



Pseudocharacters of the modular group from linking numbers of modular knots

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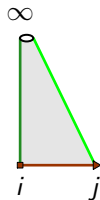
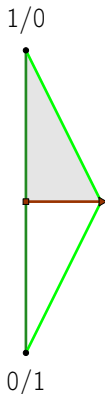
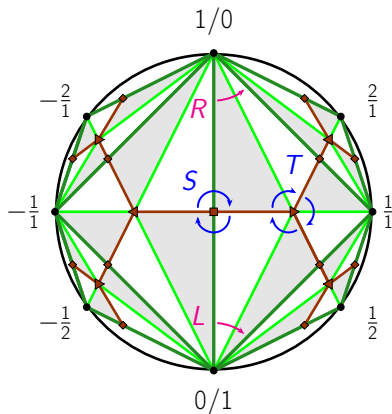
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Texas AMU, 2025-08-27

- 1 Conjugacy classes of the modular group
- 2 The topological meaning of the Rademacher cocycle
- 3 Pseudocharacter of the modular group from Brooks cocycles
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Modular group $\mathrm{PSL}_2(\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{H}\mathbb{P}$

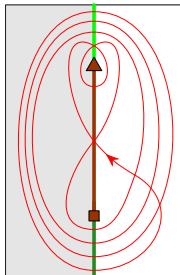
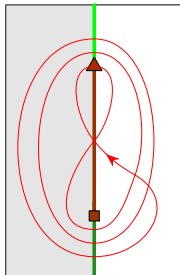
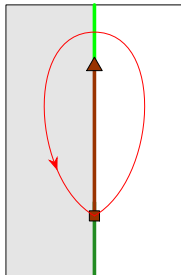
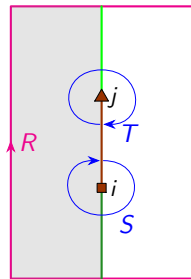
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



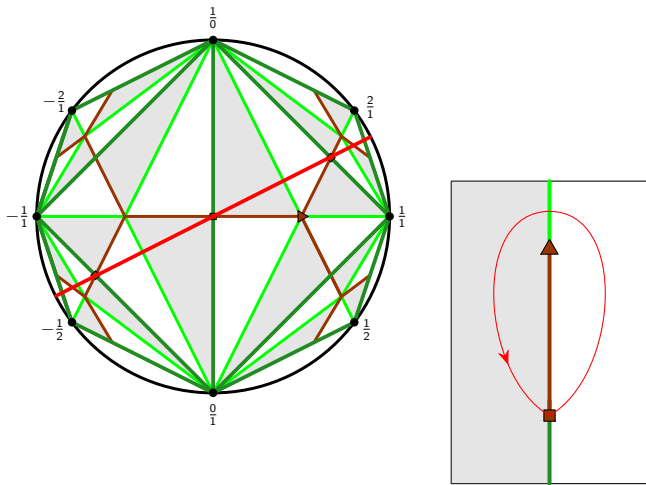
Tiling $\mathbb{H}\mathbb{P}$ under the action of the modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$, the basic fundamental tile, and the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}\mathbb{P}$.

Loops in the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{HP}$

Free homotopy classes of oriented loops in \mathbb{M}	Conjugacy classes in $\pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z})$
Around conic singularity i or j	Elliptic: S or $T^{\pm 1}$
Surround n times the cusp ∞	Parabolic: R^n , $n \in \mathbb{Z}$
$\exists!$ geodesic representative $[\alpha]$ of length λ_A	Hyperbolic: $\mathrm{disc}(A) = \left(2 \sinh \frac{\lambda_A}{2}\right)^2$

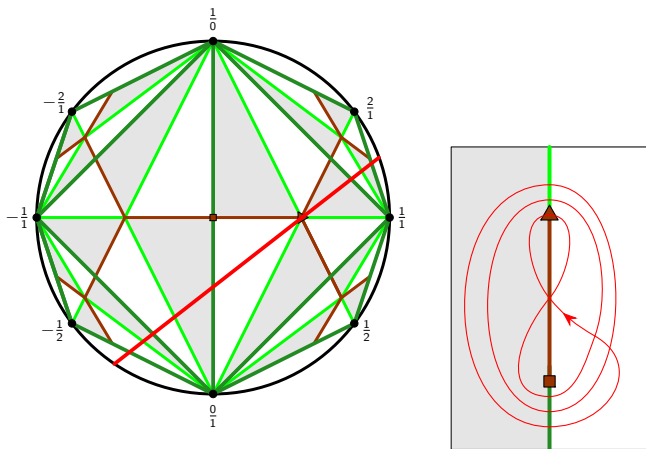


Modular geodesics as projections of hyperbolic axes



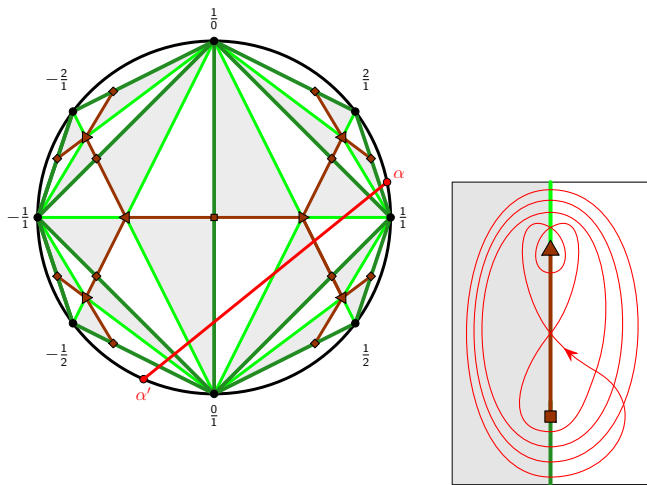
The axis of $A = RL$ in \mathbb{H}^1 from $\alpha^- = -1/\overline{[1,1]}$ to $\alpha^+ = \overline{[1,1]}$ in \mathbb{RP}^1 .

Modular geodesics as projections of hyperbolic axes



The axis of $A = RLL$ in \mathbb{HP} from $\alpha^- = -1/\overline{[2,1]}$ to $\alpha^+ = \overline{[1,2]}$ in \mathbb{RP}^1 .

Modular geodesics as projections of hyperbolic axes



The axis of $A = RLLL$ in \mathbb{HP} from $\alpha^- = -1/\overline{[3,1]}$ to $\alpha^+ = \overline{[1,3]}$ in \mathbb{RP}^1 .

Conjugacy classes and cyclic binary words

The Euclidean monoid $\mathrm{PSL}_2(\mathbb{N}) = \{L, R\}^*$

- The monoid $\mathrm{PSL}_2(\mathbb{N})$ is freely generated by $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- For $A \in \mathrm{PSL}_2(\mathbb{Z})$ we have $A \in \mathrm{PSL}_2(\mathbb{N}) \iff \alpha^- < 0 < \alpha^+$.
- For $A \in \mathrm{PSL}_2(\mathbb{N})$ its $\{L, R\}$ -factorisation has exponents yielding the convergents in the continued fraction expansion of α^+ .

The conjugacy class $[A]$ of an infinite order $A \in \mathrm{PSL}_2(\mathbb{Z})$ satisfies:

- $[A] \cap \mathrm{PSL}_2(\mathbb{N}) =$ cyclic permutations of an L & R -word $\neq \emptyset$.
- Class is primitive \iff cyclic word is primitive.
- Class is hyperbolic $\iff \#L > 0$ and $\#R > 0$.

- denote its *combinatorial length* by $\text{len}(A) = \#R + \#L$
- define its *Rademacher cocycle* as $\text{Rad}(A) = \#R - \#L$

The Rademacher invariant in arithmetic, topology, physics...

The Rademacher invariant function $\text{Rad}: \text{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ is the

- 3 Cocycle for the Eisenstein series E_2 (logarithm of Dedekind function η)
- 2 Signature defect of the torus bundle with monodromy A
- 5 Special value at 0 of the Shimizu L -function for $L_A(s) = \sum \frac{\text{sign } Q_A(m,n)}{|Q_A(m,n)|^s}$

Logarithm of the Dedekind η -Funktion

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It may now be convenient to summarize all our results in an omnibus theorem:

(5.60) **Theorem.** *Let $A \in \text{SL}(2, \mathbb{Z})$ be hyperbolic. Then the following invariants of A coincide.*

- 1) Meyer's signature invariant $\phi(A)$ (see (5.3)).
- 2) Hirzebruch's signature defect $\delta(A)$.
- 3) The invariant $\chi(A)$ describing the transformation properties of $\log \eta(\tau)$ under A (see (5.22)).
- 4) $\mu(A)$ the logarithmic monodromy (divided by πi) of Quillen's determinant line-bundle \mathcal{L} .
- 5) The value $L_A(0)$ of the Shimizu L -function (see (5.49)).
- 6) The Atiyah-Patodi-Singer invariant $\eta(A)$.
- 7) The "adiabatic limit" $\eta^0(A)$.

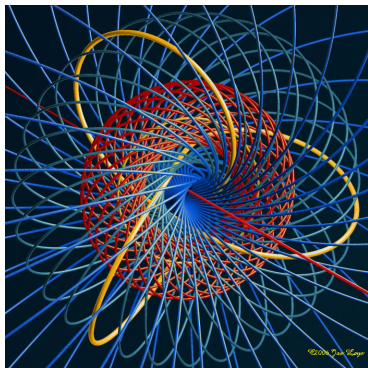
Atiyah's omnibus theorem [Ati87]

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Unit tangent bundle \mathbb{U} of the modular orbifold \mathbb{M}

$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{R}) & \xrightarrow{\mathrm{PSL}_2(\mathbb{Z})} & \mathbb{U} \\ \downarrow \mathbb{S}^1 & & \downarrow \mathbb{S}^1 \\ \mathbb{H}^{\mathrm{P}} & \xrightarrow{\mathrm{PSL}_2(\mathbb{Z})} & \mathbb{M} \end{array}$$

The Seifert fibration $\mathbb{U} \rightarrow \mathbb{M}$ reveals that $\mathbb{U} \simeq \mathbb{S}^3 \setminus \text{trefoil}$.

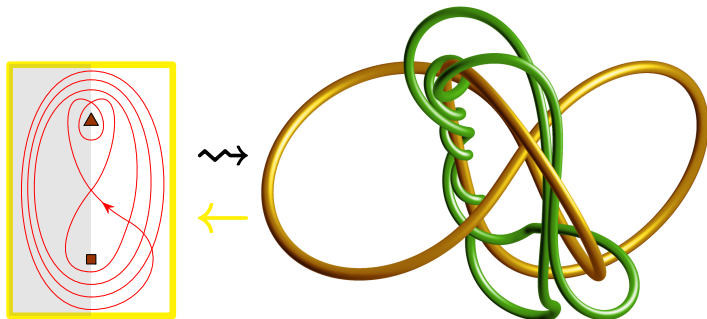


The extension of fundamental group $\mathbb{S}^1 \hookrightarrow \pi_1(\mathbb{U}) \twoheadrightarrow \pi_1(\mathbb{M})$

yields the universal central extension $\mathbb{Z} \hookrightarrow \mathcal{B}_3 \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z})$ which is the quotient of the braid group $\mathcal{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$ by its center generated by $(\sigma_1 \sigma_2)^6$ sending $\sigma_1^{-1} \mapsto L$ and $\sigma_2 \mapsto R$.

Primitive modular geodesics in \mathbb{M} lift to modular knots in \mathbb{U}

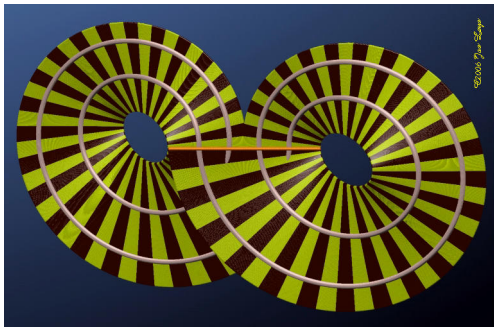
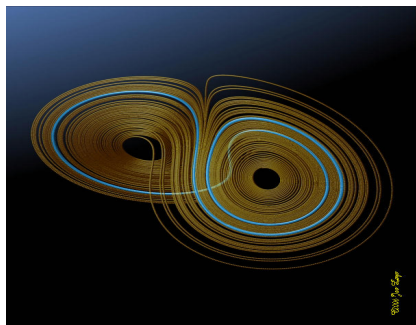
Hyperbolic classes in $\mathrm{PSL}_2(\mathbb{Z})$	Modular geodesics in \mathbb{M}	Periodic orbits in \mathbb{U}
trace form	intersection form	linking form



The primitive modular geodesic $[\alpha]$ lifts to the modular knot $\vec{\alpha}$

Master modular link isotopes into the Lorenz template

The **master modular link** isotopes into the **Lorenz template** so that the periodic orbit associated to $A \in \mathrm{PSL}_2(\mathbb{N})$ follows the sequence follows that sequence of left and right turns according to its $\{L, R\}$ -factorisation.



This is **shown** in [Ghy07] by opening the cusp of \mathbb{M} , deforming the Fuchsian representation $\rho_q: \pi_1(\mathbb{M}) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ to the boundary, so that the unit tangent bundle of \mathbb{M}_q isotopes to that of the Lorenz template.

Linking the trefoil recovers the Rademacher cocycle

Theorems [BG92, Ghy07] :

For hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ we have $\mathrm{Rad}(A) = \mathrm{lk}(\text{trefoil}, k_A)$.

The function $\mathrm{Rad}: \mathrm{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ is the **pseudocharacter** associated to the central extension $\mathbb{Z} \hookrightarrow \mathcal{B}_3 \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z})$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(\mathbb{S}^1) = \mathbb{Z} & \longrightarrow & H_1(\mathbb{S}^1) = \mathbb{Z} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_1(\mathbb{M}') = \mathcal{F}_2 & \longrightarrow & \pi_1(\mathbb{U}) = \mathcal{B}_3 & \xrightarrow{\mathrm{lk}} & H_1(\mathbb{U}) = \mathbb{Z} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_1(\mathbb{M}') = \mathcal{F}_2 & \longrightarrow & \pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z}) & \xrightarrow{\mathrm{Ab}} & H_1(\mathbb{M}) = \mathbb{Z}/6 & & \end{array}$$

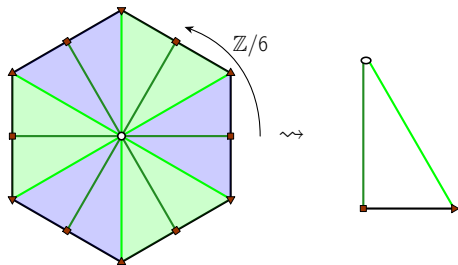
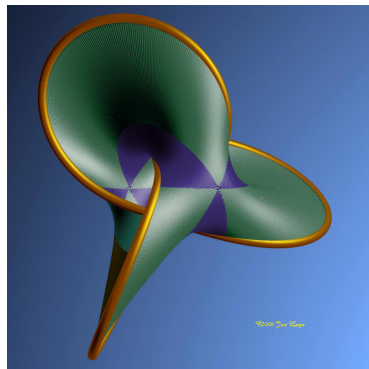
The Euler class in $H^2(\mathrm{PSL}_2(\mathbb{Z}); \mathbb{Z})$ classifying the central extension of the middle column is the pull back by the abelianisation $\mathrm{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{Z}/6$ of the Euler class $H^2(\mathbb{Z}/6; \mathbb{Z})$ classifying the central extension of the right column.

Linking the trefoil recovers the Rademacher cocycle

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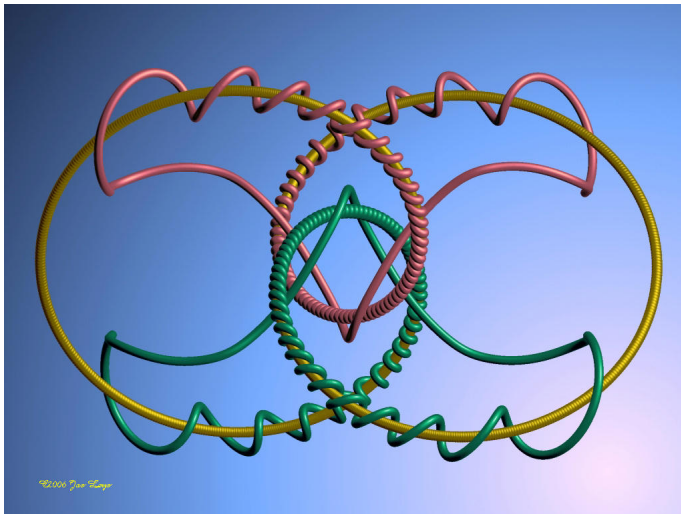
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Pages of the open book decomposition ; the universal abelian cover $\mathbb{M}' \rightarrow \mathbb{M}$.

What about the linking of modular knots ?



Another question would be to give an arithmetical or combinatorial computation of the linking numbers of two knots k_A and k_B as a function of A, B in $\mathrm{PSL}(2, \mathbb{Z})$

Picture from [GL16] and Question from [Ghy07].

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Pseudocharacters of a group Γ and their Banach quotient

Definition: Pseudocharacters are homogeneous quasicharacters

A function $\phi: \Gamma \rightarrow \mathbb{R}$ is a **quasicharacter** when it has a bounded derivative:

$$d\phi: \Gamma \times \Gamma \rightarrow \mathbb{R} \quad d\phi(A, B) = \phi(B) - \phi(AB) + \phi(A).$$

It is **homogeneous** when $\phi(A^n) = n\phi(A)$ for all $A \in \Gamma$ and $n \in \mathbb{Z}$.

Theorem ([MM85, Iva88]) : a Banach space

The \mathbb{R} -vector space $PX(\Gamma)$ has weak* topology (pointwise convergence).

- Semi-norm $\|d\phi\|_\infty$ with $\ker \|d\phi\|_\infty = H^1(\Gamma; \mathbb{R}) \subset PX(\Gamma)$
- The quotient is $H_{b,0}^2(\Gamma) = PX(\Gamma)/H^1(\Gamma; \mathbb{R}) \dots$ a Banach space!

Rotation numbers: from commutative to non-commutative

Characters $H^1(\Gamma; \mathbb{Z}) = \text{Hom}(\Gamma; \mathbb{S}^1)$.

Pseudocharacters $H_b^2(\Gamma; \mathbb{Z}) \supset \text{Hom}(\Gamma; \text{Homeo}(\mathbb{S}^1)) \bmod \text{semi-conj}$ [Ghy87].

Brooks cocycles for groups acting on trees

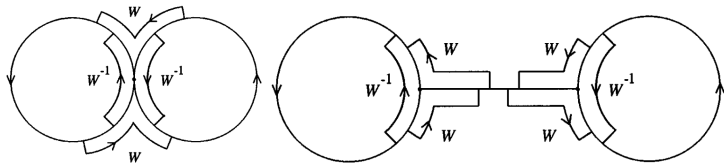
Brooks-Series cocycles ([BS84, BG91]) and [Gri95, Lemma 5.3]

For a *non-overlapping* pattern $P \in \text{PSL}_2(\mathbb{N})$, the measure of P -asymmetry is the conjugacy invariant function $\text{mas}_P: \text{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ defined:

- $A \in \text{PSL}_2(\mathbb{N}) \mapsto \text{mas}_P(A) = \text{occ}_P(A) - \text{occ}_P(A^*)$
(In particular for $P = R$ we have $\text{Rad}(A) = \text{mas}_R(A)$.)
- $A \in \text{Tors}(\text{PSL}_2(\mathbb{Z})) \mapsto \text{mas}_P(A) = 0$.

We have $\text{mas}_P \in \text{PX}(\text{PSL}_2(\mathbb{Z}))$ and $\|d \text{mas}_P\|_\infty \leq 6$.

If $P \neq P^*$ then $d \text{mas}_P$ yields a non-zero class in $H_{b,0}^2(\text{PSL}_2(\mathbb{Z}))$.



From [Gri95]: some cases in the proof of $\|d \text{mas}_P\|_\infty \leq 6$.

From computing the dimension to describing a basis

The dimension of $H_b^2(\Gamma)$ is roughly known for many groups:

- 1 If Γ is amenable then $H_{b,0}^*(\Gamma; \mathbb{R}) = 0$
- 2 If Γ is word hyperbolic then $\dim H_b^2(\Gamma) = \infty$ [EF97]
- 3 If $\Gamma = \text{Mod}(\mathbb{F})$ then $\dim H_b^2(\Gamma) = \infty$ [BF02, Fuj09].

free product of two copies of an infinite cyclic group; that is a free group with two generators. However in [9] there is no discussion about the generation of $H_b^{(2)}(\mathbb{Z} * \mathbb{Z})$ but only the statement that the dimension of $H_b^{(2)}(\mathbb{Z} * \mathbb{Z})$ is infinite. Let us analyze this statement.

From [Gri95]: what about generating $PX(\Gamma)$?

Assertion 5.1 *The bounded 2-cocycles $\delta^{(1)}f_W$ are all non-trivial cocycles, and are all linearly independent.*

But the last statement of the assertion is not correct as the following ex-

From [Gri95]: beware of independence!

A Schauder basis for $H_b^2(\Gamma)$

Definition Schauder basis:

For a topological \mathbb{R} -vector space V , a sequence $(x_n)_{n \in \mathbb{N}}$ is a *basis* when

$$\forall v \in V, \quad \exists! (c_n(v)) \in \mathbb{R}^{\mathbb{N}}, \quad v = \sum c_n(v) \cdot x_n,$$

and a *Schauder basis* when the coefficient functionals c_n are continuous.

$e_W(g)$ = number of times W occurs in \bar{g} – number of times W^- occurs in \bar{g} ,

where \bar{g} is the reduced cyclic word corresponding to the element $g \in N$.

Repeating the arguments used when proving Theorem 5.7 we get

Theorem 5.11 *The space $H_{b,2}^{(2)}(N)$ is infinite dimensional. It is isomorphic to the space $PX(N)$ and every element $f \in PX(N)$ can be uniquely written in the form*

$$f = \sum_{W \in P^+} \alpha_W e_W.$$

for some coefficients $\alpha_W \in \mathbb{R}$.

The Schauder base of $PX(\mathrm{PSL}_2(\mathbb{Z}))$ of [Gri95, Theorem 5.11]

Schauder base: $\mathrm{mas}_P \in PX(\mathrm{PSL}_2(\mathbb{Z}))$ for non-overlapping $P \in \mathrm{PSL}_2(\mathbb{N})/\star$.

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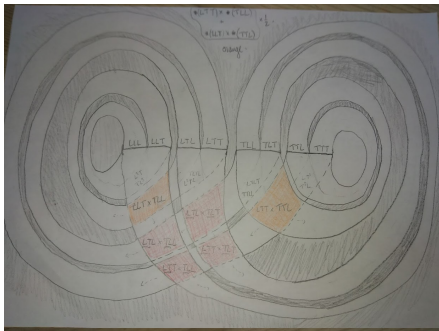
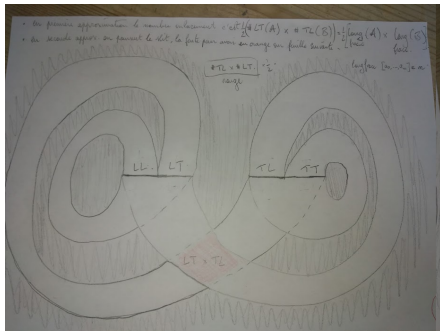
Linking number: sum crossings indexed by linked patterns

Lemma [Sim22, 4.34] & [Sim24, 7.1]: linking counts sum of crossings

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{N})$ the corresponding modular knots have:

$$\mathrm{lk}(A, B) = \frac{1}{2} \sum_w \left(\begin{array}{c} \mathrm{occ}_{RwL}(A) \cdot \mathrm{occ}_{LwR}(B) \\ + \\ \mathrm{occ}_{LwR}(A) \cdot \mathrm{occ}_{RwL}(B) \end{array} \right)$$

where the summation extends over all words $w \in \mathrm{PSL}_2(\mathbb{N})$.



Pseudocharacters from asymmetric linking numbers

Definition [Sim24, 7.6]: The cosign functions

For $A \in \mathrm{PSL}_2(\mathbb{Z})$, define the cosign function $C_A: \mathrm{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ as the antisymmetric linking number: $C_A(B) := \frac{1}{2} (\mathrm{lk}(A, B) - \mathrm{lk}(A^{-1}, B))$

Proposition [Sim24, 7.6]: decompose C_A in the Schauder basis (mas_P)

For $A, B \in \mathrm{PSL}_2(\mathbb{N})$, we may decompose C_A as:

$$C_A(B) = \mathrm{lk}(A, B) - \mathrm{lk}(A, B^*) = \frac{1}{2} \sum_w \left(\begin{array}{c} \mathrm{occ}_{RwL}(A) \cdot \mathrm{mas}_{LwR}(B) \\ + \\ \mathrm{occ}_{LwR}(A) \cdot \mathrm{mas}_{RwL}(B) \end{array} \right)$$

the sum over all words $w \in \mathrm{PSL}_2(\mathbb{N})$ with $\mathrm{len}(w) < \max\{\mathrm{len} A, \mathrm{len} B\}$.

Theorem [Sim24, 7.6, 7.7]: Pseudocharacters from linking numbers

- For all $A \in \mathrm{PSL}_2(\mathbb{N})$ we have $C_A \in \mathrm{PX}(\mathrm{PSL}_2(\mathbb{Z}))$ and $\|d C_A\|_\infty \leq 6$.
- Linear relations spanned by $C_{A^n} = n C_A$ for $A \in \mathrm{PSL}_2(\mathbb{Z})$ and $n \in \mathbb{Z}$.
(In particular C_A is trivial if and only if A is conjugate to A^{-1} .)

Link equivalence

To prove the non-triviality and linear independence of the C_A for $A \in \mathcal{P}_+$, we were led to show the non-degeneracy of the linking form.

Lemma [Sim24] : The linking pairing is non-degenerate

If hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ are link equivalent, then they are conjugate.
(Link equivalence: $\forall X \in \mathrm{PSL}_2(\mathbb{Z}), \mathrm{lk}(A, X) = \mathrm{lk}(B, X)$.)

Linking recovers intersection

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have $\frac{1}{2}I(A, B) = \mathrm{lk}(A, B) + \mathrm{lk}(A^{-1}, B)$.

Intersection from modular cocycles

For hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$, [DIT17] construct a modular function whose symbol recovers $I(A, \cdot) = 2(\mathrm{lk}(A, \cdot) + \mathrm{lk}(A^{-1}, \cdot))$.

What about the cosign $C_A(\cdot) = \mathrm{lk}(A, \cdot) - \mathrm{lk}(A^{-1}, \cdot)$?

Schauder basis of $PX(\mathrm{PSL}_2(\mathbb{Z}))$ from linking numbers

Definition: conjugacy classes modulo powers

The set \mathcal{P} of primitive infinite order conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$ contains the subset \mathcal{P}_0 of those which are stable under inversion. Partition $\mathcal{P} \setminus \mathcal{P}_0 = \mathcal{P}_- \sqcup \mathcal{P}_+$ in two subsets in bijection by inversion. We may choose $R \in \mathcal{P}_+$, and note that $C_R = \mathrm{lk}(\text{trefoil}, \cdot)$.

Theorem [Sim24]: Schauder basis for $PX(\mathrm{PSL}_2(\mathbb{Z}))$

The collection of $C_A \in PX(\Gamma; \mathbb{R})$ for $A \in \mathcal{P}_+$ is a Schauder basis:

- $\forall f \in PX(\mathrm{PSL}_2(\mathbb{Z}); \mathbb{R}), \quad \exists! (c_A(f))_A \in \mathbb{R}^{\mathcal{P}_+}, \quad f = \sum_A c_A(f) \cdot C_A$
- The period coefficients $c_A: f \mapsto c_A(f)$ are continuous

\implies Fourier theory of pseudocharacters, with a natural basis of *cosigns*.

Problem 4.10 To construct a basis for $H_b^{(2)}(B_n)$.

Posed in [Gri95].

Sketch of a project to find Schauder bases of $PX(\text{Mod}(\mathbb{F}))$

Mapping class group $\text{Mod}(\mathbb{F})$ acts on curve complex $\mathcal{C}(\mathbb{F})$:

- Metric space $\mathcal{C}(\mathbb{F})$ is hyperbolic and $\partial\mathcal{C}(\mathbb{F})$ are ending laminations
- The hyperbolic isometries of $\mathcal{C}(\mathbb{F})$ are pseudo-Anosov elements A
- Fixed points of A in $\partial\mathcal{C}(\mathbb{F})$ are stable-unstable measured laminations.

Schauder bases for $PX(\text{Mod}(\mathbb{F}))$ from *combinatorics of monoids*:


- Define Brooks–Series Barge–Ghys cocycles mas_P in $H_b^2(\text{Mod}(\mathbb{F}); \mathbb{R})$.
- Find basis of mas_P for P in “train track monoid” of Dehn-twists.


Schauder basis for $PX(\text{Mod}(\mathbb{F}))$ from *asymetric linking periods*:


- Define $\text{lk}(A, B)$ as sum cross-ratio of fixed points over double cosets.
- Show basis of C_A for classes of pseudo-Anosov A modulo powers.


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Moral of the story...

So many mysteries are concealed within a simple trefoil !



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Thank you for your attention and feel free to ask unbounded questions.