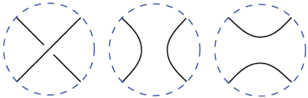
Valuations on the Character Variety

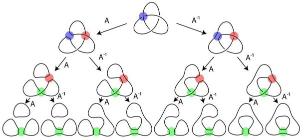
Newton Polytopes and Residual Poisson bracket

Christopher-Lloyd Simon
(in collaboration with Julien Marché)

Laboratoire Paul Painlevé, Université de Lille

17/05/2022





The $\mathsf{SL}_2(\mathbb{C})$-character variety of a finitely generated group π

Definition: Character variety $X(\pi)$ of a finitely generated group π

- *Representation variety* $\mathsf{Hom}(\pi, \mathsf{SL}_2(\mathbb{C}))$, it admits an algebraic action $\mathsf{SL}_2(\mathbb{C})$ by conjugacy at the target.
- *Character variety* $X(\pi) = \mathsf{Hom}(\pi, \mathsf{SL}_2(\mathbb{C}))//\,\mathsf{SL}_2(\mathbb{C})$ is the the algebraic quotient $= \mathsf{Spec}(\mathrm{Invariant\ Functions})$
- For $\alpha \in \pi$, invariant function: $t_\alpha \colon \rho \mapsto \mathsf{Tr}(\rho(\alpha))$

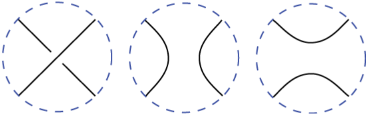
Theorem [CP17]: Presentation of the algebra $\mathbb{C}[X(\pi)]$ of characters

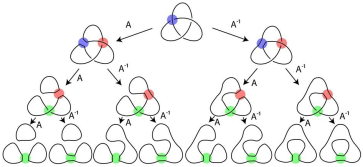
The algebra $\mathbb{C}[X(\pi)]$ of invariant functions has

- generators the $t_\alpha \colon [\rho] \mapsto \mathsf{Tr}(\rho(\alpha))$ for $\alpha \in \pi$
- ideal of relations generated by $t_1 - 2$ and $t_\alpha t_\beta - t_{\alpha\beta} - t_{\alpha\beta^{-1}}$

For π the fundamental group of a closed oriented surface Σ

For loop γ with s self-intersections, apply trace relations to decompose :





$$(-t_\alpha)(-t_\beta)+(-t_{\alpha\beta})+(-t_{\alpha\beta^{-1}})=0$$

$$\boxed{-t_\gamma=(-1)^s\sum_{2^s\text{ states}}\prod_{\text{circles}}-t_{\mu_j}}$$

Theorem [PS00]: Linear basis for the algebra $\mathbb{C}[X(\Sigma)]$

- *Multicurve* $\mu\subset\Sigma$: disjoint union of simples curves μ_j and $t_\mu=\prod t_{\mu_j}$.
- The t_μ for $\mu\in\mathrm{MC}$ form a linear basis of the algebra $\mathbb{C}[X(\Sigma)]$.

Question: what does the decomposition of t_γ look like ?

Example (Tchebychev): inside an immersed annulus Σ_0^2

- Fundamental group $\pi = \mathbb{Z}$ is free on one generator α

$$\mathbb{C}[X(\Sigma_0^2)] = \mathbb{C}[x]$$

- $\mathrm{Tr}(\alpha^n) = 2T_n(x/2)$ Tchecbychev polynomial of the first kind.

Example (Fricke): inside an embedded three holed sphere Σ_0^3

- Fundamental group $\pi = \langle a, b, c \mid abc = 1 \rangle$ is free on two generators,

$$\mathbb{C}[X(\Sigma_0^3)] = \mathbb{C}[t_a, t_b, t_c] = \mathbb{C}[x, y, z]$$

- Diagram computation: $\mathrm{Tr}([a, b]) = x^2 + y^2 + z^2 - xyz - 2$

Theorem [MS22]: Trace functions of multiloops are unitary

For all $\alpha_j \in \pi_1(\Sigma)$, the polynomial $\prod t_{\alpha_j} \in \bigoplus_{MC} \mathbb{Z} \cdot t_\mu$ is *unitary*.

Valuations and simple valuations

Strategy to study decomposition of functions in the linear basis

- Define "monomial" valuations with respect to the linear basis MC
- Define the Newton set of f as the "extremal points" in its support

Definition [MS21]: Valuations on $\mathbb{C}[X(\Sigma)]$ centred at infinity

A valuation is $v\colon \mathbb{C}[X] \to \{-\infty\} \cup \mathbb{R}_+$ satisfying for all f, g:

$$v(f) = -\infty \iff f = 0$$

$$v(fg) = v(f) + v(g)$$

$$v(f + g) \leq \max\{v(f), v(g)\}$$

Weak topology: pointwize convergence of the $v(f)$ for $f \in \mathbb{C}[X]$.

Definition [MS21]: Simple valuation ("monomial" w.r.t. linear basis)

A valuation $v\colon \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup \mathbb{R}_+$ is *simple* when for all $f = \sum m_\mu t_\mu$:

$$v(f) = \max\{v(t_\mu) \mid m_\mu \neq 0\}$$

Simple valuations are measured laminations

Proposition [MS21]: Simple valuations are the completion of $\mathbb{Q}\mathrm{MC}$

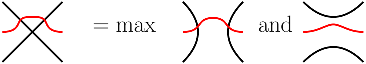
For $\lambda \in \mathrm{MC}$, there exists a unique simple valuation v_λ such that

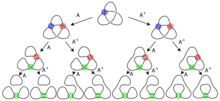
$$\forall \alpha \in \pi_1(\Sigma): \quad v_\lambda(t_\alpha) = i(\lambda, \alpha)$$

The set of simple valuations ML is equal to the completion of $\mathbb{Q}\mathrm{MC}$.

- Well defined by D. Thurston intersection formula:

$$i(\lambda, \alpha) = \bigvee_{\mu} \sum_{\mu_j} i(\lambda, \mu_j) = \max\{i(\lambda, \mu) \mid \text{states}\, \mu\} = v_\lambda(t_\alpha)$$





- Morphism $v(fg) = v(f) + v(g)$ deduced from integrality of $\bigoplus_{n\in\mathbb{N}} F_n/F_{n-1}$ where $F_n = \mathrm{Span}\{t_\alpha \mid \alpha \in \pi_1(\Sigma),\ i(\lambda, \alpha) \le n\}$.

Most simple valuations are strict

Thurston-Masur volume on the space ML

The topological space ML admits (a PL-structure of dim $6g-6$ and) a unique $\mathrm{Mod}(\Sigma)$-invariant Borelian measure up to scaling.

Defined on open subsets $U \subset \mathrm{ML}$ by: $\quad \mathrm{Vol}(U) = \lim_{r\to\infty} \frac{\mathrm{Card}(r \cdot U \cap \mathrm{MC})}{r^{6g-6}}$

Definition [MS21]: Strict valuations (implies simple and positive)

A valuation $v \colon \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup \mathbb{R}_+$ is *strict* when for all $\mu, \nu \in \mathrm{MC}$:

$$\mu \neq \nu \implies v(t_\mu) \neq v(t_\nu)$$

This implies in particular that it is simple, and that $v(t_\mu) > 0$ for all $\mu \neq \emptyset$.

Proposition [MS21]: Most simple valuations are strict.

The set of strict valuations has full measure in ML.

Newton set of a function

Definitions: Support, Extremal multicurve, Newton Set

The *support* of $f = \sum m_\mu t_\mu \in \mathbb{C}[X(\Sigma)]$ is $\mathrm{Supp}(f) = \{\mu \in \mathrm{MC}, m_\mu \neq 0\}$.

- A multicurve $\mu \in \mathrm{Supp}(f)$ is *extremal* in f if there exists a multicurve λ such that $i(\lambda, \mu) > i(\lambda, \nu)$ for all $\nu \in \mathrm{Supp}(f)$ distinct from μ.

  Equivalently, there exists a strict $v \in \mathrm{ML}$ such that $v(t_\mu) = v(f)$.
- The *Newton set* $\Delta(f)$ of f is the set of extremal multicurves in f.

Theorem [MS22]: Trace functions of multiloops are unitary

For all $\alpha_j \in \pi_1(\Sigma)$, the polynomial $f = \prod t_{\alpha_j} \in \mathbb{C}[X(\Sigma)]$ is *unitary*:

$$\forall \mu \in \Delta(f): \quad m_\mu = \pm 1$$

(Proof: Show that for strict valuation v and multiloop α with smoothings α_-, α_+ at an intersection we show $v(t_{\alpha_-}) \neq v(t_{\alpha_+})$.)

Residual value of a function at a strict valuation

Extend $v \in \mathrm{ML}$ to $v \colon \mathbb{C}(X) \to \{-\infty\} \cup \mathbb{R}$ by $v(f/g) = v(f) - v(g)$.

- *Group of values* $\Lambda_v = v(\mathbb{C}(X))$, and *rational rank* $\dim \mathbb{Q} \otimes \Lambda_v$.
- The *transcendence degree* of its *residue field* $k_v = \mathcal{O}_v / \mathcal{M}_v$.

Abhyankar inequality: $\mathrm{rat.\,rk}(v) + \mathrm{tr.\,deg}(k_v) \leq \dim(X) = 6g - 6$

Proposition [MS22]: $\mathrm{strict} \iff \mathrm{tr.\,deg} = 0 \iff \mathrm{rat.\,rk} = 6g - 6$

For a simple valuation $v \in \mathrm{ML}$ the following properties are equivalent:

strict, that is $\forall \mu, \nu \in \mathrm{MC} : \ \mu \neq \nu \implies v(t_\mu) \neq v(t_\nu)$

minimal transcendence degree: $\mathrm{tr.\,deg}(k_v) = 0$, or $k_v = \mathbb{C}$.

maximal rational rank: $\mathrm{rat.\,rk}(v) = 6g - 6 = \dim(X) = \dim(\mathrm{ML})$.

Definition: residual value at a strict valuation $v \in \mathrm{ML}$ of $f \in \mathcal{O}_v$

The *residual value* $f_v \in \mathbb{C}$ is defined as $(f \bmod \mathcal{M}_v) \in k_v$.
It equals the coefficient m_μ of t_μ for $\mu \in \Delta(f)$ such that $v(f) = v(t_\mu)$.

Quest: study the structure constants for multiplication

The linear basis of MC is not monomial: $i(\mu,\nu) \neq 0 \implies t_\mu t_\nu \neq t_\xi$.
What are the structure constants $c_{\mu\nu}^{\xi}$ for multiplication ?

$$\mathbb{C}[X(\Sigma)] = \bigoplus_{\mu \in \mathrm{MC}} \mathbb{C} \cdot t_\mu \qquad t_\mu t_\nu = \sum_{\xi \in \mathrm{MC}} c_{\mu\nu}^{\xi} t_\xi$$

Example [FG00]: In the torus Σ_1 with the bracelet basis

- Fundamental group $\pi = \langle a, b \mid [a,b] = 1 \rangle \simeq \mathbb{Z}^2$ is abelian.
- Characters $\simeq$ representations $a \mapsto \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}$ and $b \mapsto \begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix}$.
- Triangular change of basis from multicurves to bracelets $T_{p,q} = \mathrm{Tr}(a^p b^q)$ for $p \wedge q = 1$ and $T_{np,nq} = Tcheb_n(T_{p,q})$:

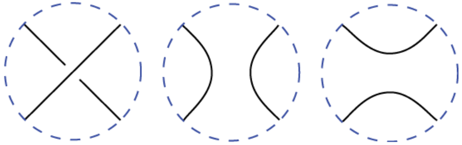
$$\mathbb{C}[X(\Sigma_1)] = \mathbb{C}\left[x^{\pm 1}, y^{\pm 1}\right]^{\sigma} = \bigoplus_{\mathbb{Z}^2/\pm 1} \mathbb{C} \cdot T_{p,q}$$

- Product to sum: $T_{p,q} \cdot T_{r,s} = T_{p+r,q+s} + T_{p-r,q-s}$

The Luo products are extremal multicurves

Definition [Luo10]: Luo product of multicurves

For $\mu, \nu \in \mathrm{MC}$, define $L_\mu(\nu)$ from $\mu \cup \nu$ by smoothing intersections with left turns as we travel along segments of μ which meet segments of ν.



$\mu \cup \nu$ $L_\mu(\nu)$ $L_\nu(\mu)$

Proposition [MS22]: Luo products are extremal multicurves of $t_\mu t_\nu$

For all $\mu, \nu \in \mathrm{MC}$ such that $i(\mu, \nu) > 0$, the Luo products $L_\mu(\nu)$ and $L_\nu(\mu)$ are distinct, and both belong to $\Delta(t_\mu t_\nu)$, with coefficients $(-1)^{i(\mu,\nu)}$.

Poisson algebra structure on $\mathbb{C}[X(\Sigma)]$

Theorem [Gol86]: Poisson bracket on $\mathbb{C}[X(\Sigma)]$

The Atiyah-Bott-Weil-Petersson-Goldman symplectic structure on X defines a Poisson bracket on $\mathbb{C}[X(\Sigma)]$. For $\alpha, \beta \in \pi_1(S)$ it is given by

$$\{t_\alpha, t_\beta\} = \sum_{p \in \alpha \cap \beta} \epsilon_p \left(t_{\alpha_p \beta_p} - t_{\alpha_p \beta_p^{-1}} \right)$$

where the sum ranges over all intersection points p between transverse representatives for $\alpha \cup \beta$ and ϵ_p is the sign of such an intersection, while α_p, β_p denote the homotopy classes of α, β based at p.

$$\{t_\alpha, t_\beta\} = \sum_\xi w_\xi t_\xi = \sum_\xi \left(\sum_{\sigma_\xi} \prod_p \sigma_\xi(p) \right) t_\xi \qquad \text{(PB-state-sum)}$$

where $w_\xi = \sum_{\sigma_\xi} \prod_p \sigma_\xi(p)$ is the sum over the smoothings $\sigma_\xi \colon \alpha \cap \beta \to \{\pm 1\}$ of $\alpha \cup \beta$ yielding the multiloop ξ.

Newton set of the Poisson bracket

Corollary [MS22]: "$\Delta(\{f, g\}) \subset \Delta(fg)$"

For $f, g \in \mathbb{C}[X]$, we have $v(\{f, g\}) \leq v(fg)$ for all $v \in \mathrm{ML}$.
This property amounts to the inverse inclusion of the dual polytopes:

$$\Delta^*(\{f, g\}) \supset \Delta^*(fg)$$

Hence the Goldman Poisson bracket induces a residual Poisson bracket at any strict valuation v. (This endows $T_v\mathrm{ML}$ with a symplectic structure...)

Proof: Apply unitarity of $t_\alpha t_\beta$ and (PB-state-sum) formula for $\{t_\alpha, t_\beta\}$.

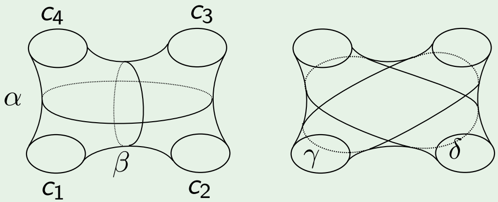
Corollary [MS22]: Luo products are extremal multicurves of $\{t_\mu, t_\nu\}$

For all $\mu, \nu \in \mathrm{MC}$ such that $i(\mu, \nu) > 0$, the Luo products $L_\mu(\nu)$ and $L_\nu(\mu)$ are distinct, and both belong to $\Delta(\{t_\mu, t_\nu\})$, with coefficients $\pm i(\mu, \nu)$.

Proof: The (PB-state-sum) formula implies $L_\mu(\nu), L_\nu(\mu) \in \Delta(\{t_\mu, t_\nu\})$.

Poisson algebra structure on $\mathbb{C}[X(\Sigma_0^4)]$

Example: Product and Poisson bracket of $\alpha, \beta \subset \Sigma_0^4$ with $i(\alpha, \beta) = 2$



The Luo product are $L_\alpha(\beta) = \delta$ and $L_\beta(\alpha) = \gamma$ and

$$t_\alpha t_\beta = t_{c_1} t_{c_3} + t_{c_2} t_{c_4} - t_\gamma - t_\delta \qquad \{t_\alpha, t_\beta\} = 2t_\delta - 2t_\gamma$$

$$\Delta(t_\alpha t_\beta) = \{c_1 \cup c_3, c_2 \cup c_4, \gamma, \delta\} \qquad \Delta(\{t_\alpha, t_\beta\}) = \{\gamma, \delta\}$$

The Newton set of $t_\alpha t_\beta$ decomposes ML into 4 domains where $i(\lambda, \alpha \cup \beta)$ equals the intersection of λ with $c_1 \cup c_3$ or $c_2 \cup c_4$ or γ or δ respectively. In the interior of these domains $\{t_\alpha, t_\beta\}$ has residual values $0, 0, -2, 2$.

Mirzakhani asymptotics as volumes of Newton Polytopes*

Topological interpretation of $\operatorname{Vol}\Delta^*(t_\alpha)$.

For a multiloop α, can we give a topological interpretation for the Thurston-Masur volume $\operatorname{Vol}\Delta^*(t_\alpha)$?
It vanishes unless α is *filling*, meaning it intersects every simple curve, in which case for every other filling multiloop β we have:

$$\lim_{r\to\infty} \frac{\operatorname{Card}\{\varphi \in \operatorname{Mod}(S) \mid i(\lambda, \varphi(\alpha)) \leq r\}}{r^{6g-6}} = \frac{\operatorname{Vol}\Delta^*(t_\beta)\operatorname{Vol}\Delta^*(t_\alpha)}{m_g}$$

Computation in terms of elementary cones in ML indexed by $\Delta(f)$

Identification between measured laminations and simple valuations implies

$$\forall f \in \mathbb{C}[X(\Sigma)]: \quad \Delta^*(f) = \bigcap_{\mu\in\operatorname{Supp}(f)} \Delta^*(t_\mu) = \bigcap_{\mu\in\Delta(f)} \Delta^*(t_\mu)$$

The $\Delta^*(t_\mu)$ are described by explicit sets of linear inequalities in any PL chart of ML, and the volume of their intersection is computable.

Bibliography

C. De Concini and C. Procesi.
The invariant theory of matrices, volume 69.
AMS, 2017.

C. Frohman and R. Gelca.
Skein modules and the noncommutative torus.
Trans. AMS, 352:4877–4888, 2000.

W. Goldman.
Invariant functions on Lie groups and Hamiltonian flows of surface group representations.
Inventiones Mathematicae, 85(2):263–302, 1986.

F. Luo.
Simple loops on surfaces and their intersection numbers.
J. Differential Geometry, 85:73–115, 2010.

Julien Marché and Christopher-Lloyd Simon.
Automorphisms of character varieties.
Annales Henri Lebesgue, 4:591–603, 2021.

J. Marché and C.-L. Simon.
Valuations on the character variety: Newton polytopes and Residual Poisson Bracket, 2022.
Submitted for publication.

J. Przytycki and A. Sikora.
On skein algebras and $SL_2(\mathbb{C})$-character varieties.
Topology, 39(1):115–148, 2000.

Thank you for your attention and feel free to ask (m)any questions !

Teichmüller space embeds in real locus of character variety

1. The Teichmüller space of Σ is the space of complex structures on Σ.
2. By the uniformisation theorem, every complex structure on Σ is conformal to a unique hyperbolic structure.
3. A hyperbolic structure on Σ is uniquely determined by its holonomy representation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$, well defined up to conjugacy.
4. These correspond to the Fuchsian representations, or by Milnor-Wood inequalities to those with extremal euler class $\pm\chi = \pm(2-2g)$.
5. If $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ has even euler class then it lifts to $\mathrm{SL}_2(\mathbb{R})$, so there are 2×2^{2g} copies of Teichmüller space in $X(\Sigma)$.
6. Teichmüller space of Σ is Zariski dense in the character variety $X(\Sigma)$ (as Fuchsian representations form open subset of $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{R}))$, which quotient to open subset of $X(\pi, \mathrm{SL}_2(\mathbb{R}))$.)
7. Trace function of loop $\leftrightarrow$ length of the unique geodesic:

$$t_\alpha([\rho]) = 2\cosh(l_\alpha(m)/2)$$