E-unitary inverse semigroups

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Overview

E-unitary inverse semigroups

All semigroups mentioned in this talk will be inverse semigroups. The subsemigroup of idempotents of a semigroup S is denoted by $E(S)$ or simply E.

Definition

An inverse semigroup S is *E-unitary* if for all e in E and s in S we have

$$es \in E \implies s \in E,$$

or, equivalently

$$se \in E \implies s \in E.$$

Examples: bands, groups, semidirect products of semilattices by groups, the bicyclic monoid, free inverse monoids.

Posets and semilattices

Definition

A binary relation $\leq$ over a set P is a *partial order* if it is reflexive, anti-symmetric and transitive.

Definition

A partially ordered set $(P, \leq)$ is a *join-semilattice* if there is a binary operation $\vee$ called the join which returns the least upper bound of two elements with respect to the partial order.

A join-semilattice $(P, \vee)$ is a commutative band semigroup.

Examples

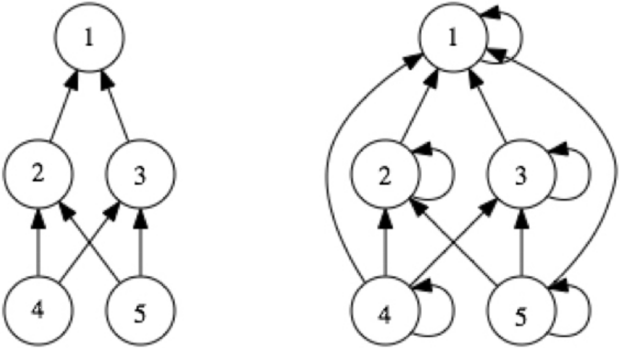

Figure: Partial orders can be represented as digraphs. The picture becomes easier to view if we remove the edges implied by reflexivity and transitivity.

Examples

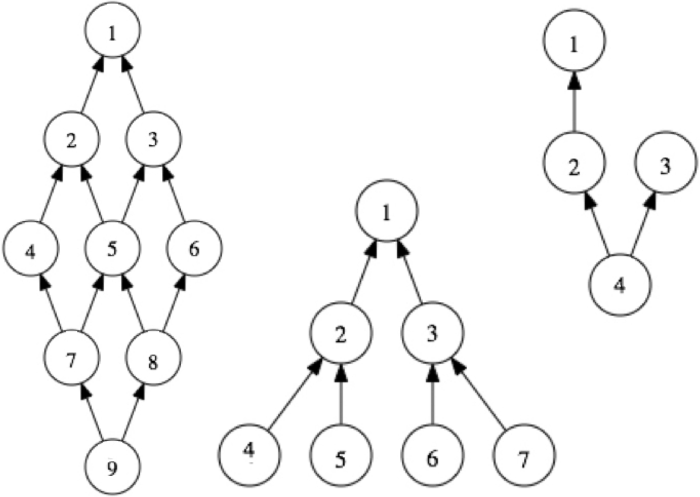

Figure: Left to right: a lattice, a join-semillatice, and a poset.

McAlister triples

McAlister triple

Let $(\mathcal{X}, \leq)$ be a partial order and let $(\mathcal{Y}, \leq)$ be a sub-order of $\mathcal{X}$ which is a join-semilattice under the induced ordering. Let $\mathcal{G}$ be a group which acts on $\mathcal{X}$ (on the left) by means of order automorphisms. In addition, if the following three conditions hold:

- $$(\forall A, B \in \mathcal{X})(A \in \mathcal{Y} \text{ and } A \leq B) \implies B \in \mathcal{Y};$$
- $\bigcup_{g \in \mathcal{G}} g\mathcal{Y} = \mathcal{X}$,
- $(\forall g \in \mathcal{G}) g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$ - always true for finite $\mathcal{Y}$,

then the triple $(\mathcal{G}, \mathcal{X}, \mathcal{Y})$ is called a *McAlister triple*.

McAlister triple semigroups

Given a McAlister triple $(\mathcal{G}, \mathcal{X}, \mathcal{Y})$ let

$$\mathcal{M}(\mathcal{G}, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in \mathcal{Y} \times \mathcal{G} : g^{-1}A \in \mathcal{Y}\},$$

and define a multiplication on this set by

$$(A, g)(B, h) = (A \vee gB, gh).$$

With this multiplication the set $\mathcal{M}(\mathcal{G}, \mathcal{X}, \mathcal{Y})$ is an E-unitary inverse semigroup.

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Theorem

Let $(\mathcal{G}, \mathcal{X}, \mathcal{Y})$ be a McAlister triple. Then $\mathcal{M}(\mathcal{G}, \mathcal{X}, \mathcal{Y})$ is an E-unitary inverse semigroup. Conversely, every E-unitary inverse semigroup is isomorphic to one of this kind.

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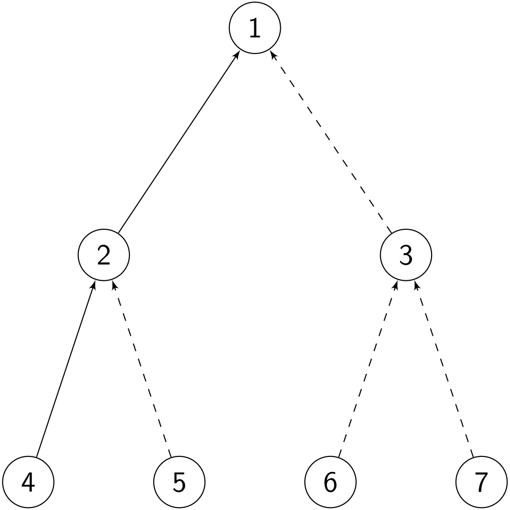
However something we might think that we would like to be true is not:

$$\mathcal{G} \cong \mathcal{G}', \mathcal{X} \cong \mathcal{X}', \mathcal{Y} \cong \mathcal{Y}' \not\Rightarrow \mathcal{M}(\mathcal{G}, \mathcal{X}, \mathcal{Y}) \cong \mathcal{M}(\mathcal{G}', \mathcal{X}', \mathcal{Y}')$$

but this is because the triple does not explicity describe the action of the group yet the same group may have different actions. If we also have that the actions are "*the same*" then the two triples represent isomorphic semigroups.

An example of a McAlister triple

$\mathcal{G} = \langle (4,5), (6,7), (2,3)(4,6)(5,7) \rangle$



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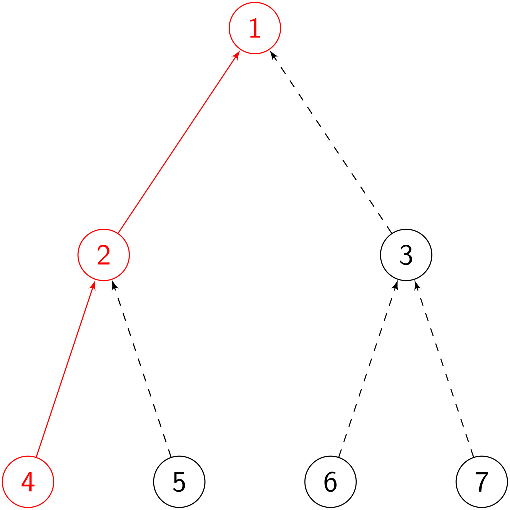

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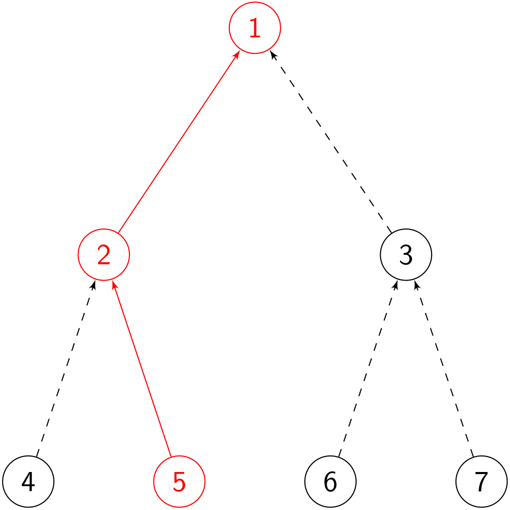

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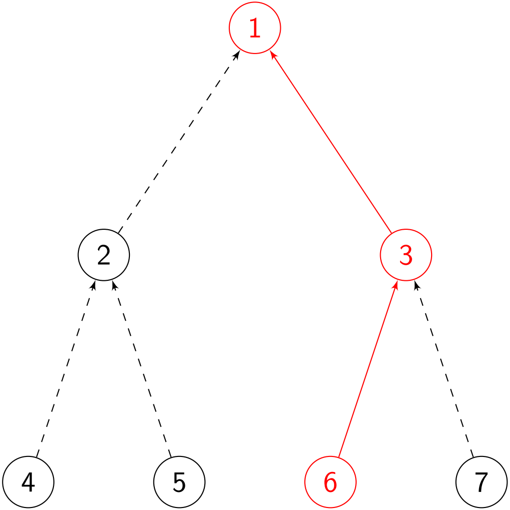

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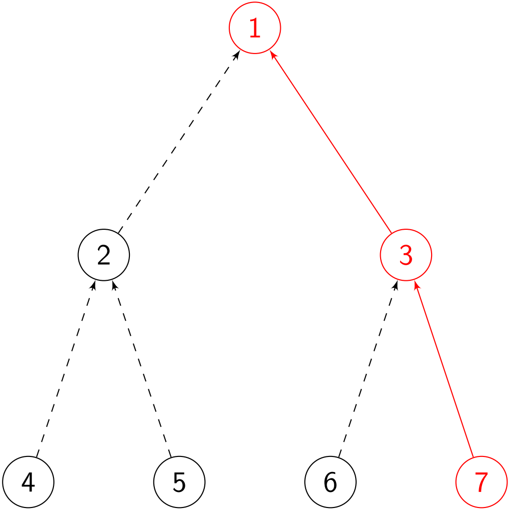


An example of a McAlister triple

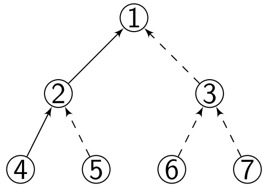
What are the elements?

They are the pairs (A, g) in $\mathcal{Y} \times \mathcal{G}$ such that $g^{-1}A$ is in $\mathcal{Y}$.

The elements are:

$$\{(1, g) : g \in \mathcal{G}\} \cup \{(2, g) : g \in Stab_{\mathcal{G}}(2)\} \cup \{(4, g) : g \in Stab_{\mathcal{G}}(4)\}$$

since 1 is always fixed; the orbit of 2 is $\{2, 3\}$ but 3 is not in $\mathcal{Y}$; and the orbit of 4 is $\{4, 5, 6, 7\}$ but only 4 is in $\mathcal{Y}$.



$$\mathcal{G} = \langle (4, 5), (6, 7), (2, 3)(4, 6)(5, 7) \rangle$$

Why study E-unitary inverse semigroups?

The following is known as McAlister's covering theorem.

Theorem

For every (finite) inverse semigroup S there is a (finite) E-unitary inverse semigroup P and a surjective, idempotent separating homomorphism $\theta : P \to S$.

It is generally seen as a key result in inverse semigroup theory, although it is not very clear what it tells us about inverse semigroups or where its applications may lie.

McAlister triples in the Semigroups package

When creating McAlister triples in GAP the partial order and the join-semilattice are represented as digraphs, using the Digraphs package. These digraphs are always labeled by the natural numbers $\{1, \ldots, n\}$. The function McAlisterTriple can be used with various arguments. All arguments must be of finite size.

- McAlisterTriple(G, X, Y) - where G is a finite permutation group.
- McAlisterTriple$(G, X, subset)$ - where subset is a subset of the vertices of X such that the induced subdigraph is a semilattice.
- McAlisterTriple(G, act, X, Y) - G is any finite group object and act is a function which is an action of G on X.
- McAlisterTriple$(G, act, X, subset)$
- McAlisterTriple(S) where S is an E-unitary inverse semigroup.

McAlister triples in the semigroups package

What could you do in GAP:

```
gap> x := Digraph([[1],[1,2],[1,3]]);
<digraph with 3 vertices, 5 edges>
gap> G := AutomorphismGroup(x);
Group([ (2,3) ])
gap> McAlisterTriple(G,x,x);
<McAlister triple over Group([ (2,3) ])>
gap> Size(M);
6
gap> Elements(M);
[ (1, ()), (1, (2,3)), (2, ()), (2, (2,3)),
(3, ()), (3, (2,3)) ]
gap> Elements(M)[3] * Elements(M)[6]
(1, (2,3))
```

McAlister triples in the semigroups package

What could you do in GAP:

```
gap> Subsemigroup(M, [Elements(M)[2]]);
<commutative semigroup with 1 generator>
gap> P := AsSemigroup(IsPartialPermSemigroup, M);
<inverse partial perm semigroup of size 6,
rank 6 with 6 generators>
gap> iso := McAlisterTriple(P);; #this returns an isomorphism
```

Future plans

- Add and improve features for McAlister triples
- Finitely presented semigroups
- Automatic semigroups

Future plans: building a database

A database of E-unitary inverse semigroups could be useful to researchers for finding examples and checking hypotheses. This should be feasible for low orders. There is currently a gap database of all semigroups of order less than or equal to 8. The number of inverse semigroups of a given order is significantly smaller than the number of semigroups although it still grows quickly.

Order	**# Semigroups**	**# Inverse semigroups**
2	4	2
3	18	5
4	126	16
5	1160	52
6	15973	208
7	836021	911
8	1843120128	4637
9	52989400714478	26422
10	12418001077381302684	169163

Future plans: building a database

Finding the E-unitary inverse semigroups of a given order is equivalent to finding the McAlister triple semigroups of a given order. GAP contains a library of small groups however finding all the semilattices and partial orders is a difficult task.

Order	# Inverse Semigroups	# E-unitary inverse semigroups
2	2	2
3	5	4
4	16	11
5	52	27
6	208	92
7	911	324
8	4637	1561

Future plans: F-inverse semigroups

F-inverse semigroups are a class of E-unitary inverse semigroups with which are interesting for many of the same reasons. For example there is also a covering theorem for F-inverse semigroups. They are precisely those McAlister triple semigroups $\mathcal{M}(\mathcal{G}, \mathcal{X}, \mathcal{Y})$ where $\mathcal{X}$ is a join-semilattice. Building a database of these should be simpler than building a database of E-unitary inverse semigroups.

References

M. V. Lawson (1994)
Inverse Semigroups: The Theory of Partial Symmetries

J. M. Howie (1995)
Fundamentals of Semigroup Theory

A. Distler (2010)
Classification and enumeration of finite semigroups

A. Distler, C. Jefferson, T. Kelsey, and L. Hotthoff (2013)
The semigroups of order 10