Chapter 5 Transmission Elements

5-1 What are transmission elements

Transmission elements transmit force and motion from one shaft to another through contact and relative motion. Gears, belts, chains, and pulleys, etc., are all transmission elements. Gears are rigid elements for short-distance, high power density transmissions, while belts and chains are flexible elements for long-distance power transmissions. We have seen one of the simplest transmissions, a gear reducer, in Chapter 1. The pre-class short video showed another one; here it is again, but in the form of a mechanism diagram, in **Figure 5-1**, with a belt-pulley set that delivers power from an electric motor to the reducer. The arrows point to the transmission elements, which are four gears and a belt-pulley set. In this chapter, we will learn most of these elements. However, our focus will be on gears due to the time constraint.

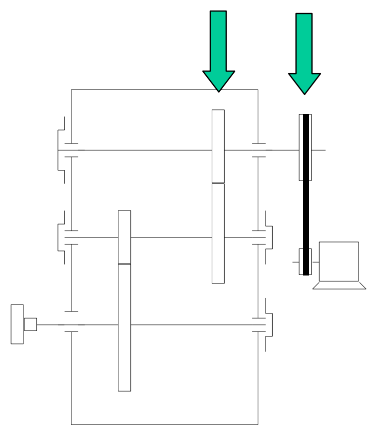


Figure 5-1. Two-stage gear reducer with an input belt-pulley set connected to an electrical motor.

5-2 Concepts of flexible transmission elements

Two types of commonly seen flexible transmission elements are belts and chains. They are excellent for long-distance power transmission and permit flexible and convenient shaft position adjustment. They are also simple in structure and convenient for maintenance.

Belts and belt drives

There are several types of belt drives, classified based on the shapes of belts, such as the flat belt, V-belt, and timing belt. **Figure 5-2** shows a typical flat-belt drive, which consists of a belt and two pulleys. One of the pulleys is the input pulley, or the driver, rotated by an input shaft, such as that of a motor. The other is the output pulley, or the driven, powered by the belt. Guess what the driving force is? Friction! Therefore, belt drives are also called traction drives.

Friction is related to the friction coefficient at the interface formed by the surfaces of the belt-pulley pair and to the stretching tension that produces the normal force between the surfaces of the pulley and the belt in contact. It is also related to the angle that the belt wraps over the pulley, or the wrap angle, because the wrap area corresponding to the wrap angle is the nominal contact area. When a belt drive works, its belt is stretched more at one side than at the other. That is why we usually see a slack side (or the loose side, with a lower tension). The other side is the tight side (with a higher tension). Where should we place the slack side in a belt drive design? Should it be at the upper side or the lower side? The upper side should be a good choice for the slack side because it can help increase the wrap angle, which is good for friction enhancement.

The velocity ratio, i_{12}, is defined as the ratio of the angular speed of pulleys 1 and 2, as shown by Equation (5-1). Note that the pulley surface speeds and the belt speed are nominally the same although the belt creeps on the pulley surfaces.

$$i_{12} = \frac{n_1}{n_2} = \frac{\omega_1}{\omega_2} = \frac{D_2}{D_1} \tag{5-1}$$

The length of a belt is purely a geometry issue. Equation (5-2) takes care of the length calculation.

$$L = \sqrt{(2c)^2 + (D_2 - D_1)^2} + \frac{\pi}{2}(D_2 + D_1) + \frac{\pi}{180}(D_2 - D_1)\sin^{-1}\left(\frac{(D_2 + D_1)}{2c}\right) \tag{5-2}$$

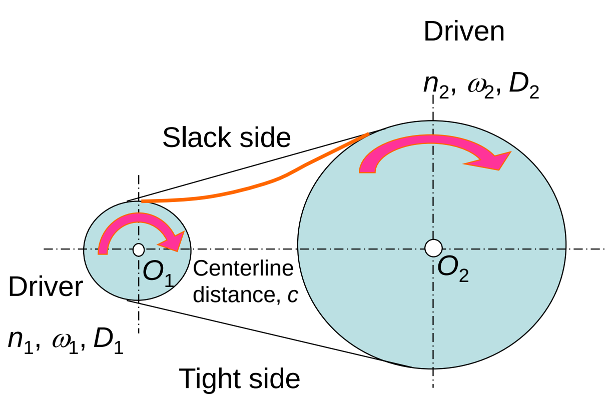


Figure 5-2. A typical flat belt drive.

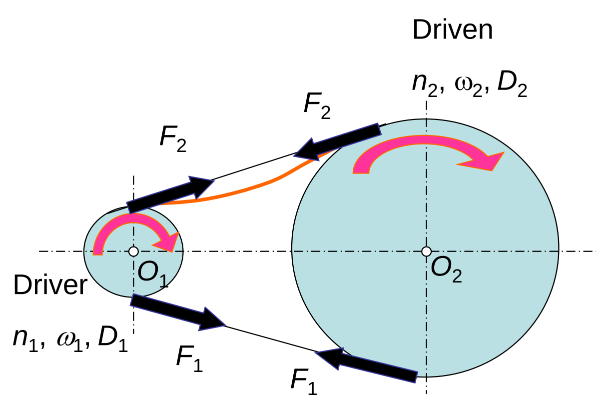


Figure 5-3. Tensions in a belt.

Figure 5-3 shows the tensions in a belt, F_1 and F_2, where $F_1 = F_2$ + friction. Clearly, we have $F_1 > F_2$. The torque transmitted, T, is

$$T = (F_1 - F_2)D/2 \tag{5-3}$$

The power transmitted, h, is

$$h = (F_1 - F_2)u \tag{5-4}$$

Here, u is the speed of the belt, which is again normally the same as the speed of the pulley surfaces.

The relationship between the two tensions when they reach the maximum without belt slip, if the centrifugal force is neglected, can be expressed as follows using the static friction coefficient, μ_s, and the wrap angle, θ. The static friction is the largest friction that the belt-pulley interface can take without belt slip.

$$\frac{F_1}{F_2} = e^{\mu_s \theta}$$

If the centrifugal force is considered as mv^2, with m as the belt mass per unit length and u the belt speed, the above becomes,

$$\frac{F_1 - mu^2}{F_2 - mu^2} = e^{\mu_s \theta}$$

For the belt segment getting into and leaving the contact region (the wrap angle), the tensions can be related to initial tension F_i as $F_1 = F_i + \Delta F$ and $F_2 = F_i - \Delta F$, leading to $F_i = \frac{F_1 + F_2}{2}$. The lower limit of F_2 is zero, and then the limit for the tight side tension is, $F_1 = 2F_i$. The only way to increase the load capacity is to increase the initial tension.

Belt drives have many advantages. The very elastic belt element makes the transmission noise level low. We mentioned the long-distance power transmission and the flexibility in shaft position adjustment. The belt arrangement is also very flexible. **Figure 5-4** shows two possible belt arrangements. In one arrangement the shafts rotate in the same direction (**Figure 5-4**, left), but in the other, they rotate oppositely (**Figure 5-4**, right).

Figure 5-4 Two belt arrangements, left, the pulley speeds are the same, right, the pulley speeds are opposite.

However, there are some disadvantages here too. 1) Elastic creep is inevitable due to the tension difference, which is intrinsic in belt drives. 2) An initial tension is needed, and the tension should be maintained. 3) The belt may slip over the pulley surfaces. 4) Creep and slip make the transmission ratio slightly vary. 5) The belt may completely slip over its pulleys when it is overloaded. Actually, the last point is not a bad thing. Belt slip is a natural over-load protection to the machine powered by the belt drive; it prevents failure in one stage of the machine to propagate to the next one.

The V-belt drive is an improved version of the flat-belt drive. It is also a traction drive; however, the belt has a V (trapezoidal) cross-section, and the V-belt drive uses a grooved pulley, or a sheave. This V shape makes the belt-pulley contact occur on inclined surfaces, or the wedged surfaces, as shown in **Figure 5-5**, which greatly increases the friction at the belt-pulley interface due to much higher normal forces there. The wedge increases traction; however, belt bending in a V-belt drive is more severe than that in a flat-belt drive due to the increased belt thickness. A larger sheave is preferred for a smaller belt-bending effect on strength.

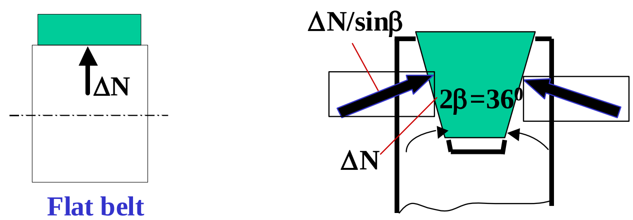


Figure 5-5 Comparison of the normal force per unit length of the belt, Δ*N*, on a V-belt with that on a flat belt.

The V-belt drive also needs an initial tension, but it does not require frequent tension adjustment because its belt usually has fiberglass-reinforced neoprene cores that can well maintain the tension.

The timing belt is actually a toothed drive, as shown in **Figure 5-6**, left. It is also called the synchronous belt drive. It uses toothed wheels, or sprockets. No slippage occurs, and a constant velocity ratio is guaranteed. In addition, no initial tension is needed.

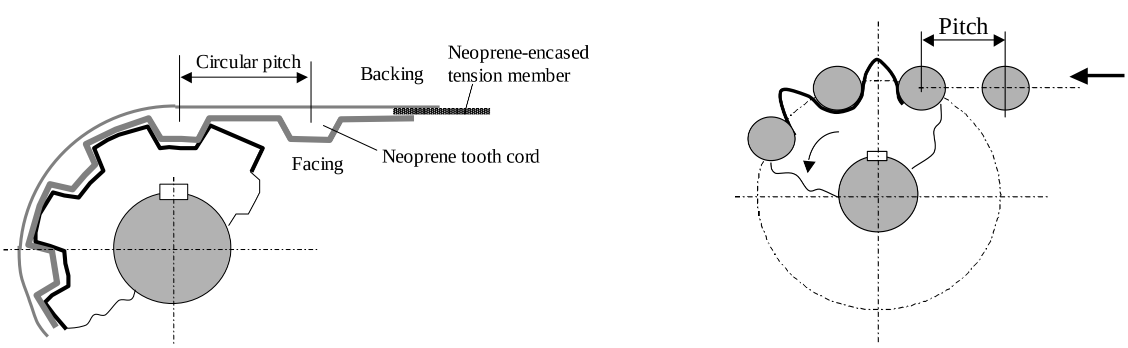


Figure 5-6 A timing belt drive (left) and a chain drive (right).

Chain drives

We see chain drives very often; they are on our bicycles! A chain drive uses toothed wheels, or sprockets, **Figure 5-6**, right. There is no slippage of the chain on the sprocket, and a constant velocity ratio is guaranteed. Likewise, no initial tension is needed. Because chains are usually constructed with metal pieces, a long service life is expected. However, the noise level in a chain drive may be higher than that in a belt drive. Why?

5-3 Gears

Gears are named for a family of toothed elements, usually made of metallic materials of great strengths, but nylon, polymer gears are increasingly used in many applications. Transmission is accomplished through contact of one or more teeth of one gear directly with those of another. They are used for secured power transmissions with accurate transmission ratios. Based on the meshing types, gears can be classified into external gears, internal gears, and gear-rack systems, as shown in **Figure 5-7**. Gears can also be classified as spur gears, helical gears, worm gears, and bevel gears based on tooth shapes, as shown in **Figure 5-8**. These figures also show the corresponding mechanical symbols for each of the gear pairs and the feature of gear meshes.

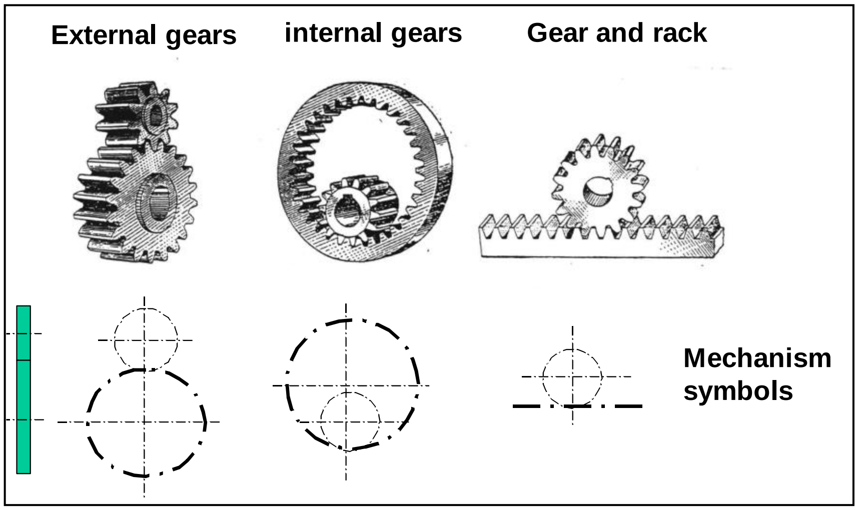


Figure 5-7 Classification of gears based on the mesh type.

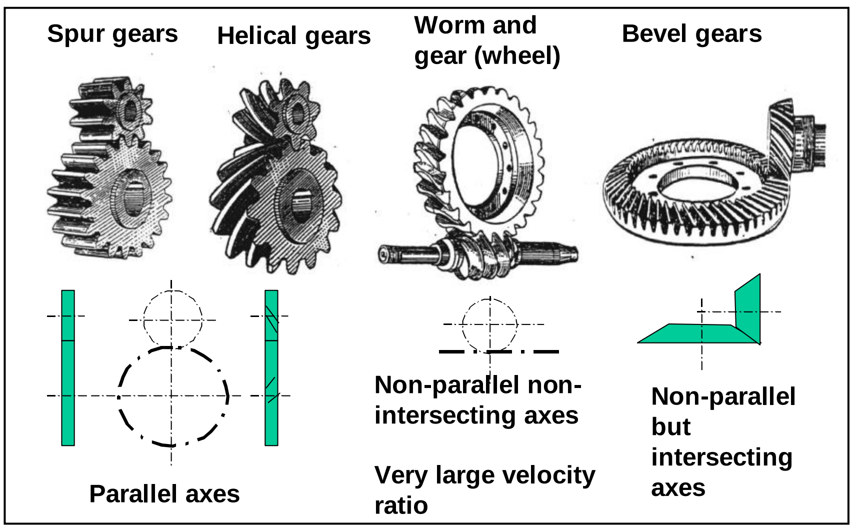


Figure 5-8 Classification of gears based on tooth shape.

The axes of a spur-gear pair and a helical-gear pair are parallel, while those of a worm-gear pair and a bevel-gear pair are generally perpendicular. The axes of a worm-gear pair do not intersect, but those of a bevel-gear pair intersect.

Let's first look at the overall geometry of gear teeth and that of a pair of gears. **Figure 5-9** shows a couple of gear teeth and terminologies that define the teeth. We will come back to this later when we know why the teeth are created this way. One item that is very important to know now is the gear pitch circle, the red dash-dot line in the figure below. The gear ratio, or speed ratio, is defined based on the speeds of the pitch circles of two meshing gears.

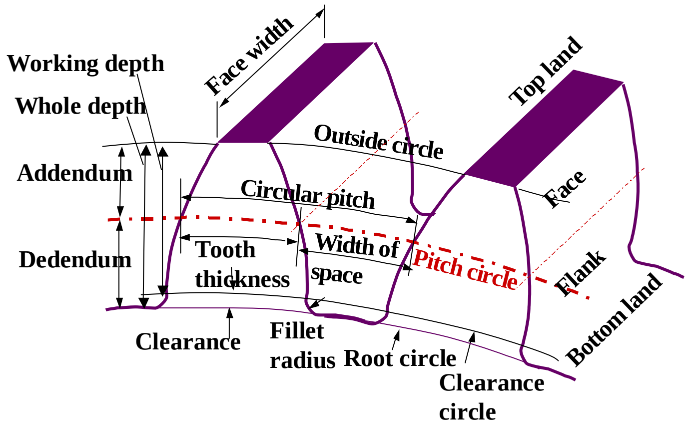


Figure 5-9 Gear tooth geometry.

Figure 5-10 shows two gears in a meshing process. The driver is usually the smaller one, called the pinion. Subscript *p* means the pinion, and subscript *g* means the gear. Locations of the gear and the pinion are determined by the centerline distance, c_d, which is the sum of the pitch radii of the two gears. The pitch-circle radius is defined by the number of teeth, *N*, times module *m* for a metric gear or by the number of teeth, *N*, divided by diametrical pitch *P* for a US custom gear, as given in Equation (5-5) below.

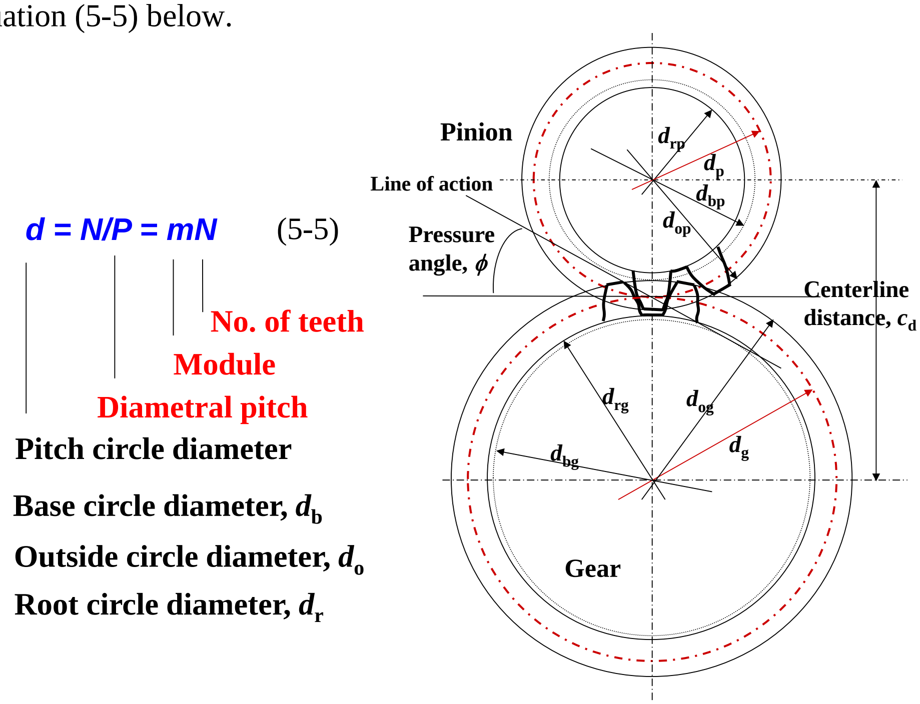


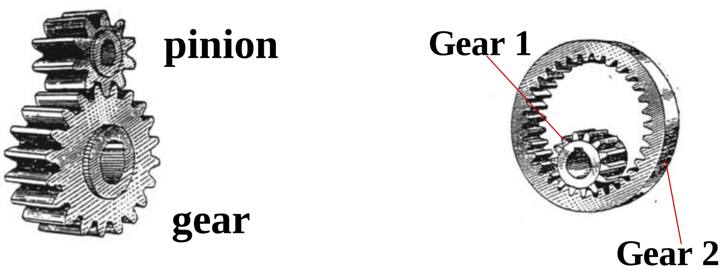
Figure 5-10 A pair of gears in meshing and diameters of circles.

5-4 Simple gear trains

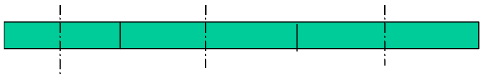
Gear trains with fixed axes. A gear train contains multiple gears in a certain arrangement and is used to achieve a desired angular velocity change between the input and output shaft. We have already had some knowledge of gear trains, like the one shown below in **Figure 5-11**. This time we opened the cover and exposed the components inside.

Figure 5-11 Gear train in a gear reducer (Courtesy of Chicago Gears).

Gears in a gear train may be a set of external gears, as shown in **Figure 5-12** (a), or may be an external and an internal gear, namely an internal gear pair, as shown in **Figure 5-12** (b). A gear train may have more than one gear pair, as shown by the three gears in **Figure 5-12** (c) plotted with gear symbols and their center lines.



(a) (b)



(c)

Figure 5-12. External (a), internal (b), and multiple gear pairs (c) in gear trains.

The gear ratio, or the train value, is the ratio of the rotation speeds of the gears in a set, which are actually determined by the diameters of the gears involved. By means of Equation (5-5), the train value becomes the ratio of the numbers of teeth.

$$i_{pg} = \pm\frac{n_p}{n_g} = \pm\frac{\omega_p}{\omega_g} = \frac{d_g}{d_p} = \frac{mN_g}{mN_p} = \frac{N_g}{N_p} \qquad (5\text{-}6)$$

We have to specify the direction of rotation of each gear. Two external gears rotate oppositely, therefore,

$$\frac{\omega_1}{-\omega_2} = \frac{n_1}{-n_2} = \frac{d_2}{d_1} = \frac{N_2}{N_1} \qquad (5\text{-}7)$$

Here, we use 1 and 2 for two gears, sometimes, we also use "i" and "o" for input and output, or "p" and "g" for the pinion and gear. For an internal gear set, the two gears rotate in the same direction, and the gear ratio is given below.

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{d_2}{d_1} = \frac{N_2}{N_1} \qquad (5\text{-}8)$$

In general, the ratio of the speed of an input (or driver) gear (subscript *i*) to that of an output (or driven) gear (subscript *o*) is given by Equation (5-9), assuming that all the gears have the same module (The same module? we will look into this issue later in this chapter).

$$\frac{\omega_i}{\omega_o} = (-1)^{em}\,\frac{\textbf{Product of No. teeth of driven gears}}{\textbf{Product of No. teeth of driver gears}} \qquad (5\text{-}9)$$

em: No. of external mesh

ω_o or n_o : speed of gear o, the driven, or the output gear

ω_i or n_i: speed of gear i, the driver, the input gear.

Let's determine the train values of each pair of the gears in the trains of the gear reducer in **Figure 5-1**, replotted in **Figure 5-13**, knowing that the numbers of teeth are, N_1, N_2, N_3, and N_4, and assuming that all gears have the same module. Note that all gears are mounted on their own shafts and rotate together with the shafts.

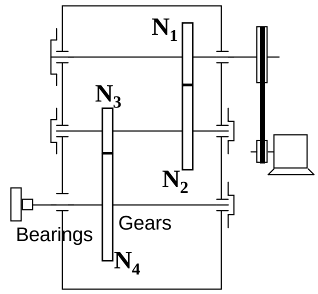
Equation (5-9) is used and the solution to this problem is shown below in **Figure 5-13**. We have two external gear sets, so that the direction of the rotation is changed twice. Here, $n_2 = n_3$, because these two gears are mounted on the same shaft.

Gears 1 and 2 $i_{12} = \dfrac{n_1}{n_2} = -\dfrac{N_2}{N_1}$

Gears 3 and 2 $n_2 = n_3$

Gears 3 and 4 $i_{34} = \dfrac{n_3}{n_4} = -\dfrac{N_4}{N_3}$

Gears 1 and 4

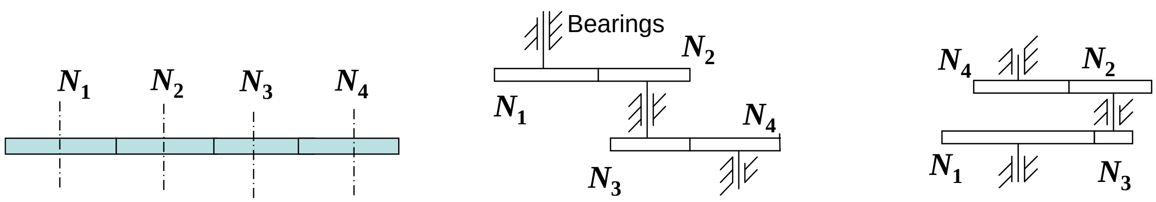


$$i_{14} = \frac{n_1}{n_4} = \frac{n_1}{n_2}\frac{n_2}{n_4} = \frac{n_1}{n_2}\frac{n_3}{n_4} = (-1)^2\frac{N_2 N_4}{N_1 N_3}$$

$$= (-1)^{em}\frac{\textbf{Product of No. teeth of driven gears}}{\textbf{Product of No. teeth of driver gears}}$$

Figure 5-13. Solutions to the train value of each gear pair and the overall train value of the gear reducer mentioned at the beginning of this chapter.

Example 5-1. More practice. There are four gears in each gear train below, **Figure 5-14**, and N_1 =50, N_4 = 40, N_2 = 35, and N_3 = 25. Gear 1 is the input and gear 4 the output. Determine the output velocity of each gear trains if the input speed is n_1 = 1000 rpm (clockwise).



(a) **(b)** **(c)**

Figure 5-14 Example 5-1.

(a) This is a simple train.

$$\frac{n_1}{n_4} = -\frac{n_1 n_2 n_3}{n_2 n_3 n_4} = (-1)^3\frac{N_2 N_3 N_4}{N_1 N_2 N_3} = -\frac{N_4}{N_1} = -\frac{40}{50}\text{, therefore } n_4 = -\frac{5}{4}(1000) = -1250rpm$$

Here, the direction of n_4 is opposite to the rotation direction of gear 1. Note that gears 2 and 3 are idlers; they do not change the train value but alter the direction of the output speed.

(b) This is a compound train.

$$\frac{n_1}{n_4} = (-1)^2\frac{N_2 N_4}{N_1 N_3} = \frac{(35)40}{50(25)}\text{, therefore, } n_4 = \frac{50(25)}{35(40)}(1000) = 892.9rpm$$

Here, the rotation direction of n_4 is the same as that of gear 1.

(c) Can you try this one? Is there any major difference between b) and c)?

Planetary (Epicyclic) gear trains. Special train values can be obtained if one (or more) of the gear axes can rotate about others. That means a planetary gear train, which may include a sum gear, one or several planet gears, and a planet carriage (or the arm, a, the handle, h) functioning like a carrier, as shown in the figure below. **Figure 5-15** shows a simple planetary gear system, which has a sun gear, a planetary gear on a carrier (arm), and a ring gear (an internal gear). The numbers of teeth (N_i) and rotation speeds (n_i, or ω_i) are all shown in the figure, with index i for the ring gear, planet gear, or the planet gear. If we could have "Jumped" onto the carrier of the system, we should virtually see a gear train of fixed axes. Then we can use the train value formula, Equation (5-9), mentioned before to solve this type of problem, by plugging in relative motions, such as $n_i - n_a$, in the analysis, as shown below. Once the speed ratio is written in terms of relative motions, an unknown speed can be solved.

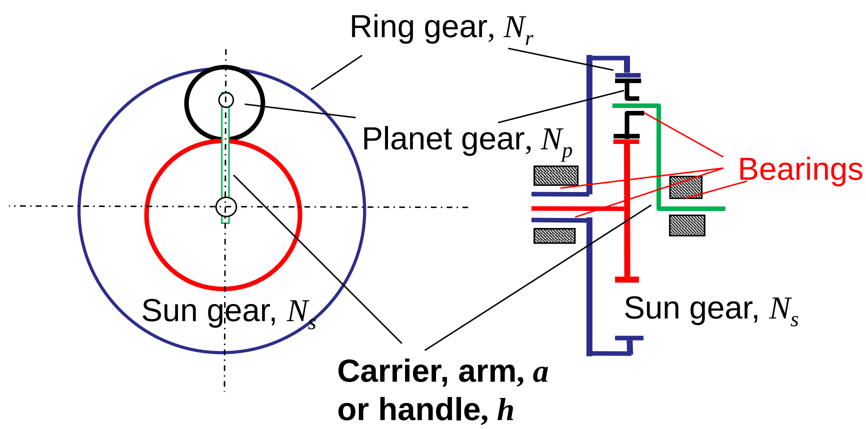


Figure 5-15. Simple planetary gear system.

$$\frac{n_r - n_a}{n_s - n_a} = \frac{n_r - n_a}{n_p - n_a}\frac{n_p - n_a}{n_s - n_a} = (-1)^{em}\frac{N_p}{N_r}\frac{N_s}{N_p} = -\frac{N_s}{N_r}$$

A slight modification of Equation (5-9) leads to Equation (5-10) below. Again, the only extra thing for us to do is to use the relative motions, $n_i - n_a$, or $\omega_i - \omega_a$ and $n_o - n_a$, or $\omega_o - \omega_a$ to replace ω_i and ω_o.

$$i_{io} = \frac{\omega_i - \omega_a}{\omega_o - \omega_a} = (-1)^{em}\frac{\text{Product of No. teeth of driven gears}}{\text{Product of No. teeth of driver gears}} \qquad (5\text{-}10)$$

em: No. of external mesh

ω_o-ω_a or n_o-n_a: relative speed of gear o, the driven, or the output gear

ω_i-ω_a or n_i-n_a : relative speed of gear i, the driver, the input gear.

Example 5-2. In the reducer below, the input shaft, *in*, is aligned with the output shaft, *b*. The input gear speed is n_2. Given, $N_2 = 24$, $N_3 = 18$, $N_6 = 64$. Find the ratio of the output speed to the input speed.

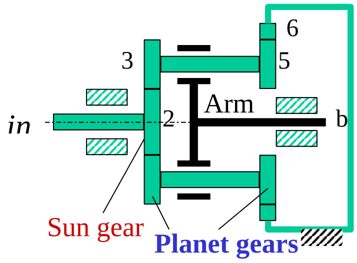


Figure 5-16 Planetary gear system of Example 5-2. Note here in this problem, N_6, N_3 and N_5 are related through the geometry of the sets of N_2 - N_3 and N_6 - N_5, by $m_1 (N_3+N_2)/2 = m_2(N_6/2-N_5/2)$, with m_1 and m_2 the module for the two sets, respectively. For $m_1 = m_2$ the result for N_5 is 22.

Solution

From $\dfrac{n_2 - n_a}{n_6 - n_a} = (-1)^1 \dfrac{N_3}{N_2}\dfrac{N_6}{N_5}$, and the fixed ring gear, we have $\dfrac{n_2 - n_a}{0 - n_a} = (-1)\dfrac{N_3}{N_2}\dfrac{N_6}{N_5}$

Thus, $1 - \dfrac{n_2}{n_a} = (-1)\dfrac{N_3}{N_2}\dfrac{N_6}{N_5}$ $\qquad \dfrac{n_2}{n_a} = 1 + \dfrac{N_3}{N_2}\dfrac{N_6}{N_5} = 1 + \dfrac{18}{24}\dfrac{64}{22} = 3.18$

$$\frac{n_a}{n_2} = \frac{1}{3.18} = 0.314$$

Gear train systems in the engineering world. Gear train systems are in every corner of the modern engineering world for high efficiency, high reliability, and high stiffness power transmissions. Typical examples are robot drives, vehicle transmissions, machine tool transmissions, and windmill power delivery (Figure below) systems. Gear transmission systems can also be found in toys, yard tools, dental tools, robots, etc. Again, they are indeed everywhere.

Figure 5-17. A windmill in Cleveland.

5-5 Introduction to the theory of gearing

What is the shape of a gear tooth profile? Usually, it is not a piece of an arc even though it looks like an arc. In most cases, it is a segment of an involute curve. Why is that profile chosen? We need to answer this question from the basic meshing kinematics, or specifically, the theory of gearing.

Conjugate action. Let's first learn something called the conjugate action, which is the meshing action with a constant angular velocity. Our gear meshing motion should be a conjugate action because the gears should run at constant angular velocities.

Assume that we have two gears, shown in **Figure 5-18** below, each with an unknown profile. The two gears are rotating about their own instantaneous centers of rotation, o_1 and o_2, respectively, with ω_1 and ω_2. Let's assume that gear 1 is the driver. The two profiles are in contact at point k at this moment. The velocity of gear 1 at k is $\mathbf{v_1}$, and its magnitude is $v_1 = \omega_1 \overline{o_1 k}$. The velocity of gear 2 at k is $\mathbf{v_2}$, and its magnitude is $v_2 = \omega_2 \overline{o_2 k}$ (Equation (5-11)). Let's use nn for the common normal of the two profiles at the location when they are in contact. We can draw two line-segments that are each perpendicular to common normal nn, one from o_1 and the other from o_2. Now we have o_1A and o_2B.

The velocities projected onto line nn must be the same because the two teeth should not run into each other, nor should they separate.

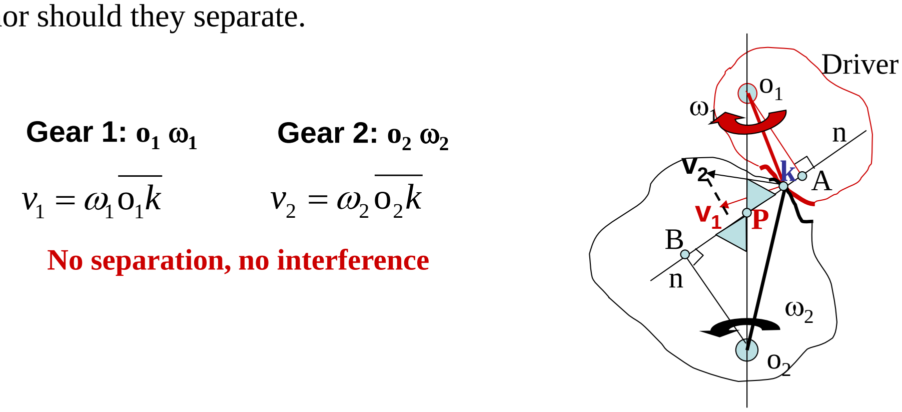


Figure 5-18. Two gear profiles are in contact at k, and o_1 and o_2 are their rotation centers.

Again, the definition of contact means that no separation or interference of the surfaces is allowed. Therefore, projections of the velocities along the normal direction should be the same (Equation (5-12)), although some relative sliding along the tangential direction is permitted.

$$v_1 \cos(\angle Ao_1k) = v_2 \cos(\angle Bo_2k) \quad (5\text{-}12)$$

Substituting Equation (5-11) into (5-12), we have

$$\omega_1 \overline{o_1 k} \cos(\angle Ao_1 k) = \omega_2 \overline{o_2 k} \cos(\angle Bo_2 k), \text{ or}$$

$$\omega_1 \overline{o_1 A} = \omega_2 \overline{o_2 B} \tag{5-13}$$

Here, $\overline{o_1 k}\cos(\angle Ao_1 k) = \overline{o_1 A}$ and $\overline{o_2 k}\cos(\angle Bo_2 k) = \overline{o_2 B}$. Now we can take a look at the vertical triangles o_1AP and o_2BP. Naturally we can derive Equation (5-14) from (5-13).

$$\omega_1 \overline{o_1 P} = \omega_2 \overline{o_2 P} \tag{5-14}$$

The ratio of the angular velocities becomes

$$\frac{\omega_1}{\omega_2} = \frac{\overline{o_2 P}}{\overline{o_1 P}} \tag{5-15}$$

Conjugate profiles. In order to make the ratio, ω_1/ω_2, a constant, the necessary and sufficient conditions are that P is a fixed point of the centerline, o_1o_2. Profiles whose common normal at any contact passes through a fixed point of o_1o_2 (or intersect with o_1o_2 at a fixed point) should result in constant angular velocities of both. These profiles are the conjugate profiles.

Once P is a fixed point at centerline o_1o_2, we can define two fundamental sets of circles for gears, see **Figure 5-19** below, namely the pitch circles by means of P and the base circles by means of A and B, respectively. Point P is a very important point for gears; it is the **pitch point**, based on which the following can be defined.

Shown in **Figure 5-20** are the pitch-circle radii:

$$r_1 = \overline{o_1 P} \tag{5-16-1}$$

$$r_2 = \overline{o_2 P} \tag{5-16-2}$$

And the base-circle radii:

$$r_{b1} = \overline{o_1 A} = r_1 \cos\phi \tag{5-16-3}$$

$$r_{b2} = \overline{o_2 B} = r_2 \cos\phi \tag{5-16-4}$$

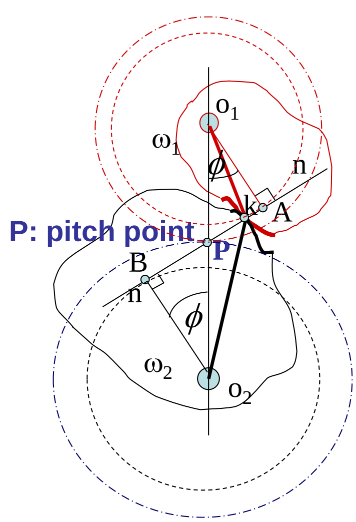


Figure 5-19 Pitch point, P, pitch and base circles, and gear centers o_1 and o_2.

Law of gearing. The derivation above states the law of gearing, which is that the ratio of the angular velocities of the gears should be constant if P is a fixed point on the centerline, i.e.

$v_{p1} = v_{p2}$ and $v_{b1} = v_{b2}$

and

$$\frac{\omega_1}{\omega_2} = \frac{\overline{o_2P}}{\overline{o_1P}} = \frac{r_2}{r_1} = \frac{r_{b2}}{r_{b1}} \tag{5-17}$$

With the law of gearing and conjugate action in mind, we can picture the mesh of two conjugate profiles through the action of the circles we have defined: the pitch and base circles. Keep in mind that these circles are invisible after gears are made. However, these conceptual circles are the key to the understanding of gear contact and motion, and to gear tooth design as well. The two pitch circles are in "contact" and in a pure-rolling motion. Note that the pure-rolling motion means $v_{p1} = v_{p2}$. The two base circles are not in "contact", but they are in a pure-rolling motion, too, $v_{b1} = v_{b2}$.

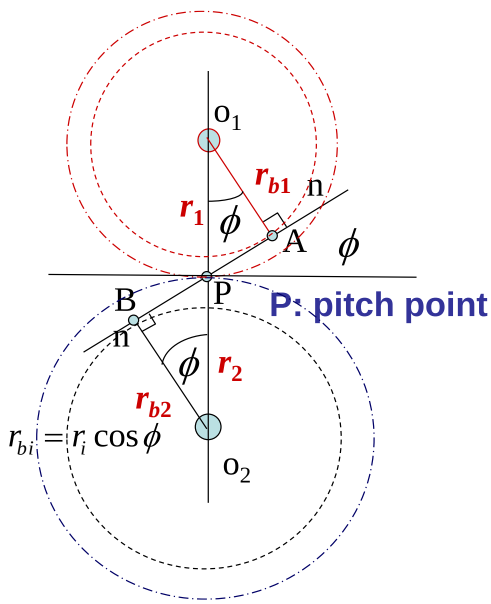


Figure 5-20 "Contact" and motion of pitch circles and base circles; gear triangles.

Figure 5-20 contains all basic quantities for gearing, which are all in two triangles, Po_1A and Po_2B, called the gear triangles. These quantities are:

Radii for the base and pitch circles
Key points, P, A, B, o_1 and o_2.
A very important angle: the pressure angle, ϕ.
A very important line: nn, or AB, which is the common normal of contacting profiles, the base-circle tangent, the line of gear mesh, the line of action, and the line of profile generation. Its use as the common normal has been shown, its other functions will be demonstrated one by one later.

Involute profiles. Now we need to find the profile that satisfies the law of gearing. The involute is one of the curves that meet such a need. It belongs to a family of curves called the roulette curves. Let's take a disk as the base circle, and a piece of rope as the line of generation, to create an involute curve. We wrap the rope on the surface of the disk, attach a pen at the end of the rope, and then gradually release the rope while keeping it tangent to the disk. The pen draws an involute curve, k_0k, shown in **Figure 5-21**. Here, k_i is the pen point, B_i the point of contact, or the point of tangency, between the rope and the base circle. The rope **is tangent to the base circle** but **normal to the involute curve at all the time of the plotting**.

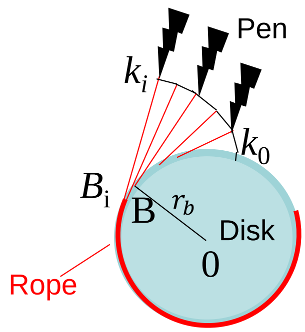


Figure 5-21 Involute curve generation.

Now we have two such curves generated from two base circles, and let them be in contact at P, or any other point, *k'*, as shown in **Figure 5-22** below. The generation line, BP, or B*k*', belongs to one gear, while the generation line AP, or A*k'*, belongs to the other gear. However, they share P or *k*' in common, and for P, meaning that AP and BP belong to the same line, or the line of the common normal, AB. Likewise the same is true for *k*'. On the other hand, these two lines are tangent to their own base circles. Therefore, AB is also the common tangent to both base circles. If the distance of the two base circles is kept fixed, AB as the common tangent is unique to these base circles. It intersects the centerline at a fixed point. Therefore, the common normal of the two involute curves intersects with the centerline at a fixed point. We can now conclude that involute curves are conjugate profiles, and that they satisfy the law of gearing.

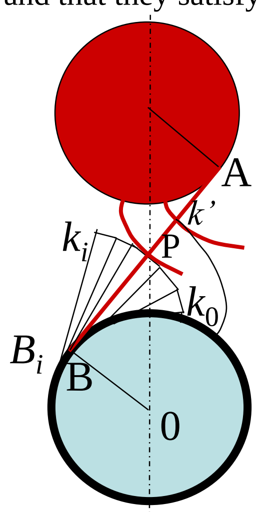


Figure 5-22 Two involutes in contact. Line AB, the common normal of the involute curves, are also the common tangent of base circles.

Properties of the involute profile. The generation process brings about several very important properties of involute profiles.

(1) Segment B_ik_i on the normal (or the base-circle tangent) equals the arc length B_ik_0 on the base circle. B_ik_i is the unwrapped length of the rope (**Figure 5-21**).

$$\overline{B_ik_i} = \text{arc}(B_ik_0) \tag{5-18}$$

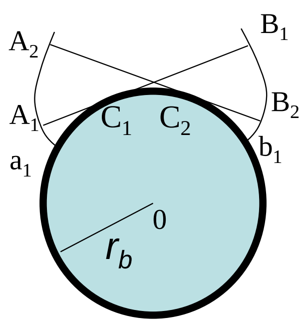
(2) The normal to an involute is the tangent line to the base circle (**Figure 5-21**).

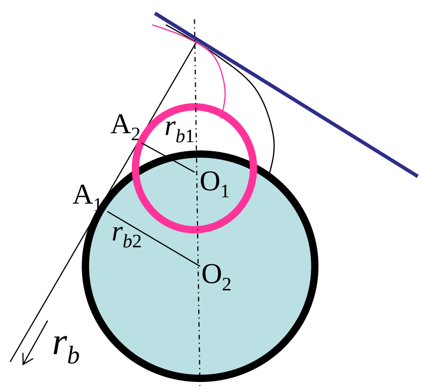
(3) B_i and A_i (the latter not shown) are the centers of curvature of the involutes (**Figure 5-22**). When two involutes are in contact at k', Bk' is the radius of curvature of one involute, and likewise, Ak' is the radius of curvature of the other involute.

(4) No involute is inside the base circle (**Figure 5-21**).

(5) The lengths of common normal lines between two opposite involutes are equal, or $A_1B_1 = A_2B_2$ (**Figure 5-23**, left). Can you prove this from what we have learned?

(6) The shape of an involute profile depends on the radius of its base circle, r_b. When r_b approaches infinite, the involute approaches a straight line (**Figure 5-23**, right). Therefore, racks of straight-line profiles are also gears.





Equal lengths of lines normal to opposite involutes. Effect of base-circle radius.

Figure 5-23 Normal lines and influence of base-circle radius on involute curvature.

5-6 Gear geometry

Although the base circle and the pitch circle of a gear have provided a lot of important geometric information, we still need more parameters to define the shape of a gear and a gear tooth. Let's first look at the circles. We need to define the outside and root circles. Now we are talking about four circles in total, as shown in **Figure 5-24**. The tooth geometry is defined in **Figure 2-25**.

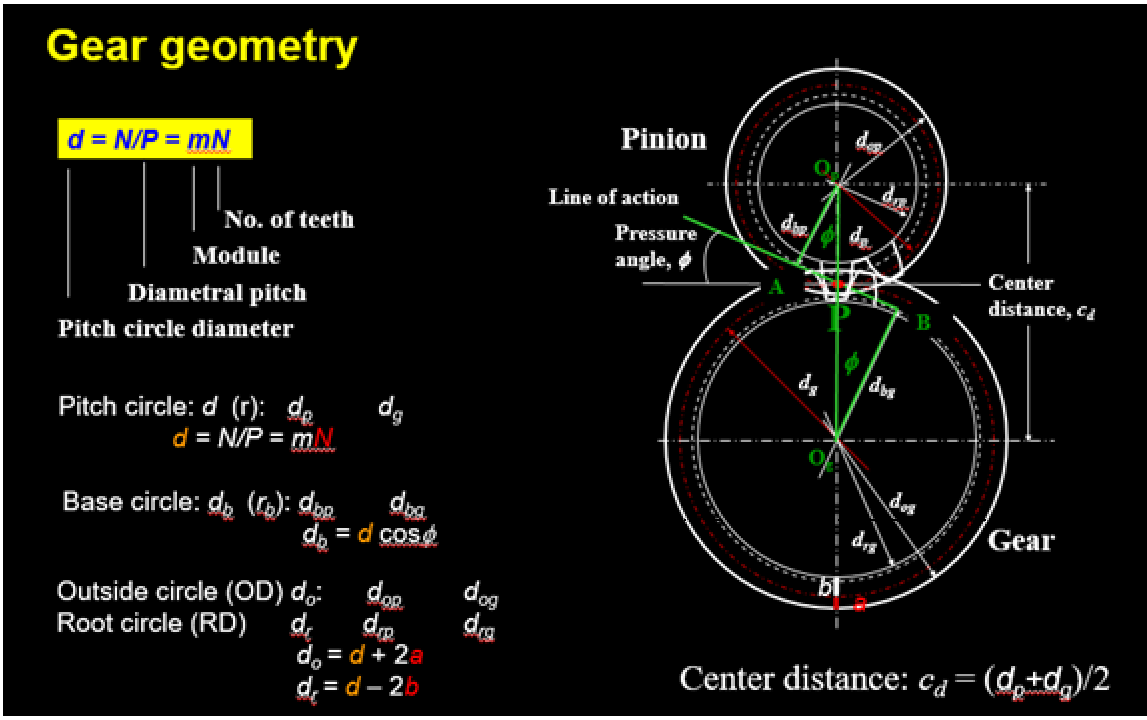


Figure 5-24 Four circles of a gear, centerline, line of action, and pressure angle. Subscripts *p* is for pinion, *g* (or *G*) for gear, *o* for outside, *r* for root.

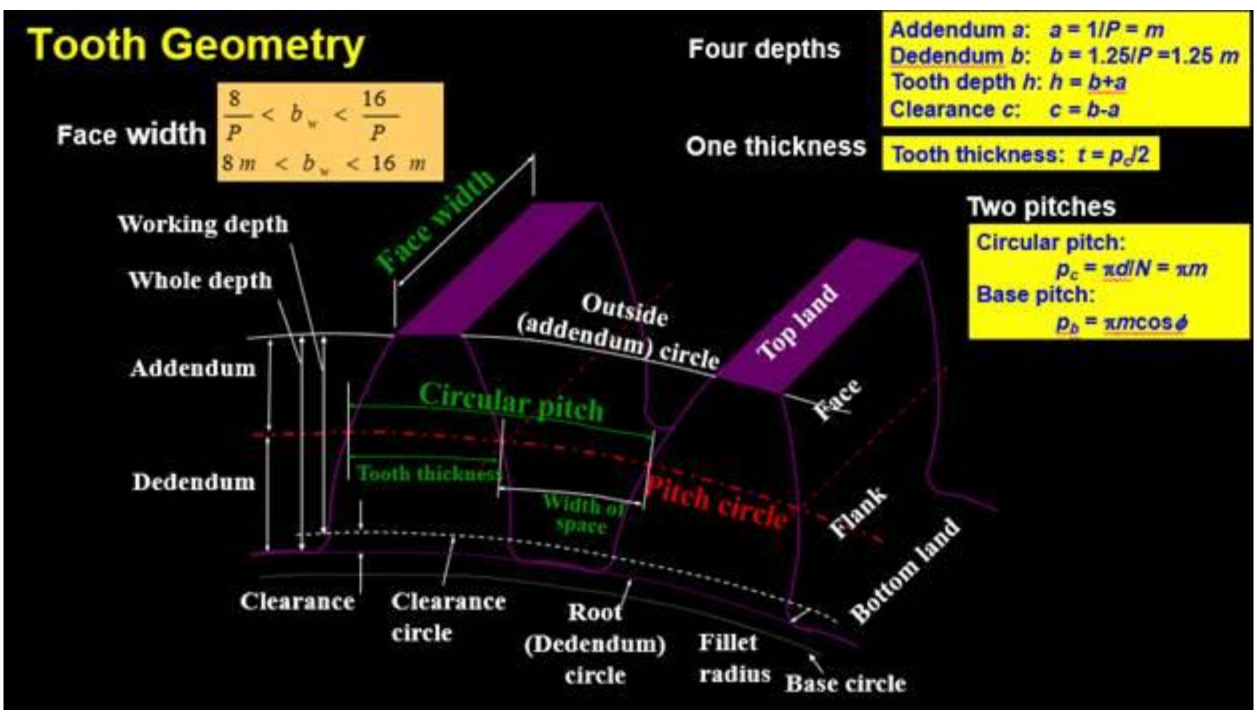


Figure 5-25 Four depths and other items defining gear teeth. Subscripts *c* is for circular and *b* for base circle.

Spur-gear tooth system. **Tables 5-1** and **5-2** list the commonly used diametral pitches and modules. **Table 5-3** lists the standard and commonly used values for *a*, *b*, and ϕ. Generally, a=m, b=1.25m, and $\phi = 20^0$ for regular ISO gears, which corresponds to a=1/P, b=1.25/P, and $\phi = 20^0$ for regular US custom gears.

Table 5-1 Commonly used diametral pitches

Coarse	2	2 ¼	2 ½ 3	4	6	8	10	12	16	
Fine	20	24	32	40	48	64	80	96	120	150

Table 5-2 Commonly used modules

Preferred	1, 1.25, 1.5, 2, 2.5, 3, 4, 5, 6, 8, 10, 12, 16, 20, 25, 32, 40, 50
Next choice	1.125, 1.375, 1.75, 2.25, 2.75, 3.5, 4.5, 5.5, 7, 9, 11, 14, 18, 22, 28, 36

Table 5-3 Standard and commonly used values for *a*, *b*, and ϕ

	Pressure angle ϕ (degree)	Addendum, *a*	Dedendum, *b*
Full depth	20	1/P or1m	1.25/P or 1.25m 1.35/P or 1.35m
	22.5	1/P or 1m	1.25/P or 1.25m 1.35/P or 1.35m
	25	1/P or 1m	1.25/P or 1.25m 1.35/P or 1.35m
Stub	20	0.8/P or 0.8m	1 /P or 1 m

How should we choose the number of teeth? Usually, for hobbed spur gears, the minimum number of teeth should be 17. For a pair of gears, it is preferred that the tooth ratio, N_2/N_1 is an irrational number in order to avoid repeated tooth wear pattern. One of N_2 and N_1 may be a prime number, at least 17.

Example 5-3 Let's work on a pair of metric gears with a module of 4 mm. The pinion has 17 teeth, and the gear has 40 teeth. The pressure angle is 20 degrees. Calculate: 1) Gear ratio of speed reduction, 2) pitch diameters of the pinion and the gear, 3) addendum and dedendum, 4) tooth depth and clearance, 5) centerline distance, and 6) circular and base pitches, and then do the 7) plot of the gear triangles.

Solution

1) The ratio of speed reduction $i = N_G/N_p = 40/17 = 2.353$

2) The pitch diameters of the pinion and the gear $d_p = m\,N_p = 4\ (17) = 68$ mm

$d_G = m\,N_G = 4\ (40) = 160$ mm

3) The addendum and dedendum $a = m = 4$ mm

$b = 1.25m = 1.25\ (4) = 5$ mm

4) The tooth depth and clearance $h = b + a = 5 + 4 = 9$ mm

$c = b - a = 5 - 4 = 1$ mm

5) The centerline distance $c_d = 0.5\ (d_p + d_G) = 0.5\ (68 + 160) = 114$ mm

6) The circular and base pitch $p_c = \pi m = \pi(4) = 12.566$mm

$p_b = p_c \cos\phi = 12.566 \cos(20) = 11.808$mm

7) Gear triangles are plotted in **Figure 5-26**. Circles are shown for clarity; actually, they are not needed.

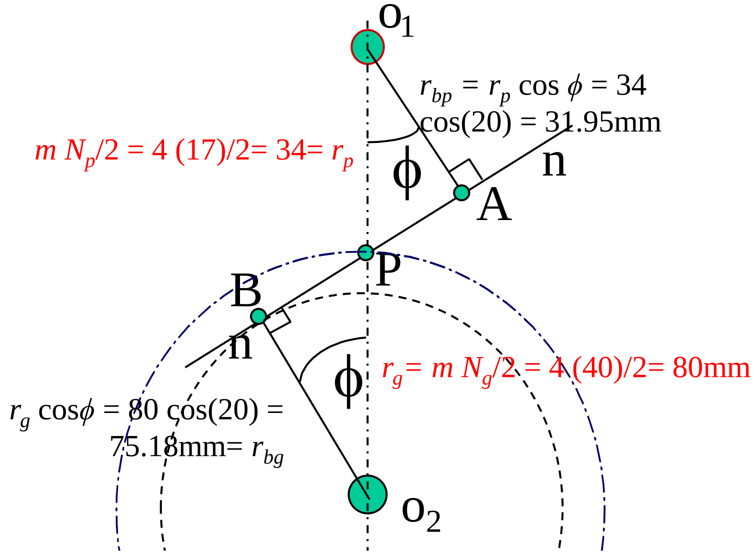


Figure 5-26 Gear triangles for the gears in **Example 5-3**. Subscripts *p* is for pinion, *g* (or *G*) for gear, *b* is for base circle, and *d* for centerline distance.

5-7 Meshing process and conditions, contact ratio, and interference.

With the understanding of the properties of involute profiles, we are ready to look at the meshing process of a pair of gear teeth, drawn below in **Figure 5-27**, where the gear triangles are also shown. A and B are the points of tangency of the common tangent and the base circles. The rotation directions are given as well in the same figure. The meshing process is along line AB, why? It is the line of the common normal to the involutes and the common tangent to the base circle. When the gear surface profiles are in contact, they have one normal, but the normal to the involute tooth profiles is the tangent of the base circles.

Meshing process. A pair of teeth starts to mesh at C, where the outside circle of the gear contacts the flank of the mating pinion tooth, or where the outside circle of the gear intersects line AB. The mesh ends at D, where the outside circle of the pinion contacts the flank of a gear tooth, or where the outside circle of the pinion intersects line AB. CD is a segment of line AB, and in the meshing process, the contacting teeth move together from C to D and then separate at D. Clearly, the length of CD should be shorter than that of AB.

Gear meshing conditions. One question immediately comes to our minds: what are the basic conditions that the meshing teeth should satisfy? From C to D, this pair of teeth should not run into each other, and they should not be separated from each other either. This means that their pitch in line AB should be the same. What is the name of that pitch? The base pitch! Remember the straight-line length along line AB (or the line of generation) equals the arc length on the base circle (**Figures 5-28, 5-21**); therefore,

$$p_{b1} = p_{b2}$$

We know that

$$p_b = p_C \cos\phi = (\pi d/N)\cos\phi = \pi/P\cos\phi = \pi m \cos\phi$$

Note that module (or diametral pitch) and pressure angle are two independent parameters. Therefore, gear mesh conditions are

English:

$$P_p = P_g = P \qquad (5\text{-}19\text{-}1)$$
$$\phi_p = \phi_g = \phi \qquad (5\text{-}19\text{-}2)$$

Metric:

$$m_p = m_g = m \qquad (5\text{-}20\text{-}1)$$
$$\phi_p = \phi_g = \phi \qquad (5\text{-}20\text{-}2)$$

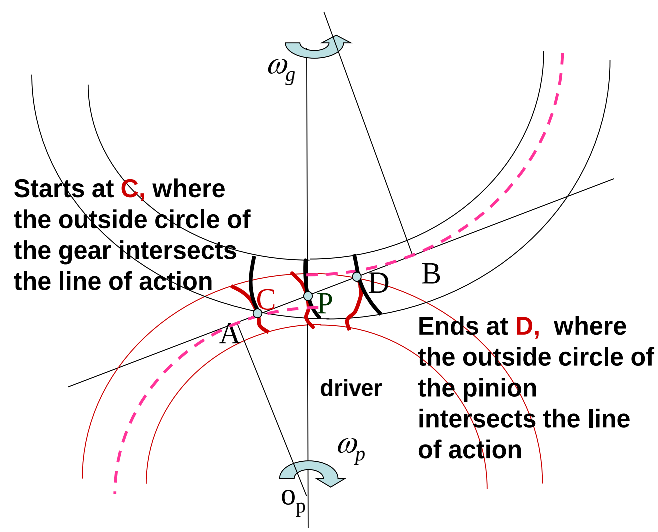


Figure 5-27 Meshing process of a pair of teeth. Subscripts *p* is for pinion, *g* for gear.

Contact ratio. Is there another pair of teeth in contact while the current pair of teeth is about to separate at D? In **Figure 5-28**, we should have a continuous gearing motion if another pair of teeth has already started the contact at C. Otherwise, the gear would stop for a while until the next pinion tooth finds its mating gear tooth. This argument means that within the length of CD, there should be at least one pair of teeth, or CD should at least have the length of the base pitch measured along line AB. How many pairs of teeth are in CD? That is simple a division: CD/p_b

The contact ratio, C_r,is defined as

$$C_r = CD/p_b \tag{5-21}$$

In many cases, we can use the following equation to calculate the contact ratio. Actually, contact ratio is a complicated issue. An accurate analysis should be referred to details in the theory of gearing, not included in this class. The meaning of the contact ratio will be further explained in **Example 5-4**.

$$C_r = \frac{1}{p_c \cos\phi}\left[\sqrt{r_{op}^2 - r_{bp}^2} + \sqrt{r_{og}^2 - r_{bg}^2}\right] - \frac{c_d \tan\phi}{p_c} \tag{5-22}$$

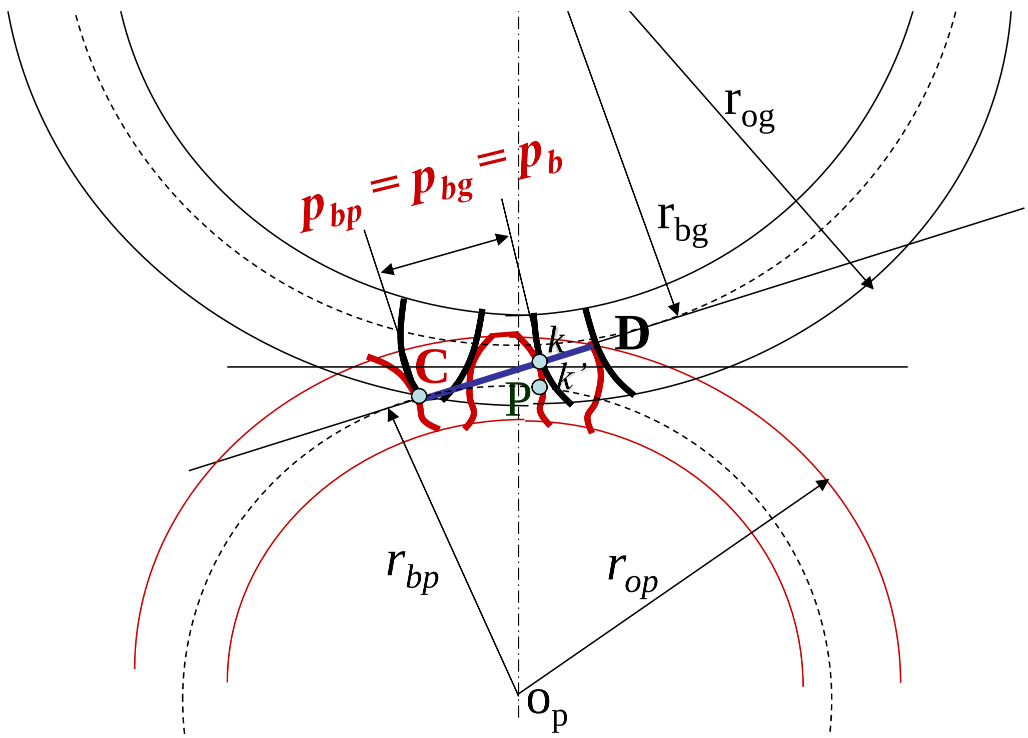


Figure 5-28 Base pitch and CD, subscripts 1 and 2, *p* and *g* are for the pinion and the gear. The straight-line length of C*k* equals the arc length of C*k'*, the base pitch.

Figure 5-29 illustrates three cases of mesh contact with $C_r = 1$, $C_r > 1$ and $C_r < 1$. For the case of $C_r = 1$, there is exactly one pair of teeth in contact in *CD*. For the case of $C_r > 1$, there may be more than one pair of teeth in contact in *CD*, depending on the tooth meshing location. When one pair separates at D, another pair has already come in from C. For the case of $C_r < 1$, there is no more than one pair of teeth in contact along *CD*. When the separation occurs at D, there is no other pair in CD. In practice, C_r should be larger than 1.1 because we need to consider the tolerance effect.

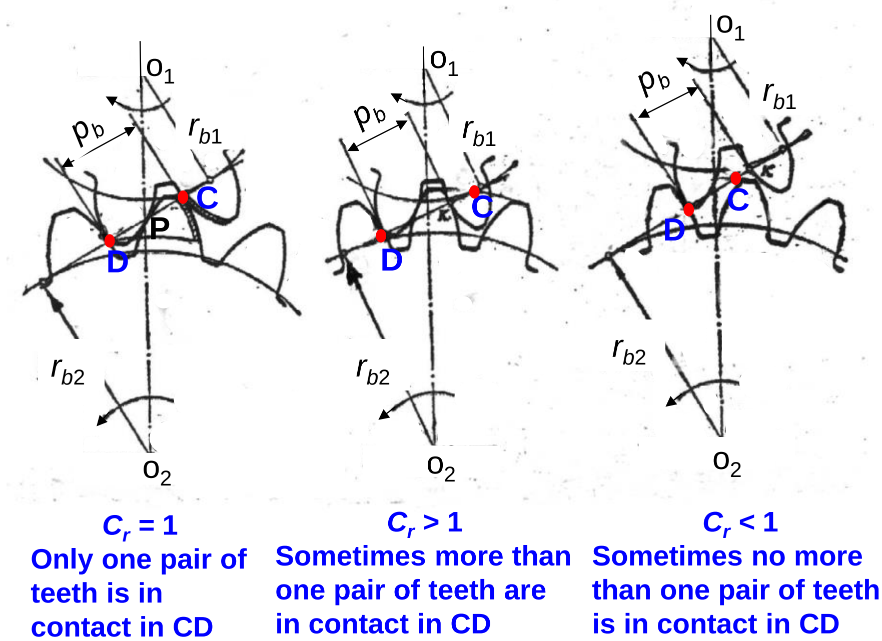


Figure 5-29 Effect of the contact ratio on gear tooth contact. C_r is the contact ratio. Modified from Pu et al. (2013).

A larger contact ratio results in smoother transmission. However, the contact ratio for a pair of spur gears is always smaller than two. Helical gears may have larger contact ratios. That is why helical gears are commonly seen in power transmission equipment.

Interference. Segment CD defines the mesh range, but segment AB determines the mesh limits. CD cannot be outside of AB. What will happen if C or D is outside of AB? Tooth contact will not be in the involute portion of the tooth profile, and this is called the mesh interference. This is because no involute is inside the base circle! Gear meshing with a non-involute curve occurs if C or D is outside of AB, and if this does occur, the conjugate action cannot be maintained. If this such interference appears in manufacturing, it becomes undercutting, meaning that the roots of the teeth will be cut inward. As mentioned before, for hobbed gears, the minimum number of teeth to avoid this interference is 17 for spur gears with regular teeth and 20-degree pressure angle. We can prove this in a homework assignment.

Example 5-4 A pair of US custom gears has a diametral pitch of 3 teeth/in. The pinion has 18 teeth, and the gear has 21 teeth. The pressure angle is 25 degrees. Calculate:

Pitch and base diameters of the pinion and the gear
Addendum and dedendum
Tooth depth and clearance

Centerline distance
Base pitch
OD and RD of the pinion and the gear
Contact ratio

Draw the gear triangles and outside circles. Will interference occur or not?

Solution Note that we have used capital P for diamentral pitch (*P*, Italic) and for the label of the pitch point (P, not Italic), and lower-case *p* for pitches and the label of a pinion. Do not confuse them.

The pitch diameters of the pinion and the gear

$d_p = N_p/P = 18/3 = 6"$

$d_g = N_g/P = 21/3 = 7"$

The base diameters of the pinion and the gear

$d_{bp} = (N_p/P)\cos 25 = 18/3 \cos 25 = 5.437"$

$d_{bg} = (N_g/P)\cos 25 = 21/3 \cos 25 = 6.344"$

The addendum and dedendum

$a = 1/P = 1/3 = 0.3333"$

$b = 1.25/P = 1.25/3 = 0.4167"$

The tooth depth and clearance

$h = b + a = 0.4167 + 0.3333 = 0.75"$

$c = b - a = 0.4167 - 0.3333 = 0.083"$

The center distance

$c_d = 0.5\,(d_p + d_g) = 0.5\,(6+7) = 6.5"$

The circular and base pitches

$p_c = \pi/P = \pi/3 = 1.047"$

$p_b = p_c \cos\phi = (\pi/P)\cos\phi 25) = \pi/3 \cos\phi 25) = 0.949"$

The outer diameters of the pinion and the gear

$d_{op} = d_p + 2a = 6 + 2(0.3333) = 6.667"$

$d_{og} = d_g + 2a = 7 + 2(0.3333) = 7.667"$

The root diameters of the pinion and the gear

$d_{rp} = d_p - 2b = 6 - 2(0.4167) = 5.166"$

$d_{rg} = d_g - 2b = 7 - 2(0.4167) = 6.166"$

The contact ratio

$$C_r = \frac{1}{p_c\cos\phi}\left[\sqrt{r_{op}^2 - r_{bp}^2} + \sqrt{r_{og}^2 - r_{bg}^2}\right] - \frac{c_d\tan\phi}{p_c}$$

$$= \frac{1}{1.047\cos 25}\left[\sqrt{\left(\frac{6.667}{2}\right)^2 - \left(\frac{5.437}{2}\right)^2} + \sqrt{\left(\frac{7.667}{2}\right)^2 - \left(\frac{6.344}{2}\right)^2}\right] - \frac{6.5\tan 25}{1.047} = 1.41$$

The meaning of $C_r = 1.41$ is that 41% of the duration has double-tooth contacts, occurring when a pair travels after C and before D, and that 59% of the duration has single-tooth contact, in the middle, before and after the pitch point. So, at the pitch point, we have on pair to take the load.

The gear triangles are plotted in **Figure 5-30**, where CD and AB are measurable. AB>CD. There is no interference.

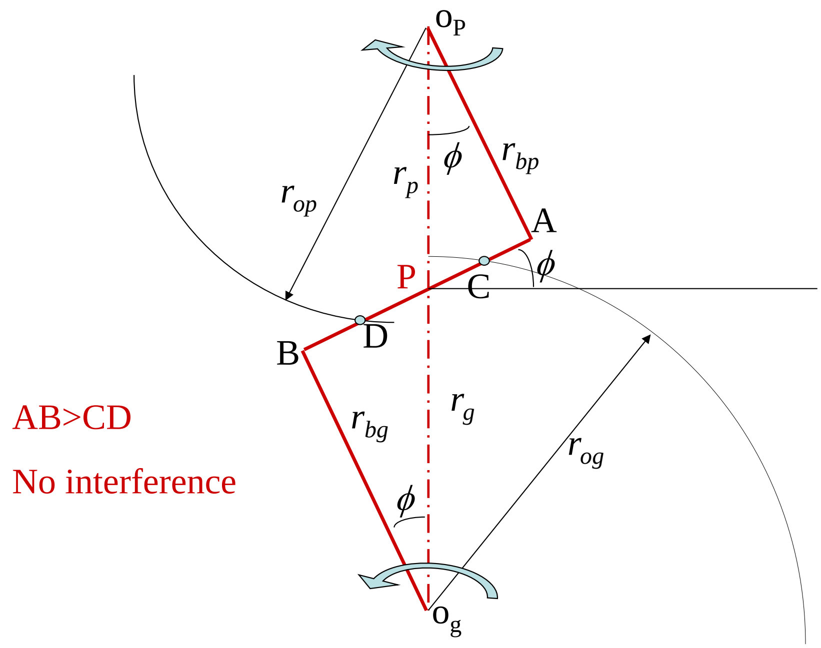


Figure 5-30. Gear triangles for **Example 5-4**. Subscripts r *p* is for pinion, *g* for gear, *o* for outside, *b* for base.

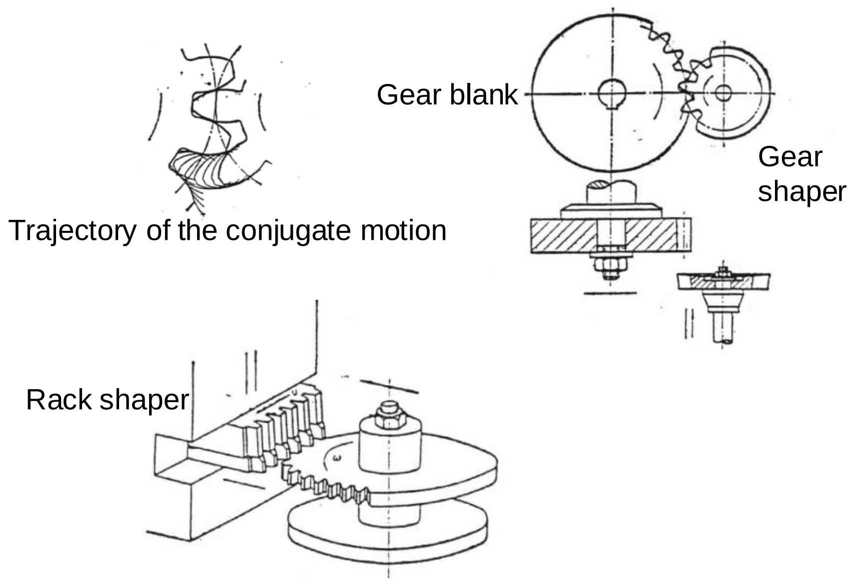
5-8 Gear manufacturing and gear structures

Gear manufacturing. Conjugate motion is utilized for gear manufacturing. Two methods are commonly used, cutting by a shaper or by a hob. The gear shaper is a gear that is made of a tool steel and machined with cutting edges. In gear shaping, the gear blank and the cutter (gear shaper) are arranged in such a way that their pitch circles are in a pure rolling motion (**Figure 5-31**). The shaper should also have a feed motion (radial) and a cutting motion (moving upward and downward). The shaper may also be a rack cutter. However, there is a disadvantage of using a rack shaper: it has a finite length, and the machine operator has to move the cutter back and forth.

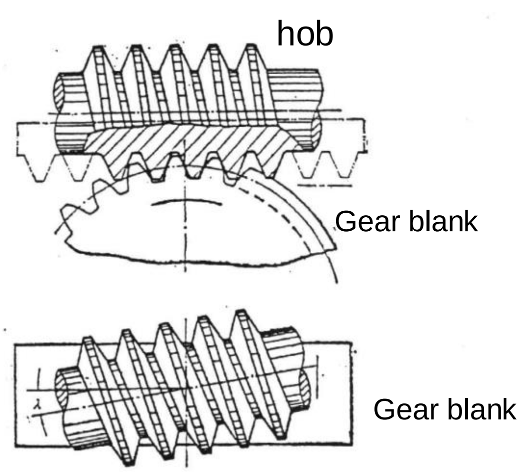
The tooth shape of a rack resembles that of a worm gear in the axial cross section. A worm gear is a continuous "rack." A worm gear cutter is called a hob. **Figure 5-31** also shows gear cutting by a hob.

Gear structures. **Figures 5-32** and **5-33** show some commonly used structures of gears. When gears are small, the gear shaft (a gear is integrated with a shaft) and the solid-disk gear structures are good choices. If the gears are large, the structure shown in **Figure 5-33** is preferred. The holes in the web are very important. They reduce weight, and they also facilitate processing and assembly.

Gear materials. Steel, cast iron, and malleable and nodular irons are the most common choices for gears. Bronzes are often used if there is a need for low friction and/or high corrosion resistance. Gray cast irons have the advantages of low cost, ease machining, and high wear resistance. However, they have low tensile strength. Steels have superior tensile strength and are cost-competitive in their low-alloy forms. Non-metallic materials, such as nylon and acetal, may also be used for gear materials. Gears bodies may be made of composite materials to reduce weight, which is then assembled with the gear tooth ring. The top material of gear teeth may be different from that of the body, and coating technologies are used to make this type of layered tooth surfaces.



Gear cutting by gear shapers.

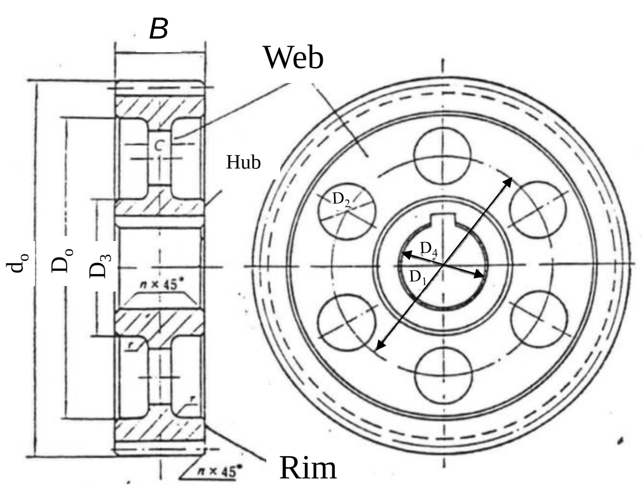


Gear cutting by a hob.

Figure 5-31 Gear cutting illustration. Plots are from Pu et al.

Figure 5-32 Structures of small gears (photos from internet).

Rim-web gear



d_o<500mm
$D_o = d_o - 10m_n$
$D_3 = 1.6D_4$
$D_2 = (0.25\sim0.35)(D_o - D_3)$
$D_1 = (D_o + D_3)/2$
$C = (0.2\sim0.3)B$
$n \sim 0.5m_n$
$r \sim 5$mm

Data from Pu et al.

Figure 5-33 Structure of medium-size and large gears with suggested dimensions.

5-9 Spur-gear force analysis

The force on a spur gear tooth is applied through the surface of its mating tooth. It is normal to the tooth surface. The force acts along line AB, now, the line of action, because it physically has to be perpendicular to the surface, and the line of action is the line of the common normal. The uniqueness of the line of action (normal of the contacting surfaces) ensures that the force is always along the same direction no matter where the contact occurs on the surface during the gear mesh. As a result of the force, the root of the gear tooth is subjected to a cyclic bending stress, while the tooth surface is under a cyclic contact stress, which we will talk about later. **Figure 5-34** shows the force on a gear tooth and the stress contours, where the tooth looks like a cantilever beam of a non-uniform cross section.

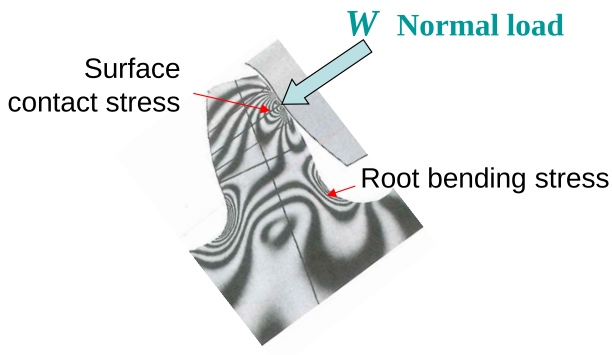


Figure 5-34 Normal force on a tooth surface. Photo from Hamrock et al.

Example 5-4 told us that, at the pitch point, we have only one pair of tooth teeth to take the load. That is the place for our force analysis. The normal force, W, on a spur-gear tooth can be easily resolved into orthogonal components, which are the tangential and radial components. Remember, the tangential force times the radius of the pitch circle results in the transmitted torque. The normal force and its orthogonal components are given in **Figure 5-35**. Equation (5-23-1) is the vector form of the force relationship, while the other equations in equation set (5-23) are for the magnitudes of forces. Here, superscripts t and r are used for tangential and radial.

Force relationship $$\vec{W} = \vec{W}^t + \vec{W}^r \qquad (5\text{-}23\text{-}1)$$

Tangential force $$W^t = \frac{T}{d/2} = \frac{60H}{\pi d\ n} \qquad (5\text{-}23\text{-}2)$$

Normal force $$W = \frac{W^t}{\cos\phi} \qquad (5\text{-}23\text{-}3)$$

Radial force $$W^r = W \sin\phi = W^t \tan\phi \qquad (5\text{-}23\text{-}4)$$

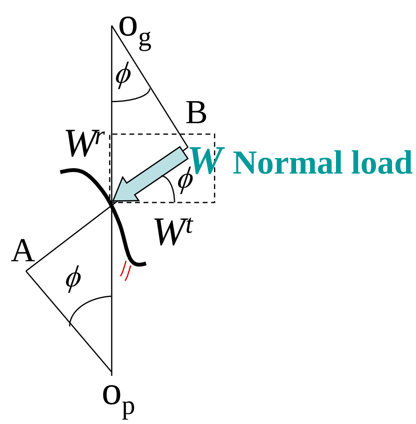


Figure 5-35. Normal force on a spur-gear tooth surface and its orthogonal components. Superscripts t and r means the transverse plane and the radial plane.

Gear force analyses are necessary preparations for gear design. Moreover, they will provide the force information for shaft design and supporting bearing selection. We will see how gear forces are considered in shaft force analyses through **example 5-5**.

Example 5-5 A gearbox is shown below in **Figure 5-36**. Four gears are mounted on three shafts to form a compound gear train. A is the input end and G the output end. The shaft outlets at A and G are connected to couplings.

Known: The power transmitted, H (unit: W, or kW), n_1, d_1, d_2, d_3, and d_4

Find: Torques and forces on the gears and the shafts.

Note:
Gears 1 and 2 must have the same module and pressure angle

Gears 3 and 4 must have the same module and pressure angle

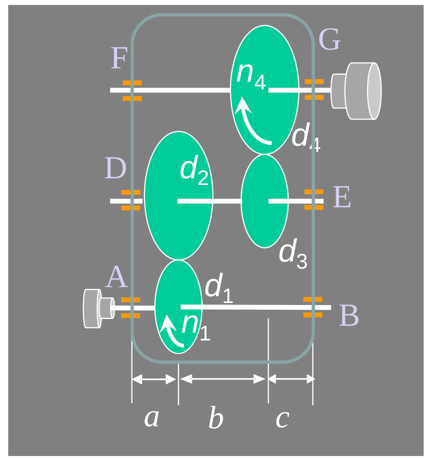


Figure 5-36 Gears and shafts in the gear box, O_1 through O_4 are the centers of the gears.

Solution

We need to analyze the shafts one by one. Let's start from shaft AB. Why? We know the input power and n_1, so that we can easily get the torque there. This shaft has a gear and a coupling. The coupling only transmits torque. Gear 1 on this shaft is the driver of the system. Its teeth push the teeth of the driven gear, which is gear 2 in this case. Therefore, the force on the teeth of gear 1 should produce a torque that is against the rotating direction of shaft AB. In other words, gear 1 "feels" the resistance from gear 2.

The forces on gear 1 and shaft AB are shown in **Figure 5-37**. The normal force on the gear tooth, which can be resolved to the tangential and radial components, must be "shifted" to the center of the shaft for the shaft force analysis. The radial force can be directly moved along its line of application, but torque should be added when moving the tangential force. This torque is balanced by the torque at the coupling. Bearing reactions should be calculated carefully based on the forces on the shaft. From the gear radial load, we can obtain the bearing reactions in the y direction, and from the gear tangential load, we can obtain the bearing reactions in the z direction. Thus, we obtained the free-body diagram, FBD, for shaft AB.

The force from gear 2 to gear 1 is marked as 21 in the subscript, its two components are marked as t and r in the superscripts. Reactions at bearings A and B are labeled by A and B with components in y and z. Spur gears do not bear axial force along the shaft direction.

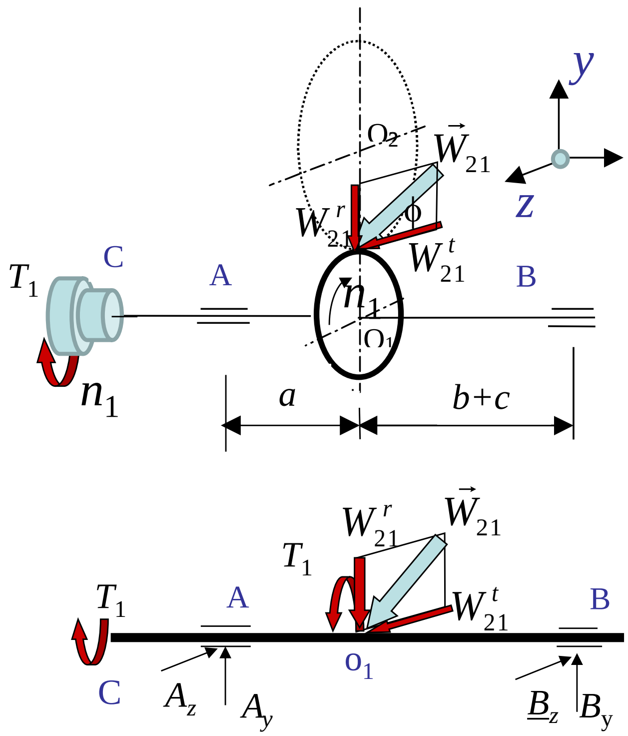


Figure 5-37 Force analysis for gear 1 and the FBD for shaft AB.

The gear torque, the normal force, and the force components are

$$\textit{Torque} \qquad T_1 = \frac{H}{\omega_1} = \frac{60H}{2\pi n_1}$$

$$\textit{Force relation} \qquad \vec{W}_{21} = \vec{W}_{21}^{\,t} + \vec{W}_{21}^{\,r}$$

$$\textit{Tangential force} \qquad W_{21}^{t} = \frac{T_1}{d_1/2} = \frac{60\,H}{\pi d_1 n_1}$$

$$\textit{Normal force} \qquad W_{21} = \frac{W_{21}^{t}}{\cos\phi}$$

$$\textit{Radial force} \qquad W_{21}^{r} = W_{21}\sin\phi = W_{21}^{t}\tan\phi$$

The bearing reactions can be found accordingly. Note that HP=550ft lbf S = 745.7 Watt and

$$H = \frac{W^t V}{33000}\,(\text{HP,lbf.ftmin,US}),\, W^t V\,(\text{Watt,ISO}),\, T\omega(\text{Watt,ISO})$$

Shaft DE (**Figure 5-38**) has two gears, and the mechanics is slightly more complicated than that for shaft AB. We will do the same thing, analyzing the force components on a gear tooth and then the force equilibrium of the shaft, or its free-body diagram, FBD. We need to pay attention to the action and reaction. The force on gear 2 is opposite to that on gear 1. The torque on gear 2 is balanced by the torque on gear 3. Therefore, the direction of the normal force on gear 3 is determined.

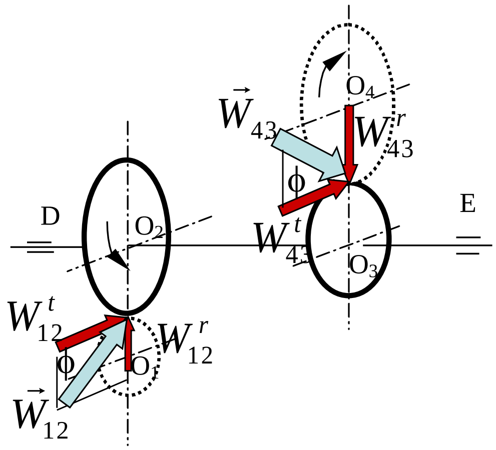


Figure 5-38 Force analysis for gears 2 and 3.

The normal force, or the components of the normal force, and the torque on gear 2 are

Force relationship $\vec{W}_{12} = -\vec{W}_{21}$

Tangential force $W^t_{12} = -W^t_{21}$

Radial force $W^r_{12} = -W^r_{21}$

Torque $T_2 = W^t_{12}(d_2/2)$

The normal force (or components of the normal force) and the torque on gear 3 are

Force relationship $T_2 = -T_3$ $\vec{W}_{34} = -\vec{W}_{43}$

Tangential force $W^t_{43} = \dfrac{T_3}{d_3/2}$

Normal force $W_{43} = \dfrac{W^t_{43}}{\cos\phi}$

Radial force $W^r_{43} = W_{43}\sin\phi = W^t_{43}\tan\phi$

With these forces analyzed, the bearing reactions can be easily calculated. Detailed force diagrams for the gears and the free-body diagram for shaft DE are shown in **Figure 5-39**.

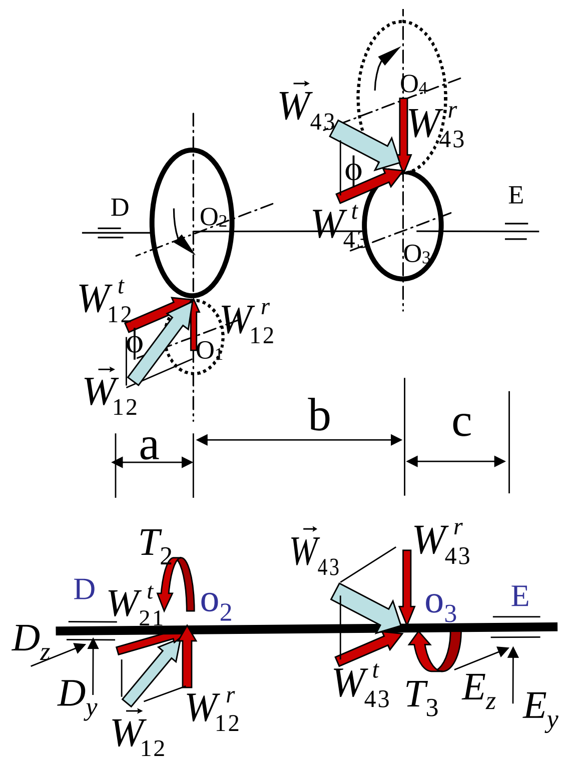


Figure 5-39 Force analysis for gears 2 and 3 and for shaft DE.

Note that on a driver gear, the directions of the torque and speed are different, but on a driven gear, the directions of the torque and speed are the same. Can you do the force components for gear 4 and the force analysis for shaft FG, together with its FBD?

Example 5-6. In the spur-gear train in **Figure 5-40**, gear 0 is the input gear. The power is H = 100KW (assuming no power loss), and the rotation speed of shaft AB is n = 1500 rpm. Gear parameters are, m_1 = 5, N_1 = 20, m_2 = 6, N_2 = 80, and pressure angles are all 20 degree. Gear centers are marked by O_0, O_1, O_2, and O_3. If a = 50 mm, b = 40 mm, c = 40 mm, please

(1) Calculate the torque transmitted to shaft AB, the forces on gears 1 and 2, and the forces on shaft AB.

(2) Determine the magnitude and location of the maximum bending moment for the shaft.

Solution.

$$d_1 = m_1 N_1 = 5(20) = 100 \text{ mm}$$

$$d_2 = m_2 N_2 = 6(80) = 480 \text{ mm}$$

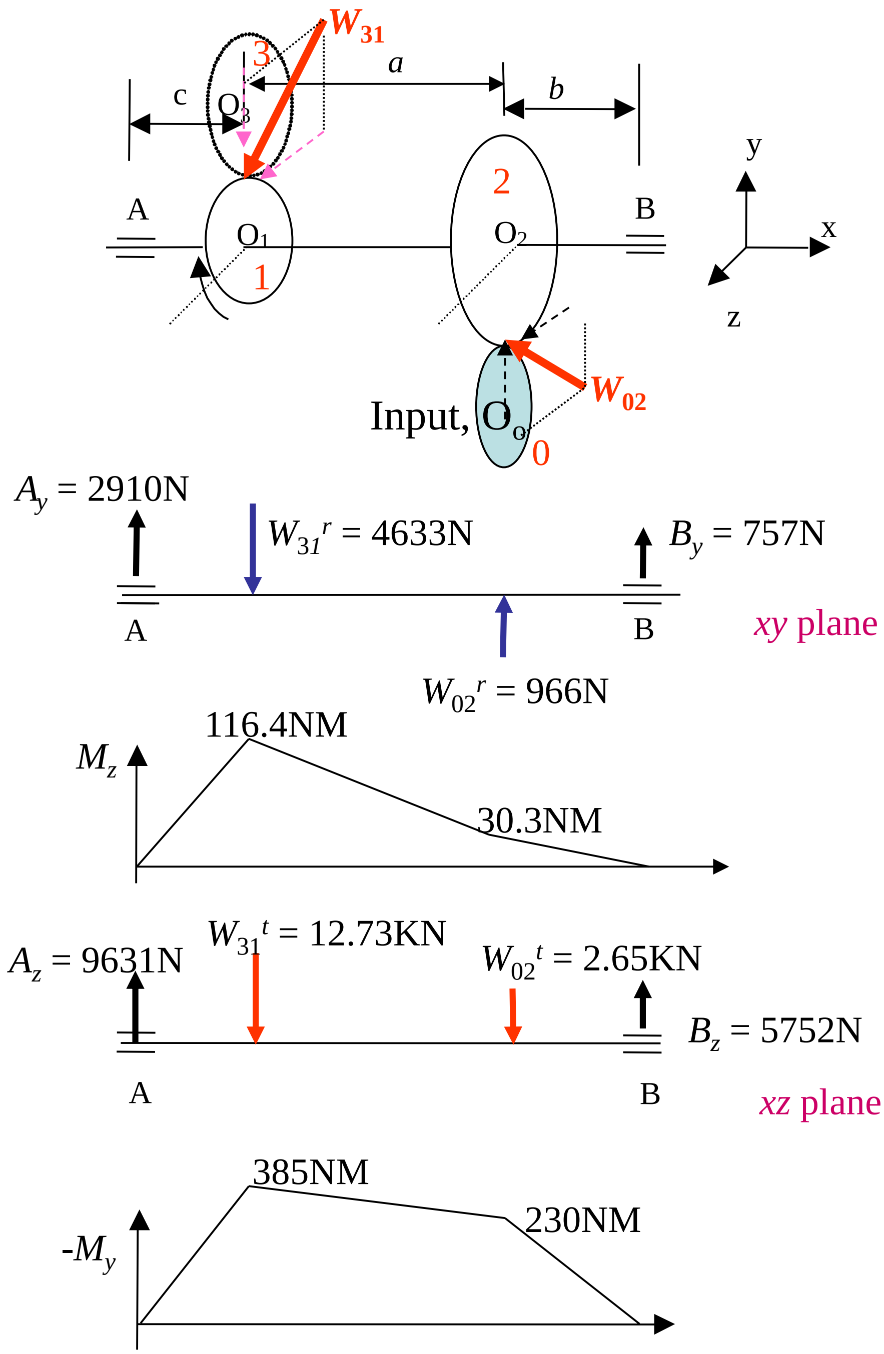


Figure 5-40 The shaft and gears in **Example 5-6**. Note that M_y is negative simply due to the coordinate selection and the moment-diagram sign convention.

Torque
$$T_2 = T_1 = \frac{H}{\omega} = \frac{10{,}000}{2\pi n/60} = \frac{100{,}000}{2\pi(1500)/60} = 636.6Nm$$

Gear forces $$W_{02}^{t} = \frac{T_2}{d_2/2} = \frac{636.6}{0.48/2} = 2653N$$

$$W_{02}^{r} = W_{02}^{t}\tan\phi = 2653(\tan 20) = 966N$$

$$W_{31}^{t} = \frac{T_1}{d_1/2} = \frac{636.6}{0.1/2} = 12.73KN$$

$$W_{31}^{r} = W_{31}^{t}\tan\phi = 12.73(\tan 20) = 4633N$$

Bearing reactions

$\Sigma M_A = 0$ $$B_y = \frac{W_{31}^{r}(c) - W_{02}^{r}(a+c)}{a+b+c} = \frac{4633(0.04) - 966(0.05+0.04)}{0.05+0.04+0.04} = 757N$$

$$A_y = W_{31}^{r} - B_y - W_{02}^{r} = 4633 - 757 - 966 = 2910N$$

Similarly $$B_z = \frac{W_{31}^{t}(c) + W_{02}^{t}(a+c)}{a+b+c} = \frac{12730(0.04) - 2653(0.05+0.04)}{0.05+0.04+0.04} = 5752N$$

$$A_z = W_{31}^{t} - B_z + W_{02}^{t} = 12730 - 5752 - 2653 = 9631N$$

Moments

The largest moments are at shaft cross section O_1:

$M_{ymax} = A_y\,(c) = 2910(0.04) = 116.4$ NM

$M_{zmax} = A_z\,(c) = 9631(0.04) = 384.5$ NM

Resultant moment $$M_{O_1} = \sqrt{M_y^2 + M_z^2} = \sqrt{116.4^2 + 384.5^2} = 402.2Nm$$

5-10 Gear failure mechanisms

Cyclic loading on gear teeth may cause fatigue failures of the teeth, which are mainly fatigue fracture at tooth roots due to bending and surface pitting due to contact fatigue. Therefore, the design criteria for gears should be formulated against root fracture and surface pitting. Actually, various types of surface failure, such as pitting, severe wear, and scuffing, account for 70-90% of total damage to heavy-duty gears. However, the analyses for wear and scuffing are less developed due to the lack of in-depth mechanism understanding and insufficient data. The simulation and analysis of wear and scuffing are much more complicated, and therefore, criteria have been less developed. **Figure 5-41** shows several photos of tooth surface failures.

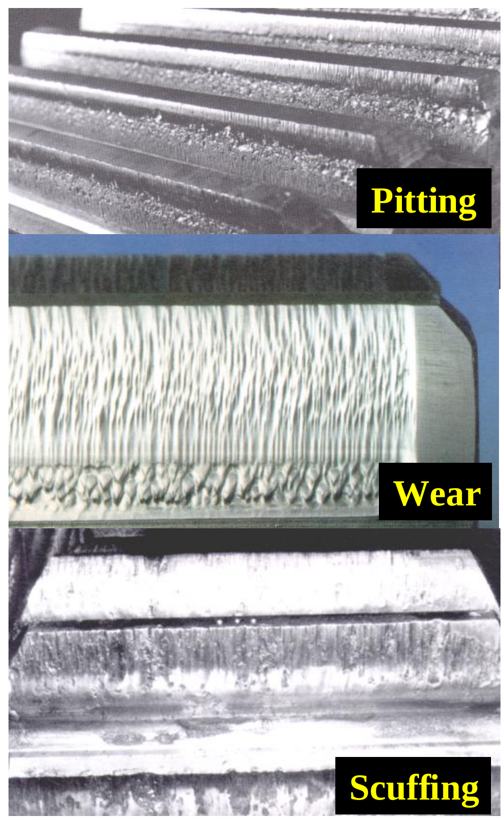


Figure 5-41 Gear tooth surface failures.

5-11 Spur gear design analyses

Similar to the shaft design we have studied, the gear design is actually the analysis of gear strength, or the control of stresses to be well below their limits to failures. The failure mechanisms of gears are root fracture and surfaces pitting, severe wear, and scuffing, as mentioned before. However, mature theories for gear failures are the root bending stress/strength analyses and surface contact-stress/strength analyses, which lead to the design criteria against root fracture and some types of surface failure. Our approach will be similar to what we previously used: first creating a structure, then making a mechanics model, and then conducting stress analyses, finding the strength of the material used, and finally analyzing factors of safety.

1 Root bending stress

A gear tooth is a cantilever beam of a varying cross section, as shown in **Figure 5-42**, which may be simplified as an equivalent beam of height l, thickness t, and width b_w. We consider the worst possible situation: a single pair of teeth is in contact and transmits the entire load at the top of the addendum. The bending force, F, is the normal-load component perpendicular to the centerline of the tooth, which is the tangential load, W^t. Equation (5-24-1) starts from the bending normal stress of a rectangular beam and indicates that the root bending stress is related to tangential load W^t, face width b_w., tooth thickness t, and tooth depth l. However, we need to take into account the real tooth shape somehow. It is noticed that among the key geometric parameters, such as module, pressure angle, and face width, only face width b_w is accurately involved here. Both thickness and tooth depth are related to the module of the gear. Therefore, the effects of the tooth depth and the thickness may be reflected by a shape factor multiplied by the module of the gear, as shown in Equation (5-24-2).

$$\sigma_b \rightarrow \frac{Fl(t/2)}{b_w t^3/12} \rightarrow \frac{W^t l}{b_w t^2} \qquad (5\text{-}24\text{-}1)$$

$$\sigma_b \rightarrow \frac{W^t l}{b_w t^2} \rightarrow \frac{W^t}{b_w m}\boxed{\frac{ml}{t^2\,??}} \rightarrow \frac{W^t}{b_w m Y_j} \qquad (5\text{-}24\text{-}2)$$

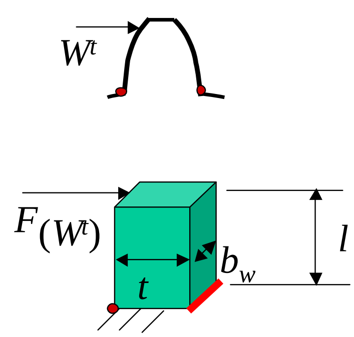


Figure 5-42 A tooth as a cantilever beam.

There are several ways to analyze the bending strength. A well-accepted approach is the AGMA approach. AGMA stands for the American Gear Manufacturing Association.

The AGMA approach may be simply expressed by Equation (5-25-1 and 2), i.e., the root beading stress, σ_b, should be smaller than the allowable stress for bending, $\sigma_{all,b}$. Thus, the factor of safety against root bending fatigue failure, n_b, can be defined by Equation (5-25-2).

$$\sigma_b \le \sigma_{all,b} \qquad (5\text{-}25\text{-}1)$$

$$n_b = \sigma_{all,b} / \sigma_b \qquad (5\text{-}25\text{-}2)$$

The AGMA approach also considers the effect of gear size, loading smoothness, load distribution, as well as different applications, and it introduces a series of factors to modify the stress equation. Equations (5-26) and (25-27) are the finalized root bending stress formulas for metric and US custom gears, respectively, using the relationship *P=1/m* to convert diametral pitch and module.

$$\sigma_b = \frac{W^t}{b_w m Y_j} K_a K_s K_m K_v K_i K_b \tag{5-26}$$

$$\sigma_b = \frac{W^t P}{b_w Y_j} K_a K_s K_m K_v K_i K_b \tag{5-27}$$

Rim thickness factor
Idler factor
Dynamic factor
Load distribution factor
Size factor
Application factor
Geometry factor

2 Surface contact stress

Contact stress is a new stress concept. It is referred to the stress state of the material right below the point of load application. Such a stress was avoided in the entire study of Mechanics of Materials. However, we cannot neglect its existence anymore when surface failure is a concern.

The contact of two cylinders of curved surfaces may be equivalent to that of two cylinders whose radii are the radii of curvature of the curved bodies at the point of contact. Such a simplification is proven to be valid because the region influenced by the contact is much smaller than the sizes of the bodies. The contact of two cylinders can be further modeled by the contact between an equivalent cylinder and a rigid, flat surface. The equivalence means that we add the curvature of both mathematically and material properties of both mechanically as if they were two springs in series. As a result, the equivalent cylinder takes care of the contact geometry and the elastic properties of the two tooth materials. **Figure 5-43** shows the gear triangles, which represent the teeth, and the simplification of the contact to that of two different cylinders, and further to that of an equivalent cylinder and a rigid flat. The radius of this equivalent cylinder, *R*, which is from the result of curvature summation at the contact point, is given in Equation (5-28). The elastic properties of the materials are summed into an equivalent Young's modulus, *E'* (Equation (5-29).

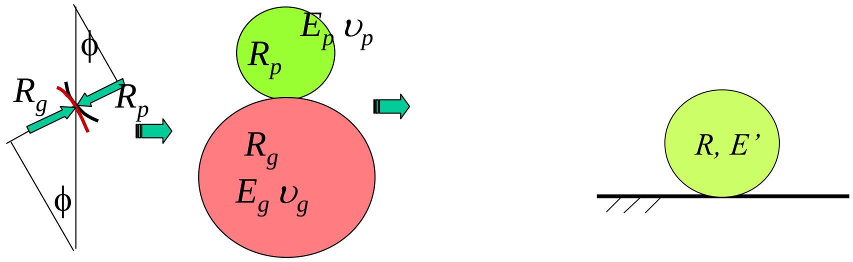


Figure 5-43. Tooth contact and equivalent radius of curvature.

$$R = \frac{1}{\dfrac{1}{R_p} \pm \dfrac{1}{R_g}} = \frac{1}{\left(\dfrac{1}{d_p} \pm \dfrac{1}{d_g}\right)\dfrac{2}{\sin\phi}} \tag{5-28}$$

$$E' = \frac{2}{\dfrac{1-\nu_p^2}{E_p} + \dfrac{1-\nu_g^2}{E_g}} \tag{5-29}$$

A dimensionless load is defined as

$$W' = \frac{W/b_w}{E'R} = \frac{W/b_w}{E'R} \tag{5-30}$$

Then the maximum contact stress is the following, which is also called the maximum Herzian contact pressure, p_H, for the contact of cylinders.

$$\sigma_c \to p_H = E'\left(\frac{W'}{2\pi}\right)^{1/2} = E'\left(\frac{W}{2\pi b_w RE'}\right)^{1/2} \tag{5-31}$$

Again, the AGMA approach is used for the surface strength analysis, i.e. the maximum contact stress should be smaller than the allowable contact stress. Likewise, the factor of safety against contact pitting failure, n_c, can be defined by (5-33).

$$\sigma_c \le \sigma_{all,c} \tag{5-32}$$

$$n_c = \sigma_{all,c} / \sigma_c \tag{5-33}$$

Similar to the bending stress formula, the AGMA approach for surface strength also considers the effects of loading, size, load distribution, as well as different applications, and a series of factors are needed to modify the contact stress equation (Equations (5-34)). Here, K_a, K_s, K_m, and K_v are the same as those in the bending stress formula. In addition, an elastic factor, K_e, is introduced for the equivalent modulus, calculated from Equation (5-35), and I is defined as the geometry factor, accounting the curvature effect and others, expressed by Equation (5-36).

$$\sigma_c = E^{'}\left(\frac{W}{2\pi b_w RE'}K_a K_s K_m K_v\right)^{1/2} = \sqrt{E'}\left(\frac{W^t d_p / \cos\phi}{b_w d_p 2\pi R}K_a K_s K_m K_v\right)^{1/2}$$
$$= K_e\left(\frac{W^t}{b_w d_p}\frac{1}{I}K_a K_s K_m K_v\right)^{1/2} \tag{5-34}$$

$$K_E = \sqrt{E'} = \sqrt{\frac{2}{\dfrac{1-\nu_p^2}{E_p} + \dfrac{1-\nu_g^2}{E_g}}} \tag{5-35}$$

$$I = \frac{2\pi R\cos\phi}{d_p} = \frac{2\pi\cos\phi}{d_p\left(\dfrac{1}{d_p}\pm\dfrac{1}{d_g}\right)\dfrac{2}{\sin\phi}} = \frac{\pi\cos\phi\sin\phi}{\left(1\pm\dfrac{d_p}{d_g}\right)} \tag{5-36}$$

Note that here K_E is defined differently from that in the Shigley book, and so is I, in the way “2” and “π” are placed. We will learn how to use these equations and how to design gears with sufficient strength from an example later.

3. Gear allowable strength

Gear allowable stresses are determined by modifying gear material strengths, as shown below.

Allowable bending stress $$\sigma_{all,b} = \frac{S_b Y_N}{K_T K_R} \tag{5-37}$$

Allowable contact stress $$\sigma_{all,c} = \frac{S_c Z_N C_H}{K_T K_R} \tag{5-38}$$

Here, S_b and S_c are for the bending and contact strengths of a gear material, K_T is the temperature factor, K_R the reliability factor, Y_N the stress cycle factor for bending, Z_N the stress cycle factor for contact, and C_H the hardness ratio factor. The example below will help explain the use of these factors.

Example 5-7

Given: A gear train of a pinion and a gear and solution steps are given in **Figure 5-44**.

The train value: 3.5:1

N_p = 18, N_g = (3.5) 18 = 63

Pressure angle; 20°

Diametral pitch P = 6

Power and speed of the pinion shaft:

H = 20 HP, n_P = 2500 rpm

Driven by an electric motor, no shock.

Life: 5 years, 1 shift per day, 99% reliability.

Find: Major gear geometry and factors of safety

What should we do? We will solve this problem in the following steps

(1) Gear geometry
(2) Tangential force
(3) Materials and allowable stresses
(4) Factors for stresses
(5) Stresses
(6) Factors of safety

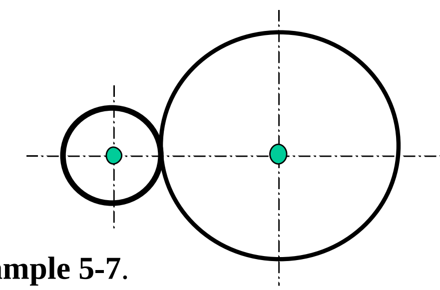


Figure 5-44 Things to solve for **Example 5-7**.

Solution

(1) Gear geometry

$$d_p = \frac{N_p}{P} = \frac{18}{6} = 3"$$

$$d_g = \frac{N_g}{P} = \frac{63}{6} = 10.5"$$

$$a = \frac{1}{P} = \frac{1}{6}"$$

$$b = \frac{1.25}{P} = \frac{1.25}{6} = 0.208"$$

Let's choose the middle value from the following empirical formula for the face width. Why the tooth face cannot be very wide or very narrow? Think about shaft deflection and load distribution issues.

$$\frac{8}{P} < b_w < \frac{16}{P}$$

$$b_w \approx \frac{12}{P} = \frac{12}{6} = 2"$$

The face width here is nominal. Actually, the pinion is made slightly wider (2-5 mm). Why? Think about assembly tolerance.

(2) Tangential force

$$W^t = \frac{H(33000)}{V} = \frac{H(33000)}{2\pi\, n \, {}^{d_p}\!/_{2}} = \frac{20(33000)}{2\pi(2500)(3/2)\,/_{12}} = 336\text{lb} \qquad (5\text{-}39)$$

Pitch circle velocity, V = 1963feet/min.

(3) Materials and Allowable stresses

Materials. We choose steel gears with Young's modulus E = 30 MPsi and Poisson's ratio ν = 0.28, through hardened.

Figures 14. 24, 25 in Hamrock's book (2005) are used here as **Figure 5-45**. We also need to select hardness values for each gear, which leads us to the strengths of the materials.

Piion – 300 HB, grade 2	S_{Pb} = 0.102HB+16.4 = 47 Ksi
	S_{Pc} = 0.349HB + 34.3 = 139 Ksi
Gear – 250 HB, grade 1	S_{Gb} = 0.0773HB+12.8 = 32.1 Ksi
	S_{Gc} = 0.322HB +29.1 = 109.6 Ksi

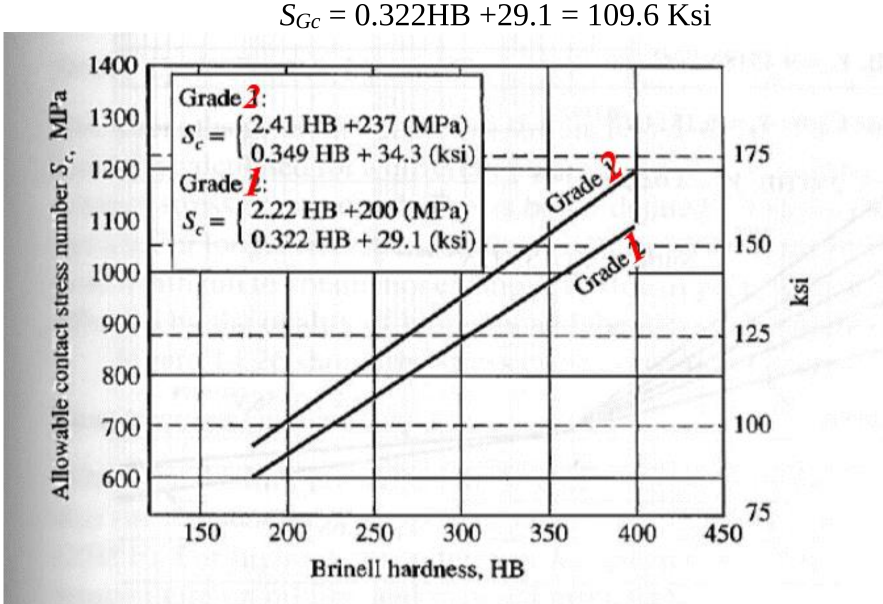
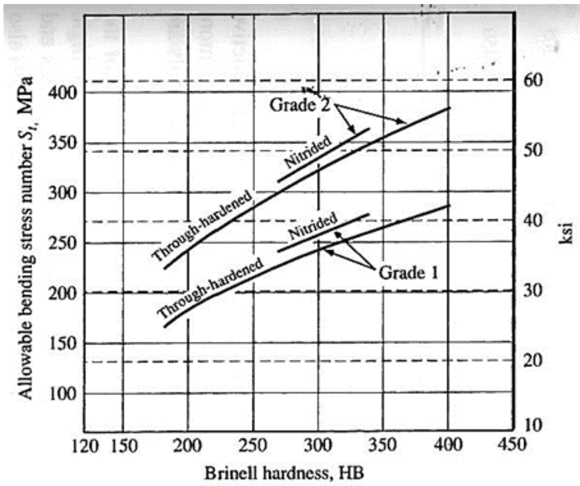


Figure 14.25 Effect of Brinell hardness on allowable contact stress number for two grades of through-hardened steel. [AGMA (1999).]

(Note here corrections are made in red in the figure and equations.)



Material	Grade	Allowable bending stress number MPa	Allowable bending stress number ksi
Through-hardened steels	1	0.533HB+88.3	0.0773 HB + 12.8
	2	0.703HB+113	0.102 HB + 16.4
Nitriding through-hardened steels	1	0.0823 HB + 12.15	0.568 HB + 83.8
	2	0.1086 HB + 15.89	0.749 HB + 110

(Note corrections of the equations are made, in red.)

Figure 5-45 Allowable stresses. These are for 10 million cycles, 99% reliability (Hamrock et al. 2005).

Here the hardness values of the two gears are different. Why? The pinion teeth and tooth surfaces are subjected to more cycles, and therefore, should be made stronger and harder. A hardness difference of 50-100HB is suggested as the starting point. However, this suggestion is not quite precise. Wait to see how the hardness factor is determined.

Factors for strengths.

Temperature factor, K_T

K_T, the temperature factor, is 1 if the environmental temperature is not higher than 125^{0}C.

Reliability factor, K_R

K_R, the reliability factor, is higher if the reliability requirement is higher. For the reliability of survival to be 99%, K_R = 1, because this is one of the conditions at which the material tests were done; for the reliability of survival to be 99.9%, K_R = 1.2. It becomes K_R = 0.85 if the reliability of survival is reduced to 90%. See **Table 5-4**.

Table 5-4 Reliability factor, K_R

Percentage reliability of survival	Reliability factor, K_R
50	0.70
90	0.85
99	1.00
99.9	1.25
99.99	1.50

(AGMA 2101-C95 (1999)

Stress cycle factors

Stress cycle factors are determined using the following figures. We need to get the cycle number first as follows.

Cycle number of the pinion = 5 (365x8)(60) (2500rpm) = 2190Million cycles
Cycle number of the gear = 5 (365x8)(60) (2500rpm/3.5) = 625.71Million cycles

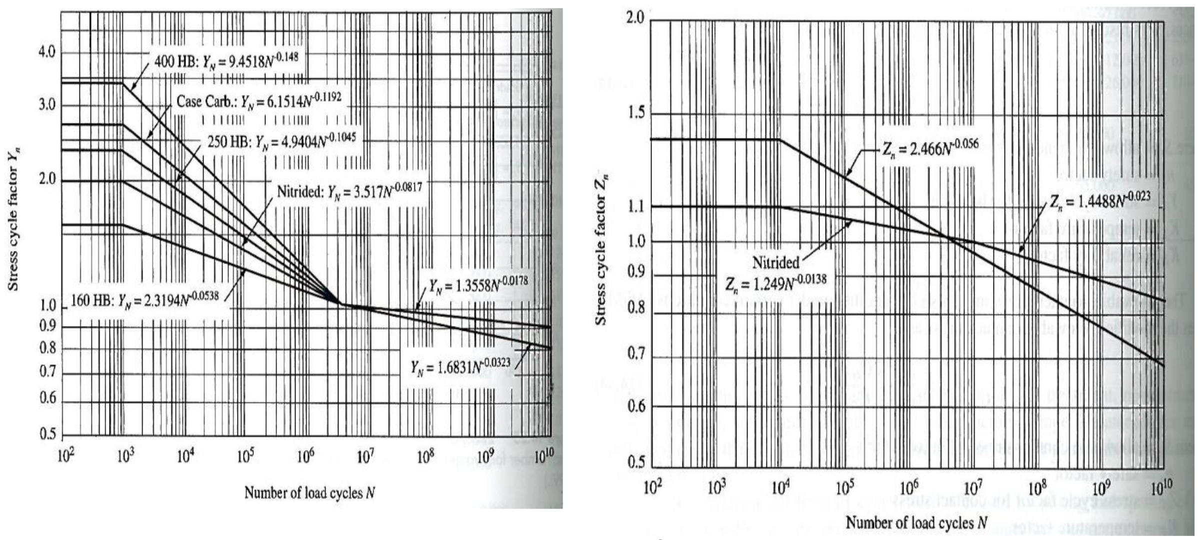


Figure 5-46 Stress cycle factors (Hamrock et al. 2005).

$Z_N = 2.466N^{-0.056}$, therefore, $Z_{Np} = 0.739$ $Z_{Ng} = 0.793$
$Y_N = 1.6831N^{-0.0323}$, therefore, $Y_{Np} = 0.840$ $Y_{Ng} = 0.875$

Hardness factor, C_H

Hardness factor, C_H, considers the difference of the hardness of the mating surfaces; it only applies to the gear, so that C_H becomes C_{Hg}. Here, the hardness is the Brinell hardness. Either the following equations or the figure below can be used to determine the hardness factor.

$$C_{Hg} = 1.0 + A'(N_g/N_p - 1.0) \quad (5\text{-}40)$$

$A' = 0$ if $HB_p/HB_g < 1.2$
$A' = (8.89\times10^{-3})(HB_p/HB_g) - 8.29\times10^{-3}$ if $1.2 \leq HB_p/HB_g \leq 1.7$
$A' = 0.00698$ if $HB_p/HB_g > 1.7$

The A' values for Equation (5-40) tell us that if that the proper range of the hardness ratio is precisely $1.2 \leq HB_p/HB_g \leq 1.7$, below which the hardness different brings no gain in the gear strength, and the strengthening effect is bonded by that given by $HB_p/HB_g = 1.7$.

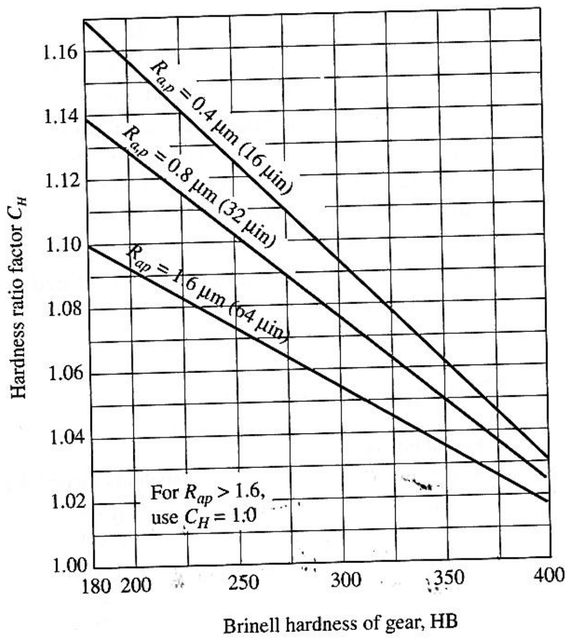


Figure 14.27 Hardness ratio factor C_H for surface hardened pinions and through-hardened gears. [AGMA Standard 2001-C95, (1999).]

Figure 5-47 Hardness factor for surface hardened gears (Hamrock et al. 2005).

In our problem, HB_P/HB_G =300/250 = *1.2*, therefore

$$C_{HG} = 1.0 + A'(N_g/N_p - 1.0) = 1.0 + 0.00238(3.5 - 1.0) = 1.006$$

The highest hardness the pinion can do is 1.7x250 = 425HB.

As a result of the factors, the allowable stresses are the following.

Allowable bending stress for the pinion $$\sigma_{all,bp} = \frac{S_b Y_N}{K_T K_R} = \frac{(47)(0.84)}{(1)(1)} = 39.84 Ksi$$

Allowable bending stress for the gear $$\sigma_{all,bg} = \frac{S_b Y_N}{K_T K_R} = \frac{(32.1)(0.875)}{(1)(1)} = 28.09 Ksi$$

Allowable contact stress for the pinion $$\sigma_{all,cp} = \frac{S_c Z_N (\cancel{C_H})}{K_T K_R} = \frac{(139)(0.739)(1)}{(1)(1)} = 102.72 Ksi$$

Allowable contat stress for the gear $$\sigma_{all,cg} = \frac{S_c Z_N C_{Hg}}{K_T K_R} = \frac{(109.6)(0.793)(1.00621)}{(1)(1)} = 87.43 Ksi$$

(4) Factors for stresses

Now we come to determine factors for bending and contact stresses

$$\sigma_b = \frac{W^t P}{b_w Y_j} K_a K_s K_m K_v K_i K_b \qquad \sigma_c = K_e \left(\frac{W^t}{b_w d_p} \frac{1}{I} K_a K_s K_m K_v \right)^{1/2}$$

Application factor: K_a.

The application factor considers the effects of system speed variation and shock. **Table 5-5** lists some values of the application factor. For a system powered by an electric motor, the load and speed are usually smooth without obvious shock. Therefore, $K_a = 1$ may be the choice for our example. For a system powered by an internal combustion engine, a higher value of K_a is preferred.

Again, in the current problem, we have an electric motor as the power supplier without shock. Therefore, $K_a = 1$

Table 5-5 Application factor

Power source	Application factor, K_a
No shock	1
Light shock	1.2
Moderate shock	1.3

Load distribution factor: K_m.

The load distribution factor increases with the face width of a gear. It is difficult to make the contacting teeth of a gear set completely parallel due to manufacturing tolerance, assembly inaccuracy, and shaft deflection, etc., if the gears have large face widths.

This gear set is not a high precision gear set. However, it cannot be made too rough. It has the commercial quality. Use b_w/d_p in Fig. 14.23 in Hamrock's book (1999), which is shown below, to determine K_m, and use the same K_m for both gears because their load distributions are the same.

lgure 14.23 Load distrib-
ion factor as function of face
idth and ratio of face width to
tch diameters. Commercial
jality gears assumed. [*Machine
'ements in Mechanical Design*
/ Mott, ©1992. Reprinted by
ermission of Prentice-Hall, Inc.,
pper Saddle River, NJ.]

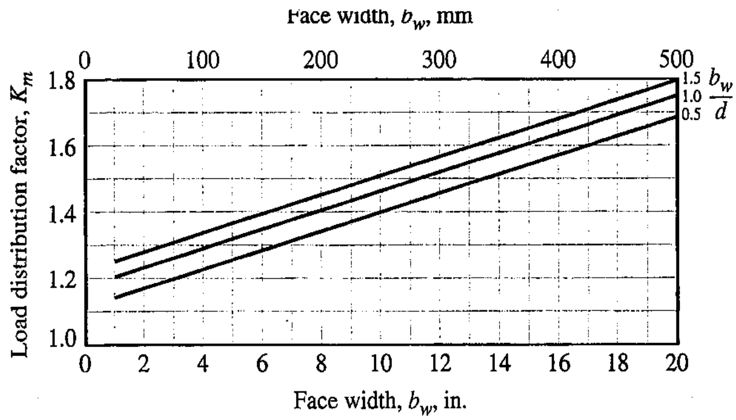


Figure 5-48. Simplified chart for load distribution factor K_m. (Hamrock et al.,1999),

More accurate considerations of load distribution suggest a more complicated determination of the load distribution factor, where both transverse and face distribution factors, C_{mt}, C_{mf}, should be evaluated. For gears with $b_w/d_P \leq 2$ with the face width less than 1m, mounted between bearings (equivalent to a simply supported beam), and subjected to contact across the full width of the narrower member when loaded, K_m may be approached roughly with only the face load distribution factor, C_{mf}.

$$K_m = f(C_{mt}, C_{mf}) \sim C_{mf} = 1.0 + C_{mc}(C_{pf}C_{pm} + C_{ma}C_e) \tag{5-41}$$

Here, C_{mc} is the lead correction factor, for uncrowned teeth, $C_{mc} = 1$
C_{pf} is the pinion proportion factor.

$$C_{pf} = \frac{b_w}{10d_d} - 0.025 \qquad \text{If } b_w \leq 1 \text{ in}$$

$$C_{pf} = \frac{b_w}{10d_d} - 0.0375 + 0.0125 b_w \qquad \text{If } 1 < b_w \leq 17 \text{ in} \tag{5-42}$$

$$C_{pf} = \frac{b_w}{10d_d} - 0.1109 + 0.02907 b_w - (2.28)(10^{-4}) b_w^2 \qquad \text{If } 17 < b_w \leq 40 \text{ in}$$

C_{pf} in the ISO unit should be

$$C_{pf} = \frac{b_w}{10d_d} - 0.025 \qquad \text{If } b_w \leq 25 \text{ mm}$$

$$C_{pf} = \frac{b_w}{10d_d} - 0.0375 + 0.000492 b_w \qquad \text{If } 25 < b_w \leq 432 \text{ mm} \tag{5-42'}$$

$$C_{pf} = \frac{b_w}{10d_d} - 0.1109 + 0.000815 b_w - (3.53)(10^{-7}) b_w^2 \qquad \text{If } 432 < b_w \leq 1020 \text{ mm}$$

For this problem, b_w = 2", and

$$C_{pf} = \frac{b_w}{10d_d} - 0.0375 + 0.0125b_w$$

$$= \frac{2}{10(3)} - 0.0375 + 0.0125(2)$$

$$= 0.0542$$

C_{pm} is the pinion proportion modifier; it is affected by the gear mounting location (see the figure below), and

C_{pm} = 1.0 if $S_1/S < 0.175$
C_{pm} = 1.1 if $S_1/S \geq 0.175$

We will make S_1 small and use $C_{pm} = 1$.

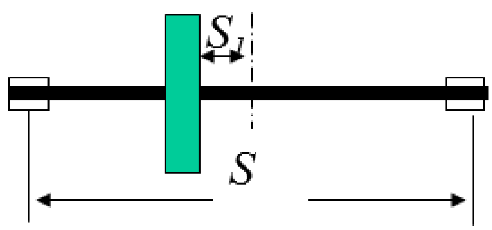


Figure 5-49 Gear mounting location, indicated by S_1 and S.

C_{ma} is the mesh alignment factor, and $C_{ma} = A + Bb_w + C\,B_w^2$. Coefficients of this equation is given in the table below.

Table 5-6 Factors A, B, and C for C_{ma}

b_w in US custom units

Condition	A	B	C
Open gearing	0.247	0.0167	$-0.765(10)^{-4}$
Commercial gears	0.127	0.0158	$-1.093\,(10)^{-4}$
Precision gears	0.0675	0.0128	$-0.926(10)^{-4}$
Extra-precision gears	0.00038	0.0102	$-0.822(10)^{-4}$

b_w in mm

Condition	A	B	C
Open gearing	0.247	$6.57(10)^{-4}$	$-1.186(10)^{-7}$
Commercial gears	0.127	$6.22(10)^{-4}$	$-1.69\,(10)^{-7}$
Precision gears	0.0675	$5.04(10)^{-4}$	$-1.44(10)^{-7}$
Extra-precision gears	0.00036	$4.02(10)^{-4}$	$-1.27(10)^{-7}$

For this example, we try a commercial unit, use

$C_{ma} = A + Bb_w + Cb_w^2$

$= 0.127 + 0.0158(2) - 0.0001093(2)^2 = \mathit{0.1582}$ (5-43)

C_e is the mesh alignment correction factor. It is 0.80 if gearing is adjusted at assembly and compatibility is improved by lapping; it is 1.0 for all other conditions.

Finally, we obtain, $K_m = C_{mf} = 1.0 + C_{mc}(C_{pf}C_{pm} + C_{ma}C_e)$
$= \mathit{1.0} + (1)[(0.0542)(1) + (0.1582)(1)] = 1.212$

If we use **Figure 5-48** for $b_w/d_p = 2/3$ and $b_w = 2$", $K_m = 1.19$~1.20. Not a bad choice.

Size factor: K_s.

The size factor accounts for the non-uniformity of material properties due to size. $K_s = 1$ if the gear modulus is no larger than 5 mm or diametral pitch no smaller than 5 (1/") for gears used in general-purpose machines.

Therefore, we use $K_s = 1$.

Generally, this factor can be found in the table below.

Table 5-6 Size factor

Diametral pitch (1/in)	Module (mm)	Size factor, K_s
$\geq$ 5	$\leq$ 5	1
4	6	1.05
3	8	1.15
3	12	1.25
1.25	20	1.40

Dynamic factor: K_v.

The dynamic factor is introduced to consider the impact loading that the gear set may experience. The value of the dynamic factor is related to the accuracy of the teeth, the properties of the material, and the speed of the gear. K_v can be calculated from the equations below.

$$K_v = \begin{cases} \left(\dfrac{A+\sqrt{V}}{A}\right)^B & V \text{ in } \mathrm{ft/min} \\[2ex] \left(\dfrac{A+\sqrt{200V}}{A}\right)^B & V \text{ in } \mathrm{m/s} \end{cases} \tag{5-44}$$

with

$$B = \frac{(12 - Q_v)^{2/3}}{4} \qquad (5\text{-}45)$$

$$A = 50 + 56(1 - B)$$

An accuracy number (Q_v) is needed in Equation (5-42). Its value may be chosen as follows.

$Q_v = 3 \sim 7$ commercial gears
$Q_v = 8 \sim 12$ precision gears

We can use $Q_v = 6$ for a hig- quality commercial gear. Therefore,

$$B = \frac{(12 - Q_v)^{2/3}}{4} = \frac{(12-6)^{2/3}}{4} = 0.826$$

$$A = 50 + 56(1 - B) = 50 + 56(1 - 0.826) = 59.7$$

$$K_v = \left(\frac{A + \sqrt{V}}{A}\right)^B = \left(\frac{59.7 + \sqrt{1963}}{59.7}\right)^{0.826} = 1.58$$

Elastic factor: K_E. The elastic factor is the coefficient for the elastic properties of the gear materials, which is defined by Equation (5-35) and used in the contact stress equation.

$$K_E = \sqrt{E'} = \sqrt{\frac{2}{\dfrac{1-\nu_p^2}{E_p} + \dfrac{1-\nu_g^2}{E_g}}} = \sqrt{\frac{2}{2\left(\dfrac{1-0.28^2}{30\times10^6}\right)}}$$

Geometric factors: Y_j and I.

The geometric factors consider the effect of the involute curve on the stresses. The geometry factor for bending, Y_j, is related to the tooth shape. Therefore, different gears should have different values of Y_j (Hamrock et al., 1999, 2005, the figure below). For the current problem, the values for Y_j are

$Y_{jp} = 0.32$ and $Y_{jg} = 0.41$

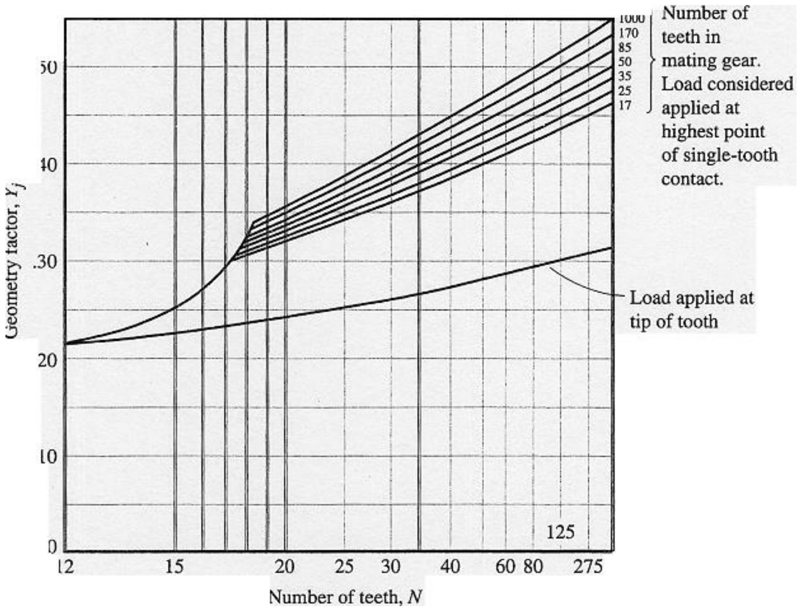


Figure 5-50 Geometric factor for bending stress (Hamrock et al., 2005). Note that the horizontal coordinate is about the gear under consideration, and the tooth number marked for each curve is for the mating gear.

However, there is only one value for the geometry factor for contact, *I*, because it is about the equivalent curvature and the contact geometry of two gear teeth. *I* can be calculated with Equation (5-36).

$$I = \frac{\pi \cos\phi \sin\phi}{\left(1 + \dfrac{d_p}{d_g}\right)} = \frac{\pi \cos 20 \sin 20}{\left(1 + \dfrac{3}{10.5}\right)} = 0.785$$

(5) Stresses

The bending stress of the pinion is different from that of the gear.

$$\sigma_b = \frac{W^t P}{b_w Y_j} K_a K_s K_m K_v K_i K_b$$

For the pinion, $\sigma_{bp} = \dfrac{336(6)}{2\left(\boldsymbol{0.32}\right)} \cdot 1(1)\left(1.212\right)(1.58) = 6032 \text{ psi}$

For the gear, $\sigma_{bg} = \dfrac{336(6)}{2\left(\boldsymbol{0.412}\right)} \cdot 1(1)\left(1.212\right)(1.58) = 4708 \text{ psi}$

The italicized numbers mean that they may be different for the pinion and the gear.

The maximum surface contact stresses (pressures) for the gear and for the pinion are the same (action and reaction), which is

$$\sigma_c = K_e \left(\frac{W^t}{b_w d_p} \frac{1}{I} K_a K_s K_m K_v \right)^{1/2}$$

$$\sigma_\mathrm{c} = \mathbf{5}705 \sqrt{\frac{336}{2(\boldsymbol{3})} \cdot \frac{1}{0.785} (1)(1)(1.212)(1.58)} = 66680 \text{ psi}$$

(6) Factors of safety

The factors of safety for bending for the pinion and the gear are

$$n_{b,p} = \frac{\sigma_{all,bp}}{\sigma_{bp}} = \frac{39840}{6032} = 6.60$$

$$n_{b,g} = \frac{\sigma_{all,bg}}{\sigma_{bg}} = \frac{28090}{4685} = 5.97$$

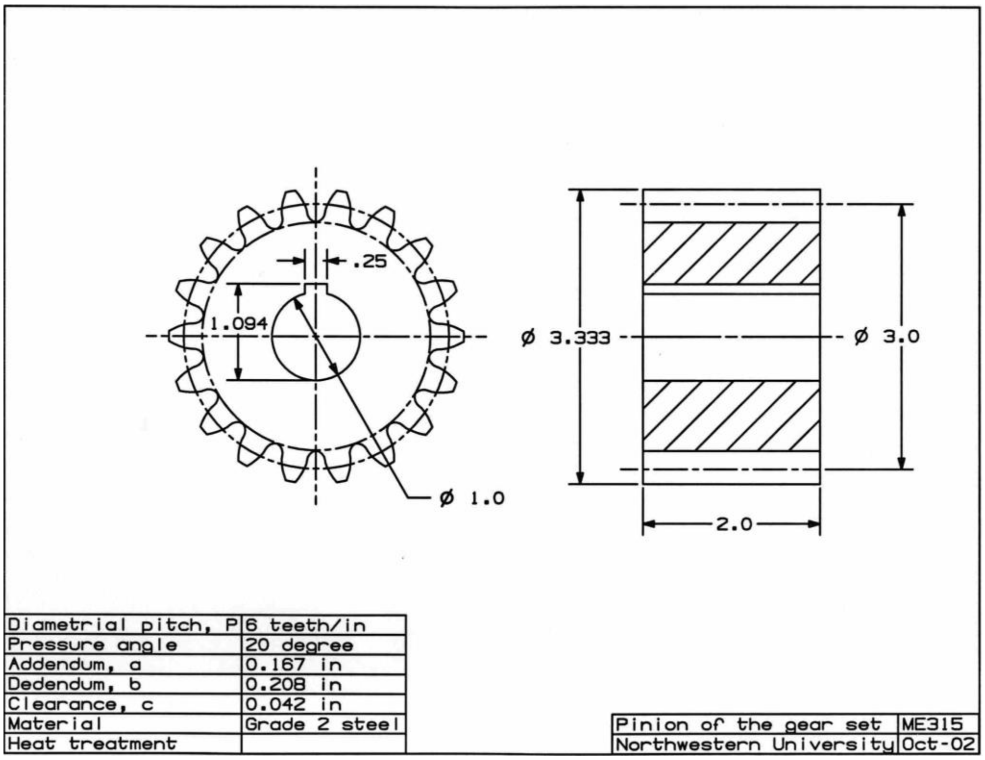
The factors of safety for contact for the pinion and the gear are

$$n_{c,p} = \frac{\sigma_{all,cp}}{\sigma_c} = \frac{102720}{66680} = 1.54$$

$$n_{c,g} = \frac{\sigma_{all,cg}}{\sigma_c} = \frac{87430}{66680} = 1.31$$

Although the contact stresses are the same, the factors of safety for surface-contact strengths are different, because the allowable stresses for the pinion and the gear are not the same.

The figure below shows the pinion of the gear set in this example.



Diametrial pitch, P	6 teeth/in
Pressure angle	20 degree
Addendum, a	0.167 in
Dedendum, b	0.208 in
Clearance, c	0.042 in
Material	Grade 2 steel
Heat treatment	

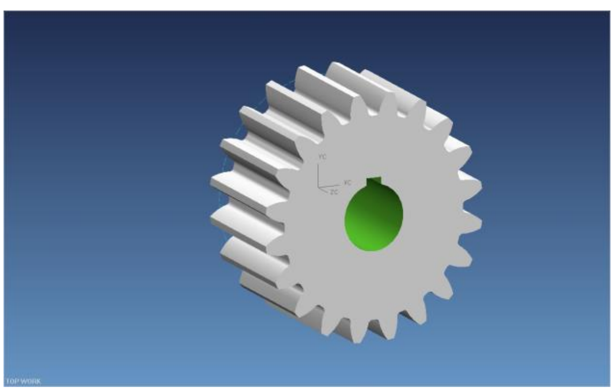


Figure 5-51 Pinion of the gear set of **Example 5-7**.

5-12 Helical gears

Spur gears have an intrinsic problem: the contact ratio is less than 2, which means that single-tooth pair meshing cannot be avoided. Helical gears are created to overcome this problem. Actually, helical gears have many advantages over spur gears. Because of the higher contact ratio, more pairs of teeth are in contact, making helical gear mesh smoother. For the same face with, the tooth of a helical gear is longer, and for the same gear parameters, the tooth of a helical gear is stronger because it is curved.

1 Helical gear geometry

Helical tooth surface generation. Instead of releasing a wire on a circle, let's release a piece of paper that covers a cylinder (**Figure 5-52**). The trajectory of an inclined line on that paper will draw an involute surface along a helical direction. Obviously, the profile viewed from the front (the transverse plane) is a true involute profile.

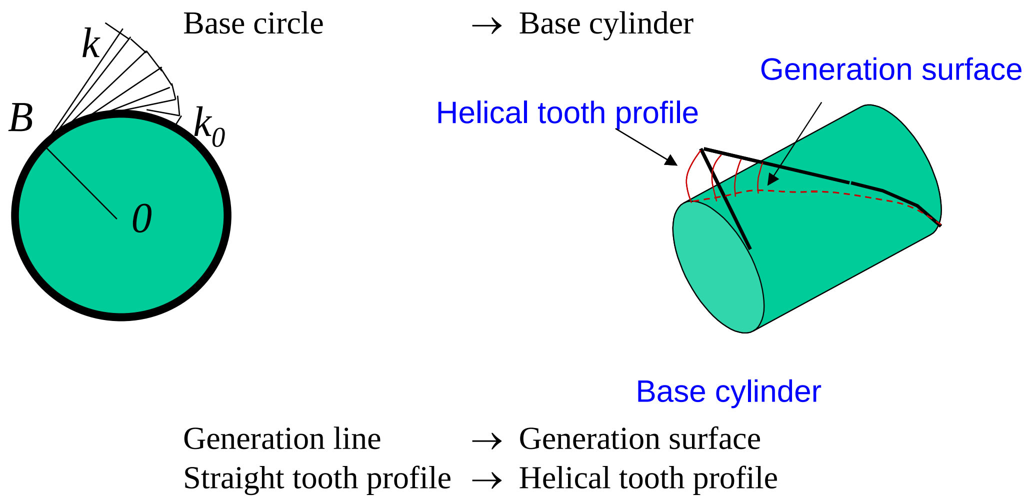


Figure 5-52 Comparison of tooth surface generation.

Figure 5-53 further compares a helical tooth with a spur-gear tooth from generation, meshing and pressure angles. Racks are illustrated for a more direct view. The helical angle is defined between the tooth tangent and the direction of the centerline. Accordingly, three planes should be defined: the transverse plane (or the front plane, or the plane of rotation), the normal plane (normal to the tooth profile), and the axial plane (parallel to the rotation axis). We will use subscripts *n* for the normal plane, *t* for the transverse plane, and *x* for the axial plane and their related parameters.

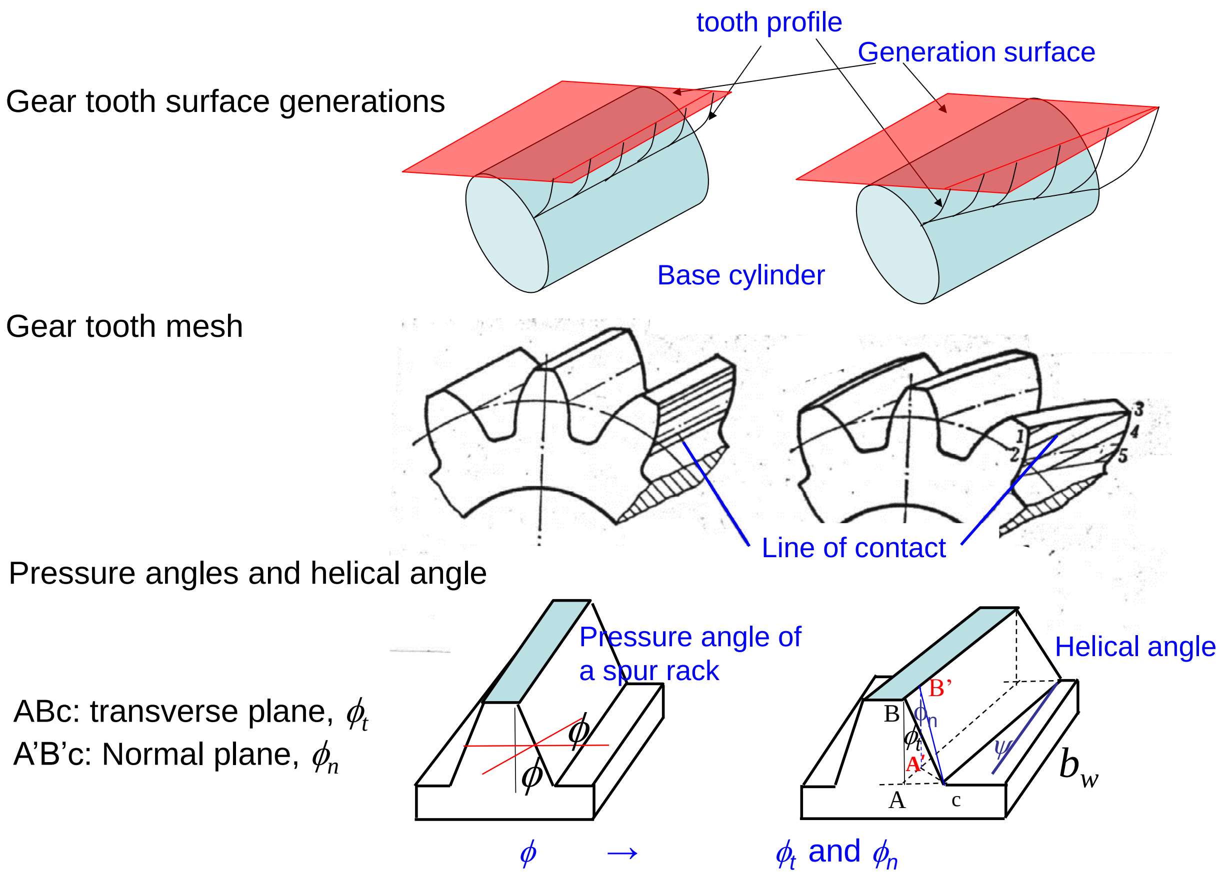


Figure 5-53 Comparison between spur and helical gear teeth. Subscripts *n*, *t*, and *x* are for the normal, transverse, and axial plane parameters. Middle plot modified from Pu et al.

New parameters. More parameters are needed to define a helical gear tooth. The helical angle, Ψ, is the one to know, and all other additional ones are due to this angle and are related to it.

Three pitches can be determined, **Figure 5-54**, the transverse pitch, p_t = ac, to be defined later the normal pitch, p_n = a'c, and the axial pitch, p_x = cd, as shown in the figure below. These pitches have the following geometric relationships.

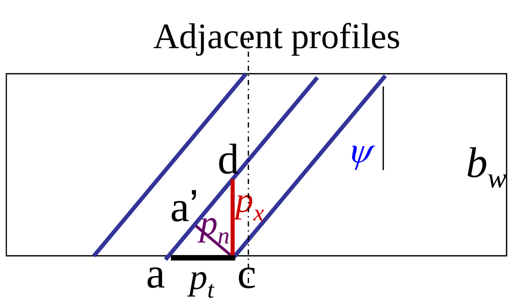


Figure 5-54 Helical gear pitches demonstrated by helical rack profiles.

$$p_n = p_t \cos \Psi \quad (5\text{-}46\text{-}1)$$

$$p_x = p_t / \tan\Psi \qquad (5\text{-}46\text{-}2)$$

The pressure angles measured in the transverse and normal planes, ϕ_n, and ϕ_t, are also different. The relationship connecting Ψ, ϕ_n, and ϕ_t is given below, which can be easily proven using the **Figure 5-53** above.

$$\cos\Psi = \tan\phi_n / \tan\phi_t \qquad (5\text{-}47)$$

Module and diametral pitch. Although the tooth profile in the transverse plane is truly involute, with transverse module, m_t, and the transverse diametral pitch, P_t, the tooth standard is based on the normal plane to accommodate most machining needs. The normal module, m_n, and the normal diametral pitch, P_n, are standardized.

Because

$$p_n = p_t \cos\Psi$$

and like what we have done for spur gears

$$p_n = \pi/P_n = \pi\, m_n \qquad p_t = \pi/P_t = \pi\, m_t \qquad (5\text{-}48)$$

We have

$$P_t = P_n \cos\Psi \qquad (5\text{-}49\text{-}1)$$

$$m_n = m_t \cos\Psi \qquad (5\text{-}49\text{-}2)$$

Addendum and dedendum are defined in the normal plane

$$a = 1/P_n = m_n \qquad (5\text{-}50\text{-}1)$$

$$b = 1.25/P_n = 1.25 m_n \qquad (5\text{-}50\text{-}2)$$

Gear diameters are defined at the rotation (transverse) plane, which are:

$$d = d_t = N/P_t = N/(P_n \cos\Psi) = N\, m_t \qquad (5\text{-}51\text{-}1)$$

$$d_{bt} = d \cos\phi_t \qquad (5\text{-}51\text{-}2)$$

$$d_{ot} = d_t + 2a = d_t + 2m_n \qquad (5\text{-}51\text{-}3)$$

$$d_{rt} = d_t - 2b = d_t - 2.5m_n \qquad (5\text{-}51\text{-}4)$$

Conditions for helical gear meshing. Now we have one more independent parameter, the helical angle. The conditions for spur gears are modified and enriched as follows.

US custom:

$$P_{pn} = P_{gn} = P_n \qquad \phi_{pn} = \phi_{gn} = \phi_n \qquad (5\text{-}52)$$

Metric:

$$M_{pn} = m_{gm} = m_n \qquad \phi_{pn} = \phi_{gn} = \phi_n \qquad (5\text{-}53)$$

And $\Psi_p = -\Psi_g$ for external gear pair (5-54)

$\Psi_p = \Psi_g$ for internal gear pair (5-55)

In general, for external gears, $\psi_p + \psi_g$ = shaft angle. For our worm and gear set in Chapter 1, $\psi_p + \psi_g$ = 90 degree.

Equivalent pitch circle and virtual number of teeth. These are for the need of gear manufacturing. The cutting force is along the tooth-surface normal direction. The normal cross section of a helical gear by "cutting" through the tooth at its pitch point along its normal direction across the gear cylinder gets an elliptical shape, as shown in **Figure 5-55**. An equivalent pitch circle, as if it were the cutter circle, is defined at the pitch point using the radius of curvature there, as shown in the figure below by the equivalent radius of curvature, r_e. For the ellipse with semi major and minor axes *a* and *b*, the equivalent radius of curvature at the pitch point, *p*, is

$$r_e = a^2/b = (r/(\cos\Psi)^2/r = r/(\cos\Psi)^2 \quad (5\text{-}56)$$

Note that *r* is the radius of the pitch cylinder (or pitch circle) of the gear. Correspondingly, more teeth, *N'*, are needed to cover the equivalent pitch circle, and N' is defined as the virtual number of teeth:

$$N' = \frac{2\pi r_e}{\pi m_n} = \frac{2r}{m_n \cos^2\Psi} = \frac{2}{m_n \cos^2\Psi}\left(\frac{m_t N}{2}\right) = \frac{N}{m_n \cos^2\Psi}\left(\frac{m_n}{\cos\Psi}\right) = \frac{N}{\cos^3\Psi} \quad (5\text{-}57)$$

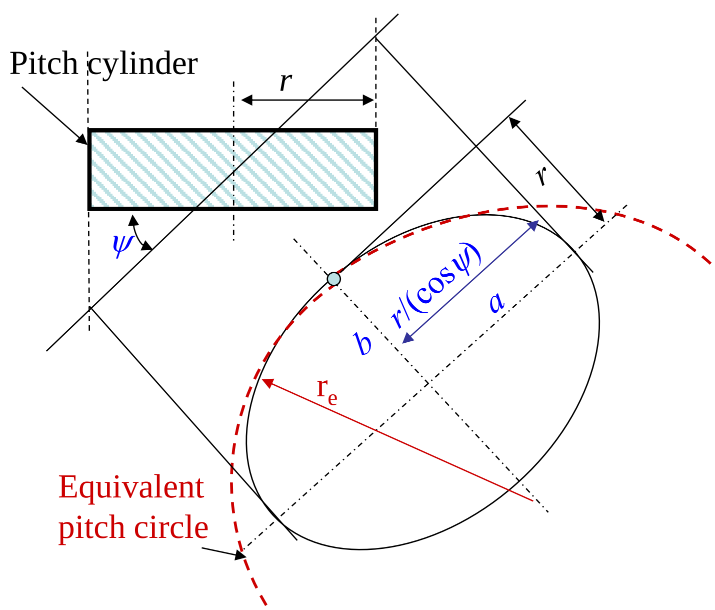


Figure 5-55 Equivalent pitch circle and the ellipse from the cut cylinder.

Contact ratio. Helical gear contact ratio comes from two parts, transverse and axial. The former is the same as that of a spur gear, C_r (defined in the transverse plane), and in the equation below, p_c is p_t. The axial contact ratio, C_{ra}, which is the second term in the equation below, significantly enhances the overall contact ratio. It corresponds to the move of contact, across the face width axially, from one pair of teeth to the next, which is b_w/p_x, and $p_x = p_n/\sin\psi$.

$$C_{rh} = C_r + C_{ra} = C_r + \frac{b_w}{p_x} = \left[\frac{1}{p_c\cos\phi}\left[\sqrt{r_{op}^2 - r_{bp}^2} + \sqrt{r_{og}^2 - r_{bg}^2}\right] - \frac{c_d\tan\phi}{p_c}\right] + \frac{b_w\sin\psi}{p_n} \qquad (5\text{-}58)$$

Example 5-8. A helical gear of 18 teeth is given, its normal pressure angle is 20^0, its helical angle is 12^0, and its transverse diametral pitch is 6 teeth/in. Determine: (1) the pitch diameter, (2) the normal diametral pitch, (3) the transverse, normal, and axial circular pitches, (4) the transverse pressure angle, (5) outside diameter, (6) radius of the equivalent pitch circle, and (7) contact ratio if the mating gear has 41 teeth.

Solution

(1) $d_t = N/P_t = 18/6 = 3$"

(2) $P_n = P_t/\cos\Psi = 6/\cos 12^0 = 6.134$"

(3) $p_t = \pi/P_t = \pi/(6) = 0.5235$"

$p_n = p_t\cos\Psi = (0.5235)\cos 12^0 = 0.5122$"

$p_x = p_t/\tan\Psi = (0.5235)/\tan 12^0 = 2.4633$"

(4) $\phi_t = \tan^{-1}(\tan\phi_n/\cos\Psi) = \tan^{-1}(\tan 20^0/\cos 12^0) = 20.41^0$

(5) $d_{ot} = d_t + 2a = 3 + 2/P_n = 3 + 2/6.134 = 3.326$"

(6) $r_e = r/(\cos\Psi)^2 = (d_t/2)/(\cos 12^0)^2 = (3/2)/(\cos 12^0)^2 = 1.5677$"

(7) $$C_{rh} = C_{rt} + C_{ra} = C_{rt} + \frac{b_w}{p_x} = \left[\frac{1}{p_t\cos\phi t}\left[\sqrt{r_{opt}^2 - r_{bpt}^2} + \sqrt{r_{ogt}^2 - r_{bgt}^2}\right] - \frac{c_{dt}\tan\phi t}{p_t}\right] + \frac{b_w\sin\psi}{p_n}$$

$$\mathrm{Cr} := \frac{\sqrt{\left(\frac{\mathrm{dot1}}{2}\right)^2 - \left(\frac{\mathrm{dbt1}}{2}\right)^2} + \sqrt{\left(\frac{\mathrm{dot2}}{2}\right)^2 - \left(\frac{\mathrm{dbt2}}{2}\right)^2}}{\mathrm{pt}\cdot\cos(\phi t)} - \left[\left(\frac{\mathrm{dt1}}{2} + \frac{\mathrm{dt2}}{2}\right)\frac{\tan(\phi t)}{\mathrm{pt}}\right] + \mathrm{bw}\cdot\frac{\sin(\psi)}{\mathrm{pn}}$$

=2.389

While the contact ratio in the transverse plane (corresponding to that for a spur gear) is only

$$\mathrm{Cr1} := \frac{\sqrt{\left(\frac{\mathrm{dot1}}{2}\right)^2 - \left(\frac{\mathrm{dbt1}}{2}\right)^2} + \sqrt{\left(\frac{\mathrm{dot2}}{2}\right)^2 - \left(\frac{\mathrm{dbt2}}{2}\right)^2}}{\mathrm{pt}\cdot\cos(\phi t)} - \left(\frac{\mathrm{dt1}}{2} + \frac{\mathrm{dt2}}{2}\right)\frac{\tan(\phi t)}{\mathrm{pt}}$$

= 1.577

What about all these if the normal diametral pitch = 6 teeth/in?

2 Helical gear design

Helical gear force. Like the spur gears, the force on a helical gear tooth is applied through the surface of its mating tooth. It is normal to the tooth surface. The following figure and equations show the force on a helical gear tooth and its resolution to three orthogonal components.

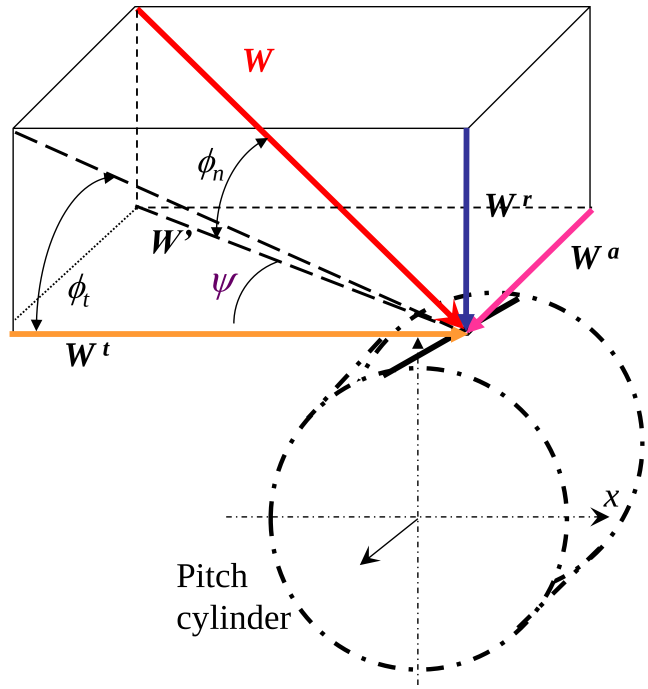


Figure 5-56 Helical gear force and components, subscripts *n*, *r*, *a*, and *t* are for the normal, radial, axial, and transverse (tangential) directions.

$W^r = W \sin \phi_n$ (5-59-1)

$W^t = W \cos \phi_n \cos\psi$ (5-59-2)

$W^a = W \cos \phi_n \sin\psi$ (5-59-3)

Torque: $T = W^t (d/2)$ (5-60)

Bending stress. Again, the well-accepted AGMA approach is used, i.e., the root beading stress, σ_b, should be smaller than an allowable bending stress, $\sigma_{all,b}$.

$$\sigma_b \le \sigma_{all,b} \quad (5\text{-}61)$$

$$n_b = \sigma_{all,b} / \sigma_b \quad (5\text{-}62)$$

The bending stress is defined in the following formulae similar to those for spur gears (Juvinall and Marshek, 2000).

$$\sigma_b = \frac{W^t}{b_w m Y_j} K_a K_s (0.93 K_m) K_v K_i K_b \qquad \text{Metric} \tag{5-63}$$

$$\sigma_b = \frac{W^t P}{b_w Y_j} K_a K_s (0.93 K_m) K_v K_i K_b \qquad \text{English} \tag{5-64}$$

Here, $m = m_t$, the module in the transverse plane, and $P = P_t$, the diametral pitch in the transverse plane. The load distribution factor, K_m, is the value for spur gears reduced by 0.93 (Juvinall and Marshek, 2000). The geometry factor, Y_j, is different from that for spur gears. All other factors, K_a, K_s, and K_v, K_i, K_b,, are the same as those for spur gears.

Contact stress. Again, the AGMA approach is used for the surface strength analysis, i.e. the maximum contact stress should be smaller than an allowable contact stress.

$$\sigma_c \le \sigma_{all,c} \tag{5-65}$$

$$n_c = \sigma_{all,c} / \sigma_c \tag{5-66}$$

The contact stress is (Juvinall and Marshek, 2000).

$$\sigma_c = K_e \left(\frac{W^t}{b_w d_{pt}} \frac{1}{I} \left(\frac{\cos \Psi}{0.95 C_r} \right) K_a K_s (0.93 K_m) K_v \right)^{1/2} \tag{5-67}$$

Contact ratio in the transverse plane, C_r, for $\Psi = 8 \sim 30^0$, may be roughly 1.64~1.38. This is only for an approximate calculation. For an accurate design, we need to use the accurate equation for the transverse contact ratio or AGMA charts. Note again K_E is defined differently from that in the Shigley book, and so is I, in the way "2" and "π" are placed.

$$K_E = \sqrt{E'} = \sqrt{\frac{2}{\frac{1-\nu_p^2}{E_p} + \frac{1-\nu_g^2}{E_g}}} \tag{5-68}$$

$$I = \frac{\pi \cos \phi_t \sin \phi_t}{\left(1 + \frac{d_p}{d_g} \right)} \tag{5-69}$$

Example 5-9

Given: A gear train of a pinion and a gear.

The train value: 3.5:1

$N_P = 18$, $N_g = (3.5)\,18 = 63$

Transverse module $m_t = 4$ mm
Helical angle $\Psi = 15^0$

Power and speed on the pinion shaft:

$H = 20$ HP, $n_P = 2500$ rpm

Driven by an electric motor, no shock, 5 years, one shift, 99% reliability.

Solution

(1) Gear geometry

$$d_{pt} = m_t N_p = 4(18) = 72\text{mm}$$
$$d_{Gt} = m_t N_G = 4(63) = 252\text{mm}$$
$$m_n = m_t \cos\Psi = 4\cos 15^0 = 3.863\text{mm}$$
$$a = m_n = 3.863\text{mm}$$
$$b = 1.25 m_n = 4.829\text{mm}$$

Let's choose the middle value from the following empirical formula for the face width.

$$\frac{8}{P} < b_w < \frac{16}{P}$$
$$8m < b_w < 16m$$
$$\text{b}_w \approx 12 m_t = 12(4) = 48mm$$

The face width here is nominal. Again, the pinion is made slightly wider (2-5 mm).

(2) Tangential force

$$W^t = \frac{H(33000)}{V} = \frac{H(33000)}{2\pi\, n \, d_{pt}/2(60)} = \frac{20(33000)}{2\pi(2500)\, 72/2(60)} = 1582\text{N}$$

Note, the pitch circle velocity is V= 9.425m/s.

(3) Materials and Allowable stresses

Steel gears	E = 210 MPa; ν = 0.28	through hardened
Pinion – 300 HB	Grade 2	$\sigma_{all,Pb}$ = 327MPa, $\sigma_{all,Pc}$ = 945 MPa
Gear – 250 HB	Grade 1	$\sigma_{all,Gb}$ = 223MPa, $\sigma_{all,Gc}$ = 786 MPa

Similarly, a hardness difference of 50-100 HB should be maintained.

Cycle of the pinion = 5 (365x8)(60) (2500rpm) = 2190Million
Cycle of the gear = 5 (365x8)(60) (2500rpm/3.5) = 625.71Million

$Y_N = 1.6831N^{-0.0323}$, $Y_{NP} = 0.840$ $Y_{NG} = 0.875$
$Z_N = 2.466N^{-0.056}$, $Z_{NP} = 0.739$ $Z_{NG} = 0.793$

$HB_P / HB_G = 1.2$
$C_{HG} = 1.0 + A'(N_G / N_P - 1.0) = 1.0 + 0.00249(3.5 - 1.0) = 1.00621$

Therefore,

$$\sigma_{all,bp} = \frac{S_b Y_N}{K_T K_R} = \frac{(327)(0.84)}{(1)(1)} = 274.7MPa$$

$$\sigma_{all,bg} = \frac{S_b Y_N}{K_T K_R} = \frac{(223)(0.875)}{(1)(1)} = 195.1MPa$$

$$\sigma_{all,cp} = \frac{S_c Z_N C_H}{K_T K_R} = \frac{(945)(0.739)(1)}{(1)(1)} = 698.4MPa$$

$$\sigma_{all,cg} = \frac{S_c Z_N C_H}{K_T K_R} = \frac{(786)(0.793)(1.00621)}{(1)(1)} = 627.2MPa$$

(4) Factors for stresses

The factors for stress are included in the formulas for the bending and contact stress.

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Application factor: K_a. Table 5-4 lists some values of the application factor. Again, for a system powered by an electric motor, the load and speed are usually smooth without obvious shock. Therefore, $K_a = 1$ should be the choice.

For this problem, an electric motor without shock is used as the power supplier. Therefore, $K_a = 1$

Size factor: K_s. We use $K_s = 1$.

Dynamic factor: K_v.

$$K_v = \begin{cases} \left(\dfrac{A+\sqrt{V}}{A}\right)^B & V \text{ in } \text{ft}/\text{min} \\ \left(\dfrac{A+\sqrt{200V}}{A}\right)^B & V \text{ in } \text{m}/\text{s} \end{cases}$$

$$B = \frac{(12 - Q_v)^{2/3}}{4}$$

$$A = 50 + 56(1 - B)$$

$Q_v = 3 \sim 7$ commercial gears
$Q_v = 8 \sim 12$ precision gears

We also use $Q_v = 6$ for a high-quality commercial gear. Therefore,

$$B = \frac{(12 - Q_v)^{2/3}}{4} = \frac{(12 - 6)^{2/3}}{4} = 0.826$$

$$A = 50 + 56(1 - B) = 50 + 56(1 - 0.826) = 59.7$$

$$K_v = \left(\frac{A + \sqrt{V}}{A}\right)^B = \left(\frac{59.7 + \sqrt{200(9.425)}}{59.7}\right)^{0.826} = 1.72$$

Elastic factor: K_E. The elastic factor is the coefficient for the elastic properties of the gear materials.

$$K_E = \sqrt{E'} = \sqrt{\frac{2}{\frac{1-\nu_p^2}{E_p} + \frac{1-\nu_p^2}{E_g}}} = \sqrt{\frac{2}{2\left(\frac{1-0.28^2}{210 \times 10^9}\right)}} = 477352\sqrt{Pa}$$

Geometric factors: Y_j and I. The geometric factors consider the effect of the involute curve on stresses. The geometry factor for bending, Y_j, is related to the tooth shape. Therefore, different gears should have different values of Y_j. For helical gears, each Y_j comes from two portion, the value for mating with a 75-tooth gear and a correction factor (Juvinall and Marshek, 2000, **Figure 5-57**. For the current problem, the values for Y_j are

$Y_{jp} = 0.5 \text{x} 0.995 = 0.498$
$Y_{jg} = 0.6 \text{x} 0.93 = 0.558$

However, there is only one value for the geometry factor for contact, I, because it is about the equivalent contact geometry of the two gears.

$$I = \frac{\pi \cos\phi_t \sin\phi_t}{\left(1 + \frac{d_p}{d_g}\right)} = \frac{\pi \cos 20 \sin 20}{\left(1 + \frac{18}{63}\right)} = 0.785$$

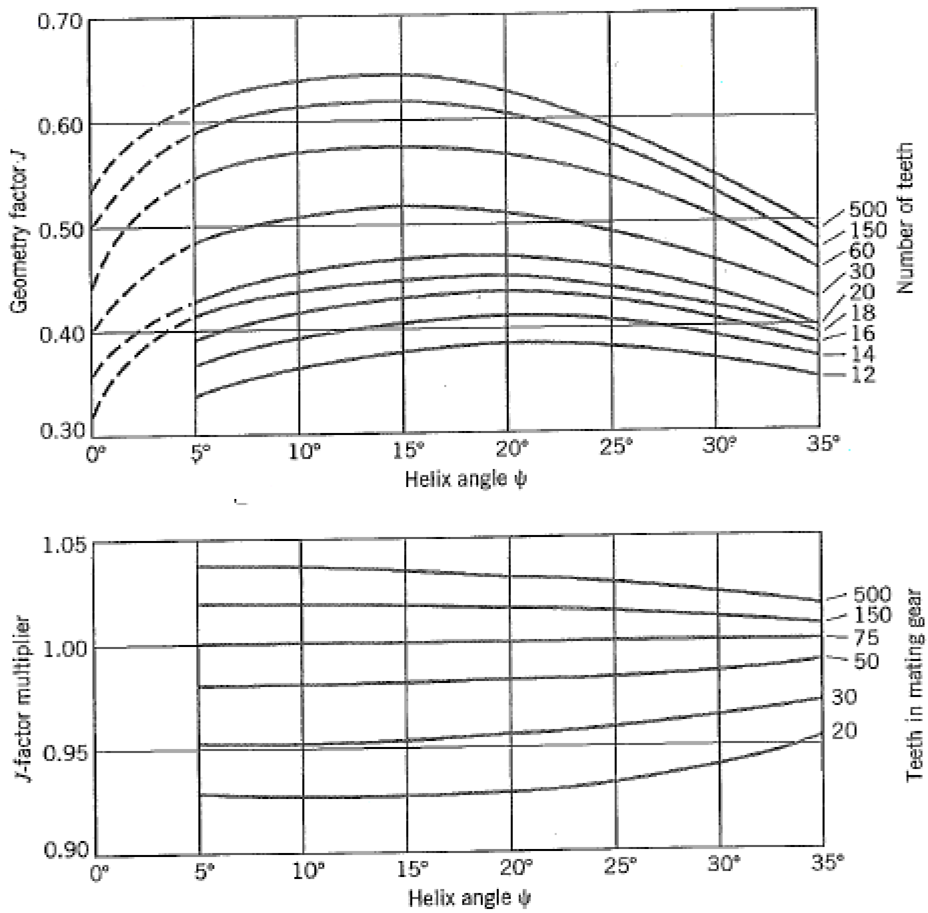
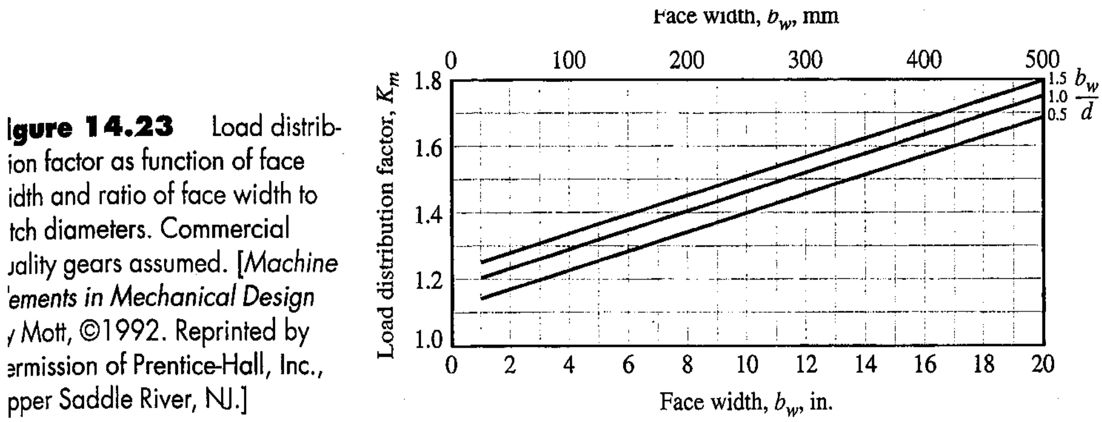


FIGURE 16.8
For helical gears having $\phi_n = 20°$, standard addendum of $1/P_n$, and shaved teeth: (*a*) geometry factor *J* for use with a 75-tooth mating gear (spur gear values from Figure 15.23 are shown at $\psi = 0°$ for comparison); (*b*) the *J*-factor multipliers for use when the mating gear has other than 75 teeth. (From AGMA Information Sheet 226.01, which also gives *J* factors for $\phi_n = 14.5°$, 15°, and 20°, for unequal gear and pinion addendum, and for ground- and hobbed-tooth surfaces; also see AGMA 908-B89.)

Figure 5-57. Geometry factor Y_j for helical gears. Here J is Y_j we used.

Load distribution factor: $K_{m.}$ Let's use the simpler way. Use b_w/d_p in Fig. 14.23 in Hamrock's book (1999) to determine K_m and use the same K_m for both gears because their load distributions are the same.



Copy of Figure 5-48 Simplified load distribution factor chart.

For simplicity, we select a proper value, $K_m = 1.19$.

(5) Stresses

The bending stress of the pinion is different from that of the gear.

$$\sigma_b = \frac{W^t}{b_w m Y_j} K_a K_s (0.93 K_m) K_v K_i K_b$$

$$\sigma_{\mathrm{Pb}} = \frac{1582}{0.048(0.004)(\boldsymbol{\mathit{0.498}})} \cdot 1(1)\big(0.93)(1.19\big)(1.72)(1)(1) = 31.5\,\mathrm{MPa}$$

$$\sigma_{\mathrm{Gb}} = \frac{1582}{0.048(0.004)(\boldsymbol{\mathit{0.558}})} \cdot 1(1)\big(0.93)(1.19\big)(1.72)(1)(1) = 28.1\,\mathrm{MPa}$$

The italicized numbers mean that they may be different for the pinion and the gear.

The maximum surface contact stresses (pressures) for the gear and for the pinion are the same (action and reaction).

$$\sigma_c = K_e \left(\frac{W^t}{b_w d_{pt}} \frac{1}{I} \left(\frac{\cos\Psi}{0.95 C_r} \right) K_a K_s (0.93 K_m) K_v \right)^{1/2}$$

$$\sigma_{\mathrm{c}} = 477352 \sqrt{\frac{1582}{0.048(0.072)} \cdot \frac{1}{0.785} \left(\frac{\cos 15^0}{0.95(1.56)} \right) (1)(1)(0.93)(1.19)(1.72)} = 406 MPa$$

Here, $C_r \sim 1.56$.

(6) Factors of safety

The factors of safety for bending for the pinion and the gear are

$$np = \frac{\sigma_{all,bp}}{\sigma_P} = \frac{274.7}{31.5} = 8.7$$

$$n_{b,G} = \frac{\sigma_{all,bg}}{\sigma_G} = \frac{195.1}{28.1} = 6.9$$

The factors of safety for contact for the pinion and the gear are

$$n_{c,p} = \frac{\sigma_{c,\,all,cp}}{\sigma_c} = \frac{698.4}{406} = 1.7$$

$$n_{c,g} = \frac{\sigma_{all,cg}}{\sigma_c} = \frac{627.2}{406} = 1.5$$

Although the contact stresses are the same, the factors of safety for surface contact strength are different because the allowable stresses for the pinion and the gear are not the same.

This gear set has nearly the same geometry as that of the set in **Example 5-7**. Clearly, helical gears have better strength.

5-13 Example of gear transmission

An automobile automatic transmission may be constructed with a hydraulic coupling system and double-row planetary gear systems, or compound epicyclic planetary gear sets, usually referred to as the Simpson planetary gear system. A torque converter, which is a fluid coupling system that hydraulically connects the engine to the transmission, is usually used with the gear system. A gear pump is mounted between the torque converter and the planetary gear system. Vehicles conforming to the US Government standards must have the modes ordered of P-R-N-D-L, which means parking, reverse, neutral, drive, and low (or 1) for winter drive, etc, although each company has its own drive modes in different detail.

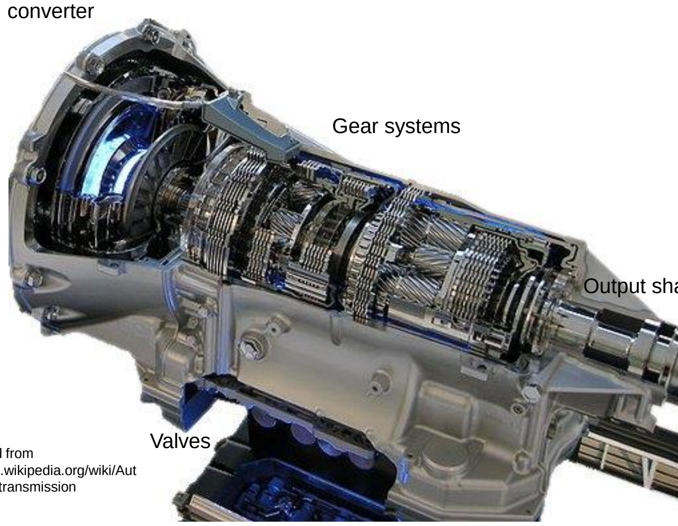
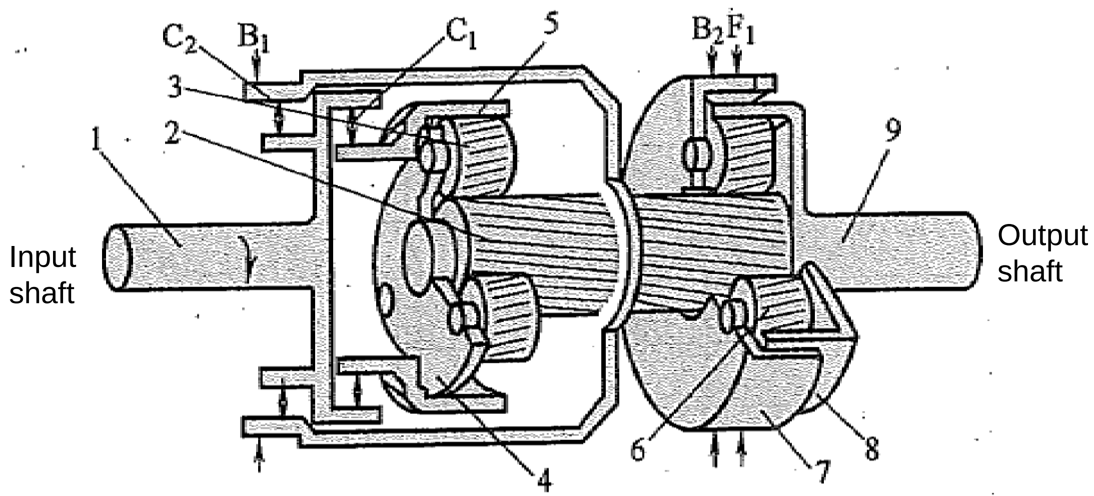


Figure 5-58 A typical automatic transmission with planetary gear systems.

The following figure shows the gear train set of a typical automatic transmission, the Simpson planetary gear system, which is a duel planetary gear system. Three couplings, one of them is a one-directional coupling, and one brake are involved in the control of 6 modes, namely P, R, N, D, 2, and L (1) the low-speed mode. D has three forward speeds, 2 has two that allow speed changes from 1~2, while L has only one speed. In this system, front carrier 4 is linked to back ring gear 8, leading to $n_{armf} = n_{rb}$.



1- input shaft 2 – sun 3- front planetary gearset 4- front carriage
5 – front ring gear 6 –back planetary gearset 7- back carriage
8- Back ring gear 9- output shaft
C1: forward clutch C2: high back clutch B1: brake
B2: low back clutch F1: one-direction clutch

Speed control position	Mode	Execution element						
		C_1	C_2	B_1	B_2	B_3	F_1	F_2
D	1	★						★
	2	★		★			★	
	3	★	★					
2	1	★				★		★
	2	★		★	★		★	

Figure 5-59. Simpson system, its drive modes and execution element (Modified from Yao [2009]). We may call the input side the front system and the output side the back system.

At the D1 mode, C1 is engaged to make the front ring gear the input device, F1 is fixed so that the back system becomes a fixed-axis system (arm *b*, or the carrier of the back system, 7, is fixed).

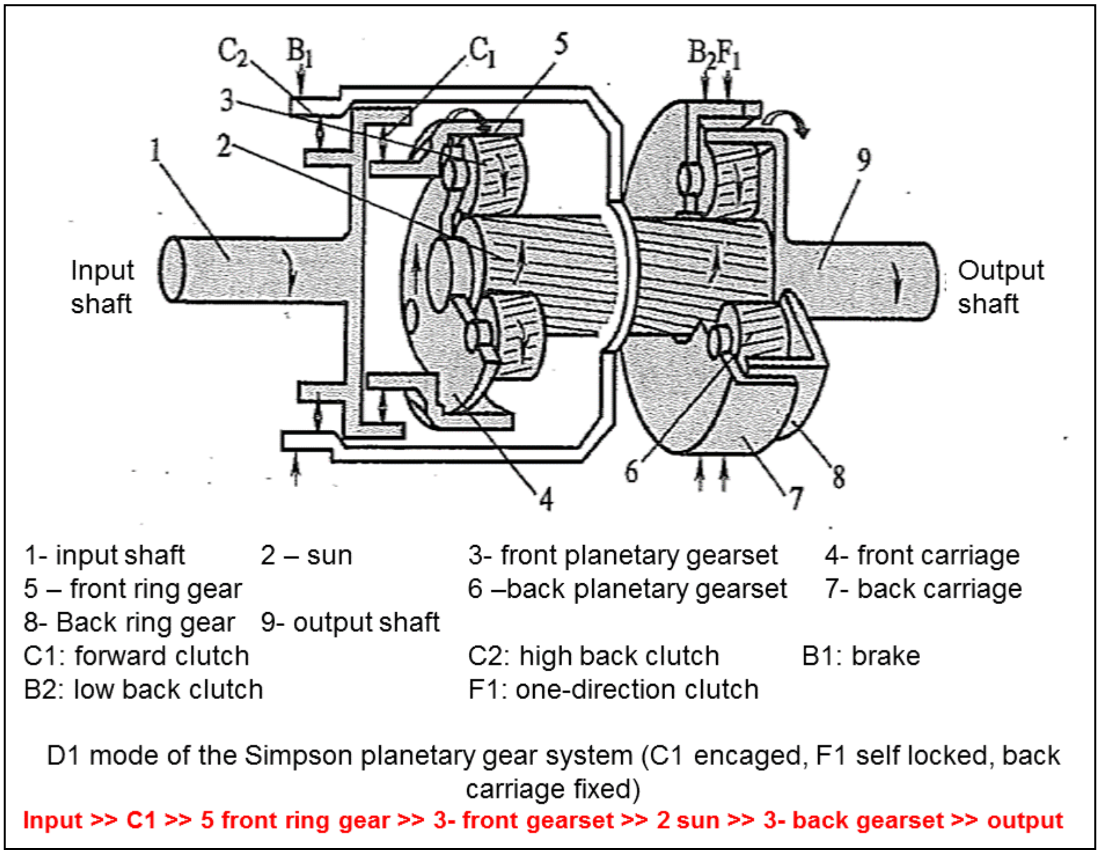


Figure 5-60. Simpson system again with motions shown for the D1 mode (Modified from Yao [2009]

The above system can be simplified and illustrated by the following diagram of two planetary systems coupled with the front carrier - back ring link and the common sums on the same shaft.

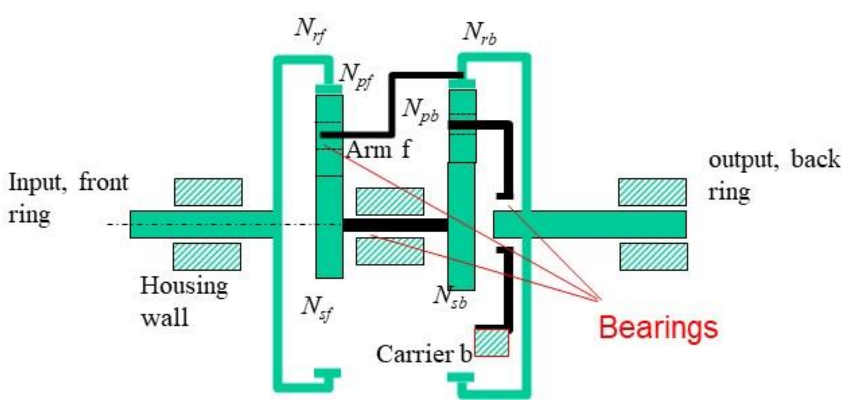


Fig. 5-61 D1 mode diagram of the two coupled planetary systems

Knowing the tooth numbers, N_{sf}, N_{pf}, N_{rf}, N_{sb}, N_{pb}, N_{rb}, and $n_{sf} = n_{sb}$, $n_{armf} = n_{rb}$, we can determine the ratio of the input and the output, n_{rf}/n_{rb}. Arranging the solution in terms of N_{rf}/N_{sf} and N_{rb}/N_{sb}

$$\frac{n_{sf}-n_{rb}}{n_{rf}-n_{rb}}=(-1)^1\frac{N_{rf}}{N_{sf}} \qquad \frac{N_{rf}}{N_{sf}}n_{rf}=n_{rb}\frac{N_{rb}}{N_{sb}}+n_{rb}+\frac{N_{rf}}{N_{sf}}n_{rb}$$

$$n_{sf}-n_{rb}=-\frac{N_{rf}}{N_{sf}}\left(n_{rf}-n_{rb}\right)$$

$$n_{sf}-n_{rb}=-\frac{N_{rf}}{N_{sf}}\left(n_{rf}-n_{rb}\right)$$

$$\frac{N_{rf}}{N_{sf}}n_{rf}=n_{rb}\frac{N_{rb}}{N_{sb}}+n_{rb}+\frac{N_{rf}}{N_{sf}}n_{rb} \qquad i_{D1}=\frac{n_{rf}}{n_{rb}}=\frac{1+\frac{N_{rf}}{N_{sf}}+\frac{N_{rb}}{N_{sb}}}{\frac{N_{rf}}{N_{sf}}}$$

If C1 is engaged to make the front ring gear the input device, F1 is not fixed so that the back arm (back carrier) has speed n_{ab}. Knowing the tooth numbers, N_{sf}, N_{pf}, N_{rf}, N_{sb}, N_{pb}, N_{rb}, we can determine the output speed n_{rb} in terms of the input speed, n_{rf}, the back carriage speed, n_{ab}, and N_{rf}/N_{sf} and N_{rb}/N_{sb}, etc. Note that the front arm (or front carrier) is connected to the back ring ($n_{af} = n_{rb}$). This time, we have two inputs n_{rb} and n_{rf}, each in one planetary system.

$$n_{sf}=n_{sb},\ n_{armf}=n_{rb}$$

Front $\dfrac{n_{sf}-n_{rb}}{n_{rf}-n_{rb}}=(-1)^1\dfrac{N_{rf}}{N_{sf}}$ back $\dfrac{n_{sb}-n_{ab}}{n_{rb}-n_{ab}}=(-1)^1\dfrac{N_{rb}}{N_{sb}}$

$$n_{sb}=n_{sf}=-\frac{N_{rb}}{N_{sb}}\left(n_{rb}-n_{ab}\right)+n_{ab} \qquad -\frac{N_{rb}}{N_{sb}}\left(n_{rb}-n_{ab}\right)+n_{ab}-n_{rb}=-\frac{N_{rf}}{N_{sf}}\left(n_{rf}-n_{rb}\right)$$

$$\frac{N_{rf}}{N_{sf}}n_{rf}=\left(\frac{N_{rf}}{N_{sf}}+\frac{N_{rb}}{N_{sb}}+1\right)n_{rb}-\left(\frac{N_{rb}}{N_{sb}}+1\right)n_{ab}$$

Result $$n_{rb}=\frac{\left(\frac{N_{rb}}{N_{sb}}+1\right)n_{ab}+\frac{N_{rf}}{N_{sf}}n_{rf}}{\left(\frac{N_{rf}}{N_{sf}}+\frac{N_{rb}}{N_{sb}}+1\right)}$$

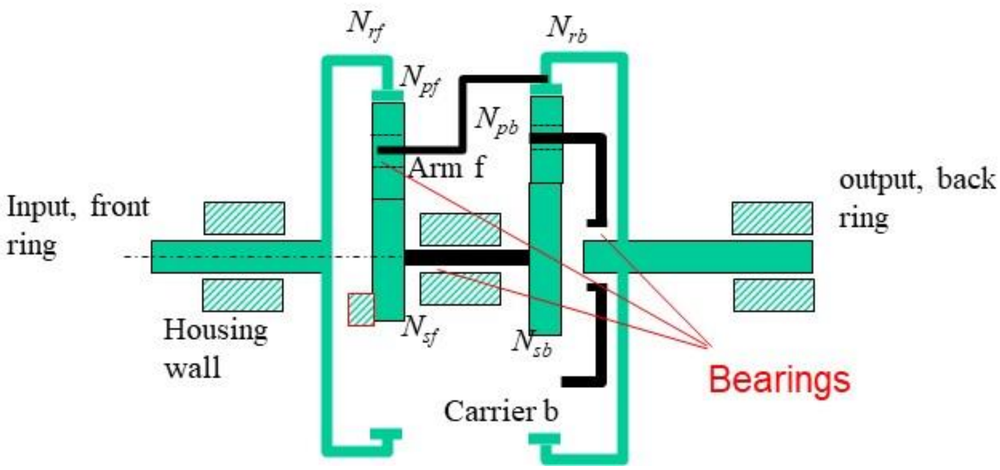
The above is Forward 2 (D2), where C1 is engaged, so that the front ring gear inputs.

If B1 is engaged, the common suns are fixed, but F1 is freed, the back system is a free system (the back carriage is released). Now

$$n_{sf}=n_{sb}=0,\ n_{armf}=n_{rb}$$

Front $\dfrac{n_{sf}-n_{rb}}{n_{rf}-n_{rb}}=(-1)^1\dfrac{N_{rf}}{N_{sf}}$ back $\dfrac{n_{sb}-n_{ab}}{n_{rb}-n_{ab}}=(-1)^1\dfrac{N_{rb}}{N_{sb}}$

$$n_{sf}-n_{rb}=-\frac{N_{rf}}{N_{sf}}\left(n_{rf}-n_{rb}\right) \qquad n_{sb}-n_{ab}=-\frac{N_{rb}}{N_{sb}}\left(n_{rb}-n_{ab}\right)$$



$$n_{rb} - \frac{N_{rf}}{N_{sf}}\left(n_{rf} - n_{rb}\right) = n_{ab} - \frac{N_{rb}}{N_{sb}}\left(n_{rb} - n_{ab}\right)$$

$$n_{rb} + \frac{N_{rf}}{N_{sf}} n_{rb} + \frac{N_{rb}}{N_{sb}} n_{rb} = \frac{N_{rf}}{N_{sf}} n_{rf} + n_{ab} + \frac{N_{rb}}{N_{sb}} n_{ab} \qquad n_{rb} = \frac{\left(\frac{N_{rb}}{N_{sb}} + 1\right) n_{ab} + \frac{N_{rf}}{N_{sf}} n_{rf}}{\left(\frac{N_{rf}}{N_{sf}} + \frac{N_{rb}}{N_{sb}} + 1\right)}$$

Chapter summary

This chapter introduces transmission elements; however, the focus is on gears and gearing mechanisms. Both spur and helical gears are studied, and gear-train analysis methods are formulated. Moreover, the concepts of conjugate action, conjugate profiles, and involute curves are introduced; gear geometries are explained, and gearing processes are discussed. This chapter also deals with the methods for gear force analyses and tooth bending and contact stress calculations. Gear design analyses are also explained and practiced.

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Media: **Several U-Tube movies are on line. 315 students should watch the short U-Tube movies before the classes.**