Solutions

1. (a) $f(x) = e^{\cos x}$

$f(0) = e^{\cos 0} = e$

$f'(x) = e^{\cos x}(-\sin x) \Rightarrow f'(0) = 0$

$f''(x) = e^{\cos x}(\sin^2 x) + e^{\cos x}(-\cos x) \Rightarrow f''(0) = -e$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots$$

$$= e + 0 - \frac{x^2}{2}e$$

Thus, $e^{\cos x} = e\left(1 - \frac{x^2}{2} + \cdots\right)$

(b) $f(x) = \sin(\cos x) \Rightarrow f(0) = \sin 1$

$f'(x) = \cos(\cos x)(-\sin x) \Rightarrow f'(0) = 0$

$f''(x) = -\sin(\cos x)(\sin^2 x) - \cos(\cos x)\cos x \Rightarrow f''(0) = -\cos 1$

Thus, $\sin(\cos x) = (\sin 1) - (\cos 1)\left(\frac{x^2}{2}\right) + \cdots$

2. $f(x) = \ln(1-x) \Rightarrow f(0) = 0$

$f'(x) = -\frac{1}{1-x} \Rightarrow f'(0) = -1$

$f''(x) = -\frac{1}{(1-x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = -\frac{2}{(1-x)^3} \Rightarrow f'''(0) = -2$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$$= 0 - x - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{6x^4}{4!} - \cdots$$

$$f(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

2. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots$

$$\ln\left(\frac{1+x}{1-x}\right) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right]$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\cdots\right]$$

$$= 2\sum_{k=0}^{\infty} \frac{x^{2k-1}}{(2k-1)}$$

3. $f(x+h) = \sin(x-3h)$

$\Rightarrow$ $f(x) = \sin x$ and $h = -3h$

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots\cdots$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$$f(x+h) = \sin(x-3h) = \sin x + (-3h)\cos x + \frac{(-3h)^2}{2!}(-\sin x) + \frac{(-3h)^3}{3!}(-\cos x) + \cdots\cdots$$

$$\sin(x-3h) = \sin x - 3h\cos x - \frac{(3h)^2}{2!}\sin x + \frac{(3h)^3}{3!}\cos x + \cdots\cdots$$

4. $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

When $x=1$, we get $\ln(1+1) = \ln 2$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)^{n-1}}{n} = S_n$$

Using Alternating Series theorem,

$$|S - S_n| \leq \frac{1}{n+1} < \frac{1}{2} \times 10^{-6}$$

$$\Rightarrow \quad n > 2 \times 10^{6} - 1$$

This means that more than two million terms will be needed which is not practical.