P-values: Motivation

Suppose the test rule is "reject H_0 for large values of the TS T", RR: $\{T > k_\alpha\}$. $k_\alpha = Q(1-\alpha)$ for cdf of TS under H_0.

$$\left.\begin{array}{ll} \alpha_1 = 0.05, & k_{\alpha_1} = Q(0.95) \\ \alpha_2 = 0.01, & k_{\alpha_2} = Q(0.99) \end{array}\right\} \Rightarrow \alpha_1 > \alpha_2 \text{ implies } Q(1-\alpha_1) < Q(1-\alpha_2)$$

Often we can reject H_0 at level $\alpha = 0.05$ but fail to reject H_0 at level $\alpha = 0.01$ (when $T \in (k_{\alpha_1}, k_{\alpha_2})$).

Practical concern: since different people have different desired levels of significance α, how to report the outcome/result of the test (in order to make everyone happy)?

p-value: "observed level of significance".

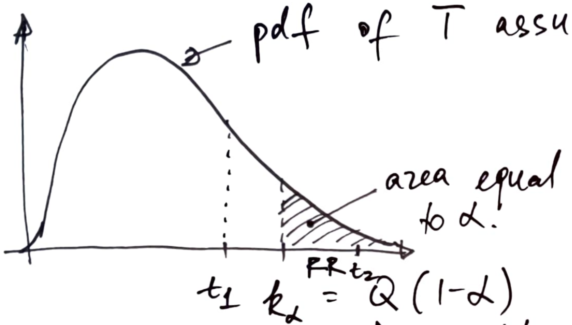
P-value for a right-sided test

For right-sided test with a RR $[T > k_\alpha]$, p-value is

observed value of T.

$$\alpha_R(t) = \Pr(T \geq t \mid \mathrm{H}_0 \text{ is true}).$$

Notice $t \in \mathrm{RR}$ if and only if $\alpha_R(t) < \alpha$.



If $t = t_1 < k_\alpha$, fail to reject H_0.

$\alpha_R(t_1) > \alpha$. — " —

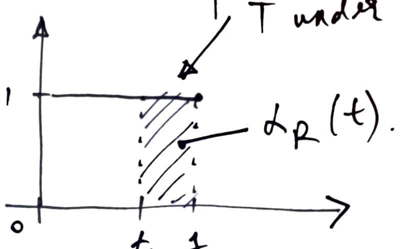
If $t = t_2 > k_\alpha$, reject H_0.

$\alpha_R(t_2) < \alpha$, — " —.

Example: $X \sim$ Uniform$(0, \theta)$. $\mathrm{H}_0: \theta = 1$ versus $\mathrm{H_A}: \theta = \theta_a > 1$.

Test rule: "reject H_0 for large values of X". $T \equiv X$.

p-value: $\alpha_R(t) = \Pr(X \geq t \mid \mathrm{H}_0 \text{ is true})$



let $t \in (0, 1)$.

$\alpha_R(t) = 1 - t$.

If $t = 0.9$, $\alpha_R(t) = 0.1$

If $\alpha = 0.05$, fail to reject H_0.

If $\alpha = 0.2$, reject H_0.

P-values: definitions

for right-sided test with a RR $[T > k_\alpha]$, p-value is

$$\alpha_R(t) = \Pr(T \geq t \mid \mathrm{H}_0 \text{ is true}).$$

Notice $t \in \mathrm{RR}$ if and only if $\alpha_R(t) < \alpha$.

p-value for left-sided test with
$\mathrm{RR} = \{T < k_\alpha\}, \alpha_L = \Pr(T \leq t \mid \mathrm{H}_0 \text{ is true})$.
a fun. of t.

p-value for a 2-sided test: $\mathrm{RR} = \{T < Q(\frac{\alpha}{2})\} \cup \{T > Q(1 - \frac{\alpha}{2})\}$.
Reject H_0 for extremely small or extremely large values of TS T.
p-value: $\alpha(t) = 2\min\{\alpha_L(t), \alpha_R(t)\}$

Notice that $t \in \mathrm{RR}$ if and only if $\alpha(t) < \alpha$.