# Multivariate cdf and pmf

If $X_1, X_2, \ldots, X_n$ are rvs, then their *joint cdf* is defined as

$$F(x_1, x_2, \ldots, x_n) = Pr\left(\bigcap_{i=1}^{n}[X_i \leq x_i]\right)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

If $X_1, X_2, \ldots, X_n$ are *discrete* rvs, then their *joint pmf* is defined as

$$f(x_1, x_2, \ldots, x_n) = Pr\left(\bigcap_{i=1}^{n}[X_i = x_i]\right)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

# Connections between multivariate cdf and pmf

$$F(0.5,\ 1.1) = Pr(\underbrace{Y_1 \le 0.5,\ Y_2 \le 1.1})$$

$$Pr\left( [Y_1 = 0] \cap \{ [Y_2 = 0] \cup [Y_2 = 1] \} \right)$$

$$Pr\left( \underbrace{\{[Y_1 = 0] \cap [Y_2 = 0]\}}_{A_1} \cup \underbrace{\{[Y_1 = 0] \cap [Y_2 = 1]\}}_{A_2} \right)$$

$A_1 \cap A_2 = \emptyset$

$$= Pr(A_1) + Pr(A_2) - \underbrace{Pr(A_1 \cap A_2)}_{0} = 1/9 + 2/9 = 3/9$$

| | $y_1$ | | |
|---|---|---|---|
| $y_2$ | 0 | 1 | 2 |
| 0 | 1/9 | 2/9 | 1/9 |
| 1 | 2/9 | 2/9 | 0 |
| 2 | 1/9 | 0 | 0 |

# Connection between multivariate pdf and cdf

If $X_1, X_2, \ldots, X_n$ are *continuous* rvs with differentiable joint cdf $F$, then their *joint pdf* is defined as

$$f(x_1, x_2, \ldots, x_n) = \frac{\partial^n F(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Q: How to obtain the joint cdf from the joint pdf? Integrate:

$$F(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(t_1, t_2, \ldots, t_n)\, dt_1 \cdots dt_n$$

Exercise: Find the joint cdf of $X_1$ and $X_2$ if the joint pdf is $f(x_1, x_2) = e^{-x_1} e^{-x_2} \cdot \mathbb{I}\{x_1 > 0\} \cdot \mathbb{I}\{x_2 > 0\}$.

*Marginal cdf/pmf/pdf* of $X_1$ is just the univariate cdf/pmf/pdf of $X_1$. This wording is used when the cdf/pmf/pdf of $X_1$ is obtained from the joint cdf/pmf/pdf of $X_1, X_2, \ldots, X_n$.

*Marginal cdf from the joint cdf*

Let $F_{12}(x_1, x_2)$ ~~de~~ be the joint cdf of $X_1$ and $X_2$.

$$F_1(x_1) = Pr(X_1 \le x_1) = Pr([X_1 \le x_1] \cap [X_2 \le \infty])$$
$$= F_{12}(x_1, \infty).$$

Exercise: in the previous exercise, find the marginal cdf of $X_1$ from the joint cdf of $X_1$ and $X_2$.

# Marginal pmf from the joint pmf

Exercise: find the marginal pmf of $Y_1$ if the joint pmf of $Y_1$ and $Y_2$ is given in the table below.

let $A_{ij} = [Y_1 = i] \cap [Y_2 = j]$

| | $y_1$ | | | |
|---|---|---|---|---|
| $y_2$ | 0 | 1 | 2 | |
| 0 | 1/9 | 2/9 | 1/9 | 4/9 |
| 1 | 2/9 | 2/9 | 0 | 4/9 |
| 2 | 1/9 | 0 | 0 | 1/9 |
| | 4/9 | 4/9 | 1/9 | |

$Pr(Y_1 = 0) = Pr([Y_1 = 0] \cap \{[Y_2 = 0] \cup [Y_2 = 1] \cup [Y_2 = 2]\})$

$Pr(A_{00} \cup A_{01} \cup A_{02}) = Pr(A_{00}) + Pr(A_{01}) + Pr(A_{0}$

$= 1/9 + 2/9 + 1/9 = 4/9$

$Pr(Y_1 = 1) = 4/9$

$Pr(Y_1 = 2) = 1/9$

## Marginal pdf from the joint pdf

Let $f_{12}$ be the joint pdf of $X_1$ and $X_2$.

Find $f_1$, the marginal pdf of $X_1$:

$$f_1(x_1) = \int_{-\infty}^{\infty} f_{12}(x_1, t)\, dt$$

Exercise: find the marginal pdf of $X_1$ if the joint pdf of $X_1$ and $X_2$ is $f(x_1, x_2) = e^{-x_1} e^{-x_2} \cdot \mathbb{I}\{x_1 > 0\} \cdot \mathbb{I}\{x_2 > 0\}$.

*Conditional pdf/pmf* $f_C$ of $X_{k+1}, X_{k+2}, \ldots X_n$ given $X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k$ is defined as

$$f_C(x_{k+1}, \ldots, x_n | x_1, \ldots, x_k) = \frac{f_J(x_1, \ldots, x_k, \ldots, x_n)}{f_M(x_1, \ldots, x_k)},$$

assuming the denominator is positive. Here, $f_J$ is the joint pdf/pmf of $X_1, \ldots, X_n$ and $f_M$ is the joint pdf/pmf of $X_1, \ldots, X_k$.

If $n = 2$ and $k = 1$, this becomes $f_C(x_2|x_1) = f_J(x_1, x_2)/f_M(x_1)$.

If $A$ and $B$ are random events, $Pr(A \mid B) = \dfrac{Pr(A \cap B)}{Pr(B)}$

Q: Where does this definition come from?

If $X_1$ and $X_2$ are discrete,

let $B = [X_1 = x_1]$; $A = [X_2 = x_2]$;

$f_c(x_2 \mid x_1) = Pr(X_2 = x_2 \mid X_1 = x_1) = \dfrac{Pr([X_1 = x_1] \cap [X_2 = x_2])}{Pr(X_1 = x_1)}$

If $X_1$ and $X_2$ are continuous rvs,

let $B = [X_1 \in (x_1 - \delta, x_1 + \delta)]$ so that $Pr(B) > 0$

$A = [X_2 \in (x_2 - \delta, x_2 + \delta)]$

# Statistical Independence

The rvs $X_1, X_2, \ldots, X_n$ with respective marginal cdfs $F_1, F_2, \ldots, F_n$ and joint cdf $F$ are *mutually independent* iff

$$\underbrace{F(x_1, x_2, \ldots, x_n)}_{\text{LHS}} = \underbrace{\prod_{i=1}^{n} F_i(x_i)}_{\text{RHS}}$$

for every $x_1, x_2, \ldots, x_n \in \mathbb{R}$. This is often referred to as *factorization criterion for independence*.

$$\text{LHS} = \Pr\left(\bigcap_{i=1}^{n} [X_i \le x_i]\right) = \prod_{i=1}^{n} \Pr(X_i \le x_i) = \text{RHS}$$

Recall

*Σ-notation*: $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

*Π-notation*: $\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \cdots \cdot a_n$.

# Factorization criterion for independence via pdfs

Let $X_1, X_2, \ldots, X_n$ be continuous rvs with respective marginal pdfs $f_i$ and joint pdf $f$. Show that

$$
\begin{aligned}
F(x_1, x_2, \ldots, x_n) &= \textstyle\prod_{i=1}^{n} F_i(x_i) \qquad \text{if and only if} \\
f(x_1, x_2, \ldots, x_n) &= \textstyle\prod_{i=1}^{n} f_i(x_i).
\end{aligned}
$$

# Mutual independence vs pairwise independence

The rvs $X_1, X_2, \ldots, X_n$ with respective marginal cdfs $F_1, F_2, \ldots, F_n$ and joint cdf $F$ are *pairwise independent* iff

$$Pr(X_i \leq x_i, X_j \leq x_j) = Pr(X_i \leq x_i) \cdot Pr(X_j \leq x_j)$$

for every pair $i \neq j$ and all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Q: Which is stronger, mutual or pairwise independence?

# Bivariate normal distribution

**Bivariate Normal pdfs**

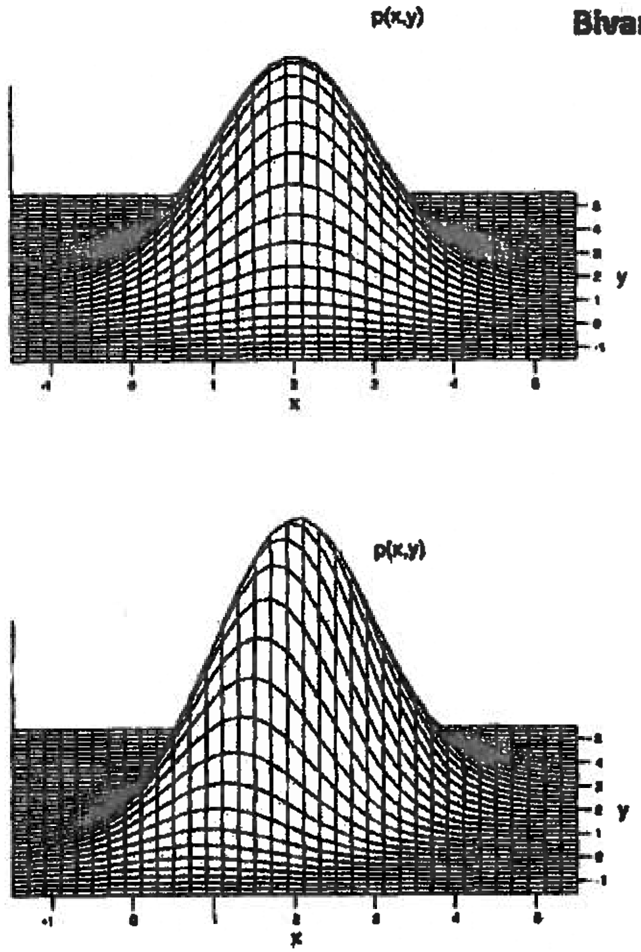


**Figure B.4.1** A plot of the bivariate normal density $p(x, y)$ for $\mu_1 = \mu_2 = 2$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0$ (top) and $\rho = 0.5$ (bottom).

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x-\mu_1)}{\sigma_1}\frac{(y-\mu_2)}{\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}\right].$$

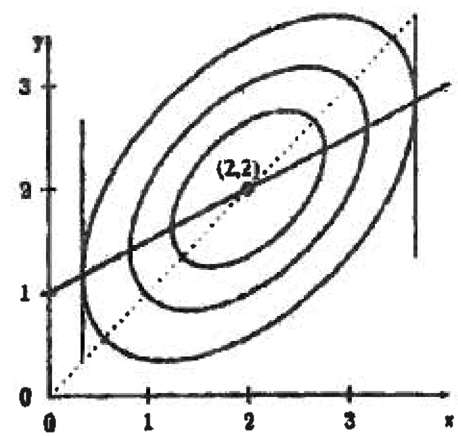


**Figure B.4.2** 25%, 50%, and 75% probability level-curves, the regression line (solid line), and major axis (dotted line) for the $\mathcal{N}(2, 2, 1, 1, 0.5)$ density.

# Some characteristics of random variables/vectors

1. joint, marginal and conditional cdfs
2. joint, marginal and conditional pdfs or pmfs (if the rvs are continuous or discrete)
3. moments - if they are well-defined ("exist"); e.g., expectation

   functions of moments; e.g., variance, covariance, correlation
4. moment-generating function (mgf) - if it is defined in a neighborhood of zero (i.e., "exists").
5. quantile function (for a rv but not a vector); loosely, this can be thought of as the inverse of the cdf

# Population moments

Let $X$ be a rv with a pdf or pmf $f$ and let $k$ be a positive integer. The *k-th (population) moment* of $X$, denoted by $\mathrm{E}\left(X^k\right)$, is defined as follows:

If $X$ is a discrete rv and

$$\sum_{x \in supp(f)} |x|^k f(x) < \infty, \qquad \text{then} \qquad \mathrm{E}\left(X^k\right) = \sum_{x \in supp(f)} x^k f(x).$$

If $X$ is a continuous rv and

$$\int_{-\infty}^{\infty} |x|^k f(x) dx < \infty, \qquad \text{then} \qquad \mathrm{E}\left(X^k\right) = \int_{-\infty}^{\infty} x^k f(x) dx.$$

Important: in general, rvs need not have all or any moments to "exist" (i.e., be well-defined).

# Example: non-existence of moments

Consider a *Cauchy* distribution with pdf

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad \text{defined for all } x \in \mathbb{R}.$$