ΑΣΚΗΣΕΙΣ ΠΡΟΣ ΛΥΣΗ

ΑΝΑΛΥΣΗ ΙΙ

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ασκ 2/35

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \frac{1}{\sqrt{25-x^2-y^2}}$$

πρέπει $25 - x^2 - y^2 > 0$

$x^2 + y^2 < 25$

είναι το εσωτερικό κύκλου κέντρου $(0,0)$ και $\rho = 5$.

ασκ 4/35

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \sqrt{\sin(x^2+y^2)}$

πρέπει $\sin(x^2+y^2) \geq 0 \iff 2\kappa\pi \leq x^2+y^2 \leq (2\kappa+1)\pi$, $\kappa \in \mathbb{N}$

δηλ. ένωση δακτυλίων.

ασκ 7/35

$f: \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = \ln(x^2+y^2+z^2)$

$f(x,y,z) = c \iff \ln(x^2+y^2+z^2) = c \iff x^2+y^2+z^2 = e^c$

δηλ. σφαίρα κέντρου $(0,0,0)$ και $\rho = e^{c/2}$

ασκ 3/47

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+4} - 2}{x^2+y^2}$$

θέτω $t = x^2+y^2$ όταν $t \to 0$, $(x,y) \to (0,0)$

άρα $\displaystyle\lim_{t\to 0} \frac{\sqrt{t+4}-2}{t} = \lim_{t\to 0} \frac{(\sqrt{t+4}-2)(\sqrt{t+4}+2)}{t(\sqrt{t+4}+2)} = \lim_{t\to 0} \frac{t+4-4}{t(\sqrt{t+4}+2)} = \frac{1}{4}$

ασκ 5/47

$$f(x,y) = \frac{xy}{x^2+y^2}$$

• $$f(x,\lambda x) = \frac{\lambda x^2}{x^2+\lambda x^2} = \frac{\lambda}{1+\lambda^2}$$

Επομένως, όταν $(x,y)\to(0,0)$ κατά μήκος της ευθείας $y=\lambda x$,

τότε το όριο είναι $\frac{\lambda}{1+\lambda^2}$ δηλ. μεταβάλλεται με το λ, άρα $\nexists$ το όριο.

• $\lim_{y\to 0} f(x,y) = 0$ | $\lim_{x\to 0} f(x,y) = 0$

$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0$ | $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0$

ασκ 7/47

$f:\mathbb{R}^2\to\mathbb{R}^2$

$$f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{\sin(x^2+2y^2)}{x^2+y^2}\right)$$

$y=\lambda x$, $\quad \frac{x}{\sqrt{x^2+\lambda^2x^2}} = \frac{x}{|x|\sqrt{1+\lambda^2}} = \frac{1}{\sqrt{1+\lambda^2}}$ Άρα $\nexists$

ασκ 6/69

$$f(x,y) = \begin{cases} 0 & , x=0 \ \& \ y=0 \\ x\sin\frac{1}{y} & , y\neq 0 \end{cases}$$

Για να είναι συνεχής, θα πρέπει $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \Leftrightarrow \lim_{(x,y)\to(0,0)} x\sin\frac{1}{y} = 0$

$$\left|x\cdot\sin\frac{1}{y}\right| = |x|\cdot\left|\sin\frac{1}{y}\right| \leq |x| \xrightarrow{(x,y)\to(0,0)} 0$$

Άρα $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ και άρα, συνεχής.

ασκ 8/69

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} \tan(x+y) & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Η παράσταση $\dfrac{xy}{x^2+y^2}$ είναι φραγμένη κοντά στο $(0,0)$ διότι ισχύει

$2|x|\cdot|y| \leq x^2+y^2$ και έχουμε ότι $\dfrac{|x|\cdot|y|}{x^2+y^2} \leq \dfrac{1}{2}$.

Επίσης $\lim\limits_{(x,y)\to(0,0)} \tan(x+y) = 0$

Άρα $\lim \left[\dfrac{xy}{x^2+y^2} \cdot \tan(x+y) \right] = 0$, άρα συνεχής.

ασκ 7/69

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{(1+y^2)\sin x}{x} & , x \neq 0 \\ a & , x = 0 \end{cases}$$

• $\lim\limits_{(x,y)\to(0,0)} \dfrac{(1+y^2)\cdot \sin x}{x} = \lim\limits_{(x,y)\to(0,0)} (1+y^2)\cdot \dfrac{\sin x}{x} = \lim\limits_{y\to 0} (1+y^2)\cdot\left(\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right) =$

$= \lim\limits_{y\to 0} (1+y^2)\cdot 1 = 1$

Άρα $a = 1$

ασκ 9/69

$f:\mathbb{R}^3\setminus\{0,0,0\}\to\mathbb{R}$

$$f(x,y,z)=\frac{x\sin x+y\sin y+z\cdot\sin z}{x^2+y^2+z^2}$$

(Υπάρχει η λύση στο βιβλίο)

ασκ 1/153

$f:\mathbb{R}^2\to\mathbb{R}$

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & ,\ (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$$

- $\dfrac{\partial F}{\partial x}(0,0)=\lim\limits_{x\to0}\dfrac{F(x,0)-F(0,0)}{x}=0$

$$\frac{\partial F}{\partial x}=\frac{\partial}{\partial x}\left(xy\cdot\frac{x^2-y^2}{x^2+y^2}\right)=y\cdot\frac{x^2-y^2}{x^2+y^2}+xy\cdot\frac{2x(x^2+y^2)-(x^2-y^2)\cdot2x}{(x^2+y^2)^2}=$$

$$=\frac{y(x^2-y^2)(x^2+y^2)+xy2x(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2}=\frac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2}$$

- $\dfrac{\partial F}{\partial y}(0,0)=0$, $\dfrac{\partial F}{\partial y}=\dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$

- $\dfrac{\partial^2F}{\partial y\partial x}(0,0)=\lim\limits_{y\to0}\dfrac{\frac{\partial F}{\partial x}(0,y)-\frac{\partial F}{\partial x}(0,0)}{y}=\lim\limits_{y\to0}\dfrac{-y}{y}=-1$

- $\dfrac{\partial^2F}{\partial x\partial y}(0,0)=\lim\limits_{x\to0}\dfrac{\frac{\partial F}{\partial y}(x,0)-\frac{\partial F}{\partial y}(0,0)}{x}=\lim\limits_{x\to0}\dfrac{x}{x}=1$

ασκ 11/154

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = x^2 + xy + y^2$

$$\frac{\partial F}{\partial x} = 2x + y$$

$$\frac{\partial F}{\partial y} = 2y + x$$

Οι μερικές παράγωγοι $\exists$ και είναι συνεχείς.
Επομένως $f(x,y)$ είναι διαφορίσιμη σε κάθε σημείο του $\mathbb{R}^2$.

ασκ 16/155

Θα πρέπει $\dfrac{\partial g}{\partial x} = \dfrac{x+ky}{(x-y)^3}$ (1) και $\dfrac{\partial g}{\partial y} = \dfrac{kx+y}{(x-y)^3}$ (2)

$$G(x,y) \overset{(1)}{=} \int \frac{x+ky}{(x-y)^3}dx = \int \left(\frac{x}{(x-y)^3} + \frac{ky}{(x-y)^3} \right) dx$$

▷ $\displaystyle\int \frac{1}{(x-y)^3}dx = \int \frac{(x-y)'}{(x-y)^3}dx = -\frac{1}{2(x-y)^2}$

▷ $\displaystyle\int \frac{x}{(x-y)^3}dx = -\frac{x}{2(x-y)^2} + \frac{1}{2}\int \frac{1}{(x-y)^2}dx =$

$$= -\frac{x}{2(x-y)^2} - \frac{1}{2}\cdot\frac{1}{x-y} = \frac{-x-x+y}{2(x-y)^2} = \frac{-2x+y}{2(x-y)^2}$$

Άρα, $G(x,y) = \dfrac{-2x+y}{2(x-y)^2} - \dfrac{ky}{2(x-y)^2} = \dfrac{-2x+y-ky}{2(x-y)^2} = \dfrac{-2x+(1-k)y}{2(x-y)^2} + c(y)$

Παραγωγίζω ως προς y, $G_y = \dfrac{(1-k)\cdot 2(x-y)^2 + (-2x+(1-k)y)\cdot 4(x-y)}{4(x-y)^4} + c'(y)$

$$\Rightarrow G_y = \frac{2(1-k)(x-y) + 4\left(-2x+(1-k)y\right)}{4(x-y)^3} + c'(y) \overset{(2)}{\Longrightarrow}$$

$$\Rightarrow \frac{kx+y}{(x-y)^3} = \frac{(1-k)(x-y) + 2(-2x+(1-k)y)}{2(x-y)^3} + c'(y) \Rightarrow$$

$$\Rightarrow \frac{2(kx+y) - (1-k)(x-y) - 2(-2x + (1-k)y)}{2(x-y)^3} = c'(y)$$

$$\Rightarrow \frac{2kx + \cancel{2y} - x + y + kx - ky + 4x - \cancel{2y} + 2ky}{2(x-y)^3} = c'(y)$$

$$\Rightarrow \frac{3kx + ky + 3x + y}{2(x-y)^3} = c'(y)$$

$$\Rightarrow \frac{3x(k+1) + y(k+1)}{2(x-y)^3} = c'(y)$$

$$\Rightarrow \frac{(k+1)(3x+y)}{2(x-y)^3} = c'(y)$$

Βρίσκω το ολοκλήρωμα και τη συνέχεια, την g.

ασκ 17/155

$z = f(x + g(y))$

Λύσ. $z = f(\varphi(x,y))$, $\varphi(x,y) = x + g(y)$

$$\frac{\partial z}{\partial x} = f'(\varphi(x,y))\frac{\partial \varphi}{\partial x} = f'(x + g(y))$$

$$\frac{\partial z}{\partial y} = f'(\varphi(x,y))\frac{\partial \varphi}{\partial y} = f'(x+g(y))\, g'(y)$$

$$\frac{\partial z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(f'(\varphi(x,y))\, g'(y)\right) = g'(y)\frac{\partial}{\partial x} f'(\varphi(x,y)) =$$

$$= g'(y) f''(\varphi(x,y))\frac{\partial \varphi}{\partial x} = f''(x+g(y))\, g'(y)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(f'(\varphi(x,y))\right) = f''(\varphi(x,y))\frac{\partial \varphi}{\partial x} = f''(\varphi(x,y))$$

▷ $\frac{\partial z}{\partial x} \cdot \frac{\partial^2 z}{\partial x \partial y} = f'(x+g(y))\, f''(x+g(y)) \cdot g'(y)$

▷ $\frac{\partial z}{\partial y} \cdot \frac{\partial^2 z}{\partial x^2} = f'(x+g(y))\, g'(y) \cdot f''(\varphi(x,y))$

ασκ 20/155

• $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial w}$

• $\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w}\right) =$

$$= \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right) \cdot \frac{\partial z}{\partial x} + \frac{\partial}{\partial w}\left(\frac{\partial F}{\partial z}\right) \cdot \frac{\partial w}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial w}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial w}\left(\frac{\partial F}{\partial w}\right)\frac{\partial w}{\partial x} =$$

$$= \frac{\partial^2 F}{\partial z^2} + \frac{\partial^2 F}{\partial w \partial z} + \frac{\partial^2 F}{\partial z \partial w} + \frac{\partial^2 F}{\partial w^2} =$$

$$= \frac{\partial^2 F}{\partial z^2} + 2\frac{\partial^2 F}{\partial w \partial z} + \frac{\partial^2 F}{\partial w^2}$$

$$\bullet\ \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial F}{\partial w}\cdot\frac{\partial w}{\partial y} = k\frac{\partial F}{\partial z} - k\frac{\partial F}{\partial w}$$

$$\bullet\ \frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(k\frac{\partial F}{\partial z}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial F}{\partial w}\right) =$$

$$= k\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right)\frac{\partial z}{\partial y} + k\frac{\partial}{\partial w}\left(\frac{\partial F}{\partial z}\right)\frac{\partial w}{\partial y} + (-k)\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial w}\right)\frac{\partial z}{\partial y} + (-k)\frac{\partial}{\partial w}\left(\frac{\partial F}{\partial w}\right)\frac{\partial w}{\partial y} =$$

$$= k^2\frac{\partial^2 F}{\partial z^2} - k^2\frac{\partial^2 F}{\partial w\partial z} - k^2\frac{\partial^2 F}{\partial z\partial w} + k^2\frac{\partial^2 F}{\partial w^2} =$$

$$= k^2\frac{\partial^2 F}{\partial z^2} - 2k^2\frac{\partial^2 F}{\partial w\partial z} + k^2\frac{\partial^2 F}{\partial w^2}$$

$$= k^2\left(\frac{\partial^2 F}{\partial z^2} - 2\frac{\partial^2 F}{\partial w\partial z} + \frac{\partial^2 F}{\partial w^2}\right)$$

Άρα, $k^2\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0 \Leftrightarrow$

$$\Leftrightarrow k^2\left(\cancel{\frac{\partial^2 F}{\partial z^2}} + 2\frac{\partial^2 F}{\partial w\partial z} + \cancel{\frac{\partial^2 F}{\partial w^2}}\right) - k^2\left(\cancel{\frac{\partial^2 F}{\partial z^2}} - 2\frac{\partial^2 F}{\partial w\partial z} + \cancel{\frac{\partial^2 F}{\partial w^2}}\right) = 0$$

$$\Leftrightarrow 4k^2\frac{\partial^2 F}{\partial w\partial z} = 0 \Leftrightarrow \frac{\partial^2 F}{\partial w\partial z} = 0$$

άσκ 21 | 155

$$\bullet\ \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$

$$\bullet\ \frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial u}\left(v\frac{\partial f}{\partial y}\right) =$$

$$= \frac{\partial}{\partial u}(u)\frac{\partial f}{\partial x} + u\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + v\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) =$$

$$= \frac{\partial f}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial u} + u\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial u} + u\cancel{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)}^{0}\frac{\partial z}{\partial u} +$$

$$+ v\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial u} + v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial u} + v\cancel{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)}^{0}\frac{\partial z}{\partial u} =$$

$$= \frac{\partial f}{\partial x} + u^2\frac{\partial^2 f}{\partial x^2} + uv\frac{\partial^2 f}{\partial y\partial x} + vu\frac{\partial^2 f}{\partial x\partial y} + v^2\frac{\partial^2 f}{\partial y^2} =$$

$$= \frac{\partial f}{\partial x} + u^2\frac{\partial^2 f}{\partial x^2} + 2uv\frac{\partial^2 f}{\partial x\partial y} + v^2\frac{\partial^2 f}{\partial y^2}$$

- $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}(-v) + \frac{\partial f}{\partial y}\cdot u$$

- $$\frac{\partial^2 f}{\partial v^2} = \frac{\partial}{\partial v}\left(-v\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial v}\left(u\frac{\partial f}{\partial y}\right) =$$

$$= \frac{\partial}{\partial v}(-v)\cdot\frac{\partial f}{\partial x} - v\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right) + u\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right) =$$

$$= -\frac{\partial f}{\partial x} - v\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial v} - v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial v} - v\cancel{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\frac{\partial z}{\partial v}}^{0} +$$

$$+ u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial v} + u\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial v} + u\cancel{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\frac{\partial z}{\partial v}}^{0}$$

$$= -\frac{\partial f}{\partial x} + v^2\frac{\partial^2 f}{\partial x^2} - u\cdot v\frac{\partial^2 f}{\partial y\partial x} - u\cdot v\frac{\partial^2 f}{\partial x\partial y} + u^2\frac{\partial^2 f}{\partial y^2}.$$

$$= -\frac{\partial f}{\partial x} + v^2\frac{\partial^2 f}{\partial x^2} - 2uv\frac{\partial^2 f}{\partial x\partial y} + u^2\frac{\partial^2 f}{\partial y^2}$$

- $$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial w} = \frac{\partial f}{\partial z}$$

- $$\frac{\partial^2 f}{\partial w^2} = \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial z}\right) = \cancel{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\frac{\partial x}{\partial w}}^{0} + \cancel{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\frac{\partial y}{\partial w}}^{0} + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)\cdot\frac{\partial z}{\partial w} =$$

$$= \frac{\partial^2 f}{\partial z^2}\cdot 1 = \frac{\partial^2 f}{\partial z^2}$$

Άρα, $$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} + (u^2+v^2)\frac{\partial^2 f}{\partial w^2} = \cancel{\frac{\partial f}{\partial x}} + u^2\frac{\partial^2 f}{\partial x^2} + \cancel{2uv\frac{\partial^2 f}{\partial x\partial y}} + v^2\frac{\partial^2 f}{\partial y^2} +$$

$$+\left(-\cancel{\frac{\partial f}{\partial x}} + v^2\frac{\partial^2 f}{\partial x^2} - \cancel{2uv\frac{\partial^2 f}{\partial x\partial y}} + u^2\frac{\partial^2 f}{\partial y^2}\right) +$$

$$+(u^2+v^2)\cdot\frac{\partial^2 f}{\partial z^2} = (u^2+v^2)\frac{\partial^2 f}{\partial x^2} + (u^2+v^2)\frac{\partial^2 f}{\partial y^2} + (u^2+v^2)\frac{\partial^2 f}{\partial z^2} =$$

$$= (u^2+v^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = (u^2+v^2)\cdot 0 = 0$$

άσκηση

Αν $xu = yv = z \cdot w = x^2+y^2+z^2$

Να υπολογιστεί η Jacobian των u, v, w ως προς x, y, z.

Λύση

$$u = \frac{x^2+y^2+z^2}{x}, \quad v = \frac{x^2+y^2+z^2}{y}, \quad w = \frac{x^2+y^2+z^2}{z}$$

$$\frac{\partial u}{\partial x} = \frac{2x^2-(x^2+y^2+z^2)}{x^2} = \frac{x^2-y^2-z^2}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x}, \quad \frac{\partial u}{\partial z} = \frac{2z}{x}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{x^2-y^2-z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{2x}{y} & \frac{y^2-x^2-z^2}{y^2} & \frac{2z}{y} \\ \frac{2x}{z} & \frac{2y}{z} & \frac{z^2-x^2-y^2}{z^2} \end{vmatrix} =$$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} x^2-y^2-z^2 & 2xy & 2xz \\ 2xy & y^2-x^2-z^2 & 2yz \\ 2xz & 2yz & z^2-x^2-y^2 \end{vmatrix} = \ldots$$

(με πράξεις – κανόνας Sarrus προκύπτει το αποτέλεσμα)