Inverse semigroups (ft. GAP)

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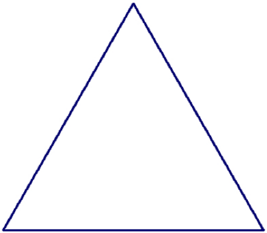
Symmetry

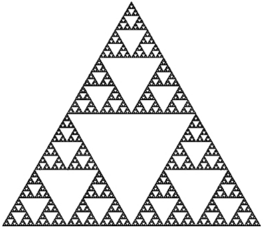
What is symmetry?

(a) "A sense of harmonious and beautiful proportion and balance."

(b) "Invariant to any of various transformations; including reflection, rotation or scaling."

Groups are a well appreciated type of semigroup which can be used to describe symmetry. However groups are concerned with symmetries of the whole object, which doesn't cover everything we might think of as symmetry. Consider the following:





Semigroups

Definition

A binary operation $*$ on some set is said to be *assoicative* if it satisfies

$$(a * b) * c = a * (b * c)$$

for all a, b, c in the set.

Definition

A *semigroup* is a set S with an associative binary operation $*$. It may be denoted by $(S, *)$ or simply by S.

Semigroups

Examples of binary operations on the real numbers are addition, multiplication, subtraction and division. However the latter two are not associative. E.g. subtraction is not associative because:

$$3 - (2 - 1) = 2 \neq 0 = (3 - 2) - 1.$$

Examples of semigroups are

- Natural numbers with addition, $(\mathbb{N}, +)$,
- Reals with multiplication $(\mathbb{R}, *)$,
- The set $\{0, 1, 2, 3\}$ with multiplication modulo 4,
- The set $\{0, a, b, c, d\}$ with $x * y = 0$ for all x, y.

Inverse semigroups

Definition: inverse elements

Let x and y be elements of a semigroup $(S, *)$. We say that y is an inverse of x if

$$xyx = x \text{ and } yxy = y.$$

This is a generalisation of group inverses. In general, semigroup inverses need not be unique and semigroups do not need to have an identity element.

Definition: inverse semigroups

A semigroup $(S, *)$ is said to be an *inverse semigroup* if every element has a unique inverse.

Example of an inverse semigroup

Inverse semigroups are commonly represented as semigroups of partial permutations. A partial permutation on a set X is a bijection between two subsets of X. For example:

$$a = \begin{pmatrix} 1 & 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}, a^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}$$

An inverse semigroup containing a is:

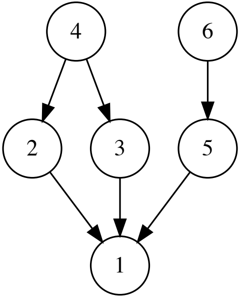
$$\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 5 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 4 \end{pmatrix} \}$$

In terms of the generators this is:

$$\{a^2, aa^{-1}, a^{-1}a, a, a^{-1}\}$$

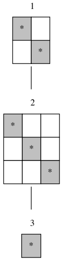
Tools for studying inverse semigroups

An *idempotent* is an element x such that $x * x = x$. Groups are the inverse semigroups with precisely one idempotent. The collection of idempotents are key to the structure of an inverse semigroup. They form a inverse subsemigroup which is a *semilattice*:



Tools for studying inverse semigroups

The idempotents are like the skeleton of an inverse semigroup. A very useful set of tools in semigroup theory are Green's Relations. These are equivalence relations and so they partition the elements of a semigroup. When encountering a new semigroup often the first thing I want to do is know the $\mathcal{D}$-classes structure.

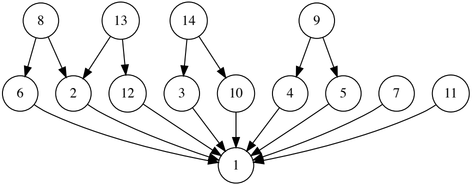


Tools for studying inverse semigroups

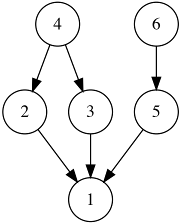
The *natural partial order* of an inverse semigroup is defined by

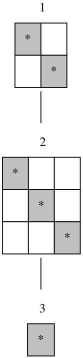
$$a \leq b \iff a = eb \text{ for some idempotent } b.$$

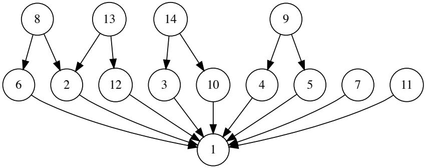
We can visualise this as a digraph.



Pictures made with GAP







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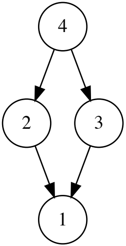

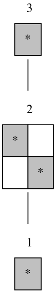

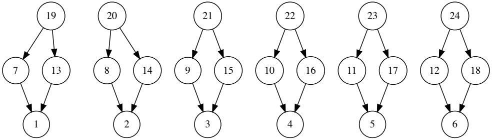

Computing with semigroups

Order	# Semigroups	# Inverse semigroups	# Groups
1	1	1	1
2	4	2	1
3	18	5	1
4	126	16	2
5	1160	52	1
6	15973	208	2
7	836021	911	1
8	1843120128	4637	5
9	52989400714478	26422	2
10	12418001077381302684	169163	2