Chapter 2 Basic Failure Theories for Static Loading

2-1 Why do we need to study failure theories?

Machine components may fail for many reasons. The most important goal of mechanical design is to make sure that failure of a component does not happen in the desired machine operation life and under required operating conditions. Therefore, it is important to know when and how a part fails. Mechanical failures may be in the form of structural fracture, they may be due to a surface damage if surface stresses are too high, and/or if friction and wear are too much. Excessive plastic deformation is also a failure because it may alter part dimensions.

Failures may be caused by static loading, which will be discussed in this chapter. Failure may be a result of variable loading and may be in the form of fatigue, which will be discussed in the next chapter. We need to define several terminologies first.

A static load is a stationary load, which may be a force, a moment, or torque, applied on a component; it results in static stresses. The magnitude, the point of application, and the direction of the load do not vary. Strictly speaking, a static load does not exist. Here, we have to make some simplifications and consider some cases as static if the loads do not vary much. Corresponding to failure, we need to define strength. **The strength of a component** is a built-in mechanical property of a mechanical component. It is usually measured by a quality named the factor of safety (F.S., or n_s), which is the margin of a stress to failure. A mechanical component should be designed with a sufficient strength, or a sufficiently large factor of safety. Note that the allowable stress is not necessarily the strength of the material used; it may be the strength of a component. A safe design requires $n_s \geq 1$, and be safer, we can practice $n_s \geq 1.1$.

Factor of safety, F.S. = n_s = Allowable stress/design stress = σ_{all}/σ_d

The strength of a component and the strength of a material are two different things although they are related. So far from Mechanics of Materials, we know a lot about the strength of materials. Commonly used properties are the yield limit, S_y, and the ultimate limit, S_u. The values of these strengths are obtained from tensile tests in a controlled loading environment. The test results are summarized in terms of stress-strain relationships, such as the plot shown in **Figure 2-1**. Most engineering materials for structural design are either ductile, such as most metals (not including cast metals) and polymers, or brittle, such as most cast metals and ceramics. The yield strength is more important for a ductile material than for other materials, and for the former, the yield strengths under tension (S_{yt}) and compression (S_{yc}) are about the same, $S_{yt} = S_{yc} = S_y$. Such ductile materials are not very sensitive to stress raisers (notches). Brittle materials, on the other hand, are likely to fail due to fracture and are sensitive to stress raisers. The compression strength of a brittle material is higher than its tensile strength, which means $S_{ut} < S_{uc}$.

We can calculate the designed stress, σ_d, of a part based on mechanics. However, we know little about the allowable stresses of a part, σ_{all}. What we have available are the properties of materials from a materials handbook or database obtained from standard strength tests. The real operating conditions are much more complicated than those defined in laboratory tensile tests.

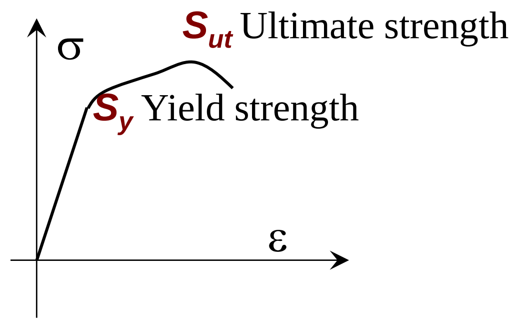


Figure 2-1. A stress-strain diagram.

Figure 2-2 compares the loading and stresses for a specimen in a tensile test and those for a real mechanical component in a machine. The specimen in the tensile test is under a uniaxial loading with only one non-zero principle stress component. A real mechanical component has complicated part geometry and is operated under complicated conditions in a true engineering environment. It is likely to be subjected to combined loading modes that can cause complicated three-dimensional stresses. What are the links between the two?

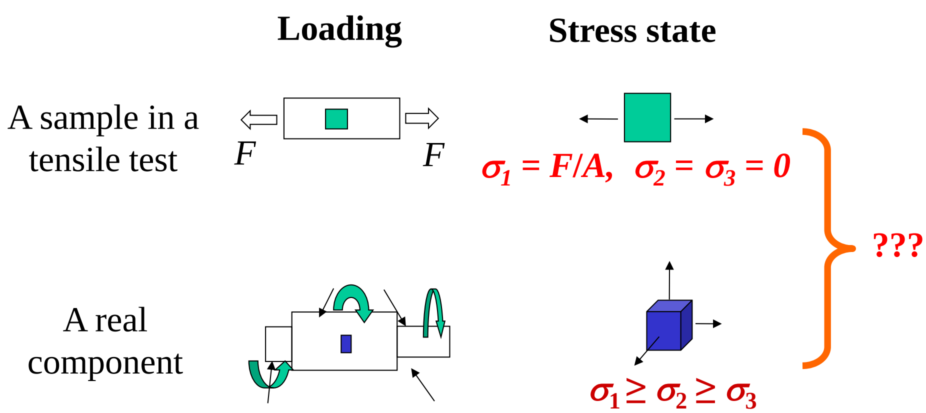


Figure 2-2. Comparison of loading and stresses for a bar specimen in a tensile test and those for a mechanical component in a machine.

For critical parts with extremely high safety and reliability requirements, such as those used in an airplane or a space shuttle, designers have to test the prototypes of a designed product. However, for general-purpose parts used in general applications, one may utilize the data published in a materials handbook. For this purpose, a link between the laboratory material test results and three-dimensional stresses in mechanical component should be developed. That is the objective of this chapter and that is the use of design theories. Here, we will study failure mechanisms and failure theories, develop design criteria using material properties, and later modify the criteria with necessary factors that account for the mechanical differences between specimens in material tests and real parts.

2-2 Summary of failure theories and design criteria

2-2-1 **Maximum-normal-stress theory** (MNST): The maximum-normal-stress theory states that failure occurs when one of the principal stresses reaches the maximum normal stress in a tensile specimen (minimum for compression) when the specimen begins to fail [(1)]. **Figure 2-3** illustrates this theory, where the safe region for design defined by the ultimate strengths in tension and compression is plotted in terms of σ_A and σ_B. σ_A and σ_B are the principal stresses that can be either in tension or compression, we can treat them as variable x, y.

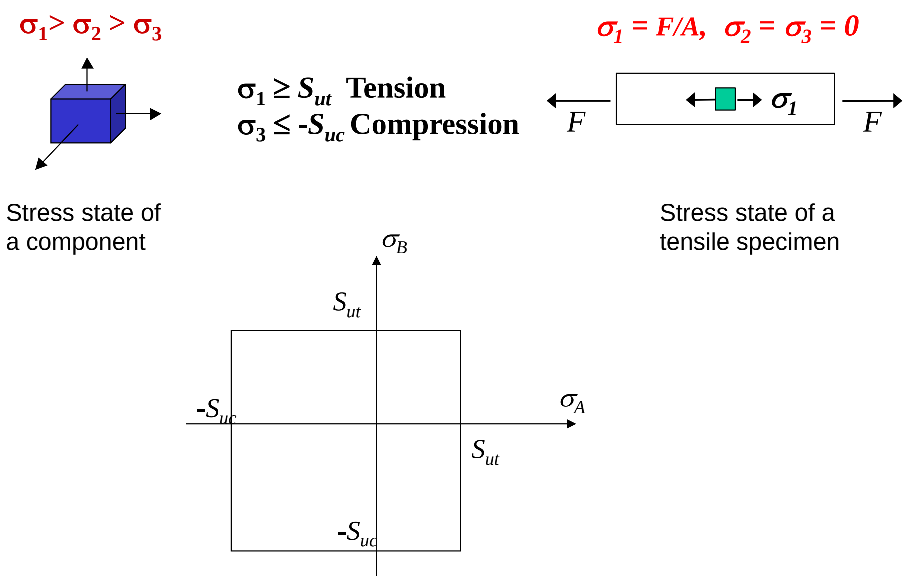


Figure 2-3. Illustration of the maximum-normal-stress theory. The square enclosed by the ultimate strengths is the safe design region.

The corresponding **design criteria** based on the maximum-normal-stress theory can be described by the following. Here, n_s is the factor of safety, and the allowable stress is the ultimate strength of the material.

$$\sigma_1 \leq S_{ut}/n_s \quad \text{Tension} \tag{2-1-1}$$

$$\sigma_3 \geq -S_{uc}/n_s \quad \text{Compression} \tag{2-1-2}$$

2-2-2 **Maximum-shear-stress theory** (MSST, or the Tresca yield criterion): The maximum-shear-stress theory states that yield occurs when the maximum shear stress reaches the maximum shear stress in a tensile specimen when the specimen begins to yield [(1)]. Note, this is for failure due to yield, not fracture! **Figure 2-4** illustrates this theory for a 2D case for clarity, where the safe region for design defined by the yield strengths in tension and compression is plotted in terms of σ_A and σ_B. σ_A and σ_B are principal stresses and can be either in tension or compression. The safe region based on the maximum-shear-stress theory is a combination of the criteria based on the maximum shear stress consideration, which is in the second and fourth quadrants, and yield due to normal stresses, which is in the first and third quadrants.

The corresponding **design criteria** based on the maximum-shear-stress theory can be described by the following.

$$(\sigma_1 - \sigma_3) \leq S_y / n_s \tag{2-2}$$

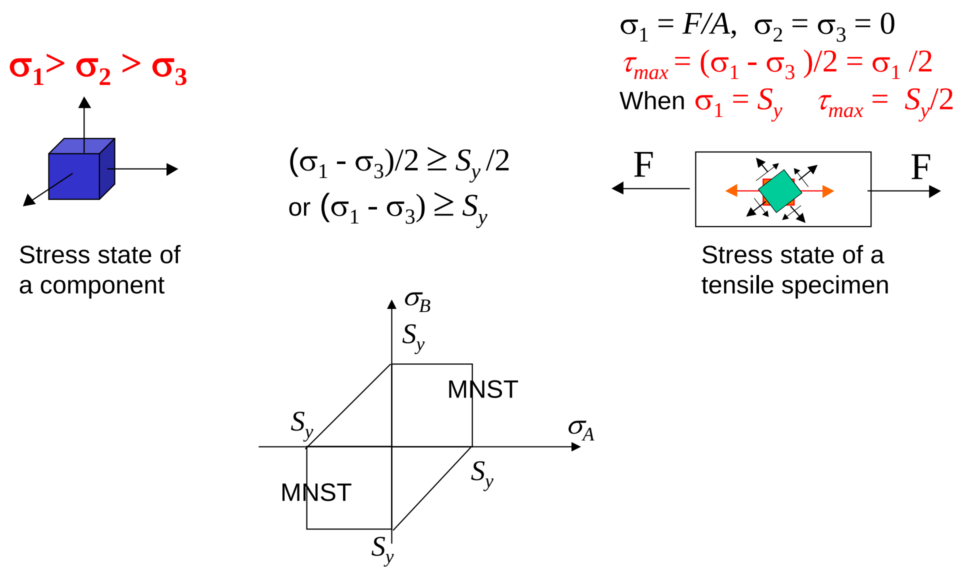


Figure 2-4. Illustration of the maximum-shear-stress theory. In the tensile specimen, upper right, the maximum shear stress is $\tau_{\max} = (\sigma_1 - \sigma_3)/2 = \sigma_1/2 = F/(2A)$, and at yield failure, $\sigma_1 = S_y$. Therefore, $\tau_{\max} = S_y/2$, which is the shear yield strength.

We may now define the shear yield strength of a material from a pure shear stress state based on the maximum shear stress theory as follows.

$$S_{sy} = S_y/2 \tag{2-3}$$

Let's spend some time to see the nature of the maximum-shear-stress theory.

Hooke's law for uniaxial stress: $\sigma_1 = E\varepsilon_1$ or $\varepsilon_1 = \sigma_1 / E$ (2-4-1)

Hooke's law for a general 3D case (principal stress state):

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{E}\left(\sigma_1 - \nu(\sigma_2 + \sigma_3)\right) \\
\varepsilon_2 &= \frac{1}{E}\left(\sigma_2 - \nu(\sigma_1 + \sigma_3)\right) \\
\varepsilon_3 &= \frac{1}{E}\left(\sigma_3 - \nu(\sigma_2 + \sigma_1)\right)
\end{aligned} \tag{2-4-2}$$

Defining *e* as $e = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, which is the total strain, plugging in the equations in the above, then $e = \frac{(1-2\nu)}{E}(\sigma_1 + \sigma_2 + \sigma_3)$. Solving (2-4-2) leads to the generalized Hooke's law that links principal stresses (σ_i) with principal strains (ε_i), i = 1, 2, 3.

$$\sigma_1 = e\frac{E}{(1-2\nu)} - (\sigma_2 + \sigma_3) = \frac{Ee}{(1-2\nu)} + \nu(E\varepsilon_1 - \sigma_1)$$

Which is $\sigma_1 = \frac{\nu Ee}{(1-2\nu)(1+\nu)} + \frac{E\varepsilon_1}{(1+\nu)}$

Defining the first term in the above as σ_0 and the second as σ_{n1}, do the same for σ_{n2}, σ_{n3}, then

$$\sigma_1 = \frac{\nu Ee}{(1+\nu)(1-2\nu)} + \frac{E\varepsilon_1}{1+\nu} = \sigma_0 + \sigma_{n1}$$

$$\sigma_2 = \frac{\nu Ee}{(1+\nu)(1-2\nu)} + \frac{E\varepsilon_2}{1+\nu} = \sigma_0 + \sigma_{n2} \qquad (2\text{-}4\text{-}3)$$

$$\sigma_3 = \frac{\nu Ee}{(1+\nu)(1-2\nu)} + \frac{E\varepsilon_3}{1+\nu} = \sigma_0 + \sigma_{n3}$$

Note that σ_0 appears to be the same for all three principle stresses, which is usually referred to as the hydrostatic component of a stress. However, σ_{n1}, σ_{n3} and σ_{n3} are non-hydrostatic.

We find $\tau_{max} = (\sigma_1 - \sigma_3)/2 = (\sigma_{n1} - \sigma_{n3})/2$

For hydrostatic loading, $\sigma_1 = \sigma_2 = \sigma_3$, which means that all non-hydrostatic stresses are zero, therefore, $\tau_{max} = (\sigma_1 - \sigma_3)/2 = 0$, the material will never fail due to yield theoretically! This argument suggests that only the non-hydrostatic portion of the stress components would cause failure. This discussion leads to the next design criterion.

2-2-3 Distortion-energy theory (**DET, or the von Mises criterion**): The distortion-energy theory states that yield occurs when the total distortion energy in a unit volume reaches the distortion energy in the same volume in a tensile specimen when the specimen begins to yield (1). Note again, this is also for yield! **Figure 2-5** illustrates this theory, where the safe region for design defined by the yield strengths in tension and compression is plotted in terms of principal stresses σ_A and σ_B. Again, σ_A and σ_B can be either in tension or compression.

This theory is motivated by the suggestion from the maximum-shear-stress theory. Let's see the strain energy in a unit-volume material in a tensile specimen.

$$U_{tot} = \frac{1}{2}\sigma_1\varepsilon_1$$

For the case of principal stresses, $U_{tot} = \frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right)$. Using (2-4-2) leads to

$$U_{tot} = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{v}{E}\left(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_3\sigma_2\right)$$

The average stress energy for the hydrostatic portion is (counting the same along the three directions),

$$U_{ave} = \frac{3}{2}\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)\left(\frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3}\right) = \frac{3(1-2v)}{18E}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 = \frac{(1-2v)}{6E}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2$$

The distortion energy can be approached by the difference between the total energy, u_{tot}, and the energy produced by the average stress.

$$U_d = U_{tot} - U_{ave} = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{v}{E}\left(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_3\sigma_2\right) - \frac{(1-2v)}{6E}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2$$
$$= \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{v}{E}\left(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_3\sigma_2\right) - \frac{(1-2v)}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{2(1-2v)}{6E}\left(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_3\sigma_2\right)$$
$$= \frac{(1+v)}{3E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{(1+v)}{3E}\left(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_3\sigma_2\right) = \frac{1+\upsilon}{3E}\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2}{2}\right]$$

Therefore,

$$U_d = U_{tot} - U_{ave} = \frac{1+\upsilon}{3E}\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2}{2}\right] = \frac{1+\upsilon}{3E}\sigma_{VM}{}^2$$

Thus, the concept of a new stress is defined, which is usually called the von Mises stress,

$$\sigma_{VM} = \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2}{2}} \qquad \text{for a general 3-D case} \qquad (2\text{-}5\text{-}1)$$

$$\sigma_{VM} = \sqrt{(\sigma_1)^2 + (\sigma_2)^2 - \sigma_1\sigma_2} \qquad \text{for a 2-D case} \qquad (2\text{-}5\text{-}2)$$

$$\sigma_{VM} = \sqrt{(\sigma_x)^2 + 3(\tau_{xy})^2} \qquad \text{for a 2-D case if } \sigma_y = 0 \qquad (2\text{-}5\text{-}3)$$

Note that for the last equation,

$\sigma_1 = (\sigma_x)/2 + [(\sigma_x/2)^2 + (\tau_{xy})^2]^{1/2}$ and $\sigma_2 = (\sigma_x)/2 - [(\sigma_x/2)^2 + (\tau_{xy})^2]^{1/2}$

Bring these to Equation (2-5-2), then

$$\sigma_{VM} = \sqrt{(\sigma_1)^2 + (\sigma_2)^2 - \sigma_1\sigma_2} = \sqrt{\left(\frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}\right)^2 + \left(\frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}\right)^2 - \left(\frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}\right)\left(\frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}\right)}$$

$$= \sqrt{\left(\frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}\right)^2 + \left(\frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}\right)^2 - \left(\frac{\sigma_x}{2}\right)^2 + \left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$= \sqrt{2\left(\frac{\sigma_x}{2}\right)^2 + 2\left\{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^{\ 2}\right\} + \tau_{xy}^{\ 2}} = \sqrt{\sigma_x^{\ 2} + 3\tau_{xy}^{\ 2}}$$

Figure 2-5 illustrates the criterion for failure prediction and the safe region based on this theory, which is an elliptical region, assuming two non-zero principal stresses.

$$u_d = u_{tot} - u(\sigma_{ave}) = \frac{1+\upsilon}{3E}\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2}{2}\right]$$

$$= \frac{1+\upsilon}{3E}\sigma_{VM}^{\ 2} \qquad\qquad = \frac{1+\upsilon}{3E}\sigma_1^{\ 2} \Rightarrow \frac{1+\upsilon}{3E}S_y^{\ 2}$$

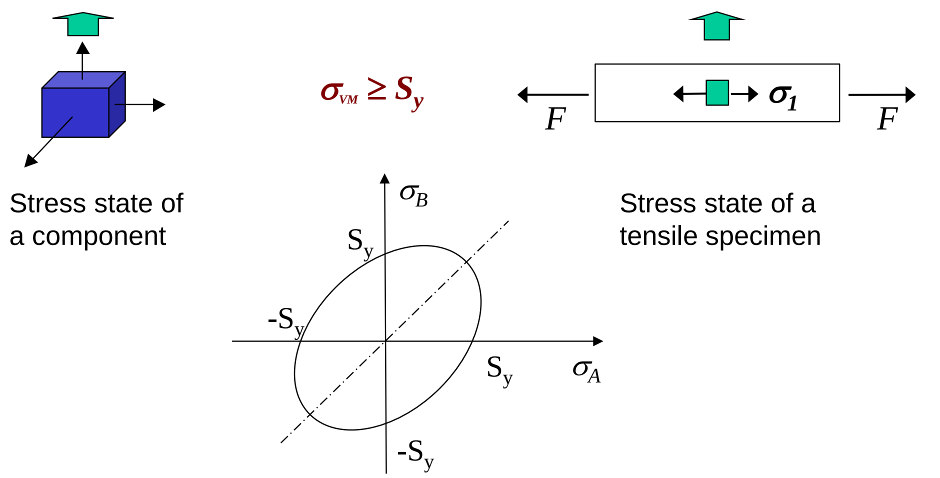


Figure 2-5. Illustration of the distortion-energy theory.

The corresponding **design criterion** is

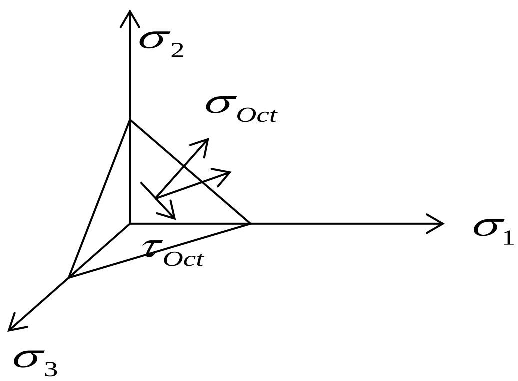
$$\sigma_{VM} \leq S_y/n_s \qquad (2\text{-}6)$$

We may also define the shear yield strength from a pure shear stress state, but this time, based on the distortion energy theory, which is given below. How? Is this the same as or different from that defined with the maximum shear stress theory?

$$S_{sy} = \frac{S_y}{\sqrt{3}} \qquad (2\text{-}6)$$

The von Mises stress is related to the octahedral shear stress, defined in the plane equally intersecting the axes of the three principal stresses (**Figure 2-6**).

The octahedral normal stress is the hydrostatic stress, or the first invariant of normal stress, I_1, while the octahedral shear stress is related to the second deviatoric stress invariant, J_2.



$$\sigma_{Oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = I_1$$

$$\tau_{Oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$

$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]$$

Figure 2-6. Octahedral stresses.

2-2-4 **Coulomb-Mohr design criterion** (Internal friction theory, IFT): The maximum-normal-stress theory may be risky if the working stresses do not have the same sign, i.e., one is positive while the other is negative. The Coulomb-Mohr theory modifies the description of failure in the second and fourth quadrants and yields the following design criteria.

$$\frac{\sigma_1}{S_{ut}} - \frac{\sigma_3}{S_{uc}} = \frac{1}{n_s}$$ In the second and fourth quadrants (2-7-1)

$$\sigma_1 = \frac{S_{ut}}{n_s}$$ In the first quadrant (2-7-2)

$$\sigma_3 = \frac{S_{uc}}{n_s}$$ In the third quadrant (2-7-3)

2-2-5 **Modified Mohr design criterion** (MMT): This theory is another modification of the maximum-normal-stress theory. The failure boundaries in the second and fourth quadrants are redefined, which yields the following design criteria. **Figure 2-7** compares the modifications of the Coulomb-Mohr design criterion and the modified Mohr design criterion.

$$\frac{\sigma_1}{\frac{S_{ut}S_{uc}}{S_{uc} - S_{ut}}} - \frac{\sigma_3}{S_{uc}} = \frac{1}{n_s}$$ If $\sigma_1 > 0$ and $\sigma_3 < -S_{ut}$ (2-8-1)

$$\sigma_1 = \frac{S_{ut}}{n_s}$$ If $\sigma_3 > -S_{ut}$ (2-8-2)

$$\sigma_3 = \frac{S_{uc}}{n_s} \qquad \text{If } \sigma_1 < 0 \qquad (2\text{-}8\text{-}3)$$

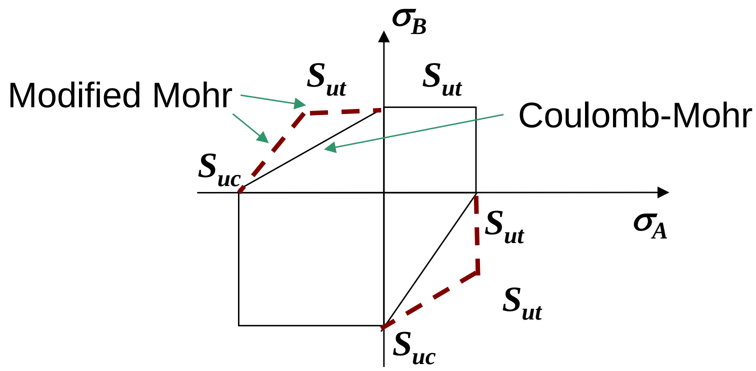


Figure 2-7. Comparison of the Coulomb-Mohr (solid) and the Modified Mohr (dotted) design criterion.

2-3 Applications of the design criteria

The selection of a design criterion should be based on the failure mode that a material, or a part, may experience when it is over loaded. Most ductile materials fail due to yield, and their yield strengths under tension and compression are about the same, $S_{yt} = S_{yc} = S_y$. The theories for equal strength, i. e., the maximum-shear-stress theory and the distortion energy theory, should be applied. Brittle materials, on the other hand, may fail due to fracture and are sensitive to stress raisers. The compression strength of a brittle material is higher than its tensile strength, $S_{ut} < S_{uc}$. The maximum-normal-stress theory and the Modified Mohr design criterion are usually applicable.

Example 2-1. The mechanics analysis of a shaft indicates that the stress components of the most critical point are: σ = 116 MPa, and τ = 174MPa. The designer has two materials available, AISI 1040, which is a medium-carbon steel (yield strength: S_y = 350 MPa), and an ASTM No. 25 cast iron (Tensile strength: S_{ut} = 170 MPa, compressive strength: S_{uc} = 660 MPa). Let's compare the factors of safety obtained from different design criteria.

Solution

Step 1: mechanics analyses to get each individual stresses. Done

Step 2: Principal stresses

$$\sigma_{max} = (\sigma)/2 + [((\sigma_x)/2)^2 + (\tau)^2\ ]^{1/2} = 116/2 + [(116/2)^2 + 174^2]\ ^{1/2} = 241.4 \text{ MPa}$$
$$\sigma_{min} = (\sigma)/2 - [((\sigma\ )/2)^2 + (\tau)^2\ ]^{1/2} = 116/2 - [(116/2)^2 + 174^2]\ ^{1/2} = -125.4 \text{ MPa}$$

σ_1 =241.4 MPa $\qquad \sigma_2 = 0 \qquad \sigma_3$ = -125.4 MPa

Step 3 Factors of safety

Criteria	**Equations**	**Results**
Maximum-shear-stress theory	$n_s = \dfrac{S_y}{\sigma_1 - \sigma_3} = \dfrac{350}{241.4 + 125.4} = \dfrac{350}{366.8}$	$= 0.95$
Distortion-energy theory	$n_s = \dfrac{S_y}{\sigma_{vm}} = \dfrac{S_y}{\sqrt{\sigma_x^2 + 3\tau_{xy}^2}} = \dfrac{350}{\sqrt{116^2 + 3(174)^2}} = \dfrac{350}{323}$	$= 1.08$
Maximum-normal-stress theory	n_S (Tension) = S_{ut} /σ_1 = 170/241.4	= 0.70
	n_S (Compression) = - S_{uc} /σ_3 = - S_{uc} /σ_3 = 660/125.4	= 5.26
Modified Mohr theory	n_S = S_{ut}/σ_1 = S_{ut} / σ_1 = 170/241.4 (σ_3 > - S_{ut})	= 0.70

The results are plotted in **Figures 2-8** and **2-9**.

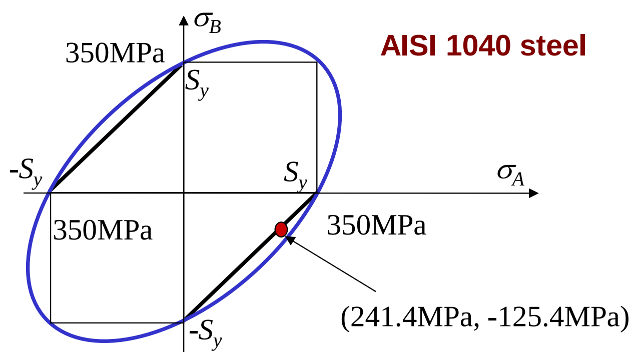


Figure 2-8. Example 2-1, results for using AISI 1040. The red dot shows the point of the given stress status, which is just outside of the safe region defined by the maximum shear stress theory but inside that by the distortion energy theory.

What can we conclude for the use of this AISI steel? AISI 1040 steel marginally satisfies the need based on the distortion-energy criterion. The maximum-shear-stress criterion is more conservative. What should the factor of safety n_s be the for distortion energy theory if we require $n_s \geq 1.1$ for the maximum shear stress theory?

What can we conclude for the cast iron? ASTM No. 25 cast iron is not a proper choice because its tensile strength is too low. However, this material can be well used for compression loading.

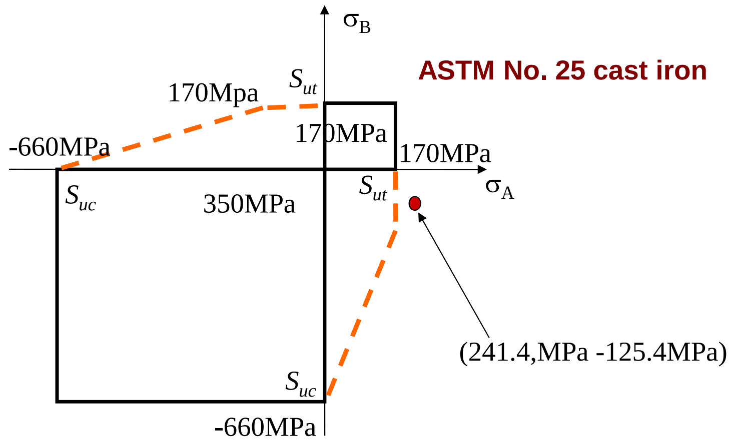


Figure 2-9. Example 2-1, results for using AISI 1040. The working stress marked by the red dot is away from the safe region

One more question. Example 2-1 is for a shaft, what kinds of loading would lead to these stresses, σ = 116 MPa, and τ = 174MPa, at the most critical point? Bending moments? Tensile forces? Torques? Transverse forces? Or combination of some of them?

Example 2-2. Let's now analyze the problem from Chapter 1, which we summarized in Example 1-3, to see which point of 1, 3, and 4 is more critical. Based on the maximum shear stress theory, all these three points are critical. In order to use the distortion energy theory, we need to calculate the von Mises stress, which is σ_{VM} = 51.2 MPa at points 1 and 3, but σ_{VM} = 50.2 MPa at point 4. Then point 4 is out. We need an additional condition to examine points 1 and 3, which is that, a large tensile stress intends to open a crack. Therefore, we use the maximum normal stress theory for tensile principal stresses as the secondary consideration. As a result, Point 3 is what we are looking for.

Is that all? Suppose the stresses are not static, which is most likely the case, then?

Chapter summary. This chapter studies several basic design theories, the maximum normal stress theory, the maximum shear stress theory, the distortion energy theory, the Coulomb-Mohr theory, and the modified Mohr theory. The maximum shear stress theory and the distortion-energy theory are widely used for the design of components using ductile materials, while the rest are for the design using brittle materials.

References

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Media: WhyCriteria, MNST, MSST are on U-Tube

315 students should watch these short movies before the class.