The Buckingham Pi Theorem in Dimensional Analysis

Reading; F. M. White *Fluid Mechanics* Sections 5.1–5.4

Historical Note

The Buckingham Pi Theorem puts the 'method of dimensions' first proposed by Lord Rayleigh in his book *"The Theory of Sound"* (1877) on a solid theoretical basis, and is based on ideas of matrix algebra and concept of the 'rank' of non-square matrices which you may see in math classes. Although it is credited to E. Buckingham (1914), in fact, White points out that the theorem has also appeared earlier in independent publications by A. Vaschy (1892) and D. Riabouchinsky (1911).

The Theorem

Let $q_1, q_2, q_3 .. q_n$ be n dimensional variables that are physically relevant in a given problem and that are inter-related by an (unknown) dimensionally homogeneous set of equations. These can be expressed via a functional relationship of the form

$F(q_1, q_2, ... q_n) = 0$ or equivalently $q_1 = f(q_2, ... q_n)$

If k is the number of fundamental dimensions required to describe the n variables, then there will be k primary variables and the remaining $j = (n-k)$ variables can be expressed as $(n-k)$ dimensionless and independent quantities or 'Pi groups', $\Pi_1, \Pi_2 ... \Pi_{n-k}$. The functional relationship can thus be reduced to the much more compact form:

$\Phi(\Pi_1, \Pi_2 \dots \Pi_{n-k}) = 0$ or equivalently $\Pi_1 = \phi(\Pi_2, \dots \Pi_{n-k})$

⇒ Note that this set of non-dimensional parameters is not unique. They are however independent and form a complete set.

Application

i) Clearly define the problem and think about which variables are important. Identify which is the main variable of interest i.e. $q_1 = f(q_2, ... q_n)$. It is important to think physically about the problem. Are there any constraints; *i.e.* 'can I vary all of these variables independently';
e.g. weight of an object $F_w = \rho g \ell^3$ (only two of these are independent, unless g is also variable)

ii) Express each of n variable in terms of its fundamental dimensions, $\{MLT\theta\}$ or $\{FLT\theta\}$
It is often useful to use one system to do problem, and then check that groups you obtain are dimensionless by converting to other system.

iii) Determine the number of Pi groups, $j = n - k$ where k is the number of reference dimensions and select k primary or repeating variables.
Typically pick variables which characterize the fluid properties, flow geometry, flow rate…

iv) Form j dimensionless Π groups and check that they are all indeed dimensionless.

v) Express result in form $\Pi_1 = \phi(\Pi_2, \dots \Pi_{n-k})$ where Π_1 contains the quantity of interest and interpret your result physically!

vi) Make sure that your groups are indeed independent; i.e. can I vary one and keep others constant.

vii) Compare with experimental data!

Tables 5.1 and 5.2 from White

Parameter	Definition	Qualitative ratio of effects	Importance
Reynolds number	$\mathrm{Re} = \dfrac{\rho U L}{\mu}$	$\dfrac{\text{Inertia}}{\text{Viscosity}}$	Always
Mach number	$\mathrm{Ma} = \dfrac{U}{a}$	$\dfrac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude number	$\mathrm{Fr} = \dfrac{U^2}{gL}$	$\dfrac{\text{Inertia}}{\text{Gravity}}$	Free-surface flow
Weber number	$\mathrm{We} = \dfrac{\rho U^2 L}{\Upsilon}$	$\dfrac{\text{Inertia}}{\text{Surface tension}}$	Free-surface flow
Cavitation number (Euler number)	$\mathrm{Ca} = \dfrac{p - p_v}{\rho U^2}$	$\dfrac{\text{Pressure}}{\text{Inertia}}$	Cavitation
Prandtl number	$\mathrm{Pr} = \dfrac{\mu c_p}{k}$	$\dfrac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$\mathrm{Ec} = \dfrac{U^2}{c_p T_0}$	$\dfrac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific-heat ratio	$k = \dfrac{c_p}{c_v}$	$\dfrac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Strouhal number	$\mathrm{St} = \dfrac{\omega L}{U}$	$\dfrac{\text{Oscillation}}{\text{Mean speed}}$	Oscillating flow
Roughness ratio	$\dfrac{\epsilon}{L}$	$\dfrac{\text{Wall roughness}}{\text{Body length}}$	Turbulent, rough walls
Grashof number	$\mathrm{Gr} = \dfrac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\dfrac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\dfrac{T_w}{T_0}$	$\dfrac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \dfrac{p - p_\infty}{\frac{1}{2}\rho U^2}$	$\dfrac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aerodynamics, hydrodynamics
Lift coefficient	$C_L = \dfrac{L}{\frac{1}{2}\rho U^2 A}$	$\dfrac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Drag coefficient	$C_D = \dfrac{D}{\frac{1}{2}\rho U^2 A}$	$\dfrac{\text{Drag force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics

Quantity	Symbol	Dimensions $MLT\Theta$	Dimensions $FLT\Theta$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	m	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	ϵ	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Expansion coefficient	β	Θ^{-1}	Θ^{-1}