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ACCELEROMETRY—A TECHNIQUE FOR THE MEASUREMENT OF HUMAN BODY MOVEMENTS*

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Abstract—A summary of the indications for new systems of measurement is given, with particular reference to the advantages and potential hazards in the use of accelerometers. A study of the movement of the shank, or lower leg, using accelerometers is reported. The paper concludes that improved transducers will allow this method to be extended to the study of the movement of other parts of the body. An Appendix shows how the signals from six accelerometers may be used to define completely the movement of a body in space.

INTRODUCTION

MANY bioengineers involved with the study of human movement have at some time attempted to use an accelerometer for that quantitative measure of that movement. Some of the attempts have been reported (Saunders *et al.*, 1953; Gage, 1964) but, certainly, by far a larger number are remembered only as failures. The probable reasons for these failures are worth examining, because, in many areas, the potential advantages of accelerometry over kinephotography (Sutherland *et al.*, 1972), electrogoniometry (Kettelkamp *et al.*, 1970) and other current methods appear numerous.

Probably the commonest cause of failure with the method is the use of an unsuitable transducer. Most applied mechanics laboratories possess a piezoelectric accelerometer designed for the study of vibration and it is usually this device which is first used by the experimenter. Since these devices might more properly be called 'jerkometers', needing as they do a charge-integrating amplifier in order to measure acceleration, the benefit of their small size is often lost. Furthermore, the absence of a true steady-state response and the low sensitivity of such devices make them wholly unsuitable for the examination of muscular-controlled movements.

Inertial guidance systems use transducers of the 'force-feedback' type with a steady-state response and high sensitivity but again the bulk of the associated electronics and the enormously high price of such accelerometers makes them unsuitable. Strain-gauge accelerometers which deform elastically due to inertial force are certainly the most suitable type of transducer, and these are available at low cost in a variety of configurations. A cantilever type with semiconductor strain elements was used in the experiments reported here.

Another reason why accelerometry has not been more widely used in biomechanics is the widespread misconception that, by analogy with inertial guidance, gyroscopes are needed for the measurement of angular movement. The true situation is quite different. Gyroscopes are used in aerospace inertial guidance precisely because the movements are largely translational, and rotations are small and slow, and therefore difficult to measure. In gait, however, the acceleration of a point of the leg is normally due largely to rotational movements with the translational components becoming large only when the system is changing the number of its degrees of freedom by contact with the external environment. It is

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sufficient that six simultaneous independent measurements are made of the translational components of the movement of a rigid body, moving unconstrained in three-dimensional space, for the movement of that body to be determined absolutely with respect to a reference coordinate system. These six measurements can all be made with accelerometers. (See Appendix).

The study of all animal movement is complex both analytically and numerically. The analysis may be simplified by approximations such as rigid-body assumptions, but the numerical effort involved is always considerable. The use of digital computation is therefore clearly indicated. Furthermore, biological data is often ill-ordered and unpredictable, and the ability of the experimenter to interact with the automatic computational process in order to make algorithmically complex decisions contributes a great saving of time and effort.

The particular study reported here involves the examination of the movement of the lower leg, or shank. The aim was to develop a system of measurement suitable for both experimental and clinical use which could be operated simply and with minimal disturbance of gait in situations outside the biomechanics laboratory.

A well recognised difficulty associated with the kinephotographic measurement of gait has been that encountered in the single and double differentiation of position data with respect to time. There are numerous sources of noise at the upper end of the frequency spectrum of the data (Gutewort, 1971). Since these noise components are preferentially increased by time-differentiation, the signal-to-noise ratio of the data is decreased. Whenever accelerations have been obtained from photographic position data, relatively severe mathematical filtering has been employed (Paul, 1965) so that the transfer function of the differentiation process has a frequency spectrum as shown in Fig. 1. The break-point, ω_0, is chosen as the highest frequency compatible with subjectively noise-free velocity and acceleration. The value of ω_0, which would allow true differentiation of the whole signal band is almost certainly higher than that normally used.

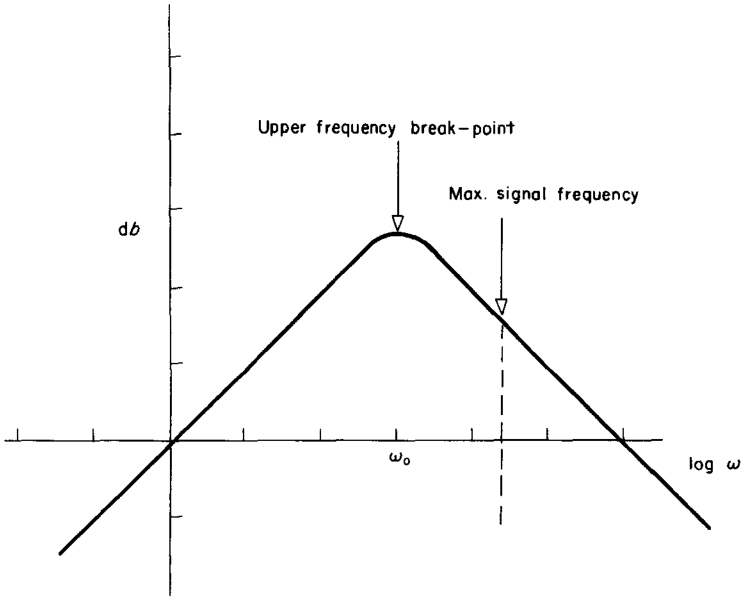

Fig. 1. Band-limited Time-differentiation, showing a possible discrepancy between the differentiation break-point and the maximum significant signal frequency.

An analogous noise problem occurs when acceleration is measured and then integrated to determine velocity and position. The noise frequency which is then the most significant is the lowest. However, in gait analysis, the lowest signal frequency of interest is known to be the double-step frequency. Cutting-off frequency components below this value prevents the determination of the moving body's absolute position in space. But more important is the knowledge of movements within a double-step and this can be determined with considerable accuracy.

METHOD

A detailed algebraic description of the analysis procedure is given in the Appendix. Only such details as are necessary to give a description of the technique are included in this section. Accelerometers of the type shown in Fig. 2 are used to obtain data on the accelerations of the leg between knee and ankle. The net acceleration field, due to movement and gravitation, causes the cantilever to bend in the plane of the sensitive axis. Mean signal-to-noise ratio within the signal frequency band is better than 40 db. Five accelerometers are mounted on the perspex platform shown in Fig. 3. No attempt is made to measure transverse rotations of the shank. Such measurements would require a larger dimension of the platform in the plane normal to the long axis of the accelerometer platform. Since such rotations are relatively small (Levens *et al.*, 1948), they may reasonably be assumed to be zero. However, results so far obtained with five transducers are good enough to suggest that the inclusion of a sixth to allow measurement of transverse rotations would be justified and this improvement is planned.

The platform is mounted over the flat, antero–medial surface of the tibia. Silicone rubber caulking is coated onto the contact side of the platform to provide a high friction interface and the platform is held in place by a moulded "Plastazote" cast also coated with silicone rubber. The effect of the mounting technique is to provide heavy mechanical damping between the accelerometers and the shank.

Signals from the accelerometers can be recorded either on a portable subject-carried tape recorder, or passed by a lightweight cable to a fixed recorder.

The entire analysis of the signals is done on a small interactive digital computer with analogue input facilities and a visual display. The data is first searched visually for an event of particular interest, and a period of 2·56 sec real-time data is selected and sampled at 10 msec. intervals, and digitised. One cycle of any periodic function can be clearly recognised on the computer v.d.u. and cursors set to mark the beginning and end of a cycle. Such

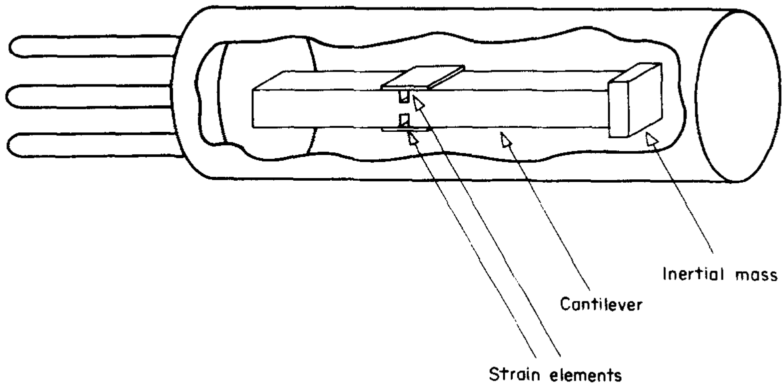

Fig. 2. A schematic representation of the type of accelerometer used. (The length of the cylindrical casing is 14 mm).

functions (Fig. 4a) are then filtered mathematically (Fig. 4b) to make their values equal at the beginning and end of the cycle. This process removes drift and sets a lower frequency limit on the signal pass-band corresponding to the double-step period.

The stance phase of the walking cycle can generally be divided into three distinct periods. The first, immediately following "heel-strike", is short and lasts until the foot is flat on the ground. The second, or "foot-flat" period lasts until "heel-off" and is of approximately the same duration as the third period, which lasts from "heel-off" until "toe-off". During the second period the shank of the leg moves with pure rotation about a point whose position is known. This point lies within the talus between the axes of the ankle and subtalar joints (Wright *et al.*, 1964). Since the position of this origin of rotation is known, the knowledge of the motion of the shank becomes mathematically redundant. The six parameters of motion which are measured or assumed to be zero are not then independent. Their interdependence allows the angular position of the leg with respect to the fixed axes to be calculated. If the start of the cycle of shank movement is chosen to occur during the 'foot-flat' period of the stance phase, the initial conditions needed to solve the simultaneous-differential equations for the instantaneous direction-cosine matrix are known. Figure 5 shows a display of the direction cosine matrix for a typical walking cycle.

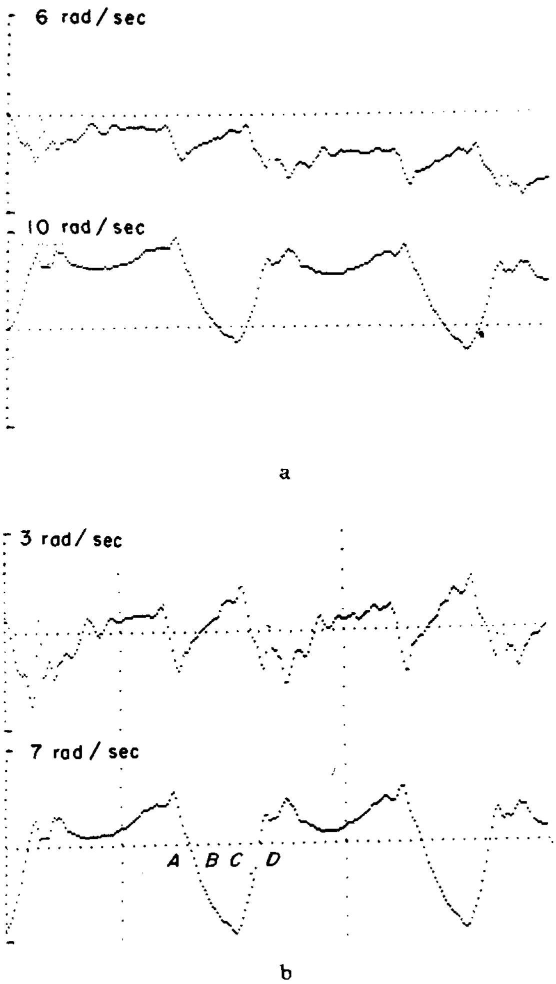

Fig. 4a, b. A period of 2·56 sec of angular velocity data before and after filtering. The upper and lower traces in each picture show coronal and sagittal plane rotations, respectively.

With the solution for the direction cosine matrix available for each sample time, it is possible to solve for the translational components of the limb's movement. The angle which the axis of each transducer makes with the vertical is known, and the component of

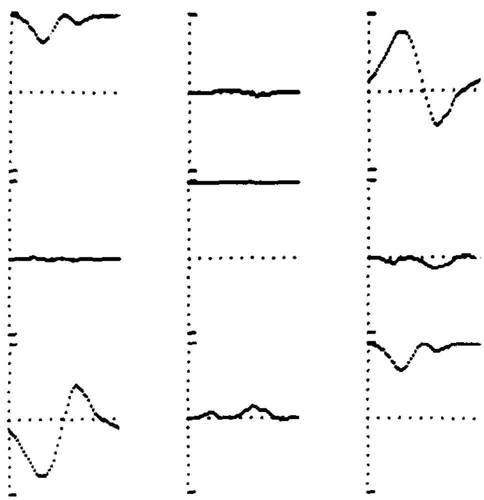

Fig. 5. The direction cosine matrix of the shank for one step, corresponding to time between the cursors in Figs. 4a and b.

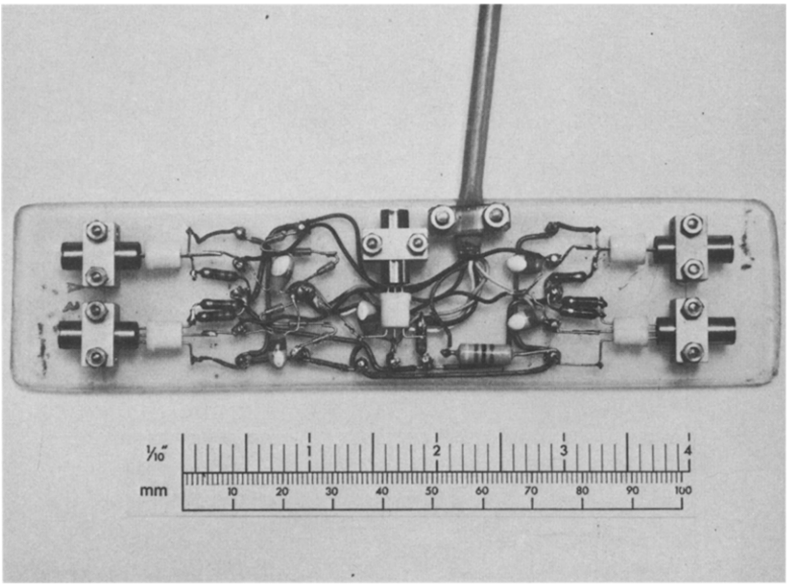

Fig. 3. The accelerometer mounting platform, showing five accelerometers and other associated electrical components.

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the gravity vector measured by each transducer can be calculated and subtracted from the net measured acceleration field. An origin is arbitrarily chosen at a fixed position in the moving body and the translational movement of this origin is calculated by double integration of the acceleration of the body referred to this point. The accelerations, velocities and translations of the mid-shank origin are shown in Fig. 6.

Besides the graphical and numerical representation of the measured movement, a true-speed moving picture of the limb element can be simulated on the computer v.d.u. A set of 26 points is used to represent a shank and foot and can be seen moving as though viewed from any desired direction. The effect is similar to viewing a photographic record of the movement of a leg mounted with marker dots. Ankle articulation is simulated simply by preventing the toe from going below the "ground", a line drawn at the lowest excursion of the heel. Otherwise the foot is kept perpendicular to the shank. (This approximation is used only to make the simulation more visually acceptable, since the actual amount of ankle dorsi and plantar flexion is unknown.)

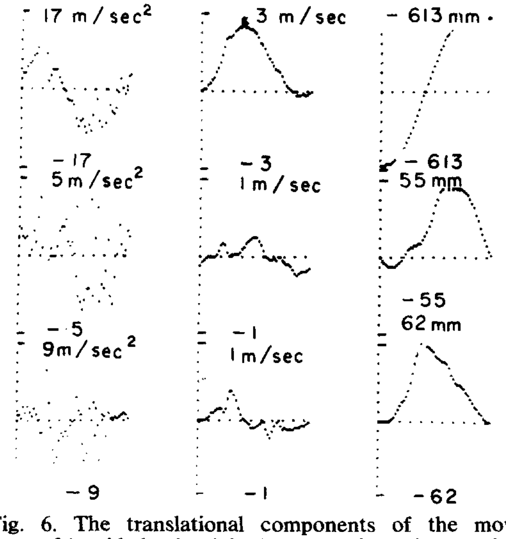

Fig. 6. The translational components of the movement of (a mid-shank origin. The rows, from the top, show forwards-backwards, sideways and vertical movements. Scales indicate units.

Simulated cinephotography is particularly useful in recognising unusual features of gait before returning to study the time-derivative functions in more detail. Figure 7 shows a time-exposure of the movement simulation over one cycle. Since the space coordinates of all 26 points are known, the movement may be displayed in any two-dimensional plane without difficulty, simply by effecting the appropriate coordinate transformation.

CONCLUSION

The purpose of gait studies is twofold. The results obtained from normal subjects have been used to make estimates of the forces in joints, muscles and ligaments (Paul, 1965; Morrison, 1970). This is a highly complex procedure both experimentally and analytically. Very little is known of the kinematics of walking in circumstances other than the laboratory. While kinematic measurements alone do not allow joint forces to be estimated, there are many areas of knowledge, for instance concerning the aetiology of osteo-arthrosis, (Radin *et al.*, 1972), which may only be clarified by gait studies made on subjects in normal environmental conditions. Portable measurement systems, such as that described here are the only way such measurements can be made.

Apart from measurements on normal subjects, gait studies infrequently find a place in clinical orthopaedics. The reason for the lack of acceptance of current methods is that they are either too complex in application, or else are simple but produce very little information. This problem is widely recognised (Sutherland, 1972), and it is hoped that this technique

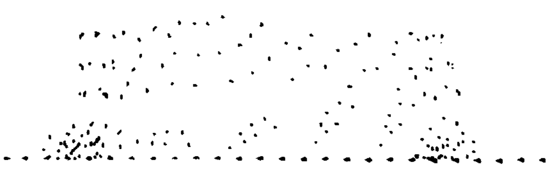

Fig. 7. Multiple exposure of movement simulation. Frame rate: rate: 12·5/sec.

will help overcome some of the current difficulties.

Figures 4b, 5 and 6 show, for the same step, the angular velocities, direction cosines, translational acceleration, velocities and positions of the shank. Comparison studies of a wide variety of types of step indicate that the angular and translational velocities are the functions which reveal most about the gait of the subject. These are shown in Fig. 4b and the centre column of three graphs in Fig. 6. The cycle lasts from foot-flat to foot-flat, passing through a smooth toe-off at A, a muscle driven initial swing-phase to point B and a sinusoidal, free swing-phase to point C. A small positive angular velocity in the saggittal plane, in conjunction with a low vertical velocity, prepares the leg for a low energy heel-strike at point D. These are clear characteristics of normal gait and each of them changes characteristically with specific abnormalities.

The purpose of the study was to show that accelerometers could be used to provide sufficient information to define the movement of a segment of the body. It has been shown that they can be used in this way. A common criticism of the technique has been that it would be impossible to predict the movement of a limb element without some form of surgical bony attachment for the transducers. Clearly, the mounting site used in this case is chosen specifically to minimise the effects of soft tissue movements. By covering the whole area adjacent to the mounting platform with the high-friction cast, the skin movements are spacially integrated and heavily damped. It is thought that, with careful site-selection, and the appropriate form of fixation, non-invasive measurements could be made on the movement of other parts of the body. New transducers, with strain-elements diffused into a semiconductor cantilever substrate, have far better drift characteristics than those at present in use. Using these transducers, it should be possible to reduce low-frequency cut-off point in the signal spectrum by several orders of magnitude. Accelerometers could then be used to study aperiodic movements, for instance of the upper limbs.

SUMMARY

A technique for using accelerometers to study the total movement of the shank of the leg has been developed. Signals from the transducers are analysed on an interactive digital computer. Graphs of angular velocity, direction cosine, translational acceleration, velocity and position are produced as well as a visual representation of the movement. It is hoped to use the technique extensively in a clinical orthopaedic environment and to develop further the method for use in the study of the movement of other parts of the body.

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NOMENCLATURE

0123 a Cartesian coordinate system fixed to a body moving in space
0′1′2′3′ a Cartesian coordinate system used as a fixed reference frame
f an acceleration vector relating point 0 to point 0′
g an acceleration vector defining the gravitational field
r, s position vectors defining the position of points in the 0123 reference frame.
αm, βm inertial measurements of acceleration taken at points in the moving body
α, β accelerations of points in the moving body
ω the angular velocity vector of 0123 w.r.t. 0′1′2′3′ expressed in 0123 coordinates
$\dot{\omega}$ time-derivative of ω
δ_{ij} Kronecker delta
ϵ_{ijk} Levi–Civita density
$[\lambda]$, λ_{pq} direction cosine transformation for vectors from 0123 to 0′1′2′3′ coordinates
$[\dot{\lambda}]$, $\dot{\lambda}_{pq}$ time-derivative of $[\lambda]$, λ_{pq}
$[\lambda]^T$ transform of matrix $[\lambda]$
$[I]$ identity matrix.

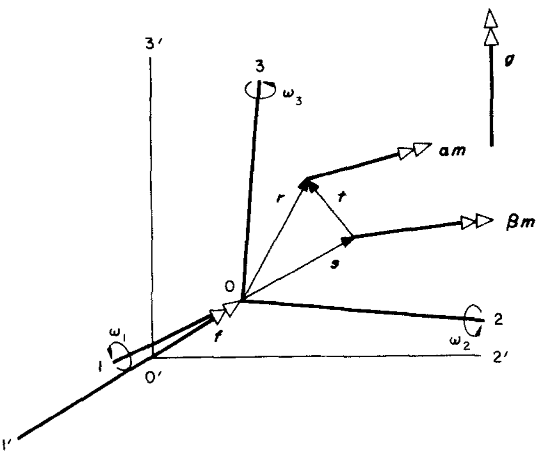

Fig. 8. Points of measurement defined w.r.t. two reference frames. The points of measurement of the components of **αm** and **βm** are not necessarily coincident. (Body-fixed frame: 0123. Fixed reference frame 0′1′2′3′).

APPENDIX

Euler's Theorem states that the general displacement of a rigid body with one point fixed can be achieved by a rotation about a suitable fixed axis through the fixed point. Since the direction of the axis can be defined by two angles and the rotation by a third, the movement should be defined by three independent measurements, for instance by orthogonal measurements of acceleration. Three further independent measurements are needed to define the movement of the 'body-fixed' point with respect to some other frame of reference. These three independent variables may also be measurements of acceleration.

We define 'fixed' reference frames as having an acceleration field, known as the gravitational field, which is a function of position. In order to determine movements within such a reference frame from unreferenced measurements of the local acceleration field, the local gravitational field must be specified. One indirect specification of the gravitational field is the assumption that it does not vary within the dimensions of the body whose movement is to be determined. This specification may be used to calculate the rotation of the body from the signals from three orthogonal parallel pairs of accelerometers.

The acceleration of a point with respect to an origin to which it is fixed is defined, in vector notation, as:

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \tag{1}$$

Since the angular velocity and acceleration do not change with position,

$$\boldsymbol{\beta} = \dot{\boldsymbol{\omega}} \times \boldsymbol{s} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{s}) \tag{2}$$

Unreferenced, or inertial, measurements of acceleration, as indicated above, contain a gravitational component, g, and a component, f, due to the acceleration of the body-fixed origin (See Fig. 8).

$$\therefore \boldsymbol{\alpha m} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \boldsymbol{g} + \boldsymbol{f} \tag{3}$$

and

$$\boldsymbol{\beta m} = \dot{\boldsymbol{\omega}} \times \boldsymbol{s} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{s}) + \boldsymbol{g} + \boldsymbol{f}. \tag{4}$$

Clearly:

$$\boldsymbol{\alpha m} - \boldsymbol{\beta m} = \dot{\boldsymbol{\omega}} \times \boldsymbol{t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{t}) \tag{5}$$

where

$$\boldsymbol{t} = \boldsymbol{r} - \boldsymbol{s}$$

$$\therefore \boldsymbol{\omega} \times \boldsymbol{t} = \int [(\boldsymbol{\alpha m} - \boldsymbol{\beta m}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{t})] + \boldsymbol{C} \tag{6}$$

where $\boldsymbol{C}$ is a constant vector, and the integral is over time. Equation (6) may be rewritten in tensor notation:

$$\omega_l\, {}_it_m\, \epsilon_{ilm} = \int [(\alpha m_i - \beta m_i) - \omega_j \omega_l\, {}_it_m\, \epsilon_{ijk}\, \epsilon_{klm}] + C_i \tag{7}$$

where: $i,j,k,l,m \in C$ (1,2,3) and the integral is over time, or matrix form:

$$\begin{bmatrix} 0 & {}_1t_3 & -{}_1t_2 \\ -{}_2t_3 & 0 & {}_2t_1 \\ {}_3t_2 & -{}_3t_1 & 0 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \int((\alpha m_1 - \beta m_1) + {}_1t_1(\omega_2^2 + \omega_3^2) - {}_1t_2\omega_1\omega_2 - {}_1t_3\omega_3\omega_1) + C_1 \\ \int((\alpha m_2 - \beta m_2) + {}_2t_2(\omega_3^2 + \omega_1^2) - {}_2t_3\omega_2\omega_3 - {}_2t_1\omega_1\omega_2) + C_2 \\ \int((\alpha m_3 - \beta m_3) + {}_3t_3(\omega_1^2 + \omega_2^2) - {}_3t_1\omega_3\omega_1 - {}_3t_2\omega_2\omega_3) + C_3 \end{Bmatrix} \tag{8}$$

where: ${}_pt_q$ is the qth component of t for measurements in pth direction and integrals are over time.

The only limitation on the points of measurement is that the measurement axes are orthogonal. Various simplifications of equations (6, 7 and 8) are possible by making certain components of t equal to zero. For instance, collinearity makes ${}_pt_q = {}_pt_q\delta_{pq}$ where δ_{pq} is the Kronecker delta. This simplifies the equations to a non-differential form, but this form, while simpler to solve, may give a less accurate solution if $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{t}) \ll \boldsymbol{g}$.

Equation (7) can be solved numerically and a description of the possible methods is not appropriate here except to say that in this particular case ${}_1t_1$, ${}_2t_2$, ${}_3t_3$, ${}_1t_2$ and ${}_2t_1$ were made equal to zero and ω_3 was also assumed to be zero. A simple, first-order, trapezoidal integration procedure was found to be adequate.

Once the angular velocity vector is known, the solution of the movement of the rigid body is that conventionally used in 'strapped-down' inertial navigation systems (Bortz, 1970). The direction cosine transformation, λ, is found from the matrix equation:

$$[\dot{\lambda}] = [\omega][\lambda] \tag{9}$$

which, again, may be rewritten:

$$\dot{\lambda}_{pq} = \omega_s \lambda_{rq} \epsilon_{psr}. \tag{10}$$

The method of solution of this set of nine equations for the components of λ is, as before, based on simple trapezoidal integration.

Knowledge of the direction cosines relating the body-fixed reference frame to the fixed frame allows the measured acceleration field from a single orthogonal triad of transducers to be transformed into the fixed coordinate system. Since the gravitational field is specified in terms of this coordinate system, this component may be subtracted from the transformed, measured field in order to determine the translational acceleration of the body-fixed origin with respect to the fixed reference frame. Simple integrations then yield the velocity and position of this origin.

Thus the movement of the body with 6 degrees-of-freedom relative to a fixed reference frame can be calculated from 6 acceleration field measurements made on the body.

Two further points of importance should be mentioned here. The first concerns the estimation of initial conditions in the integral equations. Separately determined constraints are always required for such estimations. Conventionally in inertial navigation systems, these initial conditions are found from exterior measurements of initial acceleration, velocity and position. In this case, however, such information is not complete, but, as is mentioned in the main text, the cyclic nature of the movement allows the initial angular velocity and rotation, translational acceleration and velocity to be made equal to the final values of these functions. The initial position of the leg is of importance only in a rotational sense (in order that the gravitational component may be removed from the acceleration field). As is indicated in the main text, the redundancy of the information on movement during the foot-flat phase of the walking cycle allows the initial direction cosine to be found as follows.

The origin of rotation of the shank during this phase is fixed in the stationary reference frame. The term $\boldsymbol{f}$ in equation (3) is therefore zero, so this equation becomes:

$$\boldsymbol{\alpha m} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \boldsymbol{g}. \tag{11}$$

Since $\dot{\boldsymbol{\omega}}$, $\boldsymbol{\omega}$ and $\boldsymbol{r}$ are known, $\boldsymbol{g}$ may be determined in terms of $\boldsymbol{\alpha m}$. The relative magnitudes of the components of $\boldsymbol{g}$ are related to the fixed-frame gravity vector by three terms in the direction cosine matrix. Referring to Fig. 8; $\lambda_{31} = g_1$; $\lambda_{32} = g_2$ and $\lambda_{33} = g_3$. The remaining six terms relate the directions of the three moving axes to the directions of the two fixed axes perpendicular to the gravity vector. This relationship may be defined quite arbitrarily, most conveniently by making $\lambda_{12} = 0$. This has the effect of making axis 2′ perpendicular to axis 1.

$$\lambda_{22} = +\sqrt{1-\lambda_{32}^2} \tag{12}$$
$$\lambda_{21} = -\lambda_{31}\lambda_{32}/\lambda_{22} \tag{13}$$
$$\lambda_{13} = \lambda_{21}\lambda_{32} - \lambda_{22}\lambda_{31} \tag{14}$$
$$\lambda_{23} = \lambda_{32}\lambda_{13} \tag{15}$$
$$\lambda_{11} = \lambda_{22}\lambda_{33} - \lambda_{23}\lambda_{32}. \tag{16}$$

Thus the initial direction cosine can be found.

The second point of importance is a check on the orthogonality of the λ matrix. If the matrix is orthogonal:

$$[\lambda][\lambda]^T = [I]. \tag{17}$$

If not the matrix may be corrected to orthogonality by the iterative procedure (Jordan, 1969)

$$[\lambda]_{n+1} = [\lambda]_n \left\{ [I] - \frac{1}{2}([\lambda]_n^T[\lambda]_n - [I]) \right\}. \tag{18}$$