Linear Systems
Naïve Gaussian Elimination

CS3010

Numerical Methods

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Section 2.1

Lecture 5

Linear Algebraic Equations

- An equation of the form $ax+by+c=0$ or equivalently $ax+by=-c$ is called a linear equation in x and y variables.
- $ax+by+cz=d$ is a linear equation in three variables, x, y, and z.
- Thus, a linear equation in n variables is

  $a_1x_1+a_2x_2+ \dots +a_nx_n = b$
- A solution of such an equation consists of real numbers $c_1, c_2, c_3, \dots , c_n$. If you need to work more than one linear equations, a system of linear equations must be solved simultaneously.

$$
\begin{aligned}
a_{11}x_1+a_{12}x_2+ \dots +a_{1n}x_n &= b_1 \\
a_{21}x_1+a_{22}x_2+ \dots +a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \vdots \\
a_{n1}x_1+a_{n2}x_2+ \dots +a_{nn}x_n &= b_n
\end{aligned}
$$

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (1 \le i \le n)$$

Solving Large Linear Systems

- In this chapter, we assume that the coefficient matrix $\boldsymbol{A}$ is $n \times n$ and invertible (nonsingular).
- In a pure mathematical approach, the solution to the problem $\boldsymbol{Ax} = \boldsymbol{b}$ is simply $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$, where $\boldsymbol{A}^{-1}$ is the inverse matrix. But in most applications, it is advisable to solve the system directly for the unknown vector $\boldsymbol{x}$ rather than explicitly computing the inverse matrix.
- In applied mathematics and in many applications, it can be a daunting task for even the largest and fastest computers to solve accurately extremely large systems involving thousands or millions of unknowns.

Questions to Answer

- How do we store such large systems in the computer?
- How do we know that the computed answers are correct?
- What is the precision of the computed results?
- Can the algorithm fail?
- How long will it take to compute answers?
- What is the asymptotic operation count of the algorithm?
- Will the algorithm be unstable for certain systems?
- Can instability be controlled by pivoting? (Permuting the order of the rows of the matrix is called **pivoting**.)
- Which strategy of pivoting should be used?
- How do we know whether the matrix is ill-conditioned and whether the answers are accurate?

Gaussian Elimination

- Gaussian elimination transforms a linear system into an upper triangular form, which is easier to solve.
- This process, in turn, is equivalent to finding the factorization $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L}$ is a unit lower triangular matrix and $\boldsymbol{U}$ is an upper triangular matrix.
- This factorization is especially useful when solving many linear systems involving the same coefficient matrix but different right-hand sides, which occurs in various applications.
- When the coefficient matrix has predominantly zero entries, the system is sparse and iterative methods can involve much less computer memory than Gaussian elimination

Noncomputer Methods for Solving Systems of Equations

- For small number of equations ($n \leq 3$) linear equations can be solved readily by simple techniques such as “method of elimination.”
- Linear algebra provides the tools to solve such systems of linear equations.
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.
- There are many ways to solve a system of linear equations:
  - Graphical method
  - Cramer’s rule
  - Method of elimination
  - Computer methods

For $n \leq 3$, easy to use first 3 techniques

Graphical Method

- For Two Linear System of Equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Solve both equations for $x_{2:}$

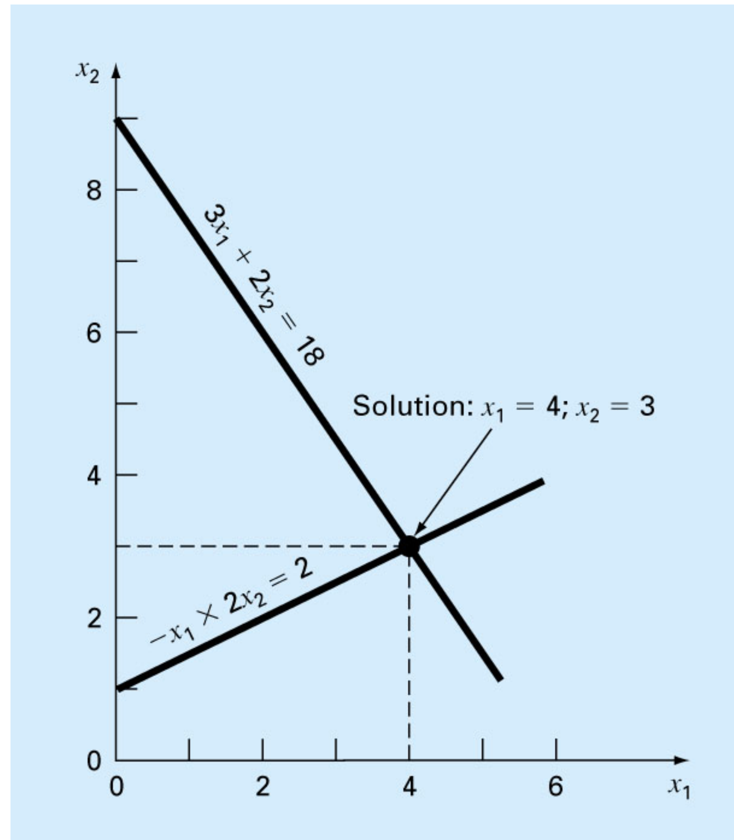
$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \left(\frac{b_1}{a_{12}}\right)$$
$$x_2 = -\left(\frac{a_{12}}{a_{22}}\right)x_1 + \left(\frac{b_2}{a_{22}}\right)$$

The are of the form of two lines

$$x_2 = (slope)x_2 + intercept$$

Intersection of two lines

- Plot x_2 vs. x_1 on rectilinear paper, the intersection of the lines present the solution.

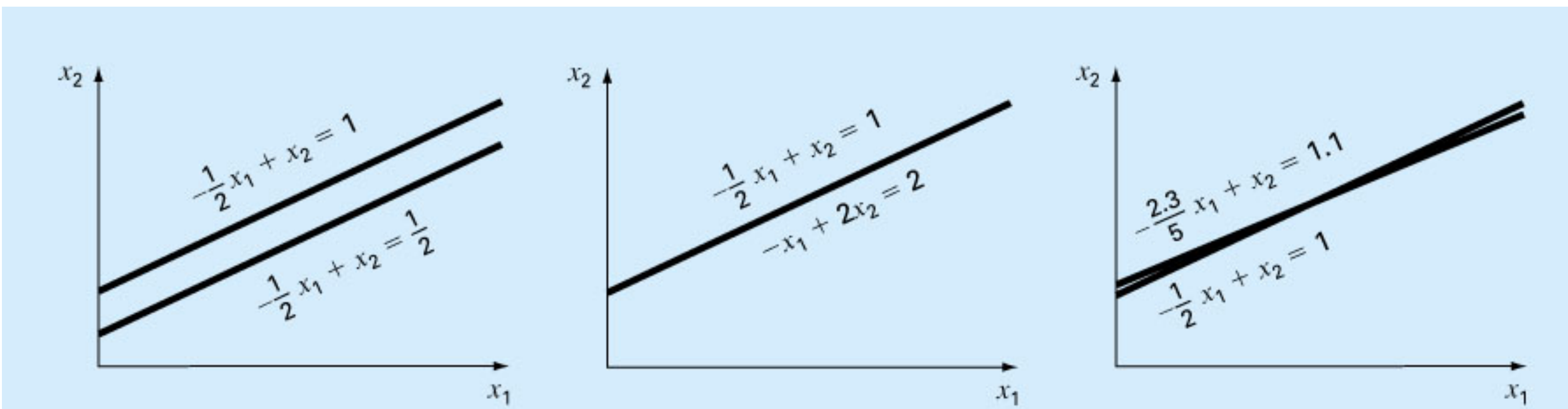


Special Cases

(a) no solution

(b) infinite solutions

(c) Point of intersection difficult to detect visually



(a) (b) (c)

Determinants and Cramer's Rule

- Determinant can be illustrated for a set of three equations:

$$[A]x = [b]$$

Where [A] is the coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

x and b are column vectors, given as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrix Determinants

- Assuming all matrices are square matrices, there is a number associated with each square matrix [A] called the determinant D
- If [A] is order 1, then [A] has one element, i.e. [A]=[a_{11}]

$$D = a_{11}$$

- For a square matrix of order 2, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$D = a_{11}a_{22} - a_{21}a_{12}$$

- For a square matrix of order 3, the ***minor*** of an element a_{ij} is the determinant of the matrix of order 2 by deleting i^{th} row and j^{th} column of [A].

Calculating Determinant for 3x3 Matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$$

$$D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

Cramer’s Rule Solution

- *Cramer’s rule* expresses the solution of a systems of linear equations in terms of ratios of determinants of the array of coefficients of the equations. For example, x_1 would be computed as:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}; \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D}; \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D};$$

Method of Elimination

- The basic strategy is to successively solve one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.
- The elimination of unknowns can be extended to systems with more than two or three equations; however, the method becomes extremely tedious to solve by hand.

Naive Gauss Elimination

- Extension of *method of elimination* to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back substitute.
- As in the case of the solution of two equations, the technique for n equations consists of two phases:
  - Forward elimination of unknowns: Results in a matrix of upper triangular form
  - Back substitution

Gaussian Elimination (1 of 4)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \bullet \\ \bullet \\ \bullet \\ x_i \\ \bullet \\ \bullet \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \bullet \\ \bullet \\ \bullet \\ b_i \\ \bullet \\ \bullet \\ b_n \end{bmatrix}$$

- Forward elimination phase: *n*-1 steps.
- First step is to use the first equation to produce *n*-1 zeros as coefficients for each x_1 in all but the first equation.
- Done by subtracting the appropriate multiples of the first equation from the others.
- First equation is referred to as the first pivot equation and to a_{11} as the first pivot element.

Gaussian Elimination (2 of 4)

- For each of the remaining equations ($2 \le i \le n$),we compute

$$a_{ij} = a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \qquad (1 \le j \le n)$$

$$b_i = b_i - \left(\frac{a_{i1}}{a_{11}}\right) b_1$$

- After this step, the system will be of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ 0 & a_{i2} & a_{i3} & \cdots & a_{in} \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \bullet \\ \bullet \\ \bullet \\ x_i \\ \bullet \\ \bullet \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \bullet \\ \bullet \\ \bullet \\ b_i \\ \bullet \\ \bullet \\ b_n \end{bmatrix}$$

Gaussian Elimination (3 of 4)

- Repeat the process for second row and second column for the remaining equations ($3 \leq i \leq n$)
- For each of the remaining equations ($3 \leq i \leq n$),we compute

$$a_{ij} = a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \qquad (2 \leq j \leq n)$$

$$b_i = b_i - \left(\frac{a_{i2}}{a_{22}}\right) b_2$$

- After this step, the system will be of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ 0 & a_{i2} & a_{i3} & \cdots & a_{in} \\ \bullet & \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & & \bullet \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \bullet \\ \bullet \\ \bullet \\ x_i \\ \bullet \\ \bullet \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \bullet \\ \bullet \\ \bullet \\ b_i \\ \bullet \\ \bullet \\ b_n \end{bmatrix}$$

Gaussian Elimination (4 of 4)

- Repeat the process for the remaining rows (i.e. a total of n-1 times)
- After this step, the system will be of the triangular form where the lower triangle is all zeros.

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & 0 & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 \cdot a_{ii} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & a_{n-1n-1} & a_{n-1n} \\
0 & 0 & 0 & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
$$

Back Substitution

- The back substitution starts from the last equation in the matrix to give a value for x_n which can be used to find the value of x_{n-1} and so on up to the value of x_1.

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n} a_{ij}x_j\right) \quad i = (n-1, n-2, \dots, 1)$$

Naive Gauss Elimination (3x3 Example)

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = c''_3 / a''_{33}$$

$$x_2 = (c'_2 - a'_{23} x_3) / a'_{22}$$

$$x_1 = (c_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$$

Back substitution

Pitfalls of Elimination Methods

- *Division by zero.* It is possible that during both elimination and back-substitution phases a division by zero can occur.
- *Round-off errors*.
- *Ill-conditioned systems*. Systems where small changes in coefficients result in large changes in the solution.
  - Alternatively, it happens when two or more equations are nearly identical, resulting a wide ranges of answers to approximately satisfy the equations.
  - Since round off errors can induce small changes in the coefficients, these changes can lead to large solution errors.

Naïve Gaussian Elimination Example (1 of 3)

- Solve the following linear system using GE

$$\begin{aligned} 6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 &= 26 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 &= -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 &= -34 \end{aligned}$$

- In the first step of the elimination procedure, certain multiples of the first equation are subtracted from the second, third, and fourth equations so as to eliminate x_1 from these equations.
- It is clear that we should subtract 2 times the first equation from the second.
- ½ times the first equation from the third.
- Finally, we should subtract −1 times the first equation from the fourth.

Naïve Gaussian Elimination Example (2 of 3)

$$
\begin{aligned}
6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\
-4x_2 + 2x_3 + 2x_4 &= -6 \\
-12x_2 + 8x_3 + x_4 &= -27 \\
2x_2 + 3x_3 - 14x_4 &= -18
\end{aligned}
$$

- Note that the first equation was not altered in this process, although it was used to produce the 0 coefficients in the other equations. In this context, it is called the **pivot equation**.
- In the second step of the process, we mentally ignore the first equation and the first column of coefficients.
- Then, Subtract 3 times the second equation from the third and subtract −1 times the second equation from the fourth to eliminate x_2

Naïve Gaussian Elimination Example (3 of 3)

$$
\begin{aligned}
6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\
-4x_2 + 2x_3 + 2x_4 &= -6 \\
2x_3 - 5x_4 &= -9 \\
4x_3 - 13x_4 &= -21
\end{aligned}
$$

The final step consists in subtracting 2 times the third equation from the fourth. The result is an upper triangular matrix

$$
\begin{aligned}
6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\
-4x_2 + 2x_3 + 2x_4 &= -6 \\
2x_3 - 5x_4 &= -9 \\
-3x_4 &= -3
\end{aligned}
$$

Back Substitution

- This completes the first phase (**forward elimination**) in the Gaussian algorithm. The second phase (**back substitution**) will solve System (5) for the unknowns *starting at the bottom*.
- Thus, from the fourth equation, we obtain the last unknown

$$x_4 = \frac{-3}{-3} = 1$$

- Put this value in the 3rd equation, we get $2x_3 - 9 = 5$,
- So, $x_3 = \frac{-4}{2} = -2$

and so on for the next two unknowns to get

$$x_1 = 3; \quad x_2 = 1; \quad x_3 = -2; \quad x_4 = 1$$

Gauss Jordan Method

- It is a variation of Gauss elimination. The major differences are:
  - When an unknown is eliminated, it is eliminated from all other equations rather than just the subsequent ones.
  - All rows are normalized by dividing them by their pivot elements.
  - Elimination step results in an identity matrix.
  - Consequently, it is not necessary to employ back substitution to obtain solution.

Gauss Jordan Method Example

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$

$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$

$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

After, doing Gaussian elimination process with all equations using the pivotal row at each step, we get a diagonal matrix and the solution directly

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2.5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$