2η Σειρά Ασκήσεων

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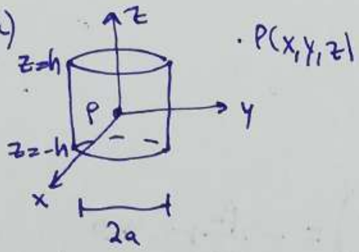
Άσκηση 1

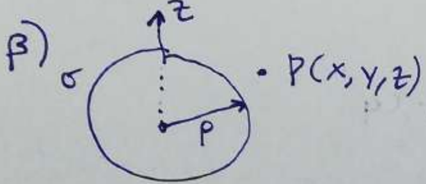
α)



$$\rho = \rho_0 \frac{r}{a} \cos\varphi$$

$$\Phi = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV = \frac{1}{4\pi\varepsilon}\int_{z=-h}^{h}\int_{\varphi=0}^{2\pi}\int_{r=0}^{a} \frac{\rho_0 \frac{r'^2}{a}\cos\varphi'\, dr'\, d\varphi'\, dz'}{\sqrt{(r'\cos\varphi' - r\cos\varphi)^2 + (r'\sin\varphi' - r\sin\varphi)^2 + (z'-z)^2}}$$

β)



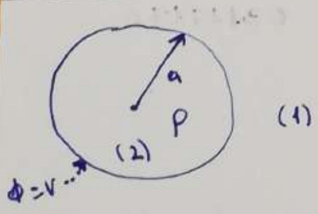
$$\rho = \rho_0 \frac{r}{a}\cos\theta\sin\varphi \quad , \quad \sigma = \sigma_0 \cos\theta$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}\,'-\vec{r}|}dV + \frac{1}{4\pi\varepsilon}\int_S \frac{\sigma(\vec{r}\,')}{|\vec{r}\,'-\vec{r}|}dS =$$

$$= \frac{1}{4\pi\varepsilon}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{r'=0}^{a} \frac{\rho_0 \frac{r'^3}{a}\cos\theta' \sin\varphi' \sin\theta'\, dr'\, d\theta'\, d\varphi'}{\sqrt{(r'\sin\theta'\cos\varphi' - r\sin\theta\cos\varphi)^2 + (r'\sin\theta'\sin\varphi' - r\sin\theta\sin\varphi)^2 + (r'\cos\theta' - r\cos\theta)^2}}$$

$$+ \frac{1}{4\pi\varepsilon}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \frac{\sigma_0 \cos\theta'\, a^2 \sin\theta'\, d\theta'\, d\varphi'}{\sqrt{(r'\sin\theta'\cos\varphi' - r\sin\theta\cos\varphi)^2 + (r'\sin\theta'\sin\varphi' - r\sin\theta\sin\varphi)^2 + (r'\cos\theta' - r\cos\theta)^2}}$$

Άσκηση 5



$$\rho = \rho_0 \frac{r}{a} \quad (0 \le r \le a)$$

a) $\nabla^2 \Phi_2 = -\frac{\rho}{\varepsilon_0} \Leftrightarrow \nabla^2 \Phi_2 = -\frac{\rho_0 \frac{r}{a}}{\varepsilon_0} \Leftrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi_2}{\partial r}\right) = -\frac{\rho_0}{\varepsilon a} r \Leftrightarrow$

$\Leftrightarrow r^2 \frac{\partial \Phi_2}{\partial r} = -\frac{\rho_0}{4\varepsilon_0 a} r^4 + C_1 \Leftrightarrow \frac{\partial \Phi_2}{\partial r} = -\frac{\rho_0}{4\varepsilon_0 a} r^2 + \frac{C_1}{r^2} \Leftrightarrow$

$\Leftrightarrow \Phi_2(r) = -\frac{\rho_0}{12\varepsilon_0 a} r^3 - \frac{C_1}{r} + C_2$

$\nabla^2 \Phi_1 = 0 \Leftrightarrow \frac{\partial \Phi_1}{\partial r} = \frac{C_3}{r^2} \Leftrightarrow \Phi_1(r) = -\frac{C_3}{r} + C_4$

- $\Phi_2(a) = \Phi_1(a) = V \Leftrightarrow \begin{cases} C_4 = V + \frac{C_3}{a} \\ C_2 - \frac{C_1}{a} - \frac{\rho_0 a^2}{12\varepsilon_0} = V \end{cases}$

- Αναφορά στο $+\infty \Rightarrow C_4 = 0$

- Οριακή συνθήκη στο 0: $\lim_{r\to 0}\left[4\pi r^2 \varepsilon_0 E_2(r)\right] = 0 \Leftrightarrow$

$\Leftrightarrow \lim_{r\to 0}\left[4\varepsilon_0 \pi r^2\left(-\frac{\rho_0}{4\varepsilon_0 a} r^2 + \frac{C_1}{r_2}\right)\right] = 0 \Leftrightarrow C_2 = 0$

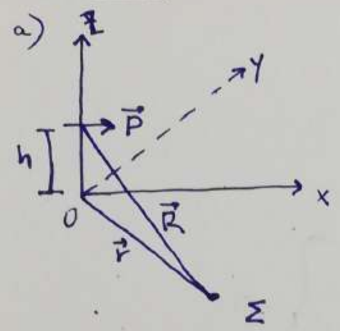
Οπότε, $\Phi(r) = \begin{cases} -\frac{\rho_0}{12\varepsilon_0 a} r^3 + \frac{\rho_0 a^2}{12\varepsilon_0} + V, & r < a \\ \frac{V \cdot a}{r}, & r > a \end{cases}$

β) Οριακή Συνθήκη στο $r = a$.

$$-\varepsilon_0 \left.\frac{\partial \Phi_1}{\partial r}\right|_{r=a} + \varepsilon_0 \cdot \left.\frac{\partial \Phi_2}{\partial r}\right|_{r=a} = \sigma \Leftrightarrow \sigma = -\frac{\rho_0 a}{4} + \frac{V \cdot \varepsilon_0}{a}$$

Άσκηση 3

α)



$$\Phi(\vec{r}) = \frac{\vec{p}\cdot\hat{R}}{4\pi\varepsilon R^2}$$

οπότε $\Phi(\vec{r}) = \frac{p\cdot\hat{x}\cdot\vec{R}}{4\pi\varepsilon R^3} = \frac{p\cdot x}{4\pi\varepsilon\sqrt{x^2+y^2+(z-h)^2}^{\,3}}$

β) Αν $\vec{p} = \hat{z}p$, τότε $\Phi(\vec{r}) = p\cdot\hat{z}\cdot\vec{R} = \frac{p(z-h)}{4\pi\varepsilon\sqrt{x^2+y^2+(z-h)^2}}$

Άσκηση 6

α) $\Phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) =$$

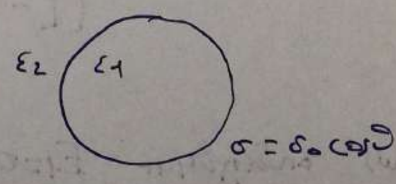
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cos\theta\left(A - \frac{2B}{r^3}\right)\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(-\sin^2\theta\right)\left(Ar + \frac{B}{r^2}\right) =$$

$$= \frac{\cos\theta}{r^2}\left(2Ar + \frac{2B}{r^2}\right) + \frac{-1}{r^2\sin\theta}\,2\sin\theta\cos\theta\left(Ar + \frac{B}{r^2}\right) = 0$$

οπότε ικανοποιείται η Laplace $\nabla^2\Phi = 0$.

β) $\Phi_1(r,\theta) = \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta, \quad r \le a$

$\Phi_2(r,\theta) = \left(A_2 r + \frac{B_2}{r^2}\right)\cos\theta, \quad r \ge a$



• Σ.Σ. στο r=a: $\Phi_1(a,\theta) = \Phi_2(a,\theta) \Rightarrow A_1 a + \frac{B_1}{a^2} = A_2 a + \frac{B_2}{a^3} \Leftrightarrow$

$\Leftrightarrow A_1 a^3 + B_1 = A_2 a^3 + B_2$

• Σ.Σ. στο r=a: $-\varepsilon_2\left.\frac{\partial\Phi_2}{\partial r}\right|_{r=a} + \varepsilon_1\left.\frac{\partial\Phi_1}{\partial r}\right|_{r=a} = \sigma_0\cos\theta \Rightarrow$

$\Leftrightarrow -\varepsilon_2\left(A_2 - \frac{2B_2}{a^3}\right) + \varepsilon_1\left(A_1 - \frac{2B_1}{a^3}\right) = \sigma_0 \Leftrightarrow$

$\Leftrightarrow \varepsilon_1 A_1 - \frac{2\varepsilon_1}{a^3}B_1 - \varepsilon_2 A_2 + \frac{2\varepsilon_2}{a^3}B_2 = \sigma_0$

- Αναφορά στο r=a: $\Phi_1(a,\theta)=0 \Leftrightarrow A_1 a^3 + B_1 = 0$

- Ισχύει ότι $\vec{E} = \hat{r}\, E(r,\theta)$ λόγω της σφαίρας, για $r<a$

οπότε, $\left.\begin{matrix} \nabla\times\vec{E}=\vec{0} \\ \nabla\cdot D = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \dfrac{\partial E_r}{\partial\theta} = 0 \Rightarrow E(r,\theta) = c_1 + f_1(r) \\ \dfrac{\partial}{\partial r}(r^2 E_r) = 0 \Rightarrow E(r,\theta) = \dfrac{c_2 + f_2(\theta)}{r^2} \end{matrix}\right\} \Rightarrow$

$$\Rightarrow f_2(\theta) = r^2(c_1 + f_1(r)) - c_2$$

Άρα, $f_2(\theta) = c, \forall\theta$ και $f_1(r) = \frac{c}{r^2} - c_1$

οπότε, $E(r,\theta) = \frac{c}{r^2}$

Από Σ.Σ. $\lim_{r\to 0}(4\pi r^2 \varepsilon_1 E_1(r,\theta)) = 0 \Rightarrow c = 0$

Όμως, $E = -\nabla\Phi_1 \Rightarrow \nabla\Phi_1 = 0 \Rightarrow \frac{\partial\Phi}{\partial r} = 0 \Rightarrow A_1 - \frac{2A_1 a^3}{r^3} = 0 \Rightarrow A_1 = 0$

οπότε $B_1 = 0$, $A_2 a^3 + B_2 = 0$ και $-\varepsilon_2 A_2 + \frac{2\varepsilon_2}{a^3} B_2 = \sigma_0$

Άρα $B_2 = -A_2 a^3$ και $A_2(-\varepsilon_2 - 2\varepsilon_2) = \sigma_0 \Rightarrow A_2 = -\frac{\sigma_0}{3\varepsilon_2}$

Τελικά, $\Phi(r,\theta) = \begin{cases} 0 & , r<a \\ \left[-\frac{\sigma_0}{3\varepsilon_2} r + \frac{\sigma_0 a^3}{3\varepsilon_2} \cdot \frac{1}{r^2}\right]\cos\theta & , r>a \end{cases}$

γ) · Όπως αποδείχθηκε $E_1 = 0, r<a$ άρα σταθερό

- Για $r>a$: $E_2 = -\nabla\Phi_2 = \frac{\partial\Phi_2}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi_2}{\partial\theta}\hat{\theta} \Leftrightarrow$

$$\Leftrightarrow E_2 = \left(-\frac{\sigma_0}{3\varepsilon_2} - \frac{2\sigma_0 a^3}{3\varepsilon_2}\cdot\frac{1}{r^3}\right)\hat{r} + \left(\frac{\sigma_0}{3\varepsilon_2} - \frac{\sigma_0 a^3}{3\varepsilon_2}\cdot\frac{1}{r^3}\right)\sin\theta\cdot\hat{\theta}.$$

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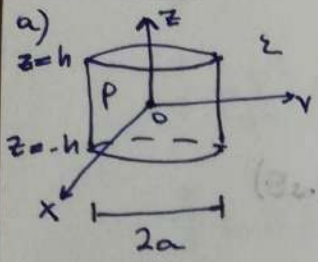
Άσκηση 2

a)



$\rho = \rho_0 \frac{r_T}{a}\cos\varphi$ $\quad 0<r_T<a$, $0<\varphi<2\pi$, $-h<z<h$

1) $\vec{P} = \int \vec{r}' dq$

$Q_{ολ} = \iiint \left(\rho_0 \frac{r_T}{a}\cos\varphi \, dr_T dz \, r_T d\varphi = \frac{\rho_0}{a}\iiint \cos\varphi \, d\varphi \, dz \, r_T^2 dr_T = 0.\right.$

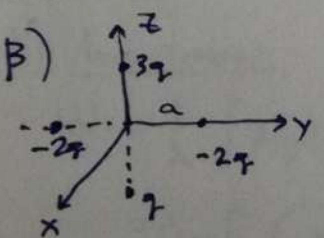
Έστω $\vec{r}$ από το O.

$\vec{P} = \int \vec{r}\rho dV = \int \left(\rho_0 \frac{r_T^2}{a}\cos^2\varphi ,\ \rho_0\frac{r_T^2}{a}\cos\varphi\sin\varphi ,\ \rho_0\frac{r_T}{a}\cos\varphi z\right) dV \Rightarrow$

$\Rightarrow \vec{P} = \frac{\pi 2h\rho_0 a^4}{4a}\hat{x} = \frac{h\rho_0 a^3\pi}{2}\hat{x}$

2) $\Phi = \frac{\cancel{Q_{ολ}}^{\,0}}{4\pi\varepsilon r} + \frac{\vec{P}\cdot\hat{r}}{4\pi\varepsilon r^2} = \frac{h\rho_0 a^3\pi}{8\pi\varepsilon r^2}\hat{x}\hat{r} = \frac{h\rho_0 a^3\pi}{8\pi\varepsilon r^2}\sin\theta\cos\varphi$

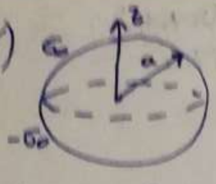
β)



1) $\vec{P} = \vec{r}'_i q_i = 3qa\hat{z} + q(-a\hat{z}) - 2qa\hat{y} + 2qa\hat{y} = 2qa\hat{z}$

$Q_{ολ} = \Sigma q_i = 3q + q + (-2q) + (-2q) = 0$

2) $\Phi = \frac{\cancel{Q_{ολ}}^{\,0}}{4\pi\varepsilon r} + \frac{\vec{P}\hat{r}}{4\pi\varepsilon r^2} = \frac{qa}{2\pi\varepsilon r^2}\hat{z}\hat{r} = \frac{qa\cos\theta}{2\pi\varepsilon r^2}$

γ) 

1) $\vec{p} = \int \vec{r}\, dq = \int a\hat{r}\, \sigma\, ds$ με $ds = a^2 \sin\theta\, d\varphi\, d\theta$

και $a\hat{r} = (a\sin\theta\cos\varphi,\ a\sin\theta\sin\varphi,\ a\cos\theta)$

(επικρατεί μόνο η z συνιστώσα)

$\vec{P}_1 = \int \sigma_0 a^3 \cos\theta \sin\theta\, d\varphi\, d\theta\, \hat{z} = \sigma_0 a^3 2\pi \cdot \frac{1}{2}\hat{z} \qquad (0<\theta<\frac{\pi}{2})$

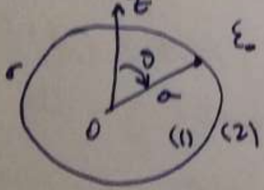
$\vec{P}_2 = \sigma_0 a^3 2\pi \left(\frac{1}{2}\right)\hat{z} \qquad (\frac{\pi}{2}<\theta<\pi)$

Άρα, $\vec{p} = \sigma_0 a^3 2\pi \hat{z}$

$Q_{ολ} = \sigma_0 2\pi a^2 - \sigma_0 2\pi a^2 = 0$

2) $\Phi = \cancel{\frac{Q_{ολ}}{4\pi\varepsilon r}}^{0} + \frac{\vec{p}\hat{r}}{4\pi\varepsilon r^2} = \frac{\sigma_0 a^3 2\pi\cos\theta}{4\pi\varepsilon r^2} = \frac{\sigma_0 a^3 \cos\theta}{2\varepsilon r^2}$

Άσκηση 4



$\Phi_1 = V\frac{r}{a}\cos\theta = V\frac{z}{a}$

$\Phi_2 = V\frac{a^2}{r^2}\cos\theta$

α). $\sigma = -\varepsilon_0 \frac{\partial \Phi_2}{\partial r}\Big|_{r=a} + \varepsilon_0 \frac{\partial \Phi_1}{\partial r}\Big|_{r=a} = \left(-\frac{2Va^2\cos\theta}{r^3}\right)(-\varepsilon_0) + \varepsilon_0 \frac{V\cos\theta}{a} \Rightarrow$

$\Rightarrow \sigma = 3\frac{\varepsilon_0 V\cos\theta}{a}$

· $\vec{E} = -\nabla\Phi$ άρα:

(1): $\vec{E} = -\frac{\partial\Phi}{\partial r}\hat{z} = -\frac{V}{a}\hat{z}$ και σε σφαιρικές: $\vec{E} = -\frac{V}{a}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

(2): $\vec{E} = -\frac{\partial\Phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\cancel{\frac{\partial\Phi}{\partial\varphi}}^{0}\hat{\varphi} = \frac{2Va^2\cos\theta}{r^3}\hat{r} + \frac{Va^2\sin\theta}{r^3}\hat{\theta}$

· $E_r^{(2)} - E_r^{(1)} = \sigma$ ικανοποιείται β) 1ος τρόπος

$$W = \frac{1}{2}\int \varepsilon_0 E^2 dV = \frac{1}{2}\int_{\sigma\varphi\alpha\acute{\iota}\rho\alpha} \frac{\varepsilon_0 V^2}{a^2} dV + \frac{1}{2}\int_{\upsilon\pi o\lambda.} \varepsilon_0 E^2 dV$$

$$= \frac{1}{2}\cdot\frac{\varepsilon_0 V^2}{a^2}\cdot\frac{4}{3}\pi a^3 + \frac{\varepsilon_0}{2}\iiint\left(\frac{4V^2a^4\cos^2\theta}{r^6} + \frac{V^2a^4\sin^2\theta}{r^6}\right) r^2\sin\theta$$

$$= \frac{2}{3}\varepsilon_0 V^2 \pi a + \varepsilon_0 V^2 \pi a^4 \int \frac{4}{r^4} dV = 2\varepsilon_0 \pi V^2 a$$

2ος τρόπος

$\rho = \nabla \vec{E}$ όμως $\nabla \vec{E} = 0 \Rightarrow \rho = 0$

$$W = \frac{1}{2}\int \Phi \sigma dS = \frac{1}{2}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \frac{V\cos\theta\, 3\varepsilon_0\, V\cos\theta\, a^2\sin\theta}{a} d\theta d\varphi =$$

$$= 3V^2\varepsilon_0 a\pi \int_{\pi}^{0}\cos^2\theta\, d(\cos\theta) = 3V^2\varepsilon_0 a\pi\left[\frac{\cos^3\theta}{3}\right]_{\pi}^{0} = 2\pi V^2\varepsilon_0 a$$

Άσκηση 5

γ) 1ος τρόπος

$$W = \frac{1}{2}\int \Phi\rho dV + \frac{1}{2}\int \Phi\sigma dS$$

$$= 2\pi\left[\int \frac{\rho_0^2 a^2}{12\varepsilon_0 a} r^3 dr - \int \frac{\rho_0^2 a^6}{12\varepsilon_0 a} + \int \frac{\rho_0 V r^3}{a} dr\right] + \frac{V\sigma 4\pi a^2}{2}$$

$$= \frac{\rho_0^2 a^5 \pi}{24\varepsilon_0} - \frac{\rho_0^2 a^5 \pi}{42\varepsilon_0} + \frac{\rho_0 V a^3 \pi}{2} + 2\pi a^2 V\left(\frac{\varepsilon_0 V}{a} - \frac{\rho_0 a}{4}\right)$$

$$= \frac{\rho_0^2 a^5 \pi}{56\varepsilon_0} + 2\pi a \varepsilon_0 V^2$$

2ος Τρόπος

$$W = \frac{1}{2}\int_{\pi\alpha\nu\tau o\acute{\upsilon}} \varepsilon_0 E^2 dV = \frac{1}{2}\int_{\sigma\varphi.} \frac{\varepsilon_0 \rho_0^2 r^4}{16a^2\varepsilon_0^2} r^2 \sin\theta d\varphi d\theta dr + \frac{1}{2}\int_{\upsilon\pi o\lambda.} \frac{\varepsilon_0 V^2 a^2}{r^4} r^2 \sin\theta d\varphi d\theta dr =$$

$$= \frac{2\pi\rho_0^2 a^7}{16a^2\varepsilon_0^7} + 2\pi\varepsilon_0 V^2 a = \frac{\pi\rho_0^2 a^5}{56\varepsilon_0} + 2\pi\varepsilon_0 V^2 a$$

3ος Τρόπος

$$W = \frac{1}{2}\oint \Phi \varepsilon_0 \vec{E}\, ds + \frac{1}{2}\int_0^a \varepsilon_0 E^2 dV$$

$$= \frac{\pi \rho_0^2 a^5}{56\varepsilon_0} + \frac{1}{2} V^2 \varepsilon_0 a 4\pi$$ (μαζί επιλέγομε την επιφάνεια με ακτίνα a^+ όπου $\Phi = V$).

4ος Τρόπος

$$W = \int\left(\int_0^{\rho} \Phi' d\rho'\right)dV + \int\left(\int_0^{\sigma} \Phi' d\sigma'\right)ds$$

(ισοδύναμος τρόπος με τον 1ο).

Άσκηση 7

ε_3, $\Phi = 0$, $z = h$

ε_2, $\rho(z) = \rho_0 \frac{z}{h}$

ε_1, $\Phi = V$, $z = 0$

$$\hat{E}(z=2h) = \hat{z} E_0$$

a) · Για $z<0$: $\nabla^2 \Phi_1 = 0 \Rightarrow \frac{\partial^2 \Phi_1}{\partial^2 z} = 0 \Rightarrow \Phi_1(z) = C_1 z + C_2$

· Για $0<z<h$: $\nabla^2 \Phi_2 = -\frac{\rho_0 z}{\varepsilon_2 h} \Rightarrow \frac{\partial^2 \Phi_2}{\partial^2 z} = -\frac{\rho_0}{\varepsilon_2 h} z \Rightarrow \Phi_2(z) = -\frac{\rho_0}{6\varepsilon_2 h} z^3 + C_3 z + C_4$

· Για $z>h$: $\nabla^2 \Phi_3 = 0 \Rightarrow \Phi_3(z) = C_5 z + C_6$

· $\Phi_1(0) = \Phi_2(0) = V \Rightarrow C_2 = C_4 = V$

· $\Phi_2(h) = \Phi_3(h) = 0 \Rightarrow C_5 h + C_6 = 0 \Rightarrow C_6 = -C_5 h$

και $C_3 = \frac{\rho_0 h}{6\varepsilon_0} - \frac{V}{h}$

· Αντισυμμετρία: $E_3 = -E_1 \Rightarrow -\nabla\Phi_3 = \nabla\Phi_1 \Rightarrow -C_5 = C_1 \Rightarrow C_1 = -C_5$.

· $\hat{E}(z=h) = \hat{z} E_0 \Rightarrow -\nabla\Phi_3\big|_{z=2h} = \hat{z} E_0 \Rightarrow -C_5 = E_0 \Rightarrow C_5 = -E_0$

Οπότε,
$$\Phi(z) = \begin{cases} E_0 z + V, & z<0 \\ -\frac{\rho_0}{6\varepsilon_2 h} z^3 + \left(\frac{\rho_0 h}{6\varepsilon_2} - \frac{V}{h}\right) z + V, & 0<z\le h \\ -E_0 z + E_0 h, & z>h \end{cases}$$

β) Συνορ. Συνθ.: $z=0$: $\quad -\varepsilon_2 \dfrac{\partial \Phi_2}{\partial z}\Big|_{z=0} + \varepsilon_1 \dfrac{\partial \Phi_1}{\partial z}\Big|_{z=0} = \sigma(0) \Rightarrow$

$$\Rightarrow \sigma(0) = +\varepsilon_1 E_0 - \varepsilon_2\left(\frac{\rho_0 h}{6\varepsilon_2} - \frac{V}{h}\right) \Rightarrow$$

$$\Rightarrow \sigma(0) = \varepsilon_1 E_0 + \frac{\varepsilon_2 V}{h} - \frac{\rho_0 h}{6}$$

Συνορ. Συνθ.: $z=h$: $\quad -\varepsilon_3 \dfrac{\partial \Phi_3}{\partial z}\Big|_{z=h} + \varepsilon_2 \dfrac{\partial \Phi_2}{\partial z}\Big|_{z=h} = \sigma(h) \Rightarrow$

$$\Rightarrow \sigma(h) = \varepsilon_3 E_0 + \varepsilon_2\left[-\frac{\rho_0 h}{2\varepsilon_2} + \frac{\rho_0 h}{6\varepsilon_2} - \frac{V}{h}\right] \Rightarrow$$

$$\Rightarrow \sigma(h) = \varepsilon_3 E_0 - \frac{\varepsilon_2 V}{h} - \frac{\rho_0 h}{3}$$

γ) $\varepsilon_2 = \varepsilon_0 h / z$

$\Phi_1(z) = C_1 z + C_2$

$$\nabla^2 \Phi_2(z) = -\frac{\rho_0 z}{4} \cdot \frac{z}{\varepsilon_0 h} = -\frac{\rho_0}{\varepsilon_0 h^2} z^2 \Rightarrow \Phi_2(z) = -\frac{\rho_0}{12\varepsilon_0 h^2} z^4 + C_3 z + C_4$$

$\Phi_3(z) = C_5 z + C_6$

- $\Phi_1(0) = \Phi_2(0) = V \Rightarrow C_2 = C_4 = V$
- $\Phi_2(h) = \Phi_3(h) = 0 \Rightarrow C_6 = -C_5 h$, $\quad C_3 = -\dfrac{V}{h} + \dfrac{\rho_0 h}{12\varepsilon_0}$
- Αντισυμμ.: $-\nabla\Phi_1 = \nabla\Phi_3 \Rightarrow C_1 = -C_5$
- $\nabla\Phi_3|_{z=2h} = \hat{z} E_0 \Rightarrow C_5 = -E_0$

Οπότε,
$$\Phi(z) = \begin{cases} E_0 z + V, & z<0 \\ -\dfrac{\rho_0}{12\varepsilon_0 h^2} z^4 + \left(\dfrac{\rho_0 h}{12\varepsilon_0} - \dfrac{V}{h}\right) z + V, & 0<z\le h \\ -E_0 z + E_0 h, & z>h \end{cases}$$