Loops in surfaces, chord diagrams, interlace graphs

Part 0 : genus from intersection, bicolouring, interlacing

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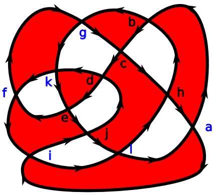

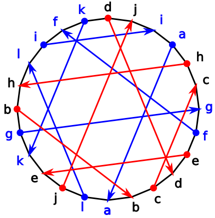

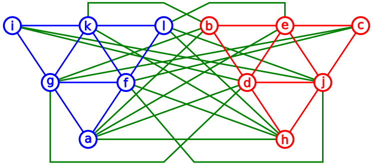

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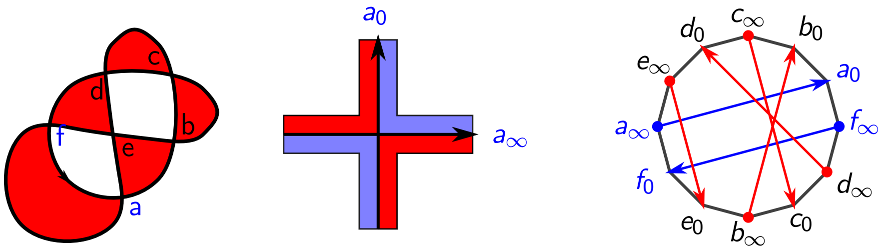


A spheriloop and its framed chord diagram.

Quest (image): describe which combinatorial *invariants* appear

Describe the *image* of a forgetful map between two combinatorial species.

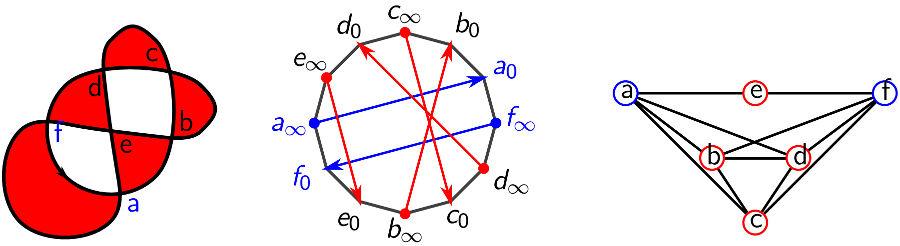
Which (un)framed chord diagrams arise from spheriloops ?

Which graphs come from chord diagrams ?

Find a certificate for checking if an element is in the image.

Find a procedure to generate (uniquely) all elements in the image.

From spheriloops to chord diagrams to graphs



A spheriloop, its framed chord diagram, and its interlace graph.

Quest (pre-images): describe which objects have the same *invariant*

Describe the *fibers* of a forgetful map between two combinatorial species.

Which spheriloops have the same (un)framed chord diagram ?

Which chord diagrams have the same (bicoloured) interlace graph ?

Which transformations leave these combinatorial *invariants*... invariant ?!

Find and present a groupoid structure whose orbits are the fibers.

The Gauss-Rosenstiehl criterion

The interlace graphs of chord diagrams will be called a *chordiagraphs*.
The chord diagrams of loops in the sphere will be called *Gaussian*.

Theorem : characterisation of Gaussian chord diagrams by Rosenstiehl

A chord diagram C arises from a spheriloop if and only if its interlace graph G with adjacency matrix $\mathfrak{e}\colon V_G \times V_G \to \{0,1\}$ satisfies:

EN1 every vertex $x \in V_G$ has an even degree

$$\forall x \in V_G, \quad \text{Card}\, N(x) \equiv 0 \tag{EN1}$$

EN2 non-adjacent vertices share an even number of neighbours

$$\forall x, y \in V_G, \quad \mathfrak{e}(x,y) = 0 \implies \text{Card}\, N(x) \cap N(y) \equiv 0 \tag{EN2}$$

RC The cocycle $\mathfrak{r}\colon E_G \to \mathbb{Z}/2$ defined by $\mathfrak{r}(x,y) = 1 + \text{Card}\, N(x) \cap N(y)$ has integral $\equiv 0$ on every cycle of G.

The Gauss-Rosenstiehl criterion

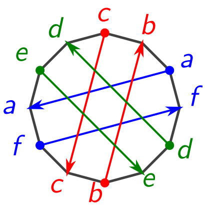
Theorem [Ros99]: characterisation of Gaussian chord diagrams

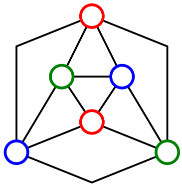
C is Gaussian $\iff$ its interlace matrix $\mathfrak{e}\colon V_G \times V_G \to \{0,1\}$ satisfies:

Loc: $\forall x, y \in V_G\colon\ \mathfrak{e}(x,y) = 0 \implies \mathfrak{e}^2(x,y) \equiv 0$.

Glo: The cocycle $\mathfrak{r} := \mathfrak{e} + \mathfrak{e}^2\colon E_G \to \mathbb{Z}/2$ is cohomologous to 0 mod 2.

Gauss found the *local conditions* EN1, EN2 as necessary but not sufficient. This yields a *criterion for checking* whether a chord diagram C is Gaussian in polynomial time, and it *depends only on its interlace graph* G.





A chordiagraph satisfying satisfying EN1 and EN2 but not RC.

From filoops to framed chord diagrams

Definition of filoops

Let F be smooth surface which is connected, oriented, closed, of genus g. A filoop is a generic immersion $\gamma\colon \mathbb{S}^1 \looparrowright F$ such that $F \setminus \gamma = \sqcup\,$discs, considered up to orientation preserving diffeomorphisms.

From filoops to framed chord diagrams

For a filoop γ, each intersection $x \in \gamma(\mathbb{S}^1)$ has two preimages $x_\infty, x_0 \in \mathbb{S}^1$ ordered so that $(\vec{\gamma}(x_\infty), \vec{\gamma}(x_0))$ forms a positive basis of the plane $T_x F$. Hence a loop yields a framed chord diagram...

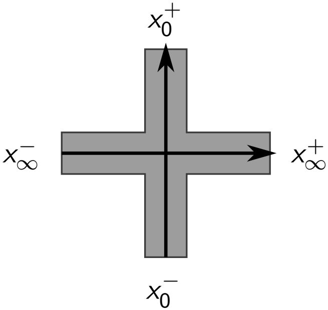
Definition of Framed chord diagrams

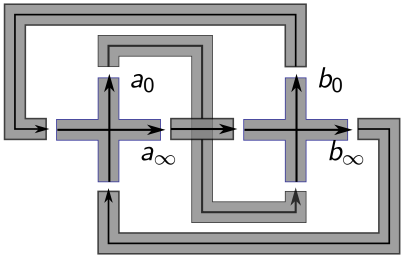
A chord diagram is a finite cyclic word C in which each letter appears exactly twice, and a framing is a map $\varphi\colon C \to \{\infty, 0\}$ which is bijective in restriction to every double occurrence letter.

From framed chord diagrams to filoops

Correspondence

Every framed chord diagram C_φ arises from a unique filoop γ.



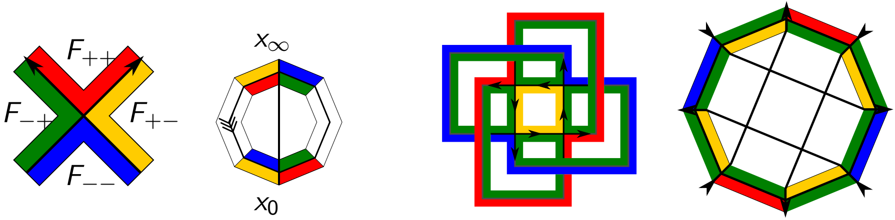


Constructing the ribbon-graph M associated to a framed chord diagram C_φ

Regions and genus of framed chord diagram

Recovering the components of $F \setminus \gamma$ from C_φ

The components of $F \setminus \gamma$ can be read off C_φ by travelling along the circle, jumping across each chord that is met along the way, and pursuing along the circle in the same or opposite direction according to the framing.



Reading the boundary components of M on $C_\varphi = a_\infty c_0 d_\infty b_0 c_\infty a_0 b_\infty d_0$.

The genus of C_φ is given by $g = 1 + (n - f)/2$

If C_φ has n chords and f regions then $2 - 2g = f - n$ by additivity of the Euler characteristic with respect to disjoint union $F = (F \setminus M) \sqcup M$.

The minimal genus of a chord diagram

Question: minimal genus among framings of a chord diagram

Compute $\mathrm{mg}(C) = \min\{\mathrm{genus}(C_\varphi) \mid \text{framings } \varphi\colon C \to \{\infty, 0\}\}$.

Definition: Interlace graph of a chord diagram

To a chord diagram C we associated its interlace graph G, with vertices V_G the set of chords and edges E_G the set of pairs of intersecting chords. A chordiagraph is the interlace graph of some chord diagram.

Theorem [?]: Minimal genus for local-even graphs

Fix a chordiagraph G satisfying EN1 and EN2 and define its Rosenstiehl cocycle $\mathfrak{r} \in Z^1(G;\mathbb{Z}/2)$ by $\mathfrak{r}(x,y) = 1 + \mathrm{Card}\, N(x) \cap N(y)$.
Every chord diagram C with interlace graph G has minimal genus:

$$\mathrm{mg}(C) = \mathrm{mg}(G) = \frac{1}{2}\min\{\mathrm{rank}(\mathfrak{r} + d\chi) \mid \chi \in Z^0(G;\mathbb{Z}/2)\}$$

namely the *minimum of the half-rank over the space* $\mathfrak{r} + B^1(G;\mathbb{Z}/2)$.

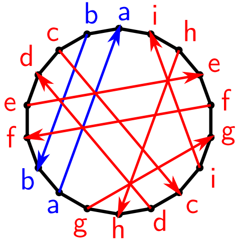
Reconstructing all spheriloops from a Gaussian graph

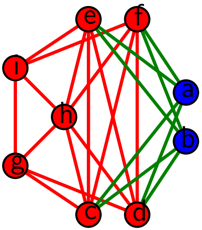
Theorem [?]: Spheriloops with prescribed chord diagram

A chord diagram with a Gaussian interlace graph G admits
Card $H_0(G;\mathbb{Z}/2) = 2^{b_0}$ framings of genus 0 obtained as follows:

Lift: Two of them are obtained by integrating the Rosenstiehl form so that consecutive letters have the same framing $\iff$ their colours differ.

Act: The others differ by locally-constant change of framings on chords associated to connected components of G.





$\mathfrak{r} \in Z^1(G;\mathbb{Z}/2)$

A chord diagram whose interlace graph is connected and Gaussian.

Reconstructing all spheriloops from a Gaussian graph

Theorem [?]: Spheriloops with prescribed chord diagram

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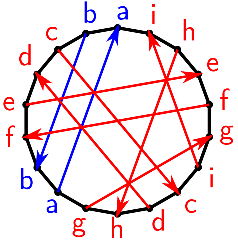
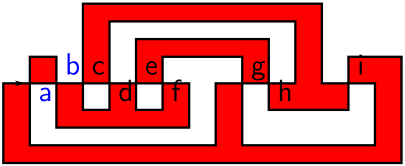




The unique framing of genus 0 and its corresponding spheriloop.

The intersection form of a filoop

The genus of F is also equal to half the rank of $H_1(F;\mathbb{Z}/2)$,
or equivalently of the symplectic intersection form on $H_1(M;\mathbb{Z}/2)$.

Definition of subloops α_x

In the surface $M \subset F$, define for every intersection $x \in \gamma$ the loops α_x and β_x running along γ from x_∞ to x_0 and x_0 to x_∞ respectively.
Since $\gamma = \alpha_x\beta_x$ in $\pi_1(M, x)$, we have $[\gamma] = [\alpha_x] + [\beta_x]$ in $H_1(M;\mathbb{Z}/2)$.

Proposition [Ghy17]: basis and intersection form of $H_1(M;\mathbb{Z}/2)$

The collections of $n+1$ cycles $[\alpha_x], [\gamma]$ forms a basis of $H_1(M;\mathbb{Z}/2)$, in which the symplectic form on $H_1(M;\mathbb{Z}/2)$ can be expressed in terms of C_φ:

$$[\gamma]\cdot[\alpha_x] = \mathrm{Card}\, N(x) \qquad [\alpha_x]\cdot[\alpha_y] = \mathfrak{i}_\varphi(x,y)$$

$$\mathfrak{i}_\varphi(x,y) = \mathfrak{e}(x,y) + \mathrm{Card}\{z_j \in C_\varphi \mid z_j \in (x_\infty, x_0) \wedge z_{1/j} \in (y_\infty, y_0)\}$$

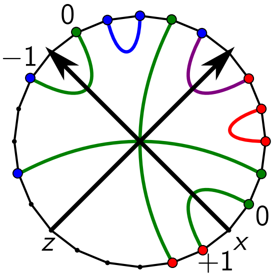
Intersection form of a framed chord diagram

Definition: Intersection form of a framed chord diagram C_φ

To a framed chord diagram C_φ with interlace graph G, we associate its *intersection form* $\mathfrak{i}_\varphi \colon V_G \times V_G \to \mathbb{Z}/2$ defined by:

$$\mathfrak{i}_\varphi(x, y) = \mathfrak{e}(x, y) + \mathrm{Card}\{z_j \in C_\phi \mid z_j \in (x_\infty, x_0) \wedge z_{1/j} \in (y_\infty, y_0)\}$$

The restriction of $\mathfrak{i}_\varphi$ to E_G is the *intersection cocycle* $\mathfrak{i}_\varphi \in Z^1(G; \mathbb{Z}/2)$.



Changing the framing φ at z affects the value of $\mathfrak{i}_\varphi(x, y)$.

The Rosenstiehl form of a graph

Definition [?]: the Rosenstiehl form

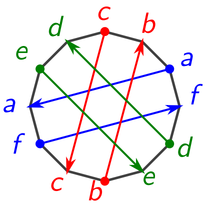
Consider a simple graph G with adjacency matrix $\mathfrak{e}\colon V_G \times V_G \to \{0,1\}$.
We define its (symmetric bilinear) Rosenstiehl form by $\mathfrak{r} = \mathfrak{e} + \mathfrak{e}^2$.
Since $\mathfrak{e}^2(x,y) = \operatorname{Card} N(x) \cap N(y)$ we have:

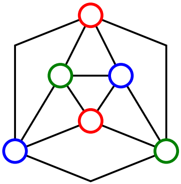
for vertices: $\mathfrak{r}(x,x) = \operatorname{Card} N(x)$

for non-edges : $\mathfrak{e}(x,y) = 0 \implies \mathfrak{r}(x,y) = \operatorname{Card} N(x) \cap N(y)$

for edges: $\mathfrak{e}(x,y) = 1 \implies \mathfrak{r}(x,y) = 1 + \operatorname{Card} N(x) \cap N(y)$

The restriction of $\mathfrak{r}$ to E_G defines the Rosenstiehl cocycle $\mathfrak{r} \in Z^1(G; \mathbb{Z}/2)$.





A chordiagraph satisfying satisfying EN1 and EN2 but not RC.

Parity conditions: local and global

Definition: local and global parity conditions

We say that a graph G with Rosenstiehl form $\mathfrak{r}$ satisfies the property:

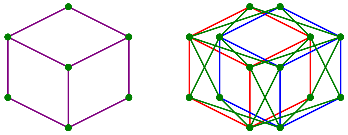
EN1 when every vertex $x \in V_G$ has an even degree, namely $\mathfrak{r}(x, x) \equiv 0$

EN2 when distinct non-adjacent vertices $x, y \in V_G$ share an even number of neighbours, namely $\mathfrak{r}(x, y) \equiv 0$

RC when the Rosenstiehl cocycle is null in cohomology: $\mathfrak{r} \in B^1(G; \mathbb{Z}/2)$, equivalently its integral along every cycle is even, namely:

$$\forall (x_1, \ldots, x_l) \in V_G^l, \quad \prod \mathfrak{e}(x_{i-1}, x_i) \neq 0 \implies \sum \mathfrak{r}(x_{i-1}, x_i) \equiv 0 \quad \text{(RC)}$$

A graph satisfying EN1 and EN2 and RC will be called *Gaussian*.



The smallest Gaussian non-chordiag graph

When bicolourings derive from framings

Remark: bicolourability only depends on the interlace graph

For filoop $\gamma \subset F$ whose chord diagram C has interlace graph G, tfae:

$\iff$ The graph G satisfies condition EN1: $\forall x \in V_G$, $\operatorname{Card} N(x) \equiv 0$.

$\iff$ In $H_1(M; \mathbb{Z}/2)$, the cycle $[\gamma]$ is symplectic-orthogonal to all $[\alpha_x]$.

$\iff$ In $H_1(F; \mathbb{Z}/2)$ the class of the filoop γ is trivial.

$\iff$ The filoop $\gamma \subset F$ admits (two) *bicolourings* $F \setminus \gamma \to \{\bullet, \circ\}$.

Proposition [CR01, ?]: Integrating bicolourings into framings

For a bicolourable chord diagram C, bicolourings $\chi \colon V_G \to \{\bullet, \circ\}$ correspond to framings $\varphi \colon C \to \{\infty, 0\}$ up to global inversion:

A framing $\varphi \colon C \to \{\infty, 0\}$ yields two bicolourings $\chi \colon V_G \to \{\bullet, \circ\}$ by choosing one for the corresponding filoop $F \setminus \gamma$ and colouring each intersection $x \in \gamma$ as the region towards $\vec{\gamma}(x_\infty) + \vec{\gamma}(x_0)$.

A bicolouring $\chi \colon V_G \to \{\bullet, \circ\}$ yields two functions $\varphi \colon C \to \{\infty, 0\}$ by choosing the value on some letter, and extending so that consecutive letters have the same value $\iff$ they have different colours.

Three bilinear forms: intersecting, bicolouring, interlacing

Bicolouring form

For a graph G with a bicolouring $\chi\colon V_G \to \{\bullet, \circ\}$, define the *colour form* $\mathfrak{c}_\chi\colon V_G \times V_G \to \mathbb{Z}/2$ as the derivative of the indicator function on $V_\bullet$ or $V_\circ$, and the *colour coboundary* $d\chi \in B^1(G, \mathbb{Z}/2)$ by restricting to the edges.

Theorem [?]: Three bilinear forms add to 0 mod 2

For a bicolourable framed chord diagram C_φ we have the following relation between its intersection form $\mathfrak{i}_\varphi$, bicolour form $\mathfrak{c}_\chi$ and Rosenstiehl form $\mathfrak{r}$:

$$\boxed{\mathfrak{i}_\varphi - \mathfrak{c}_\chi \equiv \mathfrak{r}}$$

Hence the intersection form can be computed from the bicoloured graph.

We also recover the fact that $\mathfrak{i}_\varphi$ determines $\mathfrak{c}_\chi$, whence φ up to inversions on connected components of the interlace graph.
In particular when G satisfies EN2, namely $\mathfrak{e}^2$ vanishes on the non-edges, one may compute $\mathfrak{i}_\varphi$ from $d\chi \in B^1(G, \mathbb{Z}/2)$.

Minimal genus of a bicolourable chord diagram

Corollary: Minimal genus for bicolourable graphs

Fix a graph G satisfying EN1 with Rosenstiehl form $\mathfrak{r}$.
The minimal genus of any chord diagram C with interlace graph G is:

$$\mathrm{mg}(C) = \mathrm{mg}(G) = \frac{1}{2}\min\{\mathrm{rank}(\mathfrak{r} + \mathfrak{c}_\chi) \mid \chi\colon V_G \to \{\bullet, \circ\}\}$$

the minimum of the half-rank over the space of bicolour forms $c_\chi\colon V_G \times V_G \to \{0,1\}$ translated by $\mathfrak{r}$.

Corollary: Minimal genus for local-even graphs

Fix a graph G satisfying EN1 & EN2 with Rosenstiehl cocycle $\mathfrak{r} \in Z^1(G;\mathbb{Z}/2)$.
The minimal genus of any chord diagram C with interlace graph G is:

$$\mathrm{mg}(C) = \mathrm{mg}(G) = \frac{1}{2}\min\{\mathrm{rank}(\mathfrak{r} + d\chi) \mid \chi \in Z^0(G;\mathbb{Z}/2)\}$$

namely the *minimum of the half-rank over the space* $\mathfrak{r} + B^1(G;\mathbb{Z}/2)$.

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Thank you for your presence, feel free to ask (m)any questions

Teaser...

Which chord diagrams have the same interlace graph ?

Which loops have the same chord diagram ?

How to construct all spheriloops with a given Gaussian graph ?