

1 Some counterexamples to naive degree conjectures

1.0.1 [(12, 1, 13, 2), (6, 3, 7, 4), (10, 5, 11, 6), (14, 7, 15, 8), (4, 9, 5, 10), (16, 11, 1, 12), (8, 13, 9, 14), (2, 15, 3, 16)]

PD code drawn by SnapPy: [(3, 16, 4, 1), (7, 2, 8, 3), (11, 4, 12, 5), (1, 6, 2, 7), (13, 8, 14, 9), (5, 10, 6, 11), (15, 12, 16, 13), (9, 14, 10, 15)]

Planar representation generated by plantri: -

Total optimal pinning sets: 1
Total minimal pinning sets: 6
Total pinning sets: 104
Pinning number: 4

Average optimal degree: 2.5
Average minimal degree: 2.48
Average overall degree: 2.92

Table 1: Pinning sets/average degree by cardinal

Cardinal	4	5	6	7	8	9	10	Total
Optimal pinning sets	1	0	0	0	0	0	0	1
Minimal (suboptimal) pinning sets	0	5	0	0	0	0	0	5
Nonminimal pinning sets	0	6	29	34	21	7	1	98
Average degree	2.5	2.62	2.83	2.97	3.07	3.14	3.2	

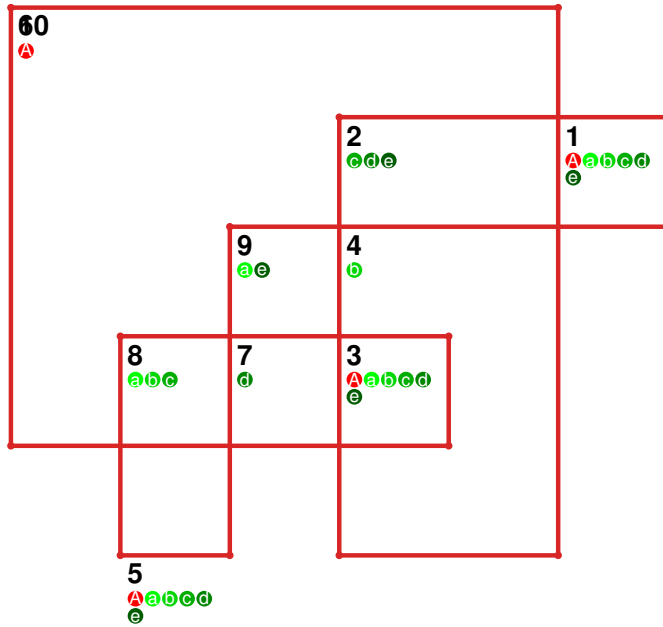


Figure 1: SnapPy multiloop plot.

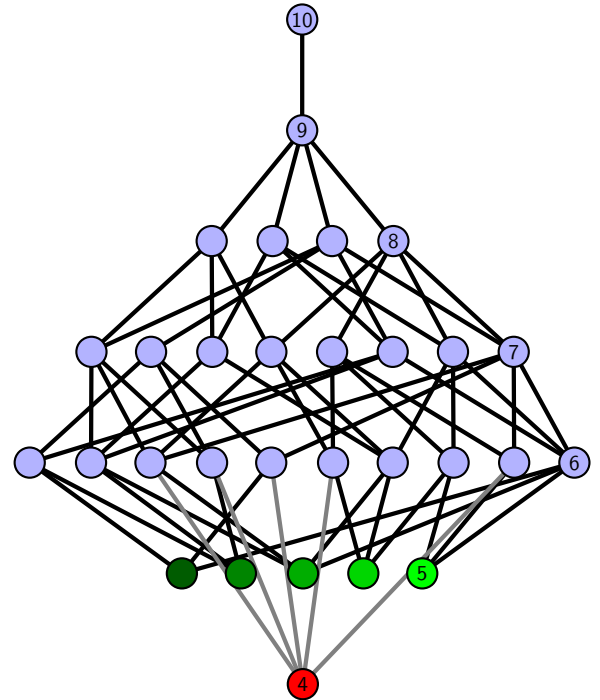


Figure 2: Minimal join sub-semi-lattice of minimal pinning sets.

Table 2: Pinning set data

Pinning set	Pindicator	Regions	Card	Degree seq	Average degree
A (optimal)	●	$\{1, 3, 5, 10\}$	4	$[2, 2, 2, 4]$	2.5
a (minimal)	●	$\{1, 3, 5, 8, 9\}$	5	$[2, 2, 2, 3, 3]$	2.4
b (minimal)	●	$\{1, 3, 4, 5, 8\}$	5	$[2, 2, 4, 2, 3]$	2.6
c (minimal)	●	$\{1, 2, 3, 5, 8\}$	5	$[2, 3, 2, 2, 3]$	2.4
d (minimal)	●	$\{1, 2, 3, 5, 7\}$	5	$[2, 3, 2, 2, 4]$	2.6
e (minimal)	●	$\{1, 2, 3, 5, 9\}$	5	$[2, 3, 2, 2, 3]$	2.4

1.0.2 $[[1, 4, 18, 3], [1, 12, 2, 13], [4, 10, 5, 9], [18, 9, 17, 8], [3, 11, 2, 12], [13, 11, 14, 10], [5, 14, 6, 15], [17, 7, 16, 8], [6, 16, 7, 15]]$

PD code drawn by **SnapPy**: $[(16, 1, 17, 2), (5, 14, 6, 15), (15, 6, 16, 7), (7, 2, 8, 3), (8, 11, 9, 12), (9, 18, 10, 1), (17, 10, 18, 11), (3, 12, 4, 13), (13, 4, 14, 5)]$

Planar representation generated by **plantri**: -

Total optimal pinning sets: 1
Total minimal pinning sets: 5
Total pinning sets: 100
Pinning number: 5

Average optimal degree: 2.4
Average minimal degree: 2.35
Average overall degree: 2.93

Table 3: Pinning sets/average degree by cardinal

Cardinal	5	6	7	8	9	10	11	Total
Optimal pinning sets	1	0	0	0	0	0	0	1
Minimal (suboptimal) pinning sets	0	4	0	0	0	0	0	4
Nonminimal pinning sets	0	6	27	33	21	7	1	95
Average degree	2.4	2.53	2.79	2.98	3.11	3.2	3.27	

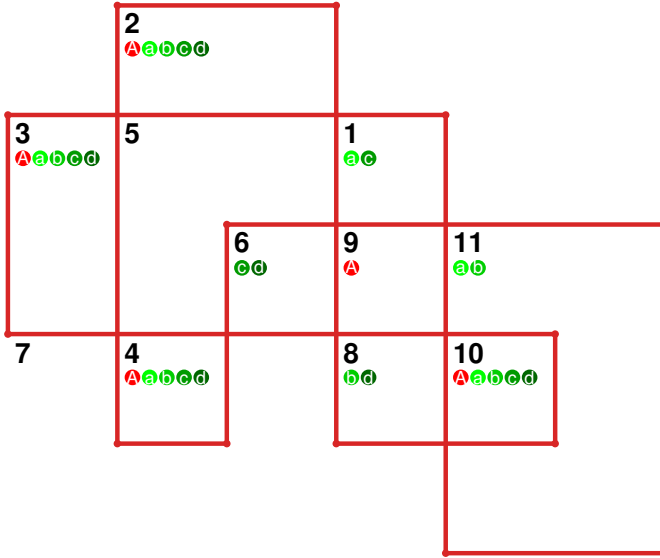


Figure 3: SnapPy multiloop plot.

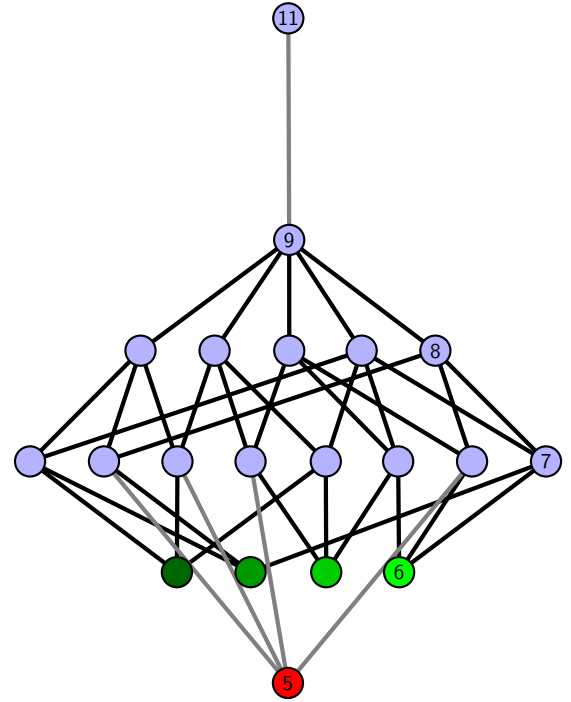


Figure 4: Minimal join sub-semi-lattice of minimal pinning sets.

Table 4: Pinning set data

Pinning set	Pindicator	Regions	Card	Degree seq	Average degree
A (optimal)	●	{2, 3, 4, 9, 10}	5	[2, 2, 2, 4, 2]	2.4
a (minimal)	●	{1, 2, 3, 4, 10, 11}	6	[3, 2, 2, 2, 2, 3]	2.33
b (minimal)	●	{2, 3, 4, 8, 10, 11}	6	[2, 2, 2, 3, 2, 3]	2.33
c (minimal)	●	{1, 2, 3, 4, 6, 10}	6	[3, 2, 2, 2, 3, 2]	2.33
d (minimal)	●	{2, 3, 4, 6, 8, 10}	6	[2, 2, 2, 3, 3, 2]	2.33

1.0.3 $[[1, 5, 20, 6], [1, 14, 2, 15], [5, 11, 4, 12], [20, 12, 19, 13], [6, 13, 7, 14], [2, 9, 3, 10], [15, 10, 16, 11], [4, 16, 3, 17], [19, 8, 18, 7], [9, 18, 8, 17]]$

PD code drawn by SnapPy: $[(3, 18, 4, 19), (16, 5, 17, 6), (20, 7, 1, 8), (13, 8, 14, 9), (9, 2, 10, 3), (10, 17, 11, 18), (4, 11, 5, 12), (19, 12, 20, 13), (14, 1, 15, 2), (6, 15, 7, 16)]$

Planar representation generated by plantri: -

Total optimal pinning sets: 1
Total minimal pinning sets: 8
Total pinning sets: 536
Pinning number: 4

Average optimal degree: 3.0
Average minimal degree: 2.83
Average overall degree: 3.19

Table 5: Pinning sets/average degree by cardinal

Cardinal	4	5	6	7	8	9	10	11	12	Total
Optimal pinning sets	1	0	0	0	0	0	0	0	0	1
Minimal (suboptimal) pinning sets	0	6	0	1	0	0	0	0	0	7
Nonminimal pinning sets	0	8	60	127	157	115	49	11	1	528
Average degree	3.0	2.97	3.06	3.14	3.21	3.26	3.3	3.32	3.33	

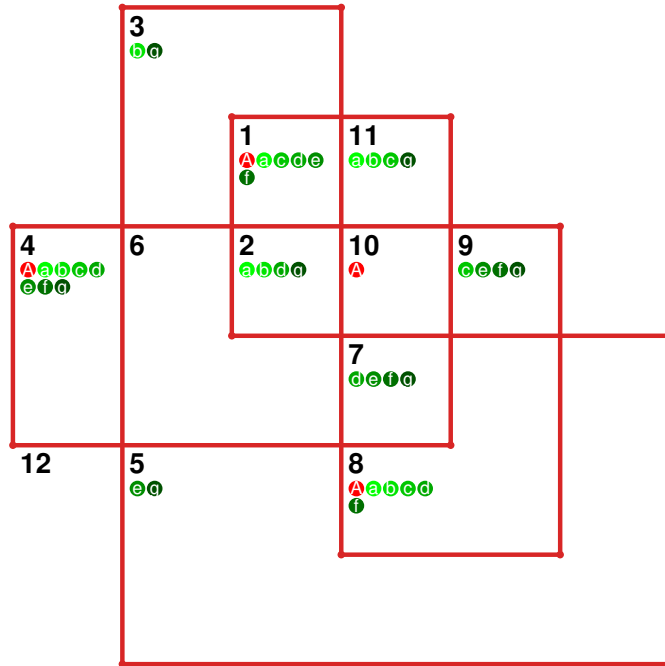


Figure 5: SnapPy multiloop plot.

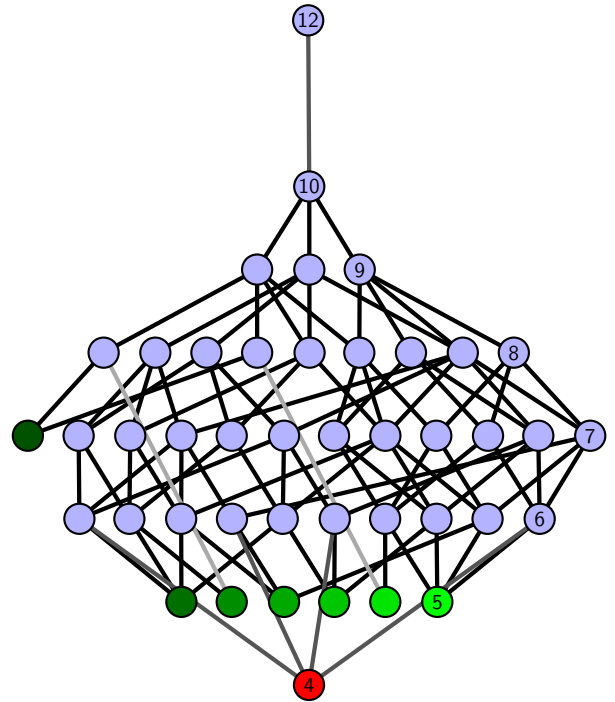


Figure 6: Minimal join sub-semi-lattice of minimal pinning sets.

Table 6: Pinning set data

Pinning set	Pindicator	Regions	Card	Degree seq	Average degree
A (optimal)	●	{1, 4, 8, 10}	4	[3, 2, 3, 4]	3.0
a (minimal)	●	{1, 2, 4, 8, 11}	5	[3, 3, 2, 3, 3]	2.8
b (minimal)	●	{2, 3, 4, 8, 11}	5	[3, 3, 2, 3, 3]	2.8
c (minimal)	●	{1, 4, 8, 9, 11}	5	[3, 2, 3, 3, 3]	2.8
d (minimal)	●	{1, 2, 4, 7, 8}	5	[3, 3, 2, 3, 3]	2.8
e (minimal)	●	{1, 4, 5, 7, 9}	5	[3, 2, 3, 3, 3]	2.8
f (minimal)	●	{1, 4, 7, 8, 9}	5	[3, 2, 3, 3, 3]	2.8
g (minimal)	●	{2, 3, 4, 5, 7, 9, 11}	7	[3, 3, 2, 3, 3, 3, 3]	2.86

1.0.4 [(5, 8, 6, 9), (2, 12, 3, 11), (1, 16, 2, 17), (12, 16, 13, 15), (4, 20, 5, 19), (9, 18, 10, 19), (6, 22, 7, 21), (7, 22, 8, 23), (20, 24, 21, 23), (3, 24, 4, 25), (10, 26, 11, 25), (17, 26, 18, 27), (30, 28, 1, 27), (13, 28, 14, 29), (14, 30, 15, 29)]

PD code drawn by SnapPy: [(2, 5, 3, 6), (8, 29, 9, 30), (12, 9, 13, 10), (28, 13, 29, 14), (6, 15, 7, 16), (16, 1, 17, 2), (18, 3, 19, 4), (4, 19, 5, 20), (20, 17, 21, 18), (30, 21, 1, 22), (22, 7, 23, 8), (14, 23, 15, 24), (24, 27, 25, 28), (10, 25, 11, 26), (26, 11, 27, 12)]

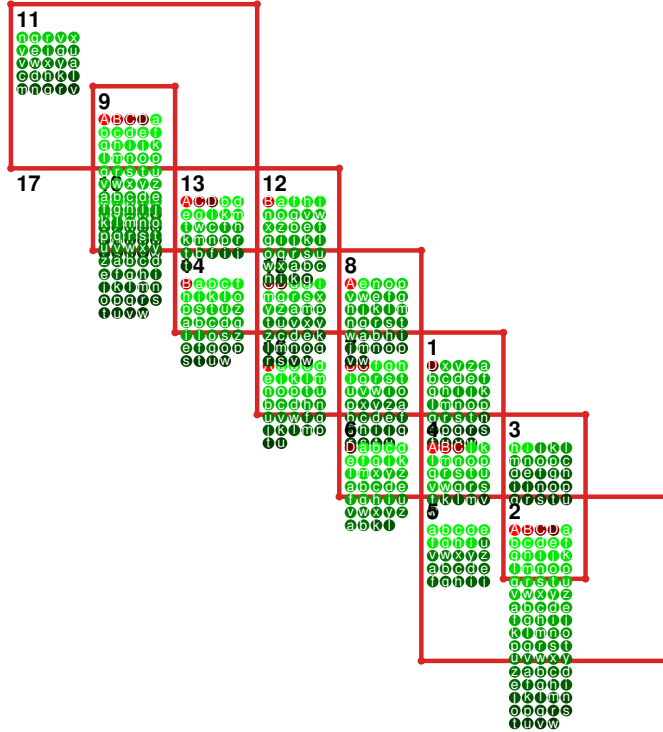
Planar representation generated by plantri: -

Total optimal pinning sets: 4
Total minimal pinning sets: 79
Total pinning sets: 12936
Pinning number: 6

Average optimal degree: 3.08
Average minimal degree: 2.93
Average overall degree: 3.36

Table 7: Pinning sets/average degree by cardinal

Cardinal	6	7	8	9	10	11	12	13	14	15	16	17	Total
Optimal pinning sets	4	0	0	0	0	0	0	0	0	0	0	0	4
Minimal (suboptimal) pinning sets	0	46	29	0	0	0	0	0	0	0	0	0	75
Nonminimal pinning sets	0	44	539	1706	2970	3300	2464	1266	447	105	15	1	12857
Average degree	3.08	3.04	3.13	3.23	3.32	3.38	3.43	3.46	3.48	3.5	3.52	3.53	



The minimal join sub-semi lattice of minimal pinning sets is too large to display.

Figure 7: SnapPy multiloop plot.

Table 8: Pinning set data

Pinning set	Pindicator	Regions	Card	Degree seq	Average degree
A (optimal)	●	{2, 4, 8, 9, 13, 16}	6	[2, 4, 3, 2, 4, 3]	3.0
B (optimal)	●	{2, 4, 7, 9, 12, 14}	6	[2, 4, 4, 2, 3, 3]	3.0
C (optimal)	●	{2, 4, 7, 9, 13, 15}	6	[2, 4, 4, 2, 4, 4]	3.33
D (optimal)	●	{1, 2, 6, 9, 13, 15}	6	[3, 2, 3, 2, 4, 4]	3.0
a (minimal)	●	{2, 5, 6, 9, 12, 14, 16}	7	[2, 3, 3, 2, 3, 3, 3]	2.71
b (minimal)	●	{2, 5, 6, 9, 13, 14, 16}	7	[2, 3, 3, 2, 4, 3, 3]	2.86
c (minimal)	●	{2, 5, 6, 9, 10, 14, 16}	7	[2, 3, 3, 2, 3, 3, 3]	2.71
d (minimal)	●	{2, 5, 6, 9, 13, 15, 16}	7	[2, 3, 3, 2, 4, 4, 3]	3.0
e (minimal)	●	{2, 5, 6, 8, 9, 13, 16}	7	[2, 3, 3, 3, 2, 4, 3]	2.86
f (minimal)	●	{2, 5, 6, 7, 9, 12, 14}	7	[2, 3, 3, 4, 2, 3, 3]	2.86
g (minimal)	●	{2, 5, 6, 7, 9, 13, 15}	7	[2, 3, 3, 4, 2, 4, 4]	3.14
h (minimal)	●	{2, 3, 5, 7, 9, 12, 14}	7	[2, 3, 3, 4, 2, 3, 3]	2.86
i (minimal)	●	{2, 3, 5, 7, 9, 13, 15}	7	[2, 3, 3, 4, 2, 4, 4]	3.14
j (minimal)	●	{2, 4, 6, 9, 12, 14, 16}	7	[2, 4, 3, 2, 3, 3, 3]	2.86
k (minimal)	●	{2, 4, 6, 9, 13, 14, 16}	7	[2, 4, 3, 2, 4, 3, 3]	3.0
l (minimal)	●	{2, 4, 6, 9, 10, 14, 16}	7	[2, 4, 3, 2, 3, 3, 3]	2.86
m (minimal)	●	{2, 4, 6, 9, 13, 15, 16}	7	[2, 4, 3, 2, 4, 4, 3]	3.14
n (minimal)	●	{2, 4, 8, 9, 11, 12, 16}	7	[2, 4, 3, 2, 3, 3, 3]	2.86
o (minimal)	●	{2, 4, 8, 9, 12, 14, 16}	7	[2, 4, 3, 2, 3, 3, 3]	2.86
p (minimal)	●	{2, 4, 8, 9, 10, 14, 16}	7	[2, 4, 3, 2, 3, 3, 3]	2.86
q (minimal)	●	{2, 4, 7, 9, 11, 12, 15}	7	[2, 4, 4, 2, 3, 3, 4]	3.14
r (minimal)	●	{2, 4, 7, 9, 10, 11, 15}	7	[2, 4, 4, 2, 3, 3, 4]	3.14
s (minimal)	●	{2, 4, 7, 9, 10, 14, 15}	7	[2, 4, 4, 2, 3, 3, 4]	3.14
t (minimal)	●	{2, 4, 7, 9, 13, 14, 16}	7	[2, 4, 4, 2, 4, 3, 3]	3.14
u (minimal)	●	{2, 4, 7, 9, 10, 14, 16}	7	[2, 4, 4, 2, 3, 3, 3]	3.0
v (minimal)	●	{2, 4, 7, 8, 9, 11, 12}	7	[2, 4, 4, 3, 2, 3, 3]	3.0
w (minimal)	●	{2, 4, 7, 8, 9, 12, 13}	7	[2, 4, 4, 3, 2, 3, 4]	3.14
x (minimal)	●	{1, 2, 6, 9, 11, 12, 15}	7	[3, 2, 3, 2, 3, 3, 4]	2.86
y (minimal)	●	{1, 2, 6, 9, 10, 11, 15}	7	[3, 2, 3, 2, 3, 3, 4]	2.86
z (minimal)	●	{1, 2, 6, 9, 12, 14, 15}	7	[3, 2, 3, 2, 3, 3, 4]	2.86
a (minimal)	●	{1, 2, 6, 9, 10, 14, 15}	7	[3, 2, 3, 2, 3, 3, 4]	2.86
b (minimal)	●	{1, 2, 6, 9, 12, 14, 16}	7	[3, 2, 3, 2, 3, 3, 3]	2.71
c (minimal)	●	{1, 2, 6, 9, 13, 14, 16}	7	[3, 2, 3, 2, 4, 3, 3]	2.86
d (minimal)	●	{1, 2, 6, 9, 10, 14, 16}	7	[3, 2, 3, 2, 3, 3, 3]	2.71
e (minimal)	●	{1, 2, 6, 8, 9, 11, 12}	7	[3, 2, 3, 3, 2, 3, 3]	2.71
f (minimal)	●	{1, 2, 6, 8, 9, 12, 13}	7	[3, 2, 3, 3, 2, 3, 4]	2.86
g (minimal)	●	{1, 2, 6, 8, 9, 12, 14}	7	[3, 2, 3, 3, 2, 3, 3]	2.71
h (minimal)	●	{1, 2, 6, 8, 9, 13, 16}	7	[3, 2, 3, 3, 2, 4, 3]	2.86
i (minimal)	●	{1, 2, 6, 7, 9, 12, 14}	7	[3, 2, 3, 4, 2, 3, 3]	2.86