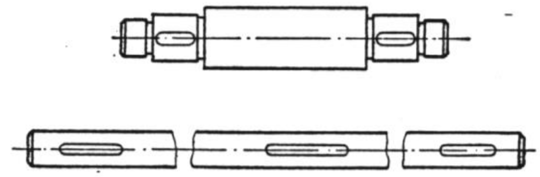
Chapter 4 Shafts and Shaft Design

4-1 Introduction to shafts

A shaft is a machine element that supports power-transmission elements, such as gears, couplings, and pulleys. It usually has several cylindrical segments of different circular cross sections. The shaft is a general term for this kind of parts. Shafts can be classified in detail based on their shapes and functions, **Figure 4-1**. In Mechanics of Materials we often see beams. A beam there may be a shaft here.

(a)



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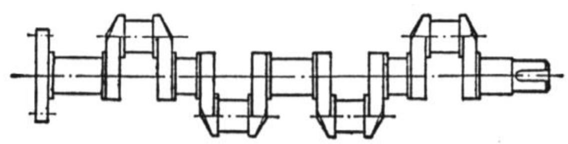

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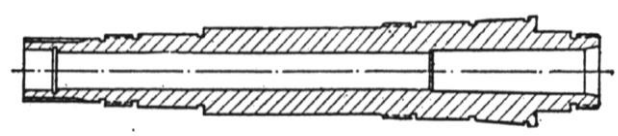


(d)

Figure 4-1. Classification of shafts based on geometry. (a) Straight and solid shafts, (b) a crank shaft, and (c) a hollow shaft. Note that the keys in (a) are shown in the same view for clarity and simplicity. Actually they should be in different circumferential positions, as shown in (d), which is our worm gear shaft in Chapter 1.

Figure 4-1 shows three types of shafts. Their differences are in their shapes, which also mean their functions. The shafts shown in **Figure 4-1** (a) are solid straight shafts. These types of shafts are very commonly used. We see them nearly everywhere and in every machine. The one shown in (b) is a crankshaft, not a straight shaft. We have this type of shafts in our cars, in the internal combustion engines. Such a crankshaft is used to convert the reciprocating motion of pistons into a rotating motion to run wheels. The shaft in (c) is a hollow shaft. This kind of shafts is often seen in machine tools. Why do we want to make a shaft hollow? Can we explain, based on mechanics and applications, the advantages of making a shaft hollow if the same material and the same volume of the material are used?

Think about the bending and torsional shear stress distributions. Where do we see the maximum stresses? Where do we see zero stresses? How about transverse shear? Is the transverse shear stress important to a shaft? See **Figure 4-2** for the plots of stress distributions, and try to answer these questions.

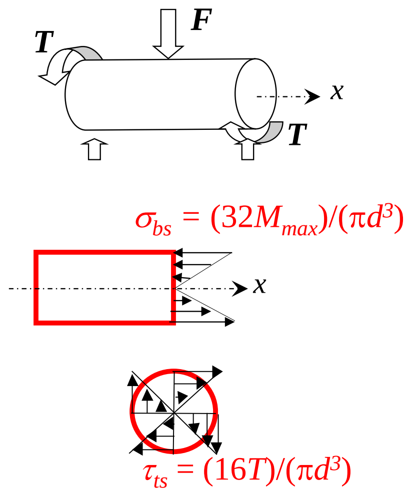


Figure 4-2. Bending and torsional shear stress distributions.

Based on functions we can tell a shaft from an axle. **Figure 4-3** shows a shaft and an axle, both are used to support the rotating gears, and the characteristics of their bending and torsional stresses marked by the red dots at the shaft surfaces. The shafts look similar except one thing, one rotates but the other does not. The rotating one is a shaft while the non-rotating one is an axle. Can you see their differences in mechanics?

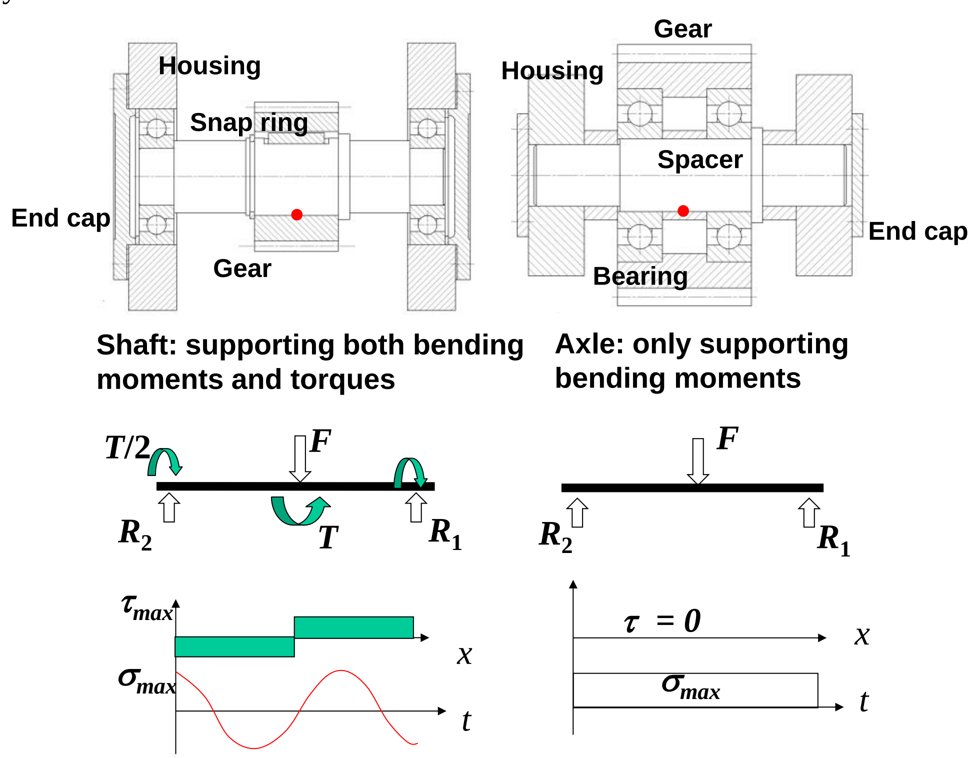


Figure 4-3 Shaft and axle. Note that end-cap bolts are not shown here in the structure drawings.

Whether a shaft is rotating or not makes a great difference. Simply look at the red dot at the surface of the shaft, or the axle. In **Figure 4-3**, the shear stress due to torsion along the shaft is not a function of time if the torque is constant, but the normal stress due to bending is a function of time. The shaft, in general, is under a combined action of torque and bending moment, and the bending stress is cyclic due to rotation, while the axle is only under bending, and the bending stress is largely static.

The structural design makes one a shaft and the other an axle although they do the same thing. The rolling element bearings are the interface between the rotating parts, such as the gear, and the stationary parts, such as the housing. In **Figure 4-3**, the axle is fixed into the housing (or a frame), while the shaft is connected to the housing though bearings.

The objective of this chapter is to learn how to design shafts – design the structure of a shaft, and conduct analyses of the stresses, strength, and life. The structure of a shaft looks very simple. It seems that we only need to consider the lengths and diameters (shoulders) of shaft segments. Yes, but it is not very easy to do all properly.

4-2 Shaft design procedure

Let's recall the shaft we saw before, which is now in **Figure 4-4**. Shaft design should be conducted hand-in-hand with the design and selection of the elements in the shaft system; otherwise we cannot decide the shape of the shaft.

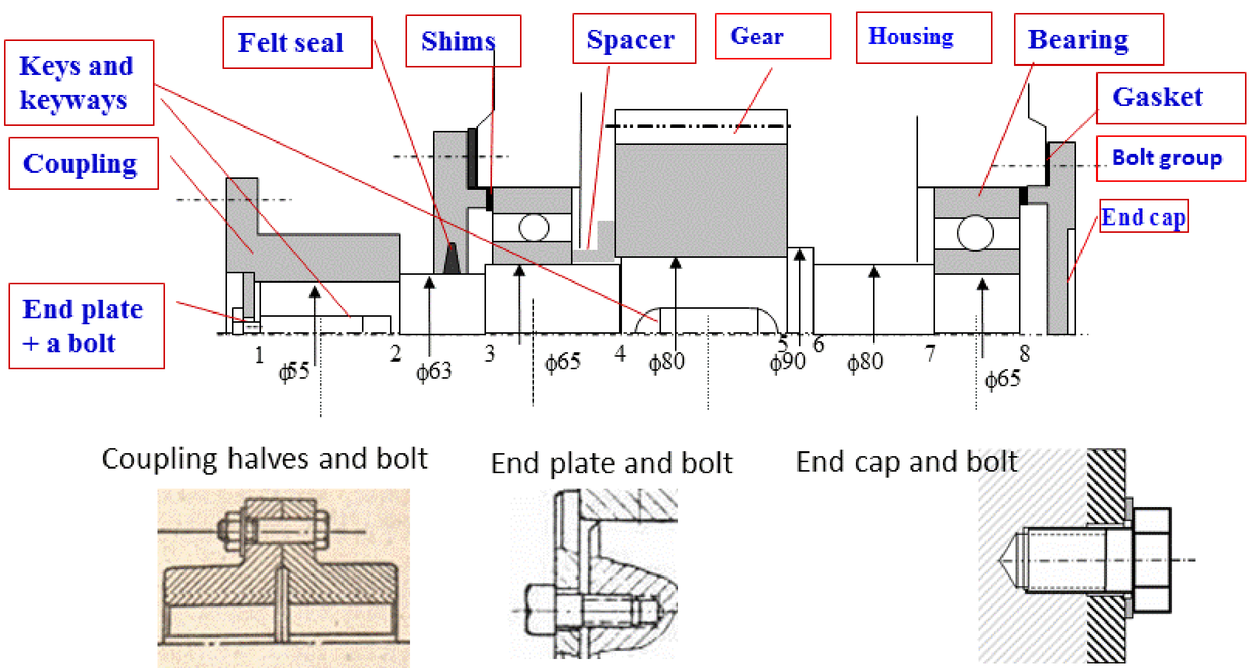


Figure 4-4 Shaft and the elements on the shaft. Dark areas mean cross-sectional cuts. The coupling, end-plate and end cap assemblies are enlarged separately for clarity. Note again, the two keyways are shown in the same view for clarity and simplicity.

The design analysis should be in the following major steps; and generally, iteration is needed.

1. Determine the minimum diameter of the shaft, which is the diameter of the thinnest shaft segment, usually at the ends of the shaft;
2. Design the structure of the shaft system: the assembly, and then the shaft;
3. Make the mechanics model for the shaft, and analyze the strength and stiffness of the shaft;
4. Conduct design modifications.

We may be given several conditions or parameters, but these are far from sufficient for the entire shaft design. We have to make decisions on the shaft shape design and do some selections. The minimum diameter is where we start. Even though design is an open-ended problem, we have to start from somewhere and stop somewhere. These four steps are discussed in the following.

1. Minimum diameter, d_{min}, of a shaft.

1-A. We need to select a material that is suitable for making a shaft. Let's try to use a ductile material because shafts are usually under cyclic stresses, and ductile materials have high fatigue strength and high endurance limit. Steels are commonly used for making industrial shafts, and we will focus our work on steel shafts.

1-B. Let's determine a **possible** minimum diameter of the shaft first. Can we directly calculate the shaft diameter for each segment using Equation (3-11-2′)? No. Why not? Think about the stress-concentration factors and strength modification parameters; some of them are related to the shaft diameter. We have to know the diameter to get to this equation. However, at the beginning, we know nothing.

Let's try to find a diameter only by considering the torsional shear stress due to the torque on the shaft. This may be at the coupling place where the shaft segment is only subjected to torsional loading, see the left end of **Figure 4-4**. Here, the allowable stress is the shear yield strength modified by a factor of safety (known from the design requirement or a selected value larger than 1.1). In Equation (4-1), the distortion energy theory is used for the shear yield strength, See Chapter 2.

$$\tau = \frac{16T}{\pi d^3} \le \tau_{all} = \frac{S_{sy}}{n_s} = \frac{0.577 S_y}{n_s} \tag{4-1}$$

From Equation (4-1), we can find a possible minimum shaft diameter

$$d_{\min} \ge \sqrt[3]{\frac{16T}{\pi \tau_{all}}} \tag{4-2}$$

1-C We have to consider other things. If there is a keyway, we should increase d_{min} by 3-5%. Use integers, or preferred numbers. Something like 12.71 mm is not a good choice; we can conveniently choose 14 mm, or even 15 mm. The minimum diameter to be used for the shaft is the larger one of the following two: 1) the value from Equation (4-2) after considering the keyway effect and rounding up to a preferred number, and 2) the value determined by a given structural requirement, for example, the match of the other half of the coupling. Note again, $d_{\min}$ may be the diameter of the shaft segment where only a torque applies, or wherever the thinnest.

2 Structure of a shaft system: shaft assembly

The only way to know the shape of a shaft is to have all elements on the shaft ready. However, that is impossible at the beginning of a design process. We must start from the minimum diameter and create the first draft of the assembly design. The word “draft” only means that the design has not been finalized yet. It does not mean a rough work. In the structural design we should design or select all elements on the shaft, and make sure that all elements have secured positions on the shaft, and that all forces are transmitted to the housing. Like grounding in an electrical circuit, we need to “ground” forces.

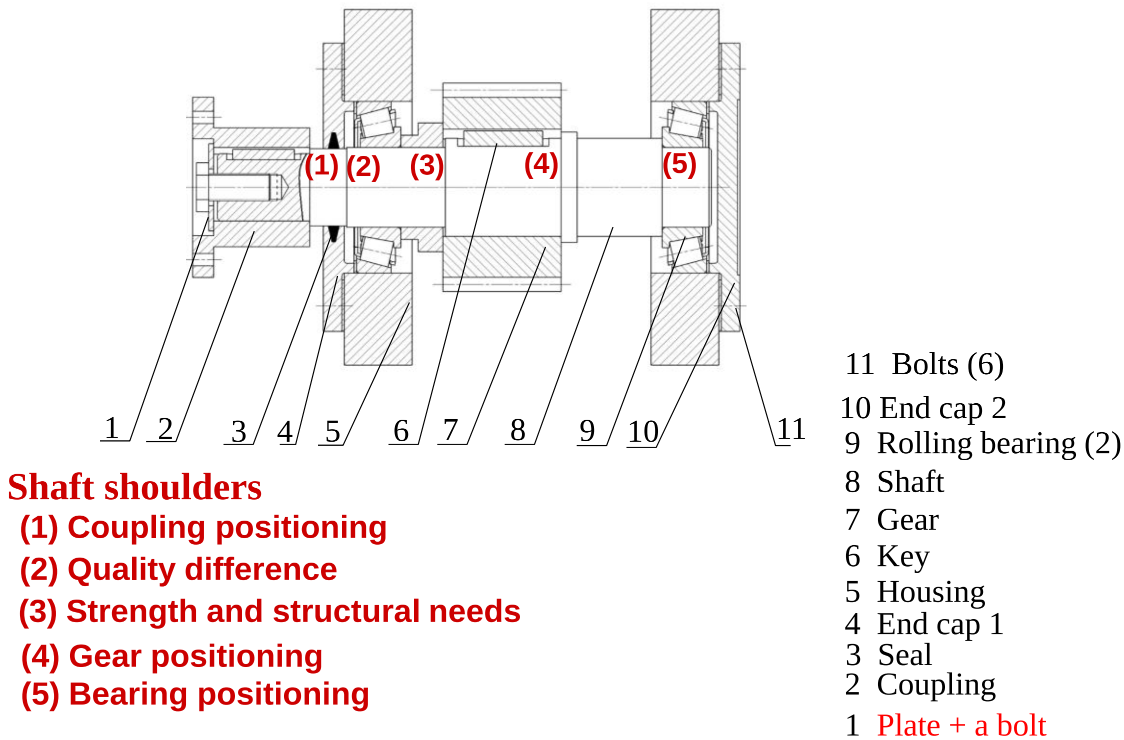
Figure 4-5 shows such a shaft assembly. Note, only the portion necessary for the shaft design is shown. From the basic knowledge of a shaft system and the minimum diameter, we have some idea about what to do: some transmission elements, a couple of bearings, etc. First, we locate these elements on the shaft. Then we decide the major dimensions: lengths and shoulders. With these dimensions we can do force and stress analyses for the strength (stiffness) of the shaft. We can then go ahead and complete the structural design if we are happy with the strength (all factors of safety are larger than 1, preferred at least 1.1). Two completed designs are shown in **Figure 4-5**.

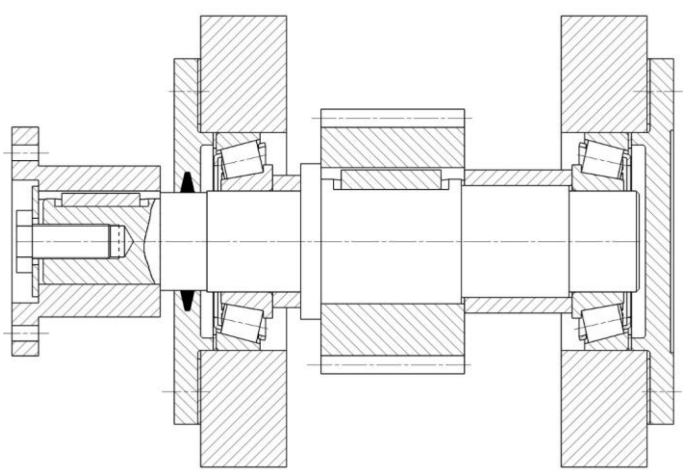
Here, the bearings are tapered roller bearings that can support both normal (radial) and axial (thrust) loads. A coupling is used as the output element, which only transmits torque. The gear here is a helical gear that transmits forces of all three directions (of course a torque, which is from the tangential force on the gear teeth).

On the shaft we have an end plate with a bolt to mount the coupling and of course the coupling, two end caps, two tapered roller bearings, a spacer, two keys, and a helical gear. Five different shaft shoulders are designed as follows: shoulder No. 1 for positioning the coupling, shoulder No. 2 for quality difference and structural needs, shoulder No. 3 for strength and structural considerations, shoulder No. 4 for positioning the helical gear, and shoulder No. 5 for positioning the bearing. Check if the forces are transmitted to the housing. Also check if all elements are securely located on the shaft and can rotate with the shaft.

The alternative design, which is the lower one in **Figure 4-5**, looks similar, and only the location of the shoulder to place the gear is different. As a result, this design uses two spacers, a short one and a long one. Can you compare the two and see the advantages and disadvantages of each design? They are both usable, but which one is more reasonable in terms of machining, assembly, operation, and maintenance?

In the assembly drawing, we need to label all parts, as shown, is a neat and ordered sequence, also list them in the part table bottom up. No need to label shoulders, which are shown here for the purpose of understanding. No part dimensions are needed here, but the overall size and assembly dimensions should be given, such as that total length from the end cover to the end of the coupling, fits of between a shaft segment and a hole (a bearing or a gear). Dimensions are not included in **Figure 4-5** for clarity of the structure design discussion, but we will practice these in our CAD work.





An alternative design

Figure 4-5 Shaft structural design.

3 Strength (stiffness) analyses: Factors of safety against fatigue and yield failure.

If the structure design is sketched, there is no problem for us to develop a mechanical model for force and stress analyses. Be sure the mechanics analysis is done right and forces are correctly located on the shaft. Here is a list of things for the strength analysis, a shortened list from that in Chapter 3.

A. Free-body diagram
B. Force analyses, bearing reactions
C. Stress analysis (bending, torsional, axial, shear, etc) to obtain the stress medium and amplitude.
D. Principal stresses (maybe not necessary)
E. Von Mises stress
F. The Goodman line, yield, and factors of safety.

4 Design modifications: Iteration

No one can guarantee that a design is made right in one trial. Some iteration is needed to make the design more reasonable, and sometimes, optimal. If the factors of safety are too small (≤1), the design is not acceptable. Improvement should be made by reducing critical stresses and/or increasing material strengths through re-selection. Think about the means to reduce stresses. If the factors of safety are too large (>>1), on the other hand, the design should be acceptable, but the structure is over designed. Improvement should be made for compact design with cost-effective considerations.

We also need to pay attention to stiffness and other structural requirements.

4-3 A design example

Figure 4-6 is a diagram of a gearbox (gear reducer) that has two pair of helical gears. Let's design the output shaft of the gear reducer with a F. S. $n_s \geq 1.3$ for infinite life. The torque is T= 960 Nm. The radial and axial loads are F_r = 1840 N and F_a= 715 N. Fatigue stress-concentration factors are K_f = 3 (bending) and K_{sf} 2.8 (torsion) for the most critical point. AISI 1020 steel is used for the shaft, S_y = 295 MPa, and S_{ut} = 395 MPa. Due to the assembly of a coupling, the diameter of the output end of the shaft cannot be smaller than 55 mm.

Solution and design

Step 1. Minimum shaft diameter based on the torque.

The possible minimum shaft diameter is at the coupling end. From Equations (4-1) and (4-2)

$$\tau = \frac{16T}{\pi d^3} \leq \tau_{all} = \frac{S_{sy}}{n_s} = \frac{0.577S_y}{n_s} = \frac{0.577(295)}{1.3} = 131MPa$$

$$d_{\min} \geq \sqrt[3]{\frac{16T}{\pi\tau_{all}}} = \sqrt[3]{\frac{16(960)}{\pi(131)(10^6)}} = 33.4mm$$

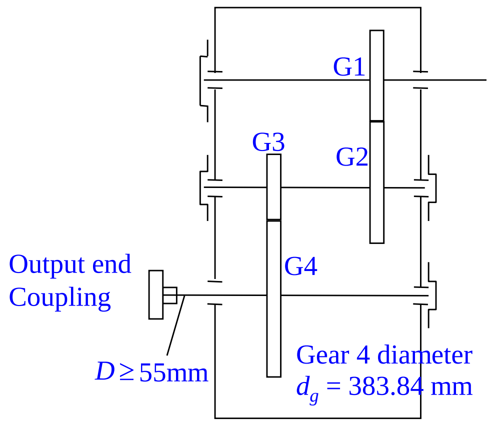


Figure 4-6 A gearbox.

There is a keyway for the coupling connection in this segment of the shaft. Therefore, we should enlarge $d_{\min}$ by about 3~5 %.

However, the structural requirement for $d_{\min}$ is 55 mm. We have to choose the larger one of them as the minimum diameter, $d_{\min}$, of the shaft. There fore $d_{\min}$ is 55 mm.

Step 2 Structure design

The gear has to be assembled on the shaft. This time we simply use ball bearings. We have already gotten some experience now. A draft of the design is shown in **Figure 4-7**. Here, the axial positions of the elements should be secured. Shoulders, lengths and diameters with fits should be determined.

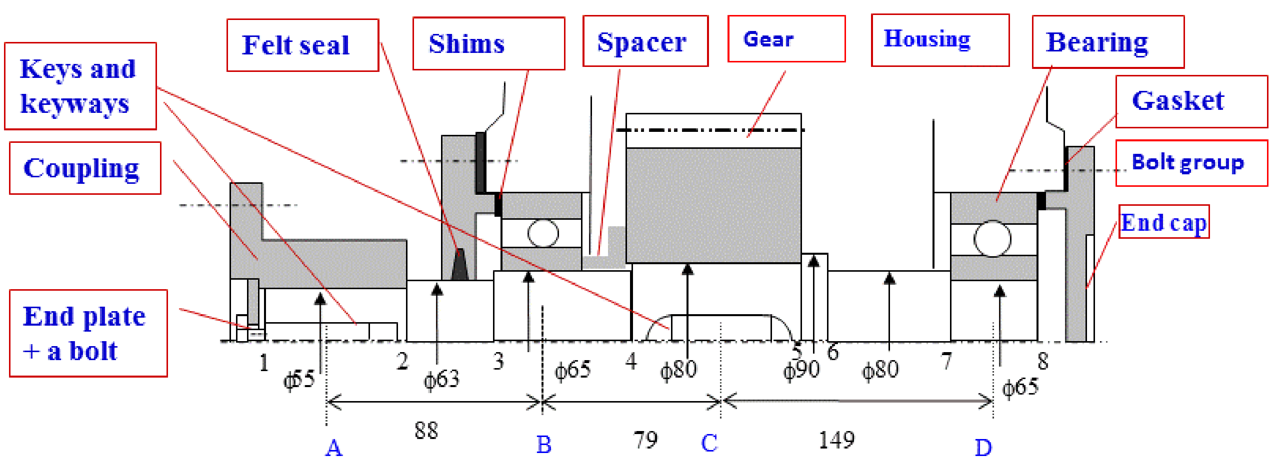


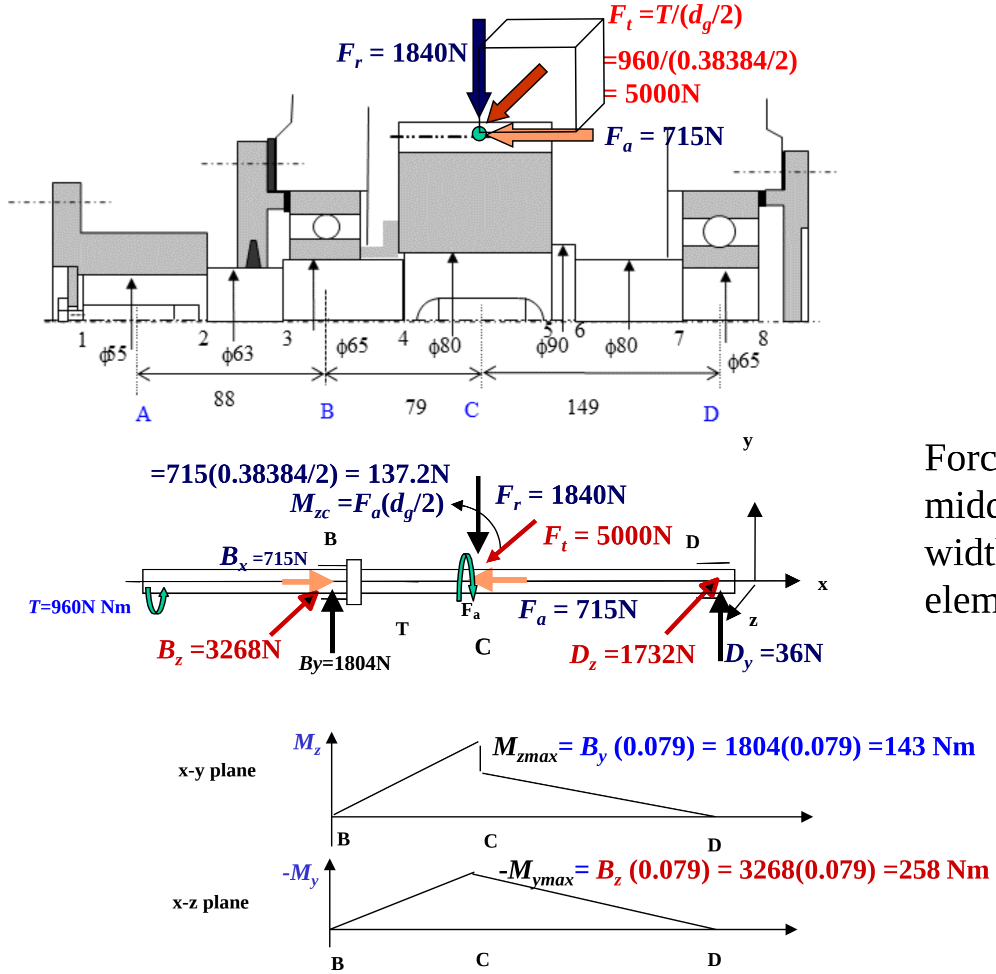
Figure 4-7 Shaft assembly for the example problem. A, B, C, and D mark the load-application locations, which are at the centers of the coupling, the bearings, and the gear, distances AB, BC, and CD are needed for the corresponding mechanics model, which is like a free-body diagram (FBD).

Step 3 Design analysis

In order to ensure that the designed shaft is sufficiently strong (and stiff), we need to conduct strength (and stiffness) analyses. We can construct a mechanical model for the shaft, or the FBD with the shaft dimensions. In this model, we need to have several critical dimensions: the distances between forces (for force positions), lengths of shaft segments and diameters (for the most critical cross section and its location).

Steps A and B of the analyses: free-body diagram and force analyses

Figure 4-8 plots the forces on the helical gear, from which we get the forces on the shaft, 3D. It also plots the bending moment diagrams in the x-y and x-z planes.



Force at the middle of the width of each element

Figure 4-8 Mechanical model (FBD) for the shaft of the example problem, as well as moment diagrams. Note that for this system, the forces are assigned at the middle of each loading element.

Step C of the analyses **Stress analyses**

Stresses to be considered are the following

Plane	Force/moment	Stress
x-y	F_r	Bending stress, transverse shear stress
	M_{zc}	Bending stress
	F_a	Compression stress
x-z	F_t	Bending stress, transverse shear stress
y-z	T	Torsional shear stress

Here, F_r and F_t can be combined to one force, which is the normal force to gear teeth, to be studied in Chapter 5.

Let's look at the cross section at C, which is likely a critical cross section. If it is, then any surface point of this cross section is the most critical point because the shaft rotates although the loads are stationary.

Combined maximum bending moment at C is:

$$M_c = \sqrt{M_{z_{\max}}{}^2 + M_{y_{\max}}{}^2} = \sqrt{143^2 + (-258)^2} = 295\text{Nm}$$

Bending stress on the surface is

$$\sigma_b = \frac{32M_c}{\pi d^3} = \frac{32(295)}{\pi(0.08)^3} = 5.87\text{MPa}$$
$$= \sigma_{ba}^0$$

σ_{ba}^0 is the amplitude of the bending stress before considering stress concentration, shown in **Figure 4-9**.

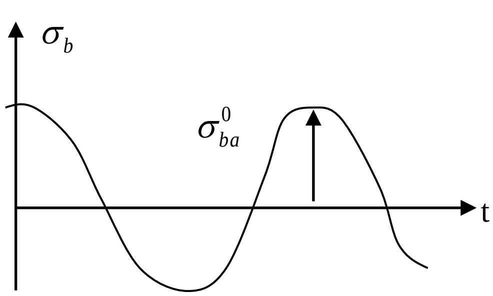


Figure 4-9 Cyclic bending stress.

The axial force, F_a, results in a constant compression stress

$$\sigma_{axial} = \frac{4F_a}{\pi d^2} = -\frac{4(715)}{\pi(0.08)^2} = -0.14\text{MPa}$$

$$= \sigma_m$$

The shear stress due to the torque is the following, which is constant if the torque does not vary.

$$\tau = \frac{16T}{\pi d^3} = \frac{16(960)}{\pi(0.08)^3} = 9.55\text{MPa}$$

$$= \tau_m$$

The shear force in this cross section is

$$V_c = \sqrt{B_z{}^2 + B_y{}^2} = \sqrt{3268^2 + (1804)^2} = 3733\text{N}$$

The surface shear stress of the neutral plane is

$$\tau_v = \frac{4V}{3A} = \frac{16V_c}{3\pi d^2} = \frac{16(3733)}{3\pi(0.08)^2} = 1.0\text{MPa}$$

$$= \tau_a^0$$

τ_a^0 is the amplitude of the shear stress variation before considering stress concentration, similar to that shown in **Figure 4-9** but with a 90^0 phase difference. Try to plot its variation with rotation.

No stress-concentration factor is applied to the midrange stresses.

Steps D - F of the analyses. We can directly use Equation (3-11-2*'*)

The endurance limit is determined through the use of modification factors as follows.

$S_e = k_f k_s k_r k_t k_m S_e'$ (Note: lower case of letter k)

$S_e' = 0.5\ S_{ut} = 0.5 \times 395 = 197.5$ MPa

k_f: surface factor

Assuming ground surface:

$k_f = eS_{ut}^f = 1.58 \times 395^{-0.085} = 0.95$

k_s: size factor

$k_s = 1.189d^{-0.112} = 1.189 \times 80^{-0.112} = 0.728$

k_r: reliability factor

For 99% survival, $k_r = 0.82$

k_t: temperature factor

$k_t = 1$ no special requirement

k_m: miscellaneous effect

$k_m = 1$ no other special problems

Therefore, $S_e = (0.95)(0.728)(0.82)(1)(1) \text{ x } 197.5 = 112$ MPa

The design against fatigue failure is done by means of the Goodman's line together with the consideration of yield.

Goodman's line, Equation (3-11-2')

$$\frac{\sqrt{(K_f \sigma_{xa})^2 + 3(K_{fs}\tau_{xya})^2}}{k_f k_s k_r k_t k_m S'_e} + \frac{\sqrt{(\sigma_{xm})^2 + 3(\tau_{xym})^2}}{S_{ut}} = 1/n_s$$

$$\frac{\sqrt{(3*5.87)^2 + 3(2.8*1.01)^2}}{(0.95)(0.728)(0.82)(1)(1)197.5} + \frac{\sqrt{(0.14)^2 + 3(9.55)^2}}{395} = 1/n_s$$

$$\frac{18.27}{112} + \frac{16.54}{395} = \frac{1}{n_s}$$

Therefore, $n_s = 4.88 > 1.3$, which is the safety factor for infinite life without fatigue. If ignoring $\sigma_{axial} = -0.14\text{MPa}$ and $\tau_v = 1.0\text{MPa}$, we gat $n_s = 5.02$. Therefore, these two stress components can be ignored in the analysis, just simply allowing a slightly larger factor of safety.

We need to check if yield would happen at the peak stress. To do so, we should calculate the peak von Mises stress without considering stress concentration factors.

$$\sigma_{VM} = \sqrt{(\sigma_{ba}^0 + \sigma_{bm})^2 + 3(\tau_m + \tau_a^0)^2}$$

$$\approx \sqrt{5.87^2 + 3 \times 9.55^2} = 17.55 \text{ MPa}$$

$$n_s = \frac{S_y}{\sigma_{VM}} = \frac{295}{17.55} = 16.8 \gg 1.3,$$

It is safe, and the design is accepatble. However, the factors of safety are very large. The shaft is over designed; and for this case, it was due to choice of d_{min} much larger than needed by the strength consideration.

The result is plotted in **Figure 4-10**.

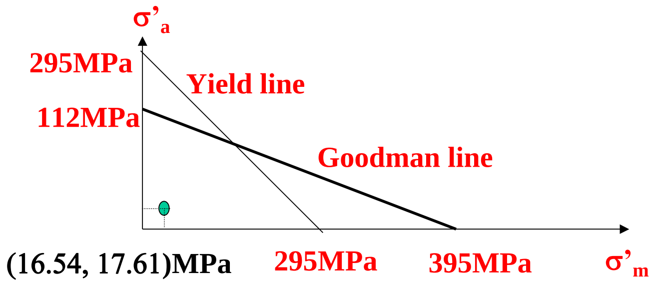


Figure 4-10 Strength lines and the working stress.

Step 4 Design modification

This is an example of an over-designed shaft due to a geometric constraint. Modifications should be conducted, such as choosing a less expensive material to lower the cost, or increasing the intensity of the loads to achieve higher power density.

Chapter summary

This is the first chapter for machine elements; it introduces the concept of shafts, shaft classification, and the shaft design process. Minimum diameter determination, structural design, design analyses, and design modifications are discussed.

References

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Shigley, J. and Mischke, C., 1989, "Mechanical Engineering Design," McGraw Hills.
Pu, L., Chen, D., and Wu, L., 2013, *Machine Design*, China Higher Education Press.

Media: **315 students should watch the short movies before the class.**