EECS 16B Final Review Session

Presented by <NAMES >(HKN)

Disclaimer

Although some of the presenters may be course staff, the material covered in the review session may not be an accurate representation of the topics covered in and difficulty of the exam.

Slides are posted at @# on Piazza.

HKN Drop-In Tutoring

- These details should be written.

Controls

Reviewing State Space

Discrete Time State Space Model:

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k]$$

Where $\vec{x}[\cdot]$ as the state vector, $u[\cdot]$ as the input vector.

Controllability

Goal: Modify $x(t)$ to be in any state we desire.

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

Expand out $x[t]$ in terms of the initial state and all inputs,

$$\vec{x}(t) = A^t\vec{x}(0) + A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + ABu(t-2) + Bu(t-1)$$

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{\triangleq R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-2) \\ u(t-1) \end{bmatrix}$$

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Given the initial condition, $x(0)$ the output of the system can be expressed in terms of the solely our inputs!

What states can we change $x(t)$ to?

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{\triangleq R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-2) \\ u(t-1) \end{bmatrix}$$

The $\mathrm{Col}(R_t)$ determines the subspace $\vec{u}(t)$ can map to.

In order to control the state to any vector in $\mathbb{R}^n$, $\mathrm{Col}(R_t) = R^n$, or it must be full rank.

i.e. The system is Controllable if and only if

$$\operatorname{rank} R_n = \operatorname{rank} \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix} = n$$

Stability

Stability in Discrete Time

A discrete system is stable iff all eigenvalues have magnitude less than 1. If any eigenvalue has magnitude greater than 1, then any state vector with a nonzero corresponding eigenvector component will have that component repeatedly magnified.

For example: $x[t+1] = 2x[t]$

Stability in Discrete Time

A discrete system is stable iff

$$\forall x \in eig(A) : |x| < 1$$

The eigenvectors form a basis (called the eigenbasis) which spans the entire space if A is full rank. (can you prove this?)

If any eigenvalue has magnitude greater than 1, then any state vector with a nonzero corresponding eigenvector component will have that component repeatedly magnified.

Stability in Discrete Time

How do the eigenvalues govern system dynamics?

If initial state is x(0), and there's no control input, the *n*th state is

$$x(n) = A^n x(0)$$

If any eigenvalue of A is larger in magnitude than 1, it "blows up" through repeated exponentiation - the system destabilizes!

Stability in Discrete Time

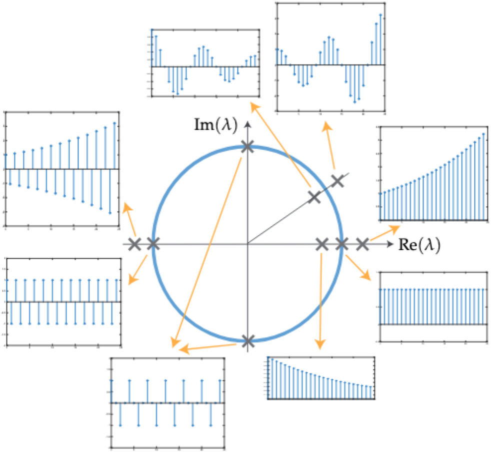


Figure 1: The real part of λ^t for various values of λ in the complex plane. It grows unbounded when $|\lambda| > 1$, decays to zero when $|\lambda| < 1$, and has constant amplitude when λ is on the unit circle ($|\lambda| = 1$).

Stability in Continuous Time

A continuous system is stable iff the real parts of all eigenvalues are negative. If any eigenvalue is positive, then any state vector with a nonzero corresponding eigenvector component will have that component grow exponentially to infinity.

For example: $\frac{\mathrm{d}}{\mathrm{d}t} x(t) = 2\,x(t)$

Stability in Continuous Time

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t)$$

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-s)}u(s)\,\mathrm{d}s$$

For scalar case, system is stable if $\mathrm{Re}\{a\} < 0$ and not stable if $\mathrm{Re}\{a\} > 0$.

By careful application of diagonalization, we get the same result for the eigenvalues of A in the matrix case.

Stability in Continuous Time

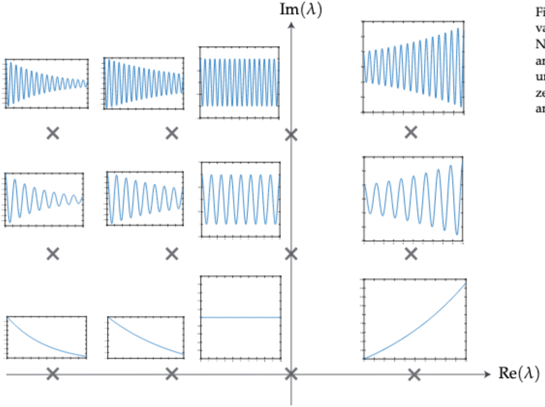


Figure 2: The real part of $e^{\lambda t}$ for various values of λ in the complex plane. Note that $e^{\lambda t}$ is oscillatory when λ has an imaginary component. It grows unbounded when $\text{Re}\{\lambda\} > 0$, decays to zero when $\text{Re}\{\lambda\} < 0$, and has constant amplitude when $\text{Re}\{\lambda\} = 0$.

Stability in Continuous Time

How do the eigenvalues govern system dynamics?

If initial state is $\vec{x}(0)$, and there's no control input, state at time t is

$$\vec{x}(t) = e^{At}\vec{x}(0)$$

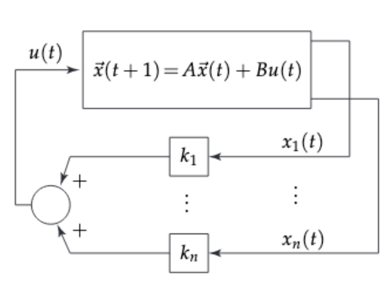
If any eigenvalue of A is larger in magnitude than 1, it "blows up" through repeated exponentiation — the system destabilizes!

Stability Through State Feedback

- If we add a feedback path (modifying the input values with the state) our state update equation changes

  $$\vec{x}(t+1) = (A + BK)\vec{x}(t)$$

- What determines the stability of this new system?



State Feedback

- By designing K, we can give our system specific dynamic properties
    - Can analyze and design the way its state changes over time
- If our "open-loop" system is unstable, choosing the right values of K can make it stable!
- Is this always possible?

Example: Controllability and Stability

$$\vec{x}[t+1] = \begin{bmatrix} -5 & 0 \\ 7 & 6 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u[t]$$

$$\vec{y}[t] = \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{x}[t]$$

Controllable?

Stable for $u[t] = 0$?

Example: Controllability and Stability

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Controllable? **Yes**

Stable for $u[t] = 0$?

Example: Controllability and Stability

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Controllable? **Yes**

Stable for $u[t] = 0$? **No**

Upper Triangularization

- Recall that not all square matrices are diagonalizable
  - An $n \times n$ matrix is diagonalizable iff has n linearly independent eigenvectors
- However, all square matrices can be brought into upper triangular form
- I'll walk through the proof from the notes
- (But I'm not sure how useful this will be / how they would ask questions about this on the test)
- (So if people want to I can instead start taking questions on SVD, time- and frequency-domain analysis of RLC circuits, and phasors)

Upper Triangularization Proof

- What are we trying to prove?
  - Remember that if M is diagonalizable, this means that there exists a matrix P such that PMP^{-1} was diagonal
  - In our case, we want to prove that for any square matrix A, there exists a matrix T such that TAT^{-1} is upper triangular
- We will proceed by induction
- First prove a base case (a 1×1 matrix must be upper triangular)
- Prove that if there exists such a matrix T_0 for a $k \times k$ matrix, then there exists the matrix T for a size $(k+1) \times (k+1)$ matrix

Upper Triangularization Proof

- Clearly a 1×1 matrix is upper triangular
- First we choose one arbitrary eigenvalue / eigenvector pair, choose an orthonormal basis for $\mathbb{R}^n$ (with Gram-Schmidt), then define V formed with those vectors.

We can upper triangularize $(k+1) \times (k+1)$ matrices if we assume that $k \times k$ matrices can be upper triangularized. To show this, let A be an arbitrary $(k+1) \times (k+1)$ matrix and let $\lambda, \vec{v}$ by an eigenvalue/vector pair: $A\vec{v} = \lambda\vec{v}$. Normalize $\vec{v}$ so that $\|\vec{v}\| = 1$ and choose k other vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^{k+1}$ such that $\{\vec{v}, \vec{v}_1, \ldots, \vec{v}_k\}$ is an orthonormal basis for $\mathbb{R}^{k+1}$. Then the $(k+1) \times (k+1)$ matrix $V \begin{bmatrix} \vec{v} & \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix}$ is orthogonal, *i.e.* $V^{-1} = V^T$.