3.36pt

Counting finite congruence free semigroups

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A good start

The size of the set $X_{m,n}$ of all $m \times n$ binary matrices with all rows distinct and all columns distinct is known. Divide this number by $m!$ to count the number up to row permutations. However we can't just divide by $m!\,n!$ because some combinations of row and column permutations fix matrices with all rows and columns unique. The following matrices are fixed by $((1,2),(1,2))$ and $((1,2,3),(1,2,3))$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Orbit counting

The orbit-counting theorem

Let X be a set and let G be a group which acts on X. Then the number of orbits of this action of G is:

$$\frac{1}{|G|}\sum_{g\in G}|\{x\in X : x^g = x\}|$$

that is to say, the average number of points fixed by an element of G.

In our case $X_{m,n}$ is the set of all $m\times n$ binary matrices with no duplicate rows and no duplicate columns and $G\cong S_m\times S_n$.

Orbit counting - refined

Let us denote the elements of a set X fixed by g with X^g.

1. If g, h are conjugate elements of G then $|X^g| = |X^h|$.
2. Elements of S_m are conjugate if and only if they have the same cycle type.
3. A conjugacy class of $S_m \times S_n$ is the product of a conjugacy class of S_m and a conjugacy class of S_n.

The number of orbits in our situation is

$$\frac{1}{m!\,n!} \sum_{i_1+2i_2+\cdots+mi_m=m} \quad \sum_{j_1+2j_2+\cdots+nj_n=n} |\text{conj class of } (\rho_i, \rho_j)||X_{m,n}^{(\rho_i,\rho_j)}|$$

where ρ_i, ρ_j are some permutations with cycle type $i = (i_1, \cdots, i_m)$ and $j = (j_1, \cdots, j_n)$.

What fixed matrices look like- a simple case.

Consider the matrices fixed by $((1, 2, \ldots, m), (1, 2, \ldots, n))$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{m-1\,n} \\ a_{m1} & \cdots & \cdots & a_{m\,n-1} & a_{mn} \end{pmatrix}$$

The permutation $((1, 2, \ldots, m), (1, 2, \ldots, n))$ permutes entries by:

$$a_{i\,j} \to a_{i+1\,j+1} \to \cdots \to a_{i-1\,j-1} \to a_{i\,j}$$

(indices taken mod m and n). If a matrix is fixed by this permutation then all the entries in this 'orbit' must be the same. There are 2 to the power of the number of 'orbits' possibilities (choose 0 or 1 for each 'orbit').

What fixed matrices look like - a simple case.

How many orbits?

The number of 'orbits' is the number of orbits of the group generated by $(1,2,\ldots,m),(1,2,\ldots,n))\rangle$ acting on $\{1,2,\ldots,m\} \times \{1,2,\ldots,n\}$. This number is the greatest common divisor of m and n. The least common multiple of m and n is the number of elements in each orbit.

Examples for 4x4, 3x4 and 4x6 are shown below:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{pmatrix}$$

What fixed matrices look like in general

Let $\rho \leq S_m$ be the permutation with cycles $c_1, \cdots, c_\alpha$ of the form

$$(1, 2, \ldots, |c_1|)(|c_1| + 1, \ldots, |c_1| + |c_2|) \ldots (m - |c_\alpha| + 1, \ldots, m).$$

Let $\sigma \leq S_n$ be the permutation with cycles $d_1, \cdots, d_\beta$ of the form

$$(1, 2, \ldots, |d_1|)(|d_1| + 1, \ldots, |d_1| + |d_2|) \ldots (n - |d_\beta| + 1, \ldots, n).$$

Then matrices fixed by (ρ, σ) will have $\alpha * \beta$ sub-matrices that act like the situation we had before. Each will be a $|c_i| \times |d_j|$ matrix fixed by (c_i, d_j).

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1|d_1|} \\ a_{21} & a_{22} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{|c_1|-1|d_1|} \\ a_{|c_1|1} & \cdots & \cdots & a_{|c_1|\,|d_1|-1} & a_{|c_1||d_1|} \end{pmatrix}$$

What fixed matrices look like in general

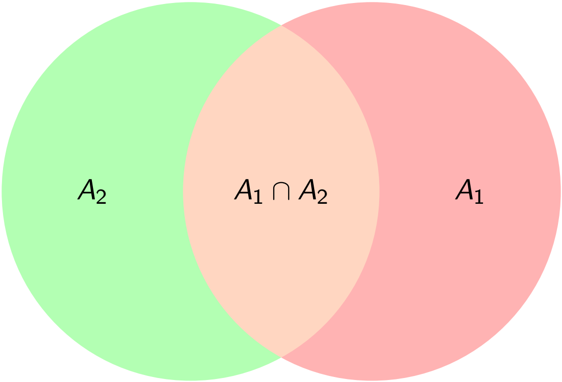
$$\rho = (1, 2, \ldots, |c_1|)(|c_1| + 1, \ldots, |c_1| + |c_2|) \ldots (m - |c_\alpha| + 1, \ldots, m).$$

$$\sigma = (1, 2, \ldots, |d_1|)(|d_1| + 1, \ldots, |d_1| + |d_2|) \ldots (n - |d_\beta| + 1, \ldots, n).$$

$$\left(\begin{array}{c|c|c|c} c_1 \times d_1 & c_1 \times d_2 & \cdots & c_1 \times d_\beta \\ \hline c_2 \times d_1 & c_2 \times d_2 & \cdots & c_2 \times d_\beta \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline c_\alpha \times d_1 & c_\alpha \times d_2 & \cdots & c_\alpha \times d_\beta \end{array}\right)$$

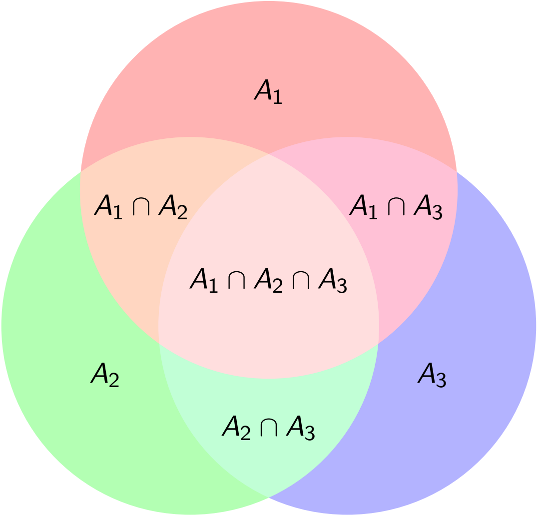
...but how many of these have all rows distinct and all columns distinct?

The inclusion-exclusion principle



$$\left|\bigcup_{i=1}^{2} A_i\right| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

The inclusion-exclusion principle



$$\left|\bigcup_{i=1}^{3} A_i\right| = |A_1|+|A_2|+|A_3|-|A_1\cap A_2|-|A_1\cap A_3|-|A_2\cap A_3|+|A_1\cap A_2\cap A_3|.$$

The inclusion-exclusion principle

Inclusion-exclusion principle

For finite sets $A_1, \ldots, A_n$ we have

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}|A_{i_1}\cap\cdots\cap A_{i_k}|\right)$$

An application of the inclusion-exclusion principle

The way to count the size of $X_{m,n}$ (also $X_{m,n}^{(1_{S_m},1_{S_n})}$) goes like this:

1. Let E_{ij} (F_{ij}) denote the binary $m \times n$ binary matrices with row (column) i equal to row (column) j.
2. $X_{m,n}$ is the complement of $\bigcup E_{ij} \cup \bigcup F_{ij}$ in the set of all $m \times n$ binary matrices.
3. Let $s(n,k)$ denote the signed Stirling number of the first kind:

$$\begin{aligned} |X_{m,n}| &= 2^{mn} - \left| \bigcup_{1 \le i < j \le m} E_{ij} \quad \cup \bigcup_{1 \le i < j \le n} F_{ij} \right| \\ &= 2^{mn} - \sum_{i=1}^{m} \sum_{j=1}^{n} s(m,i) \cdot s(n,j) \cdot 2^{ij} \end{aligned}$$

proof not obvious but not covered. Doesn't seem to generalize to counting the general case of $X_{m,n}^{(\rho,\sigma)}$.

Counting $|X_{m,n}^{(\rho,\sigma)}|$

Now we take inspiration from the previous method of counting with the inclusion-exclusion principle. Let $(\rho, \sigma) \in S_m \times S_n$ then

1. Denote the collection of matrices which are fixed by (ρ, σ) by A
2. The elements of A correspond to assigning 0 or 1 to each orbit of $[m] \times [n]$ in the group generated by (ρ, σ).
3. Denote the binary matrices which have row $i =$ row j by E_{ij}
4. The elements of E_{ij} correspond to assigning 0 or 1 to each class of the equivalence relation $\sim$ on $[m] \times [n]$ where $(i, x) \sim (j, x)$ for all x and the other classes have size 1.

The intersection $\mathcal{E}_{ij} = A \cap E_{ij}$ of matrices fixed by (ρ, σ) with row i equal to row j corresponds to the finding the join of these two equivalence relations on $[m] \times [n]$ and assigning 0 or 1 to each of it's classes. Define $\mathcal{F}_{ij}$ analogously to $\mathcal{E}_{ij}$ except for columns.

Counting $|X_{m,n}^{((1,2,3,4),(1,2,3,4))}|$

$$\mathcal{E}_{12} = \mathcal{E}_{14} = \mathcal{F}_{12} = \mathcal{F}_{14} =$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \cap \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Counting $|X_{m,n}^{((1,2,3,4),(1,2,3,4))}|$

$$\mathcal{E}_{13} = \mathcal{E}_{24} = \mathcal{F}_{13}) = \mathcal{F}_{24} =$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \bigcap \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Counting $|X_{m,n}^{((1,2,3,4),(1,2,3,4))}|$

$$\mathcal{E}_{12} \subset \mathcal{E}_{13}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \subset \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

In other words, the $A \cap E_{13}$ is finer than $A \cap E_{12}$. The lattice is trivial in this case.

Now lets try an example with more interesting cycle structure...

$$\begin{array}{c} A \\ \left(\begin{array}{cccc|cc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} \end{array}\right) \end{array}$$

Counting $|X_{m,n}^{((1,2,3,4)(5,6)(7),(1,2,3,4)(5,6))}|$

$$\mathcal{E}_{12} = \mathcal{E}_{23} = \mathcal{E}_{34} = \mathcal{E}_{14}$$

$$\left(\begin{array}{cccc|cc|c}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\
\hline
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67}
\end{array}\right)$$

Generators: $\mathcal{E}_{12}$

Not needed: $\mathcal{E}_{23}$, $\mathcal{E}_{34}$, $\mathcal{E}_{14}$

Counting $|X_{m,n}^{((1,2,3,4)(5,6)(7),(1,2,3,4)(5,6))}|$

$$\mathcal{F}_{12} = \mathcal{F}_{23} = \mathcal{F}_{34} = \mathcal{F}_{14}$$

$$\left(\begin{array}{cccc|cc|c}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\
\hline
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67}
\end{array}\right)$$

Generators: $\mathcal{E}_{12}$, $\underline{\mathcal{F}_{12}}$
Not needed: $\mathcal{E}_{23}$, $\mathcal{E}_{34}$, $\mathcal{E}_{14}$, $\underline{\mathcal{F}_{23}}$, $\underline{\mathcal{F}_{34}}$, $\underline{\mathcal{F}_{14}}$

$\mathcal{E}_{13} = \mathcal{E}_{24} = \mathcal{F}_{13} = \mathcal{F}_{24}$ is finer than $\mathcal{E}_{12}$ and $\mathcal{F}_{12}$.

$$\left(\begin{array}{cccc|cc|c}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\
\hline
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67}
\end{array}\right)$$

Generators: $\underline{\mathcal{E}_{13}}$

Not needed: $\underline{\mathcal{E}_{12}}$, $\mathcal{E}_{14}$, $\mathcal{E}_{23}$, $\underline{\mathcal{E}_{24}}$, $\mathcal{E}_{34}$, $\underline{\mathcal{F}_{12}}$, $\underline{\mathcal{F}_{13}}$, $\mathcal{F}_{14}$, $\mathcal{F}_{23}$, $\underline{\mathcal{F}_{24}}$, $\mathcal{F}_{34}$,

Counting $|X_{m,n}^{((1,2,3,4)(5,6)(7),(1,2,3,4)(5,6))}|$

$\mathcal{E}_{13}$ is finer than $\mathcal{E}_{15}, \mathcal{E}_{16}$ too.

$$\left(\begin{array}{cccc|cc|c}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\
\hline
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67}
\end{array}\right)$$

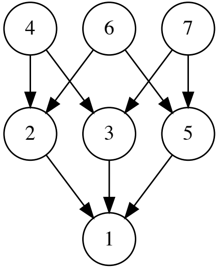
$$\left(\begin{array}{cccc|cc|c}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\
\hline
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67}
\end{array}\right)$$

Generators: $\mathcal{E}_{13}$

Not needed: $\mathcal{E}_{12}, \mathcal{E}_{14}, \mathcal{E}_{23}, \mathcal{E}_{24}, \mathcal{E}_{34}, \mathcal{F}_{12}, \mathcal{F}_{13}, \mathcal{F}_{14}, \mathcal{F}_{23}, \mathcal{F}_{24}, \mathcal{F}_{34}.$

Counting $|X_{m,n}^{((1,2,3,4)(5,6)(7),(1,2,3,4)(5,6))}|$

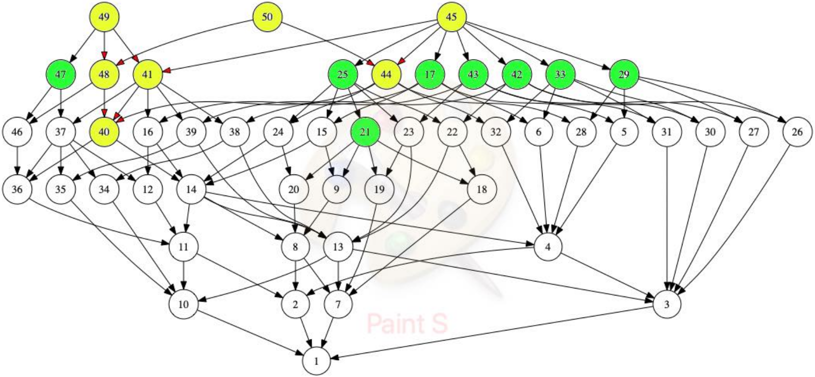
If we continue and consider all $\mathcal{E}_{ij}$ and $\mathcal{F}_{ij}$, remove the duplicates, and remove all but the finest then we are left with $\mathcal{E}_{13}$, $\mathcal{E}_{56}$, and $\mathcal{F}_{56}$. The semilattice generated by these with the join operation is pictured below:



And it is much smaller than if we hadn't filtered out generators that were coarser than other generators...

Counting $|X_{m,n}^{((1,2,3,4)(5,6)(7),(1,2,3,4)(5,6))}|$

The previous lattice can be seen (with some effort!) as the subsemilattice in yellow. Removed generators are in green.



Final points

1. I can show that $|X_{m,n}^{(\rho,\sigma)}| = 0$ if the least common multiples of the cycles lengths of ρ and σ are not equal - this is true for most conjugacy classes of $S_m \times S_n$!
2. The semilattice of equivalence relations tells us what the different intersections of the generators are but we also need to work out how many ways to write each element (if $\mathcal{E}_{ab} * \mathcal{E}_{cd} = \mathcal{E}_{ef} * \mathcal{E}_{gh}$ then we need to subtract it twice when applying IEP, and so on).
3. The method for counting $|X_{m,n}^{(1,1)}|$ is much better and I think it can utilized it in other cases which include cycles of length 1.
4. Need to remove the cases with all zero rows but this is easy to do after finding the number including zero rows.

Numbers of binary matrices with all rows distinct and all columns distinct up to row and colum permutation

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	2	0	0	0	0	0	0	0
2	0	3	3	1	0	0	0	0
3	0	3	12	19	16	9	4	1
4	0	1	19	94	250	459	649	729
5	0	0	16	250	1796	8623	32016	98097
6	0	0	9	459	8623	100494	881664	6363357
7	0	0	4	649	32016	881664	17422636	277445249
8	0	0	1	729	98097	6363357	277445249	

As before but without zero rows

We can discount those matrices which have a row of all 0's or a column or all 0's. Say the number of matrices without duplicate rows or columns is $X(m, n)$ and the number of these without any zero rows or columns in $x(m, n)$. Then
$x(m, n) = X(m, n) - x(m-1, n) - x(m, n-1) - x(m-1, n-1)$.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	0	0	0	0	0	0	0
2	0	2	1	0	0	0	0	0
3	0	1	8	10	6	3	1	0
4	0	0	10	66	168	282	363	365
5	0	0	6	168	1394	6779	24592	72777
6	0	0	3	282	6779	85542	764751	5501237
7	0	0	1	363	24592	764751	15807592	255371669
8	0	0	0	365	72777	5501237	255371669	