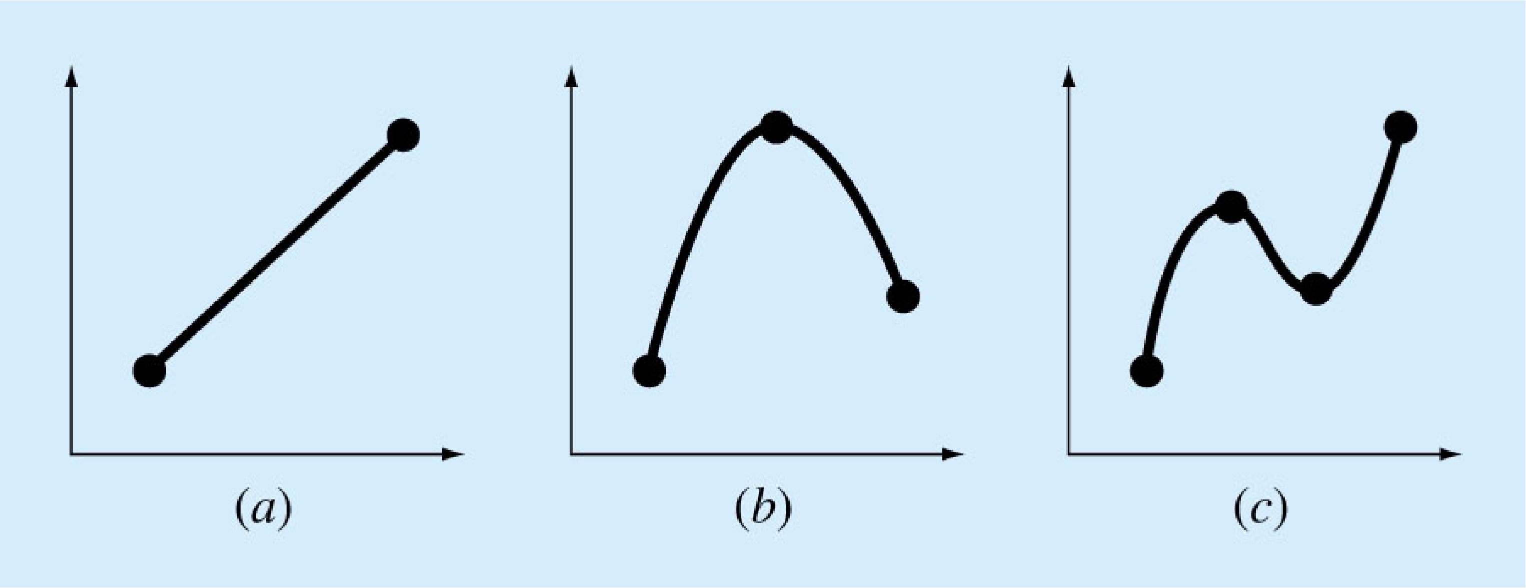
- The points x_i are generally called the nodes
- Although there is one and only one *n*th-order polynomial that fits *n+1* points, there are a variety of mathematical formats in which this polynomial can be expressed:
  - The Newton polynomial (Newton form)
  - The Lagrange polynomial (Lagrange form)

Orders of Interpolation

(a) Linear: first order interpolation connecting two points

(b) Second order, Quadratic or parabolic, connecting three points

(c) Third order, cubic interpolation, connecting four points



Direct Method for Interpolating Polynomial

- Construct interpolating polynomial

x	0	1	-1	2	-2
$f(x)$	-5	-3	-15	39	-9

5 successive polynomials in construction of final polynomial

$$p_0(x) = -5$$

$$p_1(x) = p_0(x) + c(x - x_0) = -5 + c(x - 0)$$

Condition for $p_1(x)$ is that $p_1(1) = -3 \Rightarrow -5 + c(1 - 0) = -3 \Rightarrow c = 2 \Rightarrow p_1(x) = -5 + 2x$

Similarly, $p_2(x) = p_1(x) + c\,(x - x_0)(x - x_1) = -5 + 2x + cx(x - 1)$

Given $p(-1) = -15$, we get $c = -4$

Doing these steps, one gets

$$p_4(x) = -5 + 2x - 4x(x - 1) + 8x(x - 1)(x + 1) + 3x(x - 1)(x + 1)(x - 2)$$

Polynomial in Newton's Nested Form

- Writing the polynomial $p_4(x)$ in the nested form (remember Horner's algorithm)

  $p_4(x) = -5 + x(2 + (x-1)(-4 + (x+1)(8 + (x-2))3))$

  Or $p_4(x) = -5 + x(4 + x(4 + x(-7 + x(2 + 3x))))$

- General polynomial in Newton's Form can be written as

$p(x) = a_0 + a_1[(x-x_0)] + a_2[(x-x_0)(x-x_1)] + ... + a_n[(x-x_0)(x-x_1)...(x-x_{n-1})]$

$$p(x) = a_0 + \sum_{i=1}^{n} a_i \left[\prod_{j=0}^{j=i-1} (x - x_j) \right]$$

Nested Form of $p(x)$ *is*

$p(x) = a_0 + (x-x_0)(a_1 + (x-x_1)(a_2 + ... + (x-x_{n-1})a_n))....))$