ΦΥΛΛΟ ΠΡΟΒΛ. 2

ΔΙΑΦΟΡΙΚΕΣ ΕΞΙΣΩΣΕΙΣ

ΧΡΗΣΤΟΣ ΤΣΟΥΦΗΣ
ΣΗΜΜΥ
2ο Εξάμηνο

Κεφ.5

5.3

ασκ 20/222

$$(1-x^2)y'' - 6xy' - 4y = 0, \quad x_0 = 0$$

$$y'' - \frac{6x}{1-x^2}y' - \frac{4}{1-x^2}y = 0$$

$$\left.\begin{aligned} P(x) &= -\frac{6x}{1-x^2} \\ Q(x) &= -\frac{4}{1-x^2} \end{aligned}\right\}$$ είναι αναλυτικές, άρα $x_0 = 0$ ομαλό

Αναζητούμε λύση με τη μορφή δυναμοσειράς

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) \cdot a_n \cdot x^{n-2}$$

Έχουμε,

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1) \cdot a_n \cdot x^{n-2} - 6x\sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} - 4\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 6\sum_{n=1}^{\infty} n a_n x^n - 4\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

Θέτω $v = n-2 \Rightarrow n = v+2$ και έτσι, $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{v=0}^{\infty} (v+2)(v+1)a_{v+2} \cdot x^v =$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Άρα, $\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 6\sum_{n=1}^{\infty} n \cdot a_n \cdot x^n - 4\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$

$$\Rightarrow 2\cdot 1\cdot a_2 \cdot x^0 + 3\cdot 2\cdot a_3 \cdot x + (-6\cdot 1\cdot a_1 \cdot x) + (-4\cdot a_0 \cdot x^0 - 4a_1 \cdot x) +$$

$$+ \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)\cdot a_n - 6n a_n - 4a_n\right]x^n = 0 \Rightarrow$$

$\Rightarrow 2a_2 + 6a_3 x - 6a_1 x - 4a_0 - 4a_1 x + \Sigma[\dots] = 0 \Rightarrow$

$\Rightarrow 2a_2 - 4a_0 + (6a_3 - 10a_1)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n^2+5n+4)a_n\right]x^n = 0$

πρέπει $a_2 = 2a_0$

$3a_3 = 5a_1$

$(n+2)(n+1)a_{n+2} = (n^2+5n+4)a_n \ , \ n \geq 2$

$\Delta = 5^2 - 4 \cdot 1 \cdot 4 = 25 - 16 = 9$

$n = \frac{-5 \pm 3}{2} \Big< \begin{matrix} -4 \\ -1 \end{matrix}$

Άρα, $(n+2)\cancel{(n+1)}a_{n+2} = (n+4)\cancel{(n+1)}a_n$

$(n+2)a_{n+2} = (n+4)a_n \ , \ n \geq 2$

$a_{n+2} = \frac{n+4}{n+2} \cdot a_n$

για $n=2$: $a_4 = \frac{6}{4}a_2 = \frac{3}{2} 2 \cdot a_0 = 3a_0$

για $n=4$: $a_6 = \frac{8}{6} a_4 = \frac{4}{3} \cdot 3 \cdot a_0 = 4a_0$

για $n=6$: $a_8 = \frac{10}{8} a_6 = \frac{5}{4} \cdot 4a_0 = 5a_0$

δηλ. $a_{2k} = (k+1)a_0 \ , \ k \geq 0$

και,

για $n=3$: $a_5 = \frac{7}{5}a_3 = \frac{7}{\cancel{5}} \frac{\cancel{5}}{3} \cdot a_1 = \frac{7}{3} a_1$

για $n=5$: $a_7 = \frac{9}{7} \cdot a_5 = \frac{9}{\cancel{7}} \cdot \frac{\cancel{7}}{3} a_1 = \frac{9}{3} a_1$

για $n=7$: $a_9 = \frac{11}{9} \cdot a_7 = \frac{11}{\cancel{9}} \frac{\cancel{9}}{3} a_1 = \frac{11}{3} a_1$

δηλ. $a_{2k+1} = \frac{2k+3}{3} a_1 \ , \ k \geq 0$

Άρα, $y(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} =$

$= a_0 \sum_{k=0}^{\infty} (k+1) x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2k+3}{3} x^{2k+1}$

και $R = 1$ και $(-1, 1)$ (απόσταση του $x_0 = 0$ από το πιο κοντινό ανώμαλο σημείο)

ασκ 23/222

$$y' - 2xy = 0 \ , \ x_0 = 0$$

Αναζητούμε λύση:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$$

Έχουμε, $\displaystyle\sum_{n=1}^{\infty} n a_n x^{n-1} - 2x \sum_{n=0} a_n x^n = 0 \Rightarrow$

$$\Rightarrow \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} - 2 \sum_{n=0}^{\infty} a_n \cdot x^{n+1} = 0$$

- $v = n-1 \Rightarrow n = v+1$ οπότε $\displaystyle\sum_{v=0}^{\infty} (v+1) \cdot a_{v+1} \cdot x^v = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$
- $v = n+1 \Rightarrow n = v-1$ οπότε $\displaystyle\sum_{v=1}^{\infty} a_{v-1} \cdot x^v = \sum_{n=1}^{\infty} a_{n-1} x^n$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2 \sum_{n=1} a_{n-1} x^n = 0 \Rightarrow$$

$$\Rightarrow 1 \cdot a_1 \cdot x^0 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} - 2a_{n-1}\right] x^n = 0$$

$$\Rightarrow a_1 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} - 2a_{n-1}\right] x^n = 0$$

δηλ.

$$\begin{cases} a_1 = 0 \\ (n+1) a_{n+1} = 2a_{n-1} \ , \ n \geq 1 \end{cases}$$

- $n=1$: $2a_2 = 2a_0 \Rightarrow a_2 = a_0$
- $n=3$: $4a_4 = 2a_2 \Rightarrow a_4 = \frac{1}{2} a_2 = \frac{1}{2} a_0$
- $n=5$: $6a_6 = 2a_4 \Rightarrow a_6 = \frac{1}{3} a_4 = \frac{1}{3} \cdot \frac{1}{2} \cdot a_0 = \frac{1}{6} a_0$
- $n=7$: $8a_8 = 2a_6 \Rightarrow a_8 = \frac{1}{4} a_6 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot = \frac{1}{24} a_0$

δηλ. $\displaystyle a_{2k} = \frac{1}{k!} \cdot a_0 \ , \ k \geq 0$

· για $n=2$, $3a_3 = 2a_1 \Rightarrow a_3 = \frac{2}{3}a_1 = 0$

· για $n=4$, $5a_5 = 2a_3 \Rightarrow a_5 = 0$

· για $n=6$, $7a_7 = 2a_5 \Rightarrow a_7 = 0$

δηλ. $a_{2k+1} = 0$, $k \geq 0$

Άρα $y(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} =$

$= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot a_0 \cdot x^{2k} + 0 =$

$= a_0 \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \cdot {}^{2k}$, $x \in \mathbb{R}$

Όμοια για την σελ 29/222

ασκ 30/222

$$3y'' + xy' - 4y = 0 \quad , x_0 = 0$$

$$y'' + \frac{x}{3}y' - \frac{4}{3}y = 0$$

$$\left.\begin{aligned} P(x) &= \frac{x}{3} \\ Q(x) &= -\frac{4}{3} \end{aligned}\right\} \text{αναλυτικές}$$

Αναζητούμε λύση : $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n \cdot (n-1) a_n x^{n-2}$$

Έχουμε, $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \frac{x}{3}\sum_{n=1}^{\infty} n \cdot a_n x^{n-1} - \frac{4}{3}\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \frac{1}{3}\sum_{n=1}^{\infty} n \cdot a_n \cdot x^n - \frac{4}{3}\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

Θέτουμε $v = n-2 \Rightarrow n = v+2$ δηλ. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{v=0}^{\infty}(v+2)(v+1)a_{v+2}x^v =$

$$= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Άρα, $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \frac{1}{3}\sum_{n=1}^{\infty} n a_n x^n - \frac{4}{3}\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$

$$\Rightarrow 2 \cdot 1 \cdot a_2 \cdot x^0 - \frac{4}{3}a_0 x^0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + \frac{1}{3}n \cdot a_n - \frac{4}{3}a_n\right]x^n = 0$$

$$\Rightarrow 2a_2 - \frac{4}{3}a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + \frac{1}{3}(n-4)a_n\right]x^n = 0$$

πρέπει
$$\begin{cases} 2a_2 = \frac{4}{3}a_0 \\ (n+2)(n+1)a_{n+2} = \frac{1}{3}(4-n)a_n \quad , n \geq 1 \end{cases}$$

$$\begin{cases} a_2 = \frac{2}{3}a_0 \\ (n+2)(n+1)a_{n+2} = \frac{1}{3}(4-n)a_n \, , \quad n \geq 1 \end{cases}$$

$\rightarrow \quad (n+2)(n+1)a_{n+2} = \frac{1}{3}(4-n)a_n \quad , n\geq 1$

για $n = 2k-2$, $k \geq 1$

$$2k(2k-1)c_{2k} = \frac{1}{3}(6-2k)c_{2k-2}$$

- $k=1$, $2\cdot 1 \cdot c_2 = \frac{1}{3}\cdot 4 c_0 \Rightarrow c_2 = \frac{2}{3}c_0$
- $k=2$, $4\cdot 3\cdot c_4 = \frac{1}{3}\cdot 2\cdot c_2 \Rightarrow c_4 = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot c_0 = \frac{1}{27}c_0$
- $k=3$, $6\cdot 5\, c_6 = \frac{1}{3}\cdot 0\, c_4 \Rightarrow c_6 = 0$

δηλ. $\quad c_0,\ c_2 = \frac{2}{3}c_0,\ c_4 = \frac{1}{27}c_0,\ c_{2k} = 0,\ k\geq 3$

$$\text{δηλ.} \quad y(x) = \sum_{k=0}^{\infty} c_{2k}\cdot x^{2k} + \underbrace{\sum_{k=0}^{\infty} c_{2k+1}x^{2k+1}}_{y_2(x)} =$$

$$= c_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right) + y_2(x)$$

για $n = 2k-1$, $k\geq 1$

$$(2k+1)\cdot 2k\cdot c_{2k+1} = -\frac{1}{3}\cancel{(2k-5)}\cancel{c_{2k-1}}$$

$$(2k-1)\cdot(2k-2)\cancel{c_{2k-1}} = -\frac{1}{3}\cancel{(2k-7)}\cancel{c_{2k-3}}$$

$$(2k-3)(2k-4)\cancel{c_{2k-3}} = -\frac{1}{3}\cancel{(2k-9)}\cancel{c_{2k-5}}$$

$$\cancel{(2k-5)}(2k-6)\cancel{c_{2k-5}} = -\frac{1}{3}\cancel{(2k-11)}\cancel{c_{2k-7}}$$

$$\vdots$$

$$\cancel{5}\cdot 4\cdot\cancel{c_5} = -\frac{1}{3}\cdot 1\cdot\cancel{c_3}$$

$$3\cdot 2\cdot\cancel{c_3} = -\frac{1}{3}\cdot 3\cdot c_1$$

[...]

$$y(x) = a_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right) + a_1\left(x+\frac{1}{6}x^3+\frac{1}{360}x^5+3\sum_{k=3}^{\infty}\frac{(-1)^k(2k-5)!!}{(2k+1)!\,3^k}\cdot x^{2k+1}\right)$$

$x\in\mathbb{R}$

Κεφ. 6

6.4. A.III.c

ασκ 3/325

$$x' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} x$$

δηλ. $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) + 9 = (\lambda+3)(\lambda-3) + 9 =$$
$$= \lambda^2 - 9 + 9 = \lambda^2$$

Άρα $\lambda = 0$, διπλή ιδιοτιμή

$$(A - 0I)\underset{\sim}{\xi} = \underset{\sim}{0} \Rightarrow \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi_2 = 3\xi_1$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a \\ 3a \end{pmatrix} = \underbrace{a \begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\underset{\sim}{\xi}^1}$$

Χρειαζόμαστε ένα δεύτερο ιδιοδιάνυσμα $\underset{\sim}{\xi}^2$ γραμμ. ανεξ. του $\underset{\sim}{\xi}^1$, το οποίο το βρίσκουμε:

$$(A - 0I)\underset{\sim}{\xi}^2 = \underset{\sim}{\xi}^1 \Leftrightarrow \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Leftrightarrow 3\xi_1 - \xi_2 = 1$$

Αρκεί να πάρω $\xi_1 = 0,\ \xi_2 = -1 \longrightarrow \underset{\sim}{\xi}^2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Η γεν. λύση είναι

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{\xi}^1 e^{0t} + c_2 (\underset{\sim}{\xi}^1 t + \underset{\sim}{\xi}^2) e^{0t} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 3t \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_2 t + c_1 \\ 3c_2 t + 3c_1 - c_2 \end{pmatrix}$$

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$$x' = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & -2 \\ -2 & -2 & 1 \end{pmatrix} x$$

δηλ. $\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & -2 \\ -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 1 & -2 \\ 1 & -2-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} \xrightarrow{\Sigma_1 \leftarrow \Sigma_1+\Sigma_2+\Sigma_3} \begin{vmatrix} -3-\lambda & 1 & -2 \\ -3-\lambda & -2-\lambda & -2 \\ -3-\lambda & -2 & 1-\lambda \end{vmatrix} \longrightarrow$$

$$\longrightarrow (-3-\lambda) \begin{vmatrix} 1 & 1 & -2 \\ 1 & -2-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} \xrightarrow[\Gamma_3 = \Gamma_1(-1)+\Gamma_3]{\Gamma_2 = \Gamma_1(-2)+\Gamma_2} -(\lambda+3) \begin{vmatrix} 1 & 1 & -2 \\ 0 & -3-\lambda & 0 \\ 0 & -3 & 3-\lambda \end{vmatrix} =$$

$$= -(\lambda+3)(\lambda-3)(\lambda+3) = -(\lambda+3)^2(\lambda-3).$$

• <u>για $\lambda_1 = 3$</u>

$$(A-3I)\underset{\sim}{\xi} = \underset{\sim}{0} \iff \begin{pmatrix} -5 & 1 & -2 \\ 1 & -5 & -2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\iff \begin{cases} 5\xi_1 - \xi_2 + 2\xi_3 = 0 \\ \xi_1 - 5\xi_2 - 2\xi_3 = 0 \\ \xi_1 + \xi_2 + \xi_3 = 0 \end{cases} \iff \begin{matrix} \xi_1 = \xi_2 \\ \xi_3 = -2\xi_1 \end{matrix}$$

οπότε $\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} a \\ a \\ -2a \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}_{\underset{\sim}{\xi}^1}$

• $\underline{\text{για } \lambda_2 = -3}$ (διπλή)

$$(A+3I)\underset{\sim}{\xi} = \underset{\sim}{0} \Leftrightarrow \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \xi_1 + \xi_2 - 2\xi_3 = 0$$

Άρα $\xi_2 = 2\xi_3 - \xi_1$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\beta - \alpha \\ \beta \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\underset{\sim}{\xi}^2} + \beta \underbrace{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}}_{\underset{\sim}{\xi}^3}$$

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{\xi}^1 e^{3t} + c_2 \underset{\sim}{\xi}^2 e^{-3t} + c_3 \underset{\sim}{\xi}^3 e^{-3t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} c_1 e^{3t} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_2 e^{-3t} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} c_3 e^{-3t} =$$

$$= \begin{pmatrix} c_1 e^{3t} + c_2 e^{-3t} \\ c_1 e^{3t} - c_2 e^{-3t} + 2c_3 e^{-3t} \\ -2c_1 e^{3t} + c_3 e^{-3t} \end{pmatrix}$$

$\underline{\text{Όμοια για ~~ασκ~~ 12/326}}$

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$$x' = \begin{pmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} x \quad , \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

δηλ. $\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 & 2 \\ -1 & 1-\lambda & 1 \\ -2 & 1 & 3-\lambda \end{vmatrix} =$$

$$= (-1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & 1-\lambda \\ -2 & 1 \end{vmatrix}$$

$$= -(\lambda+1)\Big((1-\lambda)(3-\lambda) - 1\Big) - (-3+\lambda+2) + 2(-1+2-2\lambda) =$$

$$= -(\lambda+1)(3-\lambda-3\lambda+\lambda^2-1) - (\lambda-1) + 2(1-2\lambda) =$$

$$= -(\lambda+1)(\lambda^2-4\lambda+2) - \lambda+1 + 2-4\lambda =$$

$$= -(\lambda+1)(\lambda^2-4\lambda+2) + 3-5\lambda =$$

$$= -\left[\lambda^3-4\lambda^2+2\lambda+\lambda^2-4\lambda+2\right] + 3-5\lambda =$$

$$= -\lambda^3+3\lambda^2-3\lambda+1 = -(\lambda^3-3\lambda^2+3\lambda-1) = -(\lambda-1)^3$$

Άρα $\lambda = 1$, τριπλή ιδιοτιμή

$$(A - I)\underset{\sim}{\xi} = \underset{\sim}{0} \Leftrightarrow \begin{pmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{array}{l} -2\xi_1 + \xi_2 + 2\xi_3 = 0 \\ -\xi_1 + \xi_3 = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \xi_1 = \xi_3 \\ \xi_2 = 2\xi_1 - 2\xi_3 = 2\xi_1 - 2\xi_1 = 0 \end{array} \right.$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\rightsquigarrow \underset{\sim}{\xi}^1$

Θα πρέπει να βρούμε όλα αυτά ιδιοδιανύσματα σε γενικευμένη μορφή.

$$(A-I)^2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad (A-I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Επομένως, κάθε γενικευμένο ιδιοδ. $\underset{\sim}{\xi}^3$ θα είναι λύση της εξίσωσης $(A-I)^3 \underset{\sim}{\xi}^3 = \underset{\sim}{0}$. Έστω λοιπόν $\underset{\sim}{\xi}^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ τότε έχουμε:

$$\underset{\sim}{\xi}^2 = (A-I)\xi^3 = \begin{pmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$$

$$\underset{\sim}{\xi}^1 = (A-I)\underset{\sim}{\xi}^2 = \begin{pmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4-1-4 \\ 2+0-2 \\ 4-1-4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

δηλ. $a=-1$ στον ορισμό του $\underset{\sim}{\xi}^1$.

$$X(t) = e^t\left(c_1\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}\frac{t^2}{2} + c_2\begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} t + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \quad (1)$$

Τώρα για $x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ βρίσκουμε το ζητούμενο.

ασκ 20/326

$$x' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x \Leftrightarrow \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Γενικά, θα δούμε τις πιθανές λύσεις αυτού του συστήματος. Το χαρακτ. πολυων. είναι: $P(\lambda) = \lambda^2 - \underbrace{(a+d)}_{TrA}\lambda + \underbrace{(ad-bc)}_{\det A}$

$\Delta > 0$, τότε διακεκριμένες πραγμ. ιδιοτιμές λ_1, λ_2 (απλές) με αντίστοιχα ιδιοδιανύσματα ξ^1, ξ^2 και γεν. λύση

$$x(t) = c_1 \xi^1 e^{\lambda_1 t} + c_2 \xi^2 e^{\lambda_2 t}$$

$$\lim_{t \to +\infty} x(t) = 0 \Leftrightarrow \lambda_1, \lambda_2 < 0 \quad \text{δηλ.} \quad a+d < 0 \text{ και } ad-bc > 0$$

$\Delta = 0$, τότε διπλή πραγμ. ιδιοτιμή λ_0 και (εκτός της τετριμμένης περίπτωσης όπου $A = \emptyset$) υπάρχει ένα ιδιοδιάνυσμα ξ^1. Το άλλο το βρίσκουμε λύνοντας το σύστημα $(A - \lambda_0 I)\xi^2 = \xi^1$. Η γεν. λύση

$$x(t) = c_1 \xi^1 e^{\lambda_0 t} + c_2 (\xi^1 t + \xi^2) e^{\lambda_0 t}$$

$$\lim_{t \to +\infty} x(t) = 0 \Leftrightarrow \lambda_0 < 0 \Leftrightarrow a+d < 0 \text{ και } ad-bc > 0$$

$\Delta < 0$, τότε συζυγείς μιγαδ. ρίζες της μορφής $\alpha \pm \beta i$

Διαλέγω μία και βρίσκω το αντίστοιχο μιγαδ. ιδιοδιάνυσμα

$\xi = \xi^1 + i\xi^2$

Η γεν. λύση είναι τότε

$$x(t) = c_1 e^{\alpha t}(\xi^1 \cos\beta t - \xi^2 \sin\beta t) + c_2 e^{\alpha t}(\xi^1 \sin\beta t + \xi^2 \cos\beta t)$$

$$\lim_{t \to +\infty} x(t) = 0 \Leftrightarrow \alpha < 0 \Leftrightarrow \begin{array}{l} a+d = 2\alpha < 0 \\ ad-bc = \alpha^2 + \beta^2 > 0 \end{array}$$

6.4.B

ασκ 7 / 332

$$x' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Θεωρώ το αντίστοιχο ομογενές σύστημα,

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ -2 & 0 & -1-\lambda \end{vmatrix} =$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} - 0 + 1 \begin{vmatrix} 1 & 1-\lambda \\ -2 & 0 \end{vmatrix} =$$

$$= (1-\lambda)(1-\lambda)(-1-\lambda) + 2(1-\lambda) =$$

$$= (1-\lambda)[-(1-\lambda)(1+\lambda) + 2] =$$

$$= (1-\lambda)(\lambda^2 - 1 + 2) =$$

$$= (1-\lambda)(\lambda^2+1)$$

$$|A - \lambda I| = 0 \Leftrightarrow \lambda = 1 \text{ ή } \lambda = i \text{ ή } \lambda = -i$$

για $\lambda=1$

$$(A-I)\underset{\sim}{\xi}=\underset{\sim}{0} \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\xi_1 = 0,\ \xi_3 = 0,\ \xi_1 = -\xi_3,\ \xi_2 \in \mathbb{R}$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

για $\lambda=i$

$$(A-iI)\underset{\sim}{\xi}=\underset{\sim}{0} \Leftrightarrow \begin{pmatrix} 1-i & 0 & 1 \\ 1 & 1-i & 0 \\ -2 & 0 & -1-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow (1-i)\xi_1 + \xi_3 = 0 \xrightarrow{\cdot(1+i)} 2\xi_1 + (1+i)\xi_3 = 0$$

$$\xi_1 + (1-i)\xi_2 = 0$$

$$-2\xi_1 - (1+i)\xi_3 = 0$$

$$\xi_2 = -\frac{1}{1-i}\xi_1 = -\frac{1+i}{2}\xi_1$$

$$\xi_3 = -\frac{2}{1+i}\xi_1 = -\frac{\cancel{2}(1-i)}{\cancel{2}}\xi_1 = (i-1)\xi_1$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} k \\ -\frac{1+i}{2}k \\ (i-1)k \end{pmatrix} = k\begin{pmatrix} 1 \\ -\frac{1+i}{2} \\ i-1 \end{pmatrix},\ \text{για } k=2 \begin{pmatrix} 2 \\ -1-i \\ 2i-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{x}(t) = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{t} + c_2 e^{0t}\left(\cos(1t)\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} - \sin(1t)\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\right) +$$

$$+ c_3 e^{0t}\left(\sin(1t)\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} + \cos(1t)\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\right).$$

Λόγω της μορφής του μη-ομογ. όρου, αναζητώ μερική λύση της μορφής $x_{\mu\tau}(t) = \begin{pmatrix} A \\ B \\ \Gamma \end{pmatrix}$, τη βάζω στο αρχικό (Σ) και προσδιορίζω A, B, Γ.

ασκ 11/332

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} 4e^t \cos t, \qquad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 5 \\ -2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (4-\lambda)(-2-\lambda) + 10 = 0$$

$$\Rightarrow -8 - 4\lambda + 2\lambda + \lambda^2 + 10 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\Delta = (-2)^2 - 4 \cdot 1 \cdot 2 = 4 - 8 = -4 < 0$$

$$\lambda_{1,2} = \frac{2 \pm i2}{2} = 1 \pm i$$

$\underline{\text{για } \lambda = \overset{a}{1} + \overset{B}{i}}$

$$(A - (1+i)I)\underset{\sim}{\xi} = \underset{\sim}{0} \Rightarrow \begin{pmatrix} 3-i & 5 \\ -2 & -3-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} (3-i)\xi_1 + 5\xi_2 = 0 \\ -2\xi_1 - (3+i)\xi_2 = 0 \end{cases} \Rightarrow \begin{cases} 10\xi_1 + 5(3+i)\xi_2 = 0 \\ -2\xi_1 - (3+i)\xi_2 = 0 \end{cases}$$

$$\rightarrow \xi_1 = -\frac{3+i}{2}\xi_2$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -\frac{3+i}{2}k \\ k \end{pmatrix} = k\begin{pmatrix} -\frac{3+i}{2} \\ 1 \end{pmatrix}, \quad k = 2 \quad \begin{pmatrix} -3-i \\ 2 \end{pmatrix} = \underbrace{\begin{pmatrix} -3 \\ 2 \end{pmatrix}}_{\xi^1} + i\underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\xi^2}$$

$$x(t) = c_1 e^t \left(\cos t \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \sin t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) + c_2 e^t \left(\sin t \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$$

Λόγω της μορφής του μη ομογ. όρου, αναζητούμε ειδική λύση της μορφής $X_{ειδ}(t) = \begin{pmatrix} A \\ B \end{pmatrix} e^t \sin t + \begin{pmatrix} \Gamma \\ \Delta \end{pmatrix} e^t \cos t$

Επειδή όμως ο μη ομογ. όρος είναι πολ/σιο και κομματιού των θεμ. λύσεων του μη ομογ. προβλ., είναι πιθανό η μέθοδος των προσδιοριστέων συντ. να αποτύχει.
Τότε, πολ/ζουμε με t την υποψήφια ειδική λύση και ξαναδοκιμάζουμε.

Όμοιο, για ασκ 16/333

Κεφ. 8

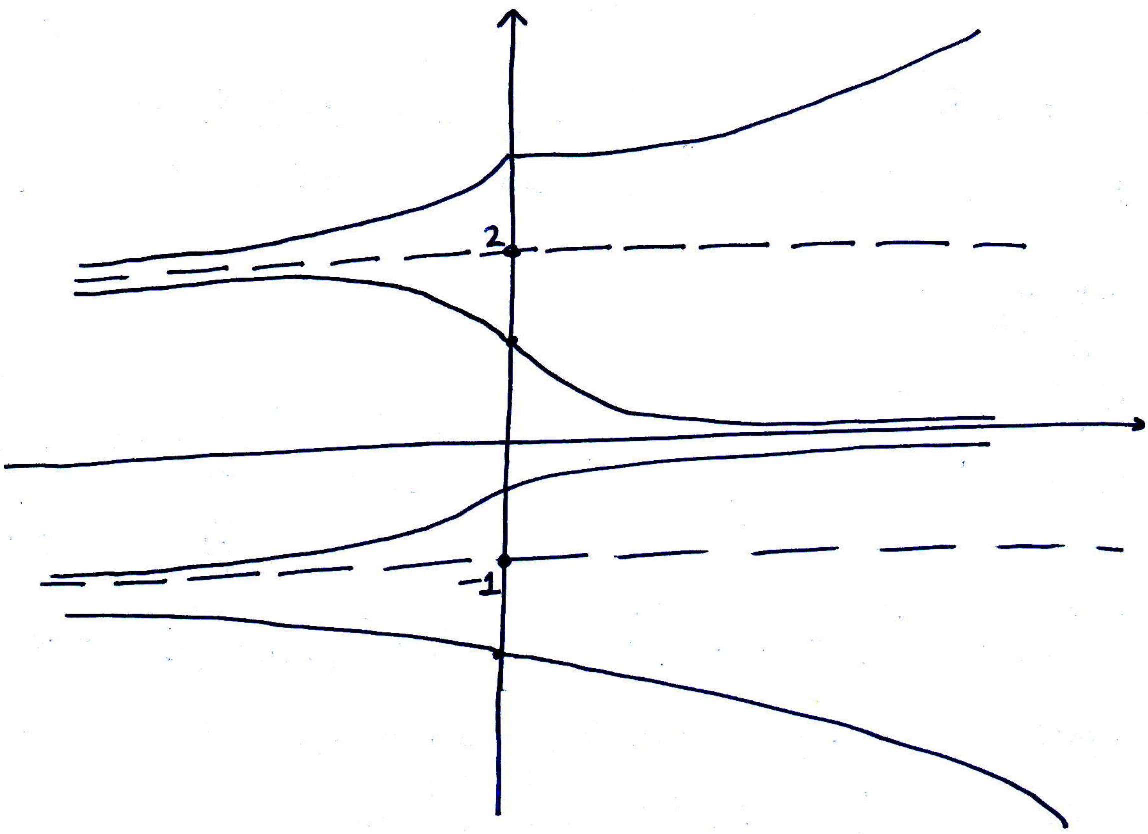
8.2

ασκ 12/431

$$y' = y(y+1)(y-2) = f(y)$$

$f(y) = 0 \Leftrightarrow y(y+1)(y-2) = 0 \Leftrightarrow y = 0$ ή $y = -1$ ή $y = 2$ σημεία ισορρ.

y		-1		0		2	
y	$-$		$-$	0	$+$		$+$
$y+1$	$-$	0	$+$		$+$		$+$
$y-2$	$-$		$-$		$-$	0	$+$
$f(y)$	$-$		$+$		$-$		$+$



Όμοιο για ασκ 14, 16/431

8.3

σελ 7/440

$$x' = 3x - y - 7$$
$$y' = -x + 3y + 5$$

Τα σημεία ισορρ. του συστ. προκύπτουν ως λύσεις του 2x2 γραμμ. συστ.

$$\begin{cases} 3x - y = 7 \\ -x + 3y = -5 \end{cases} \to \begin{array}{l} 3x - y = 7 \\ -3x + 9y = -15 \; + \\ \hline 8y = -8 \\ y = -1 \end{array}$$

$$-x - 3 = -5 \Rightarrow -x = -2 \to x = 2$$

Το $(x,y) = (2,-1)$ είναι σημείο ισορρ.

$$\begin{cases} \chi = x - 2 \\ \psi = y + 1 \end{cases} \Leftrightarrow \begin{cases} x = \chi + 2 \\ y = \psi - 1 \end{cases}$$

Άρα
$$\begin{cases} x' = 3(\chi + 2) - (\psi - 1) - 7 \\ y' = -(\chi + 2) + 3(\psi - 1) + 5 \end{cases}$$

$$\begin{cases} x' = 3\chi \cancel{+6} - \psi \cancel{+1} \cancel{-7} \\ y' = -\chi \cancel{-2} + 3\psi \cancel{-3} \cancel{+5} \end{cases}$$

$$\begin{cases} x' = 3\chi - \psi \\ y' = -\chi + 3\psi \end{cases}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \end{pmatrix}$$

$$|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow (3-\lambda)^2 - 1 = 0 \Leftrightarrow (3-\lambda) = \pm 1 \Leftrightarrow \lambda = 2 \text{ ή } \lambda = 4$$

Δύο θετ. ιδιοτιμές πραγμ.

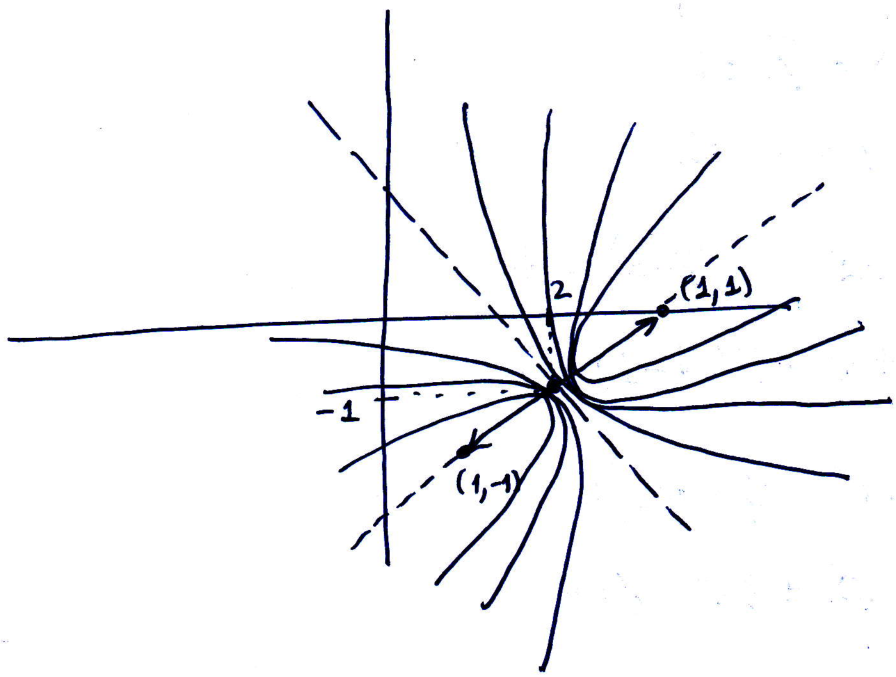
Άρα έχουμε ασταθή κόμβο.

$$(A-2I)\xi=0 \Leftrightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \xi_1=\xi_2$$

δηλ. $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} a \\ a \end{pmatrix}=a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$(A-4I)\xi=0 \Leftrightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \xi_1=-\xi_2$$

δηλ. $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} a \\ -a \end{pmatrix}=a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



<u>Και όμοια για 9,15/440-441</u>

8.4

ασκ 11/450

$$x' = y^2 - 1$$
$$y' = x^3 - y$$

$$\begin{cases} y^2 - 1 = 0 \\ x^3 - y = 0 \end{cases} \rightarrow \begin{cases} y = \pm 1 \\ x^3 = y \end{cases} \Rightarrow (-1,-1) \text{ ή } (1,1)$$

$$x' = f(x,y) = y^2 - 1$$
$$y' = g(x,y) = x^3 - y$$

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}$$

- $(1,1) \longrightarrow J(1,1) = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$

Γραμμικοποίηση:

$$\begin{cases} x' = 2y \\ y' = 3x - y \end{cases}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \lambda(1+\lambda) - 6 = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\Delta = 1^2 - 4 \cdot 1(-6) = 25$$

$$\lambda_{1,2} = \frac{-1 \pm 5}{2} = \left\langle \begin{matrix} -3 \\ 2 \end{matrix} \right\} \text{ σάγμα}$$

- $(-1,-1) \longrightarrow J(-1,-1) = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$

$$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & -2 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(1+\lambda) + 6 = 0 \Rightarrow \lambda^2 + \lambda + 6 = 0$$

$$\Delta = 1^2 - 4 \cdot 1 \cdot 6 = 1 - 24 = -23 < 0$$

$$\lambda_{1,2} = -\frac{1 \pm i\sqrt{23}}{2} \quad \Big\} \text{ Εστία ευσταθής}$$

Όμοια για τις 14,15,19/451

8.5

ασκ 8/462

$$\left.\begin{aligned} x' &= -x^3 \\ y' &= -x^2y \\ V &= x^2+y^2 \end{aligned}\right\} \quad \begin{aligned} x' &= f(x,y) = -x^3 \\ y' &= g(x,y) = -x^2y \end{aligned}$$

$$V(x,y) = x^2+y^2, \qquad \frac{\partial V}{\partial x} = 2x, \ \frac{\partial V}{\partial y} = 2y$$

$$\dot V = \frac{\partial V}{\partial x} f(x,y) + \frac{\partial V}{\partial y} g(x,y) = -2x^4 - 2x^2y^2 < 0 \quad \forall (x,y) \neq (0,0)$$

ασυμπτ. ευσταθές.

Όμοια για 12/462

ασκ 14/463

$$V = ax^{2m} + by^{2n} \longrightarrow a,b>0$$

$$x' = -x^3+y^3 = f(x,y)$$

$$y' = -x^3-y^3 = g(x,y)$$

$$\dot V = \frac{\partial V}{\partial x} f(x,y) + \frac{\partial V}{\partial y} g(x,y) =$$

$$= 2ma\,x^{2m-1}(-x^3+y^3) + 2n\beta y^{2n-1}(-x^3-y^3) =$$

$$= -2max^{2m+2} + 2max^{2m-1}y^3 - 2n\beta x^3 y^{2n-1} - 2n\beta y^{2n+2}$$

Άρα, αν πάρω $a=\beta=1$ και $m=n=2$, θα είναι

$$\dot V = -4x^6 + 4x^3y^3 - 4x^3y^3 - 4y^6 = -4(x^6+y^6) < 0$$

ασυμπτ. ευστ.

Όμοια για ασκ 19/463

Κεφ 9

9.1

ασκ 4/486

$$f(x)=\begin{cases}-1, & -2\le x\le -1\\ 0, & -1\le x\le 1\\ 3, & 1\le x\le 2\end{cases}\qquad (L=2)$$

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx=\frac{1}{2}\int_{-2}^{2}f(x)\,dx=\frac{1}{2}\int_{-2}^{-1}(-1)\,dx+\frac{1}{2}\int_{-1}^{1}0\,dx+\frac{1}{2}\int_{1}^{2}3\,dx=$$

$$=-\frac{1}{2}(-1+2)+0+\frac{1}{2}\cdot 3\cdot(2-1)=-\frac{1}{2}+\frac{3}{2}=1$$

για $n=1,2,\dots$ $\quad a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\left(\frac{n\pi x}{L}\right)dx=$

$$=\frac{1}{2}\int_{-2}^{-1}(-1)\cos\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_{1}^{2}3\cos\left(\frac{n\pi x}{2}\right)dx=$$

$$=-\frac{1}{2}\int_{-2}^{-1}\cos\left(\frac{n\pi x}{2}\right)dx+\frac{3}{2}\int_{1}^{2}\cos\left(\frac{n\pi x}{2}\right)dx=$$

$$=-\frac{1}{2}\cdot\frac{2}{n\pi}\cdot\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{-1}+\frac{3}{2}\cdot\frac{2}{n\pi}\cdot\sin\left(\frac{n\pi x}{2}\right)\Big|_{1}^{2}=$$

$$=-\frac{1}{n\pi}\left(\sin\left(-\frac{n\pi}{2}\right)-\sin(-n\pi)\right)+\frac{3}{n\pi}\left(\sin(n\pi)-\sin\left(\frac{n\pi}{2}\right)\right)=$$

$$=\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)-\frac{1}{n\pi}\sin(n\pi)+\frac{3}{n\pi}\sin(n\pi)-\frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right)=$$

$$=-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)+\cancel{\frac{2}{n\pi}\sin(n\pi)}^{\,0}\Rightarrow$$

$$\Rightarrow a_n=\begin{cases}0, & n=2k\\ \dfrac{2(-1)^k}{(2k-1)\pi}, & n=2k-1\end{cases}\qquad k=1,2,\dots$$

$$\left\{\begin{aligned}&\sin(n\pi)=0\\ &\cos(n\pi)=(-1)^n\\ &\sin\left(\frac{n\pi}{2}\right)=\begin{cases}0, & n=2k\\ (-1)^{k-1}, & n=2k-1\end{cases}\\ &k=1,2,\dots\end{aligned}\right.$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx =$$

$$= \frac{1}{2}\int_{-2}^{-1} (-1)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{1}^{2} 3\sin\left(\frac{n\pi x}{2}\right)dx =$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{n\pi}\cdot\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{-1} - \frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{1}^{2} =$$

$$= \frac{1}{n\pi}\left(\cos\left(-\frac{n\pi}{2}\right) - \cos(-n\pi)\right) - \frac{3}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) =$$

$$= \frac{1}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right) - \frac{3}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) =$$

$$= \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\cos(n\pi) \Rightarrow$$

$$\Rightarrow b_n = \begin{cases} \frac{4}{n\pi}(-1)^k - \frac{4}{n\pi}(-1)^{2k}, & n = 2k \\ -\frac{4}{n\pi}(-1)^{2k-1}, & n = 2k-1 \end{cases} =$$

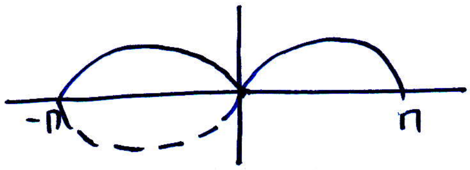
$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n = 2k-1 \\ (-1)^k, & n = 2k \end{cases}$$

$$\cos(n\pi) = (-1)^n$$

$$= \begin{cases} \frac{4}{n\pi}\left[(-1)^k - 1\right], & n = 2k \\ \frac{4}{n\pi}, & n = 2k-1 \end{cases}$$

סע 8/486

$f(x) = |\sin x|, \quad -\pi \le x \le \pi \quad (L=\pi)$



- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin x)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin x\,dx =$

$$= \frac{1}{\pi}\cos x\Big|_{-\pi}^{0} - \frac{1}{\pi}\cos x\Big|_{0}^{\pi} =$$

$$= \frac{1}{\pi}\left(\cos 0 - \cos(-\pi)\right) - \frac{1}{\pi}\left(\cos\pi - \cos 0\right) =$$

$$= \frac{1}{\pi}(1-1) - \frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\cdot\cos(nx)\,dx =$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-\sin x)\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin x\cdot\cos(nx)\,dx =$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\left[\sin((n+1)x) - \sin((1-n)x)\right]dx \; +$$

$$+ \frac{1}{2\pi}\int_{0}^{\pi}\left[\sin((n+1)x) - \sin((1-n)x)\right]dx = \dots$$

- $b_n = 0$, זוגי ע $|\sin x|$ היא פונק'.

9.2

φαι 51 / 504

$$f(x)=\begin{cases} x^2 , & -\pi \le x < 0 \\ 2 , & 0 \le x \le \pi \end{cases}$$

• Μοναδικό σημείο ασυνέχειας, το $x_0=0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2 = 2$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0$$

• $$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{2-2}{x} = 0$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{x^2-0}{x} = 0$$

• $$\lim_{x\to -L^+} f(x) = \lim_{x\to -\pi^+} x^2 = \pi^2$$

$$\lim_{x\to L^-} f(x) = \lim_{x\to \pi^-} 2 = 2$$

$$\lim_{x\to -L^+} \frac{f(x)-f(-L)}{x+L} = \lim_{x\to -\pi^+} \frac{x^2-\pi^2}{x+\pi} = \lim_{x\to -\pi^+} (x-\pi) = -2\pi$$

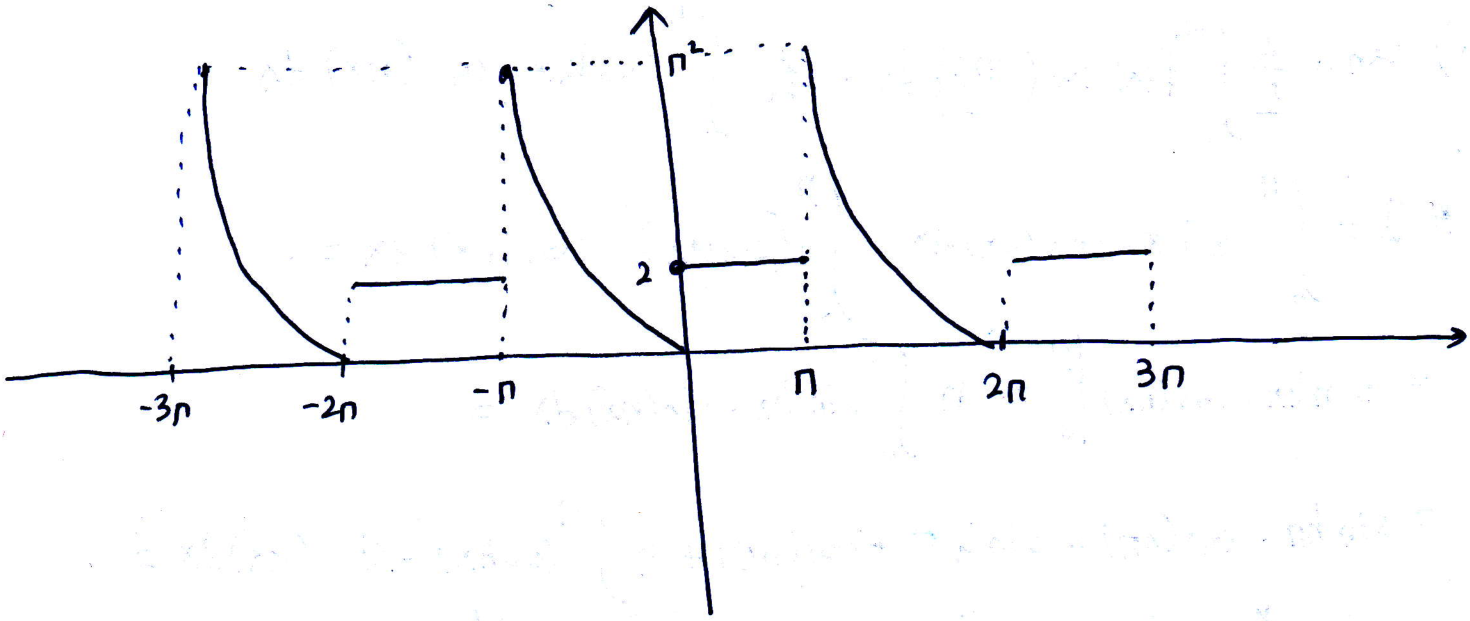
$$\lim_{x\to L^-} \frac{f(x)-f(L)}{x-L} = \lim_{x\to \pi^-} \frac{2-2}{x-\pi} = 0$$

• Κάθε $x\in(-\pi,0)$ είναι σημείο συνέχειας της f.
Άρα, η σειρά F. συγκλίνει στο $f(x)=x^2$.

• Κάθε $x\in(0,\pi)$ είναι σημείο συνέχειας της f.
Άρα, η σειρά F. συγκλίνει στο $f(x)=2$.

• Στο $x_0=0$, η σειρά F. συγκλίνει στο $\frac{f(0^+)+f(0^-)}{2} = \frac{2+0}{2} = 1$

• Στο $x_0 = \pm\pi$ συγκλίνει στο $\frac{\pi^2+2}{2}$



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$$f(x) = \begin{cases} x, & -\pi \le x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\pi - x) = \pi$$

συνεχής

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$f(x) = \cosh x \quad , \quad 0 \leq x \leq \pi$

a) $a_n = \frac{2}{L} \int_0^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{\pi} \int_0^{\pi} \cosh x \cdot \cos(nx)\, dx$

$* \; I = \int_0^{\pi} \cosh x \cdot \cos(nx)\, dx = \int_0^{\pi} (\sinh x)' \cdot \cos(nx)\, dx =$

$= \sinh x \cdot \cos(nx) \Big|_0^{\pi} + n \int_0^{\pi} \sinh x \cdot \sin(nx)\, dx =$

$= \sinh \pi \cdot \cos(n\pi) - \sinh 0 \cdot \cos(n0) + n \int_0^{\pi} (\cosh x)' \cdot \sin(nx)\, dx =$

$= (-1)^n \cdot \sinh \pi + n\left[\cosh x \cdot \sin(nx)\right]_0^{\pi} - n^2 \int_0^{\pi} \cosh x \cdot \cos(nx)\, dx =$

$= (-1)^n \cdot \sinh \pi + n\left(\cosh \pi \cdot \sin(n\pi) - \cosh 0 \cdot \sin(n0)\right) - n^2 I$

$= (-1)^n - n^2 I$

Δηλ. $\quad (n^2+1)\, I = (-1)^n \sinh \pi$

$$I = \frac{(-1)^n}{n^2+1} \sinh \pi$$

Άρα, $\quad a_n = \frac{2}{\pi} \cdot \frac{(-1)^n}{n^2+1} \cdot \sinh \pi$

b) αντίστοιχα, η ημιτονική.

Όμοια, η άσκ 69/504