MHF3202
Exam 3

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Question 1: Question 2

Lemma 0.0.1 The complement of the complement of a set in some universal set P is the original set, i.e. for some set A, $\overline{\overline{A}} = A$.

Proof: Let P be some non-empty set, and let $A \subseteq P$.
Observe $\overline{A} = P \setminus A = \{x : x \in P \text{ and } x \notin A\}$.
Then $\overline{\overline{A}} = P \setminus \overline{A} = \{y : y \in P \text{ and } y \notin \overline{A} = P \setminus A\}$.
Notice that if $y \in P$ and $y \notin \overline{A}$, then $y \in P$ and $y \in \{z : z \notin P \text{ or } z \in A\} = \{z : z \in \varnothing \cup \{a : a \in A\}\} = A$.
So $y \in P$ and $y \in A$, which by definition is the set $P \cap A = A$.
So $\overline{\overline{A}} = \{y : y \in A\} = A$. 😊

Let A be a non-empty set, and define the function $f : \mathcal{P}(A) \to \mathcal{P}(A)$ by $f(X) = \overline{X}(:= A \setminus X)$.

Theorem 0.0.1 f is injective

Proof: To show that f is injective, fix $X,\ Y \in \mathcal{P}(A)$.
Then suppose that $f(X) = f(Y)$, i.e. $\overline{X} = \overline{Y}$.
Taking the complement of both sides, we get $\overline{\overline{X}} = \overline{\overline{Y}}$ and $X = Y$ by lemma 0.0.1.
So f is injective. 😊

Theorem 0.0.2 f is surjective

Proof: To show that f is surjective, take some $C \in \mathcal{P}(A)$.
Then take $B = \overline{C}$.
Since $C \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$, then $B = C \setminus A \in \mathcal{P}(A)$.
Then we have $f(B) = \overline{\overline{C}} = C$ by lemma 0.0.1.
So f is surjective. 😊

Theorem 0.0.3 f is bijective

Proof: By theorems 0.0.1 and 0.0.2, f is both injective and surjective.
So f is injective. 😊

Question 2: Question 3

The minimum number of players that guarantee two cards of the same value/rank are dealt face up at some point in a single game of seven card stud is **four** players.

Proof: Notice that each player ends up with 4 face up cards.
With four players, there will be a total of 16 face up cards on the table.
Observe that there are only 13 ranks.
So the set of face up cards at the end of the game C has cardinality of 16 and the set of ranks R has cardinality of 13.
Define $f : C \to S$ as an arbitrary function which assigns a rank to each card.
Notice that $|C| > |S|$, and so f is not injective by the pigeonhole principle.
So there are is at least one rank that gets two cards mapped to it, so our original claim is proven. 😊

Question 3: Question 5

Proof: First, notice that we can partition $\mathbb{Z}$ into 6 equivalence classes that make up $\mathbb{Z}_7$.
Observe that we can split these equivalence classes into 4 distinct sets: [0], [1] and [6], [2] and [5], and [3] and [4], which we will label A, B, C, and D respectively.
We denote $G = \{A, B, C, D\}$.
We want to show that choosing 2 integers from equivalence classes in the same set in G will result in either their sum or difference being divisible by 7.
Notice that if we choose two distinct integers a_1 and a_2 from A (for $a_1 = 7a_m$, $a_2 = 7a_n$, $a_1 \neq a_2$ by definition of congruence of integers) we can observe $a_1 + a_2 = 7a_m + 7a_n = 7(a_m + a_n) = 7a_k$ for $a_k = a_m + a_n \in \mathbb{Z}$.
So, by definition of divisibility, $7 \mid a_1 + a_2$.
Thus, choosing any two distinct integers from the equivalence classes in A, their sum is divisible by 7.
If we choose two distinct integers b_1 and b_2 from equivalence classes from B, we have four cases:

Case 1: $b_1,\ b_2 \in [1]$

If $b_1,\ b_2 \in [1]$, then $b_1 = 7b_m + 1$ and $b_2 = 7b_n + 1$ by definition of congruence for $b_1 \neq b_2$.

Then by subtracting b_1 from b_2, then $b_1 - b_2 = 7b_m + 1 - 7b_n - 1 = 7(b_m - b_n) = 7b_k$ for $b_k = b_m - b_n \in \mathbb{Z}$.

So, by definition of divisibility, $7 \mid b_1 - b_2$

Case 2: $b_1,\ b_2 \in [6]$

If $b_1,\ b_2 \in [1]$, then $7 \mid b_1 - b_2$ by similarity to case 1.

Case 3: $b_1 \in [1],\ b_2 \in [6]$

If $b_1 \in [1]$ and $b_2 \in [6]$, then $b_1 = 7b_q + 1$ and $b_2 = 7b_r + 6$ by definition of congruence.

Then by adding b_1 and b_2, we get $b_1 + b_2 = 7b_q + 1 + 7b_r + 6 = 7(b_q + b_r + 1) = 7b_j$ for $b_j = b_q + b_r + 1 \in \mathbb{Z}$.

So, by definition of divisibility, $7 \mid b_1 + b_2$

Case 4: $b_1 \in [6],\ b_2 \in [1]$

If $b_1 \in [6]$ and $b_2 \in [1]$, then $7 \mid b_1 + b_2$ by similarity to case 3.

So for any two distinct integers from the equivalence classes in B, either their sum or their difference is divisible by 7.
Observe that we can show a similar result for C and D, since $2 + 5 = 7$ and $3 + 4 = 7$.
Hence, for any two distinct integers from the same set in G, either their sum or their difference will be divisible by 7.
Now, create a set with 5 distinct arbitrary integers $S = \{x_1, x_2, x_3, x_4, x_5\} \subset \mathbb{Z}$ with $x_i \neq x_j$ for $i \neq j$.
Define $f : S \to G$ as an arbitrary function that maps 5 distinct integers to the four defined groups.
Observe that $|S| = 5$ and $|G| = 4$, so $|S| > |G|$ and f is not injective by the pigeonhole principle.
So two of the arbitrary integers must be from the same set in G.
Thus, if we choose 5 distinct integers, two of them will add or subtract to a number divisible by 7. ☺

Question 4: Question 7

Proof: We want to show that the set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ has the same cardinality as $\mathbb{N}$, i.e. there is a bijection from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.
Consider the function $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(a, b, c) = 2^{\left(2^{a-1}(2b-1)-1\right)}(2c - 1)$.
To show that f is surjective, take some $n \in \mathbb{N}$.
By the fundamental theorem of arithmetic, n can be written by a unique prime factorization $n = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$.
Now n is either even or odd, since it is a natural number.
If n is even, then there is some p_i for $1 \leq i \leq k$ such that $p_i = 2^{m-1}$ for some $m \geq 2$ (i.e. $m - 1 \geq 1$),

since $2 \mid n$ and is thus a factor of n.
So we can then write $n = 2^{m-1} \cdot p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_{i-1} \cdot p_{i+1} \cdot .. \cdot p_k$.
If n is odd, then 2 is not a factor of n, and we can write $n = 2^{m-1} \cdot p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$ where $m = 1$ (i.e. $m - 1 = 0$).
Either way, we can take $m \in \mathbb{N}$ since $m > 0$ in either case.
We can then write $n = 2^{m-1}p$ where $p = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$ (with the p_i element removed if n is even) is a unique product by the fundamental theorem of algebra.
Recall that 2 is the only even prime.
Observe that we either factored all powers of 2 out of n in the case that n was even, or n didn't have any factors of 2 in the case that n was odd.
Thus p is odd, since all the factors of p are odd.

Note

I forget if we proved that the product of any number of odd integers is odd, but I know it's easy to do by induction. However, I have another exam I really need to study for so I'm gonna gloss over that to get this done.

Since p is odd, it can be written as $p = 2j - 1$ for some j where $j \in \mathbb{N}$ because $p > 0$ and thus $j > 0$.
So we have an m and j such that $2^{m-1}(2j-1) = n$, and we can thus we can find a pair of natural numbers to generate any given natural number.
Now take $x \in \mathbb{N}$.
We know from before that we can find a pair of natural numbers x_m and x_n that generates x.
Since x_m is a natural number, we can find a pair of natural numbers x_{m_m} and x_{m_n} that generates x_m.
So taking (x_{m_m}, x_{m_n}, x_n), we have a tuple of 3 numbers such that $f(x_{m_m}, x_{m_n}, x_n) = 2^{x_m}(2x_n - 1) = x$.

Note

I'm sure this all could've been made into a lemma and written neater but again, I"m in a rush :(.

To show that f is injective, consider some $y = (a_1, b_1, c_1), z = (a_2, b_2, c_2) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.
Suppose that $y \neq z$, i.e. $a_1 \neq a_2$, $b_1 \neq b_2$, or $c_1 \neq c_2$.

Case 1: $a_1 \neq a_2$

If $a_1 \neq a_2$, then $2^{a_1 - 1} \neq 2^{a_2 - 1}$.

Also, $2b_1 - 1$ and $2b_2 - 1$ are both odd and cannot have a 2 factored out.

Thus, $2^{a_1-1}(2b_1 - 1) = a_3 \neq a_4 = 2^{a_2-1}(2b_2 - 1)$, since the exponents of 2 are different and the resulting products will have differing unique prime factorizations.

By similar logic, since $2^{a_3-1} \neq 2^{a_4-1}$, then $f(y) = 2^{a_3}(2c_1 - 1) \neq 2^{a_4}(2c_2 - 1) = f(z)$.

Notice that this does not depend on if $b_1 = b_2$ or $c_1 = c_2$.

Case 2: $b_1 \neq b_2$

If $b_1 \neq b_2$, then $b_3 = 2b_1 - 1 \neq 2b_2 - 1 = b_4$.

By the fundamental theorem of algebra, b_3 and b_4 have different prime factorizations, and since b_3 and b_4 are odd, none of those factors are 2.

So multiplying either of them by any power of 2 cannot make them equal.

So $b_5 = 2^{a_1-1}b_3 \neq 2^{a_2-1}b_4 = b_6$.

Thus $f(y) = 2^{b_5-1}(2c_1 - 1) \neq 2^{b_6-1}(2c_2 - 1) = f(z)$.

Notice that this does not depend on if $a_1 = a_2$ or $c_1 = c_2$.

Case 3: $c_1 \neq c_2$

If $c_1 \neq c_2$, then $c_3 = 2c_1 - 1 \neq 2c_2 - 1 = c_4$.

By the fundamental theorem of algebra, c_3 and c_4 have different prime factorizations, and c_4 and c_4 are odd, none of those factors are 2.

So multiplying either of them by any power of 2 cannot make them equal.

So $f(y) = 2^{\left(2^{a_1-1}(2b_1-1)\right)}c_3 \neq 2^{\left(2^{a_2-1}(2b_2-1)\right)}c_4 = f(z)$.

Notice that this does not depend on if $a_1 = a_2$ or $b_1 = b_2$.

Since our cases are exhaustive, if $y \neq z$, then $f(y) \neq f(z)$.
So f is injective.
Since f is injective and surjective, it is bijective.
So we have a bijection f from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ which thus have the same cardinality. 😊

Question 5: Question 8

Proof: We seek to prove that a bijection must exist.
Suppose A and B are finite sets.
Suppose an injective function $f : A \to B$ and a surjective function $g : A \to B$ exist.
By the pigeonhole principle, $|A| \leq |B|$ and $|A| \geq |B|$, and so $|A| = |B|$.
Notice that since f is injective, it maps $|A|$ elements to $|A| = |B|$ distinct elements.
So f maps to every element in B, and f is surjective and thus bijective.
So a bijection must exist. 😊