RESEARCH STATEMENT

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Abstract. This is a long and illustrated version of my research statement. It is not meant to be read entirely by a single person: each one may find their interest in a different section.

I first propose a short explanation for my motivations and interest in mathematics (which may amuse a philosophically inclined person or a curious mind outside my field of research), including a disclaimer regarding the (un)importance of my work, and potential collaborations with other faculty members.

Then i survey each of my works, grouped by themes and then listed in (chrono)logical order. They are aimed at future colleagues or students who wish to familiarize themselves with part of my research, and find inspiring projects to work on. For that purpose, i provide some historical motivations, try as much as possible to distil some of the main old or new ideas underlying or emerging from my work, and provide some future directions of research.

Contents

List of Selected Works

[AOP+24] Pierre Aboulker, Nacim Oijid, Robin Petit, Mathis Rocton, and Christopher-Lloyd Simon. Computing the degreewidth of a digraph is hard, 2024. submitted for publication, arXiv version.

[GS20] Étienne Ghys and Christopher-Lloyd Simon. On the topology of a real analytic curve in the neighborhood of a singular point. *Astérisque*, Some aspects of the theory of dynamical systems: a tribute to Jean-Christophe Yoccoz. Vol. I(415):1–33, 2020. HAL version.

[MS21] Julien Marché and Christopher-Lloyd Simon. Automorphisms of character varieties. *Ann. H. Lebesgue*, 4:591–603, 2021. arXiv version.

[MS24] Julien Marché and Christopher-Lloyd Simon. Valuations on the character variety: Newton polytopes and residual Poisson bracket. *Geom. Topol.*, 28(2):593–625, 2024. arXiv version.

[Sim22a] Christopher-Lloyd Simon. *Arithmetic and Topology of Modular knots.* Thèse, Université de Lille, June 2022. HAL version.

[Sim22b] Christopher-Lloyd Simon. Topologie et dénombrement des courbes algébriques réelles. *Ann. Fac. Sci. Toulouse Math. (6)*, 31(2):383–422, 2022. arXiv version.

[Sim23a] Christopher-Lloyd Simon. Conjugacy classes in $\mathrm{PSL}_2(\mathbb{K})$. *Mathematics Research Reports*, 4:23–45, 2023. arXiv version.

[Sim23b] Christopher-Lloyd Simon. Loops in surfaces, chord diagrams, interlace graphs: operad factorisations and generating grammars, 2023. submitted for publication, arXiv version.

[Sim24] Christopher-Lloyd Simon. Linking numbers of modular knots, 2024. To appear in Geometry and Topology for publication, arXiv version.

[SS24a] Scott Schmieding and Christopher-Lloyd Simon. Isogenies of low complexity systems: Sturmian, denjoy, and interval exchange systems, 2024. In preparation.

[SS24b] Christopher-Lloyd Simon and Ben Stucky. Pin the loop taut: a one-player topologame, 2024. submitted for publication, arXiv version.

0. Presentation as a mathematician

Why do mathematics ? The universe *is*... or to describe it in a more explicit but probably less accurate way: the universe unfolds its shape in space according to certain laws (which we hope to understand). Thus, in a language of its own, we observe the formation of structures at several scales, it copies and assembles pieces of itself, according to almost symmetric and self-similar patterns.

The salient features of the language change according to the time and scale we observe. Indeed the laws of quantum physics prevail in early times and at small scales: the hierarchy of force fields (strong nuclear, electromagnetic, weak nuclear, and gravitational) matches the time and scale at which their interacting structures emerge (particles, atoms, molecules and stars).

Eventually appeared complex structures described in the language of chemistry, which led to the emergence of life on earth. After a history of evolution, certain species likes humans gathered in societies and developed the capacity to reflect: we are a model of nature which is "conscious" of its existence. The mathematical activity developed to better understand hence interact with our surroundings. It has survived so far as an evolutionary advantage.

Thus Physics leads to Chemistry leads to Biology leads to Anthropology leads to Mathematics, whose purpose is to reflect on all of the aforementioned structures. Its efficiency is far from unreasonable yet fascinating, and it culminates when it applications circle back into the universe.

Mathematical interests: pure, applied, history, philosophy... My passion for mathematics is rooted in the desire to understand the world around us, and the enjoyment found in solving intricate puzzles. These two aspects are closely connected, as broad fields (such as dynamical systems or game theory) are founded on more specialized theories (such as combinatorics, probability, group theory, differential equations, and topology). All of this fascinates me, and I take great pleasure in sharing my thoughts and listening to experts across different domains.

I would surely be classified as a pure mathematician, however i am also interested in applications and looking forward to genuine (often serendipitous) opportunities to bridge. Indeed, i am eager to focus my part of my interests towards Dynamical systems and Game theory, with applications in Physics, Biology or Computer science. These grand arching domains use the tools from and propose problems motivating the development of many branches in pure mathematics.

I am also interested in the historical development of mathematics as a whole ([Kle79, ABC$^+$10]) and of specific domains ([Die74, KLLP07, Ber09]), the psychological aspects of mathematical creativity and pedagogy ([Pol54a, Pol54b]), and the philosophy of mathematics [Man07].

Areas of research and expertise. The areas in which i have published contributions include:

- synthetic geometry (projective geometry over arbitrary fields, hyperbolic geometry) in connection with the algebraic theory of numbers [Sim23a]
- combinatorics and computational complexity (graphs, permutations) [Sim23b, SS24b, AOP$^+$24], enumerative combinatorics (analytic and symbolic methods on generating functions) [Sim22b]
- algebraic topology for low dimensions (intersection and linking) [Sim22a, Sim24, Sim23b]
- geometry and representation theory of hyperbolic groups and their automorphism groups (actions on trees, character varieties, bounded cohomology) [MS21, MS24, Sim24]
- singularity theory with a focus on real algebraic geometry [GS20, Sim22b]

Their description will be organized in independent sections which can be read in any order, but section 1 and 2 bear strong interactions, while 4 motivates both 3 and 5.

Contributions. When I wrote a paper in collaboration, the contributions and implications were balanced. This is somewhat difficult to estimate and truthfully, the product of these collaborations exceeds the sum of individual contributions.

When I wrote articles alone, all the results were the fruit of my own work. However, I often benefited from privileged interlocutors with whom to share my reflections: their interest allowed me not only to maintain constant motivation but also to correct myself and progress over the explanations; while their criticisms and suggestions have always led to a more complete work.

Indeed, the article [Sim22b] is the result of my Master's work under the supervision of Etienne Ghys, continuing the themes and open questions in [GS20]. The articles [Sim23a, Sim23a] follow from my thesis work during which my supervisors played this guiding role, and it was Sergei Tabachnikov who encouraged me to pursue the investigations in [Sim23b].

We are stepping in the footsteps of giants: contributing to this fantastic edifice obliges us to internalise, rediscover and refine our understanding of the specific methods and general principles that we have learnt. Thus we often appreciate and understand much more mathematics than what our productions suggest, especially at the beginning of our careers. However, this silent internal growth and germination often bears its fruits later in one's mathematical career. I believe this applies to my works: they are minuscule stones whose value so far lies much more in what I have learnt during their carving process, but these germs will inspire and prepare me to form greater statues.

Moreover, while the domains are multiple and diverse, they all rely on a few fundamental principles at the core of mathematics. Hence the most anodyne questions often lead to the same discoveries as the venerable problems. We therefore understand that it can be difficult to gauge the importance and predict the impact of certain mathematical works, especially at the beginning of our careers. This is not to say that they all have the same value, but that a taste for "good mathematics" requires an open mind, and that (un)importance is a subjective matter subject to trends in the present, while being difficult to foresee in the future.

Potential collaborations with faculty members. I can foresee strong collaborations with several professors and am eager to share projects with my peer juniors, pursuing the topics presented later in this research statement as well as ongoing projects briefly mentioned in this paragraph.

Of course, one can pursue only a certain number goals at a time: on the one hand i am able and willing to focus on a couple of specific projects with a few actively present and interested members, while on the other hand i will be open to initiating collaborations with other members and bridging themes.

Let me briefly mention a few specific projects that i have initiated and will not or barely allude to in this research statement, and that i would also be happy to share with members of the faculty.

Arithmetic dynamics of interval exchanges. . As mentioned in the last paragraph of this research statement, i am pursuing conjectures concerning the transcendence of simple geodesics in arithmetic surfaces. This led me to investigate the arithmetic and dynamic complexity of interval exchanges, and observe certain phenomena which i would like to further investigate. They concern on the one hand the equidistribution of a self-similar interval exchanges under powers, and on the other hand certain groups of interval exchanges associated to Hilbert class fields and Gauss composition.

Mapping class groups of mapping tori. I computed the (outer) automorphism groups of certain semi-direct products over a cyclic group, and of certain nilpotent groups. The motivation was to compute the mapping class groups of certain manifolds fibering over the circle and of certain fiber bundles. The project is almost finished, but is suffering from the lack of concrete applications, and i am sure there are interesting consequences to derive concerning fibred 3-manifolds, or the (non)-hyperbolicity of fibred manifolds in higher dimensions, the bounded cohomology of free by cyclic group.

Gauss linking forms on unit tangent bundles. I also project concerns linking numbers of Legendrian knots, and Gauss linking forms on the unit tangent bundle of certain surfaces. One of the ideas consists in defining a family of point pair invariants in the isometry group of a two dimensional geometry (sphere, plane, hyperboloid), which interpolates between the geometry at $q = 1$ and the topology as $q = \infty$. Hence the integral geometric formulae for linking numbers degenerates to a combinatorial summation of signs.

Higher intersections of geodesics in arithmetic surfaces. This project combines and furthers the investigations presented in sections 4 and 3. Given Fuchsian group Γ, and a representation $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ i associate to a pair (of conjugacy classes) $\alpha, \beta \in \Gamma$ a family of higher order symplectic intersection numbers indexed by a weight $m \in \mathbb{Z}$, denoted $\mathrm{i}_\rho^m(\alpha, \beta) \in \mathbb{R}$. They yield algebraic functions on the character variety $\rho \mapsto \mathrm{i}_\rho^m(\alpha, \beta)$, which have several interpretations from the geometric and representation theoretic viewpoints. I wish to understand their arithmetic and topological significance. For instance, i observed using PARI GP that for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ some unexpected vanishing properties at certain m, ρ, and have precise conjectures explaining them in terms of Galois theory. Besides, i expect that certain generating functions associated to the $\mathrm{i}_\rho^m(\alpha, \beta)$ yield modular forms associated to the group $\rho(\Gamma) \subset \mathrm{PSL}_2(\mathbb{R})$.

1. Topology and combinatorics of singular curves in surfaces

From the very first sentence of his book [Arn91] dedicated to the theory of singularities and its applications, Vladimir Arnold justifies the importance of studying them as follows:

> The mathematical description of the world depends on a delicate interplay between discrete and continuous objects. Discrete phenomena are perceived first, but continuous ones have a simpler description in terms of the traditional calculus. Singularity theory describes the birth of discrete objects from smooth, continuous sources.

On the topology of a real analytic curve in the neighbourhood of a singular point.

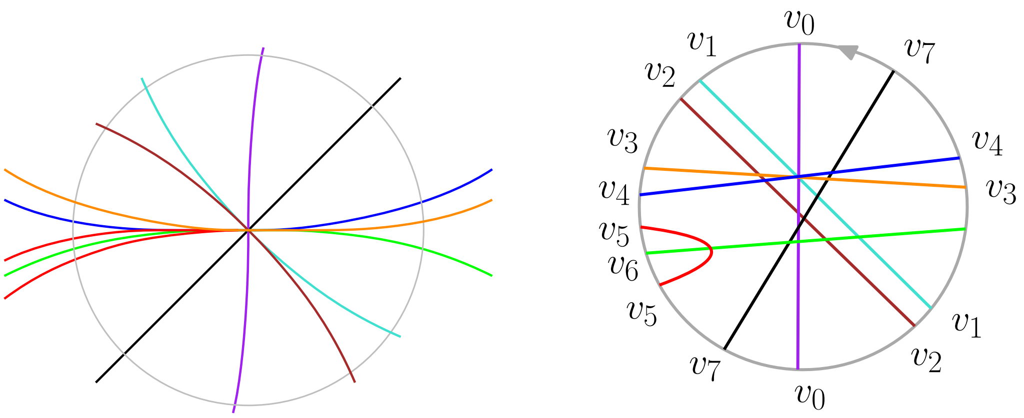
Motivation. The topological classification of algebraic or analytic singularities of curves in the complex plane has been thoroughly explored completed [Web08, EN85, Trá03]: it involves iterated torus knots, linking forms, graph manifolds, and carousels.

However, while real analytic and algebraic curves in the real plane have been thoroughly studied [BK86], the question of their singularities has been overlooked as being too simple for attention, yet their combinatorics reveals a surprising structure.

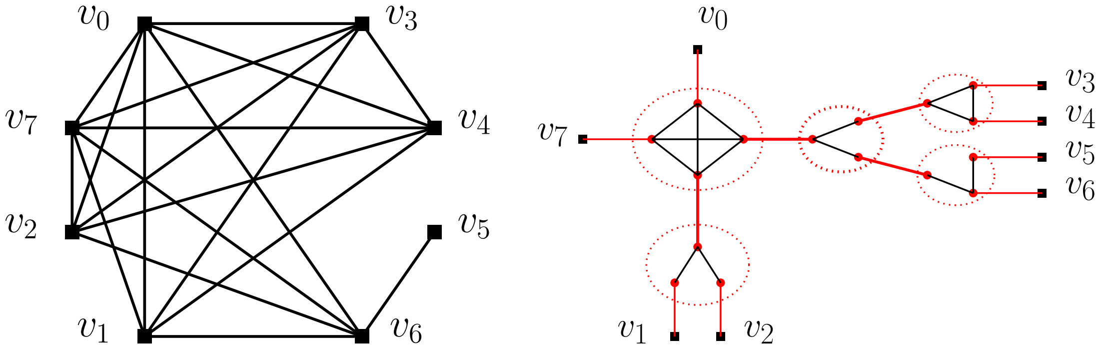
Content. The article [GS20], describes the topology of real analytic or algebraic curve singularities in terms of combinatorial invariants such as chord diagrams and their interlace graphs, encoding the linking pairing associated to the link-manifold of the plane real singularity.

The idea is to blow up the singularity, and exploit the tree-like structure of the blow up process to reveal the hyperbolic nature of the intersection form. This may be also formulated in terms of the algebraic structure of the ordered local field of real Puiseux series.

Surprisingly, the interlace graphs correspond precisely to the distance-hereditary graphs, which form a ubiquitous and well studied class in informatics. They can be described in terms of their Cunningham split-decomposition: their factorisation into a tree-of-graphs have no prime factors.



An algebraic curve singularity in the real plane and its chord diagram.



Interlace graph of the chord diagram and Cunningham factorisation.

Questions. What is the shape of a large random real algebraic singularity, namely of a large random chord diagram with star-clique interlace graph ? More precisely, describe the universal scaling limits of such chord diagrams and interlace graphs, as it is done for separable permutations and cographs in [BBF+20, BBF+22] ?

In a more algebraic direction, one may also relates this to the work of Favre and Jonsson [FJ04] about the space of all valuations on germs of complex analytic singularities. This set has the structure of a real tree, whose geometry reflects the nature of the valuations. The real analytic singularities yield a subtree: can this geometry .

Finally we wonder if our characterisation of the chord diagrams of analytic singularities can help to find new results about the deformations and morsifications or real algebraic singularities.

Topology and Combinatorics of Singular Algebraic Curves in the Real Plane.

Motivation. The global structure of singular algebraic curves in the real plane remains a wild field of current research. Indeed, even non-singular curves are the subject of Hilbert's 16th problem which remains wide open. As an attempt to understand these, Viro introduced his patchwork method [Vir84] leading to many interesting connections with combinatorics and tropical geometry [Vir08].

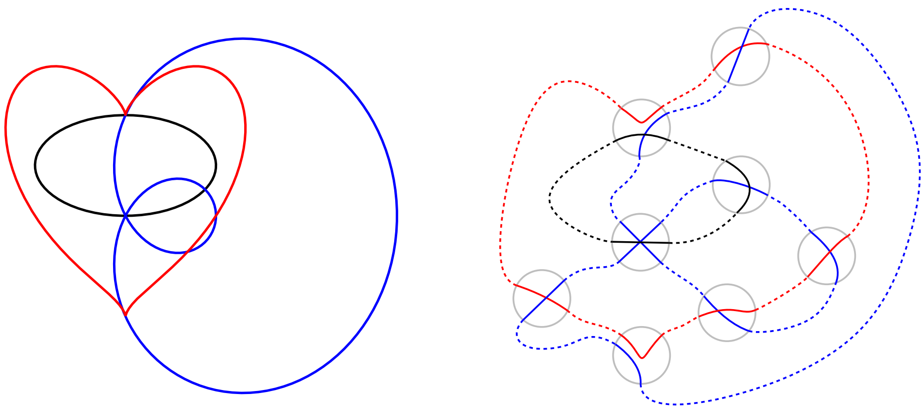
In [KO03, KO04], Kharlamov and Orevkov give an asymptotic bound on the number of isotopy classes of smooth algebraic curves of degree d in the real projective plane by expressions of the form $\exp(Cd^2 + o(d^2))$.

Content. The article [Sim22b] *Topology and combinatorics of real algebraic curves* complements the previous local study by enumerating the topological types of singularities of real analytic or algebraic curves as well as the various associated combinatorial invariants.

We first enumerate the classes of graphs and chord diagrams associated to singularities as described in the previous work. For this we extend to chord diagrams Cunningham's split-decomposition theorem providing a unique factorization of a graph into a graph-labelled-tree. The classes of structures under consideration have generating functions which are amenable to the symbolic and analytic methods: we thus derive formulas, equations or asymptotics for their coefficients.

The we characterise the possible combinatorics for the global structure of a singular curve in a real surface, both in the analytic and algebraic categories. For the latter we apply a general principle in real algebraic geometry: a smooth submanifold of a real algebraic manifold is isotopic to an algebraic model unless it represents an obvious obstruction homology class mod2. This principle finds its roots in the celebrated theorems of Nash-Tognogli [Kol17].

We finally combine the local and global topological characterisations and enumerations to bound the number of topological types of singular real planar algebraic curves based on degree.



A global singular algebraic curve in the plane and its combinatorial map.

Overview and future directions. Together, the articles [GS20, Sim22b] form a self-sufficient narrative blending algebraic geometry, topology, and combinatorics. They lay the groundwork for subsequent investigations aimed at understanding the topology of singularities of real algebraic surfaces (by intersection with a small sphere we recover a singular real algebraic curve in the sphere). This was my initial thesis project, but I soon changed my mind, so it remains an open subject for research.

Arnol'd summarizes in [Arn94, Preface to Lecture 1] a general method to study certain spaces configuration by computing the relative cohomology of their stratification by singular loci...

This is how Vassiliev introduced finite type invariants of knots (embeddings $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3$ up to isotopy): these are quantities which change as a knot passes through a singular state by undergoing a crossing. His invariants also lead to chord diagrams, and Hopf algebras.

Arnol'd initiated an analogous study of plane loops (generic immersions $\mathbb{S}^1 \looparrowright \mathbb{S}^2$ up to isotopy), and the corresponding finite type invariants now change as a loop passes through a singular state (with a cusp, self-tangency or triple point). But he notes:

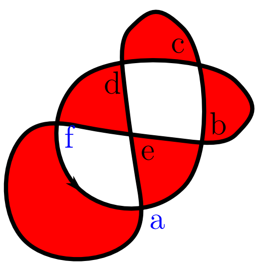
> The combinatorics of plane curves seems to be far more complicated than that of knot theory (which might be considered as a simplified "commutative" version of the combinatorics of plane curves and which is probably embedded in plane curves theory).

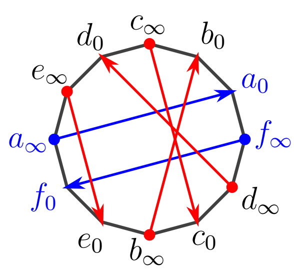
The following works about (multi)loops in the sphere or higher genus surfaces, address problems regarding their classification and the computational complexity of their topological invariants.

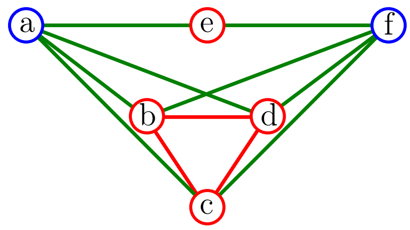
Loops, chord diagrams and graphs: operad factorizations and generating grammars.

Motivation. A plane loop is an immersion of the oriented circle in the sphere which is generic in the sense that it has no singularities except for transverse double-points. One may associate certain combinatorial invariants to its isotopy class, such as a chord diagram and its interlace graph. We are interested in describing which invariants arise from loops, which loops have the same invariants, and how to relate certain operations on loops and on their their invariants.

The combinatorial study of plane loops by their chord diagrams was initiated by Gauss in [Gau07], observing that their interlace graphs must satisfy certain parity conditions, which were not sufficient. Several solutions have been brought to the Gauss realisability problem. In Particular, Rosenstiehl showed that it only depends on the interlace graph of the chord diagram.







A plane loop, its framed chord diagram, and its interlace graph.

Content. The preprint [Sim23b] concerns the topology and combinatorics of loops in surfaces, their chord diagrams, and their interlace graphs.

First we propose independent proofs of Rosentiehl's results characterizing chord diagrams from curves in the plane, and generalize them to higher genus. Our topological proofs provide a better understanding of the links between loops and their combinatorial encodings. The main point is that a loop gives rise to a certain homology group together with a preferred basis and an intersection form, and that its (framed) chord diagram or (bicoloured) interlace graph enables to recover this more or less of this data.

Next, we relate certain operations on loops (connected sums and splicings) to those of its chord diagram (grammatical substitutions in words) and its interlace graph (generalized clique sums). These yield monoidal products and operad compositions which could be recast in terms of Hopf algebras. Thus we extend Cunningham's unique factorization theorems to loops in surfaces, completing in particular Conway's hope of describing knots as compositions of tangles [Con70]. Finally, we describe the factorizations associated with chord diagrams of plane loops, and provide a generating grammar for their interlace graphs. This contrasts Arnold's pessimism about the classification of plane loops.

Outlook. The new results of this work pave the way for numerous avenues of investigation.

Some questions concern enumeration, such as the number planar curves by number of crossings (an open problem of interest for certain theoretical physics models [SZJ04]).

Other questions concern the exploration of easily calculable or approachable invariants on the factorization of a chord diagram or interlace graph, and their asymptotic properties. In particular we believe that one may try to approximate certain properties of a graph's Laplacian spectrum from the coarse shape of its Cunningham factorisation.

We also establish a strong analogy between these structural results and the classification of 3-dimensional manifolds (particularly graphed manifolds arising in the study of complex algebraic curves). Finally, this work is closely tied to the study of real algebraic curve morsifications. These are avenues that I would like to explore further.

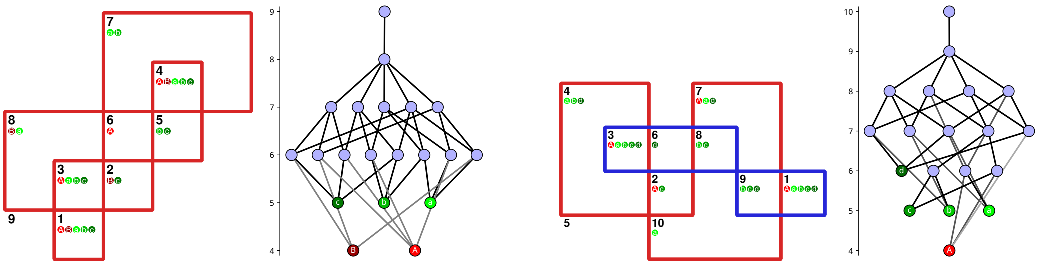
Pinning loops in surfaces: computational complexity of the pinning number.

Motivation. Once we have defined topological invariants, it is often illuminating to investigate the complexity of their computation in restriction to various classes of objects. Such questions have recently received a lot of attention for knots and links [HLP99, AHT02, Lac17, Lac21, BNBNHS24].

Content. In [SS24b] we introduce certain combinatorial invariants associated to multiloops in surfaces (the semi-lattice ideal of pinning sets, the pinning number) and determine their complexity properties. These are motivated by studying the variation in numbers of double-points under homotopies.

Fix a multiloop $\gamma\colon \sqcup_1^s \mathbb{S}^1 \looparrowright \mathbb{S}^2$, denote its number of double-points by $\#\gamma$ and its regions (connected components of its complement) by $R = \pi_0(\mathbb{S}^2 \setminus \gamma)$.

For a set of regions $P \subset R$ we define $\mathrm{si}_P(\gamma)$ as the minimal number of double-points among multiloops which are homotopic to γ in the punctured sphere $\mathbb{S}^2 \setminus P$. We say that $P \subset R$ is pinning γ when $\mathrm{si}_P(\gamma) = \#\gamma$. The set $P = R$ is always pinning but $P = \emptyset$ is not unless γ has no double-points. The pinning sets form a subposet $\mathcal{PI} \subset \mathcal{P}(R)$ absorbing under union: we call it the *pinning ideal*. We define the *pinning number* $\varpi(\gamma)$ of γ as the minimum of cardinal of its pinning sets.



The loop 9_5^1 and the multiloop 10_{16}^2 with their pinning semi-lattices.

First, we implement a polynomial algorithm to compute $\mathrm{si}_P(\gamma)$ adapting a method of Birman–Series which relies on combinatorial group theory. This enables to check if a set P is pinning γ. This enabled us to build an online catalog of pinning data for multiloops and test various conjectures. In particular, we found distinct (multi)loops with isomorphic pinning posets, the non-invariance of various quantities related to the pinning posets (such as the pinning number) under each Reidemeister move and mutations, as well as counter-examples to certain monotonicity properties of pinning numbers and their averages.

Our main result is to show that the decision problem associated to computing the pinning number of a plane loop is `NP-complete`. On the one hand we establish a cardinal-preserving correspondence between pinning sets of a loop and vertex-covers of a certain hypergraph. On the other hand, we reduce the vertex-cover problem for graphs to an instance of the pinning problem for loops. These reductions rely on geometric topology: we improve a characterisation of taut loops by Hass–Scott so that it only involves certain subloops which bound immersed discs, and such immersed discs can be computed using a polynomial algorithm of Blanc and Frish.

Further directions of research. The discovery of these pinning invariants raises many questions: how to they relate to known structures and what new can we learn from them. Moreover, we envisage applications to the study of homeomorphisms of the sphere and their approximations by braids.

3. Character varieties of Fuchsian groups

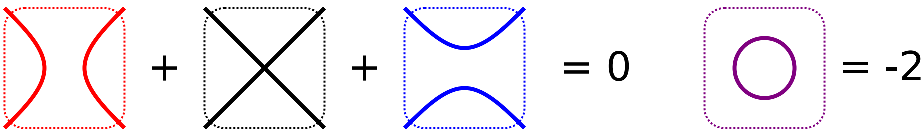
The symmetries that we observe in our universe help us to understand its patterns and formulate approximate laws governing its structure and evolution. Their mathematical study leads to the representation theory of groups. For instance the crystallographic groups describe the tessellations of the round sphere and the euclidean plane [Sha05, Mon87], as well as the spherical harmonics describing the electronic structures of atoms [Wey50, Sha05]. The finite symmetric groups also appear in quantum mechanics [Wey50], combinatorics and statistical physics [Jon16]. The representations of finite groups initiated by Frobenius and Schur, and the representations of $\pi_1(\mathbb{T}^1) = \mathbb{Z}$ which is the subject of Fourier theory, are successfully understood by introducing the algebra of abelian characters, namely the traces of homomorphisms into $\mathrm{GL}_1(\mathbb{C})$.

The study of linear differential equations of order 2 leads to Riemann surfaces, and the symmetries of their solutions lead to tessellations of the hyperbolic plane. Their deformations were investigated by Riemann, Klein and Poincaré: this leads to the space of representations $\mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))$ and motivates the study of the algebra of $\mathrm{SL}_2(\mathbb{C})$-characters of the fundamental group $\pi_1(S)$, which is the subject of the following works.

The character variety and its algebra of characters. Consider a closed orientable surface S of negative Euler characteristic. The $\mathrm{SL}_2(\mathbb{C})$-character variety (of the fundamental group) of S denoted $\mathcal{X}(S)$ is the quotient of its representation variety $\mathrm{Hom}(\pi_1(S); \mathrm{SL}_2(\mathbb{C}))$ by the relation identifying ρ_1 and ρ_2 when $\mathrm{Tr}\,\rho_1(\gamma) = \mathrm{Tr}\,\rho_2(\gamma)$ for all $\gamma \in \pi_1(S)$.

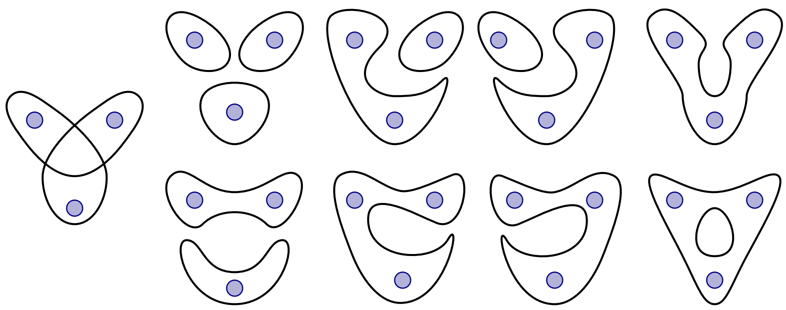
It is an affine algebraic variety defined over $\mathbb{Z}$, whose algebra of complex polynomial functions $\mathbb{C}[\mathcal{X}]$ is generated by the trace function $T_\gamma \colon [\rho] \mapsto \mathrm{Tr}([\rho])$ for $\gamma \in \pi_1(S)$ with relations generated by $T_1 = 2$ and $T_\alpha T_\beta = T_{\alpha\beta} + T_{\alpha\beta^{-1}}$ for all $\alpha, \beta \in \pi_1(S)$.

This algebra of characters $\mathbb{C}[\mathcal{X}]$ turns out to have a very concrete topological incarnation: it is isomorphic to the so called Kaufman skein-algebra of the thickened surface $S \times [-1, 1]$. In particular, it admits a linear basis indexed by the collection MS of all simple multicurves in S.



The trace relations yield the local skein relations.

Hence we may decompose the trace function of any $\gamma \in \pi_1(S)$ as $T_\gamma = \sum_{\kappa \in \mathrm{MS}} c_\kappa(\gamma) T_\kappa$. for a unique family of "Fourier" coefficients $c_\kappa(\gamma) \in \mathbb{Z}$ indexed by $\mu \in \mathrm{MS}$. This generalises to every $f \in \mathbb{C}[\mathcal{X}(S)]$.



Decomposing a (multi)curve into the basis MS of simple multicurves.

Automorphisms of Character Varieties.

Motivation. The character variety $\mathcal{X}(\mathcal{S})$ contains a Zariski-dense copy of the Teichmüller space $\mathcal{T}(S)$. Many results [AS16] show that the automorphism group of certain structures associated with the Teichmüller space $\mathcal{T}(S)$ is essentially reduced to the modular group $\mathrm{Mod}(S) = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$. Juan Souto asked what is the algebraic automorphism group $\mathrm{Aut}(\mathcal{X}(S))$ of this character variety.

Content. In the article [MS21] we answer the question by Juan Souto, and describe the automorphism group $\operatorname{Aut}\mathcal{X}(S)$ as an extension of the modular group $\operatorname{Mod}(S)$ by the finite group $H^1(S;\mathbb{Z}/2)$.

The main idea is to consider the action of $\operatorname{Mod}(S)$ on the Riemann-Zariski compactification of $\mathcal{X}(S)$ by the set of valuations centred at infinity. We define the subspace of *simple valuations* in terms of the algebra of characters: they are designed to behave monomially with respect to the linear basis of multicurves, and to detect the topology of intersections between simple curves in S. Indeed, we identify the space of simple valuations with the space measured laminations (forming Thurston's compactification of Teichmüller space). Then we show that the simple valuations that are discrete correspond to the simple multicurves, and that the topological intersection number between simple multicurves can be recovered from the mutual position of the corresponding valuation rings.

We show that this space of simple valuations is invariant under the action of $\operatorname{Aut}(\mathcal{X})$ by providing a projection from the space of all valuations to the space of simple valuations. This projection goes through the correspondence between valuations on $\mathcal{X}(S)$ and certain actions of $\pi_1(S)$ on real trees, using deep structural results by Morgan-Otal, Morgan-Shalen and Skora [MO93, MS84b, MS84a]. We deduce that $\operatorname{Aut}(\mathcal{X}(S))$ acts on the curve-complex of S, and invoke the theorem of Ivanov [Iva97] relating its automorphism group to the mapping class group.

Overview. This article is short, but the methods combine algebraic geometry (valuations), topology (measured laminations), and group theory (actions on real trees), relying on difficult results in each of these fields. It also lifts the question of whether the projection of the space of all valuations onto the space of simple valuations is a homotopy retraction.

Valuations on the character variety: Newton polytopes and residual Poisson bracket.

Motivation. The character variety $\mathcal{X}(S)$ is endowed with a symplectic form [Gol84] hence the algebra of characters $\mathbb{C}[\mathcal{X}(S)]$ has a Poisson bracket [Gol86].

The multiplication and Poisson bracket on $\mathbb{C}[\mathcal{X}(S)]$ are encoded by their structural coefficients $m_\kappa(\mu,\nu)\in\mathbb{Z}$ and $w_\kappa(\mu,\nu)\in\mathbb{Z}$ in the basis of simple multicurves, given for $\mu,\nu,\kappa\in\mathrm{MS}$ by:

$$T_\mu T_\nu = \textstyle\sum_{\kappa\in\mathrm{MS}} m_\kappa(\mu,\nu)T_\kappa \qquad\qquad \{T_\mu,T_\nu\} = \textstyle\sum_{\kappa\in\mathrm{MS}} w_\kappa(\mu,\nu)T_\kappa$$

These structure coefficients are of great interest and they should carry combinatorial information providing a bridge between the algebraic geometry of the character variety and the topology of loops in the surface. How can we evaluate them ? What are their properties ?

Content. In [MS24], we first analyse in more detail the compactification of the character variety by *simple valuations* introduced in the previous article, and how its correspondences with the space of measured laminations space ML, as well as of certain actions of the group $\pi_1(S)$ on real trees. Then we use it to define the Newton polytope of a function on the character variety and evaluate . This leads to structural results concerning the structure coefficients for the multiplication.

More precisely, we establish equivalences among algebraic properties of valuations, topological properties of (maximal) measured laminations, and combinatorial properties of (trivalent) real trees, describing the highest-dimensional stratum of smooth points of ML. We define the tangent space at such a valuation, and show how the Poisson bracket described by Goldman on the character variety induces a symplectic structure on this valuation model. We identify this symplectic space with the earlier constructions of Thurston and Bonahon.

Then, we introduce the notion of the Newton polytope of an algebraic function on the character variety: it is a subset in the dual of the piecewize linear space of simple valuations (for the sphere with 3 punctures we recover monomial space dual to the space of valuations on $\mathbb{C}^3$). We prove, among other things, that trace functions have unit coefficients at the extreme points of their Newton polytope. We also interpret the coefficients of the Poisson bracket for two functions on the character variety at the extreme points of their Newton polytope, both algebraically (residual value of the usual Poisson bracket) and combinatorially (count of signs given by cyclic orders). We deduce in particular that these coefficients also recover the topological geometric intersection number between multiloops.

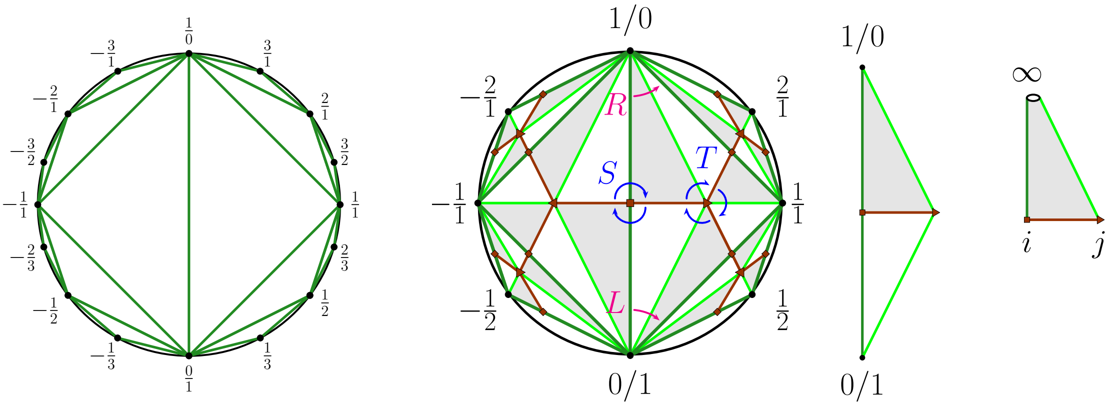
Overview. The new valuative model for the symplectic space ML paves the way for a finer algebraic study of its stratification. The work on Newton polytopes provides a step towards the study of the structure coefficients of the character algebra for multiplication and the Poisson bracket.

Background. The modular group is arguably one of the most fundamental and ubiquitous objects in mathematics. Let us recall its connection with one of the most ancient notion of mathematics: Euclidean continued fraction expansions of real numbers.

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts by conformal transformations of the hyperbolic plane $\mathbb{HP} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ with boundary $\partial\mathbb{HP} = \mathbb{RP}^1$, as the stabiliser of its ideal triangulation $\triangle$ with vertices $\mathbb{QP}^1$ and edges $\left(\frac{a}{c}, \frac{b}{d}\right)$ such that $ad - bc = 1$. Let

$$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \quad T = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \qquad L = T^{-1}S = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \quad R = TS^{-1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right).$$

The modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$ has genus zero, a cusp descending from the fixed point $\infty \in \mathbb{QP}^1 \subset \partial\mathbb{HP}$ of R, as well as two conical singularities of order 2 and 3 descending from the fixed points $i, j \in \mathbb{HP}$ of S and T. Thus $\mathrm{PSL}_2(\mathbb{Z}) = \pi_1(\mathbb{M})$ is the free amalgam of its subgroups $\mathbb{Z}/2$ and $\mathbb{Z}/3$ generated by S and T.

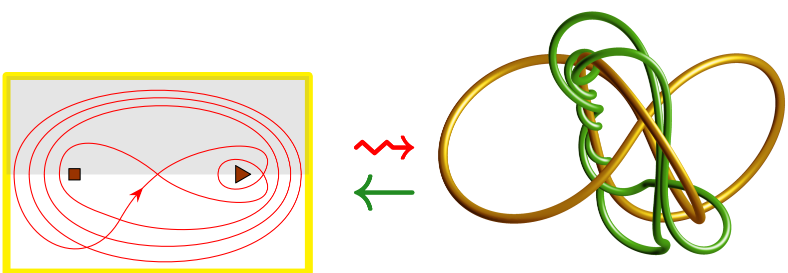


The elements of $\mathrm{PSL}_2(\mathbb{Z})$ stabilising $[0, +\infty] \subset \mathbb{RP}^1$ form the euclidean monoid $\mathrm{PSL}_2(\mathbb{N})$, which is freely generated by the parabolic matrices R and L. The subset of sequences in $\{L, R\}^{\mathbb{N}}$ which do not end with an infinite string of L are in bijective correspondence with the continued fraction expansions of positive numbers in $(0, +\infty] \subset \mathbb{RP}^1$.

A geodesic $(\alpha^-, \alpha^+) \subset \mathbb{HP}$ whose endpoints satisfy $-1 < \alpha^- < 0$ and $1 < \alpha^+ < \infty$ intersects $\triangle$ along a sequence of triangles whose $\{L, R\}^{\mathbb{Z}}$-encoding is obtained from the continued fraction expansions of α^+ and $-1/\alpha^-$ by concatenating the transpose of the latter with the former. It projects to a geodesic in $\mathbb{M}$, which is closed if and only if the $\{L, R\}$-sequence is periodic if and only if (α^-, α^+) are Galois-conjugate quadratic numbers.

Linking Numbers of Modular Knots.

Motivation. The modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the hyperbolic plane with quotient the modular surface $\mathbb{M}$, whose unit tangent bundle $\mathbb{U}$ is a 3-manifold homeomorphic to the complement of the trefoil knot in the sphere. The hyperbolic conjugacy classes of $\mathrm{PSL}_2(\mathbb{Z})$ correspond to the oriented closed geodesics of $\mathbb{M}$. Their lifts in $\mathbb{U}$ define the modular knots.



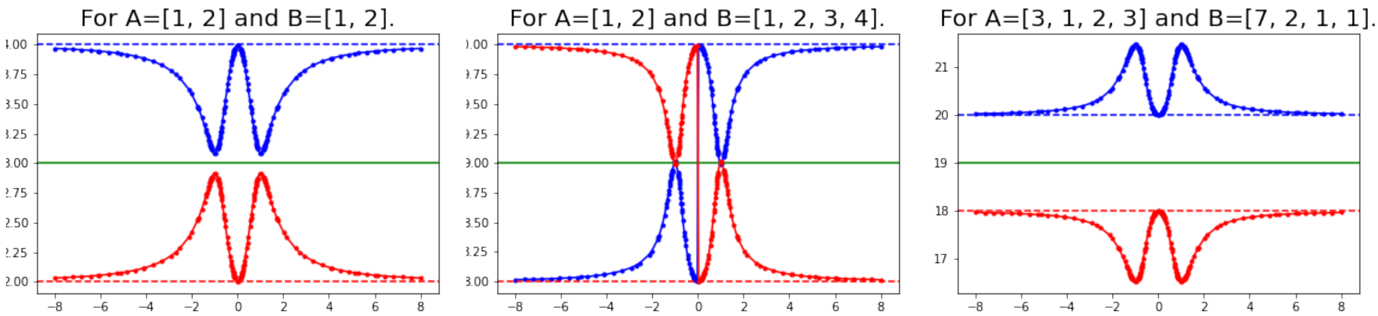
The linking between a modular knot and the trefoil knot is well understood: Étienne Ghys showed in [Ghy07] that it is given by the Rademacher invariant of the associated conjugacy class. The Rademacher function is a quasi-character of $\mathrm{PSL}_2(\mathbb{Z})$ that he recognized with Jean Barge in [BG92] as half the primitive of the bounded Euler class, illuminating Michael Atiyah's work [Ati87] on the logarithm of the Dedekind eta function and the ubiquity of the Rademacher function in various fields of mathematics.

Content. The article [Sim24], summarizes and extends the results obtained in Chapters 4 and 5 of my thesis. It addresses the last question in [Ghy07], about the linking between two modular knots, We propose several formulas of arithmetic, combinatorial, or representation-theoretic flavour.

The complete hyperbolic metrics on the orbifold $\mathbb{M}$ correspond to the discrete faithful representations $\rho\colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ modulo $\mathrm{PSL}_2(\mathbb{R})$-conjugacy at the target. They form a real algebraic variety parametrized by $q \in \mathbb{R}^*$, by fixing S and conjugating T by $\exp \frac{1}{2}\log(q)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The hyperbolic conjugacy classes of $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ correspond to geodesics in the quotient $\mathbb{M}_q = \rho_q(\mathrm{PSL}_2(\mathbb{Z}))\backslash\mathbb{HP}$ which do not only surround the cusp, and their intersection points persist by deformation: only their angles $\theta_q \in (-\pi, \pi)$ vary so we may define the sums over these intersection points:

$$\mathrm{Lk}_q(A,B) = \tfrac{1}{2}\sum \left(\cos\tfrac{\theta_q}{2}\right)^2 \qquad \text{et} \qquad \mathrm{Cos}_q(A,B) = \tfrac{1}{2}\sum\left(\cos\theta_q\right)$$

We show that as $q \to +\infty$ the limit $\mathrm{Lk}_q(A,B)$ recovers the linking number of the modular knots. Thus Lk_q interpolates between the arithmetic at $q = 1$ and the topology at $q = +\infty$. We also have that the sum $I(A,B) = \frac{1}{2}(\mathrm{Lk}_q(A,B) + \mathrm{Lk}_q(A,B))$ is constant equal to the geometric intersection number of modular geodesics, hence $\mathrm{Cos}_q(A,B) \to \frac{1}{2}\left(\mathrm{lk}(A,B) - \mathrm{lk}(A^{-1},B)\right)$.



The graphs of $q \mapsto 2\,\mathrm{Lk}_q(A,B)$ and $q \mapsto 2\,\mathrm{Lk}_q(A,B^{-1})$ along with their average $\frac{1}{2}I(A,B)$ for some pairs $A, B \in \mathrm{PSL}_2(\mathbb{N})$. The legend $A = [a_0, a_1, \dots]$ means $A = R^{a_0}L^{n_1}\dots$ has attractive fixed point $\alpha \in \mathbb{RP}^1$ with continued fraction $\alpha = a_0 + \frac{1}{a_1 + \dots}$.

Next, we use linking numbers with modular knots to provide a Shauder basis for the Banach space of quasi-characters of $\mathrm{PSL}_2(\mathbb{Z})$, which is isomorphic to $H_b^2(\mathrm{PSL}_2(\mathbb{R}), \mathbb{R})$. More precisely, the functions $\mathrm{Cos}_A\colon \beta \mapsto \frac{1}{2}\left(\mathrm{lk}(A,B) - \mathrm{lk}(A^{-1},B)\right)$ indexed by $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$ generate $H_b^2(\mathrm{PSL}_2(\mathbb{R}))$, and their only relations are given by $C_{\gamma\alpha\gamma^{-1}}$ for all $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ and $C_{\alpha^n} = n \cdot C_\alpha$ for all $n \in \mathbb{Z}$. The main philosophy emerging from this result is a duality between quasi-characters and primitive conjugacy classes of infinite order modulo inversion. This is reminiscent of the theory of characters for finite groups.

While our quasi-characters $C_A(B)$ are given by the antisymetric part of linking numbers of modular knots, while the geometric intersection number of modular geodesics is recovered by the symmetric part. This supports the idea that quasi-characters provide a measure of asymptotic achirality in a certain direction (see also [Gri95, Corollary 3.12]).

Impact and future research. The results in this paper are unique in their genre. I believe that they reveal or conceal deep insights into and the representation theory of the modular group as well as the arithmetic of elliptic curves and real quadratic fields.

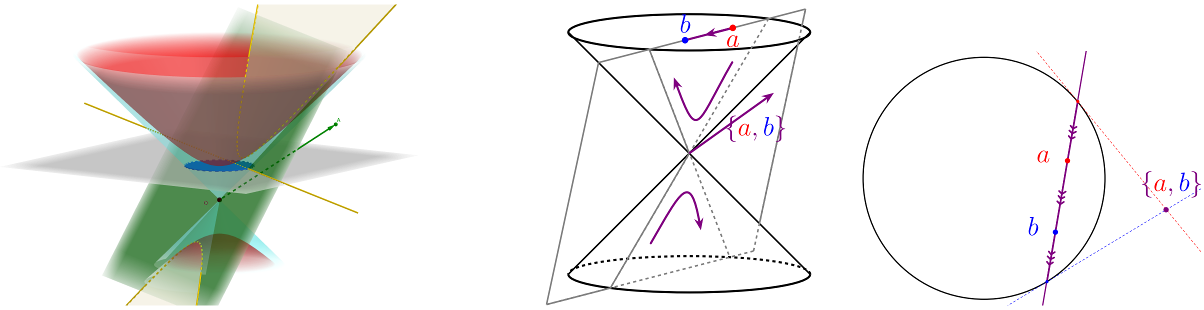
The limit formula motivated part of the arithmetic studies of the next article about conjugacy classes in $\mathrm{PSL}_2(\mathbb{K})$ and cross-ratios. It is a source of inspiration for various generalizations to Fuchsian groups, including arithmetic analogues. In particular, i started investigating other natural constructions of periods, and observed that some present special symmetries and vanishing properties, which can be explained in terms of the symmetries of the modular surface and its congruence covers.

The linking-basis for the quasi-characters of the modular group deserves attention, and I am working on its generalisations to other mapping class groups. There are very few groups for which one has a non-trivial basis of quasi-characters [Gri95], and this is the only one for which we have a description in terms of periods, which can be interpreted as a Fourier theory for quasi-characters.

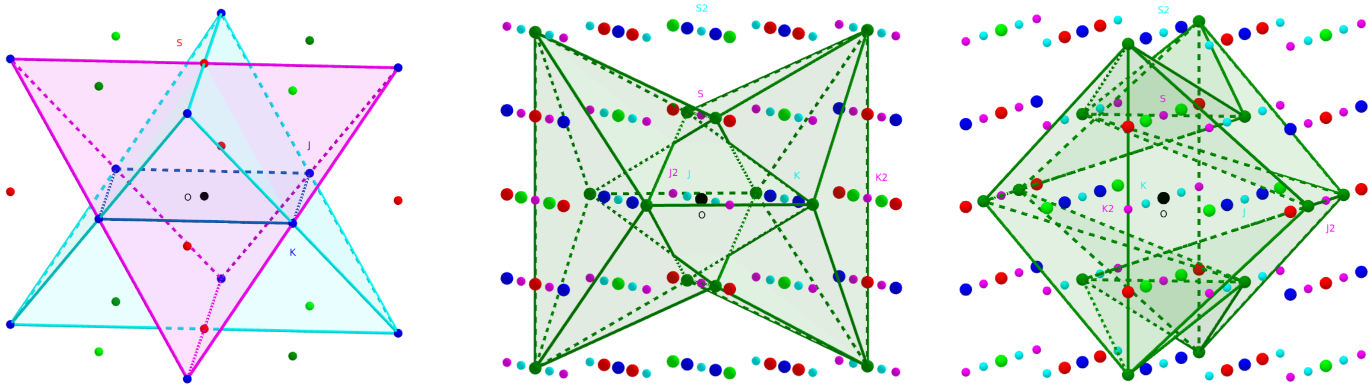
Finally, one of the combinatorial formulae derived to prove the quasi-morphism could be used to compute distribution properties for linking numbers of modular knots.

Conjugacy Classes of $\mathrm{PSL}_2(\mathbb{K})$**.** The article [Sim23a] summarizes the results obtained and proves the open conjectures in Chapter 1 of my thesis.

We begin by describing, over a field $\mathbb{K}$ of characteristic different from 2, the orbits of the adjoint action of the Lie group $\mathrm{PGL}_2(\mathbb{K})$ on its Lie algebra $\mathfrak{sl}_2(\mathbb{K})$. This leads to the solution of generalized Pell-Fermat equations $x^2 - \Delta y^2 = \chi$.



The synthetic approach leads in passing to a variant of Ptoleme's for hyperbolic ideal quadrilaterals. More importantly, it allows us to change the base field, and we illustrate it for fields with 3 and 5 elements, in connection with the geometry of the tetrahedron and icosahedron.



$\mathfrak{sl}_2(\mathbb{F}_3)$: the isotropic cone $\mathbb{X}$, the orbit $\{\det = +1\}$ and the partition of $\{\det = -1\}$ into 3 orbits: the vertices of each tetrahedron and the intersections of their edges.
$\mathfrak{sl}_2(\mathbb{F}_5)$: the orbits $\det = -2, -1, 1, 2$ and and icosahedral orbit in $\mathbb{X} \setminus \{0\}$.

We then apply these results to partition the set of $\mathrm{PSL}_2(\mathbb{Z})$ classes of integer binary quadratic forms into $\mathrm{PSL}_2(\mathbb{Q})$-classes, and establish their connection with the genera which are (see [Cox13, Hat22]).

Finally, we provide a geometric interpretation of $\mathrm{PSL}_2(\mathbb{Q})$-equivalence of closed geodesics in the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$, in terms of orthogeodesic lengths and intersection angles. In summary, this provides a synthetic and geometric interpretation for the multiplication of genera of quadratic forms: they are given by multiplying cross-ratios.

Finally, these geometric quantities are related to linking numbers between modular knots.

Applications. The characterisation orbits for the adjoint action of $\mathrm{PSL}_2(\mathbb{K})$ on $\mathfrak{sl}_2(\mathbb{K})$ may seem familiar but they are not covered in such generality or detail by the existing literature. Moreover, this geometry of the Minkowski lattice $(\mathfrak{sl}_2(\mathbb{Z}), \det)$ provides a neat framework to study the distribution properties of cross-ratios and linking numbers of modular knots, which are currently of great interest.

The results on the class group are classical, but our geometric approach has independent interest, as it leads to new geometric interpretations of Gauss composition, which could be generalized to more general global fields such as function fields.

Every positive real number γ admits a Euclidean continued fraction expansion:

$$\lfloor c_0, c_1, \dots \rfloor = c_0 + \tfrac{1}{c_1 + \dots} \quad \text{with} \quad c_j \in \mathbb{N} \quad \text{and} \quad \forall j > 0,\ c_j > 0.$$

Such an expansion is infinite if and only if γ is irrational, in which case it is unique. According to a longstanding conjecture [Sha92, Section 4], if (c_k) is bounded then γ must be either rational, quadratic or transcendental. Thus only rational and quadratic numbers are simple both from the symbolic viewpoint of continued fraction expansions and the arithmetic viewpoint of algebraic number theory.

The group $\mathrm{PGL}_2(\mathbb{Z})$ acting faithfully transitively on $\mathbb{RP}^1$ is generated by the inversion $J\colon \gamma \mapsto 1/\gamma$ and the translation $R\colon \gamma \mapsto \gamma + 1$. These two transformations generate the submonoid $\mathrm{PGL}_2(\mathbb{N})$ preserving and acting freely transitively on $(0, \infty)$. Two numbers $\alpha, \beta \in (0, \infty)$ belong to the same $\mathrm{PGL}_2(\mathbb{Z})$-orbit when they have equal tails $\alpha_i = \lfloor a_i, \dots \rfloor$ and $\beta_j = \lfloor b_i, \dots \rfloor$. Hence the action of $\mathrm{PGL}_2(\mathbb{N}) \subset \mathrm{PGL}_2(\mathbb{Z})$ on $(0, \infty) \subset \mathbb{RP}^1$ is well understood, as it is orbit equivalent to the action of the monoid $\mathbb{N}$ on $(0, 1)$ generated by the Gauss-map $\gamma \mapsto \gamma_1$ which shifts the expansion $(c_k) \mapsto (c_{k+1})$.

The multiplicative action of $\mathbb{N}^* \subset \mathbb{Q}^*$ on $(0, \infty) \subset \mathbb{R}$ is much more cryptic from the viewpoint of continued fractions. Recent results [AS18, Aka20] about multiples of quadratic irrationals indicate that the actions of $\mathbb{N}^*$ by multiplication and by the Gauss-map are asymptotically decorrelated.

The actions of $\mathrm{PGL}_2(\mathbb{Z})$ and $\mathbb{N}^*$ on $\mathbb{RP}^1$ generate the action of $\mathrm{PGL}_2(\mathbb{Q})$. This action preserves (the projective line over) each real field $\mathbb{K} \subset \mathbb{R}$. In particular the rationals consist of a single orbit, and each real quadratic field partitions in two orbits. McMullen shows [McM09] that for every square-free $\delta \in \mathbb{N}_{>1}$, there is a bound $M_\delta \in \mathbb{N}_{>1}$ such that $\mathbb{Q}(\sqrt{\delta})$ contains infinitely many irrationals $\gamma = \lfloor c_0, \dots \rfloor$ with $\max(c_k) \leq M_\delta$ (conjecturally, one may always choose $M_\delta = 2$).

Isogenies of low complexity systems: Sturm, Denjoy and Interval exchanges.

Motivation. The motivation of [SS24a] is to understand when $\alpha, \beta \in \mathbb{RP}^1$ belong to the same orbit under the action of $\mathrm{PGL}_2(\mathbb{Q})$ in terms of their Euclidean continued fraction expansions.

The diophantine approximation properties of $\gamma \in (0, 1)$ are encoded both by its sequence of partial quotients c_j for $j \in \mathbb{N}^*$ and by the sequence of fractional parts of $q\gamma$ for $q \in \mathbb{N}^*$. This leads to the dynamical approach: the interval $[0, 1)$ is endowed with the map $t \mapsto t + \gamma \bmod 1$ which exchanges the intervals $I_0 = [0, 1 - \gamma)$ and $I_\infty = [1 - \gamma, 1)$ by translations. The fractional parts of $q\gamma$ are the orbit of 0, and recording the sequence of intervals $x_q \in \{0, \infty\}$ containing them yields the *Sturmian sequence* associated to γ. The compact space $\{0, 1\}^{\mathbb{Z}}$ is endowed with the shift homeomorphism $\sigma\colon (u_n) \mapsto (u_{n+1})$, and the orbit of our Sturmian sequence x has a closure denoted $X_\gamma \subset \{0, \infty\}^{\mathbb{Z}}$. Hence the dynamical system given by the Sturmian subshift $\sigma_\gamma\colon X_\gamma \to X_\gamma$, and we are led to study the projective action of $\mathrm{PGL}_2(\mathbb{Q})$ on $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ in terms of Sturmian subshifts $X_\gamma \subset (\{0, \infty\}^{\mathbb{Z}}, \sigma)$.

Content. A subshift (X, σ) on an alphabet $\mathcal{A}$ is a compact and shift-invariant subspace of $(\mathcal{A}^{\mathbb{Z}}, \sigma)$. Its suspension is the mapping torus $(X \times \mathbb{R}) \bmod \mathbb{Z}$, where the diagonal action of $n \in \mathbb{Z}$ is by $(x, s) \to (\sigma^n(x), s - n)$ with the continuous flow $(x, s) \mapsto (x, s + s')$. Two subshifts are called *flow-equivalent* when their suspensions are related by a homeomorphism (preserving flow lines). There is a correspondence[Put88, Fok91, BW00] between $\mathrm{PGL}_2(\mathbb{Z})$-equivalence of real irrationals and flow-equivalence of Sturmian subshifts. We generalise this to subshifts encoding interval exchanges.

We show that fo $m \in \mathbb{Z}$ the $(X_{m\gamma}, \sigma_{m\gamma})$ is obtained from the m-power $(X_\gamma, \sigma_\gamma^m)$ by identifying m special pairs of asymptotic orbits. Thus we are led to consider *virtual equivalence* given by taking powers of a subshift, and to introduce the notion of *infinitesimal 2-asymptotic equivalence* between subshifts. These lead us to extend the class of Sturmian systems (encoding rotations of a torus) to the class of Denjoy systems (encoding more general homeomorphisms of the torus). We thus give a complete description of $\mathrm{PSL}_2(\mathbb{Q})$-equivalence of real numbers in terms of Sturmian subshifts and extend it to Denjoy systems. We also describe infinitesimal 2-asymptotic factors of interval exchanges.

Our methods heavily rely on the study of cohomological invariants and states [Put89, GPS01].

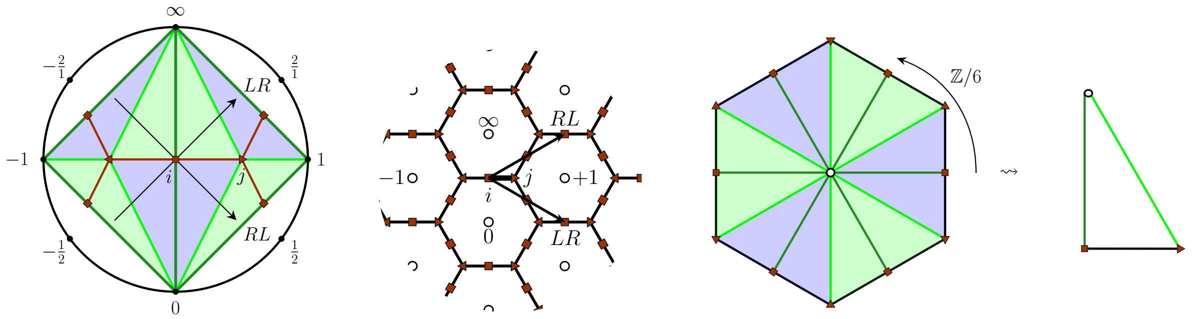
Open problems. The powers of interval exchanges are more subtle to deal with, and we propose conjectures in the case of self-similar interval exchanges (generalising quadratic irrationals) in relation to the commensurator of the Veech groups in $\mathrm{PGL}_2(\mathbb{R})$. We also propose conjectures concerning the equidistribution for such powers of self-similar interval exchange subshifts with respect to the Masur-Veech measure on Teichmüller space (generalizing what is expected from quadratic irrationals).

Arithmetic and topology of the Hexponential map.

Motivation. The modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the upper-half plane $\mathbb{HP}$ with quotient the modular orbifold $\mathbb{M}$, uniformized by the modular function $\mathrm{j}\colon \mathbb{HP} \to \mathbb{C}$. The values of j have deep arithmetic properties: $\mathrm{j}(\tau)$ is algebraic if and only if τ is complex quadratic (this relies on transcendental function theory [Sch49]), in which case it generates the maximal unramified abelian extension of $\mathbb{Q}(\tau)$, whose Galois group is the ideal class group of $\mathbb{Z}[\tau]$ (this leads to class field theory and complex multiplication[Cox13]). Attempts to generalize this to real quadratic numbers (Kronecker's Jugendtraum), have lead to computing certain special values related to the limits of j and its integrals along geodesics $(\alpha^-, \alpha^+) \subset \mathbb{HP}$. When $\alpha^\pm \in \mathbb{R}$ are quadratic conjugates, the geodesic projects in $\mathbb{M}$ to a closed geodesic, along which the integral of j is called a period (see [KZ01, DIT11]). One may ask how this period relates to the periodic continued fraction expansion of $\alpha^\pm$?

The derived subgroup $\mathrm{PSL}_2(\mathbb{Z})' \subset \mathrm{PSL}_2(\mathbb{Z})$ corresponds to the $\mathbb{Z}/6$-Galois cover of $\mathbb{M}$ by a punctured torus $\mathbb{M}'$, whose abelian differential lifts on $\mathbb{HP}$ to a weight 2 modular form $C\eta^4\colon \mathbb{HP} \to \mathbb{C}$ where $C \in \mathbb{R}$ and η is the Dedekind eta function. The geodesics in $\mathbb{M}'$ which are *simple* correspond the $\mathrm{PSL}_2(\mathbb{Z})'$-orbits of pairs $(\alpha^-, \alpha^+) \in (-1, 0) \times (1, \infty)$ whose associated continued fraction expansion yields a *Sturmian* sequence over $\{1, 2\}$. In particular, the *closed simple* geodesics in $\mathbb{M}'$ correspond to the *periodic Sturmian* sequences: these numbers yield *quadratic conjugate Markov* pairs. Thus we wonder whether limits and integral of η along lifts $(\alpha^-, \alpha^+) \subset \mathbb{HP}$ of simple geodesics in $\mathbb{M}'$ relate to the arithmetic properties of $\alpha^\pm$ and to the symbolic dynamics of its continued fraction expansion.

Content. We show that the second derived subgroup $\mathrm{PSL}_2(\mathbb{Z})''$ corresponds to a $\mathbb{Z}^2 \rtimes \mathbb{Z}/6$-Galois cover of $\mathbb{M}$ by a hyperbolic surface $\mathbb{M}''$ which is conformal to the complex plane punctured along a hexagonal lattice $\mathbb{C} \setminus \omega_0\mathbb{Z}[j]$ where $\omega_0 \in i\mathbb{R}$. This hexpunctured plane $\mathbb{M}''$ is uniformized by the hexponential map $\mathrm{hexp}\colon \mathbb{HP} \to \mathbb{C} \setminus (\omega_0\mathbb{Z}[j])$, which is a primitive of $C\eta^4$, namely $\mathrm{hexp}(\tau) = \int_\infty^\tau C\eta^4(z)dz$.



$$\mathbb{HP} \xrightarrow{\Gamma'' \simeq \mathrm{F}_\infty} \mathbb{M}'' \xrightarrow{\Gamma''/\Gamma' \simeq \mathbb{Z}^2} \mathbb{M}' \xrightarrow{\Gamma/\Gamma' \simeq \mathbb{Z}/6} \mathbb{M}.$$

We then describe the values of the *cusp-compactification* $\partial\,\mathrm{hexp}\colon \mathbb{QP}^1 \to \omega_0\mathbb{Z}[j]$: these are precisely the periods of $C\eta^4$ along simple closed geodesics in $\mathbb{M}''$, and we explicit them in terms of the continued fraction expansions of the associated Markov pairs (α^-, α^+).

When $\tau \in \mathbb{HP}$ approaches an irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the value $\mathrm{hexp}(\tau)$ diverges, but we show that the phase $\Im \log \mathrm{hexp}(\tau) \in \mathbb{S}^1$ converges when α belongs to a subset of numbers $\mathcal{R} \subset \mathbb{R}$ containing the Sturmian hence Markov numbers $\mathscr{S} \supset \mathscr{M}$. This defines the *radial-compactification* $\mathrm{Shexp}\colon \mathscr{R} \to \mathbb{S}^1$, and we construct a *simple section* $\mathrm{InSh}\colon \mathbb{S}^1 \to \mathscr{S} \bmod \mathrm{PSL}_2(\mathbb{Z})'$. We deduce from [ADQZ01] that the Sturmian values of InSh are which are not Markov quadratic irrationals are transcendental. This is analogous to the transcendent-quadratic dichotomy for the evaluations of $\mathfrak{j}$.

Finally we express hexp in terms of the $\Gamma(2)$-modular function $\lambda\colon \mathbb{HP} \to \mathbb{C} \setminus \{0, 1\}$ as a ratio of hypergeometric series to deduce a continued fraction expansion, and discuss its monodromy.

Further directions. This work motivated the following conjecture which i am currently pursuing: if a hyperbolic geodesic $(\alpha^-, \alpha^+) \subset \mathbb{HP}$ projects in some arithmetic surface $\Gamma\backslash\mathbb{HP}$ to a geodesic with a finite number of self-intersections, then $\alpha^\pm$ either lie in a quadratic extension of the invariant trace field $k\Gamma = \mathbb{Q}(\{\mathrm{disc}(\gamma) \mid \gamma \in \Gamma\})$, or else they are transcendental.

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