Σήματα & Συστήματα

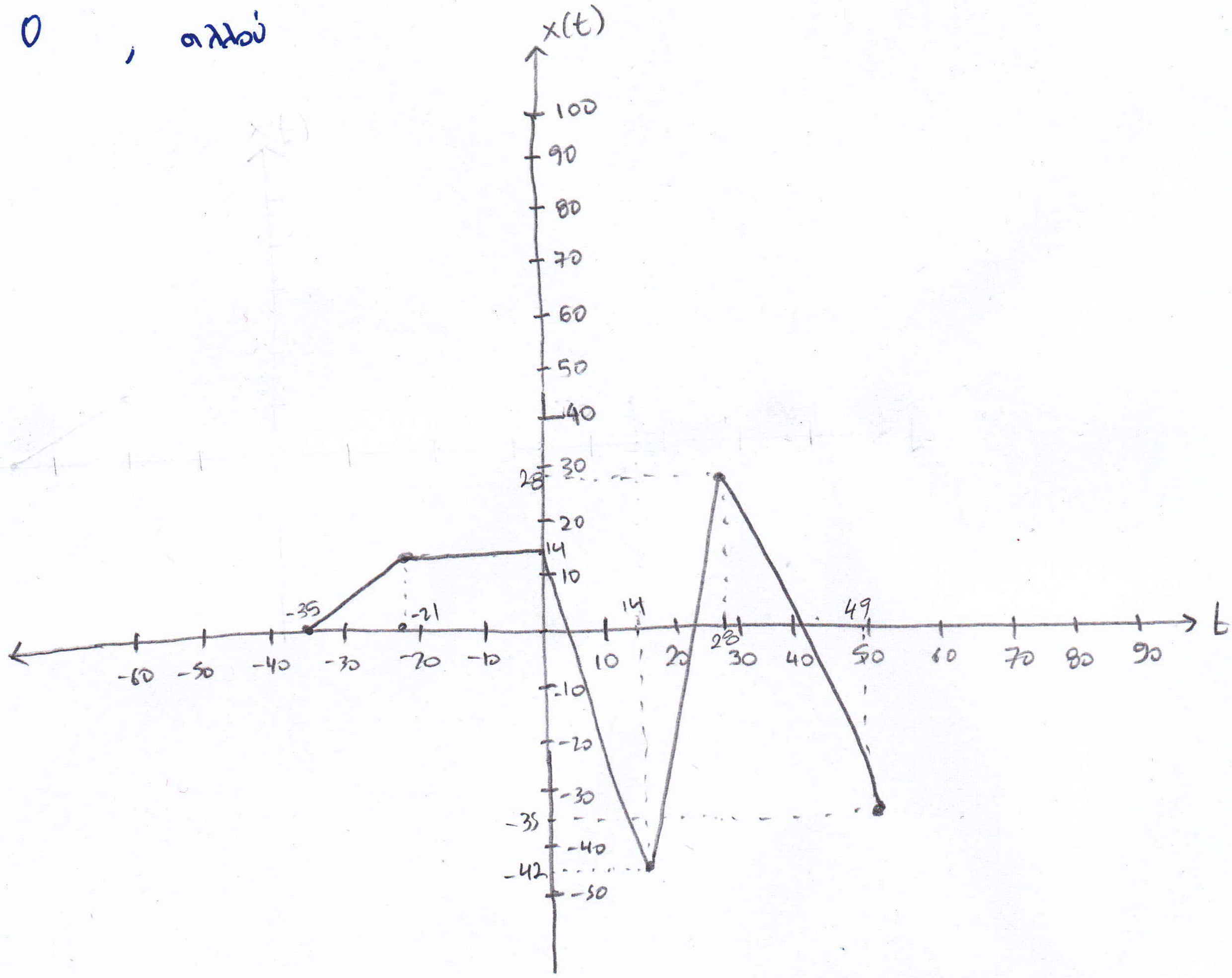
1η σειρά ασκήσεων

Χρήστος Τσούφης

$\alpha = 03117176 \bmod 10 + 1 = 6 + 1 = 7$

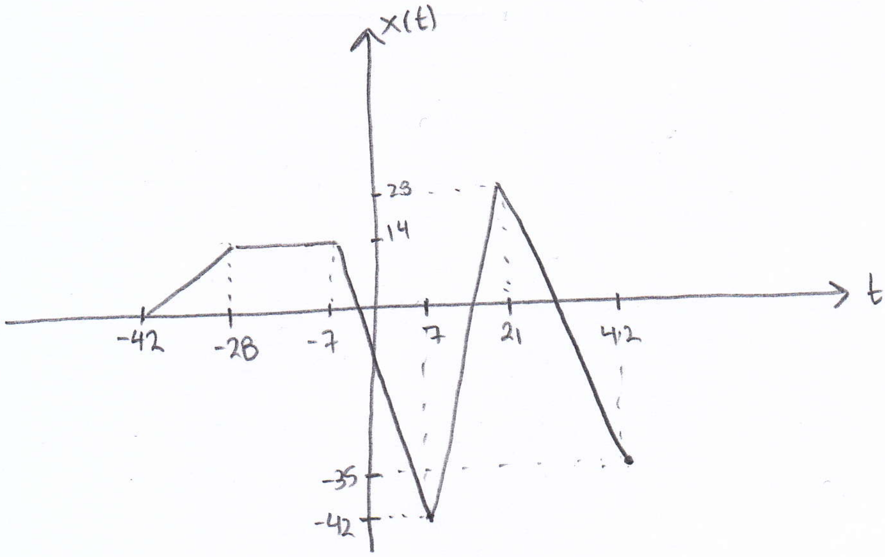
1.1

$$x(t) = \begin{cases} t + 35, & -35 \le t < -21 \\ 14, & -21 \le t < 0 \\ -4t + 14, & 0 \le t < 14 \\ 5t - 112, & 14 \le t < 28 \\ -3t + 112, & 28 \le t < 49 \\ 0, & \text{αλλού} \end{cases}$$

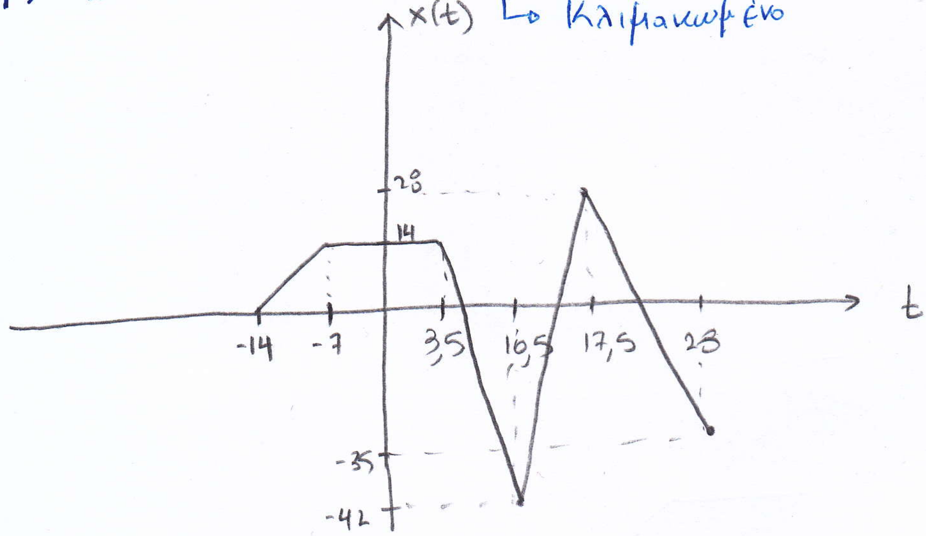


α) $x_1(t) = x(t+7)$

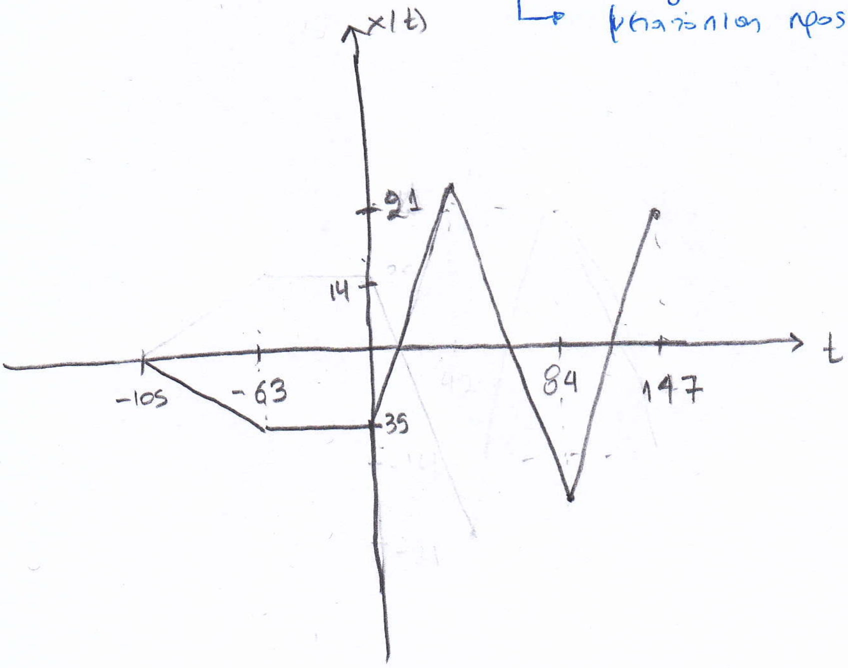
↳ Μετατοπισμένο 7 θέσεις αριστερά



β) $x_2(t) = x(2t-7)$ → Μετατοπισμένο 7 θέσεις δεξιά
→ Κλιμακωμένο

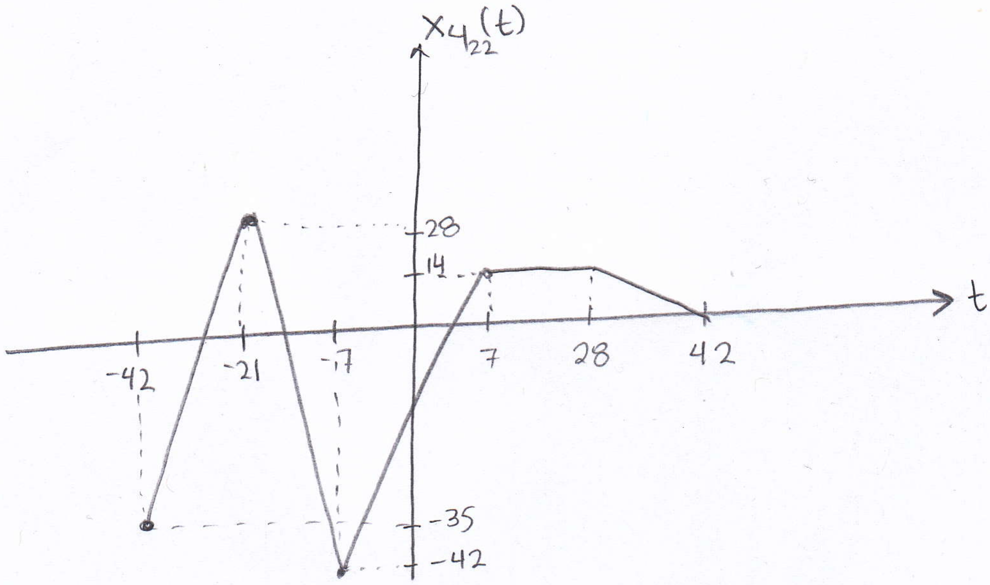


γ) $x_3(t) = -7\,x\left(\frac{t}{3}\right) + 21$ → κλιμάκωση
→ αλλαγή κλίσης
→ μετατόπιση προς τα πάνω

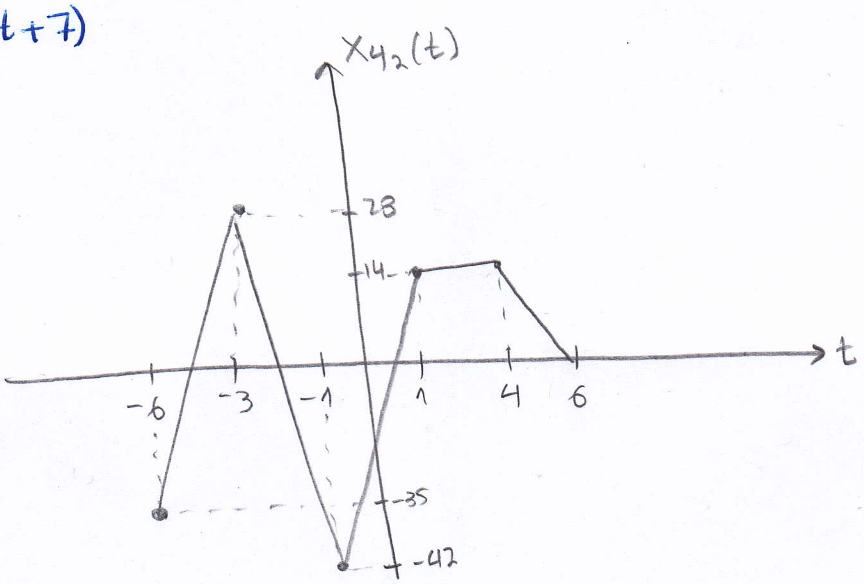


δ) $x_4(t) = x(t-7) - x(7-7t) = x(t-7) - x(-7t+7)$

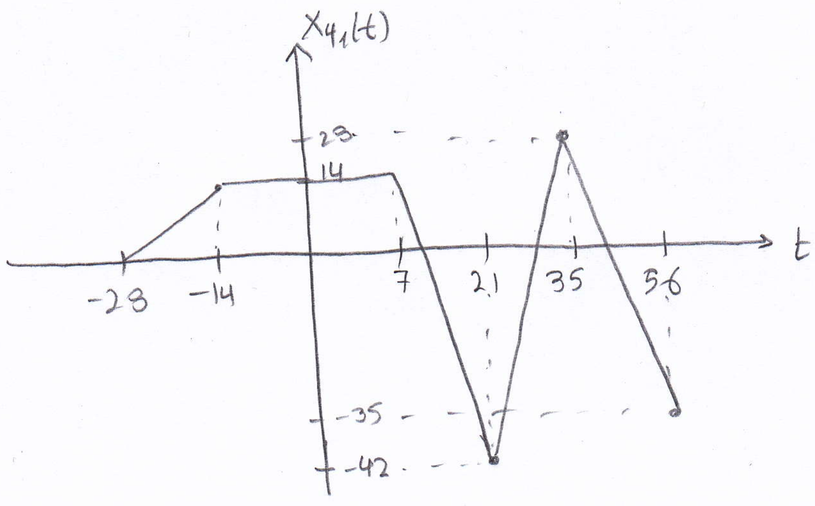
- $x_{4_1}(t) = x(t-7)$ παρόμοιο με (β)
- $x_{4_{21}}(t) = x(t+7)$ ίδιο με (α)
- $x_{4_{22}}(t) = x(-t+7)$ ανάκλαση



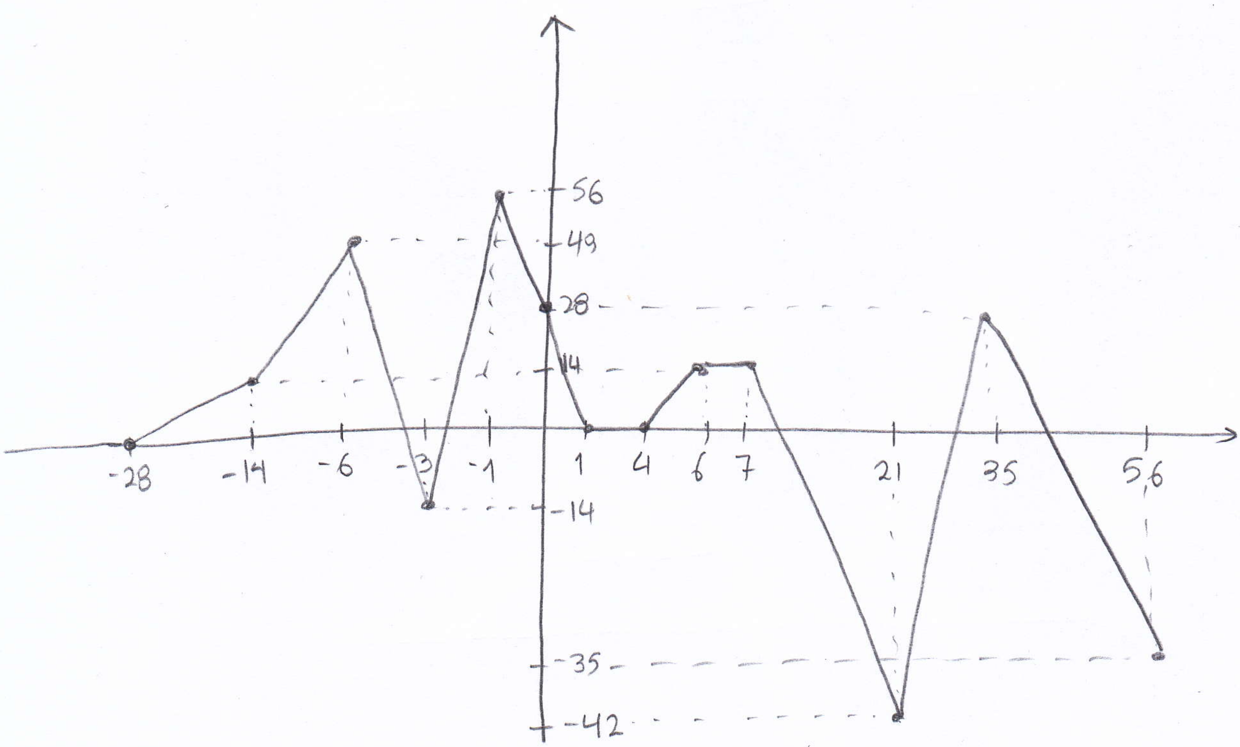
- $x_{4_2}(t) = x(-7t+7)$



- $x_{4_1}(t) = x(t-7)$



x	y_{42}	y_{41}	$y_{41} - y_{42}$
-28	0	0	0
-14	0	14	14
-6	-35	14	49
-3	28	14	-14
-1	-42	14	56
0	-14	14	28
1	14	14	0
4	14	14	0
6	0	14	14
7	0	14	14
21	0	-42	-42
35	0	28	28
56	0	-35	-35



1.2

α) $x_1(t) = \left[\cos\left(\frac{\pi}{5}t\right)\right]^1 + \left[\sin\left(\frac{\pi}{4\cdot 7}t\right)\right]^{3-1} + \cos\left(\frac{\pi}{8}t\right)\sin\left(\frac{\pi}{3\cdot 7}t\right)$

$\Rightarrow x_1(t) = \cos\left(\frac{\pi}{5}t\right) + \sin^2\left(\frac{\pi}{28}t\right) + \cos\left(\frac{\pi}{8}t\right)\cdot\sin\left(\frac{\pi}{21}t\right)$

β) $x_2(t) = \exp\left[j\left(7t + \frac{\pi}{3\cdot 7}\right)\cdot \mathrm{rect}\left(\frac{t}{2\cdot 7}\right)\right] + 7$

$\Rightarrow x_2(t) = \exp\left[j\left(7t + \frac{\pi}{21}\right)\cdot \mathrm{rect}\left(\frac{t}{14}\right)\right] + 7$

γ) $x_3(t) = \sin\left(\frac{\pi}{18\cdot 7}t\right) - \sum_{k=-\infty}^{+\infty}\delta\left(\frac{t + 6\cdot 7\cdot k}{3\cdot 7}\right)$

$\Rightarrow x_3(t) = \sin\left(\frac{\pi}{126}t\right) - \sum_{k=-\infty}^{+\infty}\delta\left(\frac{t+42k}{21}\right)$

δ) $x_4[n] = \exp\left(j\left(\frac{\pi}{6\cdot 7}n^2 + \frac{\pi}{9\cdot 7}n^3\right)\right)$

$\Rightarrow x_4[n] = \exp\left(j\left(\frac{\pi}{42}n^2 + \frac{\pi}{63}n^3\right)\right)$

ε) $x_5[n] = \sum_{k=-\infty}^{+\infty}\left[\delta(n-k(3+1)) - \delta(n-k(1+1)) + \delta(n-k(7+1))\right] + \cos\left(\frac{\pi}{3\cdot 7}n\right)$

$\Rightarrow x_5[n] = \sum_{k=-\infty}^{+\infty}\left[\delta(n-4k) - \delta(n-2k) + \delta(n-8k)\right] + \cos\left(\frac{\pi}{21}n\right)$

στ) $x_6[n] = (j)^n + \cos^3\left(\frac{\pi}{3}n\right) + 4\sin\left(\frac{\pi}{7}n\right)\cdot\cos\left(\frac{\pi}{6}n\right) + \sin^3\left(\frac{\pi}{8}n\right)$

Περιοδικότητα;

α) $x_1(t) = \cos\left(\frac{\pi}{5}t\right) + \frac{1-\cos\left(2\cdot\frac{\pi}{28}t\right)}{2} + \frac{1}{2}\left[\sin\left(\frac{\pi}{8}t+\frac{\pi}{21}t\right) - \sin\left(\frac{\pi}{8}t-\frac{\pi}{21}t\right)\right]$

$= \cos\left(\frac{\pi}{5}t\right) + \frac{1-\cos\left(\frac{\pi}{14}t\right)}{2} + \frac{1}{2}\left[\sin\left(\frac{29\pi}{168}t\right) - \sin\left(\frac{13\pi}{168}t\right)\right]$

$= \cos\left(\frac{\pi}{5}t\right) + \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi}{14}t\right) + \frac{1}{2}\sin\left(\frac{29\pi}{168}t\right) - \frac{1}{2}\sin\left(\frac{13\pi}{168}t\right)$

Η x_1 αποτελεί παράθεση ημιτονοειδών σημάτων συνεχούς χρόνου με περιόδους. $T_1 = 10s$, $T_2 = 28s$, $T_3 = \frac{336}{29}s$, $T_4 = \frac{336}{13}s$

$T = \text{ΕΚΠ}(T_1, T_2, T_3, T_4) = \text{ΕΚΠ}(10, 28, 336) = 1680s$

β) $x_2(t) = \exp\left[j\left(7t+\frac{\pi}{21}\right)\cdot \text{rect}\left(\frac{t}{14}\right)\right] + 7$

η $\text{rect}\left(\frac{t}{14}\right) = \begin{cases} 0, & \left|\frac{t}{14}\right| > \frac{1}{2} \\ \frac{1}{2}, & \left|\frac{t}{14}\right| = \frac{1}{2} \\ 1, & \left|\frac{t}{14}\right| < \frac{1}{2} \end{cases} = \begin{cases} 0, & |t| > 7 \\ \frac{1}{2}, & |t| = 7 \\ 1, & |t| < 7 \end{cases}$

- για $|t|>7$, $\text{rect}\left(\frac{t}{14}\right) = 0$ οπότε $x_2(t) = e^0 + 7 = 8$
- όμως για $|t|<7$, $\text{rect}\left(\frac{t}{14}\right) = 1$ οπότε $x_2(t) = e^{j\left(7t+\frac{\pi}{21}\right)} + 7$ η οποία όμως δεν είναι η σταθερή ίση με 8 οπότε η $x_2(t)$ δεν γίνεται να είναι περιοδική.

γ) $x_3(t) = \sin\left(\frac{\pi}{126}t\right) - \sum_{k=-\infty}^{+\infty} \delta\left(\frac{t+42k}{21}\right)$

η $\sum_{k=-\infty}^{+\infty} \delta\left(\frac{t+42k}{21}\right)$ παίρνει τιμή μόνο για τις ακέραιες t και είναι περιοδική με $T=42s$ για προφανείς λόγους.

Οπότε, η x_3 αποτελεί παράθεση περιοδικών συναρτήσεων άρα είναι περιοδική με $T = \text{ΕΚΠ}(252, 42) = 252\,s$

δ) $x_4[n] = \exp\left(j\left(\frac{\pi}{42}n^2+\frac{\pi}{63}n^3\right)\right) = \cos\left(\frac{\pi}{42}n^2+\frac{\pi}{63}n^3\right)+j\sin\left(\frac{\pi}{42}n^2+\frac{\pi}{63}n^3\right)$

Με χρήση του ορισμού για cos: $\cos\left(\frac{\pi}{42}n^2+\frac{\pi}{63}n^3\right) = \cos\left(\frac{\pi}{42}(n+N)^2+\frac{\pi}{63}(n+N)^3\right) \Rightarrow$

$\Rightarrow \cos\left(\frac{\pi}{42}n^2+\frac{\pi}{63}n^3\right) = \cos\left(\frac{\pi}{42}(n^2+2nN+N^2)+\frac{\pi}{63}(n^3+3n^2N+3nN^2+N^3)\right) \Rightarrow$

$\Rightarrow 2k\cancel{\pi} + \cancel{\frac{\pi}{42}n^2} + \cancel{\frac{\pi}{63}n^3} = \cancel{\frac{\pi}{42}}(\cancel{n^2}+2nN+N^2) + \cancel{\frac{\pi}{63}}(\cancel{n^3}+3n^2N+3nN^2+N^3)$

$\Rightarrow k = \frac{nN}{42} + \frac{N^2}{84} + \frac{n^2N}{42} + \frac{nN^2}{42} + \frac{N^3}{126}$

πρέπει να είναι όλα ακέραιοι για να είναι περιοδικό το σήμα.

$$\left.\begin{array}{l} \frac{(n^2+N)N}{42} \in \mathbb{Z} \Rightarrow \frac{N}{42} \in \mathbb{Z} \Rightarrow N = 42k \\ \frac{(1+2n)N^2}{84} \in \mathbb{Z} \Rightarrow \frac{N^2}{84} \in \mathbb{Z} \Rightarrow N = 42k \\ \frac{N^3}{126} \in \mathbb{Z} \Rightarrow N = 21k \end{array}\right\} \Rightarrow N_0 = 42$$

ε) $x_5[n] = \sum_{k=-\infty}^{+\infty}\left[\delta(n-4k) - \delta(n-2k) + \delta(n-8k)\right] + \cos\left(\frac{\pi}{21}n\right)$

Τότε $N_1 = 4$, $N_2 = 2$, $N_3 = 8$

και $N_4 = 42$

Άρα $N = ΕΚΠ(4,2,8,42) = 168$

στ) $x_6[n] = (j)^n + \cos^3\left(\frac{\pi}{3}n\right) + 4\sin\left(\frac{\pi}{7}n\right)\cdot\cos\left(\frac{\pi}{6}n\right) + \sin^3\left(\frac{\pi}{8}n\right)$

$= (j)^n + \frac{\cos\left(3\cdot\frac{\pi}{3}n\right) + 3\cos\left(\frac{\pi}{3}n\right)}{4} + 4\cdot\frac{1}{2}\left[\sin\left(\frac{\pi}{7}n+\frac{\pi}{6}n\right) + \sin\left(\frac{\pi}{7}n-\frac{\pi}{6}n\right)\right]$

$+ \frac{3\sin\left(\frac{\pi}{8}n\right) - \sin\left(3\cdot\frac{\pi}{8}n\right)}{4}$

$= (j)^n + \frac{1}{4}\cos(\pi n) + \frac{3}{4}\cos\left(\frac{\pi}{3}n\right) + 2\sin\left(\frac{13\pi}{42}n\right) - 2\sin\left(\frac{\pi n}{42}\right)$

$+ \frac{3}{4}\sin\left(\frac{\pi}{8}n\right) - \frac{1}{4}\sin\left(\frac{3\pi}{8}n\right)$

$N_1 = 4$, $N_2 = 2$, $N_3 = 6$, $N_4 = 84$, $N_5 = 84$, $N_6 = 16$, $N_7 = 16$

$N = ΕΚΠ(N_1, \dots, N_7) = 336$

1.3

α) $y(t) = \exp(j7t) \cdot x([3+1]t - 7] = \exp(j7t) \cdot x(4t-7)$

β) $y(t) = \cos\left(3\pi t + \frac{\pi}{8}\right) + \frac{dx(t-7)}{dt}$

γ) $y[n] = \sin\left(\frac{7\pi}{2}\, x[n]\right)$

δ) $y[n] = \frac{1}{2n+1}\, x[-2n] + 1$

γραμμ. , ΧΑ, μνήμη, αιτιατότ., ευσταθ. ; αντιστρεψ ;

α) $y(t) = e^{j7t} \cdot x(4t-7)$

γραμμ.

$$\left.\begin{aligned} S(a x_1(t) + b x_2(t)) &= e^{j7t}(a x_1(4t-7) + b x_2(4t-7)) \\ a S(x_1(t)) &= a e^{j7t} \cdot x_1(4t-7) \\ a S(x_2(t)) &= b e^{j7t} \cdot x_2(4t-7) \end{aligned}\right\}$$ γραμμικό

ΧΑ

$x^*(t) = x(t - t_0)$

$y^*(t) = e^{j7t} x^*(4t-7) = e^{j7t} x(4(t-t_0) - 7) \neq y(t-t_0) = e^{j7(t-t_0)} x(4(t-t_0)-7)$

↳ όχι χρον. αναλλ.

μνήμη

με μνήμη

αιτιατότητα

μη αιτιατό

π.χ. για $t = 2$ εξαρτάται από το $x(1)$

Ευστάθεια BIBO

$|x(t)| \leq M \Rightarrow |y(t)| \leq (1+i)M \Rightarrow$ ευσταθές

αντιστρ.

$y^{-1}(t) = \frac{x\left(\frac{t+7}{4}\right)}{e^{j7t}}$

β) $y(t) = \cos\left(3\pi t + \frac{\pi}{8}\right) + \dfrac{dx(t-7)}{dt}$

<u>γραμμ.</u>

$$S'[a x_1(t) + b x_2(t)] = \cos\left(3\pi t + \frac{\pi}{8}\right) + \frac{d\left(a x_1(t-7) + b x_2(t-7)\right)}{dt}$$

$$= \cos\left(3\pi t + \frac{\pi}{8}\right) + a\frac{d x_1(t-7)}{dt} + b\frac{d x_2(t-7)}{dt} \neq$$

$$\neq a S'(x_1(t)) + b S'(x_2(t)) = 2\cos\left(3\pi t + \frac{\pi}{8}\right) + a\frac{d x_1(t-7)}{dt} + b\frac{d x_2(t-7)}{dt}$$

↳ μη γραμμ.

<u>ΧΑ</u>

$x^* = x(t-t_0)$

$$y^* = \cos\left(3\pi t + \frac{\pi}{8}\right) + \frac{d x^*(t-7)}{dt} = \cos\left(3\pi t + \frac{\pi}{8}\right) + \frac{d x(t-t_0-7)}{dt} \neq$$

$$\neq y(t-t_0) = \cos\left(3\pi(t-t_0) + \frac{\pi}{8}\right) + \frac{dx(t-t_0-7)}{dt}$$

↳ όχι ΧΑ

<u>μνήμη</u>

με μνήμη $\dfrac{dx(t-7)}{dt}$

<u>αιτιατότητα</u>

ναι, εξαρτάται από το παρόν και το παρελθόν

<u>ευστάθεια BIBO</u>

$|x(t)| \leq M \Rightarrow y(t) \leq 1 + \cancel{\dfrac{dM}{dt}}\infty$ μη ευσταθές

(Βλ. ένα για είσοδο $x(t) = \sqrt{49-t^2}$ $t \in [-49, 49]$ το $y(t) = \cos\left(3\pi t + \frac{\pi}{8}\right) - \dfrac{dx(t-7)}{dt}$ απειρίζεται)

<u>αντιστρ.</u>

$\cos\Theta$ μη αντιστρ. στο $(-\infty, +\infty)$ ⟶ <u>μη αντιστρ.</u>

γ) $y[n] = \sin\left(\frac{7\pi}{2} x[n]\right)$

γραμμ.:

$S(a x_1[n] + b x_2[n]) = \sin\left(\frac{7\pi}{2}(a x_1[n] + b x_2[n])\right)$

$a S(x_1[n]) = a \sin\left(\frac{7\pi}{2} x_1[n]\right)$

$b S(x_2[n]) = b \sin\left(\frac{7\pi}{2} x_2[n]\right)$

↳ μη γραμμ.

ΧΑ

$x^*[n] = x[n-n_0]$

$y^* = \sin\left(\frac{7\pi}{2} x^*[n]\right) = \sin\left(\frac{7\pi}{2} x[n-n_0]\right) = y[n-n_0]$

↳ ΧΑ

μνήμη

↳ χωρίς μνήμη

αιτιατ.

↳ ναι, αφού είναι χωρίς μνήμη

ευσταθής BIBO

για $|x[n]| \leq M \Rightarrow |y[n]| \leq 1$ ευσταθές

αντιστρ.

όχι, γιατί το sin δεν είναι αντιστρ. στο $(-\infty, +\infty)$

δ) $y[n] = \frac{1}{2n+1} x[-2n] + 1$

γραμ.

$$S\left[a x_1[n] + b x_2[n]\right) = \frac{1}{2n+1}\left(a x_1[-2n] + b x_2[-2n]\right) + 1$$

$$S\, a x_1[n] = \frac{1}{2n+1} a x_1[-2n] + 1$$

$$S\, a x_2[n] = \frac{1}{2n+1} b x_2[-2n] + 1$$

↳ μη γραμ.

X.A.

$x^*[n] = x[n-n_0]$

$$y^*[n] = \frac{1}{2n+1} x[-2(n-n_0)] + 1 \neq y[n-n_0] = \frac{1}{2(n-n_0)+1} x[-2(n-n_0) + 1]$$

↳ όχι X.A.

μνήμη

↳ με μνήμη (λόγω $x[-2n]$)

αιτιατ.

↳ όχι, για $n < 0$ εξαρτάται από $x[n]$ με $n > 0$.

Ευστάθεια BIBO

για $|x[n]| \leq M$, $\frac{1}{2n+1}$ δεν απειρίζεται για $n \in \mathbb{N}$

$x[-2n] \leq M \rightarrow$ ευσταθές

αντιστρ.

$$y^{-1}[n] = \left(x[-\tfrac{n}{2}] - 1\right)(2n+1)$$

1.4 (a=6 την δουλεύω με)

$$x[n] = n\left[u[n+6] - u[n-10+6]\right] = n\left[u[n+6] - u[n-4]\right]$$

$$h_1[n] = (n^6+1)\cdot u[-n]$$

$$h_2[n] = \exp\left(j\frac{6\pi}{3}n\right)u[n] = \left[\cos(2\pi n) + j\sin(2\pi n)\right]u[n]$$

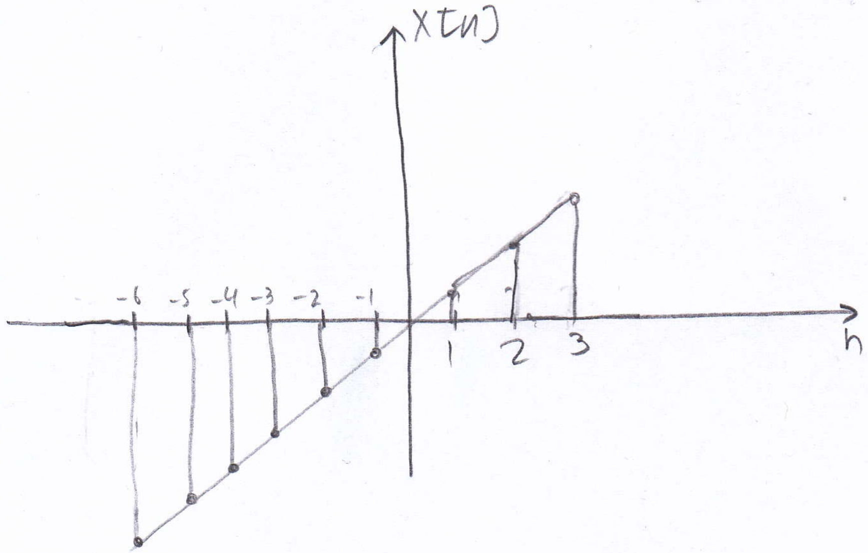
$$h_3[n] = 2u[n+6+1] - 4u[n] + 2u[n+6-5] = 2u[n+7] - 4u[n] + 2u[n+1]$$

$$h_4[n] = \sin\left(\frac{\pi}{4}n\right)\cdot\sum_{k=n-3}^{n+3}\delta[k]$$

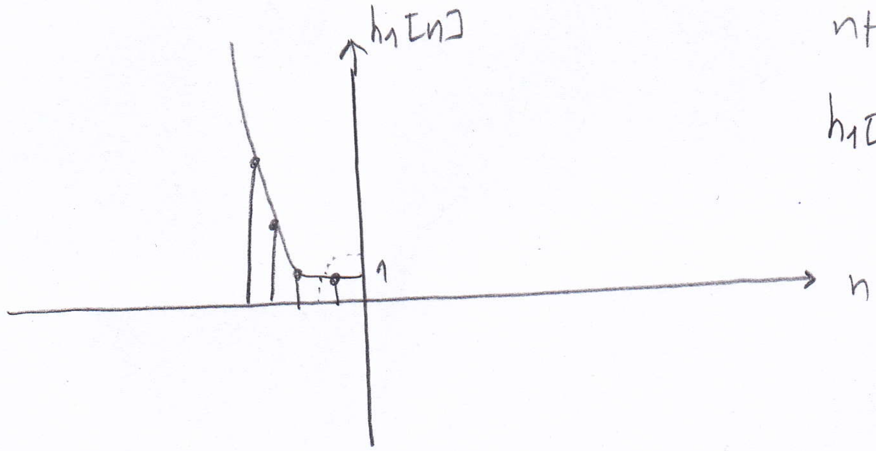
$$h_5[n] = h_3[n] * h_4[n]$$

$$x[n] = \begin{cases} 0, & n<-6 \\ n, & -6 \le n < 4 \\ 0, & n \geq 4 \end{cases}$$

$x[n]$:



$h_1[n]$:

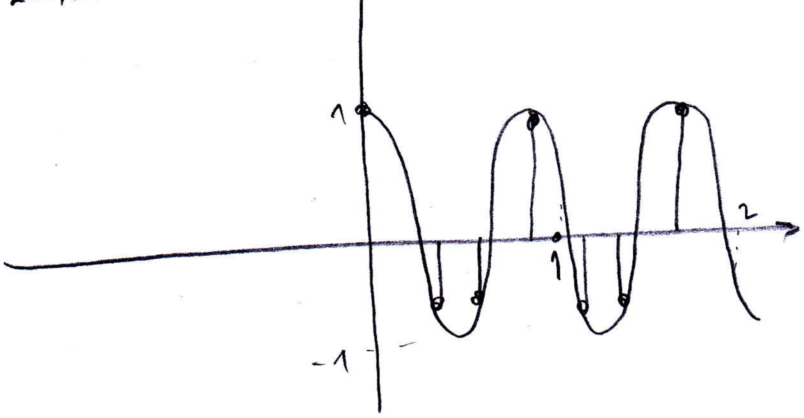


ηπλάπτιρως διάρκειας

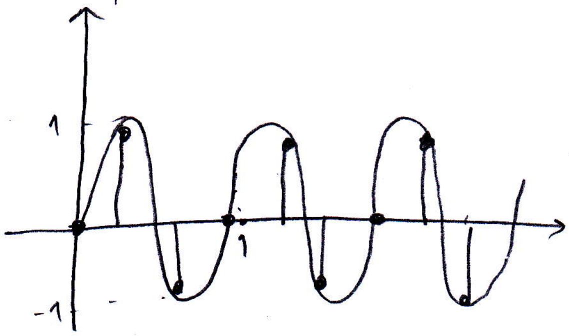
$$h_1[n] = \begin{cases} n^6+1, & n \le 0 \\ 0, & n > 0 \end{cases}$$

$h_2[n]$:

πραγματικό μέρος



φανταστικό μέρος

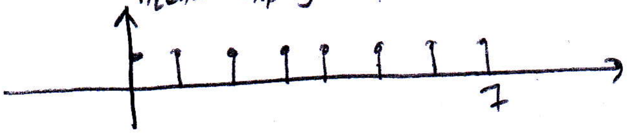


$$Re\{h_2[n]\} = \begin{cases} 1 & , n \bmod 3 = 0 \\ -0{,}5 & , \text{αλλού} \end{cases}$$

$$Im\{h_2[n]\} = \begin{cases} \frac{\sqrt{3}}{2} & , n \bmod 3 = 1 \\ -\frac{\sqrt{3}}{2} & , n \bmod 3 = 2 \\ 0 & , n \bmod 3 = 0 \end{cases}$$

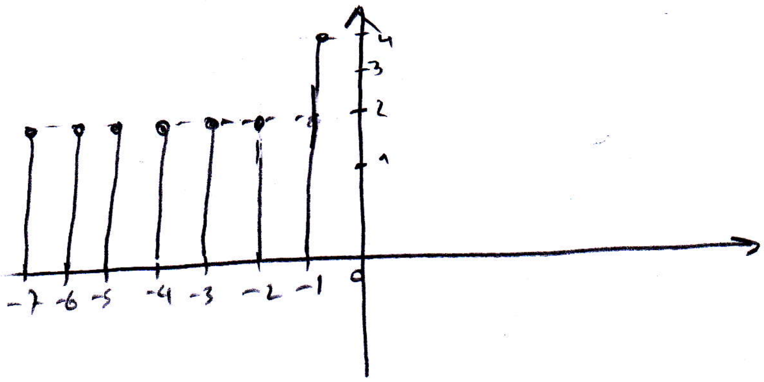
$h_2[n] = \exp(j2\pi n)$

n'



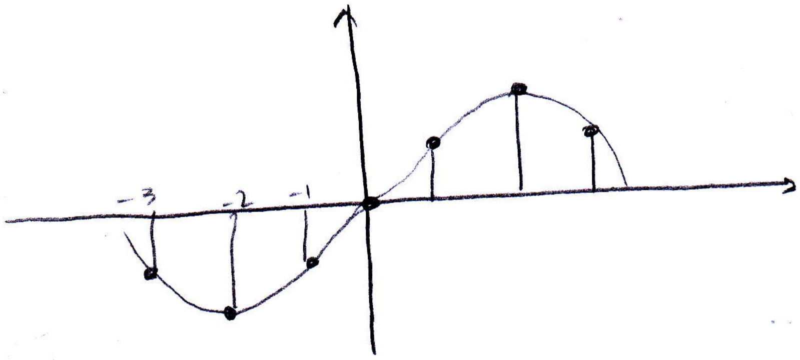
όπου $n \geq 0$

$h_3[n]$:



$$h_3[n] = \begin{cases} 0 & , n < -7 \\ 2 & , -7 \leq n < -1 \\ 4 & , -1 \leq n < 0 \\ 0 & , n \geq 0 \end{cases}$$

$h_4[n]$:



$h_5[n] = h_3[n] * h_4[n]$

$h_4[n]$ \ $h_3[n]$	$h_3[-7]$ 2	$h_3[-6]$ 2	$h_3[-5]$ 2	$h_3[-4]$ 2	$h_3[-3]$ 2	$h_3[-2]$ 2	$h_3[-1]$ 4
$h_4[-3] = -\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-2\sqrt{2}$
$h_4[-2] = -1$	-2	-2	-2	-2	-2	-2	-4
$h_4[-1] = -\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-2\sqrt{2}$
$h_4[0] = 0$	0	0	0	0	0	0	0
$h_4[1] = \sqrt{2}/2$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}$
$h_4[2] = 1$	2	2	2	2	2	2	4
$h_4[3] = \frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}$

$h_5[-10] = -\sqrt{2}$

$h_5[-9] = -2-\sqrt{2}$

$h_5[-8] = -2\sqrt{2}-2$

$h_5[-7] = -2\sqrt{2}-2$

$h_5[-6] = -2-\sqrt{2}$

$h_5[-5] = -\sqrt{2}$

$h_5[-4] = -\sqrt{2}$

$h_5[-3] = \sqrt{2}-2$

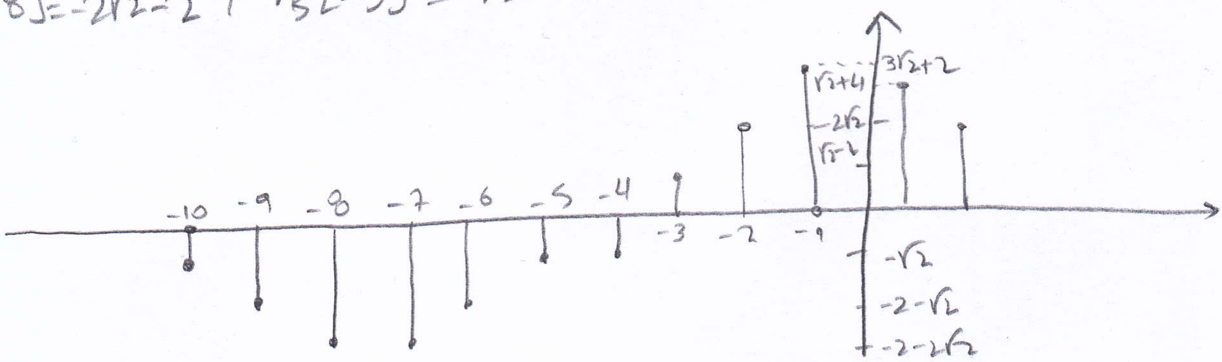
$h_5[-2] = 2$

$h_5[-1] = 2\sqrt{2}+2$

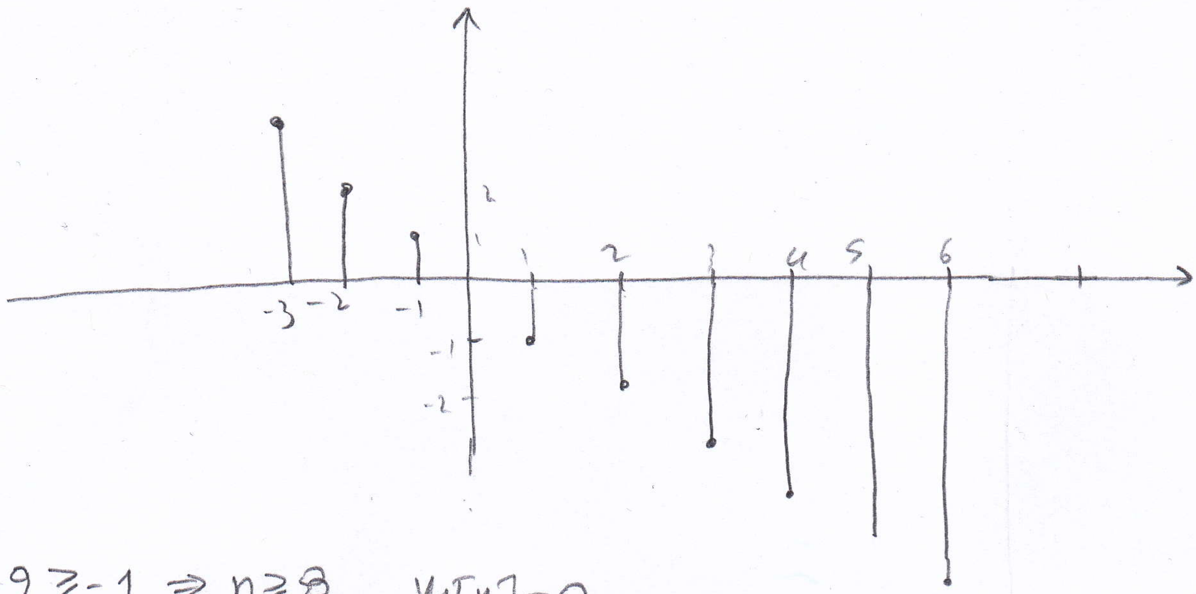
$h_5[0] = 3\sqrt{2}+2$

$h_5[1] = \sqrt{2}+4$

$h_5[2] = -2\sqrt{2}$



• $y_1[n] = x[n] * h_1[n] = (n[u[n+6] - u[n-4]]) * ((n^6+1)u[-n])$



Για $n-9 \geq -1 \Rightarrow n \geq 8$, $y_1[n]=0$

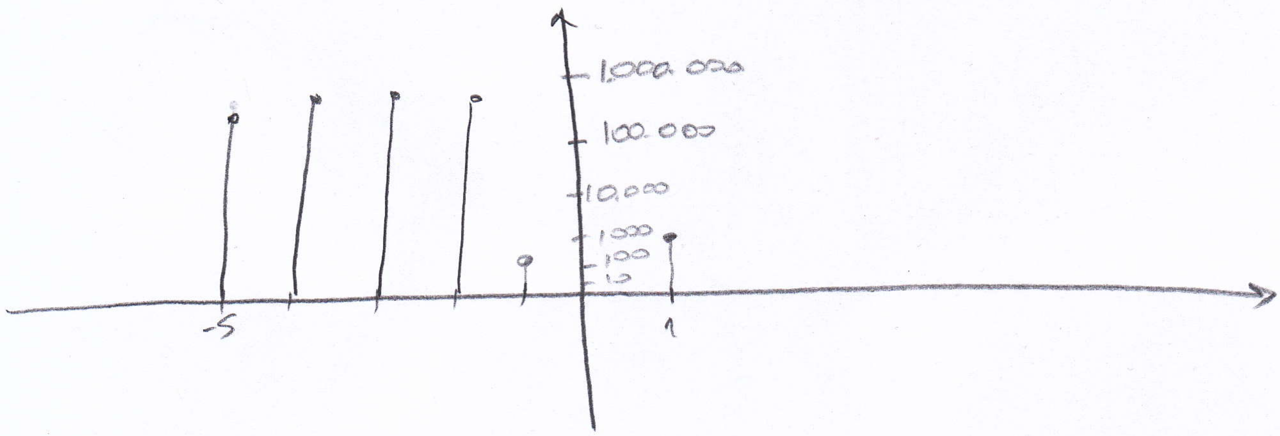
Για $-1 \leq n < 8$: $y_1[n] = \sum_{k=n-9}^{-1} x[n-k]h[k]$ ~ $y_1[1]=195$, $y_1[0]=130$, $y_1[-1]=65$

$y_1[i]=0,\ i \geq 2$

Για $n<-1$: $y_1[n] = \sum_{k=n-9}^{n} (n-k)(u[n-k+6]-u[n-k-4])(k^6+1)$

↪ $y_1[-2]=55802$, $y_1[-3]=55802$, $y_1[-4]=55072$

$y_1[-5]=46373$, $y_1[-6]=0$...



- $Y_2[n] = X[n] \cdot h_2[n]$

$$X(10) = [-6, -5, -4, -3, -2, -1, 0, 1, 2, 3]^T$$

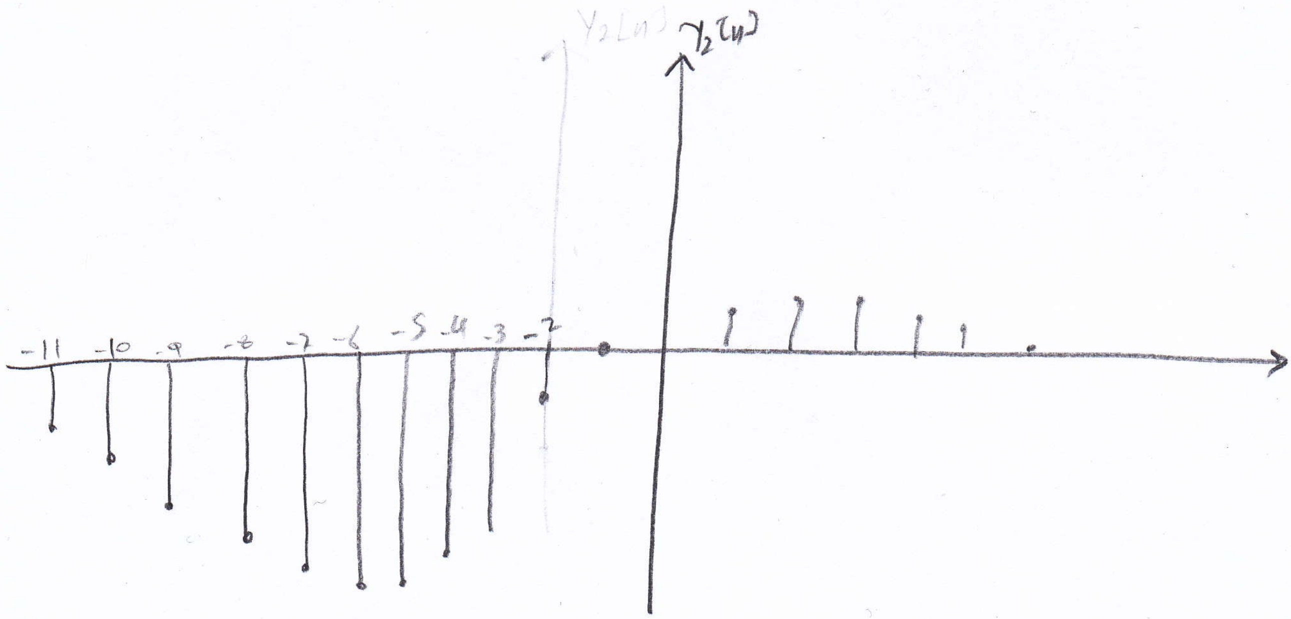
$$h_2(8) = [1, 1, 1, 1, 1, 1, 1, 1]^T$$

$$\begin{bmatrix} Y_2(0) \\ Y_2(1) \\ Y_2(2) \\ \vdots \\ Y_2(16) \end{bmatrix} = \begin{bmatrix} h(0) & & & & \\ h(1) & h(0) & & & \\ h(2) & h(1) & h(0) & \cdots & \\ \vdots & & & & \\ \cdots & & & & h(7) \end{bmatrix} \cdot \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X[9] \end{bmatrix}$$

$y(0) = -6$
$y(1) = -11$
$y(2) = -15$
$y(3) = -18$
$y(4) = -20$
$y(5) = -21$
$y(6) = -21$

$y(7) = -20$
$y(8) = -12$
$y(9) = -4$
$y(10) = 0$
$y(11) = 3$
$y(12) = 5$
$y(13) = 6$

$y(14) = 6$
$y(15) = 5$
$y(16) = 3$



$y_3[n] = x[n] * h_3[n]$

$x(10) = (-6,-5,-4,-3,-2,-1,0,1,2,3)^T$

$h_3(7) = (2,2,2,2,2,2,4)$

$$\begin{bmatrix} y_3(0) \\ y_3(1) \\ \vdots \\ y_3(15) \end{bmatrix} = \begin{bmatrix} h(0) & & & \\ h(1) & h(0) & & \\ h(2) & h(1) & h(0) & \\ & & \ddots & \\ & & & h(6) \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ \vdots \\ x(9) \end{bmatrix}$$

$y(0) = -12$

$y(1) = -22$

$y(2) = -30$

$y(3) = -36$

$y(4) = -40$

$y(5) = -42$

$y(6) = -54$

$y(7) = -38$

$y(8) = -22$

$y(9) = -6$

$y(10) = 2$

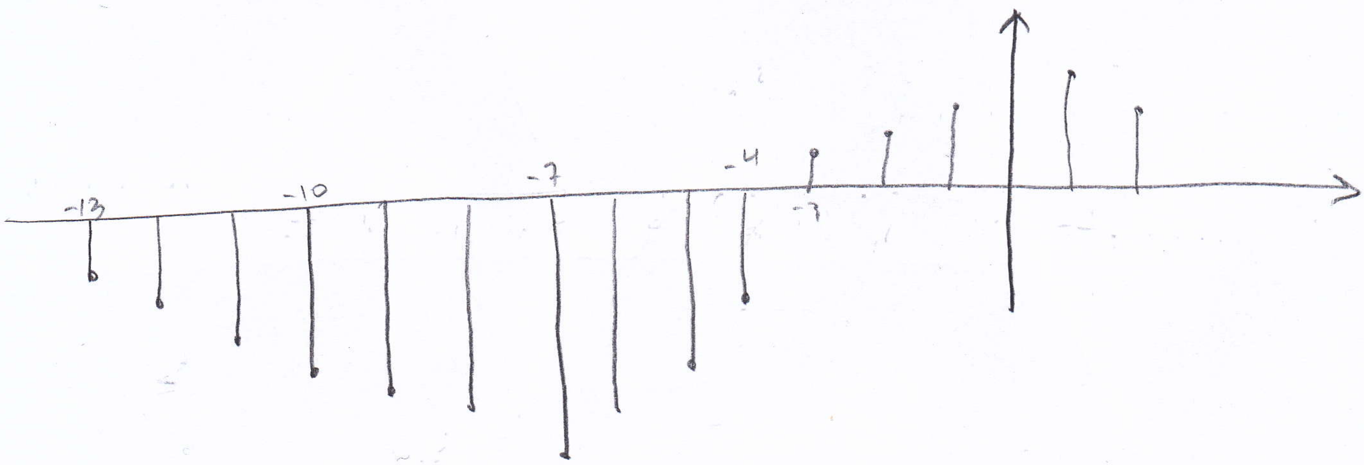
$y(11) = 8$

$y(12) = 12$

$y(13) = 14$

$y(14) = 14$

$y(15) = 12$



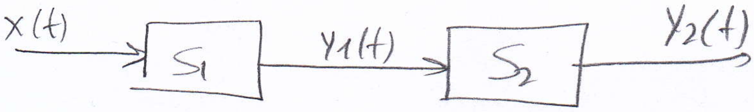
1.5

ΓΧΑ S_1 σε σειρά με ΓΧΑ S_2.

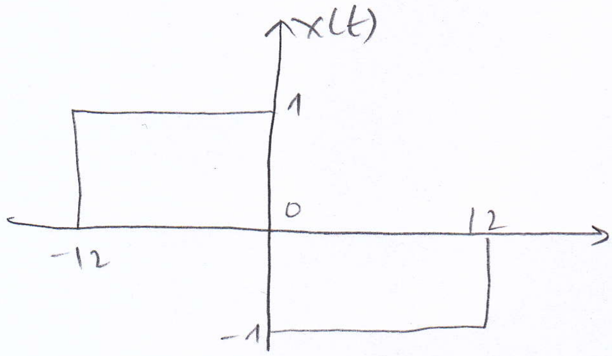
$h(t) = u(t+6) - u(t-6) + 6\,\delta(t)$

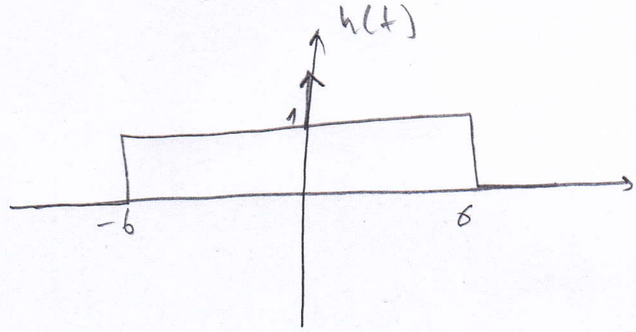
$x(t) = u(t+2\cdot 6) - 2u(t) + u(t-2\cdot 6) = u(t+12) - 2u(t) + u(t-12)$

$x'(t) = e^{-12t}\,u(t-6)$

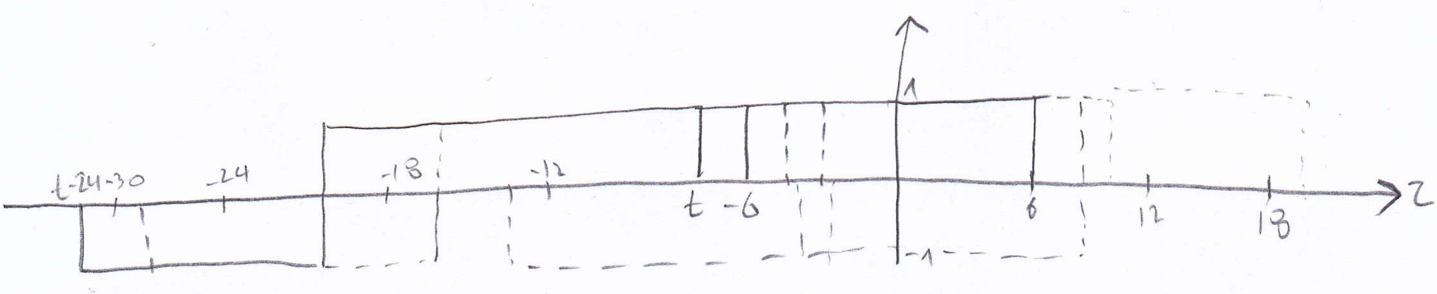


$$x(t) = \begin{cases} 0, & t < -12 \\ 1, & -12 \le t < 0 \\ -1, & 0 \le t < 12 \\ 0, & 12 \le t \end{cases}, \quad h(t) = \begin{cases} 0, & t < -6 \\ 1, & -6 \le t < 0 \\ +\infty, & t = 0 \\ 1, & 0 < t < 6 \\ 0, & 6 \le t \end{cases}$$





$y_1(t) = x(t) * h(t)$



$$) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$t<-6: \quad y(t)=0$

$-6 \le t < 0: \quad y(t) = \int_{-6}^{t} 1 \cdot 1 \, d\tau = t+6$

$0 \le t < 6: \quad y(t) = \int_{-0}^{t} 1 \cdot 1 \cdot d\tau + 6 = t+12$

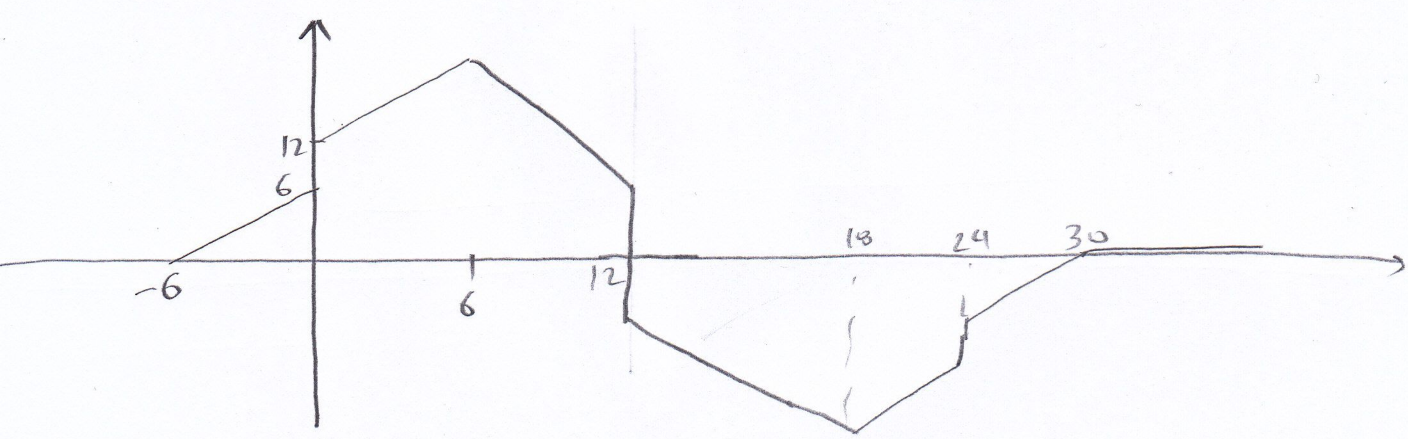
$6 \le t < 12: \quad y(t) = \int_{t-12}^{6} 1 \cdot 1 \, d\tau + \int_{6}^{t-12} (-1) \cdot 1 \, d\tau + 6 = 18 - t - (t-6) + 6 = 30 - 2t$

$12 \le t \le 18: \quad y(t) = \int_{t-12}^{6} 1 \cdot 1 \, d\tau + \int_{-6}^{t-12} 1(-1) \, d\tau - 6 = 18 - 2t$

$18 \le t \le 24: \quad y(t) = \int_{t-24}^{6} (-1) \cdot 1 \, d\tau - 6 = t - 30 - 6 = t - 36$

$24 < t \le 30: \quad y(t) = \int_{t-24}^{6} (-1) \cdot 1 \cdot d\tau = t - 30$

$t > 30: \quad y(t) = 0.$



...ω φτιλιων λάθους το σήμα $y(t)$ είναι το $y_1(t)$ μετατοπισμένο
...ρος το δεξιά κατά 12 (θεωρούμε t την δεξιά άκρη του $x(t)$ και
όχι το μέσο του)

$$y_1(t) = \begin{cases} 0, & t<-18 \\ t+6, & -18\le t<-12 \\ t+12, & -12\le t\le -6 \\ 30-2t, & -6<t<0 \\ 18-2t, & 0\le t<6 \\ t-36, & 6\le t<12 \\ t-30, & 12<t<18 \\ 6, & t>18 \end{cases}$$

Όμοια, για $y_2(t) = y_1(t) * h(t) = \dots$

Γ1α $t+6<-18 \Rightarrow t<-24 : y_2(t)=0$

Γ1α $-18\le t+6<-12 \Rightarrow -24<t<-18 : y_2(t)=\int_{-18}^{t+6} 1(\tau+6)\,d\tau = \frac{t^2}{2}+12t$

Γ1α $-12\le t+6<-6 \Rightarrow -18<t<-12 : y_2(t)=\int_{-18}^{-12} \tau+6\,d\tau + \int_{-12}^{t+6} 1(\tau+12)\,d\tau =$

$\Rightarrow y_2(t) = -(-54) + \frac{t^2+36t+324}{2}$

Γ1α $-6\le t+6<0 : -12\le t<-6 : y_2(t)=\int_{t-6}^{12} \tau+6\,d\tau + \int_{-12}^{-6} \tau+12\,d\tau + \int_{-6}^{t+6} 30-2\tau\,d\tau$

$\Rightarrow y_2(t) = -\frac{t^2-36}{2} +18 - t^2+18 +360$

Γ1α $0\le t+6<6 \Rightarrow -6\le t<0 : y_2(t) = \int_{t-6} \tau+12\,d\tau + \int_{-6}^{6} 30-2\tau\,d\tau + \int_{0}^{t+6} 13-2\tau\,d\tau$

$\Rightarrow y_2(t) = -\frac{t(t+12)}{2} +216 - t^2 + 6t+72$

Για $6 \le t+6 < 12 \Rightarrow 0 \le t < 6$: $y_2(t) = \int_{t-6}^{0} (30-2\tau)\,d\tau + \int_{0}^{6} 18-2\tau\,d\tau + \int_{6}^{t+6} \tau-36\,d\tau$

$\Rightarrow y_2(t) = t^2 - 42t + 216 + 72 + \frac{(t-60)t}{2}$

Για $12 \le t+6 < 18 \Rightarrow 6 \le t < 12$: $y_2(t) = \int_{t-6}^{6} 18-2\tau\,d\tau + \int_{6}^{12} \tau-36\,d\tau +$

$+ \int_{12}^{t+6} \tau - 30\,d\tau \Rightarrow$

$\Rightarrow y_2(t) = t^2 - 30t + 216 + (-162) +$

$+ \frac{t^2 - 48t + 252}{2}$

Για $18 \le t+6 < 24 \Rightarrow 12 \le t < 18$: $y_2(t) = \int_{t-6}^{12} (\tau-36)\,d\tau + \int_{12}^{18} \tau-30\,d\tau \Rightarrow$

$\Rightarrow y_2(t) = -\frac{t^2 - 84t + 1184}{2} + (-90)$

Για $24 \le t+6 < 30 \Rightarrow 18 \le t < 24$: $y_2(t) = \int_{t-6}^{18} \tau - 30\,d\tau \Rightarrow$

$\Rightarrow y_2(t) = -\frac{(t^2 - 72t + 1152)}{2}$

Για $t > 24$: $y_2(t) = 0$

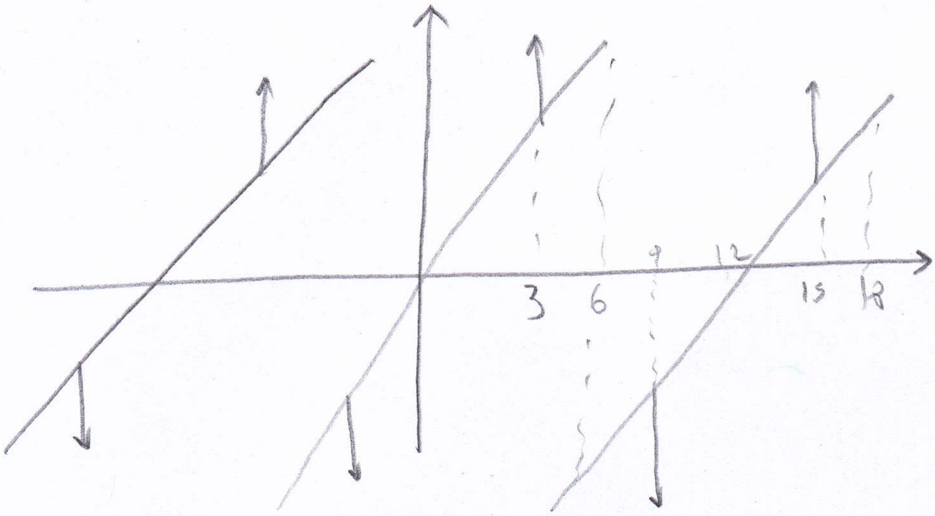
Αν τώρα $x'(t) = e^{-12t} u(t-6)$ τότε $y_1(t) = x'(t) * h(t)$

[....]

1.6

$x(t):\ T=12 \qquad x(t)=x(t+12)$

$x(t)=6t-\delta(t+3)+\delta(t-3) \quad , |t|\leq 6 \Rightarrow -6\leq t\leq 6$



$$x(t)=\sum_{m=-\infty}^{+\infty} c_m e^{jm\pi/6\, t}$$

$$c_m=\frac{1}{12}\int_{-6}^{6} x(t)e^{-jm\frac{\pi}{6}t}dt=\frac{1}{12}\int_{-6}^{0}(6t-\delta(t+3))e^{-jm\frac{\pi}{6}t}dt+$$

$$+\frac{1}{12}\int_{0}^{6}(6t+\delta(t-3))e^{-jm\frac{\pi}{6}t}dt=$$

$$=\frac{1}{12}\int_{-6}^{0}6te^{-jm\frac{\pi}{6}t}dt+\frac{1}{12}\int_{0}^{6}6te^{-jm\frac{\pi}{6}t}=$$

$$=\frac{1}{12}\cdot\frac{(216j\pi m-216)e^{j\pi m}+216}{\pi^2m^2}+\frac{1}{12}\left(-\frac{216e^{-j\pi m}(e^{j\pi m}-j\pi m-1)}{\pi^2m^2}\right)$$

$$=\ldots\Rightarrow c_m=\frac{18}{\pi^2m^2}[j\pi m(e^{j\pi m}+e^{-j\pi m})+e^{-j\pi m}-e^{j\pi m}) \quad ,m\neq 0$$

Πα $m=0$: $\frac{1}{12}\int_{-6}^{6} 6t\,dt = 0$ $\qquad C_0 = 0$

$$|c_1| = \left|\frac{18}{\pi^2}\left[i\pi\left(e^{i\pi}+e^{-i\pi}\right)+e^{-i\pi}-e^{i\pi}\right]\right| = \left|\frac{18}{\pi^2}\left[i\pi(-2)+(-1)+1\right]\right| =$$

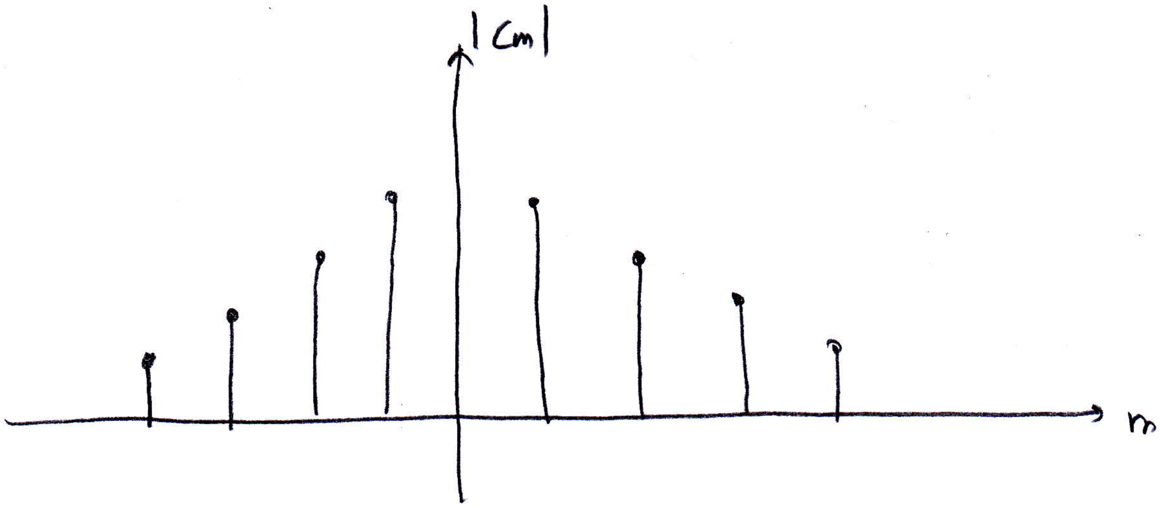
$$= \left|-\frac{36i\pi}{\pi^2}\right| = \left|-\frac{36i}{\pi}\right| = \sqrt{\frac{36^2}{\pi^2}} = \frac{36}{\pi} \simeq 11{,}46, \quad |c_1| = |c_{-1}|$$

$$|c_2| = \left|\frac{18}{4\pi^2}\left[2i\pi\left(e^{2i\pi}+e^{-2i\pi}\right)+e^{-2i\pi}-e^{2i\pi}\right]\right| =$$

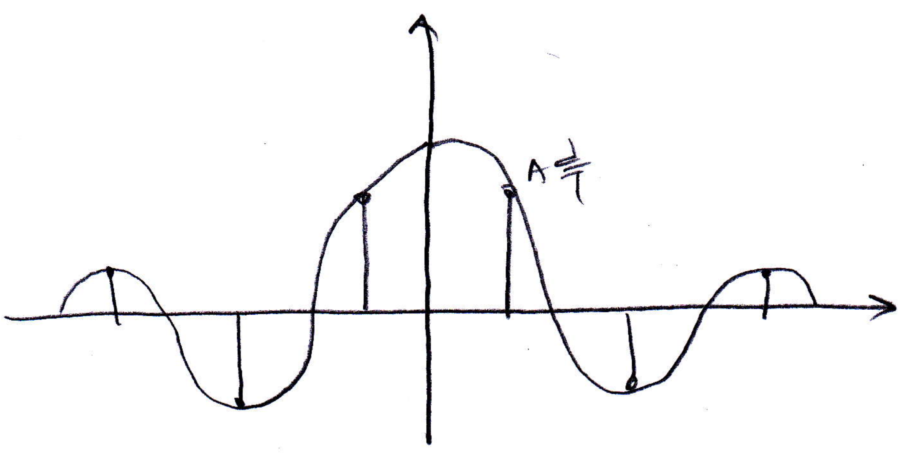
$$= \ldots \sqrt{\frac{18^2}{\pi^2}} = 5{,}72, \quad |c_2| = |c_{-2}|$$

$|c_3| = \ldots \Rightarrow \quad |c_3| = |c_{-3}|$

$|c_4| = \ldots \Rightarrow \quad |c_4| = |c_{-4}|$



β)



$$\bar{P} = \frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)\, x^*(t)\, dt =$$

$$= \frac{1}{T}\int_T \left(\sum_m c_m e^{jm\omega_0 t}\right)\left(\sum_n c_n^* e^{-jn\omega_0 t}\right) dt =$$

$$= \frac{1}{T}\sum_m \sum_n c_m c_n^* \int_T \underbrace{e^{j\omega_0 t(m-n)}}_{T\cdot\delta[m-n]} dt = \frac{1}{T}\sum_m c_m c_m^* T = \sum_{m=-\infty}^{+\infty} |c_m|^2$$

γ) C_m^x: $\overset{100Hz}{C_1^x = e^{j\frac{\pi}{3}}}$, $\overset{200Hz}{C_2^x = 2e^{j\frac{\pi}{4}}}$, $\overset{300Hz}{C_3^x = 3e^{j\frac{\pi}{10}}}$, $\overset{400Hz}{C_4 = 2e^{j\pi/3}}$

$$\frac{1}{T_x}\int_{T_x} |x(t)|^2 dt = \sum_{m=1}^{4} |c_m^x|^2 = (e^{j\frac{\pi}{3}})^2 + (2e^{j\pi/4})^2 + (3e^{j\pi/10})^2 + (2e^{j\pi/3})^2 =$$

$$= \dots = 18$$

$$\frac{1}{T_y}\int_{T_y} |y(t)|^2 dt = \sum_{m=1}^{4} |c_m^y|^2 = (3e^{j\pi})^2 + (2e^{j\pi/3})^2 + (e^{j\pi/16})^2 + (e^{j\pi/4})^2 =$$

$$= \dots = 15$$

C_m^y: $\underset{50Hz}{C_1^y = 3e^{j\pi}}$, $\underset{100Hz}{C_2^y = 2e^{j\pi/3}}$, $\underset{150Hz}{C_3^y = e^{j\pi/16}}$, $\underset{200Hz}{C_4^y = e^{j\pi/4}}$ $T_y = \frac{1}{50}$

Αρχικά θεωρούμε ότι $T_x = T_y = T = \frac{1}{50}$ και άρα έχουμε $x(t)$:

C_m^x: $C_1^x = 0$, $C_2^x = e^{j\pi/3}$, $C_3^x = 0$, $C_4^x = 2e^{j\pi/4}$, $C_5^x = 0$, $C_6^x = 3e^{j\pi/10}$,

$C_7^x = 0$, $C_8^x = 2e^{j\pi/3}$

και για το $y(t)$

C_m^y: $C_1^y = 3e^{j\pi}$, $C_2^y = 2e^{j\pi/3}$, $C_3^y = e^{j\pi/10}$, $C_4^y = e^{j\pi/4}$, $C_5^y = 0$, $C_6^y = 0$,

$C_7^y = 0$, $C_8^y = 0$

Άρα $\frac{1}{T_y}\int_{T_y} x(t)\,y(t)\,dt = \frac{1}{T_y}\int_{T_y}\left(\sum_{m=1}^{8} C_m^x e^{jm\omega_0 t}\right)\left(\sum_{m'=1}^{8} C_{m'}^y e^{jm'\omega_0 t}\right)dt =$

$$= \frac{1}{T_y}\sum_{m=1}^{8}\sum_{m'=1}^{8}\int_{T_y} e^{j\omega_0 t(m+m')}\,dt$$

Όμως για $m+m'>0$ αφού $m\geq 1$, $m'\geq 1$. Επειδή οι συναρτήσεις βάσης ορθογώνιες ισχύει $\int_{T_y} e^{j\omega_0 t(m+m')}\,dt = 0$. Άρα $\frac{1}{T_y}\int_{T_y} x(t)\,y(t)\,dt = 0$

1.7

a) $x(t) = \frac{1}{\pi t}$

Αν $x(t) = sgn(t)$ τότε $X(\omega) = \frac{2}{j\omega}$

Γενικά αν $x(t) \xrightarrow{F} X(\omega)$ (Δυϊκότητα)

τότε $X(t) \leftrightarrow 2\pi\, x(-\omega)$

Άρα $sgn(t) \leftrightarrow \frac{2}{j\omega}$

$$\frac{2}{jt} \leftrightarrow 2\pi\, sgn(-\omega)$$

$$\frac{1}{\pi t} \leftrightarrow j\pi\, sgn(-\omega)$$

$$\frac{1}{\pi t} \leftrightarrow -j\pi\, sgn(\omega)$$ Άρα $X(\omega) = \frac{\pi}{j}\, sgn(\omega)$

β) $x_2(t) = \begin{cases} -t & , -8 \le t < -6 \\ 6\sin(2\pi t) & , -6 \le t < 0 \\ t^2 & , 0 \le t < 8 \\ 0 & , \text{αλλιώς} \end{cases}$

$$X_2(\omega) = -\int_{-8}^{-6} t e^{-j\omega t} dt + 6\int_{-6}^{0} \sin(2\pi t) e^{-j\omega t} dt + \int_{0}^{8} t^2 e^{-j\omega t} dt =$$

$$= \frac{1}{j\omega}\left\{\left[t e^{-j\omega t}\right]_{-8}^{-6} - \int_{-8}^{-6} e^{-j\omega t}\right\} - 6\left(\frac{2\pi(e^{6i\omega}-1)}{\omega^2-4\pi^2}\right) + \frac{2j}{\omega^3} +$$

$$+ \frac{(64i\omega^2 + 16\omega - 2i)e^{-8i\omega}}{\omega^3} =$$

$$= -\frac{(8j\omega - 1)e^{8i\omega} + (1-6j\omega)e^{6j\omega}}{\omega^2} - \frac{12\pi(e^{6i\omega}-1)}{\omega^2 - 4\pi^2} + \frac{2j}{\omega^3} +$$

$$+ \frac{(64j\omega^2 + 16\omega - 2j)}{\omega^3} e^{-8i\omega}$$

γ) $x_3(t) = e^{-6t} u\left(\frac{t-6}{6}\right) = e^{-6t} \cdot u\left(\frac{t}{6} - 1\right)$ $\quad \frac{t}{6} - 1 \ge 0 \Rightarrow t \ge 6$

$$X_3(\omega) = \int_{-\infty}^{+\infty} e^{-6t} u\left(\frac{t}{6}-1\right) e^{-j\omega t} dt = \int_{6}^{+\infty} e^{-6t} \cdot 1 \cdot e^{-j\omega t} dt =$$

$$= \int_{6}^{+\infty} e^{-(6+j\omega)t} dt = \frac{\left[e^{-(6+j\omega)t}\right]_6^{\infty}}{-(6+j\omega)} =$$

$$= \frac{\left(e^{-\infty} - e^{-(6+j\omega)6}\right)}{-(6+j\omega)} = \frac{e^{-36-6j\omega}}{6+j\omega}$$

δ) $X_4(\omega) = \left(\frac{\sin(6\omega)}{\omega-6}\right)^2 = \frac{1}{(\omega-6)^2}\left(\frac{1}{2} - \frac{1}{2}\cos(12\omega)\right) =$

$= \frac{1}{2(\omega-6)^2}\left(1-\cos(12\omega)\right) = \frac{1}{2(\omega-6)^2} - \frac{1}{2(\omega-6)^2}\cos(12\omega) =$

$= \frac{1}{2(\omega-6)^2} - \frac{1}{2(\omega-6)^2}\cdot\frac{1}{2}\left(e^{j12\omega} + e^{-j12\omega}\right)$

$\mathrm{sgn} \rightleftharpoons \frac{2}{j\omega} \longrightarrow \frac{\mathrm{sgn}}{2} \rightleftharpoons j\frac{1}{\omega} \longrightarrow \frac{e^{j6t}}{2}\mathrm{sgn} \rightleftharpoons j\frac{1}{\omega-6}$

$\rightarrow \frac{-e^{6jt}}{2}\mathrm{sgn} \rightleftharpoons \frac{1}{(\omega-6)^2}$

Άρα $x_4(t) = -\frac{1}{4}e^{6jt}\mathrm{sgn} - \frac{1}{4}\left[-e^{6j(t-12)}(t-12)\,\mathrm{sgn}(t-12) - \right.$

$\left. - e^{(6jt+12)}(t+12)\,\mathrm{sgn}(t+12)\right]$

ε) $X_5(\omega) = \mathrm{sgn}(\omega-6)\cos(6\omega)$

$x(t) \rightleftharpoons X(\omega)$ $\qquad \frac{2}{j\omega}$ $\qquad \mathrm{sgn}(-\omega) = -\mathrm{sgn}\,\omega$

$X(t) \rightleftharpoons 2\pi X(-\omega)$

Άρα $2\pi\,\mathrm{sgn}\,\omega \rightleftharpoons \frac{2}{jt}$

$\mathrm{sgn}\,\omega \rightleftharpoons -\frac{1}{j\pi t}$

$\mathrm{sgn}(\omega-6) \rightleftharpoons \frac{je^{j6t}}{\pi t}$

και $\cos 6\omega = \frac{1}{2}\left(e^{j6\omega} + e^{-j6\omega}\right)$

Άρα $x(t) = \frac{1}{2}\left(j\frac{e^{j6(t-6)}}{\pi(t-6)} + j\frac{e^{j6(t+6)}}{\pi(t+6)}\right)$

α) $X_6(\omega) = \begin{cases} \omega^2 , & -6 \le \omega \le 6 \\ e^{-6\omega} , & \omega > 6 \\ 0 , & \text{αλλού} \end{cases}$

$$X(t) = \frac{1}{2\pi}\int_{-6}^{6} \omega^2 e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{6}^{+\infty} e^{-6\omega} e^{j\omega t}\, d\omega =$$

$$= -\frac{e^{-6it}\left((18it^2 - 6t - i)e^{12it} - 18it^2 - 6t + i\right)}{\pi t^3} - \frac{1}{2\pi}\,\frac{e^{6it-36}}{it-6}$$

(Ανολαβαν σημειώσης από πρόχειρο που όμως είχαν λυθεί με $\alpha = 2$) λόγω έλλειψης χρόνου

Άσκ. 1.8) α) $x_1[n] = 3^{2n} u[-n-2]$

$$X(\Omega) = \sum_{n=-\infty}^{+2} \left(3^2 e^{-j\Omega}\right)^n \qquad \left(\Sigma = \frac{\tau \cdot \lambda - \alpha}{\lambda - 1}\right)$$

$$\begin{cases} \alpha = 81 e^{-2j\Omega} \\ \tau = 0 \\ \lambda = 9 e^{-j\Omega} \end{cases}$$

$$= \frac{0 - 81 e^{-2j\Omega}}{9 e^{-j\Omega} - 1}$$

β) $x_2[n] = \underbrace{\cos(2n)}_{x_{21}} + \underbrace{\sin(4n) \cdot \left[u[n] - u[n-6]\right]}_{x_{22}}$

$x_{21}[n] = \dfrac{e^{j\varphi} + e^{-j\varphi}}{2}$ περιοδικό άρα

$$X_{21}(\Omega) = \pi \sum_{k=-\infty}^{+\infty} \left[\delta(\Omega - 2 - 2\pi k) + \delta(\Omega + 2 - 2\pi k)\right]$$

$$X_{22}(\Omega) = F\{u[n] - u[n-6]\} * F\{\sin(4n)\}$$

$$= e^{-3j\Omega} F\{\underbrace{u[n+3] - u[n-3]}_{rect}\} * \frac{\pi}{j} \sum_{k=-\infty}^{+\infty} \left\{\delta(\Omega - 4 - 2\pi k) - \delta(\Omega + 4 - 2\pi k)\right\}$$

$$= e^{-3j\Omega} \frac{\sin\left(\Omega \cdot \frac{7}{2}\right)}{\sin(\Omega/2)} * \frac{\pi}{j} \sum_{k=-\infty}^{+\infty} \left\{\delta(\Omega - 4 - 2\pi k) - \delta(\Omega + 4 - 2\pi k)\right\}$$

$=$

$$= \sum_{n=-\infty}^{+\infty} \left[e^{-3j(\Omega-4-2\kappa n)} \frac{\sin\left(\left(\Omega-4-2\kappa n\right)\cdot\frac{7}{2}\right)}{j\sin\left(\frac{\Omega-4-2\kappa n}{2}\right)} + e^{-3j(\Omega+4-2\kappa n)} \frac{\sin\left(\left(\Omega+4-2\kappa n\right)-7/2\right)}{j\sin\left(\frac{\Omega+4-2\kappa n}{2}\right)} \right]$$

$$X_2 = X_{21}[\Omega] + X_{22}[\Omega]$$

γ) $X_3(\Omega) = \cos^2\left(\Omega + \frac{1}{5}\right)$

$$\cos^2(\Omega) = \frac{1+\cos 2\Omega}{2}$$

$$x_3(t) = F^{-1}\left\{\cos^2(\Omega)\right\} \cdot e^{j/5}$$

$$= F^{-1}\left\{1+\cos(2\Omega)\right\} \frac{e^{j/5}}{2}$$

$$= \left[\delta[n] + n\sum_{k=-\infty}^{+\infty}\left\{\delta(\Omega-2-2\pi k) + \delta(\Omega+2-2\pi k)\right\}\right] \frac{e^{j/5}}{2}$$

δ) $$X_4(\Omega) = \frac{1-e^{-2j\Omega}}{1+\frac{e^{-j\Omega}}{2}} = \frac{1}{1-\left(-\frac{1}{2}e^{-j\Omega}\right)} - \frac{e^{-2j\Omega}}{1-\left(-\frac{1}{2}e^{-j\Omega}\right)}$$

$$x_4(t) = \left(-\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{2}\right)^{(n-2)} u[n-2]$$

ε)

$$h[n] = \left(\frac{2}{3}\right)^n \cdot \cos\left(\frac{\pi n}{2}\right) \cdot u[n]$$

$$x[n] = \cos\left(\frac{\pi n}{2}\right) + \delta[n-2]$$

$$y_3[n] = h[n] * x[n] = \left[\left(\frac{2}{3}\right)^n \cdot \cos\left(\frac{\pi n}{2}\right) \cdot u[n]\right] * \left[\cos\left(\frac{\pi n}{2}\right) + \delta[n-2]\right]$$

$$y_{32}[n] = \left(\frac{2}{3}\right)^{n-2} \cos\left(\frac{\pi(n-2)}{2}\right) \cdot u[n-2]$$

$$Y_{31}[\Omega] = \frac{1}{1-\frac{2}{3}e^{-j\Omega}} * \pi \sum_{k=-\infty}^{+\infty} \left\{\delta\left(\Omega - \frac{\pi}{2} - 2\pi k\right) + \delta\left(\Omega + \frac{\pi}{2} - 2\pi k\right)\right\} \cdot \pi \sum_{k=-\infty}^{+\infty} \left\{\delta\left(\Omega - \frac{\pi}{2} - 2\pi n\right) + \delta\left(\Omega + \frac{\pi}{2} - 2\pi k\right)\right\}$$

$$= \sum_{k=-\infty}^{+\infty} \left[\frac{\pi}{1-\frac{2}{3}e^{-j\left(\Omega-\frac{\pi}{2}-2\pi n\right)}} + \frac{\pi}{1-\frac{2}{3}e^{-j\left(\Omega+\frac{\pi}{2}-2\pi n\right)}}\right] \cdot \pi \sum_{k=-\infty}^{+\infty} \left[\delta\left(\Omega - \frac{\pi}{2} - 2\pi n\right) + \delta\left(\Omega + \frac{\pi}{2} - 2\pi n\right)\right]$$

$$= \pi \sum_{k=-\infty}^{+\infty} \left[\frac{\delta\left(\Omega-\frac{\pi}{2}-2\pi n\right)}{1-\frac{2}{3}e^{-j\left(\frac{\pi}{2}+2\pi n\right)}} + \frac{\delta\left(\Omega+\frac{\pi}{2}-2\pi n\right)}{1-\frac{2}{3}e^{-j\left(-\frac{\pi}{2}+2\pi k\right)}}\right.$$

$$\left. + \frac{\delta\left(\Omega-\frac{\pi}{2}-2\pi n\right)}{1-\frac{2}{3}e^{-j\left(\frac{\pi}{2}+2\pi n\right)}} + \frac{\delta\left(\Omega+\frac{\pi}{2}-2\pi n\right)}{1-\frac{2}{3}e^{-j\left(-\frac{\pi}{2}+2\pi n\right)}}\right]$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} \left[\frac{\delta(\Omega - \frac{\pi}{2} - 2k\pi)}{1 - \frac{2}{3} e^{-j(\frac{\pi}{2} + 2k\pi)}} + \frac{\delta(\Omega + \frac{\pi}{2} - 2k\pi)}{1 - \frac{2}{3} e^{-j(-\frac{\pi}{2} + 2k\pi)}} \right]$$

$=$

Ασκ. 1.9) $\xrightarrow{x(t)} \boxed{} \xrightarrow{y(t)}$

$$H(\omega) = \frac{3 + 3j\omega + (j\omega)^2}{4 + 8j\omega + 5(j\omega)^2 + (j\omega)^3}$$

α) $H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$ και $\dfrac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)$

$$\Rightarrow Y(\omega)\left[4 + 8j\omega + 5(j\omega)^2 + (j\omega)^3\right] = X(\omega)\left[3 + 3j\omega + (j\omega)^2\right]$$

$$\overset{\mathcal{F}^{-1}\{\}}{\Longrightarrow} \quad 4y(t) + 8\frac{dy(t)}{dt} + 5\frac{d^2y(t)}{dt^2} + \frac{d^3y(t)}{dt^3} = 3x(t) + 3\frac{dx(t)}{dt} + \frac{d^2x(t)}{dt^2}$$

Δ.Ε.

γνωστό ότι ∃ πόλος $|j\omega| = 1$

για $j\omega = -1$ $\quad 4 - 8 + 5 - 1 = 0$ δεκτό

για $j\omega = 1$ $\quad 4 + 8 + 5 + 1 \neq 0$ απορ.

κάνω διαίρεση του παρονομαστή με το $(j\omega + 1)$

$$\begin{array}{l|l}
(j\omega)^3 + 5(j\omega)^2 + 8j\omega + 4 & j\omega + 1 \\
\hline
-\left[(j\omega)^3 + (j\omega)^2\right] & (j\omega)^2 + 4j\omega + 4 \\
\quad 4(j\omega)^2 + 8j\omega + 4 & \\
\quad -\left[4(j\omega)^2 + 4j\omega\right] & \\
\qquad 4j\omega + 4 & \\
\qquad -(4j\omega + 4) & \\
\qquad\quad 0 &
\end{array}$$

$$(j\omega)^2 + 4j\omega + 4 = (j\omega + 2)^2$$

$$\Rightarrow H(\omega) = \frac{3 + 3j\omega + (j\omega)^2}{(j\omega+2)^2 (j\omega+1)} = \frac{A}{(j\omega+2)^2} + \frac{B}{j\omega+2} + \frac{\Gamma}{j\omega+1}$$

$$3 + 3j\omega + (j\omega)^2 = A(j\omega+1) + B(j\omega+2)(j\omega+1) + \Gamma(j\omega+2)^2$$

για $j\omega = -1 \Rightarrow 3 - 3 + 1 = 0 + 0 + \Gamma^2 \Rightarrow \Gamma = 1$

για $j\omega = -2 \Rightarrow 3 - 6 + 4 = -A \Rightarrow A = -1$

για $j\omega = 0 \Rightarrow 3 = -1 + 2B + 4 \Rightarrow B = 0$

$$\Rightarrow H(\omega) = \frac{-1}{(j\omega+2)^2} + \frac{1}{j\omega+1}$$

$$\overset{F^{-1}\{\}}{\Rightarrow} h(t) = -t e^{-2t} u(t) + e^{-t} u(t)$$

$$\Rightarrow h(t) = \left[e^{-t} - t e^{-2t}\right] u(t)$$

β) απο τις ιδιότητες ΓΧΑ συστήματος μέσω κρουστικής απόκρισης:

$h(t) \neq 0 \Rightarrow$ ΟΧΙ αιτιατό

$\int_{-\infty}^{+\infty} |h(t)| \, dt < \infty$ ισχύει αφού ορίζεται ο M.F. $\Rightarrow$ ευσταθές

$h(t) \neq k \cdot \delta(t) \Rightarrow$ ΜΕ μνήμη γ) $y(t) = e^{-4t} u(t+2) \Rightarrow Y(\omega) = \dfrac{\frac{1}{4+j\omega} e^{+2j\omega}}{e^{-8}}$

$= \dfrac{e^{-4(t+2)}}{e^{-8}} u(t+2) \nearrow$

$y(t) = x(t) * h(t)$

$\Rightarrow X(\omega) = \dfrac{Y(\omega)}{H(\omega)} = \dfrac{\frac{1}{4+j\omega} e^{+2j\omega}/e^{-8}}{\frac{3+3j\omega+(j\omega)^2}{(j\omega+2)^2(j\omega+1)}}$

$= \dfrac{e^{-2j\omega}(j\omega+2)^2(j\omega+1)}{(4+j\omega)((j\omega)^2+3(j\omega)+3)} \overset{(j\omega=v)}{=} e^{-2j\omega} \dfrac{v^3+5v^2+8v+4}{v^3+7v^2+15v+12}$

$\underline{\text{μακρά διαίρεση}}$

$\underline{\text{\& μερικά κλάσματα}}$ $\left[1 - \dfrac{2v+5}{7(v^2+3v+3)} - \dfrac{12}{7(v+4)}\right] \dfrac{e^{+2j\omega}}{e^{-8}}$

$= \left[1 - \dfrac{2v+5}{7\left[(v+3/2)^2 + (3-9/4)\right]} - \dfrac{12/7}{v+4}\right] e^{2j\omega}/e^{-8}$

$= \left[1 - \dfrac{\frac{2}{7}v + 5/7}{(v+3/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \dfrac{12/7}{v+4}\right] e^{2j\omega}/e^{-8}$

$\underline{\underline{(v=j\omega)}}$ $1 - \dfrac{j\omega \frac{4\sqrt{3}}{21}\left(\frac{\sqrt{3}}{2}\right)}{(j\omega+3/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \dfrac{\frac{10\sqrt{3}}{21}\cdot\frac{\sqrt{3}}{2}}{(j\omega+3/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \dfrac{12/7}{j\omega+4}$

$\Rightarrow x(t) = \dfrac{1}{e^{-8}}\left[\delta(t+2) - \dfrac{4\sqrt{3}}{21} \dfrac{d\left(e^{-\frac{3(t+2)}{2}} \sin\left(\frac{\sqrt{3}}{2}(t+2)\right)u(t+2)\right)}{dt} - \dfrac{10\sqrt{3}}{21} e^{-\frac{3(t+2)}{2}} \sin\left(\frac{\sqrt{3}}{2}(t+2)\right)u(t+2) - \dfrac{12}{7} e^{-4(t+2)}\right]$

Ασκ. 1.10) $X_1(\omega) = e^{-2\omega}\left[2u(\omega+8) - u(2\omega) - u(3\omega-30)\right]$

$X_2(\omega) = rect\left(\frac{\omega}{12}\right)$

εύρος ζώνης του $x_1(t)$: 10

εύρος ζώνης του $x_2(t)$: 6

i) $y_1 = x_1(t+2) + x_2(t-4)$

εύρος ζώνης $x_1(t+2)$: 10

εύρος ζώνης $x_2(t-4)$: 6

η ολίσθηση στο πεδίο του χρόνου δεν επηρεάζει το εύρος ζώνης πρόσθεση γίνεται ξεχωριστά για καθε

άρα εύρος ζώνης $y_1 = max(\omega_{M1}, \omega_{M2})$

$\Rightarrow$ εύρος ζώνης $y_1 : \omega_{max} = max(6,10) = 10$

πρέπει $\omega_s \geq 2\omega_{max} \Rightarrow \frac{2\pi}{T_s} \geq 2 \cdot 10 \Rightarrow T_s \leq \frac{\pi}{10}$

ii) $y_2 = [x_1(t)]^3$

εύρος ζώνης $x_1(t)$: 10

επειδή είναι γινόμενο στο πεδίο συχνότητας γίνεται συνέλιξη, το εύρος ζώνης το άθροισμα των δύο σημάτων

εύρος ζώνης y_2 : $10+10+10 = 30$

$\omega_s \geq 2\omega_{max} \Rightarrow \frac{2\pi}{T_s} \geq 2 \cdot 30 \Rightarrow T_s \leq \frac{\pi}{30}$

iii) $y_3 = [x_1(t)]^3 \cdot x_2(t)$

όπως στο ii) εύρος ζώνης $\omega_{max} = 3 \cdot 10 + 6 = 36$

$\Rightarrow T_s \leq \frac{\pi}{36}$

iv) $y_4 = x_1(t) * [x_2(t)]^3$

εύρος ζώνης $x_1(t)$: 10

εύρος ζώνης $[x_2(t)]^3$: 18

η συνέλιξη στο πεδίο της συχνότητας γίνεται πολλαπλασιασμός και στον πολλαπλασιασμό το εύρος του μεγαλύτερου σε εύρος σήματος συρρικνώνεται στο μικρότερο λόγω του επί μέρους πολλαπλασιασμού με το 0

εύρος ζώνης y_4 : $\min(10, 18) = 10$

$\Rightarrow T_s = \frac{\pi}{10}$

β)

$x_c(t) \longrightarrow \boxed{S_1} \xrightarrow{y_c(t)}$ δειγματοληψία

$x_c(t) = [\cos(4\pi t)]^2$

δ.ε.: $\frac{dy_c(t)}{dt} + 2y_c(t) = x_c(t) \xrightarrow{F} j\omega Y(\omega) + 2Y(\omega) = X(\omega)$

$\Rightarrow Y(\omega)[j\omega + 2] = X(\omega)$

$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{j\omega + 2}$

$\xrightarrow{F^{-1}} h(t) = e^{-2t} u(t)$

$$x_c(t) = \left[\cos(4\pi t)\right]^2 = \frac{1}{2} + \frac{\cos 8\pi t}{2}$$

$$\overset{F\{\}}{\Longrightarrow} X_c(\omega) = \pi\delta(\omega) + \frac{\pi}{2}\Big(\delta(\omega - 8\pi) + \delta(\omega + 8\pi)\Big)$$

$$Y_c(\omega) = X_c(\omega) \cdot H(\Omega) = \left[\pi\delta(\omega) + \frac{\pi}{2}\Big(\delta(\omega-8\pi) + \delta(\omega+8\pi)\Big)\right] \cdot \frac{1}{j\omega+2}$$

$$= \frac{\pi\delta(\omega)}{2} + \frac{\pi\,\delta(\omega-8\pi)}{2(j8\pi+2)} + \frac{\pi\,\delta(\omega+8\pi)}{2(-j8\pi+2)} \qquad \Big(x(t)\cdot\delta(t-t_0) = x(t_0)\cdot\delta(t-t_0)\Big)$$

$$= \frac{2\pi\delta(\omega)}{2\cdot 2} + \frac{2\pi\,\delta(\omega-8\pi)}{2\cdot 16\left(j\pi + \frac{1}{4}\right)} + \frac{2\pi\,\delta(\omega+8\pi)}{2\cdot(-16)\left(j\pi - \frac{1}{4}\right)}$$

$$\Rightarrow y_c(t) = \frac{1}{4} + \frac{e^{8jt}}{8(4j\pi+1)} - \frac{e^{-8jt}}{8(4j\pi-1)}$$

Εύρος φάσματος $y_c(t)$: $\omega_{max} = 8\pi$

$$\Rightarrow \quad \omega_s \geq 2\omega_{max} \Rightarrow \frac{2\pi}{T_s} \geq 2\cdot 8\pi \Rightarrow T_s \leq 1/8 = 0{,}125\text{ s}$$

$$\Rightarrow y[n] = y_c(nT_s) = \frac{1}{4} + \frac{e^{8j\pi/8}}{8(4j\pi+1)} - \frac{e^{-8j\pi/8}}{8(4j\pi-1)}$$

$$\Rightarrow y[n] = \frac{1}{4} + \frac{e^{j\pi n}}{8(4j\pi+1)} - \frac{e^{-j\pi n}}{8(4j\pi-1)}$$

sinc

iv) $z[n] = y[n] * h_2[n]$

$$h_2[n] = \frac{\sin\left(\frac{\pi n}{2} - \pi\frac{2}{2}\right)}{\frac{\pi}{\pi}(n-2)} = \pi \frac{\sin\left(\frac{\pi(n-2)}{2}\right)}{\pi(n-2)}$$

$$y[n] = \frac{1}{4} + \frac{e^{j\pi n}}{8(4j\pi+1)} - \frac{e^{-j\pi n}}{8(4j\pi-1)}$$

$$H_2[\Omega] = \pi e^{-2j\Omega} \begin{cases} 1 & 0 \leq |\Omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq |\Omega| \leq \pi \end{cases}$$ επίσης $H_2[\Omega] = H_2[\Omega + 2\pi]$

$$Y[\Omega] = \frac{\pi}{2}\sum_{k=-\infty}^{+\infty} \delta(\Omega - 2\pi k) + \frac{2\pi}{8_4}\sum_{k=-\infty}^{+\infty}\left[\frac{\delta(\Omega - 1 - 2\pi k)}{(4j\pi+1)} - \frac{\delta(\Omega + 1 - 2\pi k)}{4j\pi - 1}\right]$$

$$Z[\Omega] = H_2[\Omega] \cdot Y[\Omega]$$

για $\frac{\pi}{2} \leq |\Omega| \leq \pi$ $\quad Z(\Omega) = 0 \Rightarrow z[n] = 0$

για $0 \leq |\Omega| \leq \frac{\pi}{2}$

$$Z[\Omega] = \frac{\pi^2 e^{-2j\Omega}}{2} \frac{2\pi}{2\pi} \sum_{k=-\infty}^{+\infty}\left[\delta(\Omega - 2\pi k) + \frac{\delta(\Omega - 1 - 2\pi k)}{8j\pi + 2} - \frac{\delta(\Omega + 1 - 2\pi k)}{8j\pi - 2}\right]$$

$$z[n] = \frac{\pi}{4}\left[1 + \frac{e^{j(n-2)}}{8j\pi + 2} - \frac{e^{-j(n-2)}}{8j\pi - 2}\right]$$