EECS 16A Midterm 1 Review Session

Presented by <NAMES >(HKN)

Disclaimer

Although some of the presenters may be course staff, the material covered in the review session may not be an accurate representation of the topics covered in and difficulty of the exam.

Slides are posted at @# on Piazza.

HKN Drop-In Tutoring

- These details should be written.

Matrices and Linear Transformations

Matrices

- Matrices are **collections of vectors**.
- Typically represent **systems of equations**, where each row is an equation and each column is a variable.
- Notable matrices:
    - Identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

    - Rotation matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Augmented Matrices

Augmented matrices are a way of represening both sides of a system of equations using one matrix:

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ -2x + 2y - 2z &= -10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -2 & 2 & -2 & -10 \end{array}\right]$$

Gaussian Elimination

Gaussian Elimination

- **Gaussian elimination** is a methof for solving systems of linear equations.
- Use row operations to reduce matrix to **row echelon form** (a matrix that is all zero below the diagonal)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 3 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The row operations are:

1. **Row exchange**: reordering rows
2. **Row scaling**: scaling a row by a real number
3. **Superposition**: replace a row with the sum of itself and a scalar multiple of another row

Practice: Gaussian Elimination

Let's use Gaussian elimination to solve this system of equations!

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases}$$

Practice: Gaussian Elimination [Solution]

First, write out the system of equations into matrix-vector form:

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases} \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right]$$

Practice: Gaussian Elimination [Solution]

Now, use row operations to get the system into row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{2R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 8 \\ 0 & \boxed{6} & -1 & 7 \\ 0 & 6 & -1 & 8 \end{array}\right]$$

The values on the diagonals (boxed in the matrix above) are known as **pivots** or **leading coefficients**.

Practice: Gaussian Elimination [Solution]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ 0 & 6 & -1 & 8 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

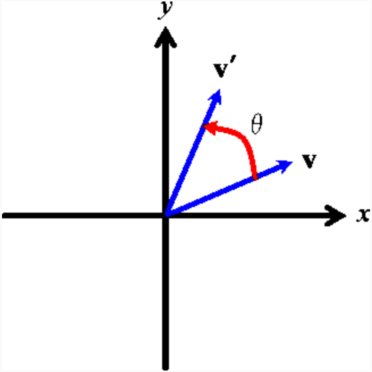
The last row says $0 = -1$, so there is **no solution** to this system of equations!

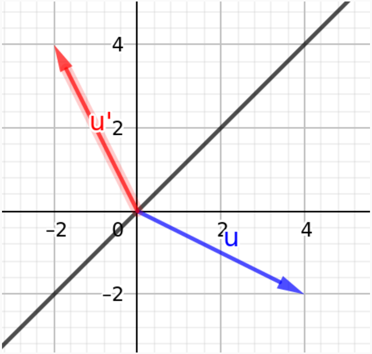
Possible results of Gaussian Elimination on $A\vec{x} = \vec{b}$

Result	Row picture	Column picture	Properties of A
Unique solution	Equations intersect at exactly one point	$\vec{b}$ is **uniquely represented** by a linear combination of the columns of A	A is **invertible**
Infinite solutions	Equations intersect along an **infinite space** (eg. line, plane, volume)	There are **multiple ways** of representing $\vec{b}$ in terms of the linear combinations of the columns of A	A has **linearly dependent columns**
No solution	Equations **do not intersect**	$\vec{b}$ is not in the span of the columns (columnspace) of A	Columnspace of A does not include $\vec{b}$. Columns of A are **linearly dependent**

Linear Transformations

- **Linear transformations** are operations that can be performed by applying a matrix to a vector.
- Some common transformations include **rotation** and **reflection**.





Common Linear Transformations: Rotation

The **rotation matrix** rotates points by a specific angle, θ:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Use this matrix by **plugging in the desired rotation angle**, then multiply it to a vector.

$$R(\theta) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotation matrices also **preserve the length** of a vector. Check it! (*Think about the eigenvalues! Real? Complex? How about the magnitude of these eigenvalues?*)

Common Linear Transformations: Reflection

- The reflection matrix **reflects vectors** across a line. (Notice that such matrix also *preserves the length of a vector*.)
- Notable reflection matrices:
    - Reflection across x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
    - Reflection across y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
    - Reflection across line $y = x$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Practice: Matrix Transformations

Create matrices to transform the vector $\vec{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ as follows:

1. Rotate by $45\,\text{deg}$
2. Reflect across $y = x$

Practice: Matrix Transformations

Create matrices to transform the vector $\vec{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ as follows:

1. Rotate by 45 deg

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix}$$

2. Reflect across $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Linear Independence

Linear Combinations

A **linear combination** of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a sum of the vectors, scaled by scalars $\{a_1, \ldots, a_n\}$:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

- If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are the *columns of a matrix* B, the set of all linear combinations is called the **columnspace** or **range** of B.
- Note: the **columnspace** of B is a **vector space**.
    - To check a subset is a vector space, it must be **closed under addition and scalar multiplication**.

Linear Independence

- Informally speaking, a set of vectors is linearly independent if *no vector in the set can be represented as a linear combination of other vectors*.
- Formal definition:
  - Given a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, if some scalars $\{c_1, c_2, \ldots, c_m\}$ $neq\{0, \ldots, 0\}$ exist such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$, then the vectors are **linearly dependent**.
  - If no such scalars exist, then the vectors are **linearly independent**.
- In other words, for a set of linearly independent vectors, $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$ implies that all $c_1 = 0, c_2 = 0, \ldots, c_m = 0$ (*useful when doing proofs*).

Practice: Linear Independence

Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Row reduce the matrix. If any of the **pivots** (numbers on the diagonal) are 0, then the columns are **linerly dependent**.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Let's look at the row-reduced matrix as the system of equations:

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = 0$$

Remember, that if there exist nonzero c_1, c_2, c_3 that satisfy that equation, the columns are **linearly dependent**.

$$\begin{aligned} c_2 &= -3c_3 \\ 2c_1 = -3c_2 - 5c_3 = 9c_3 - 5c_3 &= 4c_3 \\ c_1 &= 2c_3 \end{aligned}$$

There are infinitely many solutions, so the columns of A are **linearly dependent**.

Span and Rank

Span

- **Definition**: The **span** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$ is the set of *all possible linear combinations* of the vectors.
- The span of a collection of vectors is always a **vector space**
- Since the column space of a matrix A is the *span of its columns*, it is a vector space.

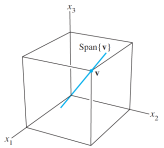


FIGURE 10 Span{**v**} as a line through the origin.

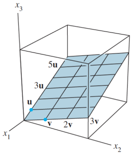
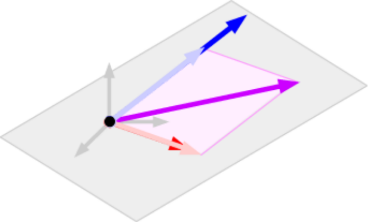


FIGURE 11 Span{**u**, **v**} as a plane through the origin.



Practice: Span

Do the following sets of vectors span $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Practice: Span [Solution]

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \implies \text{No}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \implies \text{Yes}$$

How do you know if a set of vectors spans $\mathbb{R}^3$?

- Are there **n pivots**, i.e. **n linearly independent vectors**?
- Use *Gaussian Elimination*!

Rank

- Definition: The **rank** of a matrix A is the *dimension of the column space of A*.
- Altenatively:
  - The number of rows with *nonzero leading coefficients*
  - The number of *linearly independent columns*.
- A **pivot** is the first nonzero element of a row for a matrix in *row echelon form*, and a pivot column is a column that contains a pivot.
- In fact, pivots are shared by both the columns and the rows, so $\mathbf{dim(colspace(}A\mathbf{)) = dim\ (rowspace(}A\mathbf{))}$.
- $\mathbf{rank(}A\mathbf{) = rank(}A^T\mathbf{)}$

Practice: Rank

Find the rank of B:

$$B = \begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 1 & 4 & 1 & 2 \end{bmatrix}$$

Practice: Rank [Solution]

Do Gaussian Elimination!

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 1 & 4 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 0 & 2 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 0 & 2 & 0 & 6 \end{bmatrix} \xrightarrow[R_2 \to R_3]{R_3/2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} \boxed{1} & 2 & 1 & -4 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

There are *3 pivot columns*, so $\mathbf{rank}(B) = \mathbf{3}$.

Matrix Inverses

Inverses

A matrix, A, is **invertible** if there exists a matrix B, such that

$$AB = BA = I_n \implies B = A^{-1}$$

Conditions for inverse to exist:

- The matrix must be **square** (n x n)
- The columns must be **linearly independent** (injective) and they must **span** $\mathbb{R}^n$ (surjective).

Inverse Properties

Here are some useful properties of matrix inverses:

- $AA^{-1} = A^{-1}A = I$
- $(A^{-1})^{-1} = A$
- $kA^{-1} = k^{-1}A^{-1}$ for scalar k
- $(AB)^{-1} = B^{-1}A^{-1}$ (similar to transpose)
- $(A^{-1})^T = (A^T)^{-1}$
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

Invertibility

If A is a $n \times n$ **invertible** matrix, then the following are also true:

- A has n pivot positions.
- A has a trivial nullspace ($A\vec{x} = \vec{0}$ only if $\vec{x} = \vec{0}$).
- The columns and rows of A are **linearly independent**, and **span** $\mathbb{R}^n$. As a result, they form a **basis** for $\mathbb{R}^n$.
- The columnspace of A is $\mathbb{R}^n$, and is $n-$dimensional. So, A has a **rank** of n.
- For every $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a *unique solution*.
- The determinant of A is not 0.
- A does not have an eigenvalue of 0.

For a full description of the **invertible matrix theorem** (warning: parts are out of scope), look **here**.

Computing Inverses

Use Gaussian elimination! (Who would've guessed...)

- Construct an **augmented matrix** consisting of A and the identity matrix:

$$\left[A \mid I\right]$$

- Row reduce this augmented matrix until the *left side becomes the identity*, and the *right side becomes A^{-1}*:

$$\left[A \mid I\right] \longrightarrow \left[I \mid A^{-1}\right]$$

Computing Inverses: 2×2 Matrices

For a 2×2 matrix, you can **find the inverse quickly** using the following formula:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can derive this from the general method of finding inverses, but this can be convenient.

Side Note: In real numerical computation, we generally use gaussian elimination to find the inverse of a large matrix even through there is a theoretical formula deduced from Cramer's rule which has a terrible runtime.

Practice: Computing Inverses

Find the **inverse** of

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Practice: Computing Inverses [Solution]

Make an **augmented matrix** with A and the identity, and then perform **Gaussian Elimination**:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2\to R_2]{-R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_1-3R_2\to R_1]{R_1-3R_3\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Vector Spaces

A set of elements closed under vector addition and scalar multiplication.

- *"No escape properties"* for addition and scalar multiplication
- Must also contain the **0 vector** (special case of closed scalar multiplication).

Let $\mathbb{V}$ be a vector space:

- If $\vec{u}, \vec{v}$ are vectors in $\mathbb{V}$, then $\vec{u} + \vec{v}$ must also be in $\mathbb{V}$
- If $\vec{u} \in \mathbb{V}$ and k is a real number, then $k\vec{u}$ must be in $\mathbb{V}$

Thus, any **linear combination** of vectors in $\mathbb{V}$ is also in $\mathbb{V}$.

Practice: Is it a Vector Space?

Are the following collections of vectors **vector spaces**?

- 2D plane (spanning $\mathbb{R}^2$)
- 5D space (spanning $\mathbb{R}^5$)
- n-D space (spanning $\mathbb{R}^n$)
- Line in $\mathbb{R}^2$ intersecting origin
- Line in $\mathbb{R}^2$ *not* intersecting origin
- First and third quadrant of $\mathbb{R}^2$
- Plane in $\mathbb{R}^3$ intersecting origin
- $\{\vec{0}\}$ (just the zero vector)
- $\{\vec{v}, \vec{v} \neq 0\}$
- span$(\vec{v})$

Practice: Is it a Vector Space? [Solution]

Are the following collections of vectors **vector spaces**?

- 2D plane (spanning $\mathbb{R}^2$) [yes]
- 5D space (spanning $\mathbb{R}^5$) [yes]
- n-D space (spanning $\mathbb{R}^n$) [yes]
- Line in $\mathbb{R}^2$ intersecting origin [yes]
- Line in $\mathbb{R}^2$ *not* intersecting origin [no]
- First and third quadrant of $\mathbb{R}^2$ [no]
- Plane in $\mathbb{R}^3$ intersecting origin [yes]
- $\{\vec{0}\}$ (just the zero vector) [yes]
- $\{\vec{v}, \vec{v} \neq 0\}$ [no]
- span($\vec{v}$) [yes]

Subspaces

- Definition: **A subspace is a subset of a vector space that is itself a vector space**.
- Suppose we have a vector space $\mathbb{V}$. A subset $\mathbb{S}$ of $\mathbb{V}$ is *only a subspace if the following three properties are met*:
    1. The **zero vector** of $\mathbb{V}$ is in $\mathbb{S}$
    2. $\mathbb{S}$ is closed under **vector addition**
    3. $\mathbb{S}$ is closed under **scalar multiplication**

(Same rules as before!)

Basis

Definition: A **basis** of a vector space is a **linearly independent** set of vectors that **spans** the vector space.

- **Linearly independent** (*not too big*): No vectors in a basis can be written as a linear combination of the other vectors.
- **Spanning** (*not too small*): All vectors in the vector space can be represented as a linear combination of the basis vectors.

Basis

A basis does not necessarily have to span $\mathbb{R}^n$ - it can span any vector space.

- A basis is a **minimum set of vectors** required to completely span a vector space.
- eg. Any basis of $\mathbb{R}^n$ contains **exactly n vectors**. In fact, an m-dimensional vector space must have m vectors in its basis.

Bases are **not unique**! Why? How many are there?

Practice: Basis

Are the following bases for $\mathbb{R}^n$?

$$\mathbb{V}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathbb{V}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Practice: Basis [Solution]

Are the following bases for $\mathbb{R}^n$?

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Yes! The set is linearly independent and spanning.

$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ No! The vectors are negatives of each other!

Null Space

Null Space

- **Definition**: The **null space** of a matrix (transformation) is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$.
- It is a *subspace of* $\mathbb{R}^n$.
- Solving a null space:
    1. Reduce to **reduced row echelon form**.
    2. Find solution to the system of equations.
    3. Represent the solutions in *matrix form*.

Practice: Null Space

Find the **null space** of A:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix}$$

Practice: Null Space [Solution]

First, **row reduce**.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Practice: Null Space [Solution]

- We saw that in the row-reduced matrix, there were **3 pivot columns**.
- Additionally, we know that there are 5 total “variables”
- Thus, we can say that there are **2 free variables**, and *obtain a basis for our null space in terms of these free variables*!
- The **pivot columns** occur at x_1, x_3, and x_5, so we can set $x_2 = r$ and $x_4 = s$
- Let’s find our basis in terms of r, s!

Practice: Null Space [Solution]

Let's find our basis in terms of r, s!

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = 2r + s, \;\; x_2 = r$$

$$x_3 = -2s, \;\; x_4 = s, \;\; x_5 = 0$$

In matrix form:

$$\vec{x} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition: Eigenvalues and Eigenvectors

Practice: Eigenvalues and Eigenvectors

PageRank

Graphs, Flow, and Transition Matrices

Practice: Transition Matrices

PageRank

Practice: PageRank

Practice Problems

Tomography

Bases in $\mathbb{R}^3$

Alice's Photos

Pagerank

Steady State