Linear Systems Gaussian Elimination with Scaled Partial Pivoting

CS3010

Numerical Methods

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Section 2.2

Lecture 6

Gaussian Elimination Drawback

- During elimination process, the coefficient of the diagonal element of the pivotal equation might become very small or zero.
- This creates issues with normalization to find factor to subtract from subsequent equations.
  - Normalization step leads to division by 0
- Hence, using the equations in the order presented might not be the wisest choice.
- Pivoting (partial or full) fixes this problem
  - Partial pivoting. Switching the rows so that the largest element is the pivot element.
  - Complete(full) pivoting. Searching for the largest element in all rows and columns then switching.

Partial vs. Complete Pivoting

- Gaussian elimination with **partial pivoting** selects the pivot row to be the one with the maximum pivot entry in absolute value from those in the leading column of the reduced submatrix.
- Two rows are interchanged to move the designated row into the pivot row position.
- Gaussian elimination with **complete pivoting** selects the pivot entry as the maximum pivot entry from all entries in the submatrix. (This complicates things because some of the unknowns are rearranged.)
- Two rows and two columns are interchanged to accomplish this.
- In practice, partial pivoting is almost as good as full pivoting and involves significantly less work.

Gaussian Elimination with Scaled Partial Pivoting

$$
\begin{aligned}
3x_1 - 13x_2 + 9x_3 + 3x_4 &= -19 \\
-6x_1 + 4x_2 + x_3 - 18x_4 &= -34 \\
6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\
12x_1 - 8x_2 + 6x_3 + 10x_4 &= 26
\end{aligned}
$$

$$
\begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & 18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} -19 \\ -34 \\ 16 \\ 26 \end{bmatrix}
$$

- The index vector is ℓ = [1, 2, 3, 4] at the beginning.
- The scale vector does not change throughout the procedure and is given as ***s*** = [13, 18, 6, 12] (highest absolute coefficient in each row)

Scaled Partial Pivoting (1 of 4)

- To determine the first pivot row, we look at four ratios:

$$\left\{\frac{\left|a_{\ell_{i,1}}\right|}{s_{\ell_i}} : i = 1,2,3,4\right\} = \left\{\frac{3}{13}, \frac{6}{18}, \frac{6}{6}, \frac{12}{12}\right\} \approx \{0.23, 0.33, 1.0, 1.0\}$$

- select the index j as the *first* occurrence of the largest value of these ratios. In this example, the largest of these occurs for the index j = 3.
- So row three is to be the pivot equation in step 1 (k = 1) of the elimination process. In the index vector ℓ, entries ℓ_k and ℓ_j are interchanged so that the new index vector is ℓ = [3, 2, 1, 4]

Scaled Partial Pivoting Example (2 of 4)

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -18 \\ 16 \\ -6 \end{bmatrix}$$

- In the next step (*k* = 2), we use the index vector ℓ = [3, 2, 1, 4] and scan the ratios corresponding to rows two, one, and four:

$$\left\{ \frac{|a_{\ell_{i,2}}|}{s_{\ell_i}} : i = 2,3,4 \right\} = \left\{ \frac{2}{18}, \frac{12}{13}, \frac{4}{12} \right\} \approx \{0.11, 0.92, 0.33\}$$

- The largest is the second ratio, and we therefore set *j* = 3 and interchange ℓ_k with ℓ_j in the index vector. Thus, the index vector becomes ℓ =[3,1,2,4]. The pivot equation for step 2 in the elimination is now row one, and ℓ_2=1.

Scaled Partial Pivoting Example (3 of 4)

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & -2/3 & 5/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -45/2 \\ 16 \\ 3 \end{bmatrix}$$

- The third and final step (*k* = 3) is to examine the ratios corresponding to rows two and four:

$$\left\{ \frac{\left|a_{\ell_{i,3}}\right|}{s_{\ell_i}} : i = 3,4 \right\} = \left\{ \frac{13/3}{18}, \frac{2/3}{12} \right\} \approx \{0.24, 0.06\}$$

- The larger value is the first, so we set *j* = 3. Since this is step *k* = 3, interchanging ℓ_k with ℓ_j leaves the index vector unchanged ℓ =[3,1,2,4]

Scaled Partial Pivoting Example (4 of 4)

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & 0 & -6/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -45/2 \\ 16 \\ -6/13 \end{bmatrix}$$

The solution is obtained by using equation ℓ_4 = 4 to determine x_4, and then equation ℓ_3 = 2 to find x_3, and so on. Carrying out the calculations, we have

$$x_4 = \frac{1}{-6/13}[-6/13] = 1$$

$$x_3 = \frac{1}{13/3}[(-45/2) + (83/6)(1)] = -2$$

$$x_2 = \frac{1}{-12}[-27 - 8(-2) - 1(1)] = 1$$

$$x_1 = \frac{1}{6}[16 + 2(1) - 2(-2) - 4(1)] = 3$$

Hence, the solution is $x = [3 \quad 1 \quad -2 \quad 1]^T$