Enumerating Rees 0-matrix semigroups

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Overview

Preliminaries

0-simple semigroups

Definition

A semigroup S is *0-simple* if its only proper two-sided ideal is $\{0\}$.

The 0-simple semigroups are the key ingredients in a decomposition of semigroups. Intuitively, the Greens $\mathcal{D}$-classes of a semigroup strongly resemble 0-simple semigroups.

Reminder

Greens $\mathcal{D}$-relation describes the structure of semigroups by relating elements which generate the same (two-sided) ideals.

Decomposition into 0-simple semigroups

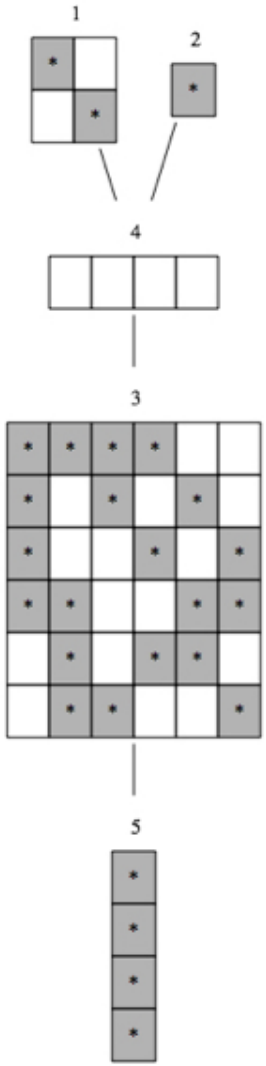
We can take the ideal generated by any $\mathcal{D}$-class and quotient by the ideal containing everything 'below' that $\mathcal{D}$-class to obtain a 0-simple semigroup (or a null semigroup).

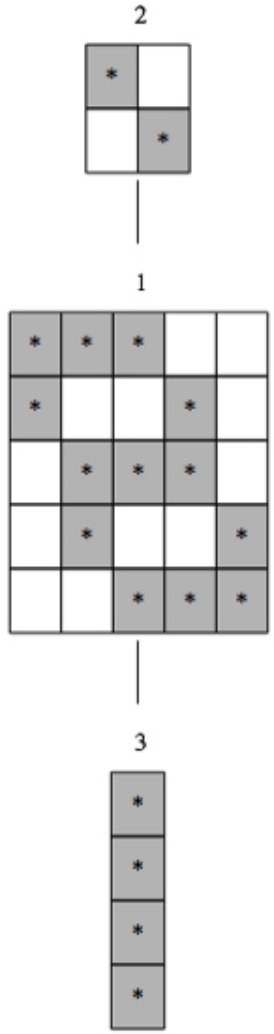
Proposition

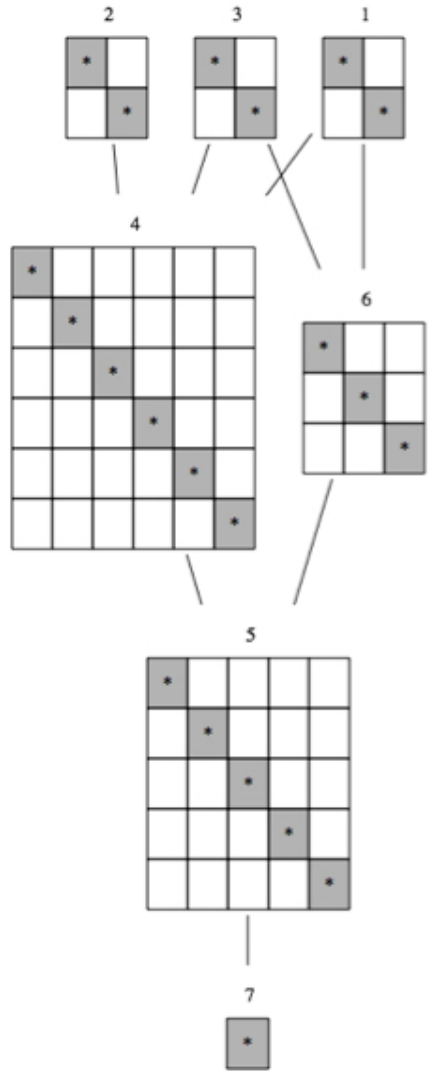
If I, J are ideals of a semigroup S such that $I \subset J$ and there is no ideal B of S such that $I \subset B \subset J$, then J/I is either 0-simple or null.

See pictures on next slide for intuition.

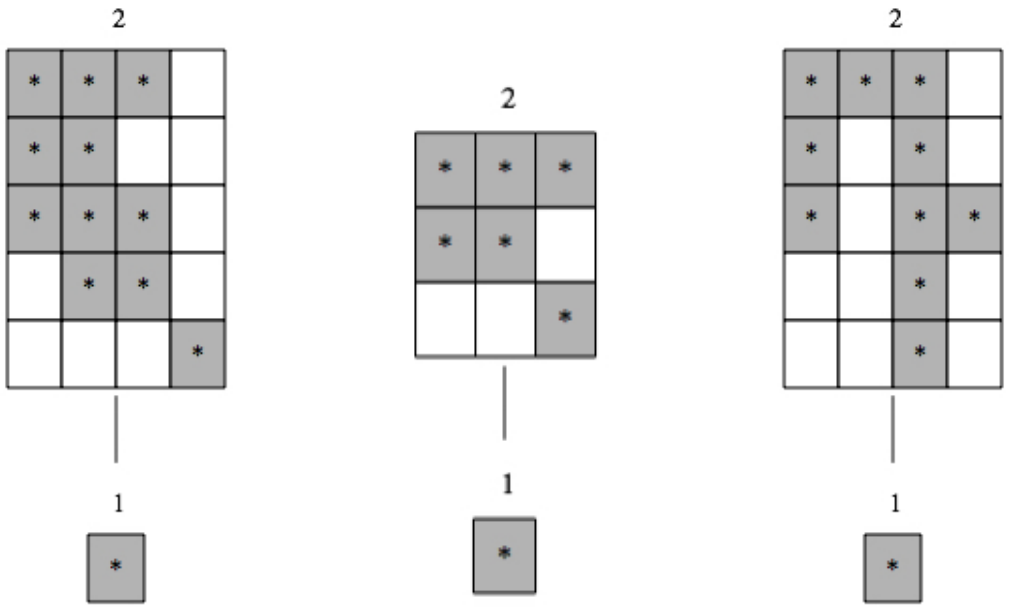
Decomposition into 0-simple semigroups







Decomposition into 0-simple semigroups



Rees 0-matrix semigroups

The importance of Rees 0-matrix semigroups is their correspondence to (completely) 0-simple semigroups. We won't mention 'completely' again since it has no added meaning for finite semigroups.

Definition

Let G be a group, $P = (p_{i,j})$ be a $m \times n$ matrix with entries in $G \cup \{0\}$. Let $S = (\{1 \cdots m\} \times G \times \{1 \cdots n\}) \cup \{0\}$, and define a multiplication on S by

$$(i, a, j)(k, b, l) = \begin{cases} (i, ap_{j,k}b, l) & \text{if } p_{j,k} \neq 0 \\ 0 & \text{if } p_{j,k} = 0 \end{cases}$$

$$(i, a, j)0 = 0(i, a, j) = 0.$$

Then S is called a *Rees 0-matrix semigroup* denoted $\mathcal{M}^0[G; P]$. If P is 'regular' (every row and column has a non-zero entry) then S is 0-simple. Conversely, every 0-simple semigroup is isomorphic an RZMS of this kind.

Enumerating RZMS

My work: a database of 0-simple semigroups

I have enumerated the 0-simple semigroups of order less than 50 and plan to release GAP code that anyone can use to generate these. Why? Databases are useful for:

1. Checking hypotheses - users can check whether a result holds for all entries in a database.
2. Obtaining examples - users can filter database entries by certain properties.

Other GAP databases include Small Groups (order < 2000 but not 1024), Primitive Groups (degree less than 4096), Transitive Groups (up to degree 30) and the SmallSemi package (semigroups of order ≤ 8).

What it boils down to (roughly)

Finding all 0-simple semigroups of order k up to isomorphism is equivalent to finding all RZMS (over regular matrices) of order k up to isomorphism.

Proposition

The order of the RZMS $\mathcal{M}^0[G;P]$ equals $|G| * |m| * |n| + 1$ where m, n are the number of rows and columns of P.

Lemma

If the RZMS $\mathcal{M}^0[G;P]$ and $\mathcal{M}^0[H;Q]$ are isomorphic then $G \cong H$ and P, Q have the same dimensions.

We may separately consider RZMS with different: groups (up to isomorphism), numbers of rows, or numbers of columns. All RZMS with over the group G and an $m \times n$ matrix are contained in the set

$$\{\mathcal{M}^0[G;P] : P \in M_{m\times n}(G^0)\}.$$

A naive approach

As an easy way to start out, I tried the following:

1. **RZMS** := All Rees 0-matrix semigroups of order k
2. for **i** in $\{1 \cdots |\mathbf{RZMS}|\}$ do
3. for **j** in $\{1 \cdots \mathbf{i} - 1\}$ do
4. if **RZMS**[**i**] is isomorphic to **RZMS**[**j**] then
5. remove **RZMS**[**j**] from **RZMS**
6. return **RZMS**

Slightly better approach

This approach avoids making as many isomorphism checks.

1. for (G, m, n) corresponding to order k do
2. **RZMS** := All Rees 0-matrix semigroups of type (G, m, n)
3. for **i** in $\{1 \cdots |\mathbf{RZMS}|\}$ do
4. for **j** in $\{1 \cdots \mathbf{i} - 1\}$ do
5. if **RZMS[i]** is isomorphic to **RZMS[j]** then
6. remove **RZMS[j]** from **RZMS**
7. Append **RZMS** to **OUT**
8. return **OUT**

Speeding things up

Strategy

We have two issues:

1. Isomorphism checks are expensive.
2. There are lots of matrices to check.

Our solutions should be:

1. Reduce number of isomorphism checks and use a better method if possible.
2. Find an easily calculable and significantly smaller subspaces of $M_{m\times n}(G^0)$ which still represent every isomorphism class.

Isomorphism Theorem

Finding an isomorphism is expensive without any specialized method because the general case is hard for semigroups.

RZMS Isomorphism Theorem (simplified)

$\mathcal{M}^0[G; P]$ is isomorphic to $\mathcal{M}^0[G; Q]$ if and only if Q can be obtained from P via:

1. Permuting the rows and columns
2. Multiplying all entries in a row or column by a group element
3. Applying automorphisms of G to every entry at once

We can use this to check whether two RZMS over the same group are isomorphic.

Isomorphism Theorem

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mapsto \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \quad \text{(swap rows 1 and 2)}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mapsto \begin{bmatrix} a*x & b*x & c*x \\ d & e & f \\ g*y & h*y & i*y \end{bmatrix} \quad \text{(multiply row 1,3 by x,y)}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mapsto \begin{bmatrix} a^{-1} & b^{-1} & c^{-1} \\ d^{-1} & e^{-1} & f^{-1} \\ g^{-1} & h^{-1} & i^{-1} \end{bmatrix}$$

Isomorphism theorem - group actions

The following are all group actions on the space $M_{m\times n}(G^0)$ of $m \times n$ matrices over G^0:

1. Permuting the rows and columns
2. Multiplying all entries in a row or column by a group element
3. Applying automorphisms of G to every entry at once

This is a promising observation - group algorithms are fast and eliminate equivalent cases quickly through symmetries.

The group action

The following group acts faithfully on the space $m \times n$ matrices over G^0.

$$\mathcal{G}(G, m, n) := (Sym(m) \times Sym(n) \times Aut(G)) \rtimes (G^m \times G^n)$$

The action can send P to Q if and only if P and Q create isomorphic RZMS. Thus finding isomorphism classes is equivalent to finding a single representative of every orbit.

1. $Sym(m)$ acts by permuting the m rows and $Sym(n)$ the n columns
2. G^m acts by multiplication on rows and G^n acts by multiplication on columns.
3. $Aut(G)$ acts by applying to every entry at once.

The semidirect product to accounts for these actions not commuting, in general.

Isomorphism Theorem

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mapsto \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$ (swap rows 1 and 2)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mapsto \begin{bmatrix} a*x & b*x & c*x \\ d & e & f \\ g & h & i \end{bmatrix}$$ (multiply row 1 by x)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mapsto \begin{bmatrix} a^{-1} & b^{-1} & c^{-1} \\ d^{-1} & e^{-1} & f^{-1} \\ g^{-1} & h^{-1} & i^{-1} \end{bmatrix}$$

CanonicalImage

The function **CanonicalImage** [Jefferson et al, 2017] takes a group action as well as an element of the set being acted upon and returns a *canonical* element of its orbit. It does this relatively quickly, without needing to find every element of the orbit.

This allows us to stop doing pairwise isomorphism checks. Now we do:

1. **RZMS** := All Rees 0-matrix semigroups of order k
2. for **i** in $\{1 \cdots |\mathbf{RZMS}|\}$ do
3. if **CanonicalImage**(**RZMS**) not in OUT then
4. add **CanonicalImage**(**RZMS**) to OUT
5. return **OUT**

Checking less matrices

The number of matrices in $M_{m\times n}(G^0)$ is $(|G|+1)^{m*n}$ which scales very quickly. There are a few immediate and easy ways to reduce the workload:

1. The case where m or n equals 1 has only a single isomorphism class.
2. The case where m or n equals 2 is fairly easy to solve and contains relatively few isomorphism classes.
3. The $M_{m\times n}(G^0)$ case corresponds exactly to transposing the $M_{n\times m}(G^0)$ case.

Checking less matrices - linked triples

Very breifly, if $\mathcal{M}^0[G;P]$ is an RZMS then for each normal subgroup N of G there exists a homomorphism:

$$\mathcal{M}^0[G;P] \to \mathcal{M}^0[G/N;P']$$

where P' corresponds to P with its entries mapped naturally into the quotient group $p_{i,j} \mapsto p_{i,j}N$. This is a small part of the theory of *linked triples* which fully describes the quotients of any RZMS. This allows us to use the solution to the case $M_{m\times n}((G/N)^0)$ to get a head start in solving the case $M_{m\times n}(G^0)$.
Actually, we will just use the $M_{m\times n}(1^0)$ case when solving any case of the form $M_{m\times n}(G^0)$.

Binary matrices

Here we illustrate the ideas of the last slide. Assume these matrices are over some 0-group G^0:

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cong \begin{bmatrix} 0 & c \\ a & b \end{bmatrix}$$

and they form isomorphic RZMS. If we replace all non-zero entries by 1 then we should have two matrices which are isomorphic over the trivial 0-group $1^0 = \{0, 1\}$. (The reverse implication does not hold.)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

We prove this by using the earlier isomorphism theorem. Then, in the case $M_{m\times n}(G^0)$ we no longer need to consider all matrices with one zero but we only consider matrices with one zero if they have that zero in the top left position.

Binary matrices

Summary

If P and Q define isomorphic RZMS over some group G and we replace all their entries with 1 then these will also define isomorphic RZMS.

We can then split the case $M_{m\times n}(G^0)$ into separate cases corresponding to the isomorphism classes of $M_{m\times n}(1^0)$. The cases where G is trivial turned out to be the hardest cases (by order).
In the case of a matrix with one zero, we reduce our work by a factor of $m * n$.

Binary matrices - performance

This table, roughly, shows the proportion of the work we are cutting out assuming we already have already solved the binary case.

Dimensions	**# binary matrices**	**# regular, up to row&col perms**
3×3	512	14
4×3	4096	37
4×4	65536	115
5×4	1048576	525

And these improvements were quite relevant to cases such as $(G, m, n) = (C_5, 3, 3), (C_4, 4, 3), (C_3, 4, 4), (C_2, 5, 4)$.

Stuck

Solving the $M_{m\times n}(1^0)$ case seems to be a hard problem. It may be rephrased as finding binary matrices up to the equivalence of row and column permutations but that doesn't help. My search ended after finding the 14289957 matrices in the 8×6 case and with me being unable to compute the 7×7 case.

Other challenges

1. Normalized RZMS.
2. Finding binary matrices up to row and column permutation equivalence.
3. Turning the group action into a permutation group action (the best GAP algorithms are for permutation groups acting on positive integers).
4. Storing solved cases and reaccessing them when needed.
5. Parallelising computations.

Results

Results

Order	# RZMS (up to anti-isomorphism)
3	2
4	2
5	7
6	2
7	10
8	2
9	21
10	18
11	16
12	2
13	75

Results

Order	# RZMS (up to anti-isomorphism)
14	2
15	24
16	88
17	185
18	2
19	247
20	2
21	599
22	320
23	46
24	2
25	2648
26	2090

Results

Order	# RZMS (up to anti-isomorphism)
27	60
28	955
29	5615
30	2
31	19159
32	2
33	15854
34	2221
35	94
36	96068
37	144146
38	2
39	117

Results

Order	# RZMS (up to anti-isomorphism)
40	4811
41	546825
42	2
43	1570709
44	2
45	199140
46	1925893
47	160
48	2
49	¿14289957