ASSIGNMENT # 1

Christopher Koepke
CS 3010.02 -1

Section 1.1 : 8b, 8d, 11a, 11d

8. show how these polynomials can be efficiently evaluated.

b) $p(x) = 3(x-1)^5 + 7(x-1)^9$

$$\boxed{= (x-5)^5(3 + 7(x-1)^4)}$$

d) $p(x) = x^{127} - 5x^{37} + 10x^{17} - 3x^7$

$$\boxed{= x^7(-3 + x^{10}(10 + x^{20}(-5 + x^{90})))}$$

11. Write segments of pseudocode to evaluate the following expressions efficiently.

a) $p(x) = \sum_{k=0}^{n-1} kx^k = 0 + x + 2x^2 + 3x^3 + \cdots (n-1)x^{n-1}$

$p(x) = x(1 + x(2 + x(3 + x(\cdots))))$

```
integer i, n ; real p, x
real array (a_i)0:n
p ← a_n
for i = n-1 to 0
    p ← a_i + xp
end for
```

d) $p(t) = \sum_{i=1}^{n} a_i \prod_{j=1}^{i-1} (t - x_j)$

```
integer i, n ; real r
real array (a_i)0:n , (b_i)0:n-1
b_n-1 ← a_n
for i = n-1 to 0
    b_i-1 ← a_i + rb_i
end for
```

Section 1.2: 4c, 4d, 4f, 6c, 13, 26, 29, 36, 41c, 45

4. Why do the following functions not possess Taylor Series expansions at $x=0$?

c) $f(x) = \arcsin(x-1)$

The function does not converge at the point of expansion.

d) $f(x) = \cot(x)$

$\cot(0)$ does not exist

f) $f(x) = \pi^x$

The taylor series converges at point of expansion but is not a correct approximation.

6. Determine the first two nonzero terms of the series expansion about zero for the following:

c) $\cos^2(x)\sin(x)$

$$* \ f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

$f(x) = \cos^2(x)\sin(x)$

$f(0) = (1)(0) = 0$

$f'(x) = -2\cos(x) + 3\cos^3(x)$

$f'(0) = -2 + 3 = 1$

$f''(x) = 2\sin(x) - 9\cos^2(x)\sin(x)$

$f''(0) = 0 - 9(1)(0) = 0$

$f'''(x) = 20\cos(x) - 27\cos^3(x)$

$f'''(0) = 20(1) - 27(1) = -7$

$$\cos^2(x)\sin(x) = \cos^2(0)\sin(0) + \frac{-2\cos(0) + 3\cos^3(0)}{1!}(x-0) + \frac{2\sin(0) - 9\cos^2(0)\sin(0)}{2!}(x-0)^2 + \frac{20\cos(0) - 27\cos^3(x)}{3!}(x-0)^3 + \ldots$$

$$= 0 + \frac{1}{1!}x + \frac{0}{2!}(x)^2 + \frac{-7}{3!}(x)^3$$

First two nonzero terms of $\cos^2(x)\sin(x)$ are $x - \frac{7}{3!}(x)^3$

13. Use the alternating series theorem to determine the number of terms in series (5) needed for computing $\ln(1.1)$ with error less than $(\frac{1}{2}) \times 10^{-8}$.

- $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \qquad (-1 < x \leq 1)$$

$$\ln(1+0.1) = 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4} + \cdots \frac{(0.1)^n}{n}$$

$$\log_{10}(2) + \log_{10}(10^{-8})$$

$$\left|\frac{(0.1)^n}{n}\right| < (\tfrac{1}{2}) \times 10^{-8} \Rightarrow |n \log_{10}(0.1) - \log_{10}(n)| < \log_{10}(1) - (\quad)$$

$$|n \log_{10}(0.1) - \log_{10}(n)| < 0 - \log_{10}(2) + 8(1)$$

$$|n(-1) - \log_{10}(n)| < -0.30102999957 + 8$$

$$|-n - \log_{10}(n)| < 7.698970004$$

When $n = 7$, $-n - \log_{10}(n)$ is greater than 7.698970004, thus we will need at least 7 terms to have an error less than $(\frac{1}{2}) \times 10^{-8}$ when computing $\ln(1.1)$.

---

26. In the taylor series for the function $3x^2 - 7 + \cos(x)$ (expanded in powers of x), what is the coefficient of x^2?

$f(x) = 3x^2 - 7 + \cos(x)$ $\qquad$ $f'(x) = 6x - \sin(x)$ $\qquad$ $f''(x) = 6 - \cos(x)$

$f(0) = -7 + 1 = -6$ $\qquad$ $f'(0) = 0$ $\qquad$ $f''(0) = 6 - 1 = 5$

$$P(x) = P(a) + \frac{P'(a)}{1!}(x-a) + \frac{P''(a)}{2!}(x-a)^2$$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= -6 + 0 + 5x^2$$

The coefficient of the x^2 term in the taylor series of the function $3x^2 - 7 + \cos(x)$ is 5.

section 1.2 : 29, 36, 41c, 45

29. Find the value of ξ that serves in Taylor's Theorem when $f(x) = \sin(x)$, with $x = \frac{\pi}{4}$, $c = 0$, and $n = 4$. * $0 < \xi \leq \frac{\pi}{4}$

$$E_{n+1} = \frac{f^{n+1}(\xi)}{(n+1)!}(x-c)^{n+1}$$

$f(x) = \sin(x)$ $f'(x) = \cos(x)$ $f''(x) = -\sin(x)$ $f'''(x) = -\cos(x)$

$f^4(x) = \sin(x)$ $f^5(x) = \cos(x)$

$$E_{4+1} = \frac{f^5(\xi)}{6!}\left(\frac{\pi}{4} - 0\right)^5 = \frac{\cos(\xi)}{6!}\left(\frac{\pi}{4}\right)^5$$

$$\frac{\cos(\frac{\pi}{4})}{6!}\left(\frac{\pi}{4}\right)^5 < \xi < \frac{\cos(0)}{6!}\left(\frac{\pi}{4}\right)^5$$

$$\boxed{4.15027 \times 10^{-4} < \xi < 4.15066 \times 10^{-4}}$$

---

36. Using the Taylor Series expansion in terms of h, determine the first three terms in the series for $e^{\sin(x+h)}$. Evaluate $e^{\sin(90.01^\circ)}$ accurately to ten decimal placer as Ce for constant C.

* $z = x+h$

$f(x) = e^{\sin(x+h)}$ $f'(x) = \cos(x+h)e^{\sin(x+h)}$

$$f''(x) = \cos^2(x+h)e^{\sin(x+h)} - \sin(x+h)e^{\sin(x+h)} = e^{\sin(x+h)}(\cos^2(x+h) - \sin(x+h))$$

$$e^{\sin(x+h)} = e^{\sin(z)} + \frac{\cos(z)e^{\sin(z)}}{1!}(h) + \frac{e^{\sin(z)}(\cos^2(z) - \sin(z))h^2}{2!}$$

* $z = 1$

$$= e^1 + (0)e^1 h + \frac{e^1(0-1)h^2}{2!}$$

$$= e^1 - \frac{e^1 h^2}{2}$$

* $h = 10^{-10}$

$$\boxed{e^{\sin(90.01)} = e^1 - \frac{e^1(10^{-10})}{2} = 2.7182818280}$$

41. Establish the Taylor Series in terms of h for the following:

c) $\ln\left[\frac{(x-h^2)}{(x+h^2)}\right] = \ln(x-h^2) - \ln(x+h^2)$ $\quad * f^n(x) = 0$?

$$* \quad f(x-h^2) = f(x) - \frac{h^2 f'(x)}{1!} + \frac{(-h^2)^2 f''(x)}{2!} + \frac{(-h^2)^3 f'''(x)}{3!} + \frac{(-h^2)^4 f^4(x)}{4!} + \cdots$$

$$= f(x) - h^2 f'(x) + \frac{h^4 f''(x)}{2!} - \frac{h^6 f'''(x)}{3!} + \frac{h^8 f^4(x)}{4!} + \cdots \frac{(-h^2)^n f^n(x)}{n!}$$

$$f(x+h^2) = f(x) + h^2 f'(x) + \frac{h^4 f''(x)}{2!} + \frac{h^6 f'''(x)}{3!} + \cdots \frac{(h^2)^n f^n(x)}{n!}$$

$$\ln\left[\frac{(x-h^2)}{(x+h^2)}\right] = f(x) + (2)\frac{h^4 f''(x)}{2!} + (2)\frac{h^8 f^4(x)}{4!} + (2)\frac{h^{12} f^6(x)}{6!}$$

$$\boxed{= f(x) + h^4 f''(x) + \frac{h^8 f^4(x)}{12} + \frac{h^{12} f^6(x)}{360} + \cdots \frac{(h)^{4n} f^{2n}(x)}{(2n)!/2}}$$

45. Determine the first three terms in the Taylor series to represent sinh(x+h). Evaluate sinh(0.0001) to 20 decimal places (rounded) using this series.

- $f(x) = \sinh(x)$ $\quad f'(x) = \cosh(x)$ $\quad f''(x) = \sinh(x)$ $\quad f'''(x) = \cosh(x)$

  $f(0) = 0$ $\quad f'(0) = 1$ $\quad f''(0) = 0$ $\quad f'''(0) = 1$

- $$\sinh(x+h) = f(x) + \frac{h f'(x)}{1!} + \frac{h^2 f''(x)}{2!} + \frac{h^3 f'''(x)}{3!} + \cdots \frac{h^n f^n(x)}{n!}$$

$$= \sin(x) + \frac{h \cosh(x)}{1!} + \frac{h^3 \cosh(x)}{3!}$$

$$\sinh(0+0.00001) = 0 + \frac{(0.00001)(1)}{1!} + \frac{(0.00001)^3(1)}{3!} = \boxed{1 \times 10^{-5}}$$

$$\sinh(0.00001) = \frac{-1+e^{0.00002}}{2e^{0.00001}} = 1 \times 10^{-5}$$