EECS 16A Final Review Session

Presented by <NAMES >(HKN)

Disclaimer

Although some of the presenters may be course staff, the material covered in the review session may not be an accurate representation of the topics covered in and difficulty of the exam.

Slides are posted at @# on Piazza.

HKN Drop-In Tutoring

- These details should be written.

Null Space

Null Space

- **Definition**: The **null space** of a matrix (transformation) is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$.
- It is a *subspace of* $\mathbb{R}^n$.
- Solving a null space:
    1. Reduce to **reduced row echelon form**.
    2. Find solution to the system of equations.
    3. Represent the solutions in *matrix form*.

Practice: Null Space

Find the **null space** of A:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix}$$

Practice: Null Space [Solution]

First, **row reduce**.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Practice: Null Space [Solution]

- We saw that in the row-reduced matrix, there were **3 pivot columns**.
- Additionally, we know that there are 5 total "variables"
- Thus, we can say that there are **2 free variables**, and *obtain a basis for our null space in terms of these free variables*!
- The **pivot columns** occur at x_1, x_3, and x_5, so we can set $x_2 = r$ and $x_4 = s$
- Let's find our basis in terms of r, s!

Practice: Null Space [Solution]

Let's find our basis in terms of r, s!

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = 2r + s, \;\; x_2 = r$$

$$x_3 = -2s, \;\; x_4 = s, \;\; x_5 = 0$$

In matrix form:

$$\vec{x} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Linear Independence

Linear Combinations

A **linear combination** of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a sum of the vectors, scaled by scalars $\{a_1, \ldots, a_n\}$:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

- If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are the *columns of a matrix* B, the set of all linear combinations is called the **columnspace** or **range** of B.
- Note: the **columnspace** of B is a **vector space**.
    - To check a subset is a vector space, it must be **closed under addition and scalar multiplication**.

Linear Independence

- Informally speaking, a set of vectors is linearly independent if *no vector in the set can be represented as a linear combination of other vectors*.
- Formal definition:
  - Given a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, if some scalars $\{c_1, c_2, \ldots, c_m\}$ $neq\{0, \ldots, 0\}$ exist such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$, then the vectors are **linearly dependent**.
  - If no such scalars exist, then the vectors are **linearly independent**.
- In other words, for a set of linearly independent vectors, $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$ implies that all $c_1 = 0, c_2 = 0, \ldots, c_m = 0$ (*useful when doing proofs*).

Practice: Linear Independence

Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Row reduce the matrix. If any of the **pivots** (numbers on the diagonal) are 0, then the columns are **linerly dependent**.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Let's look at the row-reduced matrix as the system of equations:

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = 0$$

Remember, that if there exist nonzero c_1, c_2, c_3 that satisfy that equation, the columns are **linearly dependent**.

$$\begin{aligned} c_2 &= -3c_3 \\ 2c_1 = -3c_2 - 5c_3 = 9c_3 - 5c_3 &= 4c_3 \\ c_1 &= 2c_3 \end{aligned}$$

There are infinitely many solutions, so the columns of A are **linearly dependent**.

Eigenvalues and Eigenvectors

Definition: Eigenvalues and Eigenvectors

****The most important concept in linear algebra**

An scalar-vector pair $\lambda, \vec{v}$ are an **eigenvalue-eigenvector pair** of matrix A if applying A to $\vec{v}$ produces a version of $\vec{v}$ scaled by λ.

$$A\vec{v} = \lambda\vec{v}$$

Properties:

- Eigenvectors w/ distinct eigenvalues are **linearly independent**
- A matrix is **non invertible** iff 0 is an eigenvalue
- A scalar times an eigenvector is still an eigenvector
- Eigenvalues remain the **same across transposes** - not necessarily true for eigenvectors!

$$A^{-1}\vec{v} = \lambda^{-1}\vec{v} \text{ and } A^{n}\vec{v} = \lambda^{n}\vec{v}$$

Finding Eigenvalues and Eigenvectors

To find eigenvalues and eigenvectors of A:

- Find solutions to the **characteristic polynomial**, which is:

$$\det(A - \lambda I) = 0$$

Aside: the determinant of a 2×2 *matrix is*

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- The solutions for λ give the **eigenvalues** of A.
- For each eigenvalue λ, calculate the null space of

$$A - \lambda I$$

- The basis for the null space will be your **eigenvectors**.

Practice: Eigenvalues and Eigenvectors

Find the **eigenvalues and eigenvectors** of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Practice: Eigenvalues and Eigenvectors [Solution]

First, find the solutions to the **characteristic polynomial**.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ (3-\lambda)(4-\lambda) - 2 \cdot 1 &= 0 \\ \lambda^2 - 7\lambda + 10 &= 0 \\ \lambda_1 = 2,\ \lambda_2 &= 5 \end{aligned}$$

Practice: Eigenvalues and Eigenvectors [Solution]

Then, find the **null space** of $A - \lambda I$ for each lambda:

$$\left[\begin{array}{cc|c} 3-\lambda_1 & 1 & 0 \\ 2 & 4-\lambda_1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3-\lambda_2 & 1 & 0 \\ 2 & 4-\lambda_2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

PageRank

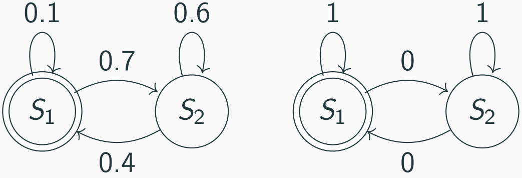
Graphs, Flow, and Transition Matrices

A **transition matrix** represents a directed graph of states and transitions.

- Edges of the graph represent *what fraction of one state moves to the next*.
- Entry **ij** in matrix means *fraction of water from node j entering node i*.
- Columns sum to ≤ 1 (*What would it mean of the sum of a column was greater than 1?*).
- Examples: Social networks, PageRank

Transition Matrix Examples

Here are some examples of **graphs** and their corresponding **transition matrices**. Notice that element *ij* represents the *fractional transition* from state *j* to state *i*.



$$T = \begin{bmatrix} 0.1 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Practice: Transition Matrices

- What do we know about the system if all *columns* each **sum to 1**?
- Less than 1? (Think about what physically happens if the system was water flows)
- Greater than 1?

Practice: Transition Matrices [Solution]

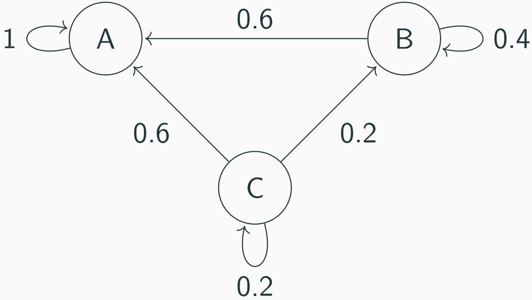
- What do we know about the system if all *columns* each **sum to 1**? **Flow is conserved: no "water" is added or lost.**
- Less than 1? (Think about what physically happens if the system was water flows) **Flow is lost.**
- Greater than 1? **Flow increases.**

PageRank

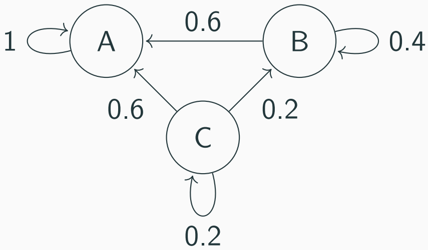
- Given a flow graph, we wish to **rank the nodes by their "importance"**, i.e. which state will hold the most "water" when the system reaches a steady state.
- Procedure:
  1. Write A, the **transition matrix** of the flow diagram.
  2. Find the eigenvector(s) of A, whose **eigenvalue is 1**. If $A\vec{v} = \vec{v}$, $\vec{v}$ is a **steady state vector**.
  3. $\vec{v}$ contains the *values the nodes will stabilize at*, and ranks the importance of each node

Practice: PageRank

Find the **transition matrix** for this flow diagram.



Practice: PageRank [Solution]



Assuming the first column represents *flow from state A*, the second represents *flow from state B*, etc. the **transition matrix** is:

$$T = \begin{bmatrix} 1 & 0.6 & 0.6 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Practice: PageRank

Find the **steady state values** of the system.

$$T = \begin{bmatrix} 1 & 0.6 & 0.6 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Practice: PageRank [Solution]

To find the **steady state values**, find the *eigenvectors corresponding to an eigenvalue of 1*.

$$A\vec{v} = \vec{v}$$

$$(A - I)\vec{v} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 0 & 0.6 & 0.6 & 0 \\ 0 & -0.6 & 0.2 & 0 \\ 0 & 0 & -0.8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$