

# Enumerating binary matrices with all rows and columns distinct up to permutations of rows and columns

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# A good start

The size of the set  $X_{m,n}$  of all  $m \times n$  binary matrices with all rows distinct and all columns distinct is known. Divide this number by  $m!$  to count the number up to row permutations. However we can't just divide by  $m! n!$  because some combinations of row and column permutations fix matrices with all rows and columns unique. The following matrices are fixed by  $((1, 2), (1, 2))$  and  $((1, 2, 3), (1, 2, 3))$ :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

## The orbit-counting theorem

Let  $X$  be a set and let  $G$  be a group which acts on  $X$ . Then the number of orbits of this action of  $G$  is:

$$\frac{1}{|G|} \sum_{g \in G} |\{x \in X : x^g = x\}|$$

that is to say, the average number of points fixed by an element of  $G$ .

In our case  $X_{m,n}$  is the set of all  $m \times n$  binary matrices with no duplicate rows and no duplicate columns, and  $G = S_m \times S_n$  which acts on matrices by permuting rows and columns in the natural way.

# Orbit counting - refined

Let us denote the elements of a set  $X$  fixed by  $g$  with  $X^g$ .

- 1 If  $g, h$  are conjugate elements of  $G$  then  $|X^g| = |X^h|$ .
- 2 Elements of  $S_m$  are conjugate if and only if they have the same cycle type.
- 3 A conjugacy class of  $S_m \times S_n$  is the product of a conjugacy class of  $S_m$  and a conjugacy class of  $S_n$ .

The number of orbits in our situation is

$$\frac{1}{m! n!} \sum_{i_1+2i_2+\dots+mi_m=m} \sum_{j_1+2j_2+\dots+nj_n=n} |\text{conj class of } (\rho_i, \rho_j)| |X_{m,n}^{(\rho_i, \rho_j)}|$$

where  $\rho_i, \rho_j$  are some permutations with cycle type  $i = (i_1, \dots, i_m)$  and  $j = (j_1, \dots, j_n)$ .

# What fixed matrices look like- a simple case.

Consider the matrices fixed by  $((1, 2, \dots, m), (1, 2, \dots, n))$ .

The diagram shows an  $m \times n$  matrix with entries  $a_{ij}$ . Three colored diagonals represent orbits of the permutation  $((1, 2, \dots, m), (1, 2, \dots, n))$ :

- Blue diagonal:**  $a_{11}, a_{22}, \dots, a_{mn}$
- Red diagonal:**  $a_{21}, a_{32}, \dots, a_{mn-1}$
- Green diagonal:**  $a_{12}, a_{23}, \dots, a_{m-1,n}$

Ellipses ( $\dots$ ) indicate intermediate entries in the matrix and along the diagonals.

The permutation  $((1, 2, \dots, m), (1, 2, \dots, n))$  permutes entries by:

$$a_{ij} \rightarrow a_{i+1,j+1} \rightarrow \dots \rightarrow a_{i-1,j-1} \rightarrow a_{ij}$$

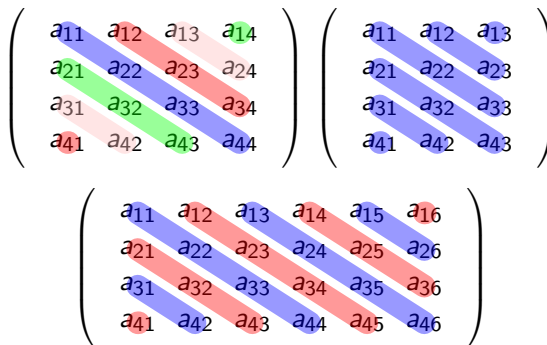
(indices taken mod  $m$  and  $n$ ). If a matrix is fixed by this permutation then all the entries in this 'orbit' must be the same. There are 2 to the power of the number of 'orbits' possibilities (choose 0 or 1 for each 'orbit').

# What fixed matrices look like - a simple case.

## How many orbits?

The number of 'orbits' is the number of orbits of the group generated by  $\langle (1, 2, \dots, m), (1, 2, \dots, n) \rangle$  acting on  $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ . This number is the greatest common divisor of  $m$  and  $n$ . The least common multiple of  $m$  and  $n$  is the number of elements in each orbit.

Examples for  $4 \times 4$ ,  $3 \times 4$  and  $4 \times 6$  are shown below:



# What fixed matrices look like in general

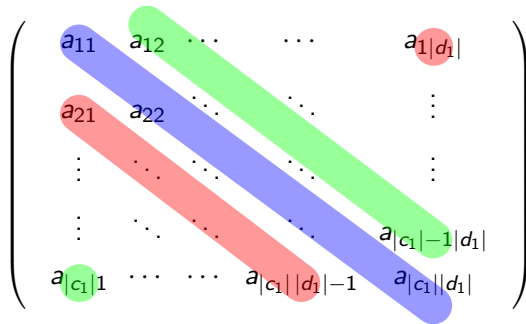
Let  $\rho \leq S_m$  be the permutation with cycles  $c_1, \dots, c_\alpha$  of the form

$$(1, 2, \dots, |c_1|)(|c_1| + 1, \dots, |c_1| + |c_2|) \dots (m - |c_\alpha| + 1, \dots, m).$$

Let  $\sigma \leq S_n$  be the permutation with cycles  $d_1, \dots, d_\beta$  of the form

$$(1, 2, \dots, |d_1|)(|d_1| + 1, \dots, |d_1| + |d_2|) \dots (n - |d_\beta| + 1, \dots, n).$$

Then matrices fixed by  $(\rho, \sigma)$  will have  $\alpha * \beta$  sub-matrices that act like the situation we had before. Each will be a  $|c_i| \times |d_j|$  matrix fixed by  $(c_i, d_j)$ .



The diagram shows a large matrix with several colored diagonals representing sub-matrices. The diagonals are labeled with elements  $a_{ij}$  and their dimensions. The colors are blue, green, and red. The blue diagonals represent sub-matrices of size  $|c_1| \times |d_1|$ . The green diagonals represent sub-matrices of size  $|c_1| \times |d_2|$ . The red diagonals represent sub-matrices of size  $|c_1| \times |d_3|$ . The matrix is shown as a large set of parentheses containing the elements  $a_{ij}$  and their dimensions.

$$\left( \begin{array}{ccccccc} a_{11} & a_{12} & \cdots & \cdots & & & a_{1|d_1|} \\ & a_{22} & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & & & \vdots \\ & & & \ddots & & & \vdots \\ & & & & \ddots & & a_{|c_1|-1|d_1|} \\ a_{|c_1|1} & \cdots & \cdots & & a_{|c_1||d_1|-1} & & a_{|c_1||d_1|} \end{array} \right)$$

# What fixed matrices look like in general

$$\rho = (1, 2, \dots, |c_1|)(|c_1| + 1, \dots, |c_1| + |c_2|) \dots (m - |c_\alpha| + 1, \dots, m).$$

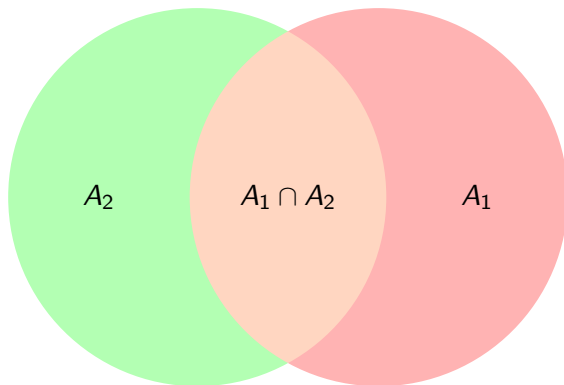
$$\sigma = (1, 2, \dots, |d_1|)(|d_1| + 1, \dots, |d_1| + |d_2|) \dots (n - |d_\beta| + 1, \dots, n).$$

$$\left( \begin{array}{c|c|c|c} c_1 \times d_1 & c_1 \times d_2 & \cdots & c_1 \times d_\beta \\ \hline c_2 \times d_1 & c_2 \times d_2 & \cdots & c_2 \times d_\beta \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline c_\alpha \times d_1 & c_\alpha \times d_2 & \cdots & c_\alpha \times d_\beta \end{array} \right)$$

... but how many of these have all rows distinct and all columns distinct?

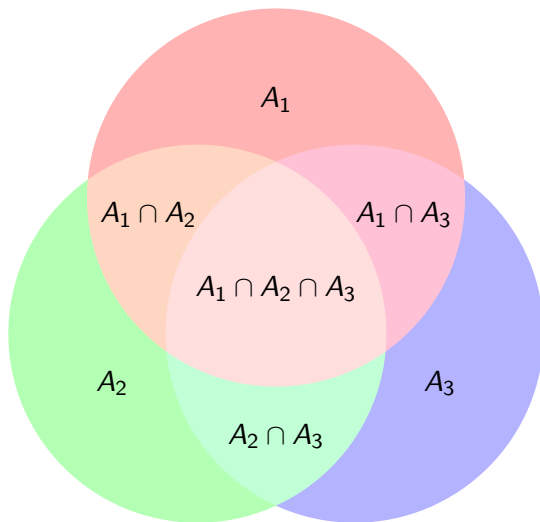


# The inclusion-exclusion principle



$$\left| \bigcup_{i=1}^2 A_i \right| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

# The inclusion-exclusion principle



$$\left| \bigcup_{i=1}^3 A_i \right| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

# The inclusion-exclusion principle

## Inclusion-exclusion principle

For finite sets  $A_1, \dots, A_n$  we have

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^{k+1} \left( \sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}| \right)$$

# An application of the inclusion-exclusion principle

The way to count the size of  $X_{m,n}$  (also  $X_{m,n}^{(1_{S_m}, 1_{S_n})}$ ) goes like this:

- 1 Let  $E_{ij}$  ( $F_{ij}$ ) denote the binary  $m \times n$  binary matrices with row (column)  $i$  equal to row (column)  $j$ .
- 2  $X_{m,n}$  is the complement of  $\bigcup E_{ij} \cup \bigcup F_{ij}$  in the set of all  $m \times n$  binary matrices.
- 3 Let  $s(n, k)$  denote the signed Stirling number of the first kind:

$$\begin{aligned} |X_{m,n}| &= 2^{mn} - \left| \bigcup_{1 \leq i < j \leq m} E_{ij} \cup \bigcup_{1 \leq i < j \leq n} F_{ij} \right| \\ &= 2^{mn} - \sum_{i=1}^m \sum_{j=1}^n s(m, i) \cdot s(n, j) \cdot 2^{ij} \end{aligned}$$

proof not obvious but not covered. Doesn't seem to generalize to counting the general case of  $X_{m,n}^{(\rho, \sigma)}$ .

Now we take inspiration from the previous method of counting with the inclusion-exclusion principle. Let  $(\rho, \sigma) \in S_m \times S_n$  then

- 1 Denote the collection of matrices which are fixed by  $(\rho, \sigma)$  by  $A$
- 2 The elements of  $A$  correspond to assigning 0 or 1 to each orbit of  $[m] \times [n]$  in the group generated by  $(\rho, \sigma)$ .
- 3 Denote the binary matrices which have row  $i =$  row  $j$  by  $E_{ij}$
- 4 The elements of  $E_{ij}$  correspond to assigning 0 or 1 to each class of the equivalence relation  $\sim$  on  $[m] \times [n]$  where  $(i, x) \sim (j, x)$  for all  $x$  and the other classes have size 1.

The intersection  $\mathcal{E}_{ij} = A \cap E_{ij}$  of matrices fixed by  $(\rho, \sigma)$  with row  $i$  equal to row  $j$  corresponds to the finding the join of these two equivalence relations on  $[m] \times [n]$  and assigning 0 or 1 to each of it's classes. Define  $\mathcal{F}_{ij}$  analogously to  $\mathcal{E}_{ij}$  except for columns.

# Numbers of binary matrices with all rows distinct and all columns distinct up to row and column permutation

	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>
<b>1</b>	2	0	0	0	0	0	0	0
<b>2</b>	0	3	3	1	0	0	0	0
<b>3</b>	0	3	12	19	16	9	4	1
<b>4</b>	0	1	19	94	250	459	649	729
<b>5</b>	0	0	16	250	1796	8623	32016	98097
<b>6</b>	0	0	9	459	8623	100494	881664	6363357
<b>7</b>	0	0	4	649	32016	881664	17422636	277445249
<b>8</b>	0	0	1	729	98097	6363357	277445249	