Arithmetic and Topology of Modular Knots

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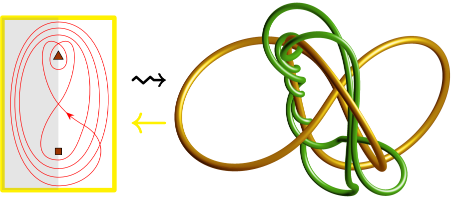

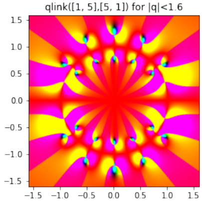

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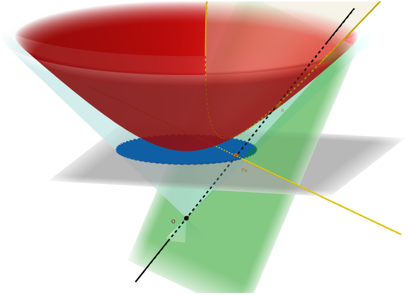
The isometry group $\mathsf{PSL}_2(\mathbb{R})$ of the hyperbolic plane $\mathbb{PH}$

$$\mathsf{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \substack{a,b,c,d\in\mathbb{R} \\ ad-bc=1} \right\} \qquad \mathfrak{sl}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \substack{a,b,c,d\in\mathbb{R} \\ a+d=0} \right\}$$

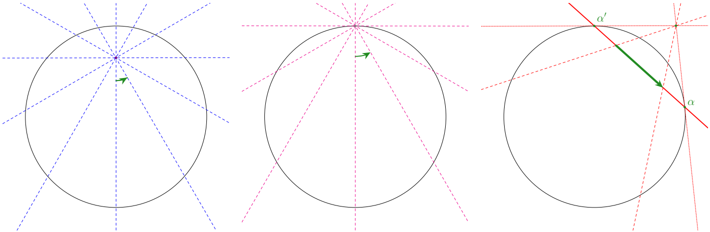
$$\mathsf{PSL}_2(\mathbb{R}) = \mathsf{SL}_2(\mathbb{R})/\{\pm \mathbb{1}\} \qquad \mathbb{H} = \{\mathfrak{a} \in \mathfrak{sl}_2(\mathbb{R}) \mid \det(\mathfrak{a}) = 1\}$$



Projectivization of the two-sheeted hyperboloid $\mathbb{H} \to \mathbb{PH}$

The isometry group $\mathrm{PSL}_2(\mathbb{R})$ of the hyperbolic plane $\mathbb{PH}$

$$A \in \mathrm{PSL}_2(\mathbb{R}) \quad \curvearrowright \quad \mathfrak{a} \in \mathbb{PH} \quad : \quad A \cdot \mathfrak{a} = A\mathfrak{a}A^{-1}$$

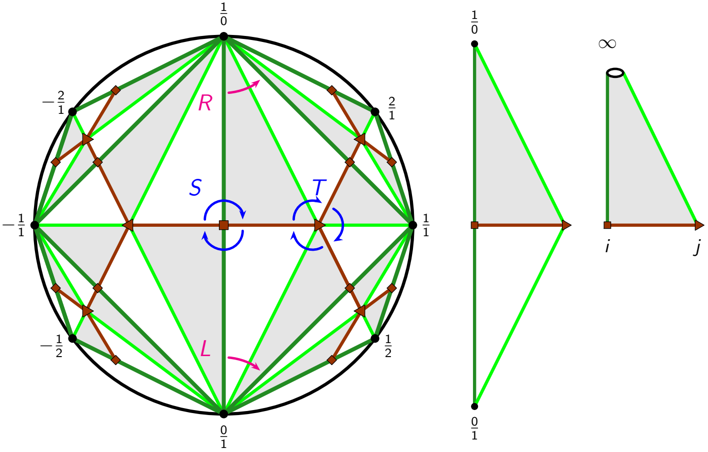


Isometries : elliptic, parabolic, hyperbolic

$$\mathrm{disc}(A) = (\mathrm{Tr}\,A)^2 - 4 \quad \in \quad [-4, 0\,[\ \sqcup\ \{0\}\ \sqcup\ ]\,0, +\infty[$$

Action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{PH}$

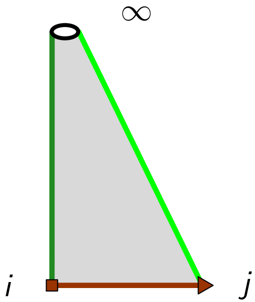
$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

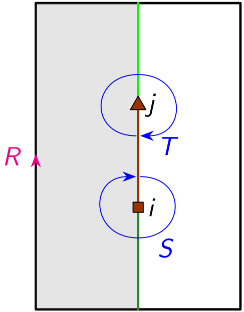


Tiling $\mathbb{PH}$ under the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$

The modular orbifold $\mathbb{M} = \mathsf{PSL}_2(\mathbb{Z})\backslash\mathbb{PH}$

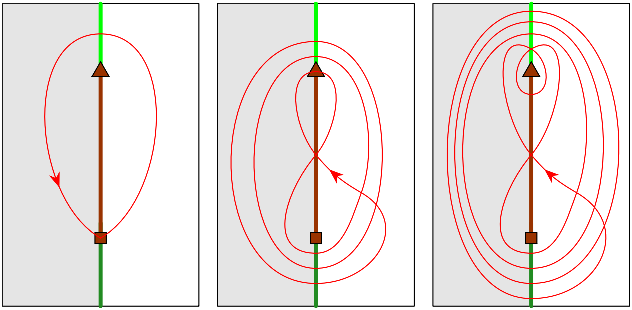
$$\pi_1(\mathbb{M}) = \mathsf{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3 \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$



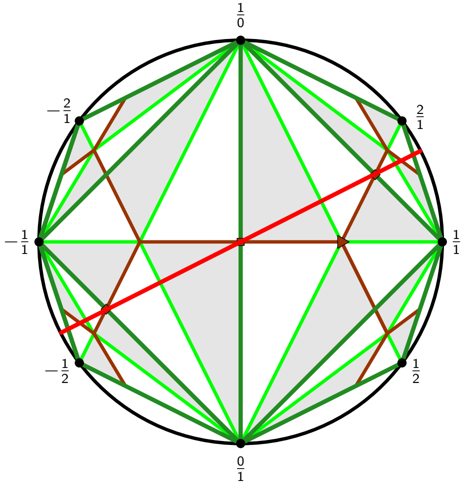


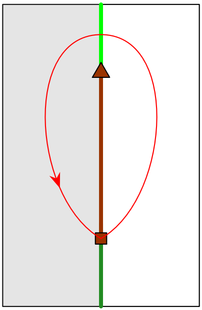
Homotopy classes of loops in the modular orbifold

Free homotopy classes of oriented loops in $\mathbb{M}$	Conjugacy classes in $\pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z})$
Around conic singularity i or j	Elliptic : S or $T^{\pm 1}$
Suround n times the cusp ∞	Parabolic : R^n, $n \in \mathbb{Z}$
$\exists!$ geodesic representative γ_A of length λ_A	Hyperbolic : $\mathrm{disc}(A) = \left(2\sinh\frac{\lambda_A}{2}\right)^2$



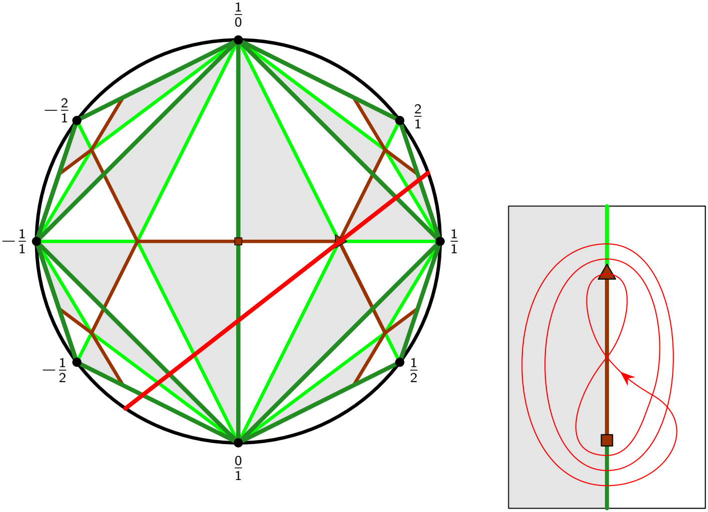
Modular geodesics : projections of the hyperbolic axes





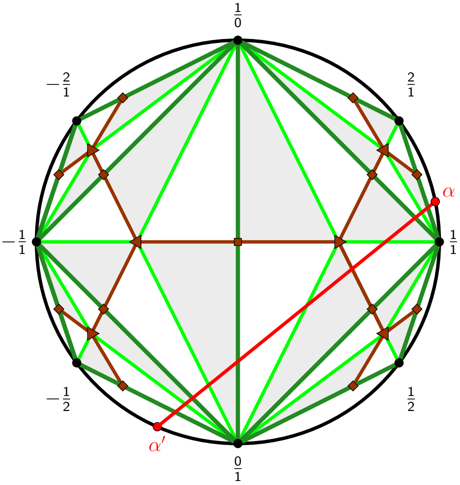
The axis of $A = RL$ in $\mathbb{PH}$ projects onto γ_A in $\mathbb{M}$.

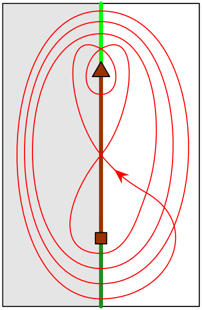
Modular geodesics : projections of the hyperbolic axes



The axis of $A = RLL$ in $\mathbb{PH}$ projects onto γ_A in $\mathbb{M}$.

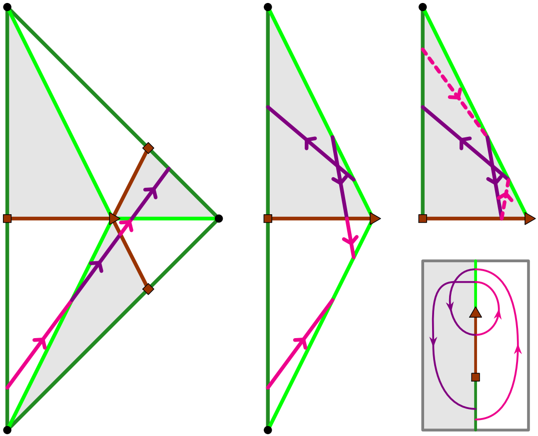
Modular geodesics : projections of the hyperbolic axes





The axis of $A = RLLL$ in $\mathbb{PH}$ projects onto γ_A in $\mathbb{M}$.

Modular geodesics : projections of the hyperbolic axes



Projecting the portion of an axis encoded by $S^{-1}T^{-2}S^{-1}$.

The modular group and its action on the hyperbolic plane

Arithmetic equivalence of modular geodesics

Linking numbers of modular knots

Class group $\mathrm{Cl}(\Delta)$ of discriminant Δ

Same length		Same discriminant		Conjugated in $\mathbb{C}$
$\lambda(\gamma_A) = \lambda(\gamma_B)$	$\iff$	$\mathrm{disc}(A) = \mathrm{disc}(B)$	$\iff$	$\exists C \in \mathrm{PSL}_2(\mathbb{C})$: $CA = BC$

The classes $\mathrm{Cl}(\Delta)$ for this equivalence relation have :

- *finite cardinals,*
  (Lagrange 1775 : reduction of quadratic forms)
- *unbounded cardinals,*
  (Horowitz 1972 : trace relations in SL_2)
- *structures of abelian groups.*
  (Gauss 1801 : composition of quadratic forms)

Arithmetic $\mathbb{K}$-equivalence

Definition :

For a field $\mathbb{K}$ extending the rationals $\mathbb{Q}$:

$$\begin{array}{c} A, B \in \mathrm{PSL}_2(\mathbb{Z}) \\ \mathbb{K}\text{-equivalent} \end{array} \quad \overset{\text{definition}}{\iff} \quad \begin{array}{c} \text{Conjugated over } \mathbb{K} \\ \exists C \in \mathrm{PSL}_2(\mathbb{K}) : \\ CA = BC \end{array}$$

Remarks and consequences :

- The $\mathbb{K}$-equivalence implies in particular $\mathrm{disc}(A) = \mathrm{disc}(B)$.
- The finest equivalence relation is $\mathbb{Q}$-equivalence.

Questions :

1. Understand the grouping of $\mathrm{PSL}_2(\mathbb{Z})$-classes into $\mathbb{K}$-classes.
2. Find geometric & arithmetic interpretations of $\mathbb{K}$-equivalence.

Arithmetico-geometric interpretation of the $\mathbb{K}$-equivalence

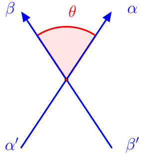
Theorem : $\mathbb{K}$-equivalence of modular geodesics

$A, B \in \mathrm{PSL}_2(\mathbb{Z})$ with discriminant $\Delta > 0$ are $\mathbb{K}$-equivalent $\iff$ $\gamma_A, \gamma_B \subset \mathbb{M}$ satisfy the following equivalent conditions :

θ: $\exists$ an intersection point with angle $\theta \in]0, \pi[$ such that :

$$\left(\cos \frac{\theta}{2}\right)^2 = X^2 - \Delta Y^2 \qquad \text{for} \quad X, Y \in \mathbb{K}$$

in which case this holds $\forall$ intersection points.



Angle well defined in $]0, \pi[$.

Arithmetico-geometric interpretation of the $\mathbb{K}$-equivalence

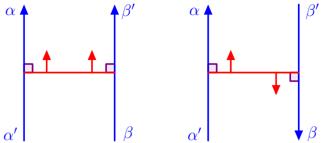
Theorem : $\mathbb{K}$-equivalence of modular geodesics

$A, B \in \mathrm{PSL}_2(\mathbb{Z})$ with discriminant $\Delta > 0$ are $\mathbb{K}$-equivalent $\iff$ $\gamma_A, \gamma_B \subset \mathbb{M}$ satisfy the following equivalent conditions :

λ: $\exists$ a co-oriented ortho-geodesic of length λ such that :

$$\left(\cosh \tfrac{\lambda}{2}\right)^2 = X^2 - \Delta Y^2 \qquad \text{for} \quad X, Y \in \mathbb{K}$$

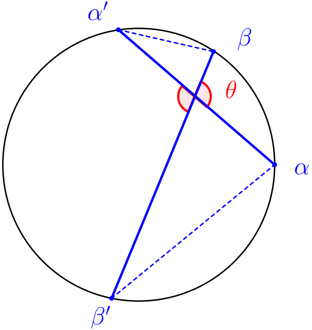
in which case this holds $\forall$ co-oriented ortho-geodesics.

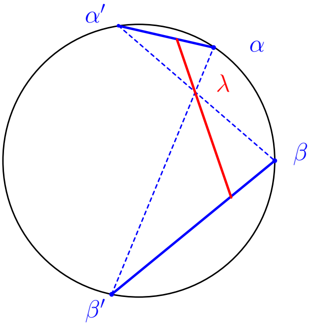


Ortho-geodesics : co-oriented and dis-co-oriented.

Geometric proof : adjoint action $\mathsf{PSL}_2(\mathbb{K}) \curvearrowright \mathbb{P}(\mathfrak{sl}_2(\mathbb{K}))$

$$C \in \mathsf{SL}_2(\mathbb{K}) \quad \longleftrightarrow \quad (x, y) \in \mathbb{K} \times \mathbb{K}$$
$$AC = CB \qquad\qquad x^2 - \tfrac{1}{4}\Delta y^2 = \chi$$





$$\frac{1}{\mathrm{bir}(\alpha',\alpha,\beta',\beta)} = \left(\cos\frac{\theta}{2}\right)^2 \qquad\qquad \frac{1}{\mathrm{bir}(\alpha',\alpha,\beta',\beta)} = \left(\cosh\frac{\lambda}{2}\right)^2$$

Remarks :

- We ask that the quantities $c^2 = 1/\operatorname{bir}$ belong to the group of norms of the quadratic extension $\mathbb{K}(\sqrt{\Delta})/\mathbb{K}$.

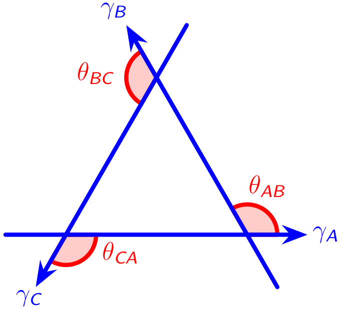
- Symmetric conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$:

$$\begin{array}{c} A = CA^{-1}C \\ \gamma_A = \gamma_{A^{-1}} \end{array} \iff \begin{array}{c} \gamma_A \text{ passes through } i \\ [i] \in \gamma_A \subset \mathbb{M} \end{array} \implies \begin{array}{c} c^2 \text{ et } 1 - c^2 \in \\ \mathrm{Norm}(\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}) \end{array}$$

Remarks :

- We ask that the quantities $c^2 = 1/\,\mathrm{bir}$ belong to the group of norms of the quadratic extension $\mathbb{K}(\sqrt{\Delta})/\mathbb{K}$.

- Equivalence relation : for every $\Delta > 0$, those properties on the intersection points and ortho-geodesics are *transitive* !



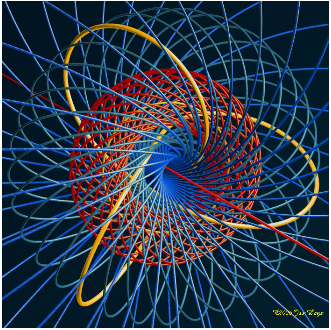
The modular group and its action on the hyperbolic plane

Arithmetic equivalence of modular geodesics

Linking numbers of modular knots

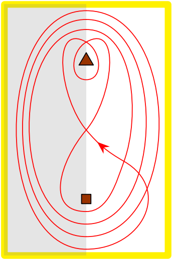
Unit tangent bundle $\mathbb{U}$ of the modular orbifold $\mathbb{M}$

$$
\begin{tikzcd}
\mathrm{PSL}_2(\mathbb{R}) \arrow[r, "\mathrm{PSL}_2(\mathbb{Z})"'] \arrow[d, "\mathbb{S}^1"'] & \mathbb{U} \arrow[d, "\mathbb{S}^1"'] \\
\mathbb{PH} \arrow[r, "\mathrm{PSL}_2(\mathbb{Z})"'] & \mathbb{M}
\end{tikzcd}
$$

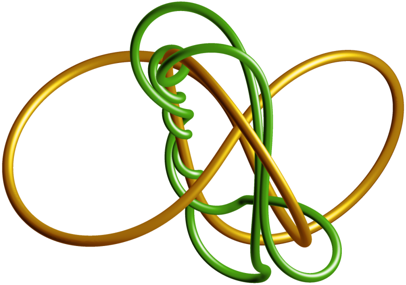


Modular knots in $\mathbb{U}$

Hyperbolic classes in $\pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z})$ primitive	Modular geodesics in $\mathbb{M}$ primitive	Periodic orbits in $\mathbb{U}$ primitive

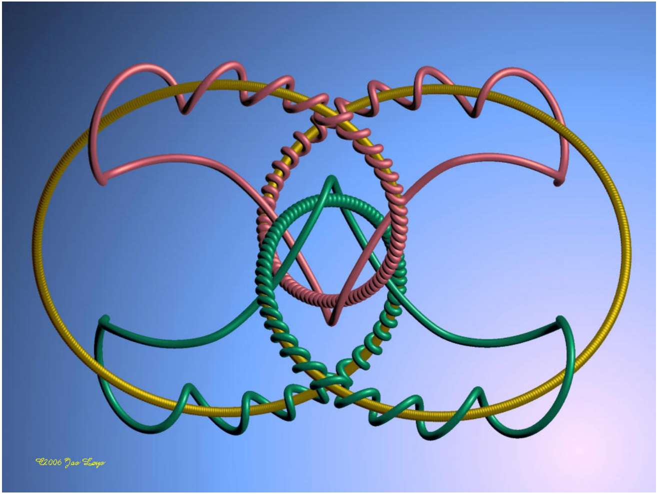


$\rightsquigarrow$
$\leftarrow$



The modular geodesics γ_A lift to modular knots k_A

Understand the topology of the *master modular link*



Two modular knots linking one another in the complement of the trefoil.

Conjugacy classes and cyclic binary words

Euclidean monoid

$$R = TS^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad L = T^{-1}S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) = \mathrm{Group}(L, R) \quad &\supset \quad \mathrm{SL}_2(\mathbb{N}) = \mathrm{Monoid}(L, R) \\ \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{Group}(L, R) \quad &\supset \quad \mathrm{PSL}_2(\mathbb{N}) = \mathrm{Monoid}(L, R) \end{aligned}$$

Conjugacy class $[A]$ of an infinite order $A \in \mathrm{PSL}_2(\mathbb{Z})$:

- $[A] \cap \mathrm{PSL}_2(\mathbb{N})$: cyclic permutations of an $L\&R$-word $\neq \emptyset$.
- Class is primitive $\iff$ cyclic word is primitive.
- Class is hyperbolic $\iff$ $\#L > 0$ and $\#R > 0$.

Combinatorics of words $\leftrightarrow$ Topology of links

Definition : combinatorial invariants

For the conjugacy class of $A \in \mathrm{PSL}_2(\mathbb{N})$ we define :

- its combinatorial length $\mathrm{len}([A]) = \#R + \#L$
- its Rademacher number $\mathrm{Rad}([A]) = \#R - \#L$

Theorem [Ghys 2006] :

For every hyperbolic conjugacy class $[A]$ in $\mathrm{PSL}_2(\mathbb{Z})$:

$$\mathrm{Rad}([A]) = \mathrm{lk}(\text{trefoil}, k_A)$$

Question [Ghys 2006] :

Arithmetic interpretation of the linking number $\mathrm{lk}(k_A, k_B)$ between two modular knots k_A, k_B ?

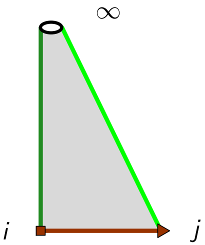
Definition : « bivariate Poincaré series »

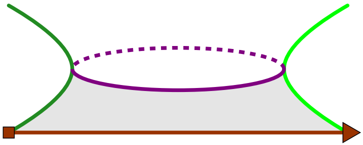
For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we defined the sum :

$$\mathsf{L}_1([A],[B]) := \sum \left(\cos\frac{\theta}{2}\right)^2 \quad \in \mathbb{R}_+^*$$

over the angles at intersection points $\gamma_A \cap \gamma_B$.

Deform the hyperbolic metric on $\mathbb{M}$ by opening the cusp...





The orbifolds $\mathbb{M} = \mathbb{M}_1$ and its deformation $\mathbb{M}_q$ with $q = (2\sinh\frac{\lambda}{2})^2$

Character variety $X(\mathrm{PSL}_2(\mathbb{Z}), \mathrm{PSL}_2(\mathbb{R}))$

Caracters of Fuchsian representations :

$$\left\{\begin{matrix}\text{Complete hyperbolic} \\ \text{metrics on } \mathbb{M}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\rho\colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R}) \\ \rho \text{ faithful \& discrete}\end{matrix}\right\} / \, \mathrm{PSL}_2(\mathbb{R})$$

- Real algebraic torus of dim 1, parametrized by $q \in \mathbb{R}^*$.
- The matrix $A_q = \rho_q(A)$ is obtained from a factorisation of A into a product of $L \& R$ by replacing $L \rightsquigarrow L_q$ and $R \rightsquigarrow R_q$ where

$$L_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix} \qquad\qquad R_q = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix}.$$

$$\rho_q \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}[q, q^{-1}])$$

The bivariate Poincaré q-series $\mathsf{L}_q(A, B)$

Conjugacy classes of infinite order (hyperbolic) in $\pi_1(\mathbb{M}_q) = \mathrm{PSL}_2(\mathbb{Z})$	Closed oriented geodesics (non peripheral) in $\mathbb{M}_q = \rho_q(\mathrm{PSL}_2(\mathbb{Z}))\backslash\mathbb{PH}$

Definition : « bivariate Poincaré q-series »

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, we define the function :

$$\mathsf{L}_q([A],[B]) := \sum \left(\cos \tfrac{1}{2}\theta_q\right)^2 \quad \in \sqrt{\mathbb{Q}(q)}$$

where the sum ranges over the intersection angles θ_q of the q-modular geodesics $\gamma_{A_q}, \gamma_{B_q} \subset \mathbb{M}_q$.

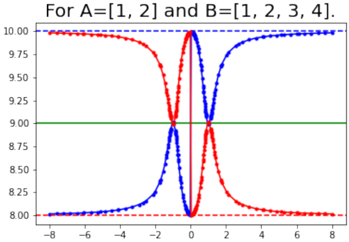
This defines a function of $q \in \mathbb{R}^*$, or on $X(\mathrm{PSL}_2(\mathbb{Z}), \mathrm{PSL}_2(\mathbb{R}))$.

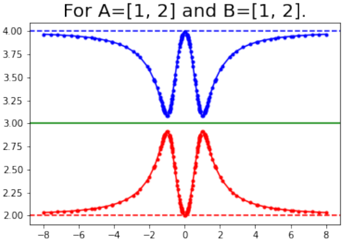
Linking function at the boundary of the character variety

Theorem : Linking number as evaluation of L_q at $+\infty \in \partial X$

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, we have the « special value » :

$$\mathrm{L}_q([A],[B]) \xrightarrow[q\to+\infty]{} 2\,\mathrm{lk}(k_A, k_B).$$

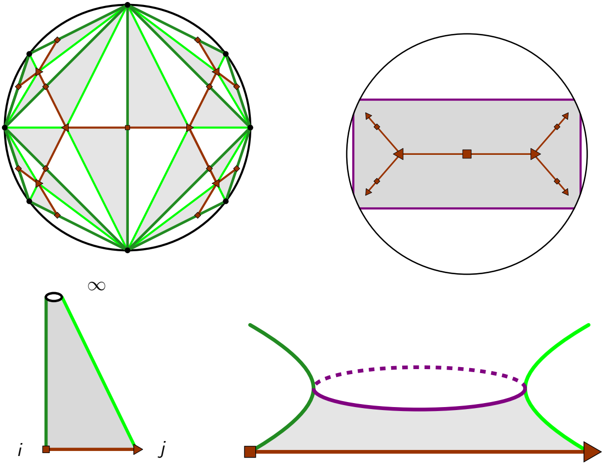




$\mathrm{L}_q(A, B)$ interpolates between arithmetic at 1 and topology at $+\infty$.

Proof using the action of $\mathsf{PSL}_2(\mathbb{Z})$ on the trivalent tree $\mathcal{T}$

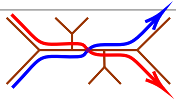
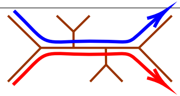
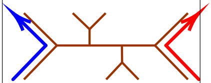
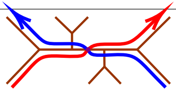
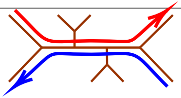
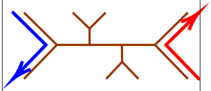
1. Lift the convex core of $\mathbb{M}_q$ in $\mathbb{PH}$: $\frac{1}{q^2}$-neighbourhood of $\mathcal{T}_q$.



2. The representation ρ_q tends to the action of $\mathsf{PSL}_2(\mathbb{Z})$ on $\mathcal{T}$.

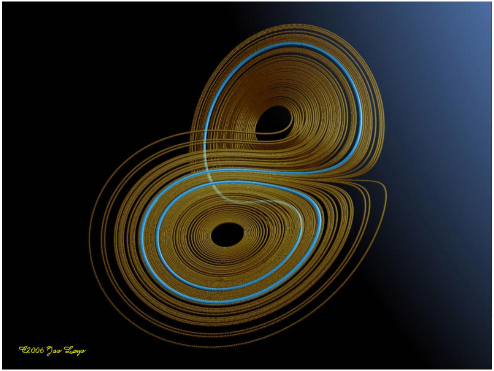
Proof using the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the trivalent tree $\mathcal{T}$

3. The angles $\theta_q \to 0 \bmod \pi$ thus $\cos(\theta_q) \to \pm 1$.
4. The sum $\mathrm{L}_q(A, B)$ counts the pairs of axes $(+1, +1)$:

cosign \ |cross|	$+1$	0	0
$+1$			
-1			

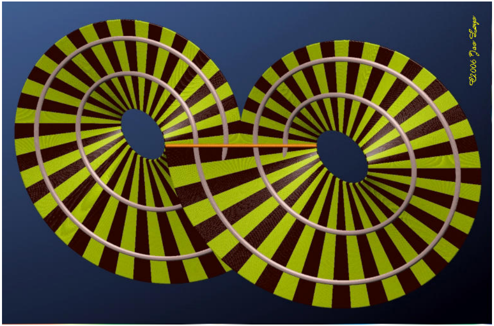
Proof using the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the trivalent tree $\mathcal{T}$

5. In the unit tangent bundle of $\mathbb{M}_q$, the master q-modular link is isotoped into a branched surface called the Lorenz template

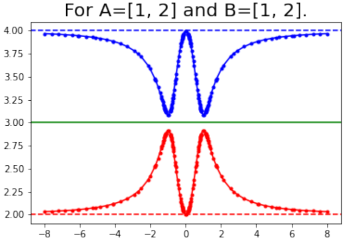


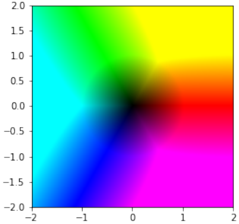
Proof using the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the trivalent tree $\mathcal{T}$

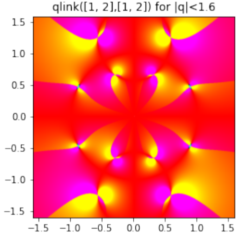
6. We recover an algorithmic formula for linking numbers in terms of the $L\&R$-cycles, using the topology of the Lorenz template.

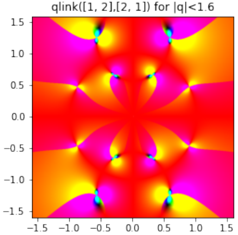


Graphs of $q \mapsto \mathrm{L}_q(A, B)$ for real and complex q



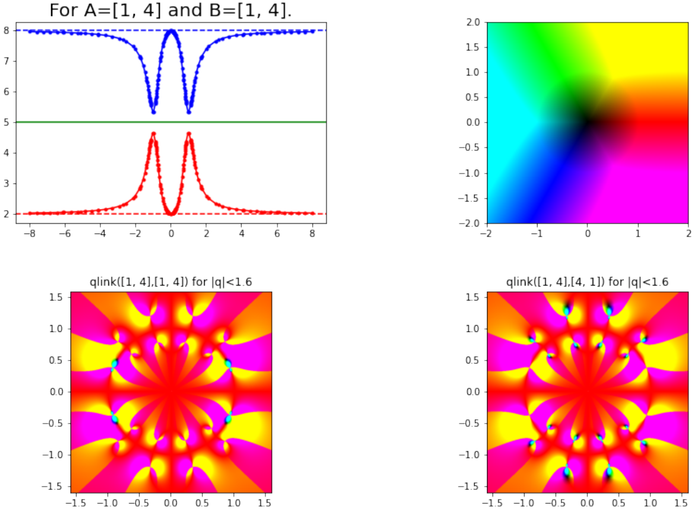






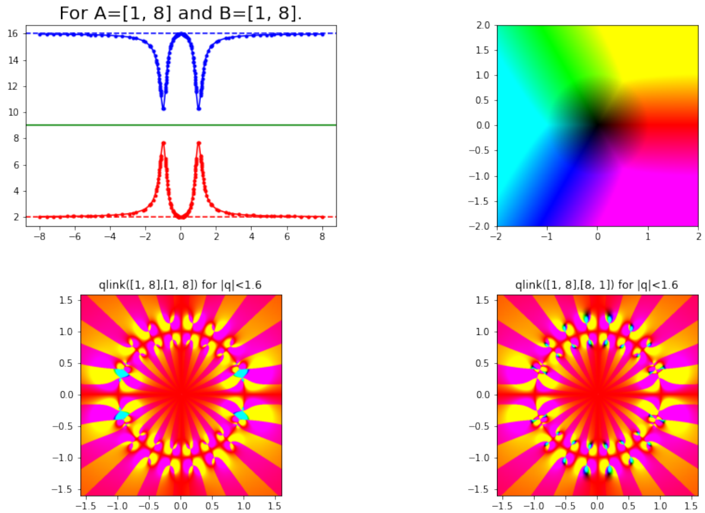
$\mathrm{L}_q(A, B)$ and $\mathrm{L}_q(A, {}^tB)$ for $A = B = RLL$ and ${}^tB = RRL$.

Graphs of $q \mapsto \mathrm{L}_q(A, B)$ for real and complex q



$\mathrm{L}_q(A, B)$ and $\mathrm{L}_q(A, {}^tB)$ for $A = B = RL^4$ and ${}^tB = R^4L$.

Graphs of $q \mapsto \mathrm{L}_q(A, B)$ for real and complex q



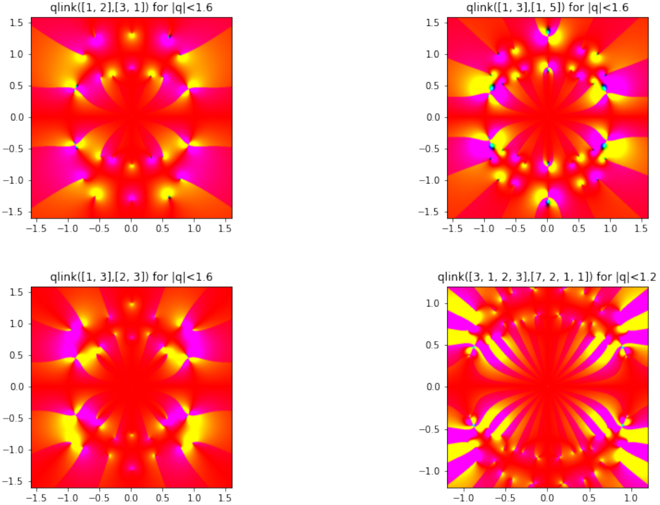
$\mathrm{L}_q(A, B)$ and $\mathrm{L}_q(A, {}^tB)$ for $A = B = RL^8$ and ${}^tB = R^8L$.

Moral of the story...

So many mysteries are concealed within a simple trefoil!

Région
Hauts-de-France

More graphs of $q \mapsto \mathrm{L}_q(A, B)$ for complex q



$\mathrm{L}_q(A, B)$ for various cycles A and B.