1η Σειρά Ασκήσεων

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1η Άσκηση

1)
α) $\vec{E} = k(xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}) = (kxy, 2kyz, 3kxz)$

β) $\vec{E} = k[y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}] = (ky^2, 2kxy + kz^2, 2kyz)$

Για την αναπαράσταση της έντασης ηλεκτροστατικού πεδίου, θα πρέπει η διανυσματική συνάρτηση να είναι αστρόβιλη.

α) $\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kxy & 2kyz & 3kxz \end{vmatrix} = -2ky\hat{x} - 3kz\hat{y} - kx\hat{z} \neq 0$

β) $\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ky^2 & 2kxy + kz^2 & 2kyz \end{vmatrix} = 0$, άρα είναι ηλεκτροστατικό πεδίο.

2) $\vec{E} = -\nabla\Phi = (ky^2, 2kxy + kz^2, 2kyz)$

οπότε $\Phi = -kxy^2 - kz^2y + c$

Για $\Phi(0,0,0) = 0 \Rightarrow c = 0$

1ος τρόπος:

$\Phi = \int_{P}^{\vec{r}_{αναφ.}} \vec{E} \cdot d\vec{r}$, $\vec{r}_{αναφ} = 0$

$\Phi = -\int_{0}^{y} (2kxy + kz^2)dy = -kxy^2 - kz^2y$

Χρησιμοποιήθηκε ο δρόμος ολοκλήρωσης από το $(0,0,0)$ μέχρι κάποιο $(x,0,z)$, όμως στην επιφ. $y=0$ υπάρχει μόνο E_y, άρα είναι ισοδυναμική επιφάνεια $\Phi(y=0) = \Phi(0,0,0) = 0$ και από το $(x,0,z)$ στο (x,y,z)

2ος Τρόπος:

- $-ky^2 = \frac{\partial \Phi}{\partial x} \Rightarrow \Phi = -ky^2x + A(y, \cdot)$

- $-2kyx - kz^2 = \frac{\partial \Phi}{\partial y} \Rightarrow \cancel{-2kyx} - kz^2 = \cancel{-2kyx} + \frac{\partial A}{\partial y} \Rightarrow$

$\Rightarrow A = -kz^2y + B(z)$

- $-2kyz = \frac{\partial \Phi}{\partial z} \Rightarrow \cancel{-2kzy} + \frac{\partial B}{\partial z} = \cancel{-2kyz} \Rightarrow B = c$

Άρα, $\Phi = -ky^2x - kz^2y + c$

$\Phi(0,0,0) = 0 \Rightarrow c = 0$

Οπότε $\Phi = -ky^2x - kz^2y$

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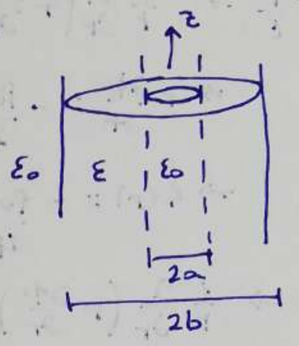
2η Άσκηση



Ν. Gauss:

$r<a: \quad \vec{E}=0$

$a<r<b: \quad \vec{E}=\left(\frac{\rho_0 r}{2\varepsilon}-\frac{\rho_0 a^2}{2\varepsilon r}\right)\hat{r}$

$r>b: \quad \vec{E}=\left(\frac{\rho_0(b^2-a^2)}{2\varepsilon_0 r}\right)\hat{r}$

▷ Για $\varepsilon=$ σταθ.

1ος Τρόπος

$a<r<b: \quad \cancel{\Phi(b)}^{\,0}-\Phi(r)=\int_b^r\left(\frac{\rho_0 r}{2\varepsilon}-\frac{\rho_0 a^2}{2\varepsilon r}\right)dr \Rightarrow \Phi(r)=\frac{\rho_0 b^2}{4\varepsilon}-\frac{\rho_0 r^2}{4\varepsilon}-\frac{\rho_0 a^2}{2\varepsilon}\ln\left(\frac{b}{r}\right)$

$r<a: \quad \Phi(a)-\Phi(r)=\int_a^r 0\,dr \Rightarrow \Phi(r)=\Phi(a) \Rightarrow \Phi(r)=\frac{\rho_0(b^2-a^2)}{4\varepsilon}-\frac{\rho_0 a^2}{2\varepsilon}\ln\left(\frac{b}{a}\right)$

$r>b: \quad \Phi(r)-\Phi(b)=\int_r^b\frac{\rho_0(b^2-a^2)}{2\varepsilon_0 r}dr \Rightarrow \Phi(r)=\frac{\rho_0(b^2-a^2)}{2\varepsilon_0}\ln\left(\frac{b}{r}\right)$

2ος Τρόπος

$a<r<b: \quad E=-\frac{\partial\Phi}{\partial r} \Rightarrow \Phi=-\int\left(\frac{\rho_0 r}{2\varepsilon}-\frac{\rho_0 a^2}{2\varepsilon r}\right)dr=-\frac{\rho_0 r^2}{4\varepsilon}+\frac{\rho_0 a^2}{2\varepsilon}\ln r+c_1$

$r>b: \quad E=-\frac{\partial\Phi}{\partial r} \Rightarrow \Phi=-\int\left(\frac{\rho_0(b^2-a^2)}{2\varepsilon_0 r}\right)dr=-\frac{\rho_0(b^2-a^2)}{2\varepsilon_0}\ln r+c_2$

• Για $\Phi(b)=0 \Rightarrow -\frac{\rho_0 b^2}{4\varepsilon}+\frac{\rho_0 a^2}{2\varepsilon}\ln b+c_1=0 \Rightarrow c_1=\frac{\rho_0 b^2}{4\varepsilon}-\frac{\rho_0 a^2\ln b}{2\varepsilon}$

και $-\frac{\rho_0(b^2-a^2)}{2\varepsilon_0}\ln b+c_2=0 \Rightarrow c_2=\frac{\rho_0(b^2-a^2)\ln b}{2\varepsilon_0}$

$r<a: \quad \Phi(r)=\Phi(a) \Rightarrow \Phi(r)=\frac{\rho_0(b^2-a^2)}{4\varepsilon}-\frac{\rho_0 a^2}{2\varepsilon}\ln\left(\frac{b}{a}\right)$

3ος Τρόπος

$a<r<b$: $\nabla^2\Phi = -\frac{\rho}{\varepsilon} \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = -\frac{\rho_0}{\varepsilon} \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = -\frac{\rho_0 r}{\varepsilon} \Rightarrow$

$\Rightarrow r\frac{\partial\Phi}{\partial r} = -\frac{\rho_0 r^2}{2\varepsilon} + a_1 \Rightarrow \frac{\partial\Phi}{\partial r} = -\frac{\rho_0 r}{2\varepsilon} + \frac{a_1}{r} \Rightarrow$

$\Rightarrow \Phi(r) = -\frac{\rho_0 r^2}{4\varepsilon} + d_1 \ln r + d_2$

$r<a$: $\nabla^2\Phi = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = 0 \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = 0 \Rightarrow r\frac{\partial\Phi}{\partial r} = c_1 \Rightarrow$

$\Rightarrow \frac{\partial\Phi}{\partial r} = \frac{c_1}{r} \Rightarrow \Phi(r) = c_1 \ln r + c_2$

$r>b$: $\nabla^2\Phi = 0 \Rightarrow \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = 0 \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = 0 \Rightarrow r\frac{\partial\Phi}{\partial r} = k_1 \Rightarrow$

$\Rightarrow \frac{\partial\Phi}{\partial r} = \frac{k_1}{r} \Rightarrow \Phi(r) = k_1 \ln r + k_2$

Ορ. Συνθ.

- $\Phi(b) = 0 \Rightarrow \begin{cases} k_1 \ln b + k_2 = 0 \\ -\frac{\rho_0 b^2}{4\varepsilon} + d_1 \ln b + d_2 = 0 \end{cases}$

- $r=a$: $\varepsilon\vec{E} = 0 \Rightarrow -\varepsilon\vec{\nabla}\Phi = 0 \Rightarrow \vec{\nabla}\Phi = 0 \Rightarrow \frac{\partial\Phi}{\partial r} = 0 \Rightarrow \frac{c_1}{a} = 0 \Rightarrow c_1 = 0$

- $a<r<b$: $\frac{d_1}{a} = \frac{\rho_0 a}{2\varepsilon} \Rightarrow d_1 = \frac{\rho_0 a^2}{2\varepsilon}$

  άρα $d_2 = \frac{\rho_0 b^2}{4\varepsilon} - \frac{\rho_0 a^2}{2\varepsilon}\ln b$

- Συνέχεια σο $\Phi(a)$:

  $c_2 = -\frac{\rho_0 a^2}{4\varepsilon} + \frac{\rho_0 a^2}{2\varepsilon}\ln a + \frac{\rho_0 b^2}{4\varepsilon} - \frac{\rho_0 a^2 \ln b}{2\varepsilon}$

- $r=b$: $-\varepsilon_0\frac{\partial\Phi_2}{\partial r} + \varepsilon\frac{\partial\Phi_1}{\partial r} = 0 \Rightarrow -\varepsilon_0\frac{k_1}{b} + \varepsilon\left(-\frac{\rho_0 b}{2\varepsilon} + \frac{\rho_0 a^2}{2\varepsilon b}\right) = 0 \Rightarrow$

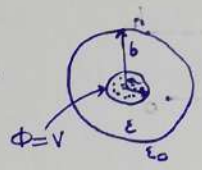
  $\Rightarrow k_1 = -\frac{\rho_0 b^2}{2\varepsilon_0} + \frac{\rho_0 a^2}{2\varepsilon_0}$

  και $k_2 = \frac{\rho_0 b^2}{2\varepsilon_0}\ln b - \frac{\rho_0 a^2}{2\varepsilon_0}\ln b$

4η Άσκηση

Για $r=a$, υπάρχει επιφ. φορτίο Q.

Ο αγωγός έχει $\vec{E}=0$.



$r<a: \quad \Phi=V$

$a<r<b: \quad \Phi=\frac{Q}{4\pi\varepsilon r}+C$

$r>b: \quad \Phi=\frac{Q}{4\pi\varepsilon_0 r}$

$$\Rightarrow \Phi(r)=\begin{cases} V, & r<a \\ \frac{Q}{4\pi\varepsilon}\left[-\frac{1}{b}+\frac{1}{r}\right]+\frac{Q}{4\pi\varepsilon_0 b}, & a<r<b \\ \frac{Q}{4\pi\varepsilon_0 r}, & r>b \end{cases}$$

1ος Τρόπος

- $\Phi(a^+)=\Phi(a^-) \Rightarrow C=V-\frac{Q}{4\pi\varepsilon a}$
- $\Phi(b^+)=\Phi(b^-) \Rightarrow \frac{Q}{4\pi\varepsilon_0 b}=\frac{Q}{4\pi\varepsilon b}+V-\frac{Q}{4\pi\varepsilon a} \Rightarrow \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b}-\frac{1}{\varepsilon b}+\frac{1}{\varepsilon a}\right)=V \Rightarrow$

$$\Rightarrow Q=\frac{4\pi\varepsilon_0 b\varepsilon a V}{\varepsilon a-\varepsilon_0 a+\varepsilon_0 b}$$

2ος Τρόπος

$a<r<b: \quad \nabla^2\Phi=0 \Rightarrow \Phi^{(1)}=\frac{a_1}{r}+a_2$

$r>b: \quad \nabla^2\Phi=0 \Rightarrow \Phi^{(2)}=\frac{b_1}{r}+b_2$

$r<a: \quad \Phi=V$ (αγωγός)

δ.σ.

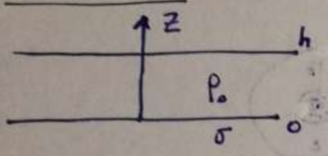
$$\begin{cases} \frac{a_1}{a}+a_2=V & (1) \\ \frac{b_1}{b}+b_2=\frac{a_1}{b}+a_2 & (2) \end{cases}$$

- $\Phi_{r\to\infty}=0 \Rightarrow b_2=0 \quad (3)$
- $\varepsilon\frac{\partial\Phi^{(1)}}{\partial r}-\varepsilon_0\frac{\partial\Phi^{(2)}}{\partial r}=\sigma \xrightarrow{r=b} -\varepsilon\frac{a_1}{b^2}+\frac{\varepsilon_0 b_1}{b^2}=\sigma \quad (4)$
- $-\varepsilon\frac{\partial\Phi^{(1)}}{\partial r}=\sigma \xrightarrow{r=a} -\frac{\varepsilon a_1}{a^2}=\sigma \Rightarrow a_1=\frac{\sigma\cdot a^2}{\varepsilon} \quad (5)$

και $\sigma=\frac{Q}{4\pi a^2} \quad (6)$

Από (1),…,(6) λύνεται το πρόβλημα.

3η Ασκηση



$$E(z) = \begin{cases} \frac{1}{2\varepsilon}\left[\sigma + \int_0^h \rho_0 dz'\right], & z>h \\ \frac{1}{2\varepsilon}\left[\sigma + \int_0^z \rho_0 dz' - \int_z^h \rho(z')dz'\right], & 0<z<h \\ -\frac{1}{2\varepsilon}\left[\sigma + \int_0^h \rho_0 dz'\right], & z<0 \end{cases}$$

Άρα, $$E(z) = \begin{cases} \frac{\sigma}{2\varepsilon} + \frac{\rho_0 h}{2\varepsilon}, & z>h \quad = E_1 \\ \frac{\sigma}{2\varepsilon} + \frac{\rho_0}{2\varepsilon}(2z-h), & 0<z<h \quad = E_2 \\ -\frac{\sigma}{2\varepsilon} - \frac{\rho_0 h}{2\varepsilon}, & z<0 \quad = E_3 \end{cases}$$

1ος Τρόπος

$$\vec{E} = -\nabla\Phi \Rightarrow \frac{\partial \Phi}{\partial z} = -E(z) \Rightarrow \begin{cases} \frac{\partial \Phi_1}{\partial z} = -E_1 \\ \frac{\partial \Phi_2}{\partial z} = -E_2 \\ \frac{\partial \Phi_3}{\partial z} = -E_3 \end{cases} \Rightarrow \begin{cases} \frac{\partial \Phi_1}{\partial z} = -\frac{\sigma}{2\varepsilon} - \frac{\rho_0 h}{2\varepsilon} \\ \frac{\partial \Phi_2}{\partial z} = -\frac{\sigma}{2\varepsilon} - \frac{\rho_0}{2\varepsilon}(2z-h) \\ \frac{\partial \Phi_3}{\partial z} = \frac{\sigma}{2\varepsilon} + \frac{\rho_0 h}{2\varepsilon} \end{cases}$$

$$\Rightarrow \begin{cases} \Phi_1 = -\left(\frac{\sigma}{2\varepsilon} + \frac{\rho_0 h}{2\varepsilon}\right)\cdot z + c_1, & z>h \\ \Phi_2 = -\frac{\sigma}{2\varepsilon}z - \frac{\rho_0}{2\varepsilon}(z^2 - hz) + c_2, & 0<z<h \\ \Phi_3 = \left(\frac{\sigma}{2\varepsilon} + \frac{\rho_0 h}{2\varepsilon}\right)z + c_3, & z<0 \end{cases}$$

· Αναφορά στο $z=h$: $\Phi_1(h) = 0 = \Phi_2(h) \Rightarrow c_1 = \left(\frac{\sigma}{2\varepsilon} + \frac{\rho_0 h}{2\varepsilon}\right)\cdot h$

$$c_2 = \frac{\sigma h}{2\varepsilon}$$

· Οριακή στο $z=0$: $\Phi_3(0) = \Phi_2(0) \Rightarrow c_3 = \frac{\sigma h}{2\varepsilon}$

Οπότε,

$$\Phi(z) = \begin{cases} \left(\frac{\sigma}{2\varepsilon} + \frac{\rho_0 h}{2\varepsilon}\right)(h-z), & z>h \\ \frac{\sigma}{2\varepsilon}(h-z) + \frac{\rho_0}{2\varepsilon}(hz - z^2), & 0<z<h \\ \frac{\sigma}{2\varepsilon}(h+z) + \frac{\rho_0 h}{2\varepsilon}z, & z<0 \end{cases}$$

2ος Τρόπος

$$\Phi(z) = \int_z^h E(z')dz' = \begin{cases} \int_z^h E_1 dz' \\ \int_z^h E_2 dz' \\ \int_z^0 E_3 dz' + \int_0^h E_2 dz' \end{cases}$$

οπότε, $$\Phi(z) = \begin{cases} \left(\frac{\sigma}{2\varepsilon} + \frac{\rho_0 h}{2\varepsilon}\right)(h-z), & z>h \\ \frac{\sigma}{2\varepsilon}(h-z) + \frac{\rho_0}{2\varepsilon}(hz - z^2), & 0<z<h \\ -\frac{\sigma}{2\varepsilon}(h+z) + \frac{\rho_0 h}{2\varepsilon}\cdot z, & z<0 \end{cases}$$

3ος Τρόπος

· Για $z>h$, $\nabla^2\Phi_1 = 0 \Rightarrow \Phi_1(z) = c_1 z + c_2$

· Για $0<z<h$, $\nabla^2\Phi_2 = -\frac{\rho_0}{\varepsilon} \Rightarrow \Phi_2(z) = -\frac{\rho_0}{2\varepsilon}z^2 + c_3\cdot z + c_4$

· Για $0>z$, $\nabla^2\Phi_3 = 0 \Rightarrow \Phi_3 = c_5\cdot z + c_6$

· Αναφορά στο $z=h$: i) $\Phi_1(h) = 0 \Rightarrow c_1 h + c_2 = 0$ (1)

· Ορ. Συνθ. στο $z=h$: · $\Phi_1(h) = \Phi_2(h) \Rightarrow -\frac{\rho_0}{2\varepsilon}h^2 + c_3\cdot h + c_4 = 0$ (2)

· $-\varepsilon \left.\frac{\partial\Phi_1}{\partial z}\right|_{z=h} + \varepsilon\cdot\left.\frac{\partial\Phi_2}{\partial z}\right|_{z=h} = 0 \Rightarrow$

$\Rightarrow -\varepsilon\cdot c_1 + \varepsilon\left(-\frac{\rho_0 h}{\varepsilon} + c_3\right) = 0 \Rightarrow$

$\Rightarrow -c_1 - \frac{\rho_0 h}{\varepsilon} + c_3 = 0$ (3)

· Ορ. Συνθ. στο $z=0$: · $\Phi_2(0) = \Phi_3(0) \Rightarrow c_6 = c_4$ (4)

· $-\varepsilon \left.\frac{\partial\Phi_2}{\partial z}\right|_{z=0} + \varepsilon\cdot\left.\frac{\partial\Phi_3}{\partial z}\right|_{z=0} = \sigma \Rightarrow$

$\Rightarrow c_5 - c_3 = \frac{\sigma}{\varepsilon}$ (5)

· Αντισυμμετρία: $E_1 = -E_3 \Rightarrow -\nabla\Phi_1 = \nabla\Phi_3 \Rightarrow -c_1 = c_5$ (6)

Από (1),…,(6) βρίσκεται το αποτέλεσμα.