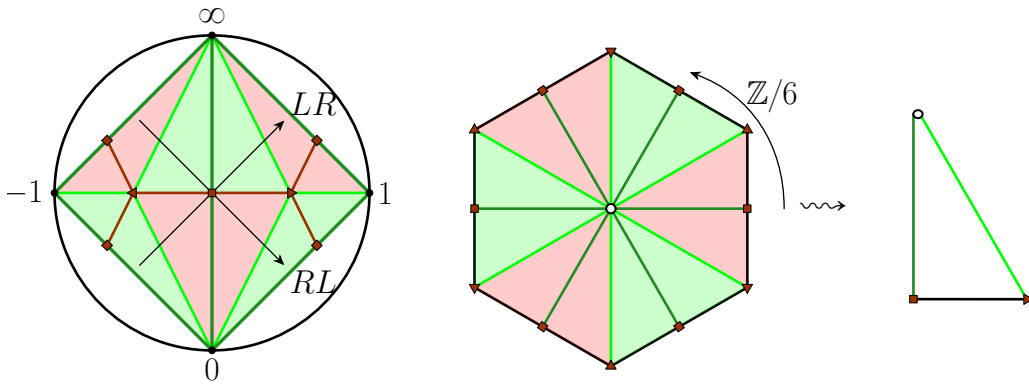# Arithmetic and Topology of the Hexponential map

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Abstract

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the upper-half plane $\mathbb{HP}$ with quotient the modular orbifold, uniformized by the j-function $\mathsf{j}\colon \mathbb{HP} \to \mathbb{C}$. We first show that second derived subgroup $\mathrm{PSL}_2(\mathbb{Z})''$ corresponds to a $\mathbb{Z}^2 \rtimes \mathbb{Z}/6$ Galois cover of the modular orbifold by a hexpunctured plane, uniformized by the hexponential map $\mathrm{hexp}\colon \mathbb{HP} \to \mathbb{C}\backslash(\omega_0\mathbb{Z}[j])$, which is a primitive of $C\eta^4$ where $\omega_0 \in i\mathbb{R}$ and $C \in \mathbb{R}$ are explicit constants and η is Dedekind eta function. We describe the values of the cusp-compactification $\partial\,\mathrm{hexp}\colon \mathbb{QP}^1 \to \omega_0\mathbb{Z}[j]$. After defining the radial-compactification $\mathrm{Shexp}\colon \mathscr{R} \to \mathbb{R}/(2\pi\mathbb{Z})$, we construct a simple section $\mathrm{InSh}\colon \mathbb{R}/(2\pi\mathbb{Z}) \to \mathscr{S}$ mod $\mathrm{PSL}_2(\mathbb{Z})'$ where $\mathscr{S} \subset \mathbb{RP}^1$ is a set of numbers whose continued fraction expansions arise from Sturmian sequences, which contains the set $\mathscr{M}$ of Markov quadratic irrationals as those numbers arising from periodic Sturmian sequences. Finally we show that the values of InSh are either Markov quadratic irrationals or transcendent.

Plan of the paper

0 Introduction

The paper unfolds the plan of this introduction, and concludes with further questions.

0.1 The modular orbifold and euclidean continued fractions

The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts by conformal transformations of the hyperbolic plane $\mathbb{HP} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ with boundary $\partial \mathbb{HP} = \mathbb{RP}^1$, as the stabiliser of its ideal triangulation $\triangle$ with vertices $\mathbb{QP}^1$ and edges $\left(\frac{a}{c}, \frac{b}{d}\right)$ such that $ad - bc = 1$. Let

$$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \quad T = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \qquad L = T^{-1}S = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \quad R = TS^{-1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right).$$

The modular orbifold $\mathbb{M} = \Gamma \backslash \mathbb{HP}$ has genus zero, a cusp associated to the fixed point $\infty \in \mathbb{QP}^1 \subset \partial \mathbb{HP}$ of R, as well as two conical singularities of order 2 and 3 associated to the fixed points $i, j \in \mathbb{HP}$ of S and T. Thus $\Gamma = \pi_1(\mathbb{M})$ is the free amalgam of its subgroups $\mathbb{Z}/2$ and $\mathbb{Z}/3$ generated by S and T.

The elements of Γ stabilising $[0, +\infty] \subset \mathbb{RP}^1$ form the euclidean monoid $\mathrm{PSL}_2(\mathbb{N})$, which is freely generated by the parabolic matrices R and L. The subset of sequences in $\{L, R\}^{\mathbb{N}}$ which do not end with an infinite string of L are in bijective correspondence with the continued fraction expansions of positive numbers in $(0, +\infty] \subset \mathbb{RP}^1$. The following result can be traced back to the works of Gauss [Gau07] and Galois [Gal29].

Proposition 0.1 (Geodesics in $\mathbb{M}$). *A geodesic in $\mathbb{M}$ has a lift $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ whose endpoints satisfy $-1 < \alpha_- < 0$ and $1 < \alpha_+ < \infty$. It intersects $\triangle$ along a sequence of triangles whose $\{L, R\}^{\mathbb{Z}}$-encoding is obtained from the continued fraction expansions of α_+ and $-1/\alpha_-$ by concatenating the transpose of the latter with the former.*

The closed geodesics in $\mathbb{M}$ correspond to the periodic sequences, hence to the Γ-orbits of pairs (α_-, α_+) of Galois conjugate quadratic irrationals.

0.2 The modular torus and Sturmian numbers

The derived subgroup Γ' of Γ, which is freely generated by $X = LR$ and $Y = RL$, corresponds to a Galois cover of $\mathbb{M}$ by a punctured torus $\mathbb{M}'$ with group $\Gamma/\Gamma' = \mathbb{Z}/6$. The simple geodesics of $\mathbb{M}'$ were described by Series in [Ser85a].

Proposition 0.2 (Simple geodesics in $\mathbb{M}'$). *Fix $-1/\alpha_-, \alpha_+ \in \mathbb{Q}_+^*$ and consider the L&R-sequence obtained by concatenating their (transposed) continued fractions.*

The geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ projects to a geodesic in $\mathbb{M}'$ which is simple if and only if the L&R can be paired to form an X&Y-sequence which is Sturmian.

The simple closed geodesics in $\mathbb{M}'$ correspond combinatorially to the periodic Sturmian X&Y-sequences, and arithmetically to the Markov quadratic irrationals.

We explain why the conformal structure on $\mathbb{M}'$ is given by a punctured elliptic curve isomorphic over $\mathbb{R}$ to the affine cubic $y^2 = (x - \omega_0 j^0)(x - \omega_0 j^1)(x - \omega_0 j^2)$ with abelian differential $dx/y = -dx/\sqrt{x^3 - |\omega_0|^3}$ having fundamental period $\omega_0 = \frac{2\pi^{1/2}}{i3^{1/4}}$.

This abelian differential on $\mathbb{M}'$ lifts on the universal cover $\mathbb{HP}$ to $C\eta^4(\tau)d\tau$ for some $C \in \mathbb{C}^*$ where η is the Dedekind eta function, whose expansion in $q = e^{i2\pi\tau}$:

$$\eta(\tau) = \sum_{n=0}^{\infty} \chi(n) q^{\frac{1}{24}n^2} \quad \text{yields} \quad \eta^4(\tau) = \sum_{n=0}^{\infty} \psi(n) q^{\frac{1}{24}n}$$

where χ is the unique primitive character mod 12 (that is the Jacobi symbol mod 12, equal to $\chi(12n \pm 1) = 1$, $\chi(12n \pm 5) = -1$ and $\chi(n) = 0$ if $2 \mid n$ or $3 \mid n$), and where $\psi(n)$ is the sum of $\chi(abcd)$ over $\{(a, b, c, d) \in \mathbb{N}^4 \mid a^2 + b^2 + c^2 + d^2 = n\}$.

0.3 The hexpunctured plane and its cusp compactifications

The second derived subgroup $\Gamma'' \subset \Gamma'$ corresponds to the universal abelian cover $\mathbb{M}'' \to \mathbb{M}'$ of the punctured torus by a lattice-punctured plane with Galois group $\Gamma'/\Gamma'' = \mathbb{Z}_X \oplus \mathbb{Z}_Y$. The action of the Galois group Γ/Γ'' on the corresponding cover $\mathbb{M}'' \to \mathbb{M}$ was described in the author's thesis [Sim22, Section 3.2].

Proposition 0.3 (Hexagonal lattice)**.** *We have a semi-direct decomposition*

$$\Gamma/\Gamma'' = \Gamma'/\Gamma'' \rtimes \Gamma/\Gamma'$$

where the action of the quotient $\Gamma/\Gamma' = \mathbb{Z}/2 \times \mathbb{Z}/3$ by outer-automophisms of the kernel $\Gamma'/\Gamma'' = \mathbb{Z}_X \oplus \mathbb{Z}_Y$ is given by $\mathrm{AD}_S = D_{S^2} = D_{-1}$ and $\mathrm{AD}_T = D_{T^2}$ since we have $SXS^{-1} = X^{-1}$, $SYS^{-1} = Y^{-1}$ and $TXT^{-1} = Y^{-1}$, $TYT^{-1} = -XY^{-1}$.

This represents Γ/Γ'' as the affine isometry group of the oriented hexagonal lattice $H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_X \oplus \mathbb{Z}_Y$ with $\mathrm{angle}(X, Y) = \frac{2\pi}{6}$, where the translation action of Γ'/Γ'' is by $\mathbb{Z}^2$-translation while the outer-automorphism action of Γ/Γ' is by $\mathbb{Z}/6$-rotation.

The integration of abelian differential on $\mathbb{M}'$ identifies its universal abelian cover $\mathbb{M}''$ with $H_1(\mathbb{M}'; \mathbb{R}) \backslash H_1(\mathbb{M}'; \mathbb{Z})$, hence with the hexpunctured plane $\mathbb{C} \backslash \omega_0 \mathbb{Z}[j]$.

The map hexp: $\mathbb{HP} \to \mathbb{M}''$ in the coordinates $\{z \in \mathbb{C} \backslash \Im(z) > 0\} \to \mathbb{C} \backslash \omega_0 \mathbb{Z}[j]$ is therefore the primitive of $C\eta(z)^4 dz$, with expansion:

$$\mathrm{hexp}(\tau) = \int_{\infty}^{\tau} C\eta^4(z)dz = \frac{12C}{i\pi} \sum_{n=1}^{\infty} \frac{\psi(n)}{n} \cdot q^{\frac{1}{24}n}$$

We deduce the value of $C = \frac{2^{10/3}}{3^{3/4}} \frac{\pi^{5/2}}{\Gamma(1/3)^3}$ from the relation $\omega_0 = \int_{\infty}^{0} C\eta^4(\tau)d\tau$.

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The main results in this paper concern arithmetic and geometric properties of the uniformizing map $\mathrm{hexp}\colon \mathbb{HP} \to \mathbb{M}''$, and its compactifications.

We first describe the values of the cusp compactification $\partial\,\mathrm{hexp}\colon \mathbb{QP}^1 \to \omega_0\mathbb{Z}[j]$.

Theorem 0.4 (Values of $\partial\,\mathrm{hexp}$). *Consider $a/c \in \mathbb{QP}^1$ with $\gcd(a,b) = 1$.*
There exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'$, and we may consider its class $A \equiv X^m Y^n \mod \Gamma''$.
The value $\partial\,\mathrm{hexp}(\frac{a}{c})$ of the improper integral and conditionally convergent series:

$$\partial\,\mathrm{hexp}(\tfrac{a}{c}) = \int_\infty^{\frac{a}{c}} C\eta^4(z)dz = \tfrac{12C}{i\pi}\sum_{n=1}^{\infty} \tfrac{\psi(n)}{n}\cdot\exp\left(\tfrac{i\pi}{12}\tfrac{a}{b}n\right) \quad where \quad C = \tfrac{2^{10/3}}{3^{3/4}}\tfrac{\pi^{5/2}}{\Gamma(1/3)^3}$$

is equal to $\quad \partial\,\mathrm{hexp}(\tfrac{a}{c}) = |\omega_0|\left(m\exp(-\tfrac{i\pi}{6}) + n\exp(+\tfrac{i\pi}{6})\right) \quad where \quad |\omega_0| = \tfrac{2\pi^{1/2}}{3^{1/4}}.$

To prove the conditional convergence, we rely on the identification of $\partial\,\mathrm{hexp}(0)$ with the special value at $s = 1$ of the L-series associated to an elliptic modular curve.

Question 0.5 (Rational correlations). *The conditional convergence of the series $\partial\,\mathrm{hexp}(r)$ at rationals $r \in \mathbb{Q}$ implies a strong correlation between $n \mapsto \psi(n)$ and periodic rotations of the circle $n \mapsto rn \mod 1$. These correlations factor through the cosets $r \in \Gamma''\backslash\mathbb{QP}^1$, and their values are organised according to the lattice $\omega_0\mathbb{Z}[j]$.*
This may be equivalent to some kind of "Dirichlet equidistribution theorem" for the arithmetic progressions represented by a χ-twisted version of the quadratic form $a^2 + b^2 + c^2 + d^2$ or the divisor function $\sum_{d|n} d$.

0.4 Transcendent arguments of the hexponential map

We define the ray compactification $\mathrm{Shexp}\colon \mathscr{R} \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ from the set of $\alpha \in \mathbb{RP}^1$ such that as $\tau \in \mathbb{HP}$ converges to $\alpha \in \partial\mathbb{HP}$, the modulus $|\mathrm{hexp}(\tau)| \in \mathbb{R}_+$ diverges while the argument $\arg\mathrm{hexp}(\tau) \in \mathbb{R}/(2\pi\mathbb{Z})$ converges to a limit $\mathrm{Shexp}(\alpha)$. Thus $\mathscr{R} \subset \mathbb{R}\backslash\mathbb{Q}$ contains the numbers α such that the geodesic $(i, \alpha) \subset \mathbb{HP}$ maps under hexp to a geodesic of $\mathbb{M}''$ which escapes in a definite direction (it follows the geodesic of $\mathcal{H}$ encoded by the $L\&R$-continued fraction expansion of α).

The surjective maps $\mathrm{Shexp}\colon \mathscr{R} \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ and $\partial\,\mathrm{hexp}\colon \mathbb{QP}^1 \to H_1(\mathbb{M}'; \mathbb{Z})$ have disjoint domains and codomains. Together, they define a continuous compactification of $\mathrm{hexp}\colon \mathbb{HP} \to \mathbb{M}''$ (for obvious topologies), which is equivariant under the left actions of Γ and Γ/Γ''. The translation action of Γ' on $\mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ is trivial so $\mathrm{Shexp}\colon \mathscr{R} \mod \Gamma' \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ is equivariant under the left actions of Γ and Γ/Γ'. The map Shexp sends uncountably many distinct Γ'-orbits in $\mathscr{R}$ to the same slope.

Remark 0.6 (Essential singularity along the boundary). *If $\tau \in \mathbb{HP}$ diverges to an irrational $\alpha \in \mathbb{RP}^1$ the function $\mathrm{hexp}(\tau)$ diverges.*

The subset $\mathscr{R} \subset \mathbb{RP}^1$ irrational numbers (at which the modulus $|\mathrm{hexp}(\tau)|$ diverges while its argument $\arg(\mathrm{hexp}(\tau))$ converges) is dense in $\mathbb{RP}^1$.

The map $\mathrm{Shexp}\colon \mathscr{R} \to \mathbb{S}H_1(\mathbb{M}', \mathbb{R})$ is uniformly continuous and surjective.

Finally, we define a simple section $\mathrm{InSh}\colon \mathbb{S}H_1(\mathbb{M}'; \mathbb{R}) \to \mathscr{S} \bmod \Gamma'$, of Shexp which is Γ/Γ'-equivariant and injective. A slope $(x, y) \in \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ corresponds to a complete simple geodesic $\alpha \subset \mathbb{M}'$: it lifts in $\mathbb{HP}$ to the Γ'-orbit of a geodesic (α_-, α_+) and we let $\mathrm{InSh}(x, y) = \Gamma' \cdot \alpha_+$.

We denote by $\mathscr{S}$ the set of *Sturmian* numbers $-1/\alpha_-, \alpha_+$ obtained in this way (namely whose continued fractions arise from a Sturmian $X^{\pm 1} \& Y^{\pm 1}$-sequence by replacing $X = LR$ and $Y = RL$). In contains the subset $\mathscr{M} \subset \mathscr{S}$ of *Markov* numbers as those arising from rational slopes $(x, y) \in \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$, namely those with periodic $X^{\pm 1} \& Y^{\pm 1}$ sequences (coinciding with the Markov quadratic irrationals).

Remark 0.7 (Cantor set). *The Sturmian numbers $\mathscr{S} \subset \mathbb{RP}^1$ form a Cantor subset: compact, perfect, nowhere dense.*

The map $\mathrm{Shexp}\colon \mathscr{S} \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ is continuous and surjective.

Question 0.8 (Sturmian correlations). *We observe that the sequence $\psi(n)/n$ bears very strong (lattice-like) correlations with rational numbers (bounded complexity), and rather strong (Cantor-like) correlations with Sturmian numbers (complexity $l + 1$), and some correlations with the numbers in $\mathscr{R}$.*

We expect that one may characterise the subset of Sturmian numbers $\mathscr{S} \subset \mathscr{R}$ by quantifying how fast $\arg \mathrm{hexp}(\theta)$ converges to $\mathrm{Shexp}(\alpha)$. We wonder if one can provide any arithmetic or diophantine interpretation of the numbers in $\mathscr{R}$.

Finally, we prove the following using [ADQZ01, Proposition 3 and Theorem 7].

Theorem 0.9 (Transcendance of the Inshection map). *For $(x, y) \in \mathbb{S}H_1(\mathbb{M}; \mathbb{R})$:*

- *If $(x, y) \in \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$ is rational then $\mathrm{InSh}(x, y) \subset \mathscr{M}$ are quadratic.*
- *If $(x, y) \notin \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$ is irrational then $\mathrm{InSh}(x, y) \subset \mathscr{S} \backslash \mathscr{M}$ are transcendent.*

Question 0.10 (Continued fraction expansion). *To understand the diophantine properties of the hexponential hexp, its compactification Shexp and its inverse InSh, we wish to find continued fraction expansions for:*

  hexp*: the complex function* $\mathrm{hexp} \circ \exp(i2\pi \cdot)\colon \mathbb{HP} \to \mathbb{C}$
  Shexp*: the real function* $\tan \circ \mathrm{Shexp}\colon \mathscr{R} \to \mathbb{RR}^1$
 InSh*: the real function* $\mathrm{InSh} \circ \arctan\colon \mathbb{RP}^1 \to \mathscr{S}$

The last Section 5 mentions a couple of directions of research and open questions. The appendix A records several well known arithmetic properties of ψ.

1 The modular orbifold and continued fractions

1.1 Modular group and its conjugacy classes

The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on the ideal triangulation $\triangle$ of $\mathbb{HP}$ with vertex set $\mathbb{QP}^1$ and edges $\left(\frac{a}{c}, \frac{b}{d}\right)$ such that $ad - bc = 1$, freely transitively on its flags, corresponding to the half-edges of its dual trivalent tree $\mathcal{T}$. The connected components of $\mathbb{HP}\backslash\mathcal{T}$ correspond to the vertices of $\triangle$, parametrized by $\Gamma/\langle R\rangle$.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
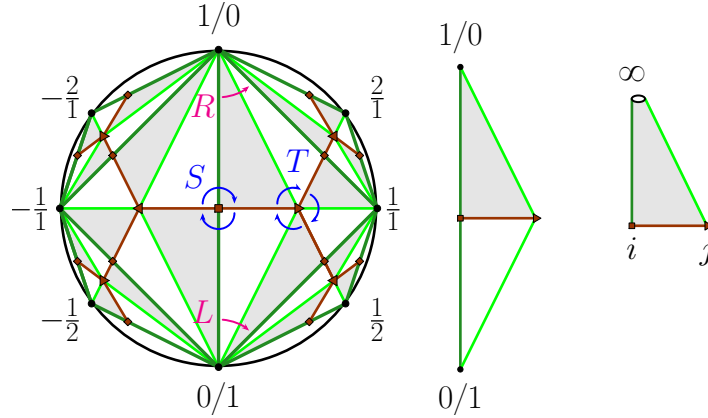
The modular orbifold $\mathbb{M} = \Gamma\backslash\mathbb{HP}$ has genus zero, a cusp associated to the fixed point $\infty \in \mathbb{QP}^1 \subset \partial\mathbb{HP}$ of R, as well as two conical singularities of order 2 and 3 associated to the fixed points $i, j \in \mathbb{HP}$ of S and T.

Thus $\Gamma = \pi_1(\mathbb{M})$ is the free amalgam of its subgroups $\mathbb{Z}/2$ and $\mathbb{Z}/3$ generated by S and T. (Hence $\mathrm{SL}_2(\mathbb{Z})$ is the amalgam of its subgroups $\mathbb{Z}/4$ and $\mathbb{Z}/6$ generated by S and T over their intersection $\mathbb{Z}/2$ generated by $S^2 = -\mathbf{1} = T^3$.)

It follows that in Γ, a finite order element is conjugate to a power of S or T; whereas an infinite order A is conjugate to $\prod_1^l T^{\epsilon_i} S^{-\epsilon_i}$ for a unique $l = \mathrm{len}\, A \in \mathbb{N}^*$ and a unique sequence of $\epsilon_i = \pm 1$ up to cyclic permutation: this corresponds to cyclic word in $R = TS^{-1}$ and $L = T^{-1}S$, and its Rademacher number is defined by

$$\mathrm{Rad}(A) = \sum_{i=1}^{\mathrm{len}(A)} \epsilon_i = \#R - \#L$$

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Conjugacy classes in Γ correspond to orbifold homotopy classes of loops in $\mathbb{M}$. The elliptic and parabolic classes correspond to loops circling around the singularities and cusp, whereas every hyperbolic class is represented by a unique closed geodesic.

An element of Γ is primitive when it generates a maximal cyclic subgroup. This notion is invariant by conjugacy. Correspondingly, a cyclic $L\&R$-word is primitive when it is not a proper power, and a closed geodesic is primitive when it does not wind several times around itself.

A complete geodesic of $\mathbb{HP}$ is uniquely determined by the sequence of triangles in $\triangle$ that it intersects: it corresponds to a geodesic of the dual tree $\mathcal{T}$ which is not horocyclic (bounding a region of the complement $\mathbb{HP}\backslash\mathcal{T}$). Such geometric and combinatorial geodesics are equally determined by their common endpoints in $\mathbb{RP}^1$.

A closed geodesic of $\mathbb{M}$ lifts to the geodesics in $\mathbb{HP}$ corresponding to the periodic geodesics of $\mathcal{T}$ whose period is given by $L\&R$-cycle of the associated conjugacy class. Their endpoints form the Γ-orbit of a pair of Galois-conjugate quadratic irrationals. Such geodesics determine a unique hyperbolic conjugacy class in Γ which is primitive.

1.2 Euclidean monoid and continued fraction expansions

The monoid $\mathrm{SL}_2(\mathbb{N}) \subset \mathrm{SL}_2(\mathbb{Z})$ is freely generated by the transvections L and R, and it identifies with its image $\mathrm{PSL}_2(\mathbb{N}) \subset \mathrm{PSL}_2(\mathbb{Z})$ which we call the euclidean monoid.

In $\mathrm{PSL}_2(\mathbb{Z})$, the conjugacy classes of torsion elements are those of $\mathbf{1}, S, T, T^{-1}$, whereas the conjugacy class of an infinite order element intersects the euclidean monoid $\mathrm{PSL}_2(\mathbb{N})$ along the cyclic permutations of a unique $L\&R$-word. Hence the hyperbolic geodesics of $\mathbb{M}$ are indexed by cyclic words in $L\&R$ containing both letters.

Observe that for $A \in \mathrm{SL}_2$, its transpose $A^{\dagger}$ and inverse A^{-1} are conjugate by S. Hence the transposition of cyclic words in $\mathrm{PSL}_2(\mathbb{N})$ corresponds to inversion of the associated conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$.

Every positive real number x admits a Euclidean continued fraction expansion:

$$\lfloor n_0, n_1, \dots \rfloor = n_0 + \cfrac{1}{n_1 + \dots} \quad \text{with} \quad n_j \in \mathbb{N} \quad \text{and} \quad \forall j > 0,\ n_j > 0.$$

Such an expansion is infinite if and only if x is irrational, in which case it is unique. A rational x has two expansions $x = \lfloor n_0, \dots, n_k \rfloor$: one for which $n_k = 1$, the other for which $n_k > 1$, and exactly one of these has even length $k + 1$.

To represent negative real numbers we apply the involution $x \mapsto S(x) = -1/x$. This corresponds to the partition $\mathbb{RP}^1 = [0, \infty) \sqcup S \cdot [0, \infty)$. Thus every $x \in \mathbb{RP}^1\backslash\mathbb{QP}^1$ admits exactly one representation of the form $x = \lfloor n_0, \dots \rfloor$ or $x = -1/\lfloor n_0, \dots \rfloor$, and every $x \in \mathbb{QP}^1$ admits exactly two such representations of the same form, including $0 = \lfloor\ \rfloor = \lfloor 0 \rfloor$ and $\infty = -1/\lfloor\ \rfloor = -1/\lfloor 0 \rfloor$.

Now consider the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{RP}^1$ and of its euclidean submonoid generated by $L\&R$ on $[0, \infty]$. If x_i denotes the ith remainder of $x \in [0, \infty]$, given by the tail $\lfloor n_i, \ldots \rfloor$ of its continued fraction expansion, then $x_0 = (R^{n_0}L^{n_1})x_2$. So the orbits of $x, y \in [0, \infty]$ under the euclidean monoid have non-empty intersection if and only if there exist even starting points i, j at which the tails x_i and y_j coincide. We deduce that $x, y \in \mathbb{RP}^1$ belong to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit if and only if there exist even starting points i, j at which the tails x_i and y_j coincide.

Proposition 1.1. *A geodesic in $\mathbb{M}$ has a lift $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ whose endpoints satisfy $-1 < \alpha_- < 0$ and $1 < \alpha_+ < \infty$. It intersects $\triangle$ along a sequence of triangles whose encoding in $\{L, R\}^{\mathbb{Z}}$ is obtained from the continued fraction expansions of α_+ and $-1/\alpha_-$ by concatenating the transpose of the latter with the former.*

The closed geodesics in $\mathbb{M}$ correspond to the periodic sequences, hence to the Γ-orbits of pairs (α_-, α_+) of Galois conjugate quadratic irrationals.

1.3 Moduli space of elliptic curves and modular J-invariant

A complex elliptic curve is isomorphic to the quotient of $\mathbb{C}$ by a lattice $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ for a unique $\tau \in \mathbb{HP}$ up to the action of $\mathrm{PSL}_2(\mathbb{Z})$. Hence the modular orbifold corresponds to the moduli space of elliptic curves.

The field of meromorphic functions on $\mathbb{C}/\Lambda_\tau$, rather of Λ_τ-invariant meromorphic functions on $\mathbb{C}$, is generated by the following Weierstrass sums over $\lambda \in \Lambda_\tau \backslash \{0\}$:

$$\wp_\tau(u) = \sum' \frac{1}{(u-\lambda)^2} - \frac{1}{\lambda^2} \quad \text{and} \quad \wp'_\tau(u) = -2\sum' \frac{1}{(u-\lambda)^3}$$

The Fourier expansion of $\wp_\tau(u) = u^{-2} + \sum_{n \geqslant 1}(2n+1)G_{2n+2}(\tau)u^{2n}$ has coefficients the Eisenstein series $G_n(\Lambda) = \sum' \lambda^{-n}$ where the sum ranges over all $\lambda \in \Lambda_\tau \backslash \{0\}$, which are modular forms of weight n.

The functions $(x, y) = (\wp_\tau, \wp'_\tau)$ lie on the plane affine cubic $\mathbb{E}_\tau \colon y^2 = 4x^3 - g_2 x - g_3$ where $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 140G_6(\tau)$, with discriminant $\Delta = g_2^3 - 27g_3^2$.

The complex differential 1-form dx/y restricts on the cubic $\mathbb{E}_\tau$ to the holomorphic 1-form $d\wp/\wp' = du$, whose integration along paths based at the origin ∞ lifts on its universal cover to the Jacobian integration map $\int^\infty du \colon H_1(\mathbb{E}_\tau; \mathbb{R}) \to \mathbb{C}$ sending its period lattice $H_1(\mathbb{E}_\tau; \mathbb{Z})$ to its lattice of periods $\mathbb{Z}\omega_0 + \mathbb{Z}\omega_1$ with $\omega_1/\omega_0 = \tau$.

Hence the isomorphism class of $\mathbb{C}/\Lambda_\tau$ is equal to that of $\mathbb{E}_\tau$, characterised by the modular invariant $\mathfrak{j}(\tau) = g_2^3(\tau)/\Delta(\tau)$. This $\mathrm{PSL}_2(\mathbb{Z})$-modular holomorphic and bijective function $\mathfrak{j} \colon \mathbb{HP} \to \mathbb{C}$ yields an isomorphism of complex curves $\mathbb{M} \simeq \mathbb{C}$.

Let us recall from [Cox97] that $\mathfrak{j}(\tau)$ is algebraic if and only if τ is complex quadratic (in which case $\mathfrak{j}(\tau)$ generates the maximal unramified abelian extension of the quadratic field $\mathbb{Q}(\tau)$ over that of $\mathbb{Q}$). For example $\mathfrak{j}(i) = 12^3$, $\mathfrak{j}(j) = 0$.

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2 The modular torus and its mapping class group

2.1 The derived modular group and geodesics in the torus

The abelianisation $\mathbb{Z}/2 * \mathbb{Z}/3 \to \mathbb{Z}/2 \times \mathbb{Z}/3$ of the modular group corresponds to the $\mathbb{Z}/6$-Galois cover $\mathbb{M}' \to \mathbb{M}$ of the modular orbifold by the modular torus, which is a punctured torus whose fundamental group $\pi_1(\mathbb{M}') = \mathrm{PSL}_2(\mathbb{Z})'$ is freely generated by

$$X = [T^{-1}, S] = LR = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad Y = [T, S^{-1}] = RL = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

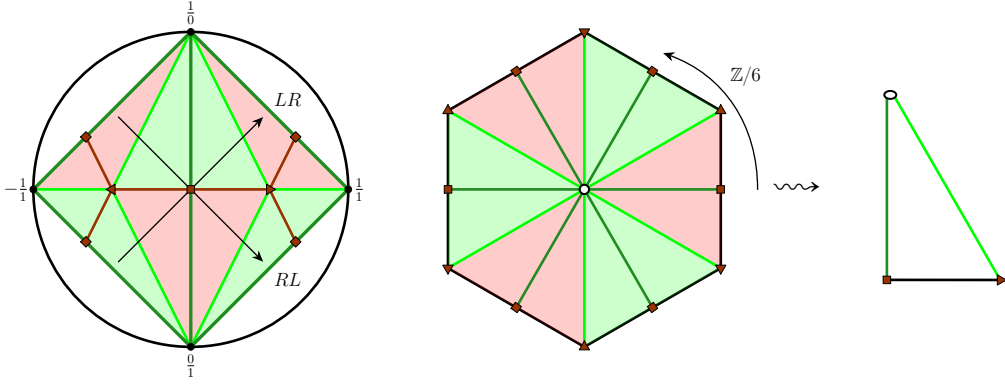Note that Γ' consists of all infinite order $A \in \Gamma$ with $\mathrm{Rad}(A) = 0 \bmod 6$.

Figure 1: The free group $\mathrm{PSL}_2(\mathbb{Z})'$ acts on $\mathbb{HP}$ with quotient a punctured torus $\mathbb{M}'$. The Galois group $\mathrm{PSL}_2(\mathbb{Z})/\mathrm{PSL}_2(\mathbb{Z})' = \mathbb{Z}/6$ acts on $\mathbb{M}'$ with quotient $\mathbb{M}$.

Homotopy classes of loops in $\mathbb{M}'$ correspond to conjugacy classes in $\pi_1(\mathbb{M}')$, hence to reduced cyclic words in X, X^{-1}, Y, Y^{-1}. The n-th power of the commutator $[X, Y]$ corresponds to the loop winding n times around the puncture, and every other nontrivial homotopy class contains a unique geodesic.

The cusp-compactification $\overline{\mathbb{M}'} = \Gamma' \backslash (\mathbb{HP} \sqcup \mathbb{QP}^1)$ of $\mathbb{M}'$ is homeomorphic to a torus, and has fundamental group $\pi_1(\overline{\mathbb{M}'}) = \Gamma'/\lhd [X, Y] \rhd = \Gamma'/\Gamma'' = H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_X \oplus \mathbb{Z}_Y$. The simple loops and geodesics in $\mathbb{M}'$ correspond to those in $\overline{\mathbb{M}'}$.

The conjugacy classes in Γ' associated to simple loops in $\mathbb{M}'$ correspond to the primitive vectors of the lattice $H_1(\Gamma'; \mathbb{Z})$. The bases of Γ' correspond to bases of $H_1(\Gamma'; \mathbb{Z})$, hence to pairs of simple loops in $\mathbb{M}'$ with one intersection point (their commutator in Γ' corresponds to a loop circling once around the puncture).

The space of simple geodesics in $\mathbb{M}'$ corresponds to the spherisation $\mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$. The basis (X, Y) of the lattice $H_1(\mathbb{M}'; \mathbb{Z}) \subset H_1(\mathbb{M}'; \mathbb{R})$ identifies $\mathbb{S}H_1(\mathbb{M}'; \mathbb{Z}) \subset \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ with $\mathbb{QS}^1 \subset \mathbb{RS}^1$.

2.2 Mapping class group action on simple geodesics

The mapping class group $\mathrm{Mod}(\overline{\mathbb{M}'})$ is identified to the outer automorphism group $\mathrm{Out}(\pi_1(\overline{\mathbb{M}'})) = \mathrm{GL}(\mathbb{Z}_X \oplus \mathbb{Z}_Y)$ by the [FM12, Theorem 8.1] of Dehn-Nielsen-Baer, and contains the orientation preserving mapping class group $\mathrm{Mod}^+(\overline{\mathbb{M}'}) = \mathrm{SL}_2(\mathbb{Z}_X \oplus \mathbb{Z}_Y)$ with index 2 and quotient generated by $D_J \colon (X, Y) \mapsto (Y, X)$.

The group $\mathrm{Mod}^+(\overline{\mathbb{M}'}) = \mathrm{SL}(\mathbb{Z}_X \oplus \mathbb{Z}_Y)$ is generated by the positive Dehn twists D_X and D_Y along the simple loops X and Y:

$$D_X \colon (X, Y) \mapsto (X, XY) \qquad D_Y \colon (X, Y) \mapsto (YX, Y)$$

whose relations are generated by $D_X D_Y^{-1} D_X = D_Y^{-1} D_X D_Y^{-1}$ and $(D_X D_Y^{-1})^6 = 1$. It contains $D_X^{-1} D_Y D_X^{-1} = D_S \colon (X, Y) \mapsto (Y, X^{-1})$, the positive rotation of order 4 whose square is the elliptic involution $D_{-1} \colon (X, Y) \mapsto (X^{-1}, Y^{-1})$.

The group $\mathrm{Mod}(\mathbb{M}')$ of diffeotopy classes of diffeomorphisms of $\mathbb{M}'$ fixing the cusp fits into Birman's short exact sequence $1 \to \pi_1(\mathbb{M}') \to \mathrm{Mod}(\mathbb{M}') \to \mathrm{Mod}(\overline{\mathbb{M}'}) \to 1$, and the associated map $\mathrm{Mod}(\overline{\mathbb{M}'}) \to \mathrm{Out}(\pi_1(\mathbb{M}'))$ is an isomorphism. This restricts to $1 \to \mathbb{Z}_X * \mathbb{Z}_Y \to \mathrm{Map}^+(\mathbb{M}') \to \mathrm{SL}_2(\mathbb{Z}_X \oplus \mathbb{Z}_Y) \to 1$.

The action of $\mathrm{Mod}(\overline{\mathbb{M}'}) = \mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{S}H_1(\mathbb{M}'; \mathbb{R}) = \mathbb{R}\mathbb{S}^1$ quotients to an action of $\mathbb{P}\mathrm{Mod}(\overline{\mathbb{M}'}) = \mathrm{PGL}_2(\mathbb{Z})$ on $\mathbb{P}H_1(\mathbb{M}'; \mathbb{R}) = \mathbb{R}\mathbb{P}^1$. (Beware not to confuse it with the action of $\mathrm{Isom}(\mathbb{HP}) = \mathrm{PGL}_2(\mathbb{Z})$ on $\partial\mathbb{HP} = \mathbb{R}\mathbb{P}^1$: we will relate them later.)

We now use this action to determine for a primitive vector $(a, c) \in \mathbb{Z}^2$ the cyclic word in $X\&Y$ associated to the unique simple geodesic of $\mathbb{M}'$ homologous to $X^a Y^c$. After acting by D_S we may assume $a, c \in \mathbb{N}^2$. Expand $\frac{a}{c} = \lfloor n_0, \ldots, n_k \rfloor$ into a continued fraction of even length, and let $A = R^{n_0} \ldots L^{n_k} \in \mathrm{SL}_2(\mathbb{N})$ so that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends $\frac{1}{0}, \frac{0}{1}$ to its last two convergents $\frac{a}{c}, \frac{b}{d}$. Then let $D_A = D_X^{n_0} \ldots D_Y^{n_k}$ act by substitution on the seed X to yield the desired $X\&Y$-word, which is primitive.

This construction adapts to a geodesic of $\mathbb{M}'$ with any slope $(x, y) \in \mathbb{S}H^1(\mathbb{M}'; \mathbb{R})$ to yield its (dual) cutting sequence with the edges of the lattice $H^1(\mathbb{M}'; \mathbb{Z}) = \mathbb{Z}_X \oplus \mathbb{Z}_Y$. First act by S to have $x, y > 0$ and expand $\frac{x}{y} = \lfloor n_0, n_1 \ldots \rfloor$ into a continued fraction. Then let $D_{(x,y)} = D_Y^{n_0} D_X^{n_1} \cdots \in \mathrm{SL}_2(\mathbb{N})$ act on the bi-infinite word $Y^\infty \in \{X, Y\}^{\mathbb{Z}}$. For rational $(x, y) \in \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$, this recovers the periodisation of the finite primitive word, or the dual cutting sequence of an affine translations of the associated line. After replacing $X = LR$ and $Y = RL$, we recover the intersection pattern of the lifted geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ with $\triangle$, whether the slope is rational or not.

For $0/1 = \lfloor\ \rfloor$, apply D_Y^0 to Y^∞ to find Y^∞, whose $L\&R$ conversion $(RL)^\infty$ corresponds to $(\frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 + \sqrt{5}))$. For $1/1 = \lfloor 0, 1 \rfloor$, apply $D_Y^0 D_X^1$ to Y^∞ to find $(YX)^\infty$, whose $L\&R$ conversion $(RLLR)^\infty$ corresponds to $(-\sqrt{2}, \sqrt{2})$. For $\cotan(2\pi/6) = \lfloor 0, 1, \overline{1, 2} \rfloor$, apply $(D_Y D_X)(D_Y D_X^2)^\infty$ to Y^∞...

Proposition 2.1 ([Ser85a]). *Fix positive irrationals* $-1/\alpha_-, \alpha_+$ *and consider the L&R-sequence obtained by concatenating their (transposed) continued fractions.*

The geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ *projects to a geodesic in* $\mathbb{M}'$ *which is simple if and only if the L&R can be paired to form an X&Y-sequence which is Sturmian.*

The simple closed geodesics in $\mathbb{M}'$ *correspond combinatorially to the periodic Sturmian X&Y-sequences, and arithmetically to the Markov quadratic irrationals.*

2.3 The Jacobian map and the Dedekind form

The punctured hyperbolic torus $\mathbb{M}'$ is conformal to the complex elliptic curve $\overline{M'}$ punctured at its origin. The $\mathbb{Z}/6$-Galois symmetry of the cover $\overline{\mathbb{M}'} \to \overline{\mathbb{M}}$ implies that $\overline{M'}$ is isomorphic to the quotient $\mathbb{C}/\mathbb{Z}[j]$ of the complex plane by its hexagonal lattice underlying the ring of Eisenstein integers, which has invariant $\mathfrak{j}(j) = 0$.

The Jacobian map of $\overline{\mathbb{M}'}$ based at the projected cusp $\infty \in \partial\mathbb{HP} \mapsto \infty \in \overline{\mathbb{M}'}$ lifts to an identification $\int_\infty du \colon H_1(\mathbb{M}'; \mathbb{Z}) \subset H_1(\mathbb{M}'; \mathbb{R}) \to \mathbb{Z}\omega_0 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$ where the periods satisfy $\omega_2/\omega_0 = \exp(i2\pi/3)$ and $\frac{1}{2}(\omega_0 \wedge \omega_2) = \pi$ thus $|\omega_0|^2 = 4\pi/\sqrt{3}$.

Note that $\overline{\mathbb{M}'}$ is isomorphic over $\mathbb{C}$ to the affine cubic $y^2 = 4(x - j^0)(x - j^1)(x - j^2)$ with abelian differential $dx/y = -dx/\sqrt{4x^3 - 4}$ whose fundamental periods satisfy:

$$\frac{\varpi_k}{\varpi_0} = j^k \quad \text{and} \quad \varpi_0 = 2\int_\infty^1 \frac{-dx}{\sqrt{4x^3 - 4}} = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} \quad \text{where} \quad \Gamma(x) = \int_0^\infty t^x \exp(-t)\frac{dt}{t}.$$

To be more precise, the complex curve $\mathbb{M}'$ with holomorphic form du is isomorphic over $\mathbb{R}$ to the affine cubic $y^2 = (x + i|\omega_0|j^0)(x + i|\omega_0|j^1)(x + i|\omega_0|j^2)$ with dx/y.

The lift on the universal cover $\mathbb{HP} \to \mathbb{M}'$ of the form du is equal to $C\eta^4(\tau)d\tau$ where $\eta(\tau) = q^{1/24}\prod_1^\infty(1 - q^n)$ is the Dedekind function, related to discriminant modular form by $\Delta = (2\pi)^{12}\eta^{24}$, and $C \in \mathbb{C}^*$. Indeed, the cover $\mathbb{M}' \to \mathbb{M}$ has Galois group $\mathbb{Z}/6$ so du^6 descends on $\mathbb{M}$ and lifts on $\mathbb{HP}$ to a Γ-modular form of weight 12, which must be a multiple of $\Delta(\tau)d\tau^6$. We will compute C later, but for now we may relate its modulus the half-area of $\mathbb{M}'$ by $\pi = \iint_{\triangle(0,1,\infty)}|C\eta^4(\tau)|^2\frac{d\tau \wedge d\overline{\tau}}{\Im(\tau)^2}$.

From the theta-series representation of the Dedkind function [BvdGHZ08, p. 30]:

$$\eta(\tau) = \sum_{n=0}^\infty \chi(n)q^{\frac{1}{24}n^2} \quad \text{we deduce} \quad \eta^4(\tau) = \sum_{n=0}^\infty \psi(n)q^{\frac{1}{24}n}$$

where χ is the unique primitive character $\bmod\, 12$ (that is the Jacobi symbol $\bmod\, 12$, equal to $\chi(12n \pm 1) = 1$, $\chi(12n \pm 5) = -1$ and $\chi(n) = 0$ if $2 \mid n$ or $3 \mid n$), and where $\psi(n)$ is the sum of $\chi(abcd)$ over $\{(a, b, c, d) \in \mathbb{N}^4 \mid a^2 + b^2 + c^2 + d^2 = n\}$.

The appendix A records arithmetic properties of the function $\psi(n)$. They rely on the fact that $\eta^4(6\tau) = q\prod_1^\infty(1 - q^{6n})^4$ is [LMF24, Modular Form 32.2.a.a], the unique normalised cusp eigenform for the group $\Gamma_0(36) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \equiv 0 \bmod 36\}$.

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3 The hexpunctured plane and its compactifications

3.1 Second derived modular group and hexpunctured plane

The abelianisation $\Gamma' \to \Gamma'/\Gamma''$ associated to Hurwicz' map $\pi_1(\mathbb{M}') \to H_1(\mathbb{M}'; \mathbb{Z})$, given by $\mathbb{Z}_X * \mathbb{Z}_Y \to \mathbb{Z}_X \times \mathbb{Z}_Y$, corresponds to a Galois cover of $\mathbb{M}'$ by a lattice-punctured-plane $\mathbb{M}''$. The Jacobian integration map of $\mathbb{M}'$ based at the projected cusp $\infty \in \partial\mathbb{HP} \mapsto \infty \in \partial\mathbb{M}'$ lifts to an identification $\mathbb{M}'' \to H_1(\mathbb{M}'; \mathbb{R})\backslash H_1(\mathbb{M}'; \mathbb{Z})$.

The kernel $\Gamma'' = \pi_1(\mathbb{M}'')$ is freely generated by an infinite set of elements (whose conjugacy classes in Γ' are) indexed by $H_1(\mathbb{M}'; \mathbb{Z}) = \Gamma'/\Gamma''$. For example, the conjugates of $[X, Y]$ by all $X^m Y^n \in \Gamma'$ with $m, n \in \mathbb{Z}$ freely generate Γ''.

We call hexp: $\mathbb{HP} \to \mathbb{M}''$ the quotient by Γ''. It maps the trivalent tree $\mathcal{T} \subset \mathbb{HP}$ to a trivalent graph $\mathcal{H} \subset \mathbb{M}''$, and the vertices $\Gamma/\langle R \rangle \subset \partial\mathbb{HP}$ of $\triangle$ dual to the connected components of $\mathbb{HP}\backslash\mathcal{T}$ to the lattice of cusps $\Gamma''\backslash\Gamma/\langle R \rangle$ in $\mathbb{M}''$. Hence the inclusion $\mathcal{H} \subset \mathbb{M}''$ is a homotopy equivalence, and the regions in the complement $\mathbb{M}''\backslash\mathcal{H}$ are punctured regular hexagons (for the hyperbolic metric).
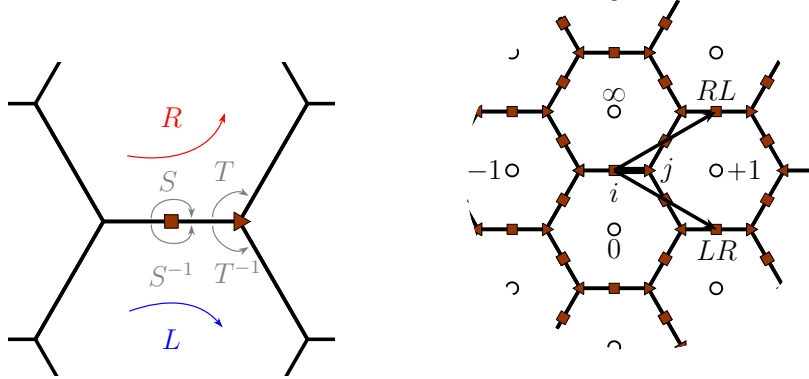
Action of Γ/Γ'' on the hexagonal graph $\mathcal{H}$ and cusps $\Gamma''\backslash\Gamma/\langle R \rangle$.

We now describe the Galois action of Γ/Γ'' on the metabelian cover $\mathbb{M}'' \to \mathbb{M}$ combining the action of Γ'/Γ'' on $\mathbb{M}'' \to \mathbb{M}'$ with the action of Γ/Γ' on $\mathbb{M}' \to \mathbb{M}$.

The group Γ/Γ'' acts freely transitively on the oriented edges of $\mathcal{H}$. More precisely the generators S and T act by rotations of order 2 and 3 around the mid-edges and vertices of $\mathcal{H}$. The subgroup Γ'/Γ'' generated by $X = LR$ and $Y = RL$ acts by translation of $\mathcal{H}$ with fundamental domain a tripod formed by three adjacent edges.

Consequently the action of $A \in \Gamma$ on $\mathcal{H}$ sends the base edge to another edge whose angle is determined by $A \bmod \Gamma'$ and whose base point is determined by $A^6 \bmod \Gamma''$. In particular Γ'/Γ'' identifies with the edges of $\mathcal{H}$ that are parallel to the base edge.

Proposition 3.1 ([Sim22]). *The group Γ/Γ'' is the semi-direct product $\Gamma'/\Gamma'' \rtimes \Gamma/\Gamma'$ where the action of the quotient $\Gamma/\Gamma' = \mathbb{Z}/2 \times \mathbb{Z}/3$ by outer-automophisms on the kernel $\Gamma'/\Gamma'' = \mathbb{Z}_X \oplus \mathbb{Z}_Y$ is given by $\mathrm{AD}_S = D_{S^2} = D_{-1}$ and $\mathrm{AD}_T = D_{T^2}$ since we have $SXS^{-1} = X^{-1}$, $SYS^{-1} = Y^{-1}$ and $TXT^{-1} = Y^{-1}$, $TYT^{-1} = -XY^{-1}$.*

This represents Γ/Γ'' as the affine isometry group of the oriented hexagonal lattice $H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_X \oplus \mathbb{Z}_Y$ with $\mathrm{angle}(X, Y) = \frac{2\pi}{6}$, where the translation action of Γ'/Γ'' is by $\mathbb{Z}^2$-translation while the outer-automorphism action of Γ/Γ' is by $\mathbb{Z}/6$-rotation.

3.2 Cusp-compactification of the hexponential map

The map $\mathrm{hexp} \colon \mathbb{HP} \to \mathbb{M}''$ in the coordinates $\{z \in \mathbb{C} \mid \Im(z) > 0\} \to \mathbb{C}\backslash \omega_0 \mathbb{Z}[j]$ is the primitive of the function $C\eta(z)^4 dz$, with expansion:

$$\mathrm{hexp}(\tau) = \int_\infty^\tau C\eta^4(z)dz = \frac{12C}{i\pi} \sum_{n=1}^\infty \frac{\psi(n)}{n} \cdot q^{\frac{1}{24}n} \tag{hexp}$$

The fundamental period of $\mathbb{M}'$ is $\omega_0 = \frac{1}{i}|\omega_0| = \int_\infty^0 C\eta^4(\tau)d\tau$, and by [nhsn] we have:

$$\int_\infty^0 \eta^4(\tau)d\tau = \frac{1}{i} \frac{3^{1/2}}{2^{7/3}\pi^2}\Gamma(1/3)^3 \quad \text{thus} \quad C = \frac{2^{10/3}}{3^{3/4}} \frac{\pi^{5/2}}{\Gamma(1/3)^3}$$

We deduce from the area of $\mathbb{M}'$ the value of $\iint_{\triangle(0,1,\infty)} |\eta^4(\tau)|^2 \frac{d\tau \wedge d\bar{\tau}}{\Im(\tau)^2} = \frac{\pi}{|C|^2}$.

Observe that the lattice relation $\partial\,\mathrm{hexp}(0) = 2\,\mathrm{hexp}(i)$ expresses the value of an conditionally convergent series and of an absolutely convergent series:

$$\sum_1^\infty \frac{\psi(n)}{n} = 2\sum_1^\infty \frac{\psi(n)}{n}\exp\left(\frac{-\pi}{12}n\right) \quad \text{as equal to} \quad \frac{\pi|\omega_0|}{12C} = \frac{1}{24}\frac{3^{1/2}}{2^{1/3}}\frac{\Gamma(1/3)^3}{2\pi}.$$

Let us generalise the computation of such special values to obtain Theorem 3.2, whose proof will justify the conditional convergence.

Recall that the cusp map $\partial\,\mathrm{hexp} \colon \Gamma/\langle R\rangle \to \Gamma''\backslash\Gamma/\langle R\rangle$ sends the vertices $\mathbb{QP}^1 \subset \triangle$ to the cusps $H_1(\mathbb{M}'; \mathbb{Z}) \subset H_1(\mathbb{M}'; \mathbb{R})$. The concatenation of even continued fraction expansions of rational numbers quotients $\mathrm{mod}\,\Gamma''$ to the addition of cusps twisted by the Γ/Γ'-action (so the twist is trivial if the first rational number belongs to $\Gamma' \cdot \infty$). This yields twisted-addition identities between the values of the $\partial\,\mathrm{hexp}$ at various $\alpha \in \mathbb{QP}^1$, given by the improper integral and conditionally convergent series (hexp).

In particular, for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'$ equal to $X^m Y^n \mod \Gamma''$, we have

$$\partial\,\mathrm{hexp}(a/c) = |\omega_0|\left(m\exp(-\tfrac{i\pi}{6}) + n\exp(+\tfrac{i\pi}{6})\right)$$
$$\partial\,\mathrm{hexp}(b/d) = |\omega_0|\left(m\exp(-\tfrac{i\pi}{6}) + n\exp(+\tfrac{i\pi}{6}) - i\right)$$

since the vectors X, Y act on on $\mathbb{M}'' = \mathbb{C}\backslash(\omega_0\mathbb{Z}[j])$ by translations of $|\omega_0|\exp(\pm\tfrac{i\pi}{6})$. Note that the left-columns of matrices in Γ' yield all rationals.

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Theorem 3.2 (Values of $\partial \operatorname{hexp}$). *Consider $a/c \in \mathbb{QP}^1$ with $\gcd(a, b) = 1$.*

There exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma'$, and we may consider its class $A \equiv X^m Y^n \mod \Gamma''$. The value $\partial \operatorname{hexp}(\frac{a}{c})$ of the improper integral and conditionally convergent series:

$$\partial \operatorname{hexp}(\tfrac{a}{c}) = \int_\infty^{\frac{a}{c}} C\eta^4(z)dz = \tfrac{12C}{i\pi} \sum_{n=1}^\infty \tfrac{\psi(n)}{n} \cdot \exp\left(\tfrac{i\pi}{12}\tfrac{a}{b}n \right) \quad where \quad C = \tfrac{2^{10/3}}{3^{3/4}} \tfrac{\pi^{5/2}}{\Gamma(1/3)^3}$$

is equal to $\quad \partial \operatorname{hexp}(\tfrac{a}{c}) = |\omega_0| \left(m \exp(-\tfrac{i\pi}{6}) + n \exp(+\tfrac{i\pi}{6}) \right) \quad where \quad |\omega_0| = \tfrac{2\pi^{1/2}}{3^{1/4}}.$

Proof. We are left with the conditional convergence of the series $\sum \frac{\psi(n)}{n} e^{i\pi r}$ for $r \in \mathbb{Q}$.

Let us first explain why they are not absolutely convergent.

The values $n \in \mathbb{N}$ such that $\psi(n) \neq 0$ are characterised in [Ser85b, Section 2.2], in terms of the ramification of n in the ring of integers of the quadratic field $\mathbb{Q}(j)$. In particular $\psi(p) \neq 0$ for primes $p \equiv 1 \mod 6$. Moreover, the Dirichlet's density theorem [Ser70, §4.1] implies that the sum $\sum \frac{1}{p}$ over primes $p \equiv 1 \mod 6$ diverges.

We now rely on past works to justify the convergence of the series $\sum_1^\infty \frac{1}{n} \psi(n)$.

By [Lig75, Théorème B (2.2.3)], the Dirichlet L-series $L(\psi, s) = \sum_1^\infty \frac{1}{n^s} \psi(n)$ attached the weight 2 cusp form $\eta^4(6\tau)$ for the group $\Gamma_0(36)$, coincides with the L-series attached to the elliptic modular curve $X_0(36)$. By [Lig75, Théorème C (2.3.2)] the function $\Xi(s) = \left(\frac{6}{2\pi} \right)^s \Gamma(s) L(\psi, s)$ admits an analytic continuation to an entire function which satisfies the functional equation $\Xi(s) = \Xi(2 - s)$. We may thus apply [Mur95, Section 2.2] to deduce that for all $\epsilon > 0$ we have $|\sum_1^N \psi(n)| = o(N^{5/6+\epsilon})$, hence the L-series converges where $\Re(s) \geqslant 5/6$. This proves convergence at $r = 0$.

The convergence at other rational $r \in \mathbb{Q}$ should follow similar lines. First, prove a functional equation for the L-series $\sum_1^\infty \frac{1}{n^s} \psi(n) e^{i2\pi n r}$ and its analytic continuation to an entire function in a vertical strip containing 1. Then, deduce a bound of the form $|\sum_1^N \psi(n) e^{i2\pi n r}| = o(N^{1-\epsilon})$ whence convergence of the L-series at $s = 1$. The closest reference we found are [Yog93, Jut87]. $\qquad\square$

Remark 3.3 (Experimental). *I computed the first 20000 terms of the partial series using PARI GP, and observed that they start to coalesce in some region, but then oscillate in that small region, at an amplitude which seemed to be neither approaching 0 (although the proof reveals they must be), nor monotonous. See A for an explanation.*

Question 3.4 (Convergence in p-adic completions). *One may also ask about convergence in other p-adic completions of $\mathbb{Q}$.*

In this direction, let us note that the function hexp is a non-congruence modular function, namely for the group for Γ''. This implies (see [LL12]) that the denominators of its coefficients $\frac{1}{n}\psi(n)$ are unbounded.

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3.3 Spheric compactification of the hexpunctured plane

A complete geodesic in $\mathbb{HP}$ or $\mathcal{T}$ encoded by a bi-infinite $L\&R$-word maps to a complete geodesic in $\mathbb{M}''$ or $\mathcal{H}$ encoded by the same bi-infinite $L\&R$-word. Observe that a euclidean straight line in $\mathbb{M}''$ determines a unique complete geodesic in $\mathbb{M}''$ both topologically (they are a homotopic relative to endpoints) and combinatorially (they intersect the triangulation $\mathrm{hexp}(\triangle)$ dual to $\mathcal{H}$ in the same bi-infinite sequence of Γ-translates of the triangle $\mathrm{hexp}(0, 1, \infty)$).

The left translation action of Γ at the source quotients to left translation-rotation action of Γ/Γ'' at the target. Hence the primitive conjugacy classes of Γ correspond to the bi-infinite periodic geodesics of $\mathcal{H}$ passing through the oriented base edge. One may construct various geometricombinatorial invariants [Sim22, Section 3.2]. For instance the Rademacher number of a hyperbolic element in Γ can be expressed as the total asymptotic winding number of its axis projected in $\mathcal{H} \subset \mathbb{M}''$.

We define a map $\mathrm{Shexp} \colon \mathscr{R} \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ from the subset of numbers $\alpha \in \mathbb{RP}^1$ such that as $\tau \in \mathbb{HP}$ converges to $\alpha \in \partial\mathbb{HP}$, the modulus $|\mathrm{hexp}(\tau)| \in \mathbb{R}_+$ diverges while the argument $\arg \mathrm{hexp}(\tau) \in \mathbb{R}/(2\pi\mathbb{Z})$ converges to a limit defining $\mathrm{Shexp}(\alpha)$.

Thus $\mathscr{R} \subset \mathbb{R}\backslash\mathbb{Q}$ contains the numbers α such that the geodesic $(i, \alpha) \subset \mathbb{HP}$ maps under hexp to a geodesic of $\mathbb{M}''$ which escapes in a definite direction (it follows the geodesic of $\mathcal{H}$ encoded by the $L\&R$-continued fraction expansion of α).

For a geometric group theorist, an irrational $\alpha = \lfloor n_0, n_1, \ldots \rfloor$ belongs to $\mathscr{R}$ if and only if the orbit of $\infty \in \Gamma''\backslash\Gamma/\langle R\rangle = H_1(\mathbb{M}'; \mathbb{Z})$ under the $A_k = R^{n_0} \ldots L^{n_{2k+1}} \in \Gamma/\Gamma''$ projectifies to a Cauchy sequence in $\mathbb{S}(\Gamma''/\Gamma') = \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$.

The surjective maps $\mathrm{Shexp} \colon \mathscr{R} \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ and $\partial \mathrm{hexp} \colon \mathbb{QP}^1 \to H_1(\mathbb{M}'; \mathbb{Z})$ have disjoint domains and codomains. Together, they define a continuous compactification of $\mathrm{hexp} \colon \mathbb{HP} \to \mathbb{M}''$ (for obvious topologies), which is equivariant under the left actions of Γ and Γ/Γ''. The translation action of Γ' on $\mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ is trivial so $\mathrm{Shexp} \colon \mathscr{R} \bmod \Gamma' \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$ is equivariant under the left actions of Γ and Γ/Γ'.

Remark 3.5 (Essential singularity along the boundary)**.** *If $\tau \in \mathbb{HP}$ diverges to an irrational $\alpha \in \mathbb{RP}^1$ the function $\mathrm{hexp}(\tau)$ diverges.*

The subset $\mathscr{R} \subset \mathbb{RP}^1$ irrational numbers (at which the modulus $|\mathrm{hexp}(\tau)|$ diverges while its argument $\arg(\mathrm{hexp}(\tau))$ converges) is dense in $\mathbb{RP}^1$.

The map $\mathrm{Shexp} \colon \mathscr{R} \to \mathbb{S}H_1(\mathbb{M}', \mathbb{R})$ is uniformly continuous and surjective.

Proof. The proof follows from the combinatorial and topological behaviour of hexp, using the fact that the symbolic encoding of real numbers by continued fractions is surjective and uniformly continuous (if $\alpha, \beta \in \mathbb{R}$ have continued fractions with the same prefix given by $A = R^{n_0} \cdots L^{n_{2k+1}}$ for some $k \in \mathbb{N}$, then $|\alpha - \beta| \leqslant 2/(F_{2k}F_{2k+1})$ where $(F_k)_{k\in\mathbb{N}}$ is the usual Fibonacci sequence). $\square$

4 Transcendant arguments of the hexponential

4.1 The Inshection map and Sturmian numbers

We defined a surjective map $\mathrm{Shexp}\colon \mathscr{R} \bmod \Gamma' \to \mathbb{S}H_1(\mathbb{M}';\mathbb{R})$ where $\mathscr{R} \subset \mathbb{RP}^1\backslash\mathbb{QP}^1$, sending uncountably many distinct Γ'-orbits in $\mathscr{R}$ to the same slope in $\mathbb{S}H_1(\mathbb{M}';\mathbb{R})$. Let us define a section $\mathrm{InSh}\colon \mathbb{S}H_1(\mathbb{M}';\mathbb{R}) \to \mathscr{S} \bmod \Gamma'$, which is Γ/Γ'-equivariant, injective and whose image will have "index at most 2" and the set $\mathscr{S}$ to be defined.

A slope $(x,y) \in \mathbb{S}H_1(\mathbb{M}';\mathbb{R})$ corresponds to a complete simple geodesic $\alpha \subset \mathbb{M}'$: it lifts in $\mathbb{HP}$ to the Γ'-orbit of a geodesic (α_-,α_+) and we let $\mathrm{InSh}(x,y) = \Gamma' \cdot \alpha_+$. The geodesic $\alpha \subset \mathbb{M}'$ lifts in $\mathbb{M}''$ to the Γ'/Γ''-orbit of geodesics which intersect the triangulation $\mathrm{hexp}(\triangle)$ according to the $X^{\pm 1}\&Y^{\pm 1}$ dual cutting sequences of the euclidean lines inside $H_1(\mathbb{M}';\mathbb{R})\backslash H_1(\mathbb{M}';\mathbb{Z})$ of slope (x,y). The $X\&Y$ cutting sequences of such lines are precisely the Sturmian sequences, and two sequences corresponds to a same Γ'/Γ''-orbit if and only if they differ by a shift. Replacing $X = LR$, $Y = RL$ recovers the bi-infinite $L\&R$-sequences giving the continued fraction expansions of the numbers $-1/\alpha_-, \alpha_+$.

Let us remark that in $\mathbb{M}''$, the lifted geodesics $(\alpha_-,\alpha_+) \bmod \Gamma''$ intersect the cusp-to-cusp geodesic $(\infty, 0) \bmod \Gamma''$ a finite number of times if and only if x/y is rational, and they approach the cusps arbitrarily close if and only if the terms in the continued fraction expansion of x/y are unbounded.

We call $\mathscr{S}$ the set of *Sturmian* numbers $\alpha_+, -1/\alpha_-$ obtained as above, namely whose continued fraction expansions arise from Sturmian $X^{\pm 1}\&Y^{\pm 1}$-sequences by replacing $X = RL$ and $Y = RL$. It is endowed with the sign reversing involutions $\sigma\colon \alpha_- \leftrightarrow \alpha_+$ and $S\colon \alpha \mapsto -1/\alpha$, which commute and compose to $\epsilon\colon \alpha_+ \mapsto -1/\alpha_-$.

Note that hexp is also invariant under ϵ, namely $\mathrm{hexp}(\alpha_+) = \mathrm{hexp}(-1/\alpha_-)$, and the map $\mathbb{S}H_1(\mathbb{M}';\mathbb{R}) \to \mathscr{S} \bmod \{\Gamma',\epsilon\}$ is bijective. In this sense the image of InSh has index at most 2 in $\mathscr{S} \bmod \Gamma'$, taking into account the fact that some $\alpha \in \mathscr{S}$ satisfy $\alpha = \epsilon(\alpha) \bmod \Gamma'$. Moreover InSh is Γ/Γ'-equivariant in the sense that

$$\mathrm{InSh\,AD}_S(x,y) = S(\mathrm{InSh}(\theta)) \quad \text{namely} \quad \mathrm{InSh}(-x,-y) = -1/\mathrm{InSh}(\theta)$$
$$\mathrm{InSh\,AD}_T(x,y) = T(\mathrm{InSh}(\theta)) \quad \text{namely} \quad \mathrm{InSh}(-y, x-y) = (\mathrm{InSh}(\theta) - 1)/\mathrm{InSh}(\theta)$$

so InSh descends to a map $\mathbb{S}H_1(\mathbb{M}';\mathbb{R}) \bmod \mathrm{AD}_\Gamma \to \mathscr{S} \bmod \Gamma$.

Remark 4.1 (Cantor set). *The Sturmian numbers $\mathscr{S} \subset \mathbb{RP}^1$ form a Cantor subset: it is compact, perfect, nowhere dense.*

The map $\mathrm{Shexp}\colon \mathscr{S} \to \mathbb{S}H_1(\mathbb{M}';\mathbb{R})$ *is continuous and surjective.*

Proof. As a subset of the 1-sided shift $\{L, R\}^{\mathbb{N}}$, the set of Sturmian sequences form a Cantor set (exercise). The proof then follows from the properties of InSh, and the continuous parametrisation of real numbers by continued fractions. $\qquad\square$

4.2 Transcendance of the Inshection map

We define the subset $\mathscr{M} \subset \mathscr{S}$ of *Markov* numbers as those arising from rational slopes $(x, y) \in \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$, namely those with periodic $X^{\pm 1} \& Y^{\pm 1}$ sequences. These coincide with the Markov quadratic irrationals [Ser85a], and the involution σ restricts to the action by Galois conjugation.

Theorem 4.2 (Transcendance of the Inshection map). *For $(x, y) \in \mathbb{S}H_1(\mathbb{M}; \mathbb{R})$,*

- *if $(x, y) \in \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$ is rational then $\mathrm{InSh}(x, y) \subset \mathscr{M}$ are quadratic*
- *if $(x, y) \notin \mathbb{S}H_1(\mathbb{M}'; \mathbb{Z})$ is irrational then $\mathrm{InSh}(x, y) \subset \mathscr{S}\backslash\mathscr{M}$ are transcendent*

Proof. The proof follows from [ADQZ01, Proposition 3 and Theorem 7], showing that our non-Markov Sturmian numbers are well approximated by Markov numbers, well enough to satisfy the hypothesis in the following theorem of Schmidt [Sch67].

The height of a quadratic irrational ξ with minimal polynomial $ax^2 + bx + c \in \mathbb{Z}[x]$ is defined as $H(\xi) = \gcd(|a|, |b|, |c|)$. A number $s \in \mathbb{R}\backslash\mathbb{Q}$ which is not quadratic must be transcendantal, if for some number $\beta > 3$ there exists infinitely many quadratic irrationals ξ_k such that $|\xi - \xi_k| < H(\xi_k)^{-\beta}$. $\qquad\square$

4.3 Continued fraction expansion of the hexponential

To understand the diophantine properties of the hexponential $\mathrm{hexp}\colon \mathbb{HP} \to \mathbb{M}''$, its compactification $\mathrm{Shexp}\colon \mathscr{R} \to \mathbb{S}H_1(\mathbb{M}', \mathbb{R})$ and its inverse $\mathrm{InSh}\colon \mathbb{S}H_1(\mathbb{M}', \mathbb{R}) \to \mathscr{S}$, we wish to find continued fraction expansions for:

hexp: the complex function $\mathrm{hexp} \circ \exp(i2\pi\cdot)\colon \mathbb{HP} \to \mathbb{C}$
Shexp: the real function $\tan \circ \mathrm{Shexp}\colon \mathscr{R} \to \mathbb{RR}^1$
InSh: the real function $\mathrm{InSh} \circ \arctan\colon \mathbb{RP}^1 \to \mathscr{S}$

The search for a continued fraction for hexp may involve functional equations involving a second companion function, as it is the case for the Rogers-Ramanujan continued fraction, which is a modular function for the congruence subgroup $\Gamma(5) \subset \Gamma$ (hence living on the dodécahedral cover $X(5) \to \mathbb{M}$).

We refer to [Duk05] for more about this.

5 Remarks and further directions of research

5.1 Complex multiplication and real multiplication

The main theorem of class field theory for complex quadratic fields states that the value $j(\tau)$ is algebraic if and only if τ is complex quadratic, in which case $j(\tau)$ generates the maximal unramified abelian extension of the quadratic field $\mathbb{Q}(\tau)$. It was Kronecker's Jugendtraum (dream of youth) to find a generalisation for real quadratic fields, which eventually became Hilbert's 12þproblem. One may consult [Sch98] for the history of that still unsolved problem.

Observe that $j\colon \mathbb{HP} \to \mathbb{M}$ factors through $\mathrm{hexp}\colon \mathbb{HP} \to \mathbb{M}''$, and the aforementionned arithmetic results about $j(\tau)$ also hold for $\mathrm{hexp}(\tau)$.

Our cusp compactification $\partial\,\mathrm{hexp}$ and ray-compactification Shexp give some sense to the limits of the function j at the boundary. One may thus view our theorem 4.2 as providing the algebraic-transcendence dichotomy for the values of InSh (the inverse to the ray-compactification Shexp of hexp). However, the analogy stops here since these values at Markov-irrationals are transcendent, so they do not generate the Hilbert class field. It may be more appropriate to consider functions built on the distances from the "straight-line" geodesics of $\mathbb{M}''$ to its cusp ([CnBG14]).

5.2 Arithmetic and dynamic correlations

Question 5.1 (Rational correlations). *The conditional convergence of the series $\partial\,\mathrm{hexp}(r)$ at rationals $r \in \mathbb{Q}$ implies a strong correlation between $n \mapsto \psi(n)$ and periodic rotations of the circle $n \mapsto rn \bmod 1$. These correlations factor through the cosets $r \in \Gamma''\backslash\mathbb{QP}^1$, and their values are organised according to the lattice $\omega_0\mathbb{Z}[j]$.*

We wonder if this is equivalent to some kind of "Dirichlet equidistribution theorem" for the arithmetic progressions represented by a χ-twisted version of the quadratic form $a^2 + b^2 + c^2 + d^2$ or the divisor function $\sum_{d|n} d$.

Question 5.2 (Sturmian correlations). *We observe that the sequence $\psi(n)/n$ bears very strong (lattice-like) correlations with rational numbers (bounded complexity), and rather strong (Cantor-like) correlations with Sturmian numbers (complexity $l + 1$), and some correlations with the numbers in $\mathscr{R}$.*

We expect that one may characterise the subset of Sturmian numbers $\mathscr{S} \subset \mathscr{R}$ by quantifying how fast $\arg\mathrm{hexp}(\theta)$ converges to $\mathrm{Shexp}(\alpha)$. We wonder if one can provide any arithmetic or diophantine interpretation of the numbers in $\mathscr{R}$.

A On the Fourier coefficients of the hexponential

Let us collect some well known facts about the sequence $\psi(n)$. Our main reference are [Lig75, Ser85b] and [LMF24].

Modularity and growth of the coefficients

The $\psi(n)$ correspond to the Fourier coefficients of the weight 2 modular form $\eta^4(6\tau)$, which is [LMF24, Modular Form 32.2.a.a], the unique normalised Hecke cusp form for the group $\Gamma_0(36) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma \mid c \equiv 0 \bmod 36\}$.

It has an associated Dirichlet series $L(\psi, s) = \sum_1^\infty \frac{1}{n^s} \psi(n)$.

Théorème B (2.2.3) The Dirichlet series $L(\psi, s)$ attached to the group $\Gamma_0(36)$ coincides with the L-series attached to the elliptic modular curve $X_0(36)$ (with Weierstrass form $y^2 = x^3 - 1$, having and complex-multicplication by $\mathbb{Z}[j]$).

Théorème C (2.3.2) The function $\Xi(s) = \left(\frac{6}{2\pi}\right)^s \Gamma(s) L(\psi, s)$ admits an analytic continuation to an entire function which is bounded in the vertical strip and satisfies the functional equation $\Xi(s) = \Xi(2 - s)$.

Théorème F (6.3.3) The analytic continuation of the function $L(\psi, s)$ converges at $s = 1$ to a constant divided by the square of the order of its group of rational points $X_0(36)(\mathbb{Q})$, which is finite by Théorème E.

In particular the sequence $\psi(n)$ satisfies the Ramanujan-Petersson conjecture [Del71] saying that $|\psi(p)| \leqslant 2\sqrt{p}$ for all prime $p \in \mathbb{N}$.

L-series and Euler product

The ring of integers $\mathbb{Z}[j]$ of the quadratic field $\mathbb{Q}(\sqrt{3})$ has Euclidean algorithms, in particular it is a principal ideal domain, hence a unique factorisation domain.

The ramification pattern of a prime $p \in \mathbb{Z}$ in $\mathbb{Z}[j]$ is given by its residue $p \bmod 3$ as follows. The prime 3 is ramified as $3 = -(1 + j)^2$. When $p = -1 \bmod 3$ it is inert, namely it remains prime in $\mathbb{Z}[j]$, and has norm p^2. When $p = +1 \bmod 3$ it splits in $\mathbb{Z}[j]$ as a product $p = \mathfrak{p}\bar{\mathfrak{p}}$ of conjugate primes with $\mathrm{Nr}(\mathfrak{p}) = p$.

The *Hecke Grossencharacter* $\mathfrak{c}$ of discriminant -3 and conductor $2\sqrt{-3}$ is a character on the multiplicative group of fractional ideals of $\mathbb{Q}(j)$ coprime to the conductor $2\sqrt{-3}$ defined as follows. For an ideal $\mathfrak{a} \subset \mathbb{Z}[j]$ coprime to $2\sqrt{-3}$, we have $\mathfrak{c}(\mathfrak{a}) = \alpha$ where α is the unique generator of $\mathfrak{a}$ such that $\alpha = 1 \bmod 2\sqrt{-3}$. In particular for a prime $p \in \mathbb{Z}$ coprime to 6 we have $\mathfrak{c}(p) = \pm 1$ according to whether $p = \pm 1 \bmod 3$.

We have the L-series representation where the product is over all primes $\mathfrak{p} \in \mathbb{Z}[j]$ coprime to the conductor $2\sqrt{-3}$:

$$L(\psi, s) = \sum_1^\infty \frac{\psi(n)}{n^s} = \prod_{\mathfrak{p}} \frac{1}{1 - \mathfrak{c}(p)\operatorname{Nr}(p)^s}$$

One may rewrite this as the product is over all primes $p \in \mathbb{Z}$ coprime to 6:

$$L(\psi, s) = \sum_1^\infty \frac{\psi(n)}{n^s} = \prod_p L_p(\psi, s)$$

where the local factors are given by $L_p(\psi, s) = (1 + p \cdot p^{-2s})^{-1}$ if $p \neq 2$ is inert and $L_p(\psi, s) = (1 - (\mathfrak{c}(\mathfrak{p}) + \mathfrak{c}(\bar{\mathfrak{p}}))p^{-s} + p \cdot p^{-2s})^{-1}$ if $p = \mathfrak{p}\bar{\mathfrak{p}}$ is split. Hence the multiplicative function $\psi \colon \mathbb{N} \to \mathbb{Z}$ is determined on powers of primes p^n for $n \in \mathbb{N}$ by:

except: If $p \in \{2, 3\}$ then $\psi(p^n) = 0$.
inert: If $p \neq 2$ is inert then $\psi(p^n) \in \{0, 1\}$ is equal to the parity $n \bmod 2$.
split: If $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits then $\psi(p^n) = \mathfrak{c}(\mathfrak{p})^n + \mathfrak{c}(\mathfrak{p})^{n-1}\mathfrak{c}(\bar{\mathfrak{p}}) + \ldots \mathfrak{c}(\mathfrak{p})\mathfrak{c}(\bar{\mathfrak{p}})^{n-1}\mathfrak{c}(\bar{\mathfrak{p}})^n$

Lacunarity

The [Ser85b, Section 2.2] characterises the values $n \in \mathbb{N}$ such that $\psi(n) = 0$ in terms of the ramification of n in the ring of integers $\mathbb{Z}[j]$ of the quadratic field $\mathbb{Q}(j)$:

> For $n \in \mathbb{N}^*$ we have $\psi(n) = 0$ if and only if there exists a prime $\mathfrak{p} \in \mathbb{Z}[j]$ with $\mathfrak{p} \equiv -1 \bmod 3$ such that the $\mathfrak{p}$-adic valuation of n is odd.

Then Serre deduces [Ser85b, Théorème 2(ii)] from an application of the Chebotarev density theorem in [Ser81, Proposition 18], that there exists $c_4 \in \mathbb{R}_+^*$ (for which an explicit value is given) such that the following asymptotic holds:

$$\frac{1}{N}\operatorname{Card}\{n \in \mathbb{N} \mid \psi(n) \neq 0, n \leqslant N\} \sim \frac{c_4}{\log(N)^{1/2}}$$

In particular the sequence ψ is lacunary in the sense that the left hand side converges to 0, which was known by [Ram00].

As Remarked in [Ser85b, page 217], this lacunarity is barely pronounced, and this is partly why we observed in Remark 3.3 a slow convergence of the exponential sums.

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