ΦΥΛΛΟ ΠΡΟΒΛΗΜΑΤΩΝ 1-Α

ΔΙΑΦΟΡΙΚΕΣ ΕΞΙΣΩΣΕΙΣ

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## Κεφ. 1

I) (1): συνήθης, γραμμ., ομογ., 3ης τάξ., 1ου βαθ.

(2): συνήθης, πλήρως μη γραμμ. 2ης τάξ., 3ου βαθ.

(3): συνήθης, σχεδόν γραμμ., 2ης τάξ.

(4): συνήθης, πλήρως μη γραμμ., 1ης τάξ.

(5): μερική

(6): συνήθης, ημιγραμμ., 4ης τάξ., 1ου βαθ.

(7): συνήθης, ημιγραμμ. ή σχεδόν γραμμ., 1ης τάξ

II) (1): $y^2 = x^2 - cx \xrightarrow{d/dx} 2yy' = 2x - c \xrightarrow{(\cdot x)} 2xyy' = 2x^2 - cx \Rightarrow$

$\Rightarrow 2xyy' = x^2 + x^2 - cx \Rightarrow 2xyy' = x^2 + y^2$

Πρέπει $x^2 - cx \geq 0 \Leftrightarrow x(x-c) \geq 0$

για $c>0$

| | $-\infty$ … 0 | 0 … c | c … |
|---|---|---|---|
| $x$ | $-$ | 0 $+$ | $+$ |
| $x-c$ | $-$ | $-$ | 0 $+$ |
| $x(x-c)$ | $+$ | $-$ | $+$ |

$A = (-\infty, 0] \cup [c, +\infty)$

για $c<0$

| | … c | c … 0 | 0 … |
|---|---|---|---|
| $x$ | $-$ | $-$ | 0 $+$ |
| $x-c$ | $-$ 0 | $+$ | $+$ |
| $x(x-c)$ | $+$ | $-$ | $+$ |

$A = (-\infty, c] \cup [0, +\infty)$

(2): $y = c_1 x \cos(\ln x) + c_2 x \sin(\ln x) \xrightarrow{d/dx}$

$\rightarrow y' = c_1\left(\cos(\ln x) + x\left(-\sin(\ln x)\cdot\frac{1}{x}\right)\right) + c_2\left(\sin(\ln x) + x\cos(\ln x)\cdot\frac{1}{x}\right) \Rightarrow$

$\Rightarrow y' = c_1\left(\cos(\ln x) - \sin(\ln x)\right) + c_2\left(\sin(\ln x) + \cos(\ln x)\right) \xrightarrow{d/dx}$

$\rightarrow y'' = c_1\left(-\sin(\ln x)\cdot\frac{1}{x} - \cos(\ln x)\cdot\frac{1}{x}\right) + c_2\left(\cos(\ln x)\cdot\frac{1}{x} - \sin(\ln x)\cdot\frac{1}{x}\right)$

Άρα $x^2 y'' - xy' + 2y =$

$$= x^2 \cdot \frac{1}{x}\left[c_1\left(-\cancel{\sin(\ln x)} - \cos(\ln x)\right) + c_2\left(-\sin(\ln x) + \cancel{\cos(\ln x)}\right)\right]$$

$$-x\left[c_1\left(\cos(\ln x) - \cancel{\sin(\ln x)}\right) + c_2\left(\sin(\ln x) + \cancel{\cos(\ln x)}\right)\right] + 2y =$$

$$= x\left[-2c_1\cos(\ln x) - 2c_2\sin(\ln x)\right] + 2y =$$

$$= -2x\left(c_1\cos(\ln x) + c_2\sin(\ln x)\right) + 2y = -2y + 2y = 0.$$

Πρέπει $x > 0$

(3): $y = \ln\left[\cos(x-a)\right] + b$

$$y' = \frac{1}{\cos(x-a)} \cdot \left(-\sin(x-a)\right) = -\frac{\sin(x-a)}{\cos(x-a)} = -\tan(x-a)$$

$$y'' = -\frac{1}{\cos^2(x-a)}$$

Άρα, $y'' + (y')^2 + 1 = -\dfrac{1}{\cos^2(x-a)} + \left(-\tan(x-a)\right)^2 + 1 =$

$$= -\frac{1}{\cos^2(x-a)} + \tan^2(x-a) + 1$$

$$= -\frac{1}{\cos^2(x-a)} + \frac{1}{\cos^2(x-a)} = 0$$

Πρέπει $\cos(x-a) > 0$

(4) : $y^2-1 = (x+2)^2 \xrightarrow{d/dx} 2yy' = 2(x+2) \overset{(1)}{\longrightarrow} yy' = x+2 \xrightarrow{(\cdot x)}$

$\longrightarrow xyy' = (x+2)x \; (2)$

Άρα $y^2-1 = (2y+xy)y' \Leftrightarrow y^2-1 = 2yy' + xyy'$

οπότε από (1)+(2) $\Rightarrow 2yy' + xyy' = 2(x+2) + x(x+2) \Rightarrow$

$\Rightarrow (2y+xy)y' = (2+x)(x+2) \Rightarrow (2y+xy)y' = (x+2)^2 \Rightarrow$

$\Rightarrow (2y+xy)y' = y^2-1$

Πρέπει, $y^2-1 = (x+2)^2 \Leftrightarrow y^2 = 1+(x+2)^2$ ισχ.

$1+(x+2)^2 \geq 0$, ισχ. $\forall x \in \mathbb{R}$

(5) $x^2y^2 - \sin x = c \xrightarrow{d/dx} 2xy^2 + x^2 \cdot 2yy' - \cos x = 0 \Rightarrow$

$\Rightarrow 2xy^2 - \cos x = -x^2 2yy' \Rightarrow 2x^2yy' = \cos x - 2xy^2 \Rightarrow$

$\Rightarrow 2y' = \dfrac{\cos x - 2xy^2}{x^2 y} \Rightarrow 2y' = x^{-2} \cdot y[\cos x - 2xy^2]$

Θα πρέπει $x \neq 0, y \neq 0$.

Κεφ.2

I) (1): $2y' - y^3\cos x = 0$

$2\dfrac{dy}{dx} = y^3 \cos x$

$\dfrac{2dy}{y^3} = \cos x \, dx$

$2\int \frac{1}{y^3}dy = \int \cos x \, dx$

$2\left[\frac{y^{-3+1}}{-3+1}\right] = \sin x + c$

$-\frac{1}{y^2} = \sin x + c$

$\frac{1}{y^2} = -\sin x + c$, γεν. λύση

Και πρέπει $y \neq 0$

Η $y=0$ είναι ιδιάζουσα λύση.

(2): $y' = \dfrac{(x-1)y^5}{x^2(2y^3-1)}$

$$\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-1)}$$

$$\frac{2y^3-1}{y^5}\,dy = \frac{x-1}{x^2}\,dx$$

$$\int\left(\frac{2}{y^2} - \frac{1}{y^5}\right)dy = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$$

και πρέπει $x \neq 0$

$y \neq \dfrac{1}{\sqrt[3]{2}}$

$x^2(2y^3-1)dy = (x-1)y^3 dx$

Άρα η $y=0$ είναι ιδιαζ. λύση

$$-\frac{2}{y} + \frac{1}{4y^4} = \ln|x| + \frac{1}{x} + c$$

(3): $y' = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0)=-1$

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$$

$$2(y-1)\,dy = (3x^2+4x+2)\,dx$$

$$2\int(y-1)\,dy = \int(3x^2+4x+2)\,dx$$

$$2\left(\frac{y^2}{2} - y\right) = x^3+2x^2+2x+c$$

$$y^2-2y = x^3+2x^2+2x+c$$

Για $x=0$, $y^2(0) - 2y(0) = c$

$1 - 2 = c \Rightarrow c = -1$

Άρα $y^2 - 2y = x^3 + 2x^2 + 2x - 1$.

(4): $2xy\,dx + (x^2+y^2)dy = 0$, $y(2)=2$

$M = 2xy \longrightarrow M_y = 2x$

$N = x^2+y^2 \longrightarrow N_x = 2x$

πλήρης

Αναζητώ $f(x,y)$ ώστε

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= M(x,y)\\ \frac{\partial F}{\partial y} &= N(x,y)\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\frac{\partial F}{\partial x} &= 2xy \quad (1)\\ \frac{\partial F}{\partial y} &= x^2+y^2 \quad (2)\end{aligned}\right.$$

Για $y(2)=2$, $c = \dfrac{32}{3}$

δηλ. $3x^2y + y^3 = 32$

$(1) \Rightarrow \int \partial F = \int 2xy\,dx \Rightarrow F(x,y) = y\int 2x\,dx = x^2y + c(y) \xrightarrow[\text{ως προς } y]{\text{Αφ/ζω}} F_y = x^2 + c'(y) \Rightarrow$

$\overset{(2)}{\Rightarrow} \cancel{x^2}+y^2 = \cancel{x^2} + c'(y) \Rightarrow c'(y) = y^2 \Rightarrow c(y) = \dfrac{y^3}{3}$

Άρα $f(x,y) = x^2y + \dfrac{y^3}{3}$ και το γεν. ολοκλ. $x^2y + y^3/3 = c$

(5): $2xe^{2y}dx + 2(1+x^2e^{2y})\,dy = 0 \quad , y(2)=0$

$$\left.\begin{array}{l} M = 2xe^{2y} \longrightarrow M_y = 2x\cdot 2\cdot e^{2y} = 4xe^{2y} \\ N = 2(1+x^2e^{2y}) \longrightarrow N_x = 2\cdot 2x\cdot e^{2y} = 4xe^{2y} \end{array}\right\} \text{πλήρης}$$

Αναζητώ $f(x,y)$ ώστε

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = M(x,y) \\ \dfrac{\partial F}{\partial y} = N(x,y) \end{array}\right\} \Rightarrow \begin{cases} \dfrac{\partial F}{\partial x} = 2xe^{2y} & (1) \\ \dfrac{\partial F}{\partial y} = 2(1+x^2e^{2y}) & (2) \end{cases}$$

$(1) \Rightarrow \int \partial F = \int 2xe^{2y}dx = e^{2y}\int 2x\,dx = x^2e^{2y} + c(y) \Rightarrow$

$\Rightarrow F(x,y) = x^2e^{2y} + c(y) \xrightarrow[\text{ως προς } y]{\text{παραγ.}} F_y = 2x^2e^{2y} + c'(y) \overset{(2)}{\Rightarrow}$

$\Rightarrow 2(1+x^2e^{2y}) = 2x^2e^{2y} + c'(y) \Rightarrow 2 + \cancel{2x^2e^{2y}} = \cancel{2x^2e^{2y}} + c'(y) \Rightarrow$

$\Rightarrow c'(y) = 2 \Rightarrow c(y) = 2y$

Άρα $f(x,y) = x^2e^{2y} + 2y$

και το γεν. ολοκλ. $x^2e^{2y} + 2y = c$

για $y(2)=0$, $\quad 2^2\cdot e^{2\cdot y(2)} + 2\cdot y(2) = c \Rightarrow$

$\Rightarrow 4e^0 + 2\cdot 0 = c \Rightarrow c = 4$

Άρα $x^2e^{2y} + 2y = 4$

(6): $\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{dy}{dx} = 0 \Rightarrow \left(3x+\frac{6}{y}\right)dx + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)dy = 0$

$$\left.\begin{array}{l} M = 3x + \dfrac{6}{y} \longrightarrow M_y = -\dfrac{6}{y^2} \\ N = \dfrac{x^2}{y} + 3\dfrac{y}{x} \longrightarrow N_x = \dfrac{2x}{y} - 3\dfrac{y}{x^2} \end{array}\right| \text{μη πλήρης}$$

Έστω $\mu = \mu(xy)$

$$\frac{M_y - N_x}{yN - xM} = \frac{-\frac{6}{y^2} - \frac{2x}{y} + \frac{3y}{x^2}}{x^2 + \frac{3y^2}{x} - 3x^2 - \frac{6x}{y}} = \frac{\frac{-6x^2 - 2x^3y + 3y^3}{x^2y^2}}{-2x^2 + \frac{3y^2}{x} - \frac{6x}{y}} = \frac{\frac{-6x^2 - 2x^3y + 3y^3}{x^2y^2}}{\frac{-2x^3y + 3y^3 - 6x^2}{xy}} =$$

$$= \frac{(\cancel{-6x^2 - 2x^3y + 3y^3})\cdot \cancel{xy}}{(\cancel{-6x^2 - 2x^3y + 3y^3})\cdot x^{\cancel{2}}y^{\cancel{2}}} = \frac{1}{xy}$$

Οπότε $\frac{d\mu}{\mu} = \frac{1}{xy}d(xy) \Rightarrow \ln|\mu| = \ln|xy| \Rightarrow |\mu| = |xy| \Rightarrow \mu = \pm xy$

Έστω $\mu = xy$

Τότε, $xy\left(3x + \frac{6}{y}\right) + xy\left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{dy}{dx} = 0 \Rightarrow$

$$\Rightarrow (3x^2y + 6x)\,dx + (x^3 + 3y^2)\,dy = 0$$

$$\left.\begin{array}{l} M' = 3x^2y + 6x \longrightarrow M'_y = 3x^2 \\ N' = x^3 + 3y^2 \longrightarrow N'_x = 3x^2 \end{array}\right\} \text{ακριβής}$$

Αναζητώ $f(x,y)$ ώστε:

$$\left\{\begin{array}{l} \dfrac{\partial F}{\partial x} = M' \\[2ex] \dfrac{\partial F}{\partial y} = N' \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} \dfrac{\partial F}{\partial x} = 3x^2y + 6x & (1) \\[2ex] \dfrac{\partial F}{\partial y} = x^3 + 3y^2 & (2) \end{array}\right\}$$

$$(2) \Rightarrow \int dF = \int (x^3 + 3y^2)\,dy \Rightarrow F(x,y) = \int x^3 dy + \int 3y^2 dy = x^3y + y^3 + c(x)$$

παραγωγίζω ως προς $x$, $F_x = 3x^2y + c'(x) \overset{(1)}{\Rightarrow} 3\cancel{x^2y} + 6x = c'(x) + \cancel{3x^2y} \Rightarrow c'(x) = 6x$

$$\Rightarrow c(x) = 3x^2$$

Άρα $f(x,y) = x^3y + y^3 + 3x^2$ και η γεν. λύση. $x^3y + y^3 + 3x^2 = C$

(7): $(x^2 - y^2 - y)\,dx + (x^2y^4 - x)\,dy = 0$

$$\left.\begin{array}{l} M = x^2 - y^2 - y \longrightarrow M_y = -2y - 1 \\ N = x^2y^4 - x \longrightarrow N_x = 2xy^4 - 1 \end{array}\right\} \text{μη ακριβής}$$

Δοκιμή για $\mu = \mu(?)$

$(8)$: $(2x-2y-x^2+2xy)dx+(2x^2-4xy-2x)dy=0$

$M=2x-2y-x^2+2xy \longrightarrow M_y=-2+2x=2(x-1)$

$N=2x^2-4xy-2x \longrightarrow N_x=4x-4y-2=2(2x-2y-1)$

Δουλεύω για $u=2y-x$, $u_x=-1$, $u_y=2$

$$\frac{1}{\mu}\frac{d\mu}{du}=\frac{M_y-N_x}{Nu_x-Mu_y}=\frac{2(2y-x)}{(2x^2-4xy-2x)\cdot(-1)-2(2x-2y-x^2+2xy)}=$$

$$=\frac{2(2y-x)}{-\cancel{2x^2}+\cancel{4xy}+2x-4x+4y+\cancel{2x^2}-\cancel{4xy}}=\frac{2(2y-x)}{2(2y-x)}=1\Rightarrow\frac{d\mu}{\mu}=du\Rightarrow$$

$\Rightarrow \ln|\mu|=u\Rightarrow|\mu|=e^u\Rightarrow\mu=\pm e^u$ και επιλέγω $\mu=e^{2y-x}$

οπότε $e^{2y-x}(2x-2y-x^2+2xy)dx+e^{2y-x}(2x^2-4xy-2x)dy=0$

$$\left.\begin{aligned}M'_y&=e^{2y-x}\left[2(2x-2y-x^2+2xy)+(-2+2x)\right]=e^{2y-x}\left[6x-4y-2x^2+4xy-2\right]\\ N'_x&=e^{2y-x}\left[(-1)(2x^2-4xy-2x)+(4x-4y-2)\right]=e^{2y-x}\left[-2x^2+4xy+6x-4y-2\right]\end{aligned}\right\}\text{ακριβής}$$

Άρα αναζητώ $f(x,y)$ με

$$\frac{\partial F}{\partial x}=M\rightarrow\frac{\partial F}{\partial x}=e^{2y-x}(2x-2y-x^2+2xy)\quad(1)$$

$$\frac{\partial F}{\partial y}=N\rightarrow\frac{\partial F}{\partial y}=e^{2y-x}(2x^2-4xy-2x)\quad(2)$$

$(1)\rightarrow F(x,y)=\int e^{2y-x}(2x-2y-x^2+2xy)dx=-\int(e^{2y-x})'(2x-2y-x^2+2xy)dx=$

$=-e^{2y-x}(2x-2y-x^2+2xy)-\int(e^{2y-x})'(2-2x+2y)dx=$

$=-e^{2y-x}(2x-2y-x^2+2xy)-e^{2y-x}(2-2x+2y)+\int e^{2y-x}(-2)dx=$

$=-e^{2y-x}(\cancel{2x}-\cancel{2y}-x^2+2xy+2-\cancel{2x}+\cancel{2y})+2\int(e^{2y-x})'dx=$

$=-e^{2y-x}(-x^2+2xy+2)+2e^{2y-x}+c(y)$

$=e^{2y-x}(x^2-2xy-\cancel{2}+\cancel{2})+c(y)=e^{2y-x}(x^2-2xy)+c(y)\xrightarrow[\text{ως προς } y]{\text{παραγωγίζω}}$

$\Rightarrow F_y=2e^{2y-x}(x^2-2xy)+e^{2y-x}(-2x)+c'(y)\Rightarrow F_y=e^{2y-x}(2x^2-4xy-2x)+c'(y)\Rightarrow$

$\Rightarrow F_y = e^{2y-x}(2x^2-4xy-2x) + c'(y) \overset{(2)}{\Rightarrow}$

$$\Rightarrow e^{2y-x}(2x^2-\cancel{4xy-2x}) = e^{2y-x}\cancel{(2x^2-4xy-2x)} + c'(y) \Rightarrow c'(y)=0$$

$$\Rightarrow c(y) = c_1$$

Άρα $f(x,y) = e^{2y-x}(x^2-2xy) + c_1$ και το γεν. ολοκλ.

$$e^{2y-x}(x^2-2xy) = c$$

(9): $y' - \frac{y}{x} + \frac{y^2}{x} = 0$

$y' - \frac{1}{x}y = -\frac{1}{x}y^2$ (*) (μορφή Bernoulli $n=2$)

Κάνουμε αλλαγή μεταβλητής $u = y^{1-2} = y^{-1} = \frac{1}{y}$, $y \neq 0$

$u' = -\frac{y'}{y^2}$

πολ/ζω την (*) με $-\frac{1}{y^2}$:

$$-\frac{y'}{y^2} + \frac{1}{x}\cdot\frac{1}{y} = \frac{1}{x} \Rightarrow u' + \frac{1}{x}\cdot u = \frac{1}{x}, \quad x>0$$

$$\Rightarrow e^{\ln x}u' + e^{\ln x}\cdot\frac{1}{x}\cdot u = \frac{1}{x}\cdot e^{\ln x} \Rightarrow (xu)' = (x)' \Rightarrow xu = x + c \Rightarrow$$

$$\Rightarrow u = 1 + \frac{c}{x} \Rightarrow \frac{1}{y} = 1 + \frac{c}{x} \Rightarrow \frac{1}{y} = \frac{x+c}{x} \Rightarrow y = \frac{x}{x+c}$$

▷ Εδώ η $y=0$ είναι ιδιάζουσα λύση της ΔΕ

(10): $x^2y' - xy = e^x y^3$

$\overset{x\neq 0}{\Rightarrow}\quad y' - \frac{1}{x}y = \frac{e^x}{x^2}\cdot y^3 \quad (*) \quad$ (Bernoulli με $n=3$)

Αλλαγή μεταβλητής $\quad u = y^{1-3} = y^{-2} = \frac{1}{y^2}\ , \ y\neq 0$

$u' = -\frac{2y'}{y^3}$

Πολ/ζω την $(*)$ με $-\frac{2}{y^3}$ :

$-\frac{2y'}{y^3} + \frac{1}{x}\cdot\left(-\frac{2}{y^2}\right) = \frac{e^x}{x^2}\cdot(-2)$

$u' + \frac{2}{x}\cdot u = -\frac{2e^x}{x^2} \Rightarrow x^2\cdot u' + 2x\cdot u = -2e^x \Rightarrow$

$\Rightarrow x^2\cdot u' + (x^2)'\cdot u = -2e^x \Rightarrow (x^2\cdot u)' = (-2e^x)' \Rightarrow$

$\Rightarrow x^2\cdot u = -2e^x + c \Rightarrow x^2\cdot\frac{1}{y^2} = -2e^x + c \Rightarrow \frac{1}{y^2} = -\frac{2e^x}{x^2} + \frac{c}{x^2}$

(11): $y' = \frac{2\cos^2 x - \sin^2 x + y^2}{2\cos x}\ , x>0 \quad , \ y_1(x) = \sin x$

$y' = \frac{1}{2\cos x}\cdot y^2 + \frac{2\cos^2 x - \sin^2 x}{2\cos x}$ (για να θυμίζει Ricatti)

Θέτω $\quad u = \frac{1}{y-\sin x} \Leftrightarrow y = \frac{1}{u} + \sin x \quad$ και $\quad y' = -\frac{u'}{u^2} + \cos x$

οπότε,

$-\frac{u'}{u^2} + \cancel{\cos x} = \frac{1}{2\cos x}\cdot\left(\frac{1}{u^2} + \cancel{\sin^2 x} + \frac{2\sin x}{u}\right) + \cancel{\cos x} - \cancel{\frac{\sin^2 x}{2\cos x}} \Rightarrow$

$\Rightarrow -\frac{u'}{u^2} = \frac{1}{2\cos x}\cdot\frac{1}{u^2} + \frac{\tan x}{u} \Rightarrow u' + \tan x\cdot u = -\frac{1}{2\cos x}$

Πολ/ζω με $\quad e^{-\ln|\cos x|} = \frac{1}{\cos x}\quad$ , για $x>0$ με $\cos x > 0$

$\frac{1}{\cos x}\cdot u' + \frac{\sin x}{\cos^2 x}\cdot u = -\frac{1}{2}\cdot\frac{1}{\cos^2 x}$

$\left(\frac{1}{\cos x}\cdot u\right)' = \left(-\frac{1}{2}\tan x\right)' \Rightarrow \frac{1}{\cos x}\cdot u = -\frac{1}{2}\tan x + c \Rightarrow$

$\Rightarrow u = -\frac{1}{2}\cdot\sin x + c\cdot\cos x \Rightarrow \frac{1}{y-\sin x} = -\frac{1}{2}\sin x + c\cdot\cos x$

(12): $y' = -8xy^2 + 4x(4x+1)y - (8x^3+4x^2-1)$, $x>0$, $y_1(x) = x$.

↳ Ricatti

$$u = \frac{1}{y-x} \Leftrightarrow y = \frac{1}{u} + x \quad \text{και} \quad y' = -\frac{u'}{u^2} + 1 \quad \text{οπότε,}$$

$$-\frac{u'}{u^2} + 1 = -8x\cdot\left(\frac{1}{u^2} + x^2 + \frac{2x}{u}\right) + 4x(4x+1)\cdot\left(\frac{1}{u} + x\right) - (8x^3+4x^2-1)$$

$$-\frac{u'}{u^2} + \cancel{1} = -\frac{8x}{u^2} - 8x^3 - \frac{16x^2}{u} + (16x^2+4x)\left(\frac{1}{u}+x\right) - 8x^3 - 4x^2 + \cancel{1}$$

$$-\frac{u'}{u^2} = -\frac{8x}{u^2} - \cancel{8x^3} - \cancel{\frac{16x^2}{u}} + \cancel{\frac{16x^2}{u}} + \cancel{16x^3} + \frac{4x}{u} + \cancel{4x^2} - \cancel{8x^3} - \cancel{4x^2}$$

$$-\frac{u'}{u^2} = -\frac{8x}{u^2} + \frac{4x}{u}$$

$$u' = 8x - 4xu$$

$$u' + 4xu = 8x \Rightarrow u' + (2x^2)'u = 8x \Rightarrow$$

$$\Rightarrow u'\cdot e^{2x^2} + (2x^2)'\cdot e^{2x^2}\cdot u = 8xe^{2x^2} \Rightarrow$$

$$\Rightarrow \left(u\cdot e^{2x^2}\right)' = \left(2\cdot e^{2x^2}\right)' \Rightarrow u\cdot e^{2x^2} = 2e^{2x^2} + c \Rightarrow$$

$$\Rightarrow u = 2 + ce^{-2x^2} \Rightarrow \frac{1}{y-x} = 2 + ce^{-2x^2}$$

(13): $y' = x^3 + \frac{2}{x}\cdot y - x^{-1}y^2$, $x>0$, $y_1(x) = -x^2$

$$y' = -\frac{1}{x}\cdot y^2 + \frac{2}{x}\cdot y + x^3 \quad (\text{Ricatti})$$

$$u = \frac{1}{y+x^2} \Leftrightarrow y + x^2 = \frac{1}{u} \Leftrightarrow y = \frac{1}{u} - x^2 \quad \text{και} \quad y' = -\frac{u'}{u^2} - 2x \quad \text{οπότε,}$$

$$-\frac{u'}{u^2} - 2x = -\frac{1}{x}\left(\frac{1}{u^2} - \frac{2x^2}{u} + x^4\right) + \frac{2}{x}\left(\frac{1}{u} - x^2\right) + x^3 \Rightarrow$$

$$\Rightarrow -\frac{u'}{u^2} - \cancel{2x} = -\frac{1}{xu^2} + \frac{2x}{u} - \cancel{x^3} + \frac{2}{xu} - \cancel{2x} + \cancel{x^3}$$

$$\Rightarrow -\frac{u'}{u^2} = -\frac{1}{x}\cdot\frac{1}{u^2} + \left(2x + \frac{2}{x}\right)\cdot\frac{1}{u} \Rightarrow -u' = -\frac{1}{x} + \left(2x+\frac{2}{x}\right)\cdot u \Rightarrow$$

$$\Rightarrow u' + \left(2x + \frac{2}{x}\right)u = \frac{1}{x} \Rightarrow u'\cdot e^{(x^2+2\ln x)} + \left(2x+\frac{2}{x}\right)\cdot e^{x^2+2\ln x}\cdot u = \frac{1}{x}\cdot e^{x^2+2\ln x} \Rightarrow$$

$$\Rightarrow \left(u\cdot e^{x^2+2\ln x}\right)' = \frac{1}{x}\cdot e^{x^2}\cdot e^{\ln x^2} = \frac{1}{\cancel{x}}e^{x^2}\cdot x^{\cancel{2}} \Rightarrow \left(u e^{x^2+2\ln x}\right)' = \left(\frac{e^{x^2}}{2}\right)' \Rightarrow$$

$$\Rightarrow u\cdot e^{x^2}\cdot x^2 = \frac{e^{x^2}}{2} + c \Rightarrow u = \frac{1}{2x^2} + \frac{ce^{-x^2}}{x^2} \Rightarrow \frac{1}{y+x^2} = \frac{1}{2x^2} + \frac{ce^{-x^2}}{x^2}$$

(14): $\frac{y}{x}\cos\frac{y}{x}\,dx - \left(\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}\right)dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}\cos\frac{y}{x}}{\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}}$$

Παρατηρούμε ότι το δεξί μέλος είναι συνάρτηση του $\frac{y}{x}$ οπότε θέτουμε $u = \frac{y}{x} \Rightarrow y = x\cdot u$

Άρα $\frac{dy}{dx} = u + x\frac{du}{dx}$ και έτσι,

$$u + x\frac{du}{dx} = \frac{u\cos u}{\frac{1}{u}\sin u + \cos u} \Leftrightarrow x\frac{du}{dx} = \frac{u\cdot\cos u}{\frac{\sin u}{u} + \cos u} - u \Leftrightarrow$$

$$\Leftrightarrow x\frac{du}{dx} = \frac{u\cos u - u(\sin u/u + \cos u)}{\frac{\sin u}{u} + \cos u} \Leftrightarrow$$

$$\Leftrightarrow x\frac{du}{dx} = \frac{\cancel{u\cos u} - \sin u - \cancel{u\cos u}}{\frac{\sin u}{u} + \cos u} \Leftrightarrow x\frac{du}{dx} = -\frac{\sin u}{\frac{\sin u}{u} + \cos u} \Leftrightarrow$$

$$\Leftrightarrow -\frac{\frac{\sin u}{u} + \cos u}{\sin u}\,du = \frac{dx}{x} \Leftrightarrow \left(\frac{1}{u} + \cot u\right)du = -\frac{dx}{x} \Leftrightarrow$$

$$\Leftrightarrow \ln|u| + \ln|\sin u| = -\ln|x| + c \Leftrightarrow \ln|u-\sin u| = \ln\frac{1}{|x|} + c \Leftrightarrow$$

$$\Leftrightarrow |u-\sin u| = \frac{e^c}{|x|} \Leftrightarrow u\sin u = \pm\frac{e^c}{x} \Rightarrow u\sin u = \frac{c}{x}$$

Άρα $\frac{y}{\cancel{x}}\cdot\sin\frac{y}{x} = \frac{c}{\cancel{x}} \Rightarrow y\cdot\sin\frac{y}{x} = c$

(15): $x^2y' - (4x^2+xy+y^2)=0$ , $y(1)=-1$

$$x^2\frac{dy}{dx} - (4x^2+xy+y^2)=0 \Rightarrow$$

$$\left(\Rightarrow x^2dy - (4x^2+xy+y^2)dx=0 \Rightarrow (4x^2+xy+y^2)\,dx - x^2dy = 0 \Rightarrow\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^2+xy+y^2}{x^2} = 4+\frac{y}{x}+\left(\frac{y}{x}\right)^2$$

Θέτω $u=\frac{y}{x} \Rightarrow y=x\cdot u$ και $\frac{dy}{dx} = u + x\frac{du}{dx}$ οπότε,

$$\cancel{u} + x\frac{du}{dx} = 4 + \cancel{u} + u^2 \Rightarrow x\frac{du}{dx} = 4+u^2 \Rightarrow \frac{du}{4+u^2} = \frac{dx}{x} \Rightarrow$$

$$\Rightarrow \int\frac{du}{u^2+4} = \int\frac{dx}{x} \Rightarrow \frac{1}{2}\cdot \text{Arctan}\,\frac{u}{2} = \ln|x| + c \Rightarrow$$

$$\Rightarrow \frac{1}{2}\text{Arctan}\,\frac{1}{2}\cdot\frac{y}{x} = \ln|x|+c$$

Για $y(1)=-1$, $\frac{1}{2}\text{Arctan}\,\frac{1}{2}\frac{y(1)}{1} = \ln|1|+c \Rightarrow \frac{1}{2}\text{Arctan}\,\frac{1}{2}(-1) = c \Rightarrow$

$$\Rightarrow c = \frac{1}{2}\text{Arctan}\left(-\frac{1}{2}\right)$$

Άρα $\text{Arctan}\,\frac{y}{2x} = 2\ln|x| + \text{Arctan}\left(-\frac{1}{2}\right) \Rightarrow$

$$\Rightarrow \frac{y}{2x} = \tan\left[2\ln|x| + \text{Arctan}\left(-\frac{1}{2}\right)\right] \Rightarrow$$

$$\Rightarrow y = 2x\cdot\tan\left[2\ln|x| + \text{Arctan}\left(-\frac{1}{2}\right)\right]$$

(16): $(x+\sqrt{y^2-xy})y' = y \quad , \quad y(\tfrac{1}{2})=1$

$$\left(x+\sqrt{y^2-xy}\right)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{x+\sqrt{y^2-xy}} = \frac{\frac{y}{x}}{1+\frac{1}{x}\sqrt{y^2-xy}} = \frac{\frac{y}{x}}{1+\sqrt{(\frac{y}{x})^2-\frac{y}{x}}}$$

Θέτω $\frac{y}{x}=u \Rightarrow y = x\cdot u$ και $\frac{dy}{dx} = u + x\cdot\frac{du}{dx}$ οπότε,

$$u + x\cdot\frac{du}{dx} = \frac{u}{1+\sqrt{u^2-u}} \Leftrightarrow x\frac{du}{dx} = \frac{u}{1+\sqrt{u^2-u}} - u \Leftrightarrow$$

$$\Leftrightarrow x\frac{du}{dx} = \frac{u\left(\cancel{1}-\cancel{1}-\sqrt{u^2-u}\right)}{1+\sqrt{u^2-u}} \Leftrightarrow x\frac{du}{dx} = -\frac{u\sqrt{u^2-u}}{1+\sqrt{u^2-u}} \Leftrightarrow$$

$$\Leftrightarrow -\frac{1+\sqrt{u^2-u}}{u\sqrt{u^2-u}}\,du = \frac{dx}{x} \Leftrightarrow \left(-\frac{1}{u\sqrt{u^2-u}} - \frac{1}{u}\right)du = \frac{dx}{x} \Leftrightarrow$$

[...]

(17): $(y')^2 + 4x^5y' - 12x^4y = 0$

$$12x^4y = (y')^2 + 4x^5y'$$

$$y = \frac{1}{12x^4}\left((y')^2 + 4x^5y'\right) \Rightarrow y = \frac{1}{12x^4}(y')^2 + \frac{x}{3}y'$$

Θέτω $p = y'$ οπότε $y = \frac{1}{12x^4}p^2 + \frac{x}{3}p$ και . .

$$y' = 2pp' \cdot \frac{1}{12x^4} + p^2 \frac{-1}{3x^5} + \frac{1}{3}p + \frac{x}{3}p'$$

$(17)'$: $x(y')^2 + (x-y)y' + 1 - y = 0$

$$x(y')^2 + xy' - yy' + 1 - y = 0$$

$$x(y')^2 + xy' + 1 = y(1+y')$$

$$y = \frac{(y')^2 + y'}{1+y'} \cdot x + \frac{1}{1+y'} = y' \cdot x + \frac{1}{1+y'} \quad (C)$$

Θέτω $p = y'$ και έχουμε, $y = x \cdot p + \dfrac{1}{1+p}$ (1), $y' = p'x + p - \dfrac{p'}{(1+p)^2}$

Άρα $\cancel{p} = p'x + \cancel{p} - \dfrac{p'}{(1+p)^2} \Rightarrow p'\left[x - \dfrac{1}{(1+p)^2}\right] = 0 \Leftrightarrow$

$\Rightarrow p' = 0$ ή $x - \dfrac{1}{(1+p)^2} = 0$

$\downarrow$

$p = C$ $\qquad\qquad x = \dfrac{1}{(1+p)^2}$ (2)

$\downarrow$ αντικαθ. στη (C)

$y = Cx + \dfrac{1}{1+C}$

γεν. λύση.

Για να βρούμε την ιδιάζουσα λύση, απαλείφουμε το $p$ από τις (1), (2):

$(2) \Rightarrow (1+p)^2 = \dfrac{1}{x} \Rightarrow 1 + p = \dfrac{1}{\sqrt{x}} \Rightarrow p = \dfrac{1}{\sqrt{x}} - 1$ $(*)$

$(1) \overset{(*)}{\Longrightarrow} y = x\left(\dfrac{1}{\sqrt{x}} - 1\right) + \sqrt{x} \Rightarrow y = \sqrt{x} - x + \sqrt{x} \Rightarrow$

$\Rightarrow (x+y)^2 = 4x.$

$(19)$: $y = xy' + \sqrt{1+(y')^2}$ $(C)$

Έχουμε ΔΕ του Clairaut:

Θέτουμε $p = y'$ και έτσι

$$y = xp + \sqrt{1+p^2} \; (1) \xrightarrow[\text{ως προς } x]{\text{παραγωγίζω}} y' = p + xp' + \frac{2pp'}{2\sqrt{1+p^2}} \longrightarrow$$

$\rightarrow \cancel{p} = p'x + \cancel{p} + \dfrac{2pp'}{2\sqrt{1+p^2}} \Rightarrow p'\left[x + \dfrac{2p}{2\sqrt{1+p^2}}\right] = 0$

Άρα $p' = 0$ ή $x + \dfrac{2p}{2\sqrt{1+p^2}} = 0$

$\downarrow$

$p = C$ $\qquad\qquad x = -\dfrac{p}{\sqrt{1+p^2}}$ (2)

$\downarrow$

$y = xC + \sqrt{1+C^2}$

γεν. λύση.

$(2) \Rightarrow x^2 = \dfrac{p^2}{1+p^2} \Rightarrow x^2 = \dfrac{p^2+1-1}{p^2+1} = 1 - \dfrac{1}{p^2+1} \Rightarrow$

$\Rightarrow \dfrac{1}{p^2+1} = 1 - x^2 \Rightarrow \dfrac{1}{1-x^2} = p^2 + 1 \Rightarrow p^2 = \dfrac{1}{1-x^2} - 1$

$\Rightarrow p = \sqrt{\dfrac{1}{1-x^2} - 1}$ $(*)$

$(1) \Rightarrow y = xp + \sqrt{1+p^2} \Rightarrow y = p\left(x - \dfrac{1}{x}\right) \overset{(*)}{=} \sqrt{\dfrac{1}{1-x^2} - 1}\left(x - \dfrac{1}{x}\right) = \ldots = \sqrt{1-x^2}$

## II)

(1): $y(8x-9y)dx + 2x(x-3y)dy = 0$

<u>1ος Τρόπος</u>

$$\left.\begin{aligned} M &= 8xy - 9y^2 \longrightarrow M_y = 8x - 18y \\ N &= 2x^2 - 6xy \longrightarrow N_x = 4x - 6y \end{aligned}\right\} \text{ μη πλήρης}$$

Δοκιμή $\mu = x$

$$\frac{M_y - N_x}{N} = \frac{8x - 18y - 4x + 6y}{2x^2 - 6xy} = \frac{4x - 12y}{2x(x-3y)} = \frac{4(x-3y)}{2x(x-3y)} = \frac{2}{x}$$

Άρα $\frac{1}{\mu}\cdot\frac{d\mu}{dx} = \frac{2}{x} \Rightarrow \frac{d\mu}{\mu} = \frac{2dx}{x} \Leftrightarrow \ln|\mu| = 2\ln|x| \Leftrightarrow |\mu| = |x|^2 \Leftrightarrow$

$\Leftrightarrow \mu = \pm x^2$ οπότε έχω $\mu = x^2$

Πολ/ζω με $x^2$ την (1):

$$\left.\begin{aligned} M_1(x,y) &= 8x^3y - 9x^2y^2 \longrightarrow M_1{}'_y = 8x^3 - 18x^2y \\ N_1(x,y) &= 2x^4 - 6x^3y \longrightarrow N_1{}'_x = 8x^3 - 18x^2y \end{aligned}\right\} \text{ πλήρης}$$

Αναζητώ $f(x,y)$ ώστε:

$$\begin{cases} \dfrac{\partial F}{\partial x} = 8x^3y - 9x^2y^2 & (2) \\[2ex] \dfrac{\partial F}{\partial y} = 2x^4 - 6x^3y & (3) \end{cases}$$

(2) $\Rightarrow \int \partial F = \int (8x^3y - 9x^2y^2)\,dx \Leftrightarrow F(x,y) = 2x^4y - 6x^3y^2 + c(y)$

παραγ/ζω ως προς $y$: $F_y = 2x^4 - 12x^3y + c'(y) \overset{(3)}{\Rightarrow} \cancel{2x^4} - 6x^3y = \cancel{2x^4} - 12x^3y + c'(y)$

$\Rightarrow c'(y) = 6x^3y$

$\Rightarrow c(y) = 3x^3y^2$

Άρα $F(x,y) = 2x^4y - 6x^3y^2 + 3x^3y^2$

και το γεν. ολοκλ. $2x^4y - 6x^3y^2 + 3x^3y^2 = c \Leftrightarrow$

$\Leftrightarrow \underline{2x^4y - 3x^3y^2 = c}$

2ος τρόπος

$$\frac{dy}{dx} = \frac{8xy - 9y^2}{6xy - 2x^2} = \frac{y^2\left(8\frac{x}{y} - 9\right)}{x^2\left(6\frac{y}{x} - 2\right)}$$

Θέτω $\frac{y}{x} = u \Rightarrow y = x \cdot u$

και $\frac{dy}{dx} = u + x\frac{du}{dx}$

Άρα, $u + x\frac{du}{dx} = u^2\frac{\left(8\frac{1}{u} - 9\right)}{(6u-2)} \Leftrightarrow u + x\frac{du}{dx} = \frac{8u - 9u^2}{6u - 2} \Leftrightarrow$

$\Leftrightarrow x\frac{du}{dx} = \frac{8u - 9u^2}{6u - 2} - u \Leftrightarrow x\frac{du}{dx} = \frac{8u - 9u^2 - 6u^2 + 2u}{6u - 2} \Leftrightarrow$

$\Leftrightarrow x\frac{du}{dx} = \frac{-15u^2 + 10u}{6u - 2} \Leftrightarrow x\frac{du}{dx} = \frac{-5u(3u-2)}{2(3u-1)} \Leftrightarrow$

$\Leftrightarrow \frac{6u-2}{-15u^2 + 10u}\,du = \frac{dx}{x}$

• $\frac{6u-2}{-5u(3u-2)} = \frac{A}{-5u} + \frac{B}{3u-2} = \frac{A(3u-2) + B(-5u)}{-5u(3u-2)} = \frac{3Au - 2A - 5Bu}{-5u(3u-2)} \Leftrightarrow$

$\Leftrightarrow 6u - 2 = (3A - 5B)u - 2A$

$$\begin{cases} 3A - 5B = 6 \\ -2A = -2 \end{cases} \Leftrightarrow \begin{cases} 5B = 3A - 6 \\ A = 1 \end{cases} \Leftrightarrow \begin{cases} 5B = -3 \\ A = 1 \end{cases} \Leftrightarrow \begin{cases} B = -\frac{3}{5} \\ A = 1 \end{cases}$$

Άρα $\int\left(-\frac{1}{5u} - \frac{3}{5}\cdot\frac{1}{3u-2}\right)du = \int\frac{dx}{x} \Leftrightarrow -\frac{1}{5}\int\frac{1}{u}\,du - \frac{1}{5}\int\frac{3}{3u-2}\,du = \int\frac{dx}{x}$

$\Leftrightarrow -\frac{1}{5}\ln|u| - \frac{1}{5}\ln|3u-2| = \ln|x|$

$\ln|u|^{-\frac{1}{5}} + \ln|3u-2|^{-\frac{1}{5}} = \ln|x|$

$\ln\left(|u|\cdot|3u-2|\right)^{-1/5} = \ln|x| \Leftrightarrow (u\cdot(3u-2))^{-1/5} = x \Leftrightarrow \underline{2x^4y - y^2x^3\cdot 3 = C}$

(2): $6y^2dx - x(2x^3+y)dy = 0$

<u>1ος τρόπος</u>

$$\left.\begin{aligned} M &= 6y^2 \longrightarrow M_y = 12y \\ N &= -2x^4 - xy \to N_x = -8x^3 - y \end{aligned}\right\}$$ μη ακριβής

Δοκιμάζω $u = 3\ln y + 8\ln x$, $u_x = \frac{8}{x}$, $u_y = \frac{3}{y}$

$$\frac{1}{\mu}\frac{d\mu}{du} = \frac{M_y - N_x}{N u_x - M u_y} \Leftrightarrow \frac{1}{\mu}\frac{d\mu}{du} = \frac{13y + 8x^3}{(-2x^4 - xy)\cdot\frac{8}{x} - 6y^2\cdot\frac{3}{y}} = \frac{13y + 8x^3}{-16x^3 - 8y - 18y} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\mu}\frac{d\mu}{du} = \frac{13y + 8x^3}{-16x^3 - 26y} \Leftrightarrow \frac{1}{\mu}\frac{d\mu}{du} = \frac{\cancel{13y+8x^3}}{-2(\cancel{13y+8x^3})} \Leftrightarrow \frac{d\mu}{\mu} = \frac{du}{-2} \Leftrightarrow$$

$$\Leftrightarrow \ln|\mu| = -\frac{1}{2}u \Leftrightarrow \ln\mu = -\frac{1}{2}(3\ln y + 8\ln x) \Leftrightarrow \ln\mu = \ln y^{-\frac{3}{2}} + \ln x^{-4} \Leftrightarrow$$

$$\Leftrightarrow \ln\mu = \ln x^{-4}\cdot y^{-\frac{3}{2}} \Leftrightarrow \mu = \frac{1}{x^4\cdot y^{3/2}}$$

Άρα
$$6y^2\cdot\frac{1}{x^4y^{3/2}}dx - \frac{1}{x^4y^{3/2}}(2x^4 + xy)dy = 0$$
$$6x^{-4}y^{\frac{1}{2}}dx - (2y^{-3/2} + x^{-3}y^{-1/2})dy = 0$$

$$\left.\begin{aligned} M_y' &= \frac{1}{2}\cdot 6x^{-4}\cdot y^{-\frac{1}{2}} = 3x^{-4}y^{-\frac{1}{2}} \\ N_x' &= -(0 - 3x^{-4}y^{-\frac{1}{2}}) = 3x^{-4}y^{-\frac{1}{2}} \end{aligned}\right\}$$ ακριβής

Αναζητώ $f(x,y)$ ώστε:

$$\left\{\begin{aligned} \frac{\partial F}{\partial x} &= 6x^{-4}y^{\frac{1}{2}} \quad (1) \\ \frac{\partial F}{\partial y} &= -(2y^{-3/2} + x^{-3}y^{-1/2}) \quad (2) \end{aligned}\right.$$

$$(1) \Rightarrow F(x,y) = \int 6x^{-4}y^{\frac{1}{2}}dx = y^{\frac{1}{2}}\int 6x^{-4}dx = y^{\frac{1}{2}}\, 6\left[\frac{x^{-4+1}}{-4+1}\right] = -2x^{-3}y^{\frac{1}{2}} + c(y)$$

$$\longrightarrow F_y = -x^{-3}y^{-\frac{1}{2}} + c'(y) \xrightarrow{(2)} -2y^{-\frac{3}{2}} - \cancel{x^{-3}y^{-\frac{1}{2}}} = -\cancel{x^{-3}y^{-\frac{1}{2}}} + c'(y) \Leftrightarrow$$

$$\Leftrightarrow c'(y) = -2y^{-3/2} \Leftrightarrow c(y) = -2\left[\frac{y^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right] = -2\,\frac{y^{-\frac{1}{2}}}{-\frac{1}{2}} = 4y^{-\frac{1}{2}}$$

Άρα $f(x,y) = -2x^{-3}y^{\frac{1}{2}} + 4y^{-\frac{1}{2}}$ και το γεν. ολοκλ. $\underbrace{-2x^{-3}y^{\frac{1}{2}} + 4y^{-\frac{1}{2}} = C}$

και ισχύουσα $y=0$.

2ος τρόπος

$$6y^2dx - x(2x^3+y)dy = 0$$

$$\frac{dx}{dy} = \frac{2x^4+xy}{6y^2} \Leftrightarrow \frac{dx}{dy} = \frac{1}{3y^2}x^4 + \frac{1}{6y}x \Rightarrow \underbrace{\frac{dx}{dy} - \frac{1}{6y}\cdot x = \frac{1}{3y^2}x^4}_{\text{Bernoulli}} \quad (*)$$

Θέτω $u = x^{1-4} = x^{-3} = \frac{1}{x^3}$, $x \neq 0$ και $u' = -\frac{3x'}{x^4}$

Πολ/ζω (*) με $-\frac{3}{x^4}$

$$-\frac{3x'}{x^4} - \frac{1}{6y}\cdot\left(-\frac{3}{x^4}\right)\cdot x = \frac{1}{3y^2}\cdot\left(-\frac{3}{\cancel{x^4}}\right)\cdot \cancel{x^4} \Leftrightarrow$$

$$\Leftrightarrow u' + \frac{1}{2y}\cdot\frac{1}{x^3} = -\frac{1}{y^2} \Leftrightarrow u' + \frac{1}{2y}\cdot u = -\frac{1}{y^2} \Leftrightarrow u' + \left(\frac{\ln y}{2}\right)' u = -\frac{1}{y^2} \Leftrightarrow$$

$$\Leftrightarrow u' + (\ln\sqrt{y})' u = -\frac{1}{y^2} \Leftrightarrow \sqrt{y}\cdot u' + \sqrt{y}\cdot(\ln\sqrt{y})'\cdot u = -\frac{1}{y^2}\sqrt{y} \Leftrightarrow$$

$$\Leftrightarrow (\sqrt{y}\cdot u)' = -y^{-\frac{3}{2}} \Leftrightarrow (\sqrt{y}\cdot u)' = -\left(\frac{y^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right)' \Leftrightarrow (\sqrt{y}\cdot u)' = (2y^{-\frac{1}{2}})'$$

$$\Leftrightarrow \sqrt{y}\cdot u = 2y^{-\frac{1}{2}} + c \Leftrightarrow \sqrt{y}\cdot u = \frac{2}{\sqrt{y}} + c \Leftrightarrow u = \frac{2}{y} + \frac{c}{\sqrt{y}} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{x^3} = \frac{2}{y} + \frac{c}{\sqrt{y}} \Leftrightarrow \frac{1}{x^3} = \frac{2 + c\sqrt{y}}{y} \Leftrightarrow x^3 = \frac{y}{2+c\sqrt{y}} \Leftrightarrow$$

$$\Leftrightarrow 2x^3 + c\,x^3 y^{\frac{1}{2}} = y$$

(3) $(x-2y-1)\,dx - (x-3)\,dy = 0 \quad (*)$

<u>1ος τρόπος</u>

$$\left.\begin{aligned} M &= x-2y-1 \longrightarrow M_y = -2 \\ N &= -x+3 \longrightarrow N_x = -1 \end{aligned}\right\} \text{μη πλήρης}$$

Δοκιμή $\mu = x$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{M_y - N_x}{N} = \frac{-2+1}{-x+3} \Leftrightarrow \frac{1}{\mu}\frac{d\mu}{dx} = \frac{-1}{-x+3} \Leftrightarrow \frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{x-3} \Leftrightarrow$$

$$\Leftrightarrow \frac{d\mu}{\mu} = \frac{dx}{x-3} \Leftrightarrow \ln|\mu| = \ln|x-3| \Leftrightarrow |\mu| = |x-3| \Leftrightarrow \mu = \pm(x-3) \quad \text{επιλέγω } \mu = x-3$$

Πολλαπλασιάζω (*) με $x-3$:

$$(x-3)(x-2y-1)\,dx - (x-3)^2\,dy = 0$$

$$(x^2 - 2xy - x - 3x + 6y + 3)\,dx - (x^2 - 6x + 9)\,dy = 0$$

$$(x^2 - 2xy - 4x + 6y + 3)\,dx + (-x^2 + 6x - 9)\,dy = 0$$

$$\left.\begin{aligned} M_y' &= -2x+6 \\ N_x' &= -2x+6 \end{aligned}\right\} \text{πλήρης}$$

Αναζητώ $f(x,y)$ τέτοια:

$$\begin{cases} \dfrac{\partial F}{\partial x} = x^2 - 2xy - 4x + 6y + 3 & (1) \\[2ex] \dfrac{\partial F}{\partial y} = -x^2 + 6x - 9 & (2) \end{cases}$$

$$(2) \Rightarrow F(x,y) = \int(-x^2+6x-9)\,dy = -x^2y + 6xy - 9y + c(x) \xrightarrow[\text{ως προς } x]{\text{παραγωγίζω}}$$

$$\Rightarrow F_x = -2xy + 6y + c'(x) \overset{(1)}{\Rightarrow} x^2 - \cancel{2xy} - 4x + \cancel{6y} + 3 = -\cancel{2xy} + \cancel{6y} + c'(x)$$

$$\Rightarrow c'(x) = x^2 - 4x + 3 \quad \Rightarrow \quad c(x) = \frac{x^3}{3} - 2x^2 + 3x$$

Άρα $F(x,y) = -x^2y + 6xy - 9y + \dfrac{x^3}{3} - 2x^2 + 3x = \dfrac{x^3}{3} - x^2y - 2x^2 + 6xy + 3x - 9y$

και το γεν. ολοκλήρωμα: $\dfrac{x^3}{3} - x^2y - 2x^2 + 6xy + 3x - 9y = c$

## 2ος τρόπος

$$(x-2y-1)\,dx-(x-3)\,dy=0$$

$$(x-2y-1)\,dx=(x-3)\,dy$$

$$\frac{dy}{dx}=\frac{x-2y-1}{x-3}$$ (σχεδόν ομογενής)

$a=1,\ b=-2,\ c=-1,\ m=1,\ e=0,\ n=-3$

$ae-bm=1\cdot 0-(-2)\cdot 1=2\neq 0$

Θέτω $x=u+k$, $y=v+\ell$

τότε $$\frac{dv}{du}=\frac{u-2v+(k-2\ell+1)}{u+(k-3)}$$

και θέλω $k-2\ell=1$, $k+0=3$ $\Rightarrow \begin{cases} k=-3 \\ \ell=-2 \end{cases}$

Έτσι, θέλω: $$\frac{dv}{du}=\frac{u+-2v}{u+0}\Leftrightarrow \frac{dv}{du}=\frac{u-2v}{u}=\frac{v\left(\frac{u}{v}-2\right)}{u}=\frac{v}{u}\left(\frac{u}{v}-2\right)$$

Θέτω $p=\frac{v}{u}\Rightarrow v=p\cdot u$ και $\frac{dv}{du}=p+u\cdot\frac{dp}{du}$ έτσι,

$$p+u\cdot\frac{dp}{du}=p\left(\frac{1}{p}-2\right)\Leftrightarrow p+u\frac{dp}{du}=1-2p\Leftrightarrow u\frac{dp}{du}=1-3p$$

$$\Leftrightarrow \frac{dp}{1-3p}=\frac{du}{u}\Leftrightarrow\left(\frac{\ln|1-3p|}{-3}\right)'=\left(\ln|u|\right)'\Leftrightarrow \ln|1-3p|=-3\ln|u|$$

$$\Leftrightarrow \ln|1-3p|=\ln|u|^{-3}\Leftrightarrow |1-3p|=|u|^{-3}\Leftrightarrow 1-3p=u^{-3}\Leftrightarrow$$

$$\Leftrightarrow 1-3\cdot\frac{v}{u}=u^{-3}\Leftrightarrow u-3v=u^{-2}\Leftrightarrow$$

$$\Leftrightarrow x+3-3(y+2)=(x+3)^{-2}\Leftrightarrow$$

$$\Leftrightarrow (x+3)^3-3(x+3)^2(y+2)=1$$

$$\Leftrightarrow (x+3)^2\left(x+3-3(y+2)\right)=1$$

$$(x+3)^2(x+3-3y-6)=1$$

$$(x+3)^2(x-3y-3)=1$$

1ος τρόπος

$$(2x-3y+1)dx - (3x+2y-4)dy = 0$$

$$(2x-3y+1)dx = (3x+2y-4)dy$$

$$\frac{dy}{dx} = \frac{2x-3y+1}{3x+2y-4}$$

$a=2$, $b=-3$, $c=1$

$m=3$, $e=2$, $h=-4$

$$ae - bm = 2\cdot 2 - 3(-3) = 4+9 = 13 \neq 0$$

Θέτουμε $x = u+k$, $y = v+\ell$ και $\frac{dy}{dx} = \frac{dv}{du}$ οπότε

$$\frac{dv}{du} = \frac{2u - 3v + (2k-3\ell+1)}{3u+2v+(3k+2\ell-4)}$$

και επιλέγουμε $k, \ell$ ώστε $\begin{cases} 2k-3\ell=-1 \\ 3k+2\ell=4 \end{cases} \Leftrightarrow \begin{cases} k=\frac{10}{13} \\ \ell=\frac{11}{13} \end{cases}$

και $$\frac{dv}{du} = \frac{2u-3v}{3u+2v} = \frac{\cancel{u}\left(2-\frac{3v}{u}\right)}{\cancel{u}\left(3+2\frac{v}{u}\right)} = \frac{2-3\frac{v}{u}}{3+2\frac{v}{u}}$$

Θέτω $p = \frac{v}{u} \Rightarrow v = p\cdot u$ και $\frac{dv}{du} = p + u\frac{dp}{du}$ έτσι,

$$p + u\frac{dp}{du} = \frac{2-3p}{3+2p} \Leftrightarrow u\frac{dp}{du} = \frac{2-3p-p(3+2p)}{3+2p} \Leftrightarrow$$

$$\Leftrightarrow u\frac{dp}{du} = \frac{2-3p-3p-2p^2}{3+2p} = \frac{-2p^2-6p+2}{3+2p} = -2\,\frac{p^2+3p-1}{3+2p}$$

$$\Leftrightarrow \frac{2p+3}{p^2+3p-1}\,dp = -2\frac{du}{u} \Leftrightarrow \frac{(p^2+3p-1)'}{p^2+3p-1}\,dp = -2\frac{du}{u} \Leftrightarrow$$

$$\Leftrightarrow \ln|p^2+3p-1| = -2\ln|u| \Leftrightarrow p^2+3p-1 = u^{-2} \Leftrightarrow \left(\frac{v}{u}\right)^2 + 3\frac{v}{u} - 1 = \frac{1}{u^2}$$

$$\Leftrightarrow \frac{v^2}{u^2} + 3\frac{v}{u} - 1 = \frac{1}{u^2} \Leftrightarrow v^2 + 3v\cdot u - u^2 = 1 \Leftrightarrow$$

$$\Leftrightarrow \left(y-\frac{11}{13}\right)^2 + 3\cdot\left(y-\frac{11}{13}\right)\cdot\left(x-\frac{10}{13}\right) - \left(x-\frac{10}{13}\right)^2 = 1$$

2ος τρόπος

Π. Euler

(5) : $2x^3y' = y\,(y^2+3x^2)$

1ος τρόπος

$$y' = \frac{1}{2}\left(\frac{y}{x}\right)^3 + \frac{3}{2}\,\frac{y}{x} \qquad (\text{ομογενής})$$

Θέτω $\frac{y}{x} = u \Rightarrow y = x\cdot u$ και $\frac{dy}{dx} = u + x\frac{du}{dx}$

οπότε $u + x\frac{du}{dx} = \frac{1}{2}u^3 + \frac{3}{2}u \Leftrightarrow x\frac{du}{dx} = \frac{1}{2}u^3 + \left(\frac{3}{2}-1\right)u \Leftrightarrow$

$\Leftrightarrow x\frac{du}{dx} = \frac{1}{2}u^3 + \frac{1}{2}u \Leftrightarrow \frac{2\,du}{u^3+u} = \frac{dx}{x}$

- $\frac{2}{u^3+u} = \frac{2}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+\Gamma}{u^2+1} \Rightarrow \ldots \Rightarrow \frac{2}{u^3+u} = \frac{2}{u} - \frac{2u}{u^2+1}$ οπότε

$\int\left(\frac{2}{u} - \frac{2u}{u^2+1}\right)du = \int\frac{dx}{x} \Leftrightarrow 2\ln|u| - \ln|u^2+1| = \ln|x| + c$

$\Leftrightarrow \ln|u|^2 - \ln|u^2+1| = \ln|x| + c \Leftrightarrow \frac{u^2}{u^2+1} = x\underbrace{e^c}_{\tilde{c}} \Leftrightarrow \frac{u^2+1-1}{u^2+1} = xc \Leftrightarrow$

$\Leftrightarrow 1 - \frac{1}{u^2+1} = xc \Leftrightarrow \frac{1}{u^2+1} = 1-xc \Leftrightarrow u^2+1 = \frac{1}{1-xc} \Leftrightarrow u^2 = \frac{1}{1-xc} - 1$

$\Leftrightarrow u^2 = \frac{1-(1-xc)}{1-xc} \Leftrightarrow u^2 = \frac{xc}{1-xc} \Leftrightarrow \frac{y^2}{x^2} = \frac{xc}{1-xc} \Leftrightarrow y^2 = \frac{x^3c}{1-xc}$

2ος τρόπος

$2x^3y' = y(y^2+3x^2)$

$y' - \frac{3}{2x}y = \frac{1}{2x^3}y^3$ (*) (Bernoulli)

Θέτω $u = y^{1-3} = y^{-2} = \frac{1}{y^2}$ $y \neq 0$ και $u' = \frac{-2y'}{y^3}$

Πολ/ζω την (*) με $-\frac{2}{y^3}$

$\frac{2y'}{-y^3} - \frac{3}{\cancel{2}x}\left(-\frac{\cancel{2}y}{y^3}\right) = \frac{1}{\cancel{2}x^3}\cdot\frac{\cancel{2}y^3}{(-y^3)}$

$\frac{-2y'}{y^3} + \frac{3}{x}\cdot\frac{1}{y^2} = -\frac{1}{x^3}$

$u' + \frac{3}{x}\cdot u = -\frac{1}{x^3}$

$u' + (3\ln x)'u = -\frac{1}{x^3}$

$u' + (\ln x^3)'u = -\frac{1}{x^3}$

$(x^3\cdot u)' = -1 \Leftrightarrow (x^3\cdot u)' = (-x)' \Leftrightarrow x^3\cdot u = -x + c$

$\Leftrightarrow x^3\cdot\frac{1}{y^2} = -x + c \Leftrightarrow x^2\cdot \;= (-x+c)y^2$