8η Σειρά Ασκήσεων

ΧΡΗΣΤΟΣ ΤΣΟΥΦΗΣ
03117176

Άσκηση 1

11.16

α) Το άθροισμα των ανά μονάδα ρευμάτων στον z θα είναι 0.

$$\int \bar{J} d\bar{s} + \int \bar{K} d\bar{l} = 0 \Rightarrow \hat{z}\int_a^b \int_0^{2\pi} J_o r dr d\varphi + \hat{z}\int_0^{2\pi} - K_o c\, d\varphi = 0 \Rightarrow$$

$$\Rightarrow J_o \cdot 2\pi \frac{b^2-a^2}{2} = 2\pi c \cdot K_o \Rightarrow K_o = \frac{b^2-a^2}{2c} J_o$$

β) Θα επιλυθεί το ανάλογο πρόβλημα ηλεκτροστατικής:

$0<r<a$: $\nabla^2\Phi = 0 \Rightarrow \Phi(r) = 0$ (λόγω αναφοράς & $\bar{E}=\bar{0}$)

$a<r<b$: $\nabla^2\Phi = -\frac{\rho_o}{\varepsilon} \Rightarrow r\frac{\partial\Phi}{\partial r} = -\frac{\rho_o}{2\varepsilon}r^2 + C_1 \Rightarrow \Phi(r) = -\frac{\rho_o}{4\varepsilon}r^2 - C_1 \ln r + C_2$

$b<r<c$: $\nabla^2\Phi = 0 \Rightarrow \Phi(r) = C_3 \ln r + C_4$

$c<r<+\infty$: $\nabla^2\Phi = 0 \Rightarrow \Phi(r) = C_5 \ln r + C_6$

Συνοριακές Συνθήκες:

$$\begin{cases} \Phi(a^-) = \Phi(a^+) \Rightarrow -\frac{\rho_o}{4\varepsilon}a^2 - C_1 \ln a + C_2 = 0 \\ \Phi(b^-) = \Phi(b^+) \Rightarrow -\frac{\rho_o}{4\varepsilon}b^2 - C_1 \ln b + C_2 = C_3 \ln b + C_4 \\ \Phi(c^-) = \Phi(c^+) \Rightarrow C_3 \ln c + C_4 = C_5 \ln c + C_6 \end{cases}$$

$$(\text{ίδιο επίπεδο})\begin{cases} -\frac{\partial\Phi}{\partial r}\Big|_{r=a^+} + \cancel{\frac{\partial\Phi}{\partial r}\Big|_{r=a^-}}^{\,0} = 0 \Rightarrow -\frac{\rho_o}{2\varepsilon}a + \frac{C_1}{a} = 0 \\ -\frac{\partial\Phi}{\partial r}\Big|_{r=b^+} + \frac{\partial\Phi}{\partial r}\Big|_{r=b^-} = 0 \Rightarrow -\frac{\rho_o}{2\varepsilon}b + \frac{C_1}{b} = \frac{C_3}{b} \\ -\frac{\partial\Phi}{\partial r}\Big|_{r=c^+} + \frac{\partial\Phi}{\partial r}\Big|_{r=c^-} = -\sigma_o \Rightarrow -\frac{C_3}{c} + \frac{C_5}{c} = -\sigma_o \end{cases}$$

Προκύπτει γραμμικό σύστημα 6×6 άρα προσδιορίζεται το Φ και έπειτα το $A_z(r)$ με τις αναλογίες $J_z \leftrightarrow \rho$, $\frac{1}{\mu} \leftrightarrow \varepsilon$, $K_o \leftrightarrow \sigma_o$

Οπότε $\bar{B} = \nabla\times\bar{A}$, $\bar{H} = \frac{1}{\mu}\bar{B}$

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· Ισχύει $\nabla^2 \bar{A} = \bar{0}$ παντού

· Για τις A_y, A_z και οι ανυπεραλλοί είναι μηδενικές άρα $A_y = A_z = 0$.

· $\nabla^2 A_x = 0 \Rightarrow$ και είναι $A_x(z) \Rightarrow \frac{\partial^2 A_x}{\partial z^2} = 0 \Rightarrow A_x(z) = \begin{cases} C_1 z + C_2, & z<0 \\ C_3 z + C_4, & 0<z<d \\ C_5 z + C_6, & d<z<h \\ C_7 z + C_8, & z>h \end{cases}$

· Αναφορά: $A_x(0) = 0 \Rightarrow C_4 = 0$

· Συνοριακές συνθήκες: $\begin{cases} A_x(0^+) = A_x(0^-) \Rightarrow C_2 = 0 \\ A_x(d^+) = A_x(d^-) \Rightarrow C_3 d = C_5 d + C_6 \\ A_x(h^+) = A_x(h^-) \Rightarrow C_5 h + C_6 = C_7 h + C_8 \end{cases}$

$$-\frac{1}{\mu}\frac{\partial A_x}{\partial z}\Big|_{z=0^+} + \frac{1}{\mu_0}\frac{\partial A_x}{\partial z}\Big|_{z=0^-} = -K_0 \Rightarrow -\frac{C_3}{\mu} + \frac{C_1}{\mu_0} = -K_0$$

$$-\frac{1}{\mu_0}\frac{\partial A_x}{\partial z}\Big|_{z=d^+} + \frac{1}{\mu}\frac{\partial A_x}{\partial z}\Big|_{z=d^-} = 0 \Rightarrow -\frac{C_5}{\mu_0} + \frac{C_3}{\mu} = 0$$

$$-\frac{1}{\mu_0}\frac{\partial A_x}{\partial z}\Big|_{z=h^+} + \frac{1}{\mu_0}\frac{\partial A_x}{\partial z}\Big|_{z=h^-} = K_0 \Rightarrow -\frac{C_7}{\mu_0} + \frac{C_5}{\mu_0} = K_0$$

· Συνθήκη στο άπειρο: $H(z\uparrow) = -H(z\downarrow) \Rightarrow B(z\uparrow) = -B(z\downarrow) \Rightarrow \frac{\partial A_x}{\partial z}(z\uparrow) = -\frac{\partial A_x}{\partial z}(z\downarrow) \Rightarrow$

$\Rightarrow C_7 = -C_1$

(όπου $z\uparrow$: για μεγάλο z
$z\downarrow$: για μικρό z)

Προκύπτει σύστημα 8×8 άρα προσδιορίζεται το A_x.

Οπότε $\bar{B} = \nabla\times\bar{A}$, $\bar{H} = \frac{1}{\mu}\bar{B}$

Άσκηση 2

$\boxed{11.20}$

Biot-Savart: $\bar{B} = \frac{\mu I}{4\pi}\int \frac{d\bar{\ell}\times\hat{R}}{R^2} = \frac{\mu I}{4\pi}\int \frac{d\bar{\ell}\times\bar{R}}{R^3}$

$\bar{R} = (x-x', y-y', z-z')$ και $d\bar{\ell} = \hat{\varphi}\, r d\varphi = \hat{\varphi}\, a d\varphi$

Το ημικύκλιο βρίσκεται στο xy άρα $z'=0$.

Στον άξονα z : $x=y=0$.

Άρα, $\bar{R} = (-x', -y', z) = -x'\hat{x} + (-y')\hat{y} + z\hat{z}$

Οπότε, $\bar{B} = \frac{\mu_0 I}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\hat{\varphi}\times\bar{R}}{R^3}d\varphi = \frac{a\mu_0 I}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\hat{\varphi}\times\bar{R}}{(x'^2+y'^2+z^2)^{3/2}}d\varphi = \frac{a\mu_0 I}{4\pi(a^2+z^2)^{3/2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{\varphi}\times\bar{R}\, d\varphi$

- $\hat{\varphi}\times\bar{R} = \hat{x}\cos\varphi\, z + \hat{y}\sin\varphi\, z + \hat{z}(\cos\varphi\, x' + \sin\varphi\, y')$ διότι $\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$

Άρα $\bar{B} = \frac{a\mu_0 I}{4\pi(a^2+z^2)^{3/2}}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\hat{x}\cos\varphi\, z\, d\varphi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\hat{y}\,\overset{\to 0}{\cancel{\sin\varphi}}\, z\, d\varphi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\hat{z}(\cos\varphi\, x' + \sin\varphi\, y')d\varphi\right] =$

$\overset{x'=a\cos\varphi}{\underset{y'=a\sin\varphi}{=}} \frac{a\mu_0 I}{4\pi(a^2+z^2)^{\frac{3}{2}}}\left[\hat{x}\cdot z\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\, d\varphi + \hat{z}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\, d\varphi\right] \Rightarrow \bar{B} = \frac{a\mu_0 I}{4\pi(a^2+z^2)^{\frac{3}{2}}}(\hat{x}2z + \hat{z}a\pi)$

Άσκηση 3

$\boxed{11.21}$

β) $\bar{B} = \frac{\mu_0 I}{4\pi}\int\frac{d\bar{\ell}\times\bar{R}}{R^3}$

▷ Χωρίζω το $\int\frac{d\bar{\ell}\times\bar{R}}{R^3}$ σε 5 μέρη, βάσει της καμπύλης. Το καθένα είναι: $S_i = \int\frac{d\bar{\ell}_i\times\bar{R}_i}{R_i^3}$

- $\bar{R}_1 = (-x', 0, 0)$, $d\bar{\ell}_1 = \hat{x}dx \longrightarrow S_1 = 0$
- $\bar{R}_2 = (-\frac{b}{2}, -y', 0)$, $d\bar{\ell}_2 = \hat{y}dy \longrightarrow S_2 = \hat{z}\int_0^a \frac{\frac{b}{2}}{(\frac{b}{2})^2\left(1+\left(\frac{y'}{\frac{b}{2}}\right)^2\right)}dy' = \hat{z}\,2\arctan\left(\frac{a}{\frac{b}{2}}\right)$
- $\bar{R}_3 = (-x', -a, 0)$, $d\bar{\ell}_3 = \hat{x}dx \longrightarrow S_3 = -\hat{z}\int_{-\frac{b}{2}}^{\frac{b}{2}}\frac{a}{a^2(1+(\frac{x'}{a})^2)}dx' = -\hat{z}\,2\arctan\left(\frac{\frac{b}{2}}{a}\right)$

- $\bar{R}_4 = (-\frac{b}{2}, -y', 0)$, $d\bar{\ell}_4 = -\hat{y}\,dy \longrightarrow S_4 = S_2 = \hat{z} \arctan\left(\frac{a}{\frac{b}{2}}\right)$

- $\bar{R}_5 = (-x', 0, 0)$, $d\bar{\ell}_5 = \hat{x}\,dx \longrightarrow S_5 = 0$

Οπότε, $\bar{B} = \hat{z} \frac{\mu_0 I}{4\pi} \left[\arctan\left(\frac{a}{b/2}\right) - \arctan\left(\frac{b/2}{a}\right)\right]$