3.5

(a) The mean function for $\{Y_t\}$ should be the same as that for $\{X_t\}$.

Besides, $\gamma_Y(t,t) = E[(Y_t-\mu)(Y_t-\mu)]$

$$= E[((X_t-\mu)+W_t)\cdot((X_t-\mu)+W_t)]$$

$$= \gamma_X(t,t) + \sigma_W^2$$

At lagh, $\gamma_Y(t,t+h) = E[(Y_t-\mu)(Y_{t+h}-\mu)]$

$$= E[((X_t-\mu)+W_t)((X_{t+h}-\mu)+W_{t+h}]$$

$$= \gamma_X(t,t+h)$$

These quantities are independent of t, hence $\{Y_t\}$ is stationary.

(b) $U_t = \phi(B)Y_t$

$$= \phi(B)(X_t+W_t)$$

$$= \phi(B)(X_t) + \phi(B)(W_t)$$

$$= Z_t + \theta Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

$$+ W_t - \phi_1 W_{t-1} - \cdots - \phi_p W_{t-p}$$

$\therefore$ U_t is r-correlated. Thus, there's $\{V_t\} \sim WN(0,\sigma_v^2)$ and a polynomial $\tilde{\theta}(z)$ of degree r s.t.

$$\phi(B)Y_t = \tilde{\theta}(B)V_t$$

$\therefore$ $\{Y_t\}$ is ARMA(p,r) process.

3.7

To show $\{W_t\}$ is WN $(0, \sigma_w^2)$

$$E(W_t) = E\left[\sum_{j=0}^{\infty} (-\theta)^{-j} X_{t-j}\right]$$
$$= \sum_{j=0}^{\infty} (-\theta)^{j} \underbrace{E[X_{t-j}]}_{0} = 0$$

$$\gamma_w(t+h, t) = E[W_{t+h} W_t] = E\left[\sum_{j=0}^{\infty} (-\theta)^{-j} X_{t+h-j} \sum_{k=0}^{\infty} (-\theta)^{-k} X_{t-k}\right]$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-\theta)^{-j} (-\theta)^{-k} E[X_{t+h-j} X_{t-k}]$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-\theta)^{-j} (-\theta)^{-k} \gamma_x(h-j+k)$$

$$= \left\{ \gamma_x(r) = \sigma^2(1+\theta^2) 1_{\{0\}}(r) + \sigma^2\theta 1_{\{1\}}(|r|) \right\}$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-\theta)^{-(j+k)} \left(\sigma^2(1+\theta^2) 1_{\{j-k\}}(h) + \sigma^2\theta 1_{\{j-k+1\}}(h) + \sigma^2\theta 1_{\{j-k-1\}}(h)\right)$$

$$= \sum_{j=h}^{\infty} (-\theta)^{-(j+j-h)} \sigma^2(1+\theta^2) + \sum_{j=h-1, j\geq 0}^{\infty} (-\theta)^{-(j+j-h+1)} \sigma^2\theta$$
$$+ \sum_{j=h+1}^{\infty} (-\theta)^{-(j+j-h-1)} \sigma^2\theta$$

$$= \sigma^2(1+\theta^2)(-\theta)^{-h} \sum_{j=h}^{\infty} (-\theta)^{-2(j-h)} + \sigma^2\theta(-\theta)^{-(h-1)} \sum_{j=h-1, j\geq 0}^{\infty} (-\theta)^{-2(j-(h-1))}$$
$$+ \sigma^2\theta(-\theta)^{-(h+1)} \sum_{j=h+1}^{\infty} (-\theta)^{-2(j-(h+1))}$$
$$= \sigma^2(-\theta)^{-h} \frac{\theta^2}{\theta^2-1} (1+\theta^2-\theta^2-1) + \sigma^2\theta^2 1_{\{0\}}(h)$$
$$= \sigma^2\theta^2 1_{\{0\}}(h)$$

therefore, $\{W_t\}$ is $WN(0, \sigma_w^2)$ w/ $\sigma_w^2 = \sigma^2\theta^2$.

$\therefore\ W_t = \sum_{j=0}^{\infty}(-\theta)^{-j}X_{t-j} = \sum_{j=0}^{\infty}\pi_j X_{t-j}$ where $\pi_j = (-\theta)^{-j}$

$\sum_{j=0}^{\infty}|\pi_j| < \infty$

$\therefore\ X_t$ is invertible.

$\therefore\ \sum_{j=0}^{\infty}\pi_j z^j = \sum_{j=0}^{\infty}\left(-\frac{z}{\theta}\right)^j = \frac{1}{1+z/\theta} = \frac{\phi(z)}{\theta(z)}$

$\therefore\ \phi(z) = 1, \quad \theta(z) = 1 + \frac{z}{\theta}$

$\therefore\ X_t = W_t + \theta^{-1}W_{t-1}$

3.9

(a) $E(Y_t) = E(\mu + Z_t + \theta_1 Z_{t-1} + \theta_{12} Z_{t-12}) = \mu$

$\gamma(h) = E(Y_t Y_{t+h}) = E[(Z_t + \theta_1 Z_{t-1} + \theta_{12} Z_{t-12}) \cdot (Z_{t+h} + \theta_1 Z_{t+h-1} + \theta_{12} Z_{t+h-12})]$

$$= \begin{cases} \sigma^2(1+\theta_1^2+\theta_{12}^2) & h=0 \\ \sigma^2\theta_1 & |h|=1 \\ \sigma^2\theta_1\theta_{12} & |h|=11 \\ \sigma^2\theta_{12} & |h|=12 \\ 0 & \text{o.w.} \end{cases}$$

(b)

$\frac{\gamma(1)}{\gamma(0)} = -0.3588$

$\frac{\gamma(11)}{\gamma(0)} = 0.1952$

$\frac{\gamma(12)}{\gamma(0)} = -0.3332$

(c) $\hat{\mu} = 28.831$

$\hat{\theta}_1 = \hat{\gamma}(11) / \hat{\gamma}(12) = 0.1952 / (-0.3332) = -0.5858$

$\hat{\theta}_{12} = \hat{\gamma}(11) / \hat{\gamma}(1) = 0.1952 / (-0.3588) = -0.5440$

$\hat{\sigma}^2 = \hat{\gamma}(1) / \hat{\theta}_1 = 92740$

5.3

(a) The auto-regressive polynomial is $\phi(z) = 1 - \phi z - \phi^2 z^2$

We can compute the roots by

$$\frac{\phi \pm \sqrt{\phi^2 - 4(-\phi^2)}}{-2\phi^2} = \frac{-1 \pm \sqrt{5}}{2\phi}$$

Causal $\iff |\phi| \le \frac{\sqrt{5}-1}{2} = 0.618$

(b) Yule-Walker:

$$\begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{bmatrix} \cdot \begin{bmatrix} \hat{\phi} \\ \hat{\phi}^2 \end{bmatrix} = \begin{bmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{bmatrix}$$

and $\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}\hat{\gamma}(1) - \hat{\phi}^2 \hat{\gamma}(2)$

$\therefore \hat{\rho}(0)\hat{\phi} + \hat{\rho}(1)\hat{\phi}^2 = \hat{\rho}(1)$

$\therefore 0.687 - \hat{\phi} - 0.687\hat{\phi}^2 = 0$

$\therefore \hat{\phi} \in \{0.509, -1.965\}$

By part (a), $\hat{\phi} = 0.509$ is preferred.

The second Yule-Walker equation then states

$\hat{\rho}(2) = \hat{\rho}(1)\hat{\phi} + \hat{\phi}^2$

$\therefore \hat{\rho}(2) = 0.687(0.509) + (0.509)^2 = 0.609$

$\therefore \hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\phi}\hat{\rho}(1) - \hat{\phi}^2\hat{\rho}(2)) = 2.985$

5.4

$\bar{X}_{200} = 3.82$. $\hat{\gamma}(0) = 1.15$, $\hat{\rho}(1) = 0.427$

$\hat{\rho}(2) = 0.475$

$\hat{\rho}(3) = 0.169$

(a) For large n the sample autocorrelations $\hat{\rho}(h) \overset{iid}{\sim} N(0, \frac{1}{n})$

$\because n = 200$

$\therefore$ the confidence lies in $\pm \frac{1.96}{\sqrt{n}} = 0.1386$

$\because$ $\hat{\rho}(1)$ $\hat{\rho}(2)$ $\hat{\rho}(3)$ are outside these levels

$\therefore$ Reject the hypothesis of W.N.

(b)

$$X_t - \mu - \phi_1(X_{t-1} - \mu) - \phi_2(X_{t-2} - \mu) = Z_t$$

$\hat{\mu} = \bar{x} = 3.83$.

Solve the Yule-Walker equations:

$$\begin{bmatrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{bmatrix} \cdot \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0.427 \\ 0.427 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} 0.427 \\ 0.475 \end{bmatrix}$$

and $\hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\phi}_1 \hat{\rho}(1) - \hat{\phi}_2 \hat{\rho}(2)) = 1.15(1 - \hat{\phi}_1(0.427) - \hat{\phi}_2(0.475))$

$\therefore \hat{\phi}_1 = 0.274$, $\hat{\phi}_2 = 0.358$, $\hat{\sigma}^2 = 0.820$.

(c) $\bar{x}_n - \mu \sim N\left(0, \frac{1}{n} \sum_{|h|<\infty} \gamma(h)\right)$

We can approximate the variance by

$$\sum_{|h|<\infty} \gamma(h) = \sum_{|h|\leq 3} \hat{\gamma}(h) = 3.61$$

$\because 3.82 > 1.96 \cdot \sqrt{\frac{3.61}{200}} = 0.26$

$\therefore$ We reject the hypothesis which is $\mu = 0$.

(d) Given $\hat{\phi} - \phi \approx N\left(0, \frac{\sigma^2}{n} \Gamma_p^{-1}\right)$

plug in estimates for σ^2, Γ_p^{-1}:

$$\hat{\phi} - \phi \approx N\left(0, \frac{\hat{\sigma}^2}{n} \hat{\Gamma}_p^{-1}\right)$$

$$= N\left(0, \frac{\hat{\sigma}^2}{n} \left(\hat{\gamma}(0) \begin{bmatrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{bmatrix}\right)^{-1}\right)$$

$$= N\left(0, \frac{0.82}{200} \left(1.15 \begin{bmatrix} 1 & 0.427 \\ 0.427 & 1 \end{bmatrix}\right)^{-1}\right)$$

$$= N\left(0, \begin{bmatrix} 0.0044 & -0.0019 \\ -0.0019 & 0.0044 \end{bmatrix}\right)$$

$$\therefore \begin{cases} \phi_1 = 0.274 \pm 1.96 \cdot \sqrt{0.0044} = [0.144, 0.404] \\ \phi_2 = 0.385 \pm 1.96 \cdot \sqrt{0.0044} = [0.228, 0.488] \end{cases}$$

(e) $\hat{\alpha}(0)=1$, $\hat{\alpha}(h)=\hat{\phi}_{hh}$ for $|h|\geq 1$.

where $\hat{\phi}_{hh}$ is the last component of $\hat{\phi}_h = \hat{R}_h^{-1}\hat{\rho}_h$

For $h=1$, $\hat{\rho}(0)\hat{\phi}_1=\hat{\rho}(1)$

$\therefore \hat{\alpha}(1)=\hat{\rho}(1)=0.427$

For $h=2$, from part (b):

$\hat{\alpha}(2)=\hat{\phi}_2=0.358$

For $h>2$, $\hat{\alpha}(h)=0$

5.11

$$X_1 - \hat{X}_1 = X_1 \sim N(0, V_0)$$
$$X_2 - \hat{X}_2 = X_2 - \phi X_1 \sim N(0, V_1)$$
$$V_0 = \sigma^2 r_0 = E[(X_1 - \hat{X}_1)^2]$$
$$V_1 = \sigma^2 r_1 = E[(X_2 - \hat{X}_2)^2]$$
$$\therefore V_0 = E[X_1^2] = \gamma(0), \quad r_0 = 1/(1-\phi^2)$$
$$V_1 = E[(X_2 - \hat{X}_2)^2] = \gamma(0) - 2\phi\gamma(1) + \phi^2\gamma(0)$$

$$\therefore r_1 = \frac{\gamma(0)\cdot(1+\phi^2) - 2\phi\gamma(1)}{\sigma^2} = \frac{1+\phi^2 - 2\phi^2}{1-\phi^2} = 1$$

where $\gamma(1) = \dfrac{\sigma^2\phi}{1-\phi^2}$

$\because$ the dist of the innovations is normal

$\therefore$ the density of $X_j - \hat{X}_j$ is:

$$f_{X_j - \hat{X}_j} = \frac{1}{\sqrt{2\pi\sigma^2 r_{j-1}}} \exp\left(-\frac{x^2}{2\sigma^2 r_{j-1}}\right)$$

$$\therefore L(\phi, \sigma^2) = \prod_{j=1}^{2} f_{X_j - \hat{X}_j} = \frac{1}{\sqrt{(2\pi\sigma^2)^2 r_0 r_1}} \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{(X_1 - \hat{X}_1)^2}{r_0} + \frac{(X_2 - \hat{X}_2)^2}{r_1}\right)\right\}$$

$$= \frac{1}{\sqrt{(2\pi\sigma^2)^2 r_0 r_1}} \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{X_1^2}{r_0} + \frac{(X_2 - \phi X_1)^2}{r_1}\right)\right\}$$

Take the Log:

$$\log L(\phi,\sigma^2) = -\frac{1}{2}\log(4\pi^2\sigma^2\gamma_0\gamma_1) - \frac{1}{2\sigma^2}\left(\frac{x_1^2}{\gamma_0} + \frac{(x_2-\phi x_1)^2}{\gamma_1}\right)$$

$$= -\frac{1}{2}\log(4\pi^2\sigma^2/(1-\phi^2)) - \frac{1}{2\sigma^2}\left(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\right)$$

$$= -\log(2\pi) - \log(\sigma^2) + \frac{1}{2}\log(1-\phi^2) - \frac{1}{2\sigma^2}\left(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\right)$$

Take the derivative:

$$\begin{cases} \dfrac{\partial l(\phi,\sigma^2)}{\partial \sigma^2} = -\dfrac{1}{\sigma^2} + \dfrac{1}{2\sigma^4}\left(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\right) = 0 \\[2ex] \dfrac{\partial l(\phi,\sigma^2)}{\partial \phi} = \dfrac{1}{2}\cdot\dfrac{-2\phi}{1-\phi^2} + \dfrac{x_1x_2}{\sigma^2} = 0 \end{cases}$$

$$\therefore\ \sigma^2 = \frac{1}{2}\left(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\right)$$

$$\phi = \frac{2x_1x_2}{x_1^2+x_2^2}$$

$$\therefore \begin{cases} \hat{\sigma}^2 = \dfrac{(x_1^2-x_2^2)^2}{2(x_1^2+x_2^2)} \\[2ex] \hat{\phi} = \dfrac{2x_1x_2}{x_1^2+x_2^2} \end{cases}$$

5.12.

We consider this question as a generalized version of "11".

Now $r_0 = \frac{1}{1-\phi^2}$, $\hat{X}_1 = 0$,

Since $V_i = X_i - \hat{X}_i$, due to the nature of AR(1)

$\therefore \hat{X}_j = \phi X_{j-1}$ and $V_{i-1} = \sigma^2$, $r_{j-1} = 1 \quad \forall j \geq 2$

Now we write the likelihood as

$$L(\Gamma_n) = \frac{1}{\sqrt{(2\pi)^n r_0 r_1 r_2 \cdots r_{n-1}}} \exp\left\{-\frac{1}{2\sigma}\sum_{j=1}^{n} \frac{(x_j - \hat{x}_j)^2}{r_{j-1}}\right\}$$

$$= \sqrt{\frac{1-\phi^2}{(2\pi)^n}} \cdot \exp\left\{\frac{1}{-2\sigma^2}\sum_{j=1}^{n} \frac{(x_j - \phi x_{j-1})^2}{r_{j-1}}\right\}$$

$$= \sqrt{\frac{1-\phi^2}{(2\pi)^n}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{j=2}^{n} (x_j - \phi x_{j-1})^2 + x_1^2(1-\phi^2)\right]\right\}$$

Take the log:

$$\ell(\phi, \sigma^2) = \frac{1}{2}\log(1-\phi^2) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\left[\sum_{j=2}^{n} (X_j - \phi X_{j-1})^2 + X_1^2(1-\phi^2)\right]$$

Take partial derivative:

$$\frac{\partial \ell(\phi, \sigma^2)}{\partial \phi} = -\frac{2\phi}{2(1-\phi^2)} - \frac{1}{2\sigma^2}\left[\sum_{j=2}^{n} 2(X_j - \phi X_{j-1})(-X_{j-1}) + X_1^2(-2\phi)\right] = 0$$

$$\therefore \frac{-\phi}{1-\phi^2} = \frac{1}{2\sigma^2}\left[\sum_{j=2}^{n} 2(x_j - \phi x_{j-1})(-x_{j-1}) - 2\phi x_1^2\right]$$

$$\therefore -\phi = \frac{1-\phi^2}{2\sigma^2}\left[\sum_{j=2}^{n} 2(x_j - \phi x_{j-1})(-x_{j-1}) - 2\phi x_1^2\right]$$

$$\underbrace{\therefore -\phi = \frac{1}{2\sigma^2}\left[\sum_{j=2}^{n} \underbrace{2(1-\phi^2)(x_j - \phi x_{j-1})(-x_{j-1})}_{\text{This is the cubic term of } \phi.} - 2\phi x_1^2\right]}_{\text{This is the cubic equation for estimation of } \phi.}$$

A3 Final Question

Christopher Zheng

04/04/2020

(a) Remove the last 12 values from the Beer data set by using

```
beer_original = dget("beer.Rput")
beer <- head(beer_original,-12)
```

(b) Find an ARIMA model for the logarithms of the beer data. Your analysis should include:

i) a logical explanation of the steps taken to choose the final model;

ii) appropriate 95% bounds for the components of φ and θ;

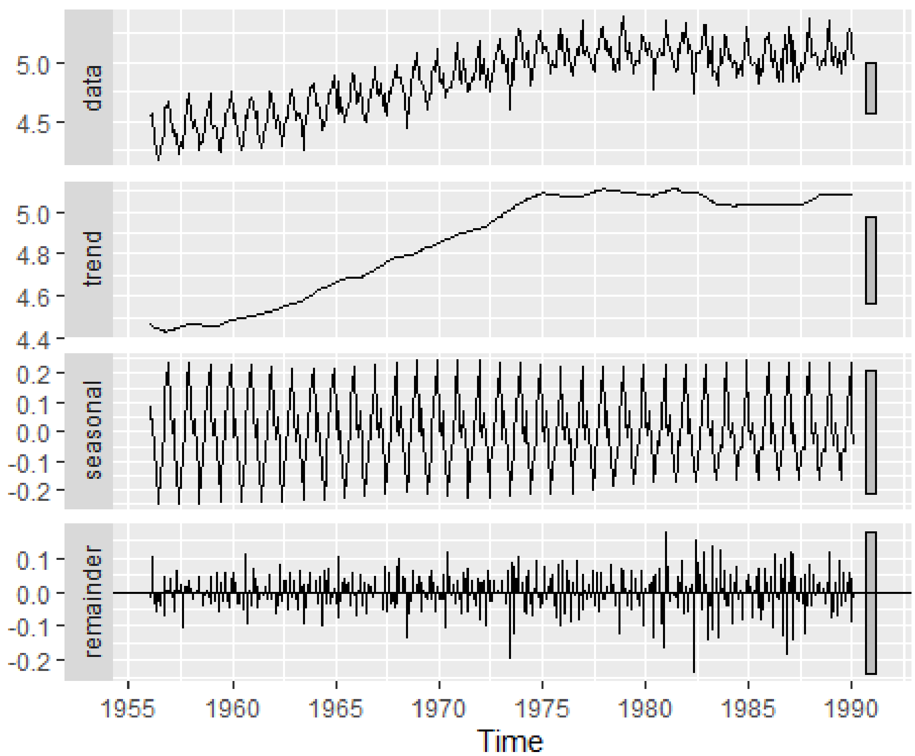
iii) an examination of the residuals to check for similarity to a white noise process;

iv) a graph of the series showing forecasts of the removed 12 values and 95% prediction bounds;

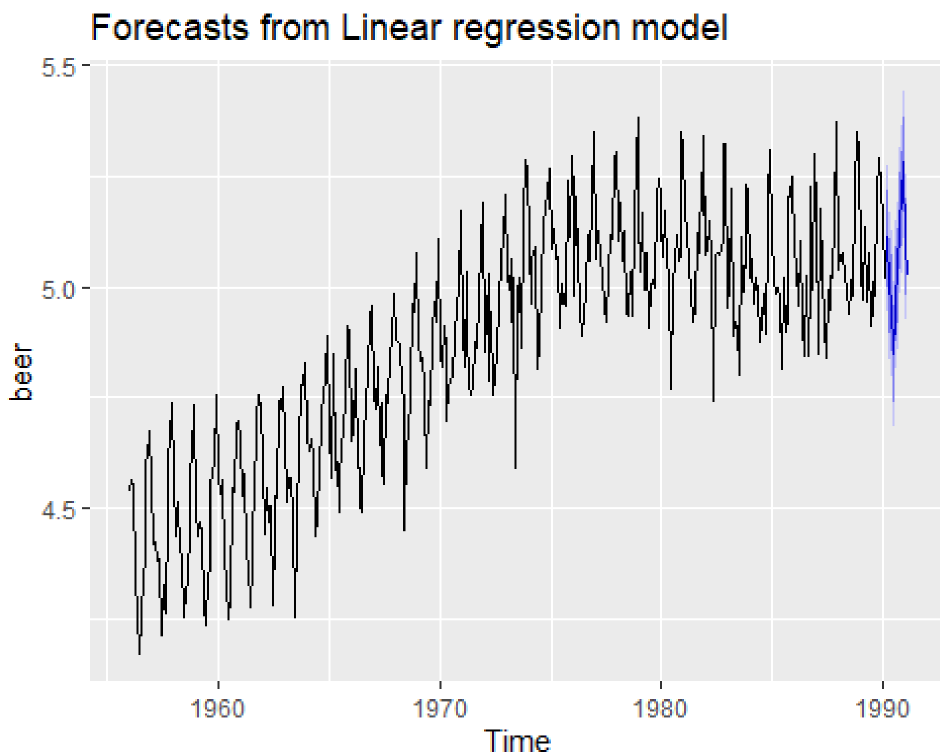
v) numerical values for the 12-step ahead forecast and the corresponding 95% prediction bounds

vi) a table of the actual forecast errors, i.e. observed - predicted, for the removed 12 values

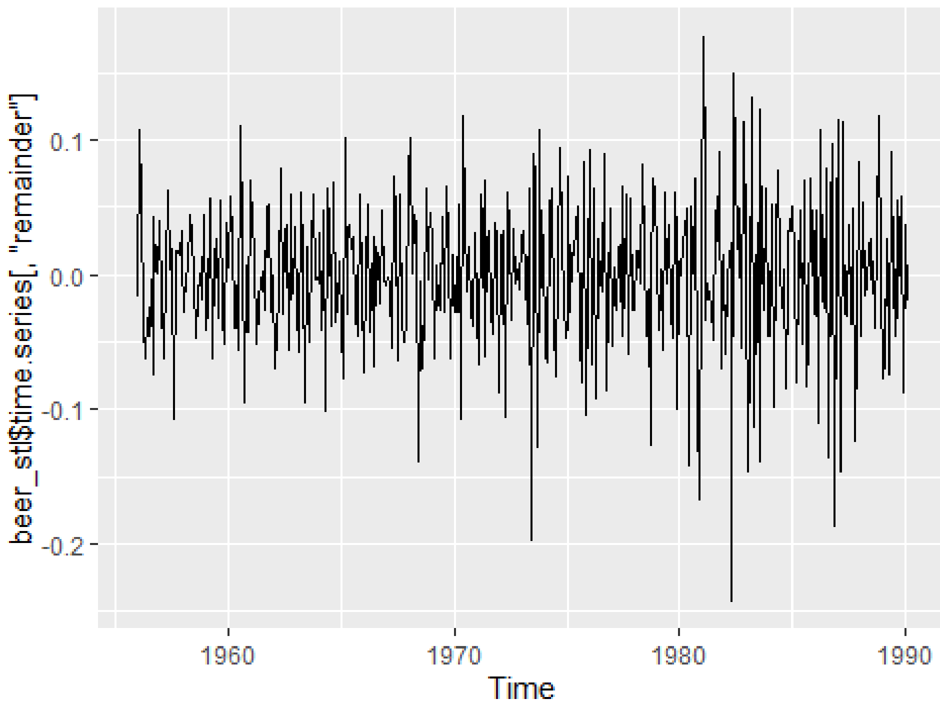
```
beer <- log(beer)
#autoplot(beer)
#acf(beer)

# Decompose w/ stl
beer_stl<-stl(beer,s.window=12)
autoplot(beer_stl)
```



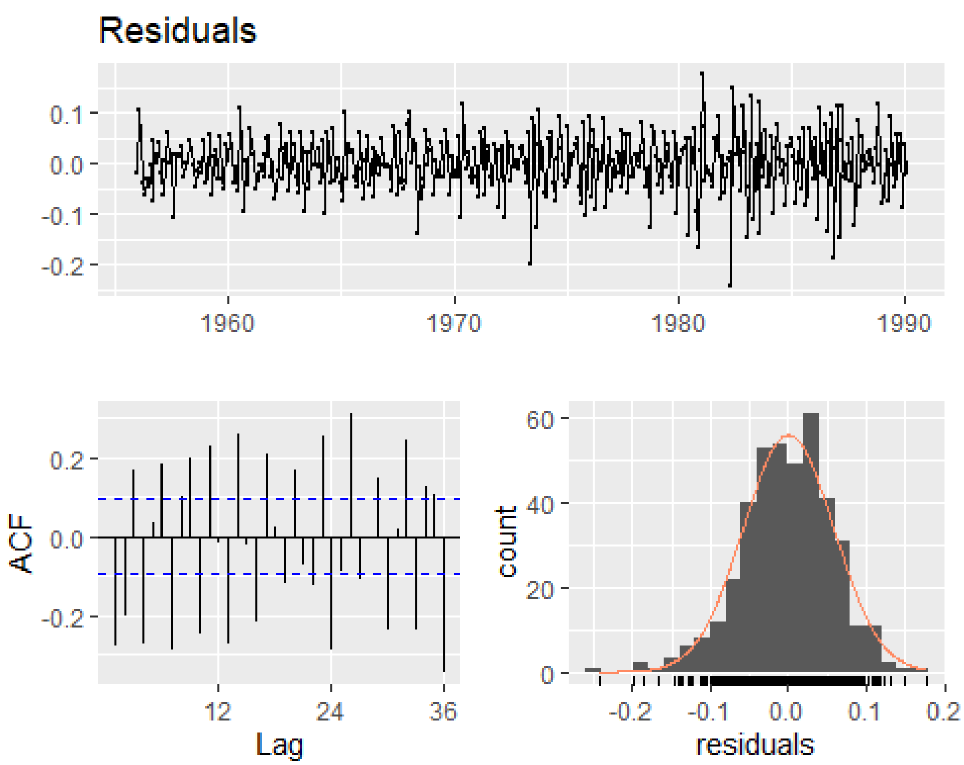

```
# Use tslm() to extract the seasonality and the quadratic trend
beer_tslm <- tslm(beer~trend + I(trend^2) + season)
beer_tslm_forecast <- forecast(beer_tslm, h = 12)
autoplot(beer_tslm_forecast)
```




```
# Focus on the remainder. Since ACF(remainder) suggests a strong autocovarian
ce out of the confidence interval band, it is not a white noise and we need a
n ARMA to fit the remainder component.
autoplot(beer_stl$time.series[,'remainder'])
```




```r
checkresiduals(beer_stl$time.series[,'remainder'])
```



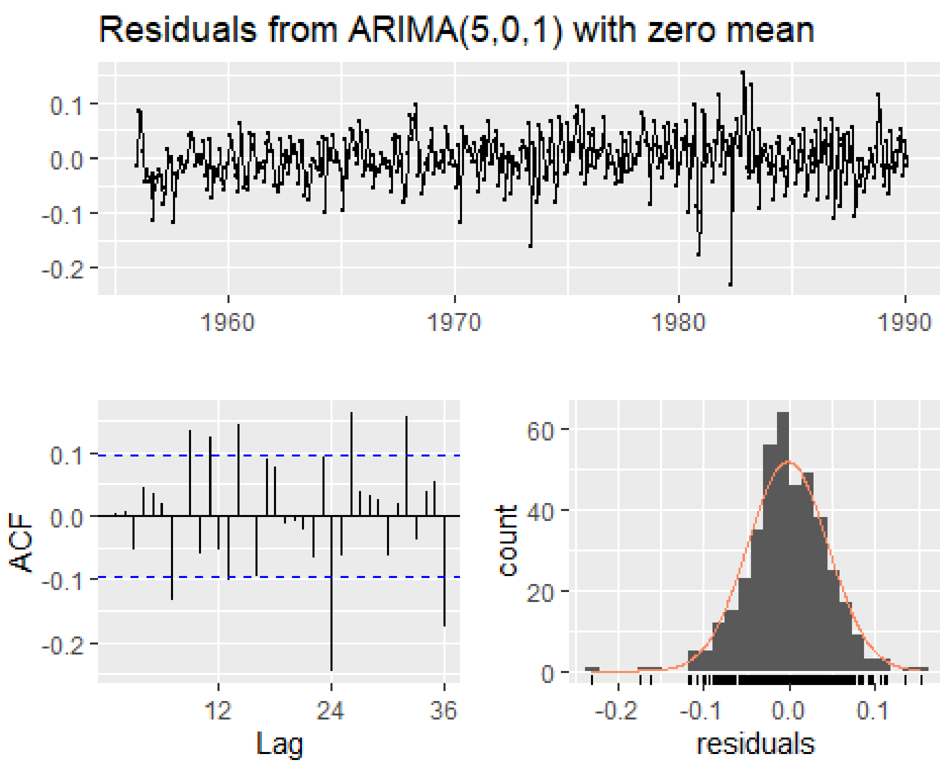
```r
# Use auto.arima() to fit data into an ARMA(5,1), which is ideal because the
resulted residuals do not witness a strong autocovariance.

beer_model_part_b <- auto.arima(beer_stl$time.series[,"remainder"], stepwise
= FALSE, seasonal = FALSE, ic = "aic", trace = TRUE, max.order = 10, max.d =
0)

##
##  Fitting models using approximations to speed things up...
##
##  ARIMA(0,0,0)            with zero mean     : -1167.818
##  ARIMA(0,0,0)            with non-zero mean : -1165.82
##  ARIMA(0,0,1)            with zero mean     : -1256.687
##  ARIMA(0,0,1)            with non-zero mean : -1254.926
##  ARIMA(0,0,2)            with zero mean     : -1312.783
##  ARIMA(0,0,2)            with non-zero mean : -1311.074
##  ARIMA(0,0,3)            with zero mean     : -1311.503
##  ARIMA(0,0,3)            with non-zero mean : -1309.798
##  ARIMA(0,0,4)            with zero mean     : -1320.639
##  ARIMA(0,0,4)            with non-zero mean : -1318.91
##  ARIMA(0,0,5)            with zero mean     : -1326.706
##  ARIMA(0,0,5)            with non-zero mean : -1325.008
##  ARIMA(1,0,0)            with zero mean     : -1197.809
##  ARIMA(1,0,0)            with non-zero mean : -1195.809
##  ARIMA(1,0,1)            with zero mean     : -1310.018
##  ARIMA(1,0,1)            with non-zero mean : -1308.069
##  ARIMA(1,0,2)            with zero mean     : -1313.66
##  ARIMA(1,0,2)            with non-zero mean : -1311.851
##  ARIMA(1,0,3)            with zero mean     : -1313.648
##  ARIMA(1,0,3)            with non-zero mean : -1311.866
##  ARIMA(1,0,4)            with zero mean     : -1322.149
##  ARIMA(1,0,4)            with non-zero mean : -1320.315
##  ARIMA(1,0,5)            with zero mean     : -1331.193
##  ARIMA(1,0,5)            with non-zero mean : -1329.193
##  ARIMA(2,0,0)            with zero mean     : -1237.671
##  ARIMA(2,0,0)            with non-zero mean : -1235.69
##  ARIMA(2,0,1)            with zero mean     : -1279.599
##  ARIMA(2,0,1)            with non-zero mean : -1277.955
##  ARIMA(2,0,2)            with zero mean     : -1300.604
##  ARIMA(2,0,2)            with non-zero mean : -1299.01
##  ARIMA(2,0,3)            with zero mean     : -1299.075
##  ARIMA(2,0,3)            with non-zero mean : -1297.477
##  ARIMA(2,0,4)            with zero mean     : -1359.852
##  ARIMA(2,0,4)            with non-zero mean : -1358.181
##  ARIMA(2,0,5)            with zero mean     : -1366.592
##  ARIMA(2,0,5)            with non-zero mean : -1364.925
##  ARIMA(3,0,0)            with zero mean     : -1237.681
##  ARIMA(3,0,0)            with non-zero mean : -1235.731
##  ARIMA(3,0,1)            with zero mean     : -1270.749
##  ARIMA(3,0,1)            with non-zero mean : -1269.054
```

```
##  ARIMA(3,0,2)            with zero mean     : -1281.815
##  ARIMA(3,0,2)            with non-zero mean : -1280.151
##  ARIMA(3,0,3)            with zero mean     : Inf
##  ARIMA(3,0,3)            with non-zero mean : Inf
##  ARIMA(3,0,4)            with zero mean     : -1314.367
##  ARIMA(3,0,4)            with non-zero mean : -1312.739
##  ARIMA(3,0,5)            with zero mean     : -1312.672
##  ARIMA(3,0,5)            with non-zero mean : -1311.035
##  ARIMA(4,0,0)            with zero mean     : -1282.117
##  ARIMA(4,0,0)            with non-zero mean : -1280.19
##  ARIMA(4,0,1)            with zero mean     : -1320.789
##  ARIMA(4,0,1)            with non-zero mean : -1319.058
##  ARIMA(4,0,2)            with zero mean     : -1338.146
##  ARIMA(4,0,2)            with non-zero mean : -1336.172
##  ARIMA(4,0,3)            with zero mean     : Inf
##  ARIMA(4,0,3)            with non-zero mean : Inf
##  ARIMA(4,0,4)            with zero mean     : Inf
##  ARIMA(4,0,4)            with non-zero mean : Inf
##  ARIMA(4,0,5)            with zero mean     : Inf
##  ARIMA(4,0,5)            with non-zero mean : Inf
##  ARIMA(5,0,0)            with zero mean     : -1289.149
##  ARIMA(5,0,0)            with non-zero mean : -1287.209
##  ARIMA(5,0,1)            with zero mean     : -1346.94
##  ARIMA(5,0,1)            with non-zero mean : -1344.991
##  ARIMA(5,0,2)            with zero mean     : -1352.495
##  ARIMA(5,0,2)            with non-zero mean : Inf
##  ARIMA(5,0,3)            with zero mean     : Inf
##  ARIMA(5,0,3)            with non-zero mean : Inf
##  ARIMA(5,0,4)            with zero mean     : Inf
##  ARIMA(5,0,4)            with non-zero mean : Inf
##  ARIMA(5,0,5)            with zero mean     : Inf
##  ARIMA(5,0,5)            with non-zero mean : Inf
##
##  Now re-fitting the best model(s) without approximations...
##
##  ARIMA(0,0,0)            with zero mean     : -1167.818
##  ARIMA(0,0,0)            with non-zero mean : -1165.82
##  ARIMA(0,0,1)            with zero mean     : -1255.194
##  ARIMA(0,0,1)            with non-zero mean : -1253.362
##  ARIMA(0,0,2)            with zero mean     : -1313.3
##  ARIMA(0,0,2)            with non-zero mean : -1311.341
##  ARIMA(0,0,3)            with zero mean     : -1311.825
##  ARIMA(0,0,3)            with non-zero mean : -1309.88
##  ARIMA(0,0,4)            with zero mean     : -1322.096
##  ARIMA(0,0,4)            with non-zero mean : -1320.097
##  ARIMA(0,0,5)            with zero mean     : -1327.382
##  ARIMA(0,0,5)            with non-zero mean : -1325.427
##  ARIMA(1,0,0)            with zero mean     : -1198.642
##  ARIMA(1,0,0)            with non-zero mean : -1196.643
##  ARIMA(1,0,1)            with zero mean     : -1308.656
```

```
##  ARIMA(1,0,1)            with non-zero mean : -1306.662
##  ARIMA(1,0,2)            with zero mean     : -1312.72
##  ARIMA(1,0,2)            with non-zero mean : -1310.79
##  ARIMA(1,0,3)            with zero mean     : -1312.975
##  ARIMA(1,0,3)            with non-zero mean : -1311.037
##  ARIMA(1,0,4)            with zero mean     : -1322.216
##  ARIMA(1,0,4)            with non-zero mean : -1320.216
##  ARIMA(1,0,5)            with zero mean     : -1326.94
##  ARIMA(1,0,5)            with non-zero mean : -1325.025
##  ARIMA(2,0,0)            with zero mean     : -1235.925
##  ARIMA(2,0,0)            with non-zero mean : -1233.931
##  ARIMA(2,0,1)            with zero mean     : -1307.916
##  ARIMA(2,0,1)            with non-zero mean : -1305.936
##  ARIMA(2,0,2)            with zero mean     : -1316.337
##  ARIMA(2,0,2)            with non-zero mean : -1314.362
##  ARIMA(2,0,3)            with zero mean     : -1315.183
##  ARIMA(2,0,3)            with non-zero mean : -1313.197
##  ARIMA(2,0,4)            with zero mean     : Inf
##  ARIMA(2,0,4)            with non-zero mean : Inf
##  ARIMA(2,0,5)            with zero mean     : Inf
##  ARIMA(2,0,5)            with non-zero mean : Inf
##  ARIMA(3,0,0)            with zero mean     : -1233.955
##  ARIMA(3,0,0)            with non-zero mean : -1231.961
##  ARIMA(3,0,1)            with zero mean     : -1307.097
##  ARIMA(3,0,1)            with non-zero mean : -1305.104
##  ARIMA(3,0,2)            with zero mean     : -1317.345
##  ARIMA(3,0,2)            with non-zero mean : -1315.348
##  ARIMA(3,0,3)            with zero mean     : Inf
##  ARIMA(3,0,3)            with non-zero mean : Inf
##  ARIMA(3,0,4)            with zero mean     : Inf
##  ARIMA(3,0,4)            with non-zero mean : Inf
##  ARIMA(3,0,5)            with zero mean     : Inf
##  ARIMA(3,0,5)            with non-zero mean : Inf
##  ARIMA(4,0,0)            with zero mean     : -1278.86
##  ARIMA(4,0,0)            with non-zero mean : -1276.883
##  ARIMA(4,0,1)            with zero mean     : -1333.842
##  ARIMA(4,0,1)            with non-zero mean : -1331.929
##  ARIMA(4,0,2)            with zero mean     : -1334.693
##  ARIMA(4,0,2)            with non-zero mean : -1332.765
##  ARIMA(4,0,3)            with zero mean     : Inf
##  ARIMA(4,0,3)            with non-zero mean : Inf
##  ARIMA(4,0,4)            with zero mean     : Inf
##  ARIMA(4,0,4)            with non-zero mean : Inf
##  ARIMA(4,0,5)            with zero mean     : Inf
##  ARIMA(4,0,5)            with non-zero mean : Inf
##  ARIMA(5,0,0)            with zero mean     : -1285.853
##  ARIMA(5,0,0)            with non-zero mean : -1283.888
##  ARIMA(5,0,1)            with zero mean     : -1335.382
##  ARIMA(5,0,1)            with non-zero mean : -1333.441
##  ARIMA(5,0,2)            with zero mean     : -1333.472
```

```
##  ARIMA(5,0,2)            with non-zero mean : -1331.532
##  ARIMA(5,0,3)            with zero mean     : Inf
##  ARIMA(5,0,3)            with non-zero mean : Inf
##  ARIMA(5,0,4)            with zero mean     : Inf
##  ARIMA(5,0,4)            with non-zero mean : Inf
##  ARIMA(5,0,5)            with zero mean     : Inf
##  ARIMA(5,0,5)            with non-zero mean : Inf
##
##
##
##
##
##  Best model: ARIMA(5,0,1)            with zero mean
```

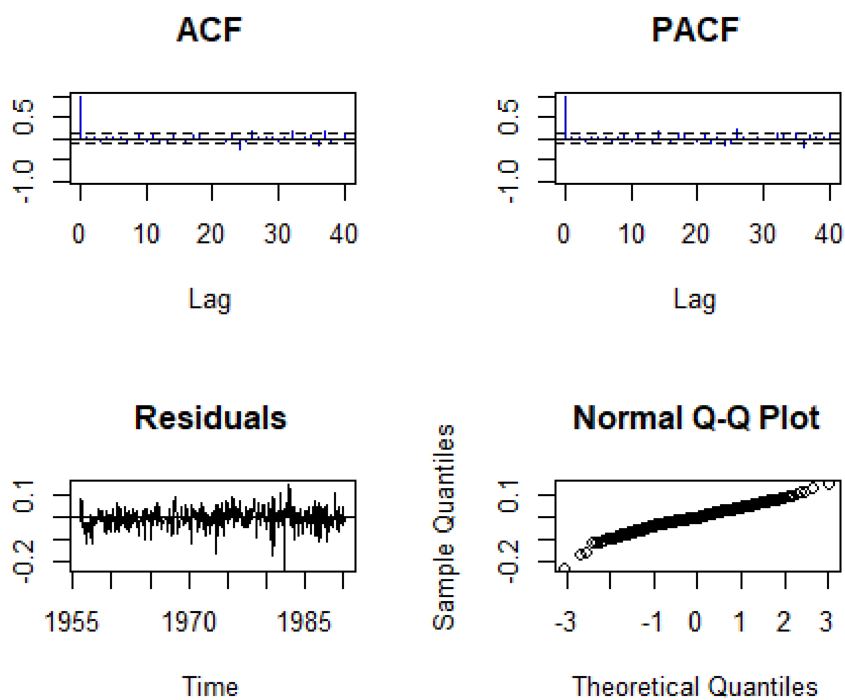
```
checkresiduals(beer_model_part_b)
```



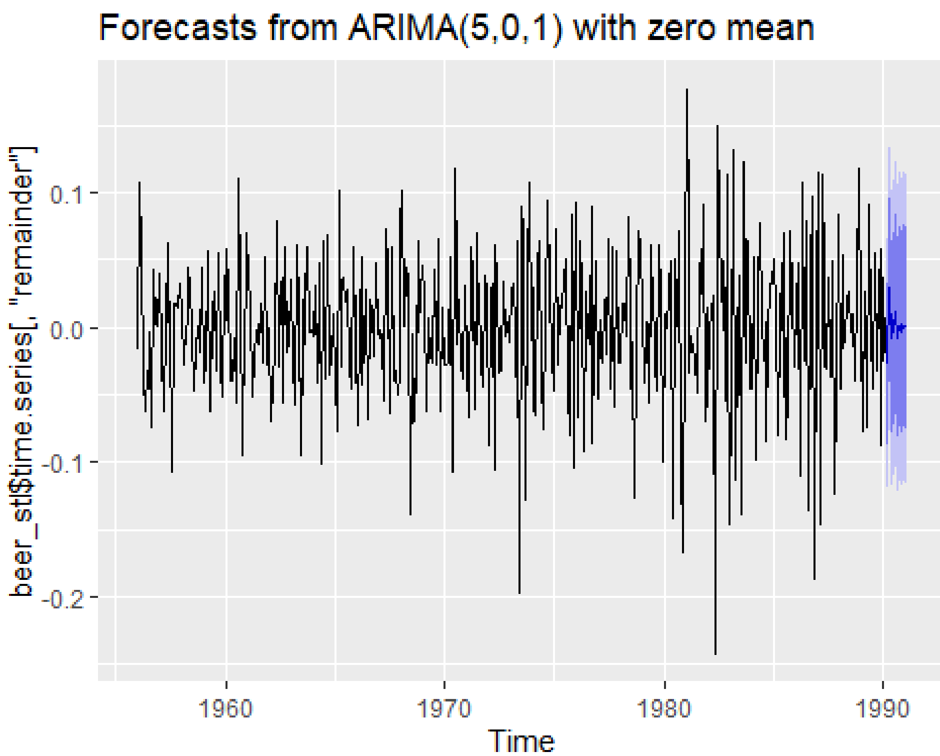

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(5,0,1) with zero mean
## Q* = 82.054, df = 18, p-value = 3.737e-10
##
## Model df: 6.   Total lags used: 24
```

```
test(residuals(beer_model_part_b))
```

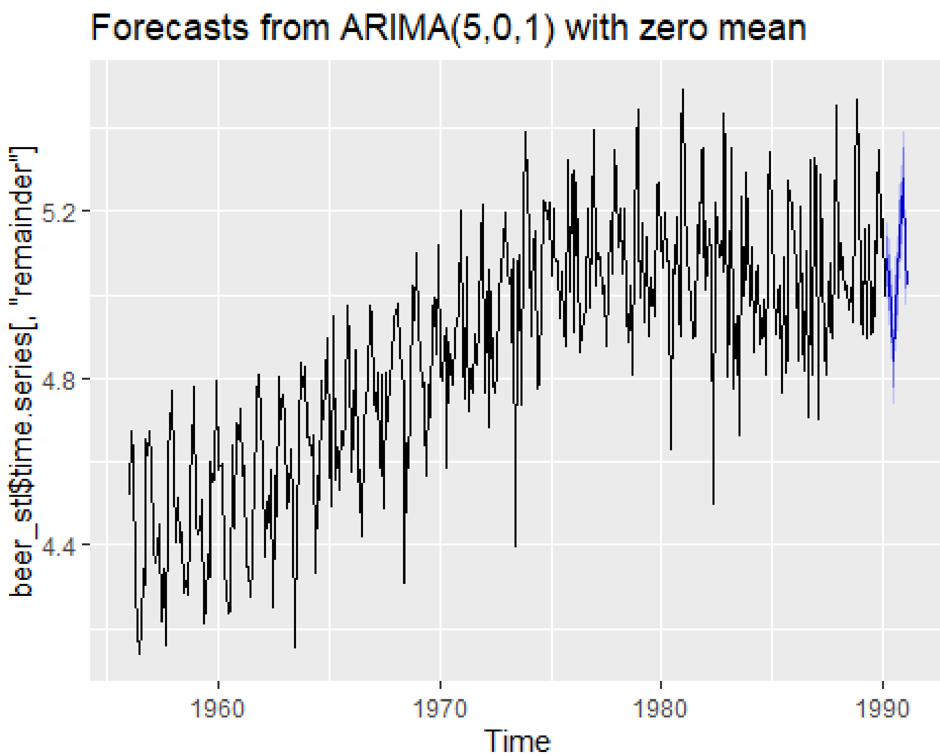
```
## Null hypothesis: Residuals are iid noise.
## Test                        Distribution Statistic   p-value
## Ljung-Box Q                Q ~ chisq(20)     50.11     2e-04 *
## McLeod-Li Q                Q ~ chisq(20)     40.29    0.0046 *
## Turning points T     (T-272)/8.5 ~ N(0,1)       293    0.0137 *
## Diff signs S       (S-204.5)/5.9 ~ N(0,1)       207    0.6692
## Rank P        (P-41922.5)/1386.2 ~ N(0,1)     45636    0.0074 *
```




```
#  Side note: ARMA(4,1), AR(4) and AR(5) are also acceptable since 0 is conta
ined in the 95% interval.
confint(beer_model_part_b)

##             2.5 %      97.5 %
## ar1  0.320861311  0.52205552
## ar2 -0.216736258 -0.01455726
## ar3  0.055485219  0.25838971
## ar4 -0.403400686 -0.20021308
## ar5 -0.004051991  0.19891227
## ma1 -0.998377015 -0.94133262

# Forecasting w/ ARMA(5,1)
beer_model_part_b_forecast <- forecast(beer_model_part_b, h = 12)
autoplot(beer_model_part_b_forecast)
```




```
# Combine the tslm() and ARMA() forecasting results.
beer_model_part_b_forecast_wmean <- beer_model_part_b_forecast
beer_model_part_b_forecast_wmean$x <- beer_tslm_forecast$x + beer_model_part_
b_forecast$x
beer_model_part_b_forecast_wmean$mean <- beer_tslm_forecast$mean + beer_model
_part_b_forecast$mean
beer_model_part_b_forecast_wmean$lower <- beer_tslm_forecast$mean + beer_mode
l_part_b_forecast$lower
beer_model_part_b_forecast_wmean$upper <- beer_tslm_forecast$mean + beer_mode
l_part_b_forecast$upper
autoplot(beer_model_part_b_forecast_wmean)
```

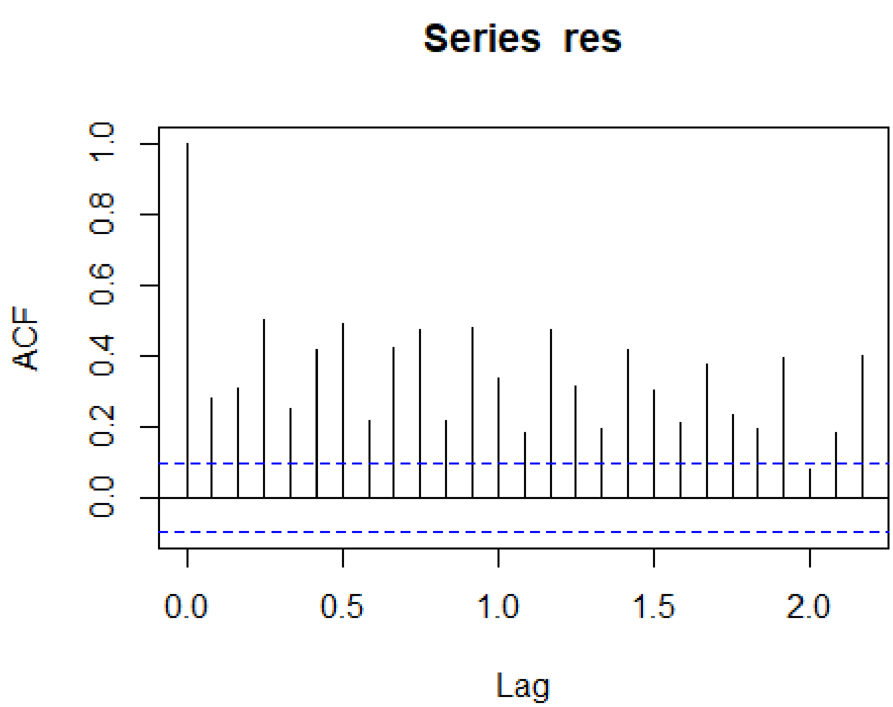



```
# Evaluation
beer_tail <- log(tail(beer_original, 12))
errors <- beer_tail - beer_model_part_b_forecast_wmean$mean

df <- data.frame(beer_model_part_b_forecast_wmean$mean, errors, beer_model_pa
rt_b_forecast_wmean$lower, beer_model_part_b_forecast_wmean$upper)
#rename
colnames(df) <- c("Estimate", "Errors", "80% C.I. Lower", "95% C.I. Lower", "
80% C.I. Upper", "95% C.I. Upper")
df

##    Estimate        Errors 80% C.I. Lower 95% C.I. Lower 80% C.I. Upper
## 1  5.084305  0.0094454226       5.024243       4.992449       5.144366
## 2  5.033403  0.0009489748       4.964903       4.928641       5.101903
## 3  4.957986  0.0711438940       4.886390       4.848489       5.029582
## 4  4.845262  0.0629713172       4.773527       4.735553       4.916997
## 5  4.939564  0.0596735517       4.865353       4.826068       5.013775
## 6  4.974467  0.0247702934       4.900149       4.860807       5.048785
## 7  5.023972 -0.1298707404       4.949260       4.909710       5.098684
## 8  5.150515  0.1163120522       5.075716       5.036120       5.225313
## 9  5.193854  0.1456048570       5.118927       5.079263       5.268782
## 10 5.280787  0.0024164039       5.205859       5.166194       5.355716
## 11 5.094439  0.0054273469       5.019457       4.979764       5.169421
## 12 5.024512 -0.0273001441       4.949520       4.909821       5.099505
##    95% C.I. Upper
## 1        5.176161
## 2        5.138165
```

```
## 3         5.067483
## 4         4.954971
## 5         5.053059
## 6         5.088127
## 7         5.138234
## 8         5.264909
## 9         5.308446
## 10        5.395381
## 11        5.209114
## 12        5.139204
```

(c) Repeat the steps in part (b), but instead use a classical decomposition approach by deseasonalizing, subtracting a quadratic trend, and then fitting and ARMA model to the residuals. Then compare your forecast errors to those in part (b).

```
# We have deseasonalized the log series in (b), so now we further remove the
quadratic trend.
b <- beer_stl$time.series[,"trend"] + beer_stl$time.series[,"remainder"]
quadratic = trend(b, p=2)
res = quadratic - b
#autoplot(res)
acf(res)
```




```
# Strong autocorrelation but no seasonality.
# An ARMA suffices.

model <- auto.arima(res, stepwise = FALSE, seasonal = FALSE, ic="aic", trace
= TRUE, max.d = 0)
```

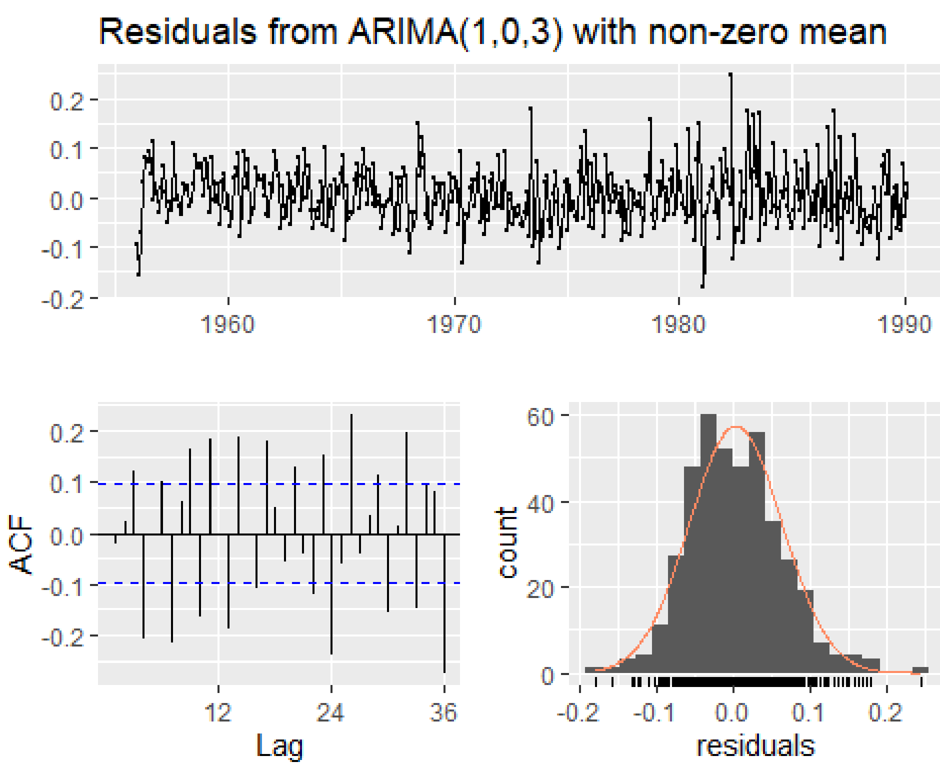
```
##
##  Fitting models using approximations to speed things up...
##
##  ARIMA(0,0,0)            with zero mean     : -931.6883
##  ARIMA(0,0,0)            with non-zero mean : -929.6883
##  ARIMA(0,0,1)            with zero mean     : -953.5766
##  ARIMA(0,0,1)            with non-zero mean : -951.5767
##  ARIMA(0,0,2)            with zero mean     : -962.6016
##  ARIMA(0,0,2)            with non-zero mean : -960.6057
##  ARIMA(0,0,3)            with zero mean     : -1011.104
##  ARIMA(0,0,3)            with non-zero mean : -1009.121
##  ARIMA(0,0,4)            with zero mean     : -1013.057
##  ARIMA(0,0,4)            with non-zero mean : -1011.086
##  ARIMA(0,0,5)            with zero mean     : -1034.401
##  ARIMA(0,0,5)            with non-zero mean : -1032.472
##  ARIMA(1,0,0)            with zero mean     : -965.4083
##  ARIMA(1,0,0)            with non-zero mean : -963.4176
##  ARIMA(1,0,1)            with zero mean     : -1093.212
##  ARIMA(1,0,1)            with non-zero mean : -1091.934
##  ARIMA(1,0,2)            with zero mean     : -1119.574
##  ARIMA(1,0,2)            with non-zero mean : -1118.265
##  ARIMA(1,0,3)            with zero mean     : -1129.924
##  ARIMA(1,0,3)            with non-zero mean : -1128.543
##  ARIMA(1,0,4)            with zero mean     : -1130.919
##  ARIMA(1,0,4)            with non-zero mean : -1129.585
##  ARIMA(2,0,0)            with zero mean     : -999.7246
##  ARIMA(2,0,0)            with non-zero mean : -997.7889
##  ARIMA(2,0,1)            with zero mean     : -1097.856
##  ARIMA(2,0,1)            with non-zero mean : -1096.739
##  ARIMA(2,0,2)            with zero mean     : -1099.108
##  ARIMA(2,0,2)            with non-zero mean : -1097.926
##  ARIMA(2,0,3)            with zero mean     : -1127.045
##  ARIMA(2,0,3)            with non-zero mean : -1125.82
##  ARIMA(3,0,0)            with zero mean     : -1091.853
##  ARIMA(3,0,0)            with non-zero mean : -1090.043
##  ARIMA(3,0,1)            with zero mean     : -1123.338
##  ARIMA(3,0,1)            with non-zero mean : -1122.065
##  ARIMA(3,0,2)            with zero mean     : -1130.027
##  ARIMA(3,0,2)            with non-zero mean : -1128.815
##  ARIMA(4,0,0)            with zero mean     : -1090.744
##  ARIMA(4,0,0)            with non-zero mean : -1088.933
##  ARIMA(4,0,1)            with zero mean     : -1137.667
##  ARIMA(4,0,1)            with non-zero mean : -1136.199
##  ARIMA(5,0,0)            with zero mean     : -1119.94
##  ARIMA(5,0,0)            with non-zero mean : -1118.134
##
##  Now re-fitting the best model(s) without approximations...
##
##  ARIMA(0,0,0)            with zero mean     : -931.6883
##  ARIMA(0,0,0)            with non-zero mean : -929.6883
```

```
##  ARIMA(0,0,1)            with zero mean     : -953.6117
##  ARIMA(0,0,1)            with non-zero mean : -951.6118
##  ARIMA(0,0,2)            with zero mean     : -963.2431
##  ARIMA(0,0,2)            with non-zero mean : -961.2456
##  ARIMA(0,0,3)            with zero mean     : -1013.309
##  ARIMA(0,0,3)            with non-zero mean : -1011.314
##  ARIMA(0,0,4)            with zero mean     : -1016.323
##  ARIMA(0,0,4)            with non-zero mean : -1014.332
##  ARIMA(0,0,5)            with zero mean     : -1039.042
##  ARIMA(0,0,5)            with non-zero mean : -1037.064
##  ARIMA(1,0,0)            with zero mean     : -962.773
##  ARIMA(1,0,0)            with non-zero mean : -960.7738
##  ARIMA(1,0,1)            with zero mean     : Inf
##  ARIMA(1,0,1)            with non-zero mean : Inf
##  ARIMA(1,0,2)            with zero mean     : Inf
##  ARIMA(1,0,2)            with non-zero mean : Inf
##  ARIMA(1,0,3)            with zero mean     : Inf
##  ARIMA(1,0,3)            with non-zero mean : -1120.07
##  ARIMA(1,0,4)            with zero mean     : Inf
##  ARIMA(1,0,4)            with non-zero mean : Inf
##  ARIMA(2,0,0)            with zero mean     : -988.8586
##  ARIMA(2,0,0)            with non-zero mean : -986.8697
##  ARIMA(2,0,1)            with zero mean     : Inf
##  ARIMA(2,0,1)            with non-zero mean : Inf
##  ARIMA(2,0,2)            with zero mean     : -1119.552
##  ARIMA(2,0,2)            with non-zero mean : -1118.042
##  ARIMA(2,0,3)            with zero mean     : Inf
##  ARIMA(2,0,3)            with non-zero mean : Inf
##  ARIMA(3,0,0)            with zero mean     : -1078.489
##  ARIMA(3,0,0)            with non-zero mean : -1076.569
##  ARIMA(3,0,1)            with zero mean     : Inf
##  ARIMA(3,0,1)            with non-zero mean : Inf
##  ARIMA(3,0,2)            with zero mean     : Inf
##  ARIMA(3,0,2)            with non-zero mean : Inf
##  ARIMA(4,0,0)            with zero mean     : -1078.247
##  ARIMA(4,0,0)            with non-zero mean : -1076.345
##  ARIMA(4,0,1)            with zero mean     : Inf
##  ARIMA(4,0,1)            with non-zero mean : Inf
##  ARIMA(5,0,0)            with zero mean     : -1105.872
##  ARIMA(5,0,0)            with non-zero mean : -1104.058
##
##
##
##
##
##  Best model: ARIMA(1,0,3)            with non-zero mean

# It says ARMA(1,3) is the best
```

```r
model_part_c <- arima(res, c(1,0,3))
summary(model_part_c)
```

```
##
## Call:
## arima(x = res, order = c(1, 0, 3))
##
## Coefficients:
##          ar1      ma1      ma2     ma3  intercept
##       0.9883  -1.0841  -0.0061  0.2352    -0.0161
## s.e.  0.0077   0.0536   0.0920  0.0619     0.0318
##
## sigma^2 estimated as 0.003683:  log likelihood = 566.03,  aic = -1120.07
##
## Training set error measures:
##                        ME       RMSE       MAE      MPE     MAPE      MASE
## Training set 0.002965673 0.06069078 0.0483887 98.44751 268.4778 0.6630548
##                    ACF1
## Training set -0.0216579
```

```r
checkresiduals(model_part_c)
```

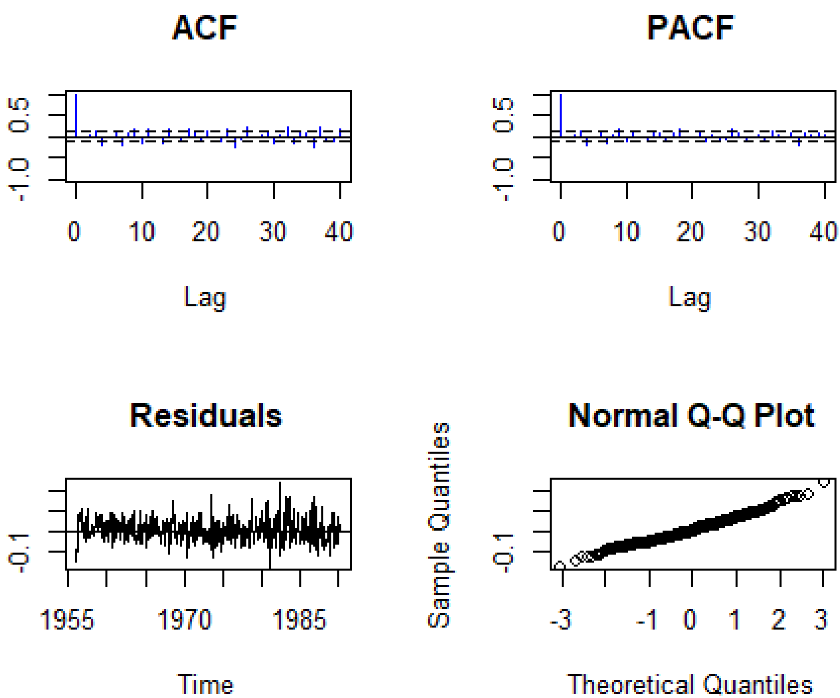



```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(1,0,3) with non-zero mean
## Q* = 187.21, df = 19, p-value < 2.2e-16
```

```
##
## Model df: 5.   Total lags used: 24
```

```r
test(residuals(model_part_c))
```

```
## Null hypothesis: Residuals are iid noise.
## Test                          Distribution Statistic   p-value
## Ljung-Box Q                  Q ~ chisq(20)    144.92         0 *
## McLeod-Li Q                  Q ~ chisq(20)      37.8    0.0094 *
## Turning points T     (T-272)/8.5 ~ N(0,1)        285     0.127
## Diff signs S         (S-204.5)/5.9 ~ N(0,1)      211    0.2667
## Rank P          (P-41922.5)/1386.2 ~ N(0,1)    38675    0.0191 *
```

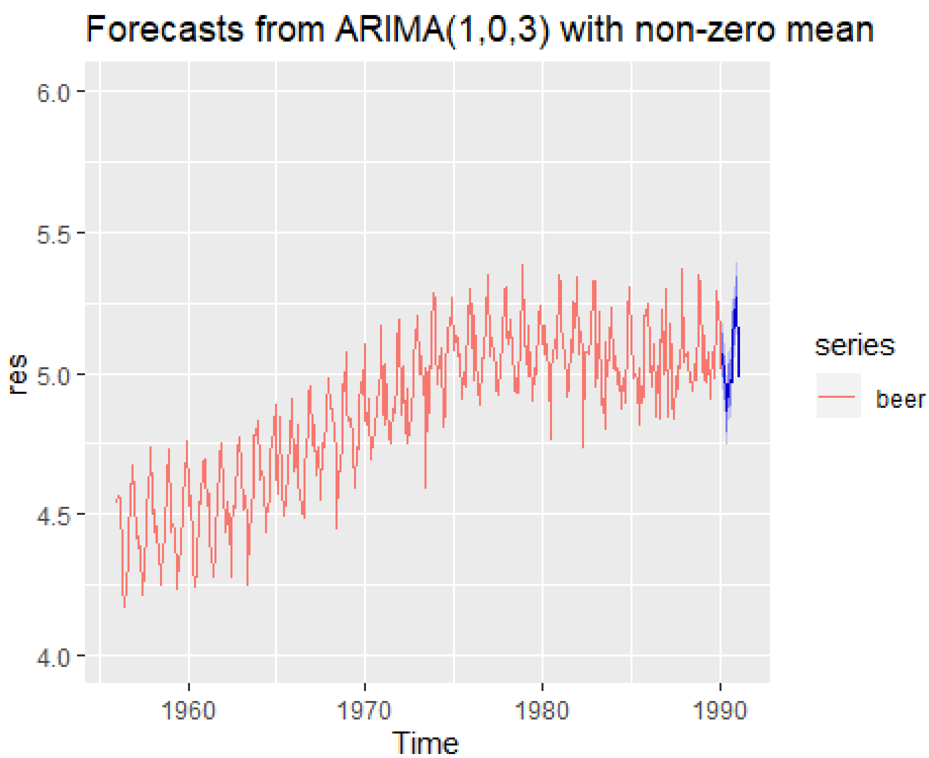



```r
confint(arima(window(res),c(1,0,3))) # These figures suggest that ARma(1,3) s
uffices.
```

```
##                 2.5 %      97.5 %
## ar1        0.97320834  1.00348210
## ma1       -1.18923496 -0.97901564
## ma2       -0.18647590  0.17417922
## ma3        0.11384685  0.35660945
## intercept -0.07841339  0.04617739
```

```r
# Forecasting w/ ARMA(1,3)
remainder <- forecast(model, h = 12)

dummy1 <- as.numeric(forecast(quadratic, h = 12)$mean)
t1 <- ts(dummy1, start = c(1990, 3), end = c(1991, 2), frequency = 12)
```

```r
dummy2 <- as.numeric(beer_stl$time.series[,"seasonal"])
t2 <- ts(tail(dummy2, 12), start = c(1990, 3), end = c(1991, 2), frequency =
12)

remainder$mean <- t1 + remainder$mean + t2
remainder$lower <- t1 + remainder$lower + t2
remainder$upper <- t1 + remainder$upper + t2

# Make sure do not use ggfortify and forecast simultaneously
autoplot(remainder, ylim=c(4,6)) + autolayer(beer)
```




```r
errors <- beer_tail - remainder$mean
df2 <- data.frame(remainder$mean, errors, remainder$lower, remainder$upper)
colnames(df2) <- c("Estimate", "Errors", "80% C.I. Lower", "95% C.I. Lower",
"80% C.I. Upper", "95% C.I. Upper")
df2

##    Estimate      Errors 80% C.I. Lower 95% C.I. Lower 80% C.I. Upper
## 1  5.066742  0.02700864       4.988485       4.947058       5.144999
## 2  4.990779  0.04357311       4.912164       4.870547       5.069394
## 3  4.939153  0.08997675       4.860143       4.818318       5.018163
## 4  4.867174  0.04105907       4.787455       4.745254       4.946894
## 5  4.953062  0.04617520       4.872656       4.830091       5.033468
## 6  4.972182  0.02705554       4.891110       4.848194       5.053253
## 7  4.965868 -0.07176689       4.884152       4.840894       5.047585
## 8  5.133099  0.13372811       5.050758       5.007169       5.215440
```

```
## 9  5.180510  0.15894923       5.097563       5.053654       5.263457
## 10 5.267113  0.01609029       5.183579       5.139359       5.350648
## 11 5.058233  0.04163369       4.974128       4.929606       5.142337
## 12 4.981379  0.01583343       4.896721       4.851906       5.066036
##    95% C.I. Upper
## 1        5.186425
## 2        5.111010
## 3        5.059989
## 4        4.989095
## 5        5.076033
## 6        5.096170
## 7        5.090842
## 8        5.259028
## 9        5.307366
## 10       5.394868
## 11       5.186859
## 12       5.110851
```

As we can see, the method of part b and the method of part c are of the same quality. This is because in part b we use tslm to estimate the trend while in part c we estimate the trend quite separately. The paths are different but the goals are the same.