Numerical Integration
Upper and Lower Sums
Trapezoid Method

CS3010
Numerical Methods
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Section 5.1, 5.2, 5.3

Trapezoid Rule: 2^n Equal Intervals

- Find the recursive trapezoid formula for 2^n equal subintervals for $\int_a^b f(x)dx$
- Start with n = 0 and then do n = 1,2 and 3 and see if you can generalize for n
- Romberg Algorithm produces a triangular array of numbers, all of which are estimates of the definite integral

$$
\begin{array}{llllll}
R(0,0) & & & & & \\
R(1,0) & R(1,1) & & & & \\
R(2,0) & R(2,1) & R(2,2) & & & \\
R(3,0) & R(3,1) & R(3,2) & R(3,3) & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
R(n,0) & R(n,1) & R(n,2) & R(n,3) & \cdots & R(n,n)
\end{array}
$$

Romberg Algorithm

- The first column contains estimates of integral obtained by recursive trapezoid formula with decreasing values of step size. *R*(*n*,0) is got by applying trapezoid rule with 2^n subintervals. *R*(0,0) is obtained with just using one trapezoid.

$$R(0,0) = \frac{1}{2}(b-a)[f(a)+f(b)]$$

R(1,0) is obtained by using two trapezoids

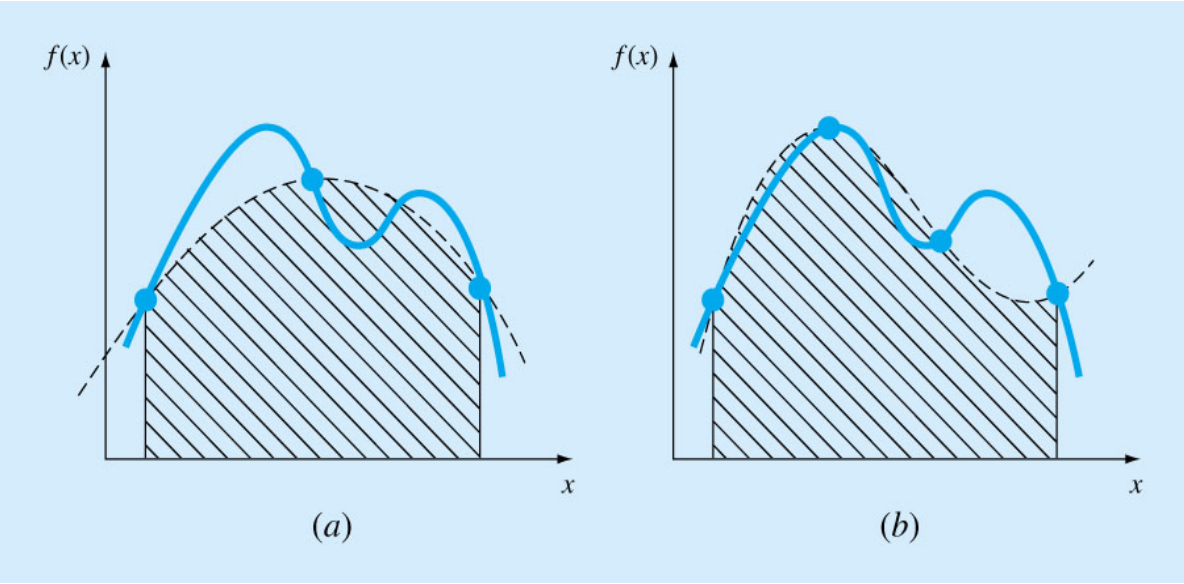
$$\begin{aligned} R(1,0) &= \frac{1}{4}(b-a)\left[f(a)+f\left(\frac{a+b}{2}\right)\right] + \frac{1}{4}(b-a)\left[f\left(\frac{a+b}{2}\right)+f(b)\right] \\ &= \frac{1}{4}(b-a)[f(a)+f(b)] + \frac{(b-a)}{2}f\left(\frac{a+b}{2}\right) \\ &= \frac{1}{2}R(0,0) + \frac{(b-a)}{2}f\left(\frac{a+b}{2}\right) \end{aligned}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\sum_{k=1}^{2^{n-1}} f[a+(2k-1)h]$$

where $h = \frac{b-a}{2^n}$ and $n \geq 1$

Simpson's Rules

- More accurate estimate of an integral is obtained if a high-order polynomial is used to connect the points. The formulas that result from taking the integrals under such polynomials are called *Simpson's rules*.



Simpson's 1/3 Rule

- Results when a second-order interpolating polynomial is used.

Simpson's 1/3 Rule

$$I = \int_a^b f(x)dx \; \cong \int_a^b f_2(x)dx \text{ and } a = x_0 \text{ and } b = \; x_2$$

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)\right] dx$$

$$\boxed{I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]} \text{ where } h = \frac{b-a}{2}$$

Simpson’s 1/3 Error

- Single segment application of Simpson’s 1/3 rule has a truncation error of:

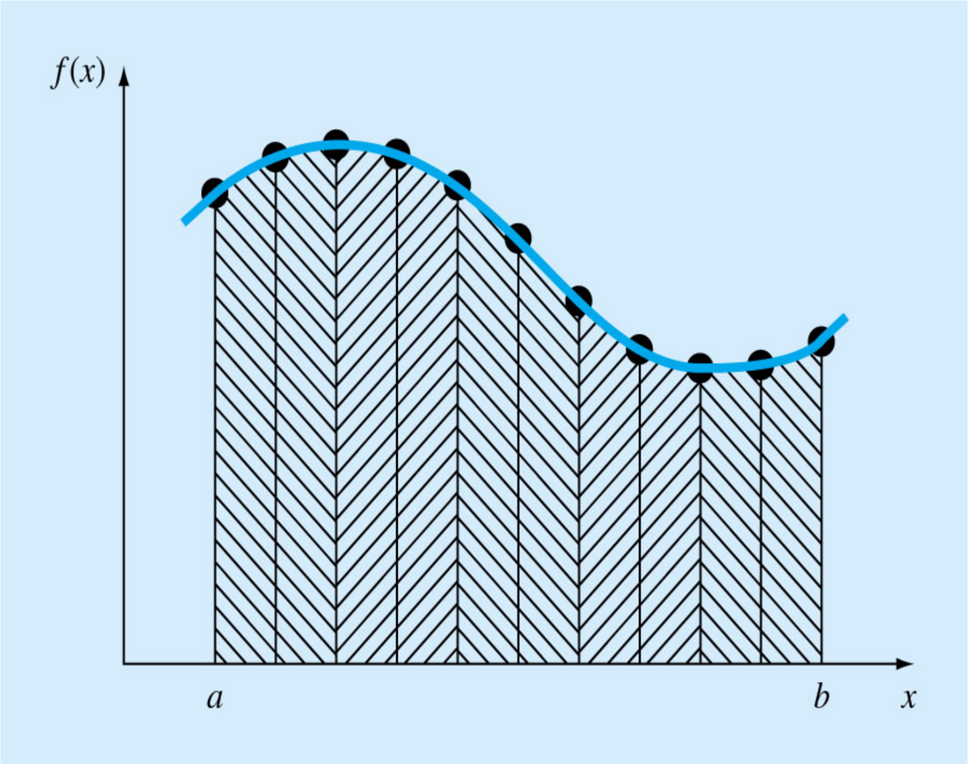
$$Error = -\frac{1}{180}(b-a)h^4 f^{(4)}(\xi)$$

where $a < \xi < b$

- Simpson’s 1/3 rule is more accurate than trapezoidal rule.

The Multiple-Application Simpson's 1/3 Rule

- Just as the trapezoidal rule, Simpson's rule can be improved by dividing the integration interval into a number of segments of equal width.
- Yields accurate results and considered superior to trapezoidal rule for most applications.
- However, it is limited to cases where values are equi-spaced.
- Further, it is limited to situations where there are an <u>even number of segments</u> and odd number of points.



Composite 1/3 Simpson's Rule

- Composite 1/3 Simpson's Rule over n (even) subintervals

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(a)+f(b)] + \frac{4h}{3}\sum_{i=1}^{n/2} f[a+(2i-1)h] + \frac{2h}{3}\sum_{i=1}^{(n-2)/2} f[a+2ih] \quad \text{where } h = \frac{b-a}{n}$$

- Example:

x	1	1.25	1.5	1.75	2
$f(x)$	10	8	7	6	5

- Use on values at 1,1.5 and 2 to find $\int_1^2 f(x)dx$

  0.5/3[10+4.7+5] = 43/6 = 7.1667
- Use all values to calculate the same integral, n = 4

  0.25/3[10+4.8+2.7+4.6+5] = 85/12 = 7.0833

Simpson's 3/8 Rule

- An odd-segment-even-point formula used in conjunction with the 1/3 rule to permit evaluation of both even and odd numbers of segments.

$$I = \int_a^b f(x)dx \cong \int_a^b f_3(x)dx$$

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$h = \frac{(b-a)}{3}$$

$$E_t = -\frac{(b-a)5}{6480} f^{(4)}(\xi)$$

More accurate

Simpson's Rule Comparison