Population moments

Let X be a rv with a pdf or pmf f and let k be a positive integer. The *k-th (population) moment* of X, denoted by $\mathrm{E}\left(X^k\right)$, is defined as follows:

If X is a discrete rv and

$$\sum_{x \in supp(f)} |x|^k f(x) < \infty, \qquad \text{then} \qquad \mathrm{E}\left(X^k\right) = \sum_{x \in \underbrace{supp(f)}_{\text{"support of } f\text{"}}} x^k f(x).$$

"support of f" $\equiv$ set of all elementar outcomes of rv X with positive prob.

If X is a continuous rv and

$$\int_{-\infty}^{\infty} |x|^k f(x) dx < \infty, \qquad \text{then} \qquad \mathrm{E}\left(X^k\right) = \int_{-\infty}^{\infty} x^k f(x) dx.$$

Important: in general, rvs need not have all or any moments to "exist" (i.e., be well-defined).

Law of large numbers: $X_1, X_2, \ldots$ be iid (indep. from distribution F)

let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

"convergence in probability"

LLN : $\underbrace{\bar{X}_n}_{\text{sample mean}} \longrightarrow \underbrace{E(X_1)}_{\text{population mean.}}$ for large n $\bar{X}_n$ acts as a const.

Q: How does $\bar{X}_n$ (a rv) behave when $E(|X_1|) = \infty$?

A: does not converge; as we collect more and more data, the sample mean $\bar{X}_n$ never stabilizes.

LLN requires that $E(|X_1|) < \infty$ (strong LLN)

& and often that $E(X_1^2) < \infty$ (weak LLN).

Example: non-existence of moments

$\equiv$ Student's t distr. with 1 degree of freedom

Consider a *Cauchy* distribution with pdf $\quad X \sim f$

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad \text{defined for all } x \in \mathbb{R}.$$

1. Check $E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2} \, dx < \infty$

by symmetry

$$E(|X|) \overset{\downarrow}{=} 2 \int_0^{\infty} \frac{1}{\pi} \cdot \frac{x}{1+x^2} \, dx = \infty$$

$$= \frac{1}{1/x + x} \simeq \frac{1}{x} \text{ for large } x.$$

since $\int_c^{\infty} \frac{1}{x} \, dx = \infty.$

$\Rightarrow$ sanity check $E(|X|) < \infty$ failed.

Hence there is no expectation. (no population mean!

(Mathematical) expectation (aka *expected value*) of a rv X is $\mathrm{E}(X)$, the first (population) moment, provided it is well-defined.

Convention/shortcut: let's denote by $\mathrm{E}\left(|X|^k\right)$ the k-th moment of the rv $|X|$, the absolute value of X, assuming that the expectation is finite.

Proposition: Let p be a positive real number. If $\mathrm{E}(|X|^p) < \infty$, then $\mathrm{E}(|X|^q) < \infty$ for every $q \in [0, p]$.

Properties of Expected Values (Expectations), I

Let $X_1, X_2, \ldots, X_n$ be rvs with well-defined expectations $\mathrm{E}(X_1), \mathrm{E}(X_2), \ldots, \mathrm{E}(X_n)$. Let $b_1, b_2, \ldots, b_n$ be any constants. Assume that the rvs X_i are continuous (have pdfs). In the discrete case, replace the integrals by sums (left as an exercise).

Linearity of expectations:
$\mathrm{E}(b_1 X_1 + b_2 X_2) = b_1 \mathrm{E}(X_1) + b_2 \mathrm{E}(X_2)$

0. $\mathrm{E}(b_1) = b_1$.

$$E(b_1) = \int_{-\infty}^{\infty} \cdots \int b_1 \cdot \cancel{}\, f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = b_1 \cdot 1 = b_1.$$

1. $\mathrm{E}(b_1 X_1) = b_1 \mathrm{E}(X_1)$.

$$E(b_1 X_1) = \int_{-\infty}^{\infty} b_1 \cdot x_1 \cdot f_1(x_1)\, dx_1$$

$$= b_1 \underbrace{\int_{-\infty}^{\infty} x_1 \cdot f_1(x_1)\, dx_1}_{= E(X_1)}.$$

Properties of Expected Values (Expectations), II

2. $\mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2)$.

joint pdf of X_1 and X_2

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 + x_2) \cdot f_{12}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f_{12}(x_1, x_2)\, dx_1\, dx_2 = \int_{-\infty}^{\infty} x_1 \cdot f_1(x_1)\, d$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f_{12}(x_1, x_2)\, dx_1\, dx_2 = \int_{-\infty}^{\infty} x_2 f_2(x_2)\, dx_2$$

$$= E(X_1) + E(X_2).$$

f_j is marginal pdf of X_j.

Properties 3 – 5 below follow from properties 0, 1 and 2.

$$Q: \quad E(X_1) \overset{\text{1st def}}{=} \int_{-\infty}^{\infty} x_1 \cdot f_1(x_1)\, dx_1$$

marginal pdf of X_1

$$E(X_1) \overset{\text{2nd def}}{=} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1 \cdot f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$= \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1 f(x_1, \ldots, x_n)\, dx_2 \ldots dx_n \right)}_{x_1 \cdot f_1(x_1)} dx_1$$

$$\int_{-\infty}^{\infty} x_1 \cdot f_1(x_1) \quad dx_1$$