In [1]:
```
import numpy as np
import pandas as pd
from scipy.stats import norm,t
import matplotlib.pyplot as plt
```

Q1

Question 1 can be found in our Rmarkdown sheet.

Q2

2.1

In [2]:
```
from IPython.display import Image
Image('2_1.jpg')
```

Out[2]:

2.1

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Q(x,y)}\,dy$$

where $Q(x,y) = \dfrac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{1-\rho^2}$

$$= \left(\frac{x-a}{b}\right)^2 + c$$

$$a = \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu)$$

$$b = \sigma_X\sqrt{1-\rho^2}$$

$$c = \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2$$

$$\Rightarrow f_Y(y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\left(\frac{x-a}{b}\right)^2 + c\right]^2}\,dx$$

$$= \frac{e^{-\frac{c}{2}}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}\,dx$$

$$= \frac{e^{-\frac{c}{2}}}{\sqrt{2\pi}\sigma_X\sigma_Y\sqrt{1-\rho^2}}\,\sigma_X\sqrt{1-\rho^2}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}$$

Similarly,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

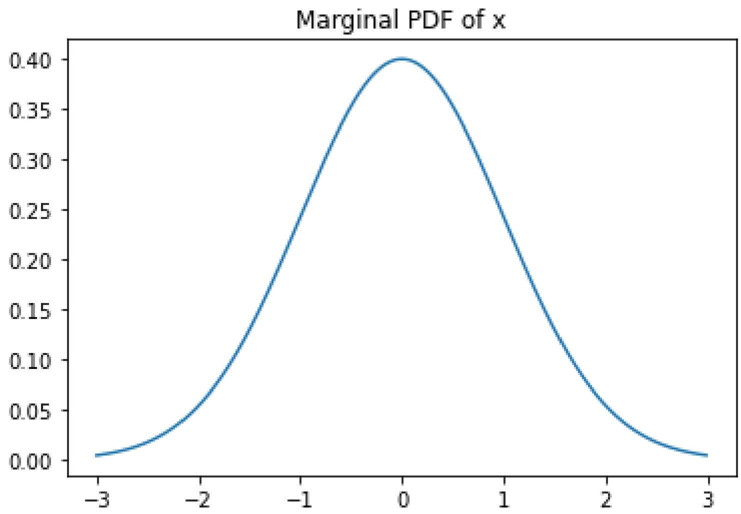
$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

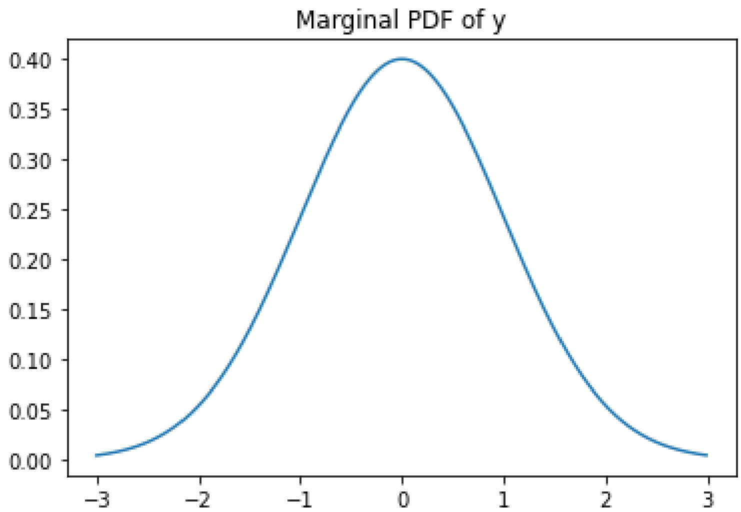
In [3]:

```
x = np.arange(-3, 3, 0.01)
y = np.arange(-3, 3, 0.01)

# Calculating mean and standard deviation
mu_x,mu_y,sig_x,sig_y = 0,0,1,1

plt.figure()
plt.plot(x, norm.pdf(x, mu_x, sig_x))
plt.title('Marginal PDF of x')
plt.show()
plt.figure()
plt.plot(y, norm.pdf(y, mu_y, sig_y))
plt.title('Marginal PDF of y')
plt.show()
```





Those marginal pdfs are identical with Normal(0,1) pdf

2.2

In [4]:
```
Image('2_2.jpg')
```

Out[4]:

2.2 If $\rho=0$,

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_Y} e^{-\frac{1}{2}[(\frac{x-\mu_x}{\sigma_x})^2+(\frac{y-\mu_Y}{\sigma_Y})^2]}$$

$$f_X(x)f_Y(y) = \frac{1}{2\pi\sigma_x\sigma_Y} e^{-\frac{1}{2}[(\frac{x-\mu_x}{\sigma_x})^2+(\frac{y-\mu_Y}{\sigma_Y})^2]}$$

$\therefore$ $f_{XY}(x,y) = f_X(x)f_Y(y)$

$\therefore$ X and Y are independent

2.3

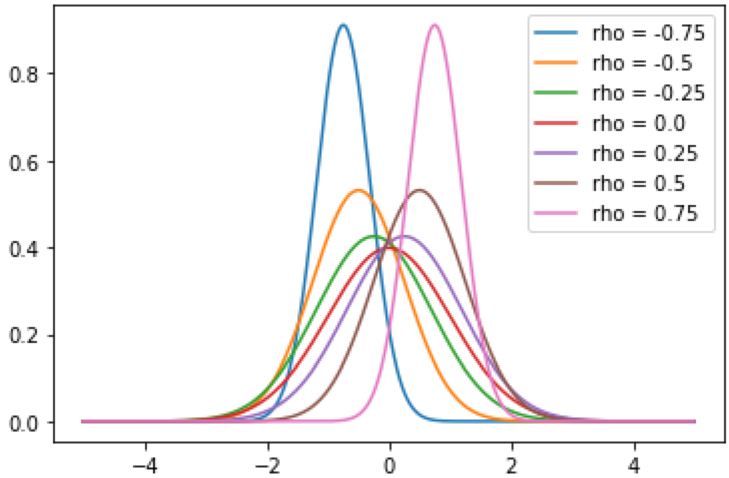
Given $Y=1$:

$E[X|Y=1]=\mu_x+\rho\sigma_x\frac{1-\mu_y}{\sigma_y}$

$Var[X|Y=1]=(1-\rho^2)\sigma_x^2$

In [5]:
```
rho_grid = np.arange(-0.75,1,0.25)
x = np.arange(-5, 5, 0.01)
mu_x,mu_y,sig_x,sig_y = 0,0,1,1
plt.figure()
```

```
for rho in rho_grid:
    mu = mu_x+rho*sig_x*(1-mu_y)/sig_y
    sig = (1-rho**2)*sig_x**2
    plt.plot(x, norm.pdf(x, mu, sig),label=f'rho = {rho}')
plt.legend()
plt.show()
```



The negative and positive ρ both help to predict X since they indicate a linear relation bwtween X and Y. Larger ρ implies a stronger linearity. It is hard to predict X if X and Y are independent, i.e. $\rho=0$.

Q3

3.1

In [6]:
```
np.random.seed(0)
lam = 10
x = np.random.exponential(1/lam,100)
```

$L = \lambda e^{(-\lambda{x_1})}\lambda e^{(-\lambda{x_1})}...\lambda e^{(-\lambda{x_100})} = \lambda^{100} e^{(-\lambda(x_1+x_2+...+x_{100}))}$

$L'=\ln{L}=100\ln{\lambda}-\lambda\sum_{i=0}^{100}x_i$

$\frac{\partial L'}{partial \lambda} = \frac{100}{\lambda}-\sum_{i=0}^{100}x_i = 0$

$\lambda = \frac{100}{\sum_{i=0}^{100}x_i}$

In [7]:
```
lam = 100/x.sum()
print(f'extimated lambda is {lam}')
```

```
extimated lambda is 10.885557365453474
```

The estimated value of λ (10.89) is close to true λ (10)

3.2

In [8]:
```
Image('3_2.jpg')
```

Out[8]:

3.2 $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$L(\lambda) = \prod_{i=1}^{N} \lambda e^{-\lambda x_i}$

Take ln :

$\Rightarrow L' = \ln L = \sum_{i=1}^{N} \ln \lambda e^{-\lambda x_i}$

$= \sum_{i=1}^{N} (\ln \lambda - \lambda x_i)$

$= N \ln \lambda - \lambda \sum_{i=1}^{N} x_i$

Take derivative:

$\Rightarrow \frac{\partial L'}{\partial \lambda} = \frac{N}{\lambda} - \sum_{i=1}^{N} x_i = 0$

$\Rightarrow \lambda = \frac{N}{\sum_{i=1}^{N} x_i}$

Check if the solution is the maximum:

$\frac{\partial^2 L'}{\partial \lambda^2} = -\frac{N}{\lambda^2}$

$N > 0$, $\lambda^2 > 0$

$\therefore \frac{\partial^2 L'}{\partial \lambda^2} = -\frac{N}{\lambda^2} < 0$

$\therefore \lambda = \frac{N}{\sum_{i=1}^{N} x_i}$, which is the reciprocal of Sample mean

Q4

Case 1

In [9]:
```
seed_grid = np.arange(1,1001)
coverage = 0

for i in seed_grid:
    np.random.seed(i)
    x = np.random.normal(0,1,4)
    mu = x.mean()
```

```
    std = x.std()
    up = mu+1.6449*std/np.sqrt(4)
    lower = mu-1.6449*std/np.sqrt(4)
    if up>=0 and lower<=0:
        coverage += 1
    else:
        pass
ef = coverage/len(seed_grid)
print(f'The empirical frequency of coverage is {ef}')
```

```
The empirical frequency of coverage is 0.749
```

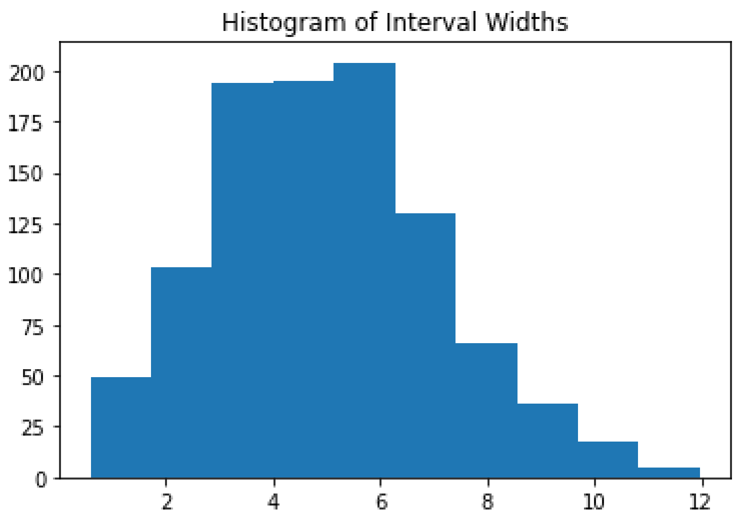
Case 2

In [10]:
```
seed_grid = np.arange(1,1001)
coverage = 0
width = []

for i in seed_grid:
    np.random.seed(i)
    x = np.random.normal(0,1,4)
    mu = x.mean()
    std = x.std()
    interval = t.interval(alpha=0.95, df=3, loc=mu, scale=std)
    if interval[1]>=0 and interval[0]<=0:
        coverage += 1
    else:
        pass
    width.append(interval[1]-interval[0])
ef = coverage/len(seed_grid)
print(f'The empirical frequency of coverage is {ef}')
```

```
C:\Users\GCM\AppData\Local\Temp\ipykernel_23480\3993943170.py:10: DeprecationWarning: Use of keyword argument `alpha` for method `interval` is deprecated. Use first positional argument or keyword argument `confidence` instead.
  interval = t.interval(alpha=0.95, df=3, loc=mu, scale=std)
The empirical frequency of coverage is 0.988
```

In [11]:
```
plt.figure()
plt.hist(width)
plt.title('Histogram of Interval Widths')
plt.show()
```



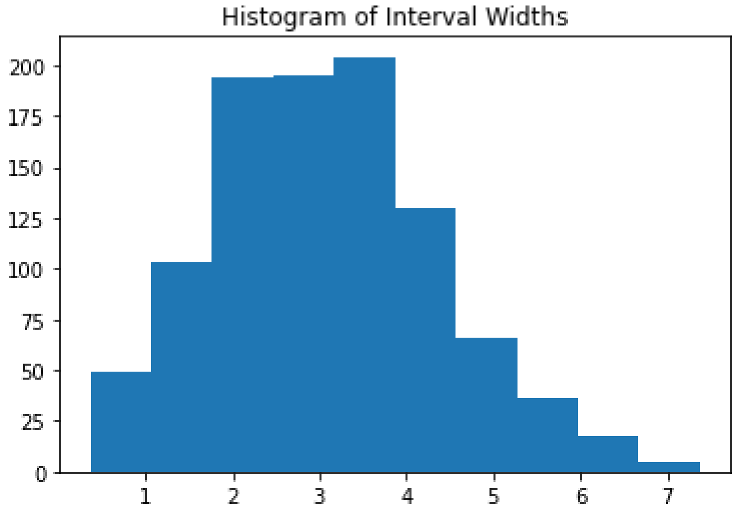
Case 3

In [12]:
```
seed_grid = np.arange(1,1001)
coverage = 0
width = []

for i in seed_grid:
    np.random.seed(i)
    x = np.random.normal(0,1,4)
    mu = x.mean()
    std = x.std()
    interval = norm.interval(alpha=0.95, loc=mu, scale=std)
    if interval[1]>=0 and interval[0]<=0:
        coverage += 1
    else:
        pass
    width.append(interval[1]-interval[0])
ef = coverage/len(seed_grid)
print(f'The empirical frequency of coverage is {ef}')
```

```
C:\Users\GCM\AppData\Local\Temp\ipykernel_23480\1891025027.py:10: DeprecationWarning: Use of keyword argument `alpha` for method `interval` is deprecated. Use first positional argument or keyword argument `confidence` instead.
  interval = norm.interval(alpha=0.95, loc=mu, scale=std)
The empirical frequency of coverage is 0.957
```

In [13]:
```
plt.figure()
plt.hist(width)
plt.title('Histogram of Interval Widths')
plt.show()
```



Case 2 has the highest empirical frequency of coverage when exact small sample CI is computed using t-distribution. That makes sense since the sample is very small (N=4). When using approximate large-sample CI in case 3, the empirical frequency decreases.

In [ ]: