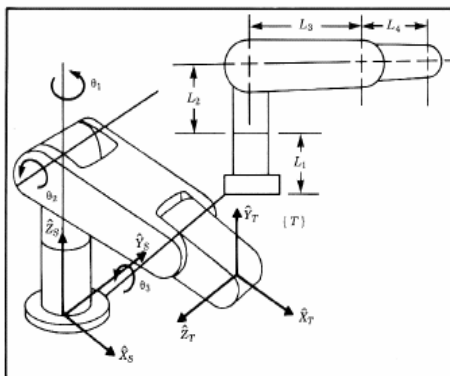


## MECH 498: Introduction to Robotics

### Inverse Manipulator Kinematics

M. O'Malley

## Manipulator Kinematics – Example – 3R



$i-1$	$i$	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
0	1				
1	2				
2	3				
3	4				

$${}^B_T T$$

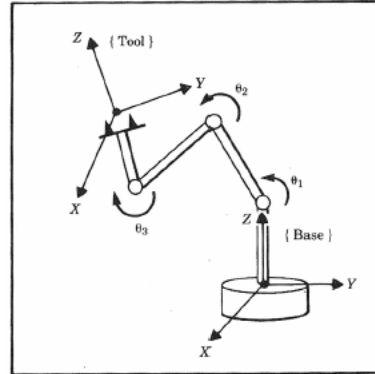
# Direct vs. Inverse Kinematics

- **Direct (Forward) Kinematics**

- Given: Joint angles and links geometry
- Compute: Position and orientation of the end effector relative to the base frame

- **Inverse Kinematics**

- Given: Position and orientation of the end effector relative to the base frame
- Compute: All possible sets of joint angles and link geometries which could be used to attain the given position and orientation of the end effector



## Solvability – PUMA 560

Given : PUMA 560 - 6 DOF,  ${}^0T_6$

Solve:  $\theta_1 \dots \theta_6$

$${}^0T_6 = {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Total Number of Equations: 12

Independent Equations: 3 - Rotation Matrix  
3 - Position Vector

Type of Equations: Non-linear

$$\begin{aligned} r_{11} &= c_1 [c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_5 c_6] + s_1 (s_4 c_5 c_6 + c_4 s_6), \\ r_{21} &= s_1 [c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_5 c_6] - c_1 (s_4 c_5 c_6 + c_4 s_6), \\ r_{31} &= -s_{23}(c_4 c_5 c_6 - s_4 s_6) - c_{23} s_5 c_6, \\ r_{12} &= c_1 [c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_5 s_6] + s_1 (c_4 c_6 - s_4 c_5 s_6), \\ r_{22} &= s_1 [c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_5 s_6] - c_1 (c_4 c_6 - s_4 c_5 s_6), \\ r_{32} &= -s_{23}(-c_4 c_5 s_6 - s_4 c_6) + c_{23} s_5 s_6, \\ r_{13} &= -c_1 (c_{23} c_4 s_5 + s_{23} c_5) - s_1 s_4 s_5, \\ r_{23} &= -s_1 (c_{23} c_4 s_5 + s_{23} c_5) + c_1 s_4 s_5, \\ r_{33} &= s_{23} c_4 s_5 - c_{23} c_5, \\ p_x &= c_1 [a_2 c_2 + a_3 c_{23} - d_4 s_{23}] - d_3 s_1, \\ p_y &= s_1 [a_2 c_2 + a_3 c_{23} - d_4 s_{23}] + d_3 c_1, \\ p_z &= -a_3 s_{23} - a_2 s_2 - d_4 c_{23}. \end{aligned}$$

# Solvability

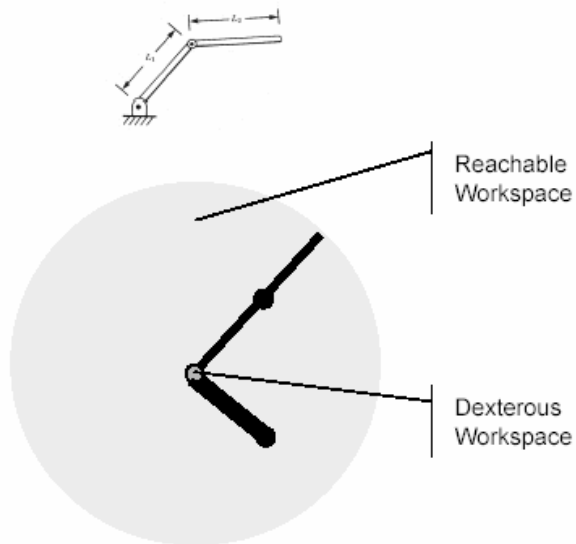
- **Existence of Solutions**
- **Multiple Solutions**
- **Method of solutions**
  - Closed form solution
    - Algebraic solution
    - Geometric solution
  - Numerical solutions

## Solvability – Existence of Solution

- For a solution to exist,  ${}^0_N T$  must be in the **workspace** of the manipulator
- **Workspace** - Definitions
  - **Dexterous Workspace (DW)**: The subset of space in which the robot end effector can reach ***in all orientations***.
  - **Reachable Workspace (RW)**: The subset of space in which the robot end effector can reach in ***at least 1 orientation***
- The Dexterous Workspace is a subset of the Reachable Workspace

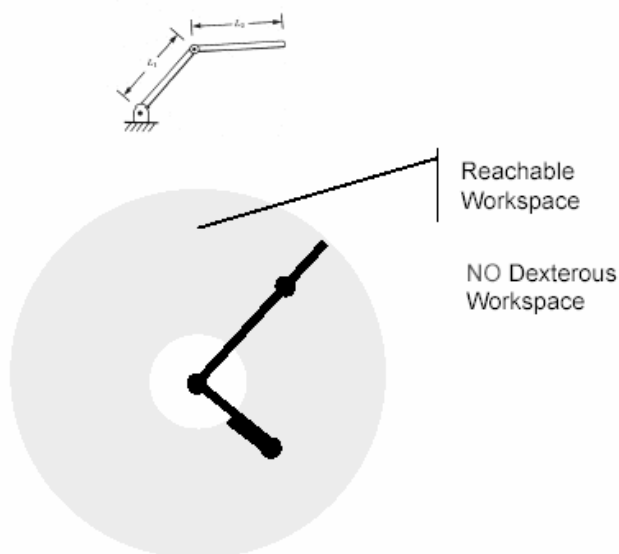
## Solvability - Existence of Solution - Workspace - 2R

Example 1 –  $L_1 = L_2$



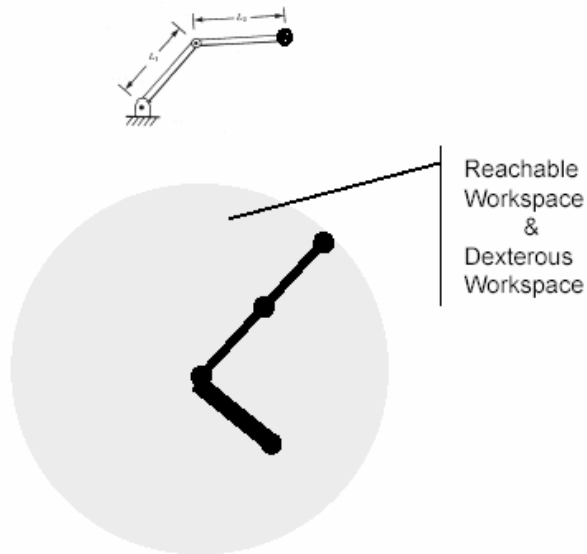
## Solvability - Existence of Solution - Workspace - 2R

Example 2 –  $L_1 \neq L_2$



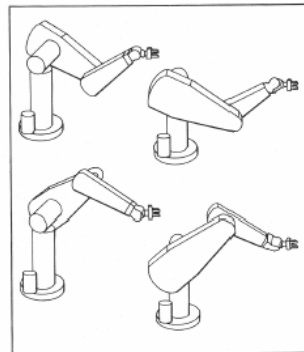
## Solvability - Existence of Solution - Workspace - 3R

### Example 3 – $L_1 = L_2$



## Solvability – Multiple Solutions

- Multiple solutions are a common problem that can occur when solving inverse kinematics because the system has to be able to choose one
- The number of solutions depends on the number of joints in the manipulator but is also a function of the link parameters ( $a_i, \alpha_i, \theta_i, d_i$ )
- Example: The PUMA 560 can reach certain goals with 8 different arm configurations (solutions)



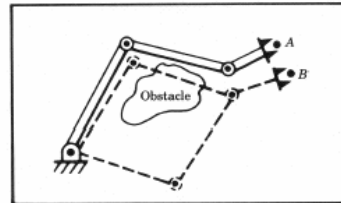
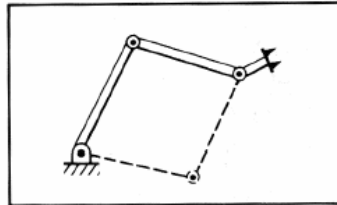
$$\theta'_4 = \theta_4 + 180^\circ$$

$$\theta'_5 = -\theta_5$$

$$\theta'_6 = \theta_6 + 180^\circ$$

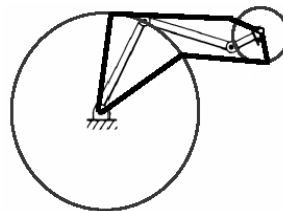
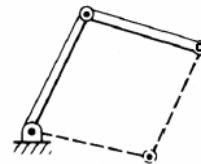
# Solvability – Multiple Solutions

- **Problem:** The fact that a manipulator has multiple solutions may cause problems because the system has to be able to choose one
- **Solution:** Decision criteria
  - The closest (geometrically)
    - Minimizing the amount that each joint is required to move
    - Note 1: input argument - present position of the manipulator
    - Note 2: Joint Weight - Moving small joints (wrist) instead of moving large joints (Shoulder & Elbow)
  - Obstacles exist in the workspace
    - Avoiding collision



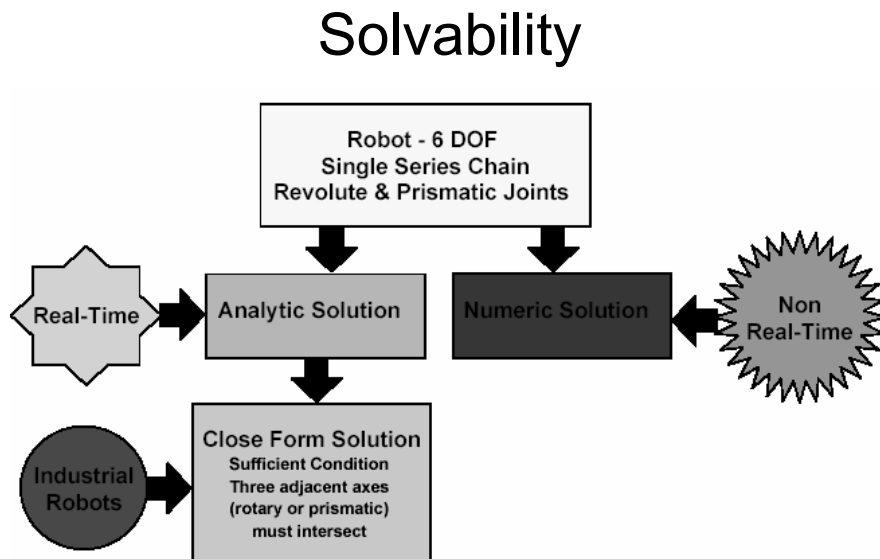
## Solvability – Multiple Solutions – Number of Solutions

- Task Definition - Position the end effector in a specific point in the plane (2D)
- No. of DOF = No. of DOF of the task
  - Number of solutions:
    - 2 (elbow up/down)
- No. of DOF > No. of DOF of the task
  - Number of solutions:  $\infty$
  - Self Motion - The robot can be moved without moving the end effector from the goal



# Solvability – Methods of Solutions

- **Solution** (Inverse Kinematics)- A “solution” is the set of joint variables associated with an end effector’s desired position and orientation.
- **No general algorithms** that lead to the solution of inverse kinematic equations.
- **Solution Strategies**
  - **Closed form Solutions** - An analytic expression includes all solution sets.
    - **Algebraic Solution** - Trigonometric (Nonlinear) equations
    - **Geometric Solution** - Reduces the larger problem to a series of plane geometry problems.
  - **Numerical Solutions** - Iterative solutions will not be considered in this course.



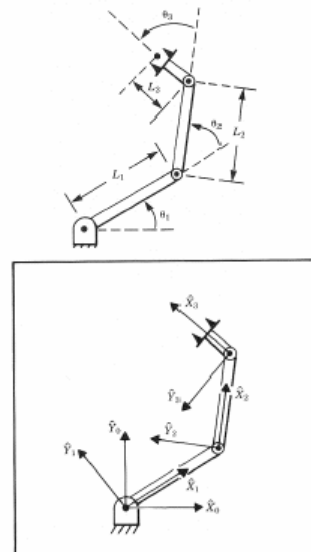
# Mathematical Equations

- Law of Sines / Cosines - For a general triangle

- Sum of Angles

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$i$	$\alpha_i - 1$	$a_i - 1$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	0	$L_1$	0	$\theta_2$
3	0	$L_2$	0	$\theta_3$





## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$${}^{i-1}T_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$i$	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	0	$L_1$	0	$\theta_2$
3	0	$L_2$	0	$\theta_3$

$${}^0T_1 = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} c2 & -s2 & 0 & L1 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2T_3 = \begin{bmatrix} c1 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$${}^B_T = {}^0T_3 = {}^0T_1 {}^1T_2 {}^2T_3 = \begin{bmatrix} c1c2c3 - c1s2s3 - s1s2c3 - s1c2s3 & -c1c2s3 - c1s2c3 + s1s2s3 - s1c2c3 & 0 & c1(L2c2 + L1) - s1s2L2 \\ s1cc3 - s1s2s3 + c1s2c3 + c1c2s3 & -s1c2s3 - s1s2c3 - c1s2s3 + c1c2c3 & 0 & s1(L2c2 + L1) + c1s2L2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Using trigonometric identities to simplify  ${}^B_T$ , the solution to the forward kinematics is:

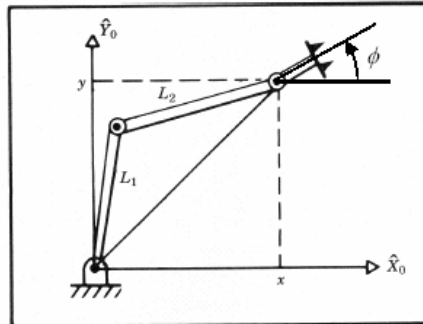
$${}^B_T = {}^0T_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1c_1 + L_2c_{12} \\ s_{123} & c_{123} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- where  $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$      $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- **Given:**

- **Direct Kinematics:** The homogenous transformation from the base to the wrist  ${}^B_w T$
- **Goal Point Definition:** For a planar manipulator, specifying the goal can be accomplished by specifying three parameters: The position of the wrist in space  $(x, y)$  and the orientation of link 3 in the plane relative to the  $\hat{X}$  axis  $(\phi)$



## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- **Problem:**

What are the joint angles  $(\theta_1, \theta_2, \theta_3)$  as a function of the wrist position and orientation  $(x, y, \phi)$

- **Solution:**

- The goal in terms of position and orientation of the wrist expressed in terms of the homogeneous transformation is defined as follows

$${}^B_w T_{Goal} = \begin{bmatrix} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^B_w T = {}^0_3 T = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1 c_1 + L_2 c_{12} \\ s_{123} & c_{123} & 0 & L_1 s_1 + L_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$${}^B T_{Goal} = {}^0 T_3$$

- A set of four nonlinear equations which must be solved for  $\theta_1, \theta_2, \theta_3$

$$c_\phi = c_{123}$$

$$s_\phi = s_{123}$$

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

- Solving for  $\theta_2$
- If we square  $x$  and  $y$  add them while making use of  $c_{12} = c_1 c_2 - s_1 s_2$  ;  $s_{12} = c_1 s_2 + s_1 c_2$  we obtain

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2$$

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Solving for  $c_2$  we obtain

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

- Note: In order for a solution to exist, the right hand side must have a value between -1 and 1. Physically if this constraints is not satisfied, then the goal point is too far away for the manipulator to reach.
- Assuming the goal is in the workspace, and making use of  $c_2^2 + s_2^2 = 1$  we write an expression for  $s_2$  as

$$s_2 = \pm \sqrt{1 - c_2^2}$$

- Note: The choice of the sign corresponds to the multiple solutions in which we can choose the "elbow-up" or the "elbow-down" solution

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Solving for  $\theta_1$
- For solving  $\theta_1$  we rewrite the the original nonlinear equations using a **change of variables** as follows

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

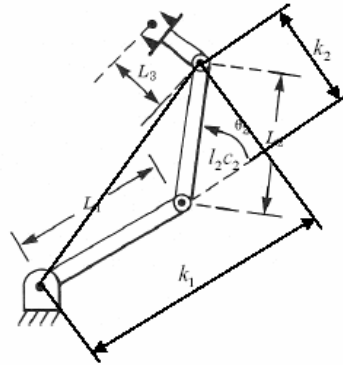
$$x = k_1 c_1 + k_2 s_1$$

$$y = k_1 s_1 + k_2 c_1$$

- where

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$



## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Finally, we compute  $\theta_2$  using the two argument arctangent function

$$\theta_2 = \text{Atan2}(s_2, c_2) = \text{Atan2}\left(\pm\sqrt{1 - c_2^2}, \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)$$

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Changing the way in which we write the constants  $k_1$  and  $k_2$

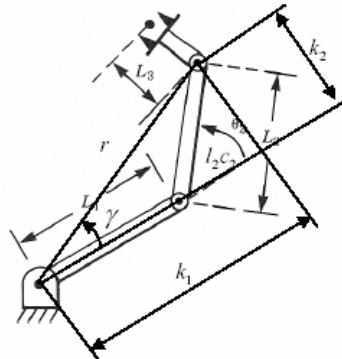
$$r = +\sqrt{k_1^2 + k_2^2}$$

$$\gamma = \text{Atan2}(k_2, k_1)$$

- Then

$$k_1 = r \cos \gamma$$

$$k_2 = r \sin \gamma$$



## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Based on the previous two transformations, the equations can be rewritten as:

$$x = r \cos \gamma \cos \theta_1 - r \sin \gamma \sin \theta_1$$

$$y = r \cos \gamma \sin \theta_1 + r \sin \gamma \cos \theta_1$$

- or

$$\frac{x}{r} = \cos \gamma \cos \theta_1 - \sin \gamma \sin \theta_1$$

$$\frac{y}{r} = \cos \gamma \sin \theta_1 + \sin \gamma \cos \theta_1$$

- or

$$\frac{x}{r} = \cos(\gamma + \theta_1)$$

$$\frac{y}{r} = \sin(\gamma + \theta_1)$$

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Using the two argument arctangent we finally get a solution for  $\theta_1$

$$\gamma + \theta_1 = A \tan 2\left(\frac{y}{r}, \frac{x}{r}\right) = A \tan 2(y, x)$$

$$\theta_1 = A \tan 2(y, x) - A \tan 2(k_2, k_1)$$

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$

- Note:

(1) When a choice of a sign is made in the solution of  $\theta_2$  above, it will cause a sign change in  $k_2$  thus affecting  $\theta_1$

(2) If  $x = y = 0$  then the solution becomes undefined - in this case  $\theta_1$  is arbitrary

## Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Solving for  $\theta_3$

- Based on the original equations,

$$c_\phi = c_{123}$$

$$s_\phi = s_{123}$$

- We can solve for the sum of  $\theta_1, \theta_2, \theta_3$

$$\theta_1 + \theta_2 + \theta_3 = A \tan 2(s_\phi, c_\phi) = \phi$$

$$\theta_3 = \phi - \theta_1 + \theta_2$$

- Note: It is typical with manipulators that have two or more links moving in a plane that in the course of a solution, expressions for sum of joint angles arise

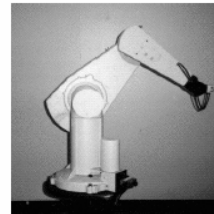
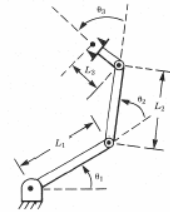
# Central Topic – Inverse Manipulator Kinematics - Examples

- **Geometric Solution – Concept**
  - Decompose spatial geometry into several plane geometry problems
  - **Examples** - Planar RRR (3R) manipulators - Geometric Solution
- **Algebraic Solution - Concept**

$${}^0_N T = {}^0_1 T \dots {}^{N-1}_N T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

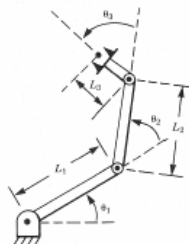
Direct Kinematics      Goal (Numeric values)

- **Examples** - PUMA 560 - Algebraic Solution



## Inverse Kinematics - Planar RRR (3R) - Geometric Solution

- **Given:**
  - **Manipulator Geometry**
  - **Goal Point Definition:** The position  $x, y$  and orientation  $\theta$  of the wrist in space
- **Problem:**  
What are the joint angles (  $\theta_1, \theta_2, \theta_3$  ) as a function of the goal (wrist position and orientation)



# Inverse Kinematics - PUMA 560 - Geometric Solution

- Solution:*

- We can apply the law of cosines to solve for  $\theta_2$

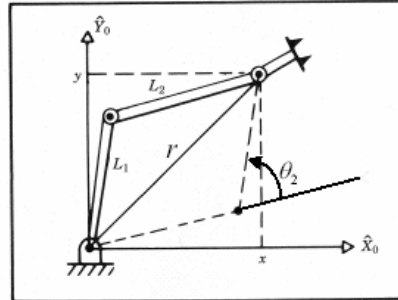
$$r^2 = x^2 + y^2 = l_1^2 + l_2^2 - 2l_1l_2 \cos(180 + \theta_2)$$

- Since

$$\cos(180 + \theta_2) = -\cos \theta_2$$

- We have

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$



# Inverse Kinematics - PUMA 560 - Geometric Solution

- Note : Condition - Should be checked by the computational algorithm to verify existence of solutions.

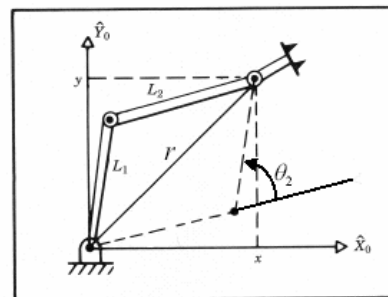
$$l_1 + l_2 \geq \sqrt{x^2 + y^2}$$

- Assuming that the solution exist it lies in the range of

$$180^\circ \leq \theta_2 \leq 0^\circ$$

- The other possible solution may found by symmetry to be

$$\theta_2' = -\theta_2$$





# Inverse Kinematics - PUMA 560 - Geometric Solution

- By definition

$$\theta_1 = \beta \pm \psi$$

$\swarrow \theta_2 < 0$   
 $\searrow \theta_2 > 0$

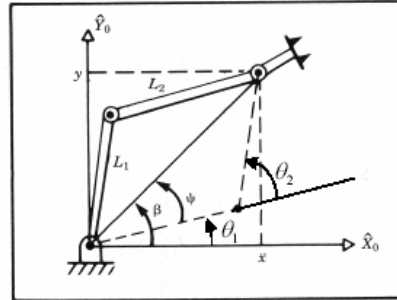
- Defining  $\beta$  as a function of  $x, y$

$$\beta = \arctan 2(y, x)$$

- Applying the law of cosine to find

$$\cos \psi = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2l_1 \sqrt{x^2 + y^2}}$$

- Note:  $0^\circ \leq \psi \leq 180^\circ$

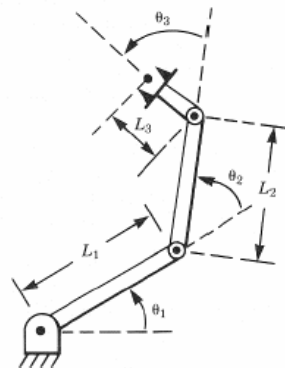


# Inverse Kinematics - PUMA 560 - Geometric Solution

- Angle in the plane add up to define the orientation of the last link

$$\phi = \theta_1 + \theta_2 + \theta_3$$

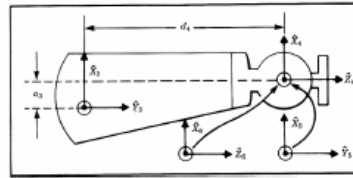
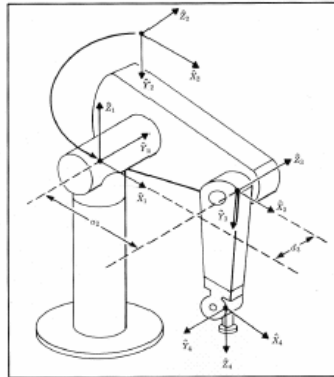
$$\theta_3 = \phi - \theta_1 + \theta_2$$



# Inverse Kinematics - PUMA 560 - Algebraic Solution

- **Given:**

- **Direct Kinematics:** The homogenous transformation from the base to the wrist  ${}^B_w T$
- **Goal Point Definition:** The position and orientation of the wrist in space



# Inverse Kinematics - PUMA 560 - Algebraic Solution

- **Problem:**

What are the joint angles  $(\theta_1 \dots \theta_6)$  as a function of the wrist position and orientation ( or when  ${}^0_6 T$  is given as numeric values)

$${}^0_6 T = \underbrace{{}^0_1 T(\theta_1) {}^1_2 T(\theta_2) {}^2_3 T(\theta_3) {}^3_4 T(\theta_4) {}^4_5 T(\theta_5)}_{\text{Direct Kinematics}} \underbrace{{}^5_6 T(\theta_6)}_{\text{Goal}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- *Solution (General Technique)*: Multiplying each side of the direct kinematics equation by a an inverse transformation matrix for separating out variables in search of solvable equation
- Put the dependence on  $\theta_1$  on the left hand side of the equation by multiplying the direct kinematics eq. with  $[{}^0T(\theta_1)]^{-1}$  gives

$$\begin{aligned}
 & I \\
 & [{}^0T(\theta_1)]^{-1} {}^0T = \boxed{[{}^0T(\theta_1)]^{-1} {}^0T(\theta_1)} {}^1T(\theta_2) {}^2T(\theta_3) {}^3T(\theta_4) {}^4T(\theta_5) {}^5T(\theta_6) \\
 & [{}^0T(\theta_1, \theta_2, \theta_3)]^{-1} {}^0T = \boxed{[{}^0T(\theta_1)]^{-1} {}^0T(\theta_1) {}^1T(\theta_2) {}^2T(\theta_3)} {}^3T(\theta_4) {}^4T(\theta_5) {}^5T(\theta_6) \\
 & [{}^0T(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} {}^0T = \boxed{[{}^0T(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} {}^0T(\theta_1) {}^1T(\theta_2) {}^2T(\theta_3) {}^3T(\theta_4)} {}^4T(\theta_5) {}^5T(\theta_6) \\
 & [{}^0T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} {}^0T = \boxed{[{}^0T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} {}^0T(\theta_1) {}^1T(\theta_2) {}^2T(\theta_3) {}^3T(\theta_4) {}^4T(\theta_5)} {}^5T(\theta_6)
 \end{aligned}$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- Put the dependence on  $\theta_1$  on the left hand side of the equation by multiplying the direct kinematics eq. with  $[{}^0T(\theta_1)]^{-1}$  gives

$$\begin{aligned}
 & [{}^0T(\theta_1)]^{-1} {}^0T = \underbrace{[{}^0T(\theta_1)]^{-1} {}^0T(\theta_1)}_I {}^1T(\theta_2) {}^2T(\theta_3) {}^3T(\theta_4) {}^4T(\theta_5) {}^5T(\theta_6) \\
 & {}^0T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [{}^A_BT]^{-1} = {}^B_A T = \begin{bmatrix} {}^A_B R^T & -{}^A_B R^T A P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 & {}^1T = [{}^0T]^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

$${}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_4 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^5_6T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1r_{11} = c_{23} [c_4 c_5 c_6 - s_4 s_6] - s_{23} s_5 s_6,$$

$${}^1r_{21} = -s_4 c_5 c_6 - c_4 s_6,$$

$${}^1r_{31} = -s_{23} [c_4 c_5 c_6 - s_4 s_6] - c_{23} s_5 c_6,$$

$${}^1r_{12} = -c_{23} [c_4 c_5 s_6 + s_4 c_6] + s_{23} s_5 s_6,$$

$${}^1r_{22} = s_4 c_5 s_6 - c_4 c_6,$$

$${}^1r_{32} = s_{23} [c_4 c_5 s_6 + s_4 c_6] + c_{23} s_5 s_6,$$

$${}^1r_{13} = -c_{23} c_4 s_5 - s_{23} c_5,$$

$${}^1r_{23} = s_4 s_5,$$

$${}^1r_{33} = s_{23} c_4 s_5 - c_{23} c_5,$$

$$\Rightarrow {}^1p_x = a_2 c_2 + a_3 c_{23} - d_4 s_{23},$$

$$\Rightarrow {}^1p_y = d_3,$$

$$\Rightarrow {}^1p_z = -a_3 s_{23} - a_2 s_2 - d_4 c_{23}.$$

## Inverse Kinematics - PUMA 560 - Algebraic Solution

$$\Rightarrow \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \stackrel{\downarrow}{=} {}^1_6T$$

- Equating the (2,4) elements from both sides of the equation we have

$$-s_1 p_x + c_1 p_y = d_3$$

- To solve the equation of this form we make the trigonometric substitution

$$p_x = \rho \cos \phi$$

$$p_y = \rho \sin \phi$$

## Inverse Kinematics - PUMA 560 - Algebraic Solution

$$\rho = \sqrt{p_x^2 + p_y^2}$$

$$\phi = \tan^{-1} \left( \frac{p_y}{p_x} \right)$$

- Substituting  $p_x, p_y$  with  $\rho, \phi$  we obtain

$$c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho}$$

- Using the difference of angles formula

$$\sin(\phi - \theta_1) = \frac{d_3}{\rho}$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- Based on  $\sin^2(\phi - \theta_1) + \cos^2(\phi - \theta_1) = 1$

$$\cos(\phi - \theta_1) = \pm \sqrt{1 - \frac{d_3^2}{\rho^2}}$$

- and so

$$\phi - \theta_1 = A \tan 2 \left( \frac{d_3}{\rho}, \pm \sqrt{1 - \frac{d_3^2}{\rho^2}} \right)$$

- The solution for  $\theta_1$  may be written

$$\theta_1 = A \tan 2(p_y, p_x) - A \tan 2 \left( \frac{d_3}{\rho}, \pm \sqrt{1 - \frac{d_3^2}{\rho^2}} \right)$$

- Note: we have found two possible solutions for  $\theta_1$  corresponding to the +/- sign

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- Equating the (1,4) element and (3,4) element

$$\begin{array}{c} \Rightarrow \\ \Rightarrow \end{array} \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \stackrel{1}{=} \stackrel{6}{T}$$

- We obtain

$$\Rightarrow c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2$$

$$\Rightarrow -p_z = a_3 c_{23} + d_4 s_{23} + a_2 c_2$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- If we square the following equations and add the resulting equations

$$\Rightarrow -s_1 p_x + c_1 p_y = d_3$$

$$\Rightarrow c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2$$

$$\Rightarrow -p_z = a_3 c_{23} + d_4 s_{23} + a_2 c_2$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- we obtain

$$a_3 c_3 - d_4 s_3 = K$$

- where

$$K = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - a_4^2}{2a_2}$$

- Note that the dependence on  $\theta_1$  has been removed. Moreover the eq. for  $\theta_3$  is of the same form as the eq. for  $\theta_1$  and so may be solved by the same kind of trigonometric substitution to yield a solution for  $\theta_3$

$$\theta_3 = A \tan 2(a_3, d_4) - A \tan 2\left(K, \pm \sqrt{a_3^2 + d_4^2 - K^2}\right)$$

- Note that the +/- sign leads to two different solutions for  $\theta_3$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

$$[{}^0_3T(\theta_1, \theta_2, \theta_3)]^{-1} {}^0_6T = [{}^0_3T(\theta_1)]^{-1} {}^0_1T(\theta_1) {}^1_2T(\theta_2) {}^2_3T(\theta_3) {}^3_4T(\theta_4) {}^4_5T(\theta_5) {}^5_6T(\theta_6)$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & -c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Equating the (1,4) element and (2,4) element we obtain
  - $c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 = a_3$
  - $-c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 = d_4$
- These equations may be solved simultaneously for  $s_{23}$  and  $c_{23}$  resulting in

# Inverse Kinematics - PUMA 560 - Algebraic Solution

$$s_{23} = \frac{(-a_3 - a_2c_2)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4)}{p_z^2 + (c_1p_x + s_1p_y)^2}$$

$$c_{23} = \frac{(a_2s_3 - d_4)p_z - (-a_3 - a_2c_2)(c_1p_x + s_1p_y)}{p_z^2 + (c_1p_x + s_1p_y)^2}$$

- Since the denominator are equal and positive, we solve for the sum of  $\theta_2$  and  $\theta_3$  as

$$\theta_{23} = A \tan 2[(-a_3 - a_2c_2)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4), (a_2s_3 - d_4)p_z - (-a_3 - a_2c_2)(c_1p_x + s_1p_y)]$$

- The equation computes four values of  $\theta_{23}$  according to the four possible combination of solutions for  $\theta_1$  and  $\theta_3$



# Inverse Kinematics - PUMA 560 - Algebraic Solution

- Then, four possible solutions for  $\theta_2$  are computed as

$$\theta_2 = \theta_{23} - \theta_3$$

- Equating the (1,3) and the (3,3) elements

$$\begin{bmatrix} c_1 c_{23} & s_1 c_{23} & -s_{23} & -a_2 c_3 \\ -c_1 s_{23} & -s_1 s_{23} & -c_{23} & a_2 s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & -c_4 s_5 & a_3 \\ s_5 c_6 & -s_5 s_6 & -c_5 & d_4 \\ -s_4 c_5 c_6 - c_4 s_6 & s_4 c_5 s_6 - c_4 c_6 & s_4 s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- we get

$$r_{13} c_1 c_{23} + r_{23} s_1 c_{23} - s_{23} r_{33} = -c_4 s_5$$

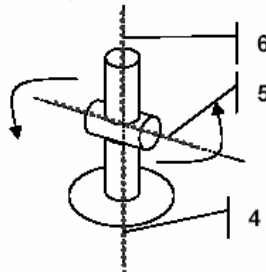
$$-r_{13} s_1 + r_{23} c_1 = s_4 s_5$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- As long as  $s_5 \neq 0$  we can solve for  $\theta_4$

$$\theta_4 = \text{Atan2}(-r_{13} s_1 + r_{23} c_1, -r_{13} c_1 c_{23} - r_{23} s_1 c_{23} + s_{23} r_{33})$$

- When  $\theta_5 = 0$  the manipulator is in a **singular configuration** in which joint axes 4 and 6 line up and cause the same motion of the last link of the robot. In this case all that can be solved for is the sum or difference of  $\theta_4$  and  $\theta_6$ . This situation is detected by checking whether both arguments of  $\text{Atan2}$  are near zero. If so  $\theta_4$  is chosen arbitrary (usually chosen to be equal to the present value of joint 4), and  $\theta_6$  is computed later, it will be computed accordingly



# Inverse Kinematics - PUMA 560 - Algebraic Solution

$$[{}^0_4T(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} {}^0_6T = [{}^0_4T(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} {}^0_1T(\theta_1) {}^1_2T(\theta_2) {}^2_3T(\theta_3) {}^3_4T(\theta_4) {}^4_5T(\theta_5) {}^5_6T(\theta_6)$$

$$\Rightarrow \begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 - a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Equating the (1,3) and the (3,3) elements we get

$$\Rightarrow r_{13}(c_1c_{23}c_4 + s_1s_4) + r_{23}(s_1c_{23}c_4 - c_1s_4) - r_{33}(s_{23}c_4) = s_5$$

$$\Rightarrow r_{13}(-c_1s_{23}) + r_{23}(-s_1s_{23}) + r_{33}(-c_{23}) = c_5$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- We can solve for  $\theta_5$

$$\theta_5 = \text{Atan2}(s_5, c_5)$$

$$[{}^0_5T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} {}^0_6T = [{}^0_5T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} {}^0_1T(\theta_1) {}^1_2T(\theta_2) {}^2_3T(\theta_3) {}^3_4T(\theta_4) {}^4_5T(\theta_5) {}^5_6T(\theta_6)$$

$$r_{11}(c_1c_{23}s_4 + s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4) = s_6$$

$$r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5] - r_{31}(s_{23}c_4c_5 + c_{23}s_5) = c_6$$

$$\theta_6 = \text{Atan2}(s_6, c_6)$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- Summary - Number of Solutions

- Four solution

$$\theta_1 = A \tan 2(p_y, p_x) - A \tan 2\left(\frac{d_3}{\rho}, \pm \sqrt{1 - \frac{d_3^2}{\rho^2}}\right)$$
$$\theta_3 = A \tan 2(a_3, d_4) - A \tan 2\left(K, \pm \sqrt{a_3^2 + d_4^2 - K^2}\right)$$

- For each of the four solutions the wrist can be flipped

$$\theta_4' = \theta_4 + 180^\circ$$

$$\theta_5' = -\theta_5$$

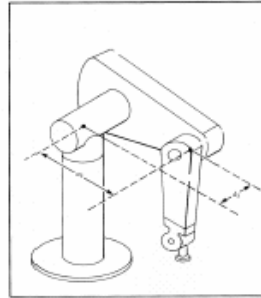
$$\theta_6' = \theta_6 + 180^\circ$$

# Inverse Kinematics - PUMA 560 - Algebraic Solution

- After all eight solutions have been computed, some or all of them may have to be discarded because of joint limit violations.
- Of the remaining valid solutions, usually the one closest to the present manipulator configuration is chosen.

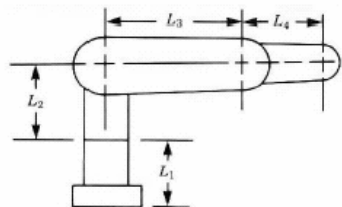
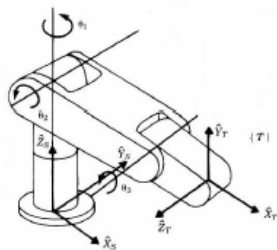
# Central Topic - Inverse Manipulator Kinematics -Examples

- **Geometric Solution - Concept**
  - Decompose spatial geometry into several plane geometry
  - **Example** - 3D - RRR (3R) manipulators - Geometric Solution
- **Algebraic Solution (closed form) –**
  - Piepers Method - Last three consecutive axes intersect at one point
  - **Example** - Puma 560



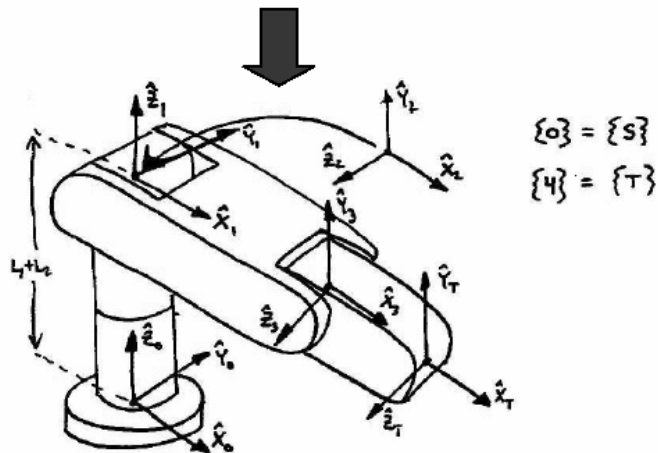
## Algebraic Solution by Reduction to Polynomial - Example

- **Given:**
  - **Manipulator Geometry**
  - **Goal Point Definition:** The position  $x_d, y_d, z_d$  of the wrist in space



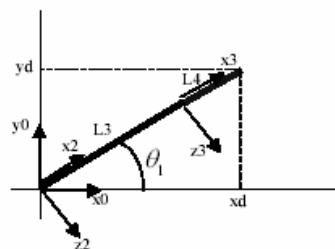
- **Problem:**  
What are the joint angles  $(\theta_1, \theta_2, \theta_3)$  as a function of the goal (wrist position and orientation)

## Algebraic Solution by Reduction to Polynomial - Example



## Algebraic Solution by Reduction to Polynomial - Example

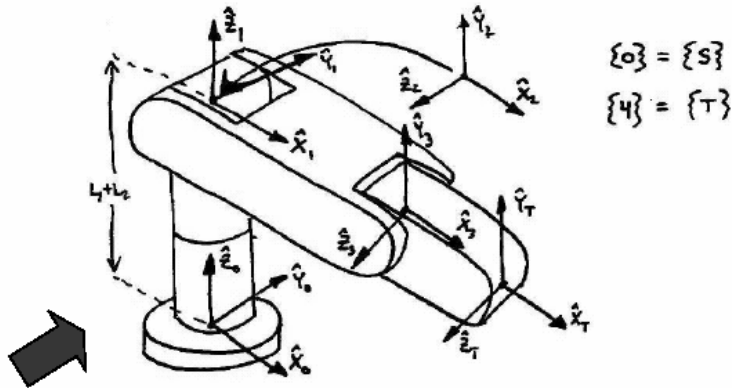
- The planar geometry - top view of the robot



$$\theta_1 = \text{Atan2}(y_d, x_d)$$

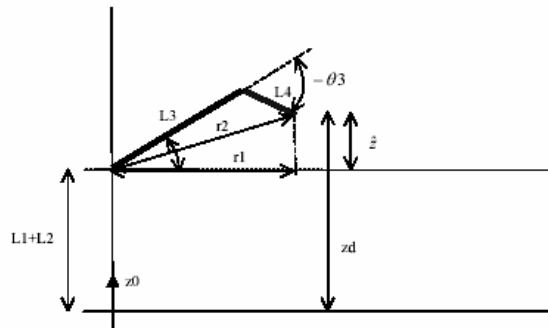
$$r_1 = \sqrt{x_d^2 + y_d^2}$$

## Algebraic Solution by Reduction to Polynomial - Example



## Algebraic Solution by Reduction to Polynomial - Example

- The planar geometry - side view of the robot:



$$r_2 = \sqrt{r_1^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2}$$

- where  $\hat{z} = z_d - (L_1 + L_2)$

# Algebraic Solution by Reduction to Polynomial - Example

- By applying the law of cosines, we get

$$r_2^2 = L_3^2 + L_4^2 - 2L_3L_4 \cos(180 + \theta_3) = L_3^2 + L_4^2 + 2L_3L_4 \cos(\theta_3)$$

- Rearranging gives

$$c_3 = \frac{r_2^2 - (L_3^2 + L_4^2)}{2L_3L_4}$$

- and

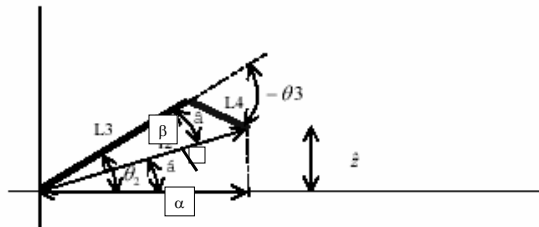
$$s_3 = \sqrt{1 - c_3^2}$$

- Solving for  $\theta_3$  we get

$$\theta_3 = \text{Atan2}(\pm\sqrt{1 - c_3^2}, c_3)$$

- Where  $c_3$  is defined above in terms of known parameters  $L_3, L_4, x_d, y_d$ , and  $z_d$

# Algebraic Solution by Reduction to Polynomial - Example



- Finally we need to solve for  $\theta_2$

$$\theta_2 = \alpha + \beta$$

$$\alpha = \text{Atan2}(\hat{z}, r_1)$$

- where

$$r_1 = \sqrt{x_d^2 + y_d^2} \quad \hat{z} = z_d - (L_1 + L_2)$$

# Algebraic Solution by Reduction to Polynomial - Example

- Based on the law of cosines we can solve for  $\beta$

$$L_4^2 = r_2^2 + L_3^2 - 2r_2L_3 \cos(\beta)$$

$$c_\beta = \frac{r_2^2 + L_3^2 - L_4^2}{2r_2L_3}$$

$$\beta = A \tan 2(\pm \sqrt{1 - c_\beta^2}, c_\beta)$$

$$\theta_2 = A \tan 2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2}) + A \tan 2(\pm \sqrt{1 - c_\beta^2}, c_\beta)$$

# Algebraic Solution by Reduction to Polynomial - Example

- Summary

$$\theta_1 = A \tan 2(y_d, x_d)$$

$$\theta_2 = A \tan 2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2}) + A \tan 2\left(\pm \sqrt{1 - \left(\frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 + L_3^2 - L_4^2}{2\sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))L_3}\right)^2}, \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 + L_3^2 - L_4^2}{2\sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))L_3}}\right)$$

$$\theta_3 = A \tan 2\left(\pm \sqrt{1 - \left[\frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 - (L_3^2 + L_4^2)}{2L_3L_4}\right]^2}, \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 - (L_3^2 + L_4^2)}{2L_3L_4}\right)$$



# Algebraic Solution by Reduction to Polynomial

- Transcendental equations are difficult to solve because they are a function of  $c\theta, s\theta$

$$f(c\theta, s\theta) = k$$

- Making the following substitutions yields an expression in terms of a single variable  $u$
- Using this substitution, transcendental equations are converted into polynomial equations

$$u = \tan \frac{\theta}{2}$$

$$\cos \theta = \frac{1-u^2}{1+u^2}$$

$$\sin \theta = \frac{2u}{1+u^2}$$

# Algebraic Solution by Reduction to Polynomial - Example

- Transcendental equation

$$ac\theta + bs\theta = c$$

- Substitute  $c\theta, s\theta$  with the following equations

$$\cos \theta = \frac{1-u^2}{1+u^2}$$

$$\sin \theta = \frac{2u}{1+u^2}$$

- yields

$$a(1-u^2) + 2bu = c(1+u^2)$$

$$(a+c)u^2 - 2bu + (c-a) = 0$$

# Algebraic Solution by Reduction to Polynomial - Example

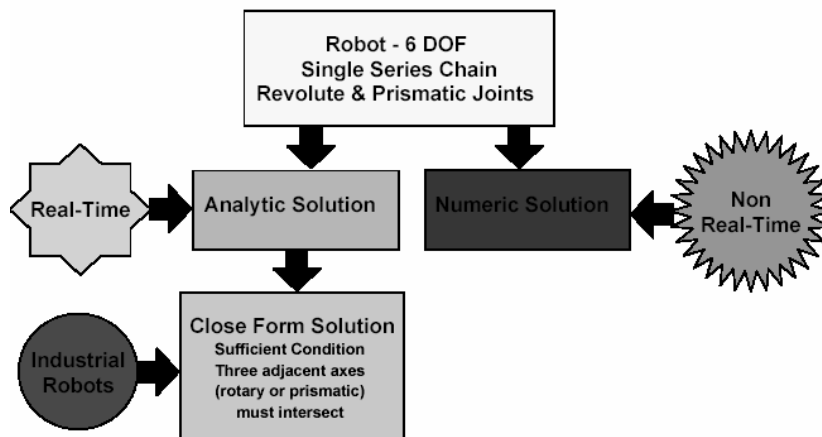
- Which is solved by the quadratic formula to be

$$u = \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a + c}$$

$$\theta = 2 \tan^{-1} \left( \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a + c} \right)$$

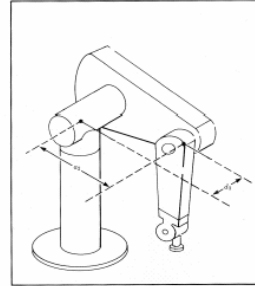
- Note
  - If  $u$  is complex there is no real solution to the original transcendental equation
  - If  $a + c = 0$  then  $\theta = 180^\circ$

## Solvability



# Pieper's Solution - Three consecutive Axes Intersect

- **Pieper's Solution**
  - Closed form solution for a serial 6 DOF in which **three consecutive axes intersect at a point** (including robots with three consecutive parallel axes, since they meet at a point at infinity)
- Pieper's method applies to the majority of commercially available industrial robots
  - Example: (Puma 560)
    - All 6 joints are revolute joints
    - The last 3 joints are intersecting



# Pieper's Solution - Three consecutive Axes Intersect

- **Given:**
  - **Manipulator Geometry:** 6 DOF & DH parameters
    - All 6 joints are revolute joints
    - The last 3 joints are intersecting
  - **Goal Point Definition:** The position and orientation of the wrist in space

$${}^0T_6 = {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Problem:**
  - What are the joint angles (  $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$  ) as a function of the goal (wrist position and orientation)

## Pieper's Solution - Three consecutive Axes Intersect

- When the last three axes of a 6 DOF robot intersect, the origins of link frame {4}, {5}, and {6} are all located at the point of intersection. This point is given in the base coordinate system as

$${}^0P_{4org} = {}^0T_1 {}^1T_2 {}^2T_3 P_{4org}$$

- From the general forward kinematics method for determining homogeneous transforms using DH parameters, we know:

$${}^{i-1}T_i = \begin{bmatrix} {}^{i-1}R_i & {}^{i-1}P_{iorg} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Pieper's Solution - Three consecutive Axes Intersect

- For  $i=4$

$${}^3T_4 = \begin{bmatrix} {}^3R_4 & {}^3P_{4org} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ s\theta_4 c\alpha_3 & c\theta_4 c\alpha_3 & -s\alpha_3 & -s\alpha_3 d_4 \\ s\theta_4 s\alpha_3 & c\theta_4 s\alpha_3 & c\alpha_3 & c\alpha_3 d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Using the fourth column and substituting for  ${}^3P_{4org}$  we find

$${}^0P_{4org} = {}^0T_1 {}^1T_2 {}^2T_3 P_{4org} = {}^0T_1 {}^1T_2 {}^2T_3 \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix} \quad {}^0P_{4org} = {}^0T_1 {}^1T_2 {}^2T_3 P_{4org} = {}^0T_1 {}^1T_2 \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

## Pieper's Solution - Three consecutive Axes Intersect

- where

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^2_3T \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 c\alpha_2 & c\theta_3 c\alpha_2 & -s\alpha_2 & -s\alpha_2 d_3 \\ s\theta_3 s\alpha_2 & c\theta_3 s\alpha_2 & c\alpha_2 & c\alpha_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}$$

$$f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2$$

$$f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2$$

$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

## Pieper's Solution - Three consecutive Axes Intersect

- Repeating the same process again

$${}^0P_{4org} = {}^0_1T {}^1_2T {}^2_3T {}^3_4P_{4org} = {}^0_1T {}^1_2T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix} = {}^1_2T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix} = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_1 \\ s\theta_2 c\alpha_1 & c\theta_2 c\alpha_2 & -s\alpha_1 & -s\alpha_1 d_2 \\ s\theta_2 s\alpha_1 & c\theta_2 s\alpha_1 & c\alpha_1 & c\alpha_1 d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

## Pieper's Solution - Three consecutive Axes Intersect

$$g_1(\theta_2) = c_2 f_1 + s_2 f_2 + a_1$$

$$g_2(\theta_2) = s_3 s \alpha_1 s_3 f_1 + c_2 c \alpha_1 f_2 - s \alpha_1 f_3 - d_2 s \alpha_1$$

$$g_3(\theta_2) = s_2 s \alpha_1 s_2 f_1 + c_2 s \alpha_1 f_2 + c \alpha_1 f_3 + d_2 c \alpha_1$$

- Repeating the same process for the last time

$${}^0P_{4org} = {}^0T_1 {}^1T_2 {}^2T_3 P_{4org} = {}^0T_1 \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix}$$

$${}^0P_{4org} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix}$$

## Pieper's Solution - Three consecutive Axes Intersect

- Frame {0}** - The frame attached to the base of the robot or link 0 called frame {0} This frame does not move and for the problem of arm kinematics can be considered as the **reference frame**.

- Assign {0} to match {1} when the first joint variable is zero

$$\theta_1 \neq 0 \quad \alpha_0 = d_1 = a_0 = 0$$

$${}^0P_{4org} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix}$$

$${}^0P_{4org} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$

## Pieper's Solution - Three consecutive Axes Intersect

- Through algebraic manipulation of these equations, we can solve for the desired joint angles (  $\theta_1, \theta_2, \theta_3$  ).
- The first step is to square the magnitude of the distance from the frame {0} origin to frame {4} origin.

$$r^2 = \left({}^0P_{4orgx}\right)^2 + \left({}^0P_{4orgy}\right)^2 + \left({}^0P_{4orgz}\right)^2 = g_1^2 + g_2^2 + g_3^2$$

- Using the previously define function for  $g_i$  we have

$$r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 + a_1(c_2f_1 - s_2f_2)$$

## Pieper's Solution - Three consecutive Axes Intersect

$$r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 + a_1(c_2f_1 - s_2f_2)$$

$$Z = {}^0P_{4orgz} = g_3$$

- Applying a substitution of temporary variables, we can write the magnitude squared term along with the z-component of the {0} frame origin to the {4} frame origin distance.

$$\begin{aligned} r^2 &= (k_1c_2 + k_2s_2)2a_1 + k_3 & k_1 &= f_1 \\ Z &= (k_1s_2 - k_2c_2)s\alpha_1 + k_4 & k_2 &= -f_2 \\ & & k_3 &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 \\ & & k_4 &= f_3c\alpha_1 + d_2c\alpha_1 \end{aligned}$$

- These equations are useful because dependence on  $\theta_1$  has been eliminated, and dependence on  $\theta_2$  takes a simple form

## Pieper's Solution - Three consecutive Axes Intersect

- Consider 3 cases while solving for  $\theta_3$  :
- Case 1** -  $a_1 = 0$

$$r^2 = k_3$$

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3$$

$$f_1(\theta_3) = a_3c_3 + d_4s\alpha_3s_3 + a_2$$

$$f_2(\theta_3) = a_3c\alpha_2s_3 - d_4s\alpha_3c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2$$

$$f_3(\theta_3) = a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2$$

- Solution Methodology - Reduction to Polynomial => Quadratic Equation

$$u = \tan \frac{\theta}{2} \quad \cos \theta = \frac{1-u^2}{1+u^2} \quad \sin \theta = \frac{2u}{1+u^2}$$

## Pieper's Solution - Three consecutive Axes Intersect

- Case 2** -  $s\alpha_1 = 0$

$$Z = k_4$$

$$k_4 = f_3c\alpha_1 + d_2c\alpha_1$$

$$f_3(\theta_3) = a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2$$

- Solution Methodology - Reduction to Polynomial => Quadratic Equation

$$u = \tan \frac{\theta}{2} \quad \cos \theta = \frac{1-u^2}{1+u^2} \quad \sin \theta = \frac{2u}{1+u^2}$$



## Pieper's Solution - Three consecutive Axes Intersect

- **Case 3 (General case)** : We can find  $\theta_3$  through the following algebraic manipulation:

$$\frac{r^2 - k_3}{2a_1} = (k_1c_2 + k_2s_2)$$

$$\frac{Z - k_4}{s\alpha_1} = (k_1s_2 - k_2c_2)$$

- squaring both sides, we find

$$\left(\frac{r^2 - k_3}{2a_1}\right)^2 = (k_1c_2 + k_2s_2)^2 = k_1^2c_2^2 + k_2^2s_2^2 + 2k_1k_2c_2s_2$$

$$\left(\frac{Z - k_4}{s\alpha_1}\right)^2 = (k_1s_2 - k_2c_2)^2 = k_1^2s_2^2 + k_2^2c_2^2 - 2k_1k_2c_2s_2$$

## Pieper's Solution - Three consecutive Axes Intersect

- Adding these two equations together and simplifying using the trigonometry identity (Reduction to Polynomial), we find a fourth order equation for  $\theta_3$

$$\left(\frac{r^2 - k_3}{2a_1}\right)^2 + \left(\frac{Z - k_4}{s\alpha_1}\right)^2 = k_1^2 + k_2^2$$

$$k_1 = f_1$$

$$k_2 = -f_2$$

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3$$

$$k_4 = f_3c\alpha_1 + d_2c\alpha_1$$

$$f_1(\theta_3) = a_3c_3 + d_4s\alpha_3s_3 + a_2$$

$$f_2(\theta_3) = a_3c\alpha_2s_3 - d_4s\alpha_3c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2$$

$$f_3(\theta_3) = a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2$$

## Pieper's Solution - Three consecutive Axes Intersect

- With  $\theta_3$  solved, substitute into  $r^2, Z$  to find  $\theta_2$

$$r^2 = (k_1 c_2 + k_2 s_2) 2a_1 + k_3$$

$$Z = (k_1 s_2 - k_2 c_2) s \alpha_1 + k_4$$

$$k_1 = f_1$$

$$k_2 = -f_2$$

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3$$

$$k_4 = f_3 c \alpha_1 + d_2 c \alpha_1$$

$$f_1(\theta_3) = a_3 c_3 + d_4 s \alpha_3 s_3 + a_2$$

$$f_2(\theta_3) = a_3 c \alpha_2 s_3 - d_4 s \alpha_3 c \alpha_2 c_3 - d_4 s \alpha_2 c \alpha_3 - d_3 s \alpha_2$$

$$f_3(\theta_3) = a_3 s \alpha_2 s_3 - d_4 s \alpha_3 s \alpha_2 c_3 + d_4 c \alpha_2 c \alpha_3 + d_3 c \alpha_2$$

## Pieper's Solution - Three consecutive Axes Intersect

- With  $\theta_2, \theta_3$  solved, substitute into  ${}^0P_{4org}$  to find

$${}^0P_{4org} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$

$${}^0P_{4orgx} = c_1 g_1 - s_1 g_2$$

$${}^0P_{4orgy} = s_1 g_1 + c_1 g_2$$

$$g_1(\theta_2) = c_2 f_1 + s_2 f_2 + a_1$$

$$g_2(\theta_2) = s_2 s \alpha_1 s_1 f_1 + c_2 c \alpha_1 f_2 - s \alpha_1 f_3 - d_2 s \alpha_1$$

$$g_3(\theta_2) = s_2 s \alpha_1 s_2 f + c_2 s \alpha_1 f_2 + c \alpha_1 f_3 + d_2 c \alpha_1$$

$$f_1(\theta_3) = a_3 c_3 + d_4 s \alpha_3 s_3 + a_2$$

$$f_2(\theta_3) = a_3 c \alpha_2 s_3 - d_4 s \alpha_3 c \alpha_2 c_3 - d_4 s \alpha_2 c \alpha_3 - d_3 s \alpha_2$$

$$f_3(\theta_3) = a_3 s \alpha_2 s_3 - d_4 s \alpha_3 s \alpha_2 c_3 + d_4 c \alpha_2 c \alpha_3 + d_3 c \alpha_2$$

- Solve for  $\theta_1$  using the reduction to polynomial method

# Pieper's Solution - Three consecutive Axes Intersect

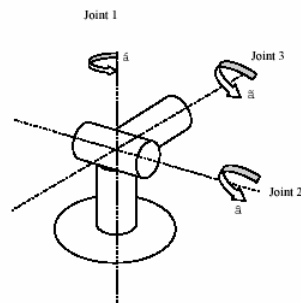
- To complete our solution we need to solve for  $\theta_4, \theta_5, \theta_6$
- Since the last three axes intersect these joint angles affect the orientation of only the last link. We can compute them based only upon the rotation portion of the specified goal  ${}^0_6R$

$${}^4_6R \Big|_{\theta_4=0} = {}^0_4R^{-1} \Big|_{\theta_4=0} \quad {}^0_6R$$

- ${}^0_4R \Big|_{\theta_4=0}$  - The orientation of link frame {4} relative to the base frame {0} when  $\theta_4 = 0$
- $\theta_4, \theta_5, \theta_6$  are the Euler angles applied to  ${}^4_6R \Big|_{\theta_4=0}$

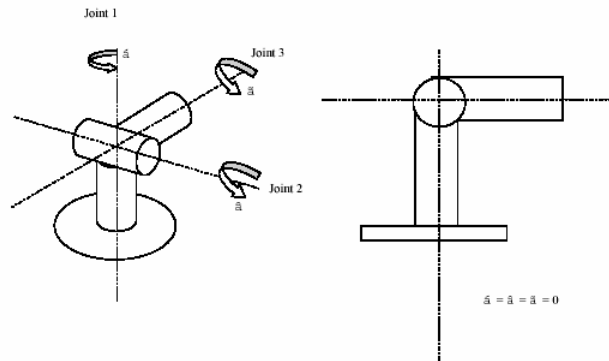
## Central Topic - Inverse Manipulator Kinematics - Examples

- **Algebraic Solution**  
(closed form) –
  - Pieper's Method  
(Continued)
  - Last three consecutive axes intersect at one point



## Three consecutive Axes Intersect - wrist

- Consider a 3 DOF non-planar robot whose axes all intersect at a point.

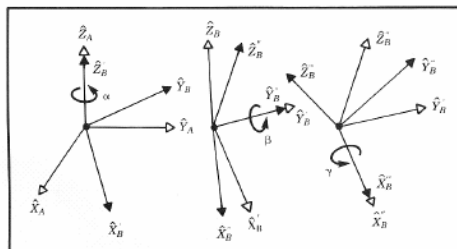


## Mapping - Rotated Frames - Z-Y-X Euler Angles

Start with frame {B} coincident with a known reference frame {A}.

- Rotate frame {B} about  $\hat{Z}_A$  by an angle  $\alpha$
  - Rotate frame {B} about  $\hat{Y}_B$  by an angle  $\beta$
  - Rotate frame {B} about  $\hat{X}_B$  by an angle  $\gamma$
- Euler Angles**

**Note** - Each rotation is performed about an axis of the moving reference frame {B}, rather than a fixed reference frame {A}.



# Mapping - Rotated Frames - ZYX Euler Angles

$${}^A_B R_{X'Y'Z'}(\alpha, \beta, \gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$${}^A_B R_{X'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$${}^A_B R_{X'Y'Z'}(\alpha, \beta, \gamma) = {}^0_3 R = {}^0_1 R {}^1_2 R {}^2_3 R$$

Three consecutive Axes Intersect - wrist

- Because, in this example, our robot can perform no translations, we can write

$${}^0_3 T = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & 0 \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & 0 \\ -s\beta & c\beta s\gamma & c\beta c\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The above transform provides the solution to the forward kinematics.

## Three consecutive Axes Intersect - wrist

- The inverse kinematics problem.
  - Given a particular rotation - Goal (again, this robot can perform no translations)
  - Solve: Find the Z-Y-X Euler angles

$${}^0_3T_d = \begin{bmatrix} & [R] & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^0_3T(\alpha, \beta, \gamma)$$

## Three consecutive Axes Intersect - wrist

$$\underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Goal}} = \underbrace{\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & 0 \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & 0 \\ -s\beta & c\beta s\gamma & c\beta c\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Direct Kinematics}}$$

## Three consecutive Axes Intersect - wrist

- Using elements  $r_{11}$  and  $r_{21}$ , we can solve for angle  $\alpha$

$$r_{11} = c\alpha c\beta$$

$$r_{21} = s\alpha c\beta$$

- when  $\beta \neq \pm \frac{n\pi}{2}$  where n is an odd integer by using the Atan2 function we obtain

$$\alpha = \begin{cases} \text{Atan2}(r_{21}, r_{11}) & \text{when } c\beta \geq 0 \\ \text{Atan2}(-r_{21}, -r_{11}) & \text{when } c\beta < 0 \end{cases}$$

## Three consecutive Axes Intersect - wrist

- Similarly, we find angle  $\gamma$  by

$$r_{32} = c\beta s\gamma$$

$$r_{33} = c\beta c\gamma$$

- when  $\beta \neq \pm \frac{n\pi}{2}$  where n is an odd integer by using the Atan2 function we obtain

$$\gamma = \begin{cases} \text{Atan2}(r_{32}, r_{33}) & \text{when } c\beta \geq 0 \\ \text{Atan2}(-r_{32}, -r_{33}) & \text{when } c\beta < 0 \end{cases}$$

## Three consecutive Axes Intersect - wrist

- The third angle,  $\beta$ , can be found from

$$r_{11}^2 + r_{21}^2 = c\beta^2(c\alpha^2 + s\alpha^2)$$

$$r_{31} = -s\beta$$

$$c\beta = \pm\sqrt{r_{11}^2 + r_{21}^2}$$

- Using the Atan2 function, we find

$$\beta = \text{Atan2}\left(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}\right)$$

## Three consecutive Axes Intersect - wrist

- Note: Two answers exist for angle  $\beta$  which will result in two answers each for angles  $\alpha$  and  $\gamma$ .

$$\alpha = \begin{cases} \text{Atan2}(r_{21}, r_{11}) & \text{when } c\beta \geq 0 \\ \text{Atan2}(-r_{21}, -r_{11}) & \text{when } c\beta < 0 \end{cases}$$

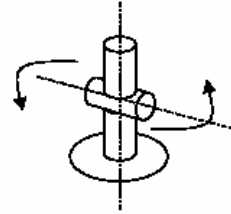
$$\beta = \text{atan2}\left(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\gamma = \begin{cases} \text{Atan2}(r_{32}, r_{33}) & \text{when } c\beta \geq 0 \\ \text{Atan2}(-r_{32}, -r_{33}) & \text{when } c\beta < 0 \end{cases}$$



## Three consecutive Axes Intersect - wrist

- What do we do if  $\beta = 90^\circ$  ?
- This is troublesome because  $\cos(90^\circ) = 0$  .
- Applying the difference of angles formula, we find:



$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & s(\gamma - \alpha) & c(\gamma - \alpha) & 0 \\ 0 & c(\gamma - \alpha) & s(\gamma - \alpha) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- We are left with  $(\gamma - \alpha)$  for every case. This means we can't solve for either, just their difference.

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- One solution is to define two cases such that

$$\gamma - \alpha = \text{Atan2}(r_{12}, r_{22}) \quad \text{if} \quad \beta = 90^\circ$$

$$\gamma + \alpha = \text{Atan2}(r_{12}, r_{22}) \quad \text{if} \quad \beta = -90^\circ$$

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- Unfortunately, while this seems like a simple solution, it is troublesome in practice because  $\alpha$  is never exactly zero. This leads to singularity problems
- For this example, the singular case results in the capability for self-rotation. That is, the middle link can rotate while the end effector's orientation never changes.

