Σήματα & Συστήματα

2η σειρά ασκήσεων

Χρήστος Τσάκμας

Ασκ. 1

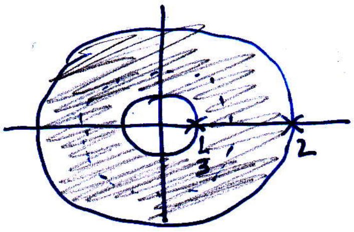
α) 1. $x_1[n] = 2^n u[-n] + \left(\frac{1}{3}\right)^n u[n-1]$

$$X(z) = \sum_{n=-\infty}^{+\infty} 2^n u[-n] z^{-n} + \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u[n-1] z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{2}{z}\right)^n + \sum_{n=1}^{+\infty} \left(\frac{1}{3z}\right)^n$$

$$= \frac{\frac{2}{z}\cdot 1 - 0}{\frac{2}{z} - 1} + \frac{0 - \frac{1}{3z}}{\frac{1}{3z} - 1} = \frac{-2}{z-2} + \frac{1}{3z-1}$$

$|z| \le 2$ $\quad$ $|z| \ge \frac{1}{3} \Rightarrow$ RoC: $\frac{1}{3} \le |z| \le 2$



$|z| = 1 \in$ RoC $\Rightarrow$ ευσταθές $\Rightarrow$ ορίζεται DTFT

2. $x_2[n] = n \underbrace{a^{n-1} u[n]}_{Y[n]}$

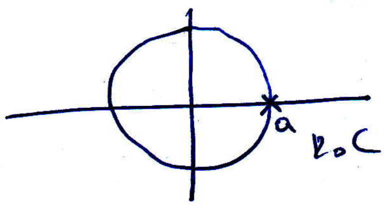
$$X(z) = -z \frac{d(Y(z))}{dz} = \frac{z}{a} \frac{d\left(\frac{z}{a-z}\right)}{dz} = \frac{z}{a}\left(\frac{a-z+z}{(a-z)^2}\right) = \frac{z}{\cancel{a}} \cdot \frac{\cancel{a}}{(a-z)^2}$$

$$= \frac{z}{(a-z)^2}$$

όπου $Y(z) = \sum_{n=0}^{\infty} a^{n-1} z^{-n} = \frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{a} \frac{0 \cdot \frac{a}{z} - 1}{\frac{a}{z} - 1} = -\frac{z}{a(a-z)}$

$\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$

RoC : $|z| \ge |a|$



για $|a| < 1 \Rightarrow$ ορίζεται DTFT

για $|a| \ge 1 \Rightarrow$ δεν ορίζεται DTFT

3. $x_3[n] = 3^n \cos\left[\frac{2\pi}{6} n + \frac{\pi}{4}\right] \cdot u[-n-1] = 3^n u[-n-1] \cdot \left[\cos\left(\frac{2\pi}{6} n\right) \overset{\frac{\sqrt{2}}{2}}{\cancel{\cos\left(\frac{\pi}{4}\right)}} - \sin\left(\frac{2\pi}{6}\right) \overset{\frac{\sqrt{2}}{2}}{\cancel{\sin\left(\frac{\pi}{4}\right)}}\right]$

$$= \frac{\sqrt{2}}{2} 3^n \cdot u[-n-1] \left[\frac{e^{j\frac{\pi n}{3}} + e^{-j\frac{\pi n}{3}}}{2} - \frac{e^{j\frac{\pi n}{3}} - e^{-j\frac{\pi n}{3}}}{2j} \right] =$$

$$= \frac{\sqrt{2}}{4} 3^n u[-n-1] \cdot \left[e^{j\frac{\pi n}{3}} + e^{-j\frac{\pi n}{3}} + j e^{j\frac{\pi n}{3}} - j e^{-j\frac{\pi n}{3}} \right] =$$

$$= \frac{\sqrt{2}}{4} 3^n u[-n-1] \left[(1+j) e^{j\frac{\pi n}{3}} + (1-j) e^{-j\frac{\pi n}{3}} \right]$$

$$X(z) = \frac{\sqrt{2}}{4}(1+j) \sum_{n=-\infty}^{-1} \left(\frac{3e^{j\frac{\pi}{3}}}{z}\right)^n + \frac{\sqrt{2}}{4}(1-j) \sum_{n=-\infty}^{-1} \left(\frac{3e^{-j\frac{\pi}{3}}}{z}\right)^n$$

$$= \frac{\sqrt{2}}{4} \left[(1+j) \cdot \frac{1-0}{\frac{3e^{j\frac{\pi}{3}}}{z} - 1} + (1-j) \cdot \frac{1-0}{\frac{3e^{-j\frac{\pi}{3}}}{z} - 1} \right]$$

$$= \frac{\sqrt{2}}{4} \left[(1+j) \left(\frac{z}{3e^{j\frac{\pi}{3}} - z} \right) + (1-j) \left(\frac{z}{3e^{-j\frac{\pi}{3}} - z} \right) \right]$$

$$\left| \frac{3e^{\pm j\frac{\pi}{3}}}{z} \right| > 1 \Rightarrow |z| < \left| 3e^{\pm j\frac{\pi}{3}} \right| \Rightarrow |z| < 3$$

RoC: $|z| < 3 \Rightarrow$ υπάρχει DTFT

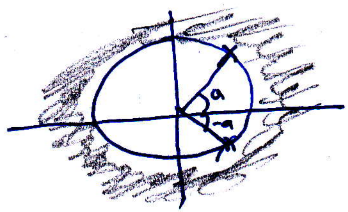
B) 1. $x[n] = \cos(an) u(n) = \frac{e^{jan} + e^{-jan}}{2} \cdot u(n) =$

$$X(z) = \frac{1}{2} \left[\sum_{n=0}^{\infty} \left(\frac{e^{ja}}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{e^{-ja}}{z}\right)^n \right] =$$

$$= \frac{1}{2} \left[\frac{0-1}{\frac{e^{ja}}{z} - 1} + \frac{0-1}{\frac{e^{-ja}}{z} - 1} \right] =$$

$$= \frac{1}{2} \left[-\frac{z}{e^{ja} - z} + \frac{-z}{e^{-ja} - z} \right] = -\frac{z}{2} \left[\frac{1}{e^{ja} - z} + \frac{1}{e^{-ja} - z} \right]$$

$$\left| \frac{e^{\pm ja}}{z} \right| < 1 \Rightarrow |z| > \left| e^{\pm ja} \right| \Rightarrow |z| > 1$$



RoC: $|z| > 1$ άρα δεν υπάρχει DTFT

2. $x(n) = n\big(u(n) - u(n-6)\big) = n \cdot u(n) - n \cdot u(n-6)$

$$X(z) = \frac{d\left(\frac{z}{z-1}\right)}{dz} - \frac{d\left(\frac{z^{-5}}{z-1}\right)}{dz}$$

$|z|>1$ $\qquad$ $|z|>1$

$$= \frac{z-1-z}{(z-1)^2} - \frac{-5z^{-6}(z-1) - z^{-5}}{(z-1)^2}$$

$$= \frac{-1}{(z-1)^2} - \frac{-5z^{-5} + 5z^{-6} - z^{-5}}{(z-1)^2} =$$

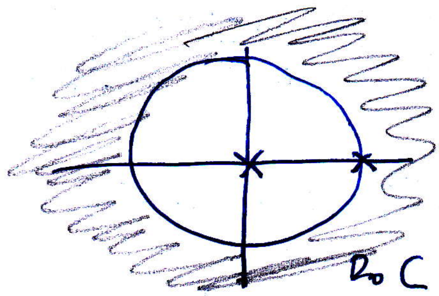
$$= \frac{-1 + 6z^{-5} - 5z^{-6}}{(z-1)^2} = \frac{-z^6 + 6z - 5}{z^6 (z-1)^2}$$

αφού $u(n-6) \overset{z}{\rightleftharpoons} z^{-6} \frac{z}{z-1} = \frac{z^{-5}}{z-1}$

$z=1$: διπλός πόλος και μηδενικό

$z=0$: εξαπλός πόλος

↳ δεν ορίζεται DTFT



Ασκ2

a) 1. $X(z) = \frac{1 - z^{-1}}{1 - \frac{1}{4}z^{-2}}, \quad |z| > \frac{1}{2}$

$$X(z) = \frac{\frac{z-1}{z}}{\frac{z^2 - \frac{1}{4}}{z^2}} = \frac{z(z-1)}{z^2 - \frac{1}{4}} = \frac{z(z-1)}{(z-\frac{1}{2})(z+\frac{1}{2})} = 1 + \frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{1}{2}} =$$

$$= 1 - \frac{3}{2(2z+1)} - \frac{1}{2(2z-1)} = 1 - \frac{3}{4} \cdot \frac{1}{z-(-\frac{1}{2})} - \frac{1}{4} \frac{1}{z - (\frac{1}{2})}$$

$$\Rightarrow x[n] = \delta[n] - \frac{3}{4}\left(-\frac{1}{2}\right)^{n-1} u[n-1] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

2. $X(z) = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$, $|z|>2$

$$\frac{X(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{\Gamma}{(z-2)^2} + \frac{\Delta}{(z-3)^2}$$

$$\lambda_k = \frac{1}{(r-k)!}\cdot\frac{d^{r-k}}{dz^{r-k}}\left[(z-p)^r X(z)\right]_{z=p}$$

$$A = \left[\frac{\cancel{(z-1)}(2z^2-11z+12)}{\cancel{(z-1)}(z-2)^3}\right]_{z=1} = \frac{3}{-1} = -3$$

$$B = \frac{1}{2!}\left[\frac{d^2}{dz^2}\left(\frac{(2z^2-11z+12)}{z-1}\right)\right]_{z=2} = \frac{1}{2}\left[\frac{d}{dz}\left(\frac{(4z-11)(z-1)-(2z^2-11z+12)}{(z-1)^2}\right)\right]_{z=2}$$

$$= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{4z^2-11z-4z+11-2z^2+11z-12}{(z-1)^2}\right)\right]_{z=2} = \frac{1}{2}\left[\frac{d}{dz}\left(\frac{2z^2-4z-1}{(z-1)^2}\right)\right]_{z=2} =$$

$$= \frac{1}{2}\left[\frac{(4z-4)(z^2-2z+1)-(2z^2-4z-1)(2z-2)}{(z-1)^2}\right]_{z=2} = \frac{1}{2}\left[\frac{4-(-2)}{1}\right] = \frac{6}{2} = 3$$

$$\Gamma = \left[\frac{d}{dz}\left(\frac{2z^2-11z+12}{z-1}\right)\right]_{z=2} = \left[\frac{(6z-11)(z-1)-(2z^2-11z+12)}{(z-1)^2}\right]_{z=2} =$$

$$= \frac{1-(-2)}{1} = 1+2 = 3$$

$$\Delta = \left[\frac{2z^2-11z+12}{z-1}\right]_{z=2} = -2$$

$$\Rightarrow X(z) = \frac{-3z}{z-1} + \frac{3z}{z-2} + \frac{3z}{(z-2)^2} - \frac{2z}{(z-2)^3}$$

$$\Rightarrow x(n) = -3u(n) + 3\cdot 2^n u(n) + 3n2^{n-1}u(n-1) - 2\cdot\frac{1}{2^2\cdot 2!}n(n-1)(n+1)2^n u(n)$$

3. $X(z) = e^{z^{-1}}$, $|z|>0$ & $x(n)=0,\ n<0$

↳ ανάπτυγμα Taylor του $e^{z^{-1}}$: $\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = \sum_{n=-\infty}^{+\infty}\frac{z^{-n}}{|n|!}$

$$\Rightarrow x(n) = \frac{1}{|n|!}$$

4. $X(z) = \dfrac{z-4}{z^2-5z+6} = \dfrac{z-4}{(z-2)(z-3)}$, $|z|>3$

$$= \frac{A}{z-2} + \frac{B}{z-3}$$

$$A = \left[\frac{z-4}{z-3}\right]_{z=2} = \frac{-2}{-1} = 2$$

$$B = \left[\frac{z-4}{z-2}\right]_{z=3} = -\frac{1}{1} = -1$$

$$X(z) = \frac{2}{z-2} - \frac{1}{z-3}$$

$$x(n) = 2\cdot 2^{n-1}u(n-1) - 3^{n-1}u(n-1) = u(n-1)\left(2^n - 3^{n-1}\right)$$

5. $X(z) = \dfrac{2z(3z+17)}{(z-1)(z^2-6z+25)}$, $|z|>5$

$$= \frac{2z(3z+17)}{(z-1)(z-(3+4j))(z-(3-4j))}$$

$$\frac{X(z)}{2z} = \frac{A}{z-1} + \frac{B}{z-(3+4j)} + \frac{\Gamma}{z-(3-4j)}$$

$$A = \left[\frac{3z+17}{(z-(3+4j))(z-(3-4j))}\right]_{z=1} = \frac{20}{(-2-4j)(-2+4j)} = \frac{20}{4+16} = \frac{20}{20} = 1$$

$$B = \left[\frac{3z+17}{(z-1)(z-(3-4j))}\right]_{z=3+4j} = \left[\frac{3(3+4j)+17}{(3+4j-1)(3+4j-3+4j)}\right] =$$

$$= \frac{9+12j+17}{(2+4j)8j} = \frac{26+12j}{16j+32j^2} = \frac{26+12j}{-32+16j} = \frac{13+6j}{-16+8j} = 2{,}39-1{,}23j$$

$$\Gamma = \left[\frac{3z+17}{(z-1)(z-(3+4j))}\right]_{z=3-4j} = \left[\frac{3(3-4j)+17}{(3-4j-1)(3-4j-3-4j)}\right] = \dots = -9{,}8+9{,}03j$$

$$X(z) = \frac{2z}{z-1} + \frac{2z(2{,}39-1{,}23j)}{z-(3+4j)} + \frac{2z(-9{,}8+9{,}03j)}{z-(3-4j)}$$

$$x(n) = 2u(n) + 2(2{,}39-1{,}23j)(3+4j)^n u(n) + 2z(-9{,}8+9{,}03j)(3-4j)^n u(n)$$

6. $X(z) = \ln\left(\frac{1}{1-\frac{z}{a}}\right) \qquad |z|<|a|$

$= \ln\left(\frac{a}{a-z}\right) = \ln a - \ln(a-z)$

για το $X_1(z) = \ln(a-z)$: Taylor του $X_1(z)$: $\sum_{n=1}^{\infty} \frac{-z^n}{n} = \sum_{-\infty}^{-1} \frac{z^n}{n}$

$x_1(t) = \frac{1}{n} u[-n-1]$

$x(t) = \ln(a)\,\delta(n) - \frac{1}{n} u[-n-1]$

Ασκ 3

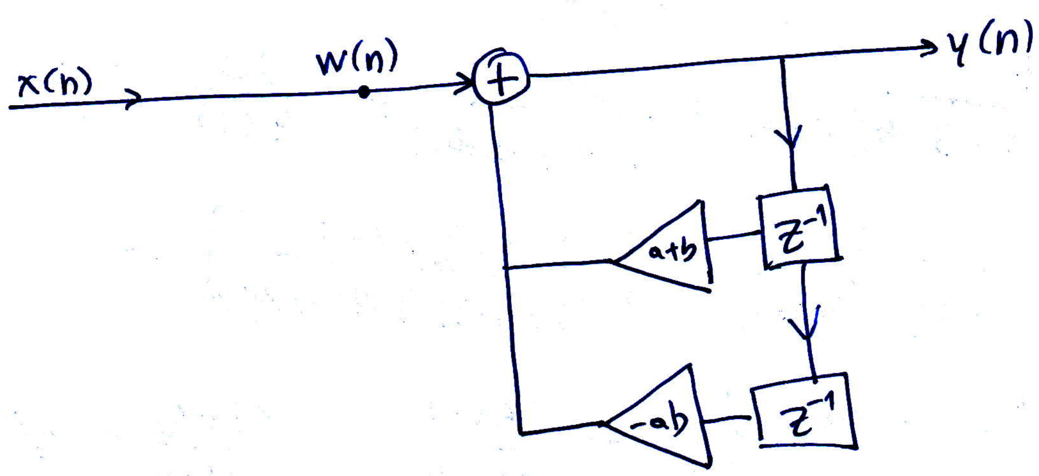
a) $y(n) - (a+b)y(n-1) + ab\,y(n-2) = x(n)$

$\Rightarrow Y(z) - (a+b)z^{-1}Y(z) + ab\,z^{-2}Y(z) = X(z)$

$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-(a+b)z^{-1}+ab\,z^{-2}}$

1. κατευθείαν μορφή 1

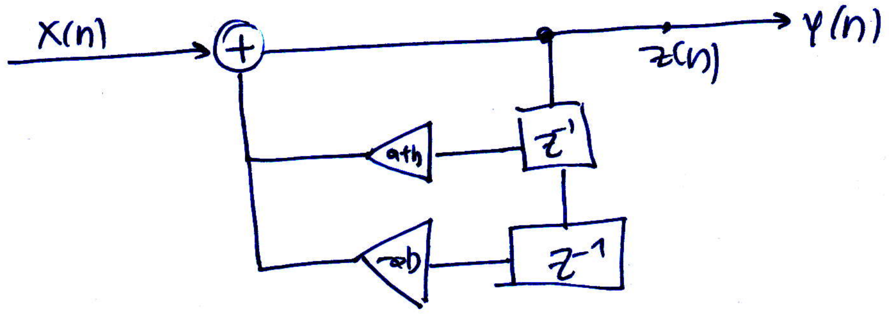
$Y[n] = [x[n]] + [(a+b)\,y(n-1) - ab\,y(n-2)]$



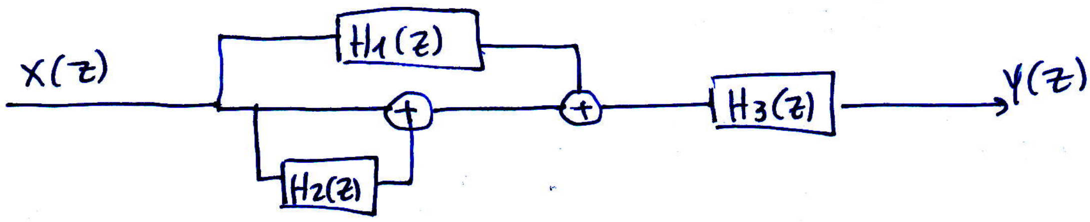
2. κατευθείαν μορφή 2

$$z(n) = x(n) + (a+b)\,z(n-1) - ab\,z(n-2)$$

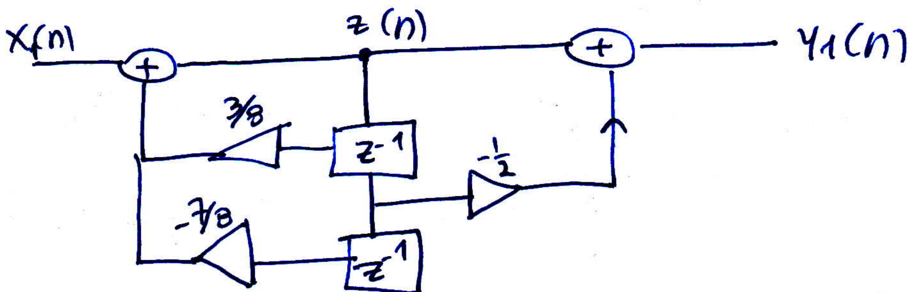
$$y(n) = z(n) = x(n) + (a+b)\,z(n-1) - ab\,z(n-2)$$



β)



για το $H_1(z)$:



$$y_1(n) = \frac{3}{8}y_1(n-1) - \frac{7}{8}y_1(n-2) + x(n) - \frac{1}{2}x(n-1)$$

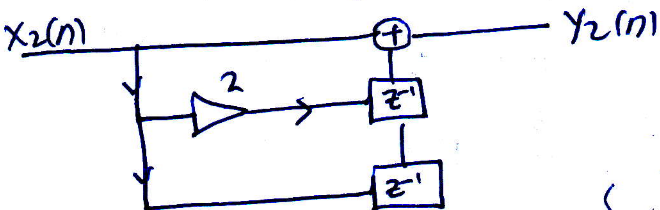
ή $z(n) = \frac{3}{8}z(n-1) - \frac{7}{8}z(n-2) + x(n)$

$$y_1(n) = z(n) - \frac{1}{2}z(n-1)$$

$\Rightarrow \frac{1}{2}z(n) - \frac{1}{2}y_1(n) = \frac{1}{2}z(n) - \frac{1}{2}y_1(n)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{8}z^{-1}-\frac{7}{8}z^{-2}}$$

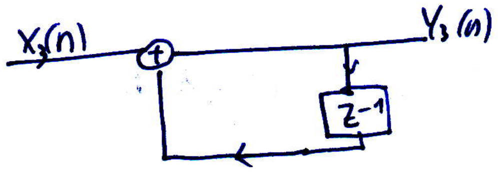
για το $H_2(z)$:



$$y_2(n) = x(n) + 2x(n-1) + x(n-2)$$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = 1+2z^{-1}+z^{-2}$$

Για το $H_3(z)$:



$$y_3(n) = x_3(n) + y_3(n-1) \quad \Bigg\} \quad H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{1}{1-z^{-1}}$$

$$Y(z) = \left(H_1(z) + H_2(z) \right) H_3(z) \cdot X(z)$$

$$[H_1(z) + H_2(z)] H_3(z) = \left(\frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{8}z^{-1}-\frac{7}{8}z^{-2}} + 1+2z^{-1}-z^{-2} \right) \cdot \frac{1}{1-z^{-1}} =$$

$$= \frac{1-\frac{1}{2}z^{-1} + \left(1+\frac{3}{8}z^{-1}-\frac{7}{8}z^{-2}\right)\left(1+2z^{-1}-z^{-2}\right)}{1+\frac{3}{8}z^{-1}-\frac{7}{8}z^{-2}} \cdot \frac{1}{1-z^{-1}}$$

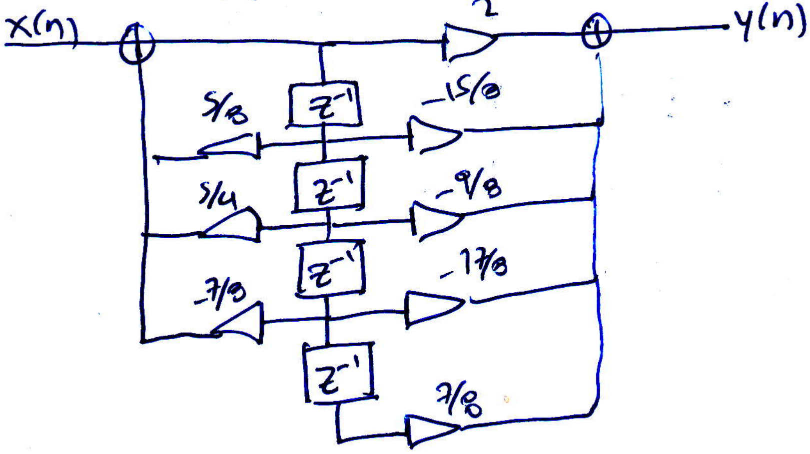
$$= \frac{1-\frac{1}{2}z^{-1} + 1+2z^{-1}-z^{-2} + \frac{3}{8}z^{-1} - \frac{3}{4}z^{-2} - \frac{3}{8}z^{-3} - \frac{7}{8}z^{-2} - \frac{7}{4}z^{-3} + \frac{7}{8}z^{-4}}{1+\frac{3}{8}z^{-1}-\frac{7}{8}z^{-2} - z^{-1} - \frac{3}{8}z^{-2} + \frac{7}{8}z^{-3}} =$$

$$= \frac{2+\frac{15}{8}z^{-1} - \frac{9}{8}z^{-2} - \frac{17}{8}z^{-3} + \frac{7}{8}z^{-4}}{1-\frac{5}{8}z^{-1} - \frac{5}{4}z^{-2} + \frac{7}{8}z^{-3}} = \frac{Y(z)}{X(z)}$$

2. $2x(n) + \frac{15}{8}x(n-1) - \frac{9}{8}x(n-2) - \frac{17}{8}x(n-3) + \frac{7}{8}x(n-4) =$

$= y(n) - \frac{5}{8}y(n-1) - \frac{5}{4}y(n-2) - \frac{7}{8}y(n-3)$

3. Για το H_1 θα το φτιάξουμε σε κανονική μορφή II (για ελάχιστα z^{-1} στοιχεία)



Ασκ 4

α) $x(n)$ μη μηδεν. τιμές στο $0 \leq n \leq N-1$, N άρτιος

1. $f(n) = x(2n)$, $g(n) = x(2n+1)$

άρτια δείγματα $x(n)$, περιττά δείγματα $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) w_N^{nk}$$ ο DFT N σημείων του $x(n)$ $W_N = e^{-j\frac{2\pi}{N}}$

$f(n) \xrightarrow{DFT} F(k)$, $g(n) \xrightarrow{DFT} G(k)$, DFT των $\frac{N}{2}$ σημείων τους

$$F(k) = \sum_{n=0}^{N-1} f(n) w^{nk} = \sum_{n=0}^{N-1} x(2n) w^{nk} =$$

$$= x(2\cdot 0) + x(2\cdot 1) w^{k} + \dots + x\left(2\left(\frac{N}{2}-1\right)\right) w^{\left(\frac{N}{2}-1\right)k} +$$ } για $0 \leq n \leq \frac{N}{2}-1$ άρα: $F(k)$: $0 \leq n \leq \frac{N}{2}-1$

$$+ x\left(2\cdot\frac{N}{2}\right) w^{k\left(\frac{N}{2}\right)} + \dots + x(2\cdot N) w^{kN}$$ } για $n \geq \frac{N}{2}$ $f(n) = 0$ αφού $x(n) = 0$ για $n > N-1$

$\Rightarrow$ $f(n)$ μηδενίζεται εκτός του $0 \leq n \leq \frac{N}{2}-1$

$$G(k) = \sum_{n=0}^{N-1} g(n) w^{nk} = \sum_{n=0}^{N-1} x(2n+1) w^{nk} =$$

$$= x(2\cdot 0+1) w^{0k} + x(2\cdot 1+1) w^{1\cdot k} + \dots + x\left(2\left(\frac{N}{2}-1\right)+1\right) w^{\left(\frac{N}{2}-1\right)k} +$$ } για $0 \leq n \leq \frac{N}{2}-1$ άρα $G(k)$: $0 \leq n \leq \frac{N}{2}-1$

$$+ \dots + x\left(2\left(\frac{N}{2}\right)+1\right) w^{\frac{N}{2}k} + \dots + x(2N+1) w^{Nk}$$ } για $n \geq \frac{N}{2}$ $g(n) = 0$ αφού $x(n) = 0$ για $n > N-1$

$\Rightarrow$ $g(n)$ μηδενίζεται εκτός του $0 \leq n \leq \frac{N}{2}-1$

2. Nδo $F[k+\frac{N}{2}] = F[k]$

$$F[k+\tfrac{N}{2}] = \sum_{n=0}^{N-1} F(n+\tfrac{N}{2})\, w^{k(n+\frac{N}{2})} = \sum_{n=0}^{N-1} x(2(n+\tfrac{N}{2}))\, w^{k(n+\frac{N}{2})} =$$

$$= \sum_{n=0}^{N-1} x(2n+N)\, w^{k(n+\frac{N}{2})} = x(N)\, w^{k(\frac{N}{2})} + x(2\cdot 1+N)\, w^{k(1+\frac{N}{2})} + x(2\cdot 2+N)\, w^{k(2+\frac{N}{2})}$$

$$+\dots+ x(2(\tfrac{N}{2}-1)+N)\, w^{k(\frac{N}{2}-1+\frac{N}{2})} =$$

$$= x(N)\, w^{k(\frac{N}{2})} + x(2+N)\, w^{k(1+\frac{N}{2})} + x(4+N)\, w^{k(2+\frac{N}{2})} + \dots +$$

$$+ x(.N-2)\, w^{k(N-1)}$$

$$\overset{x(n)=x(n+N)}{=} x(0) + x(2)\, w^{k} + x(4)\, w^{2k} + \dots + x(N-2)\, w^{(\frac{N}{2}-1)k}$$

$$\Rightarrow F(k+\tfrac{N}{2}) = F(k)$$

Nδo $G(k) = G(k+\frac{N}{2})$

$$G(k+\tfrac{N}{2}) = \sum_{n=0}^{N-1} g(n+\tfrac{N}{2})\, w^{k(n+\frac{N}{2})} = \sum_{n=0}^{N-1} x(2(n+\tfrac{N}{2})+1)\, w^{k(n+\frac{N}{2})} =$$

$$= \sum_{n=0}^{N-1} x(2n+N+1)\, w^{k(n+\frac{N}{2})}$$

$$= x(N+1)\, w^{k(\frac{N}{2})} + x(N+3)\, w^{k(1+\frac{N}{2})} + x(N+5)\, w^{k(2+\frac{N}{2})} + \dots + x(2N-1)\, w^{k(\frac{N}{2}-1+\frac{N}{2})}$$

$$= x(N+1)\, w^{k\frac{N}{2}} + x(N+3)\, w^{k(1+\frac{N}{2})} + x(N+5)\, w^{k(2+\frac{N}{2})} + \dots + x(2N-1)\, w^{k(N-1)}$$

$$\overset{x(n)=x(n+N)}{=} x(1) + x(3)\, w^{k} + x(5)\, w^{2k} + \dots + x(N-1)\, w^{k(\frac{N}{2}-1)}$$

$$\Rightarrow G(k+\tfrac{N}{2}) = G(k)$$

3. Νδο $X(k) = \frac{1}{2}\left(F(k) + W_N^k G(k)\right)$, $k = 0, 1, \ldots, N-1$

$F(k)$ για $k = 0, 1, \ldots, N-1$

$= F(k') + F(k' + \frac{N}{2})$

$= 2F(k')$, $k' = 0, \ldots, \frac{N}{2}-1$ $(k' = 2k)$

ομοίως

$G(k) = 2G(k')$

$\Rightarrow$ νδο $X(k) = \cancel{\frac{1}{2}\cdot 2}\left(F(k') + W_N^k G(k')\right)$

$BM = X(0) + X(2)W^{2k} + \ldots + X(N-2)\, W^{(N-2)k}$

$+ X(1)W^k + X(3)\, W^{3k} + \ldots + X(N-1)\, W^{(N-1)k}$

$X(k) = \sum_{n=0}^{N-1} x(n)\, W^{nk} = X(0) + X(1)W^k + \ldots + X(N-2)\, W^{(N-2)k} + X(N-1)\, W^{(N-1)k}$

$\Rightarrow X(k) = \frac{1}{2}\left(F(k) + W_N^k G(k)\right)$

$\triangleright$ $X(k) = \sum_{n=0}^{N-1} X(n)\, W_N^{nk}$ & $|X(k)| = |X(N-k)|$

$X(k) = X_r(k) + jX_i(k)$ και (3) $X(k) = \frac{1}{2}\left(F(k) + W^k G(k)\right)$

και από (2) $F(k + \frac{N}{2}) = F(k)$ & $G(k + \frac{N}{2}) = G(k)$

$\Rightarrow F_i(k) = G_i(k) = 0$

$\Rightarrow X(k) = X_r(k)$

B) a) $x(n) = \left(\frac{1}{4}\right)^n u(n+2) = \dfrac{\left(\frac{1}{4}\right)^{n+2}}{\left(\frac{1}{4}\right)^2} u(n+2)$

$$X(\Omega) = 16 e^{2j\Omega} F^{-1}\left\{ \left(\tfrac{1}{4}\right)^n u(n) \right\} = \frac{16 e^{2j\Omega}}{1 - \frac{e^{-j\Omega}}{4}}$$

β) $x(n) = \sum_{k=-\infty}^{+\infty} \left(\frac{1}{4}\right)^n \delta(n-3k) u(n) = \left(\frac{1}{4}\right)^n u(n) \sum_{k=-\infty}^{+\infty} \delta(n-3k)$

$$X(\Omega) = \frac{1}{1 - \frac{e^{-j\Omega}}{4}} * \frac{2\pi}{3} \sum_{k=-\infty}^{+\infty} \delta\left(\Omega - \frac{2\pi k}{3}\right)$$

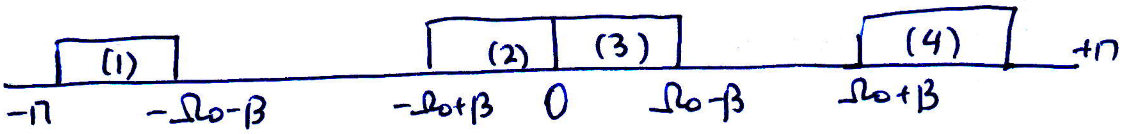
$$= \frac{1}{2\pi} \cdot \frac{2\pi}{3} \sum_{k=-\infty}^{+\infty} \frac{1}{1 - \frac{e^{-j(\Omega - \frac{2k\pi}{3})}}{4}} = 4 \sum_{k=-\infty}^{+\infty} \frac{1}{4 - e^{-j(\Omega - \frac{2\pi k}{3})}}$$

γ) $x(n) = n\left(\frac{1}{2}\right)^{|n|} = n\left(\frac{1}{2}\right)^n u(n) + n\left(\frac{1}{2}\right)^{-n} u(-n)$

$$X(\Omega) = j \frac{d\left(\frac{1}{1 - \frac{e^{-j\Omega}}{2}}\right)}{d\Omega} + j \frac{d\left(\frac{1}{1 - \frac{e^{j\Omega}}{2}}\right)}{d\Omega} \qquad \left(x(-n) \rightleftharpoons X(-\Omega) \right)$$

$$= j \frac{1 - \frac{e^{-j\Omega}}{2} + j\frac{e^{-j\Omega}}{2}}{\left(1 - \frac{e^{-j\Omega}}{2}\right)^2} + j \frac{1 - \frac{e^{j\Omega}}{2} - \frac{e^{j\Omega}}{2}}{\left(1 - \frac{e^{j\Omega}}{2}\right)^2}$$

δ) $X(\Omega) = \begin{cases} 1, & 0 \le \Omega_0-\beta < \Omega < \Omega_0+\beta < \pi \\ 1, & -\pi < -\Omega_0-\beta < \Omega < -\Omega_0+\beta < 0 \\ 0, & \text{αλλού} \quad (\text{εντός } [-\pi,\pi]) \end{cases}$



Εύρος (1) = εύρος (4) αφού $|-\pi+\Omega_0+\beta| = |\pi-\Omega_0-\beta|$

Εύρος (2) = εύρος (3) αφού $|-\Omega_0+\beta| = |\Omega_0-\beta|$

$$\left(X(\Omega) = \begin{cases} 1, & 0 \le |\Omega| \le W \\ 0, & W \le |\Omega| \le \pi \end{cases} \rightleftarrows \frac{\sin(Wn)}{\pi n} \right)$$

για τα (2) & (3): $W = \dfrac{\Omega_0-\beta}{2}$ & μετατόπιση $\pm W$ από το 0

$$X_1(n) = e^{-jW}\frac{\sin(Wn)}{\pi n} + e^{jW}\frac{\sin(Wn)}{\pi n}$$

$$\Rightarrow X_1(n) = e^{-j\frac{\Omega_0-\beta}{2}}\frac{\sin\left(\frac{\Omega_0-\beta}{2}n\right)}{\pi n} + e^{j\frac{\Omega_0-\beta}{2}}\frac{\sin\left(\frac{\Omega_0-\beta}{2}n\right)}{\pi n}$$

για τα (1) & (4): $W' = \dfrac{\pi-\Omega_0-\beta}{2}$ & μετατόπιση $\pm \Omega_0+\beta+W'$ από το 0.

$$= \pm \Omega_0+\beta+\frac{\pi}{2}-\frac{\beta}{2}-\frac{\Omega_0}{2}$$

$$= \pm \frac{\pi+\beta+\Omega_0}{2}$$

$$X_2(n) = e^{-j(\Omega_0+\beta+W')}\frac{\sin(W'n)}{\pi n} + e^{j(\Omega_0+\beta+W')}$$

$$\Rightarrow X_2(n) = e^{-\frac{j}{2}(\pi+\beta+\Omega_0)}\frac{\sin\left(\frac{\pi-\Omega_0-\beta}{2}n\right)}{\pi n} + e^{\frac{j}{2}(\pi+\beta+\Omega_0)}\frac{\sin\left(\frac{\pi-\Omega_0-\beta}{2}n\right)}{\pi n}$$

$x(n) = X_1(n) + X_2(n).$

ε) $X(\Omega) = \cos^2\Omega = \frac{1+\cos 2\Omega}{2} = \frac{1}{2} + \frac{\cos 2\Omega}{2} =$

$= \frac{1}{2} + \frac{e^{j2\Omega} + e^{-j2\Omega}}{4} = \frac{1}{2} + \frac{1}{4}e^{j2\Omega} + \frac{1}{4}e^{-j2\Omega}$

$$x(n) = \frac{\delta(n)}{2} + \frac{1}{4}\delta(n+2) + \frac{1}{4}\delta(n-2)$$

φ) $X(\Omega) = \frac{1}{(1-ae^{-j\Omega})^r}$

$x(n) = \frac{(n+r-1)!}{n!\,(r-1)!}\, a^n u(n)$ για $|a|<1$ (τύπος βιβλίου σελ. 209)

δεν ορίζεται για $|a|>0$.

Ασκ 5

α) 1. $y(n) - y(n-1) + y(n-2) = \frac{1}{2}\left(x(n) + x(n-1)\right)$

$Y(z) - z^{-1}Y(z) + z^{-2}Y(z) = \frac{1}{2}\left(X(z) + z^{-1}X(z)\right)$

$\Rightarrow Y(z)\left(1 - z^{-1} + z^{-2}\right) = X(z)\left(\frac{1}{2} + \frac{1}{2}z^{-1}\right)$

$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{\frac{1}{2} + \frac{1}{2}z^{-1}}{1 - z^{-1} + z^{-2}} = \frac{1}{2}\,\frac{\frac{z+1}{z}}{\frac{z^2-z+1}{z^2}} = \frac{1}{2}\,\frac{z^2+z}{z^2-z+1} =$

$= \frac{1}{2}\left(1 + \frac{2z-1}{z^2-z+1}\right) = \frac{1}{2} + \frac{z-\frac{1}{2}}{\left(z-\frac{1+i\sqrt{3}}{2}\right)\left(z-\frac{1-i\sqrt{3}}{2}\right)} =$

$= \frac{1}{2} + \frac{A}{z-\frac{1+i\sqrt{3}}{2}} + \frac{B}{z-\frac{1-i\sqrt{3}}{2}}$

$$A=\left[\frac{z-\frac{1}{2}}{z-\frac{1-i\sqrt{3}}{2}}\right]_{z=\frac{1+i\sqrt{3}}{2}}=\left[\frac{\frac{1}{2}+\frac{i\sqrt{3}}{2}-\frac{1}{2}}{\frac{1}{2}+\frac{i\sqrt{3}}{2}-\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right]=\frac{\frac{i\sqrt{3}}{2}}{i\sqrt{3}}=\frac{1}{2}$$

$$B=\left[\frac{z-\frac{1}{2}}{z-\frac{1+i\sqrt{3}}{2}}\right]_{z=\frac{1-i\sqrt{3}}{2}}=\left[\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}-\frac{1}{2}}{\frac{1}{2}-\frac{i\sqrt{3}}{2}-\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right]=\frac{\frac{i\sqrt{3}}{2}}{i\sqrt{3}}=\frac{1}{2}$$

$$=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{z-\frac{1+i\sqrt{3}}{2}}+\frac{1}{2}\cdot\frac{1}{z-\frac{1-i\sqrt{3}}{2}} \qquad \left\{\left|\frac{1-i\sqrt{3}}{2}\right|=\frac{1+3}{4}=1\right.$$

για $|z|>1$: $h(n)=\frac{1}{2}\delta(n)+\frac{1}{2}\left(\frac{1+i\sqrt{3}}{2}\right)^{n-1}u(n-1)+\frac{1}{2}\left(\frac{1-i\sqrt{3}}{2}\right)^{n-1}u(n-1)$

$|z|<1$: $h(n)=\frac{1}{2}\delta(n)-\frac{1}{2}\left(\frac{1+i\sqrt{3}}{2}\right)^{n-1}u(-n)-\frac{1}{2}\left(\frac{1-i\sqrt{3}}{2}\right)^{n-1}u(-n)$

2. $x(n)=\left(\frac{1}{2}\right)^{n}u(n)$ $\qquad y(-1)=\frac{3}{4},\quad y(-2)=\frac{1}{4}$

για $x(n)\rightleftarrows X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}=\frac{z}{z-\frac{1}{2}}$ $\qquad$ για $x(n-1)\rightleftarrows\frac{1}{z-\frac{1}{2}}$

ΕΔ $\Rightarrow$ $Y^{+}(z)-z^{-1}\left\{Y^{+}(z)+z\,y(-1)\right\}+z^{-2}\left\{Y^{+}(z)+y(-1)z+y(-2)z^{2}\right\}=$

$$=\frac{1}{2}\left(\frac{z}{z-\frac{1}{2}}+\frac{1}{z-\frac{1}{2}}\right)$$

$\Rightarrow$ $Y^{+}(z)-z^{-1}Y^{+}(z)-\frac{3}{4}+z^{-2}Y^{+}(z)+\frac{3}{4}z^{-1}+\frac{1}{4}=\frac{1}{2}\left(\frac{z+1}{z-\frac{1}{2}}\right)$

$\Rightarrow$ $Y^{+}(z)\left[1-z^{-1}+z^{-2}\right]=\frac{1}{2}\left(\frac{z+1}{z-\frac{1}{2}}\right)+\frac{1}{2}-\frac{3}{4}z^{-1}$

$\Rightarrow$ $Y^{+}(z)=\frac{1}{2}\left[\frac{\frac{z+1}{z-\frac{1}{2}}+\frac{1}{2}-\frac{3}{4z}}{\frac{z^{2}-z+1}{z^{2}}}\right]\Rightarrow$

$$\Rightarrow Y^+(z) = \frac{1}{2}\left[\frac{\dfrac{(z+1)z + \frac{z}{2}(z-\frac{1}{2}) - \frac{3}{4}(z-\frac{1}{2})}{\cancel{z}(z-\frac{1}{2})}}{\dfrac{z^2-z+1}{z^{\cancel{2}}}}\right] =$$

$$= \frac{z}{2}\left[\frac{z^2+z+\frac{z^2}{2}-\frac{z}{4}-\frac{3z}{4}+\frac{3}{8}}{(z-\frac{1}{2})(z^2-z+1)}\right] = \frac{z}{2}\left[\frac{\frac{3z^2}{2}+\frac{3}{8}}{(z-\frac{1}{2})(z^2-z+1)}\right]$$

$$\Rightarrow Y^+(z) = \frac{3z}{4}\left[\frac{z^2+\frac{1}{4}}{(z-\frac{1}{2})(z-\frac{1+j\sqrt{3}}{2})(z-\frac{1-j\sqrt{3}}{2})}\right]$$

$$\Rightarrow \frac{Y^+(z)}{\frac{3z}{4}} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1+j\sqrt{3}}{2}} + \frac{\Gamma}{z-\frac{1-j\sqrt{3}}{2}}$$

$$A = \frac{1}{3}, \quad B = \frac{1-j2\sqrt{3}}{6}, \quad \Gamma = \frac{1+j2\sqrt{3}}{6}$$

$$Y^+(z) = \frac{\frac{3z}{4}\cdot\frac{1}{3}}{z-\frac{1}{2}} + \frac{\frac{3z}{4}\cdot\left(\frac{1-j2\sqrt{3}}{6}\right)}{z-\frac{1+j\sqrt{3}}{2}} + \frac{\frac{3z}{4}\left(\frac{1+j2\sqrt{3}}{6}\right)}{z-\frac{1-j\sqrt{3}}{2}}$$

$$= \frac{1}{4}\frac{z}{z-\frac{1}{2}} + \frac{1-j2\sqrt{3}}{8}\cdot\frac{z}{z-\frac{1+j\sqrt{3}}{2}} + \frac{1-j2\sqrt{3}}{8}\cdot\frac{z}{z-\frac{1-j\sqrt{3}}{2}}$$

για RoC: $|z|>1$:

$$\Rightarrow y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1-j2\sqrt{3}}{8}\right)\left(\frac{1+j\sqrt{3}}{2}\right)^n u(n) +$$
$$+\left(\frac{1-j2\sqrt{3}}{8}\right)\left(\frac{1-j\sqrt{3}}{2}\right)^n u(n)$$

για RoC: $|z|<1$:

$$\rightarrow y(n) = u(-n-1)\left[-\frac{1}{4}\left(\frac{1}{2}\right)^n - \left(\frac{1-j2\sqrt{3}}{8}\right)\left(\frac{1+j\sqrt{3}}{2}\right)^n + \right.$$
$$\left. - \left(\frac{1-j2\sqrt{3}}{8}\right)\left(\frac{1-j\sqrt{3}}{2}\right)^n\right]$$

β) σύστημα A: $\frac{dz(t)}{dt} + 6z(t) = \frac{dx(t)}{dt} + 5x(t)$

$$h_b(t) = e^{-10t}u(t) \rightleftarrows H_b(t) = \frac{1}{10+j\omega}$$

α) $\xrightarrow{F\{\}}$ $j\omega Z(\omega) + 6Z(\omega) = j\omega X(\omega) + 5X(\omega)$

$$\Rightarrow Z(\omega)[j\omega+6] = X(\omega)[j\omega+5]$$

$$\Rightarrow \frac{Z(\omega)}{X(\omega)} = H(\omega) = \frac{j\omega+5}{j\omega+6} = \frac{j\omega}{j\omega+6} + \frac{5}{j\omega+6}$$

$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = H_a(\omega)\cdot H_b(\omega) = \frac{j\omega+5}{j\omega+6}\cdot\frac{1}{j\omega+10} = \frac{5}{4(j\omega+10)} - \frac{1}{4(j\omega+6)}$$

β) $\Rightarrow h(t) = \frac{5}{4}e^{-10t}u(t) - \frac{1}{4}e^{-6t}u(t)$

γ) $x(t) * h_a(t) * h_b(t) = y(t) \xrightarrow{F\{\}} X(\omega)\cdot H_a(\omega)\cdot H_b(\omega) = Y(\omega)$

$$\Rightarrow Y(\omega)\left[(j\omega)^2 + 16j\omega + 60\right] = X(j\omega+5)$$

$$\Rightarrow \frac{d^2y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 60y = \frac{dx(t)}{dt} + 5x(t)$$

δ) $x(t) = e^{-5t+5}u(t-1) = e^{-5(t-1)}u(t-1)$

$$y(t) = x(t) * h_a(t) * h_b(t) \to Y(\omega) = H(\omega)\cdot X(\omega)$$

$$X(\omega) = \frac{1}{5+j\omega}e^{-j\omega}$$

$$Y(\omega) = \frac{e^{-j\omega}}{5+j\omega}\cdot\frac{j\omega+5}{(j\omega+6)(j\omega+10)} = e^{-j\omega}\left[\frac{A}{j\omega+6} + \frac{B}{j\omega+10}\right]$$

$$\Rightarrow A = \frac{1}{4},\ B = -\frac{1}{4} \Rightarrow Y(\omega) = e^{-j\omega}\left[\frac{1}{4(j\omega+6)} - \frac{1}{4(j\omega+10)}\right]$$

$$\Rightarrow y(t) = \frac{1}{4}e^{-6(t-1)}u(t) - \frac{1}{4}e^{-10(t-1)}u(t).$$