MAC2313
Notes

Oliver Deng

Contents

Chapter 1

Geometry of Space & Vector Functions

Forgot to take notes on the first 3 lectures but if you're using this for review you should be fine.

1.1 3-Dimensional Rectangular System

1.2 Vectors

1.3 Dot Product

The dot product combines two vectors to produce a scalar.

1.4 Cross Product

The cross product combines two vectors to produce a vector.

Definition 1.4.1: Cross Product

Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the **cross product** $\vec{u} \times \vec{v}$ is defined by

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The major/minor method to compute the cross product is as follows:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \hat{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

Taking the determinant of each smaller matrix, we get the final end vector:

$$\vec{u} \times \vec{v} = \hat{i}(v_2 w_3 - v_3 w_2) - \hat{j}(v_l w_3 - v_3 w_l) + \hat{k}(v_1 w_2 - v_2 w_1)$$

1.4.1 Geometric Meaning of the Cross Product

The cross product gives a vector orthogonal to both of the original vectors.

Theorem 1.4.1

The vector $\vec{u} = \vec{v}$ is **orthogonal** to both $\hat{u}$ and $\hat{v}$.

The magnitude of the cross product is the area of the parallelogram formed by the two vectors.

Theorem 1.4.2
If θ is the angle between the nonzero vectors $\hat{u}$ and $\hat{u}$, then

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$$

1.4.2 Right Hand Rule

Let the fingers of your right hand curl from $\vec{u}$ to $\vec{v}$ and the direction of $\vec{u} \times \vec{v}$ is the direction of your thumb.

1.4.3 Co-linearity

Two nonzero vectors are **co-linear** if they are parallel. $\vec{u}$ and $\vec{v}$ are parallel $\iff \vec{u} \times \vec{u} = \vec{0}$

1.4.4 Algebraic Properties

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and c is a scalar, then

1. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$
2. $\vec{u} \times \vec{u} = \vec{0}$
3. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
4. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
5. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

1.4.5 Triple Product

The volume of a parallel piped is determined by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

1.4.6 Co-planar Vectors

Definition 1.4.2

The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are **co-planar** if they lie in the same plane. Therefore, $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

1.5 Lines & Planes

1.5.1 Lines

To define a L line in $\mathbb{R}$ we need a point $P_0(x_0, y_0, z_0)$ and a vector $\vec{v} = \langle a, b, c \rangle$ parallel to that line. Using P and $\vec{v}$, we can define a vector $\vec{r}$ on the line to be

$$\vec{r} = \vec{r_0} + \vec{v}t = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \tag{1.5.1}$$

where $\vec{r_0}$ is the vector from the origin to P_0 and t is a scalar parameter.

Note

A line segment from (x_0, y_0, z_0) to (x_1, y_1, z_1) can be parameter iced as

$$\vec{r}(t) = (1 - t)\langle x_0, y_0, z_0 \rangle + t\langle x_1, y_1, z_1 \rangle$$

Note

$\vec{v}$ describes the direction of L, and the components a, b, and c are called **direction numbers**.

1.5.2 Parametric Equations for a Line

Referencing eq.(1.5.1), we can express our line in terms of t for each dimension.

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

1.5.3 Symmetric Equations for a Line

If none of the direction numbers are 0, we can solve for t and obtain the following:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

1.5.4 Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n} = \langle a, b, c\rangle$ that is **orthogonal** to the plane.

1.5.5 Vector Equation of a Plane

Let $P(x, y, z)$ be any point on the plane. We can then form the vector $\vec{P_0P}$, and we know that if we dot it with the orthogonal vector $\vec{n}$ we'll get 0, giving us this equation:

$$\vec{P_0P} \cdot \vec{n} = 0 \tag{1.5.2}$$

1.5.6 Scalar Equation of a Plane

Expanding the LHS of the equation, the scalar equation of the plane is given by the following equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.5.3}$$

Note

Remember, the coefficients a, b, and c are components of the vector that is orthogonal to the plane.

1.5.7 Linear Equation of a Plane

Rearranging eq.(1.5.3) and considering all constants to be grouped into one constant d, the linear equation of a plane is given by:

$$ax + by + cz + d = 0 \tag{1.5.4}$$

1.5.8 Parallel and Orthogonal Planes

Two planes are said to be **parallel** if their normal vectors are parallel: $(\vec{n_1} = c\vec{n_2})$.
Two planes are said to be **orthogonal** if their normal vectors are orthogonal $(\vec{n_1} \cdot \vec{n_2} = 0)$.
If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the **acute angle** between their normal vectors, which can be found with the usual equations involving the dot product between two vectors.
In addition, remember that the straight line of the intersection is $\perp$ to both $\vec{n_1}$ and $\vec{n_2}$.

1.5.9 Distances

The distance D from any point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{1.5.5}$$

1.6 Cylinders and Quadratic Surfaces

I'm lowkey too lazy to make LaTeX graphics for all the surfaces, but just use common sense when graphing.

Definition 1.6.1: Cylinders

Given a curve C in a plane P and a line L not parallel to P, a **cylinder** is the surface consisting of all lines parallel to L and passing through C.

1.6.1 Surfaces in 3D

The graph of a 3-variable equation which can be written in the form $F(x, y, z) = 0$ or $z = f(x, y)$ is a **surface** in 3D. One technique for graphing a surface is to graph **traces** (intersections of the surface with the plane parallel to the coordinate planes).

1.6.2 Quadric Surfaces

A **quadric surface** is a 3D surface whose equation is of the second degree. The general equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + hy + Iz + J = 0$$

given that $A^2 + B^2 + C^2 + D^2 + E^2 + F^2 \neq 0$. With rotation and translation, we can simplify down to 2 distinct types:

1. $$Ax^2 + By^2 + Cz^2 + J = 0$$
   (all three quadratic terms; no linear terms)
2. $$Ax^2 + By^2 + Iz = 0$$
   (only two quadratic terms plus one linear term)

We already know of two surfaces: planes (which are **not** quadric) and spheres.
There are 6 more you need to know:

1. Ellipsoid
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
2. Hyperboloid of one sheet
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
3. Hyperboloid of two sheets
   $$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
4. Elliptic cone
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$
5. Elliptic paraboloid
   $$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$
6. Hyperbolic paraboloid
   $$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

1.7 Vector-Valued Functions

Definition 1.7.1: Vector-Valued Functions

A **vector-valued** functions is a function of the form

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

where the component functions $f(t)$, $g(t)$, $h(t)$ are real-valued functions of the parameter t.

The **domain** of $\vec{r}(t)$ is the intersection of the domains of each of the individual component functions.

Definition 1.7.2: Space Curve

A **space curve** is the set of all ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$.

To graph space curves, plug in different values for t, plot the vectors, and sketch the curve along those points.

1.7.1 Limits

The **limit** of a vector-valued function $\vec{r}(t)$ is defined as

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$$

provided the limits of the component functions exist.

1.7.2 Continuity

$\vec{r}(t)$ is **continuous** at a if

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

or

$$\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle = \langle f(a), g(a), h(a)\rangle$$

1.7.3 Derivatives

Definition 1.7.3

The **derivative** $\vec{r}\,'$ of a vector-valued function $\vec{r}$ is defined by

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

if the limit exists.

Theorem 1.7.1

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, g, and h are differentiable functions, then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Definition 1.7.4: Unit Tangent Vector

We define the **unit tangent vector** to the curve $\vec{r}(t)$ by

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Definition 1.7.5: Smoothness

A curve $\vec{r}(t)$ is **smooth** on an interval I if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$ on I. A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piece-wise smooth**.

Note

If a curve has a **cusp** at a point, then $\vec{r}\,'(t) = \vec{0}$ at that point. However, the converse is not true; it may happen that $\vec{r}\,'(t) = \vec{0}$ at a point that is not a cusp.

1.7.4 Differentiation Rules

The basic rules of differentiation apply; however, note that the product rule also applies to dot products and cross products (remember that order matters in a cross product).

1.7.5 Integrals

An **antiderivative** of the vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a function $\vec{R}(t) = \langle F(t), G(t), H(t)\rangle$ such that

$$\vec{R}\,'(t) = \vec{r}(t)$$

The **indefinite integral** of $\vec{r}(t)$ is defined by

$$\int \vec{r}(t)dt = \langle \int f(t)dt, \int g(t)dt, \int h(t)dt\rangle = \langle F(t), G(t), H(t)\rangle + \vec{C}$$

The **definite integral** of $\vec{r}(t)$ over $[a, b]$ is defined to be

$$\int_a^b \vec{r}(t)dt = \langle F(b) - F(a), G(a) - G(b), H(a) - H(b)\rangle$$

1.8 Arc Length and Curvature

Recall that the length of a curve given by parametric equations is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

The length of a curve in 3D is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt = \int_a^b |\vec{r}\,'(t)|dt$$

1.8.1 Arc Length Function

Suppose that C is a piecewise smooth curve given by a vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t)\langle$, $a \leq t \leq b$. The **arc length function** is defined by

$$s(t) = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}du \tag{1.8.1}$$

which tells us how far we've gone after time t. The Fundamental Theorem of Calculus tells us that s is differentiable function of t and

$$\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = |\vec{r}\,'(t)| \tag{1.8.2}$$

1.8.2 Reparameterization of a Curve With Respect to Arc Length

It is often useful to parameterize a curve with respect to arc length since arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. Basically, we want to know the position after we've gone a certain length, meaning that our input is s. In order to do this, there are two steps:

1. Solve for t as a function of s
2. Reparametrize the curve in terms of s by substituting for t: $\vec{r} = \vec{r}(t(s))$

1.8.3 Curvature

Remember that the unit tangent vector $\hat{T}$ points in the direction of motion of the curve. Since it's a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The magnitude of the rate of change of this vector with respect to length along the curve gives information about how quickly the curve is changing direction. This quantity is known as **curvature**.

Definition 1.8.1: Curvature

The **curvature** of a curve is

$$\kappa = |\frac{d\hat{T}}{ds}|$$

which is the rate of change of the direction with respect to distance

In general, the formal definition of curvature is not easy for calculations. We use the following three alternate formulas:

1. $$\kappa(t) = \frac{|\hat{T}(t)|}{|\vec{r}\,'(t)|}$$
2. $$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$
3. This one can be used when given a 2-D function
$$\kappa(t) = \frac{|f''(t)|}{(1 + f'(t)^2)^{\frac{3}{2}}}$$

1.9 Motion in Space

Definition 1.9.1: Principal Unit Tangent Vector

Let C be a smooth curve represented by $\vec{r}(t)$. If $\hat{T}'(t) \neq \vec{0}$, then the **principal unit normal vector** is defined by

$$\hat{N}(t) = \frac{\vec{\hat{T}}\,'(t)}{|\vec{\hat{T}}\,'(t)|}$$

This is of particular importance because it points in the direction that the curve is turning.

Note

The third direction perpendicular to $\hat{T}$ and $\hat{N}$ is called the **unit binormal vector** $\hat{B} = \hat{T} \times \hat{N}$.

1.9.1 Motion Along a Curve

- **Velocity** $= \vec{v}(t) = \vec{r}\,'(t)$
- **Speed** $= |\vec{v}(t)| = |\vec{r}\,'(t)|$
- **Velocity** $= \vec{a}(t) = \vec{r}\,''(t)$

1.9.2 Components of Acceleration

We can use $\hat{T}$ and $\hat{N}$ to gain insight into acceleration; namely if we change the speed of an object, then we accelerate the object in the direction of $\hat{T}$, and if we change the direction of the object, then we accelerate the object in the direction of $\hat{N}$. In addition, we can represent $\vec{a}$ as a linear combination of $\hat{T}$ and $\hat{N}$, so then it follows that $\vec{a}$ lies in the plane determined by the two normal vectors.

Theorem 1.9.1

If $\vec{r}(t)$ is the position vector for a smooth curve C, then the **tangential vector** a_T and **normal vector** a_N of acceleration are as follows:

$$a_T = |\vec{v}|' = \vec{a} \cdot \hat{T} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$$
$$a_N = |\vec{v}||\hat{T}'| = \vec{a} \cdot \hat{N} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

Chapter 2

Differentiation

2.1 Functions of Several Variables

Definition 2.1.1

A function of two variables $z = f(x, y)$ assigns each ordered pair (x, y) in a subset D of $\mathbb{R}^2$ a unique number z.

The graph of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and for which (x, y) is in the domain of f. The graph can be interpreted as a surface in space.

Note

The projection of the graph $z = f(x, y)$ onto the xy-plane is D, the domain of f.

2.1.1 Level Curves

Definition 2.1.2: Level Curve

Given a function $f(x, y)$ and a number k in the range of f, a **level curve** (or contour curve) of f for the value k is defined to be the set of points satisfying the equation

$$f(x, y) = k$$

Note

A collection of contour curves of f is called a **contour map**.

2.1.2 Higher Dimensions

Obviously, we can extend all of this to higher dimensions. A function $w = f(x, y, z)$ of 3 independent variables lies in $\mathbb{R}^3$, a function of 4 independent variable lies in $\mathbb{R}^4$, etc. This means that we can have **level surfaces** when $w = f(x, y, z) = k$. Of course, however, it is much harder to visualize these.

2.2 Limits and Continuity

2.2.1 Limits

The normal intuitive version of limits breaks down with 2 independent variables, as there are infinitely many paths to approach a point. Instead, we use the following limit laws:

1. $$\lim_{(x,y)\to(a,b)} c = c$$

2.
$$\lim_{(x,y)\to(a,b)} x = a$$

and

$$\lim_{(x,y)\to(a,b)} y = b$$

3.
$$\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$$

4.
$$\lim_{(x,y)\to(a,b)} [cf(x,y)] = cL$$

5.
$$\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$$

6.
$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$$

for $M \neq 0$

7.
$$\lim_{(x,y)\to(a,b)} [f(x,y)]^n = L^n$$

for an integer $n > 0$

8.
$$\lim_{(x,y)\to(a,b)} [f(x,y)]^{\frac{1}{n}} = L^{\frac{1}{n}}$$

for an integer $n > 0$ (assuming $L > 0$ if n is even)

where $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{(x,y)\to(a,b)} g(x,y) = M$, and c is a constant.

Note

Limits at $(0,0)$ may be easier to evaluate by first converting to **polar coordinates**. Remember that $(x,y) \to (0,0) \implies r \to 0$.

Example 2.2.1

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x^2+2y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(2r^2)}{r^2}$$

Applying L'Hôpital's rule:

$$\lim_{r\to 0} 2\cos(2r^2) = 2$$

2.2.2 Continuity

Definition 2.2.1

A function $f(x,y)$ is **continuous** at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say that f is continuous on D if f is continuous at every point (a,b) in D.

We can also show that a function is discontinuous if the limit of the function approaches different values from different paths, since that means the limit doesn't exist, implying that f is not continuous.

Note

By using the limit laws previously discussed, we can prove that all polynomials are continuous on $\mathbb{R}^2$ and all rational functions are continuous on their domains. Thus, we can just substitute directly to find the limit of a polynomial.

Note

The basic properties of continuous functions of a single variable carry over to multivariable functions; that is the sum, difference, product, quotient, and composition of continuous functions are continuous.

Note

Naturally, the ideas and techniques used for two variables carries over to functions with more than two variables given a few changes.

2.3 Partial Derivatives

The easiest way to remember a partial derivative is taking the derivative of the function with respect to one variable while treating the other variable like a constant.

Definition 2.3.1: Partial Derivatives

If $z = f(x, y)$, the **partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Other notations for a partial derivative include $f_x = \dfrac{\partial f}{\partial x} = D_x f$

Finding the tangent line to the curve of the intersection between a function f and a plane fixed on an axis is a matter of finding the derivative with respect to the other axis (i.e. if the plane is $x = c$ then find the partial derivative with respect to y) and plugging in points.

2.3.1 Implicit Differentiation

Example 2.3.1 (Find $\frac{\partial z}{\partial x}$ if $z^2 + xz + y^2 = 3xy + z$)

$$2z\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} = 3y + \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x}(2z + x - 1) = 3y - z$$

$$\frac{\partial z}{\partial x} = \frac{3y - z}{2z + x - 1}$$

2.3.2 Higher Order and Mixed Partial Derivatives

$f(x, y)$ has the following **second partial derivatives**:

1. Differentiating twice w.r.t. x:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = f_{xx}$$

2. Differentiating twice w.r.t. y:
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = f_{yy}$$

3. Differentiating w.r.t. x and then w.r.t. y:
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = f_{xy}$$

4. Differentiating w.r.t. y and then w.r.t. x:
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = f_{yx}$$

The last two cases are called **mixed partial derivatives**. An interesting result concerning mixed partial derivatives is given by the next theorem.

> **Theorem 2.3.1** Clairaut's Theorem
>
> Assume that f is defined on an open set D. If f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$ on D.

2.4 Tangent Planes and Linear Approximations

2.4.1 Tangent Planes

Let $z = f(x, y)$ be a function near the point (a, b). Assume that $f_x(x, y)$ and $f_y(x, y)$ exist.

1. f is **differentiable** at (a, b) if it is locally approximately linear (a more formal definition will be given in the next subsection 2.4.2).

2. In this case, the **tangent plane** to the graph z at $P(a, b, f(a, b))$ is the plane with the equation
$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$
which is the unique plane containing the tangent lines to the two traces of the x-axis and y-axis through P.

2.4.2 Differentiability

The formal definition of differentiability for a function of two variables is hard to apply, so instead we use the more user-friendly theorem that follows:

> **Theorem 2.4.1** Conditions for Differentiability
>
> If the partial derivatives f_x and f_y exist near (a, b) and both are continuous at (a, b), then f is differentiable at (a, b).

> **Theorem 2.4.2** Differentiability Implies Continuity
>
> If a function f is differentiable at (a, b), it is continuous at (a, b).

This also also means that if f is not continuous at (a, b), then f is not differentiable at (a, b).

> **Note**
>
> There is an edge case where the partial derivatives of a function exist at a point, but where the function itself is not differentiable at that point.

2.4.3 Linear Approximations and Differentials

By definition, if $f(x, y)$ is differentiable at (a, b), then it is locally linear and the **linear approximation** is

$$f(x, y) \approx L(x, y)$$

for (x, y) near (a, b) where

$$L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

The linear approximation is often expressed in terms of **differentials**:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

(this form is called the total differential). To summarize, we can write

$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)\Delta x + \frac{\partial f}{\partial y}(a, b)\Delta y \approx df + f(a, b)$$

2.5 Chain Rule

2.5.1 Single Independent Variable Case

If $z = f(x, y)$ is a differentiable function of x and y, and both x and y are differentiable functions of t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 2.5.1 ($z = f(x, y) = 4x^2 + 3y^2, x = \sin(t), y = \cos(t)$)

$$\begin{aligned}\frac{dz}{dt} &= 8x(\cos(t)) + 6y(-\sin(t)) \\ &= 8\sin(t)\cos(t) - 6\cos(t)\sin(t) \\ &= 2\sin(t)\cos(t)\end{aligned}$$

This generalizes to any number of variables in a function. For example, if $w = f(x, y, z)$ and all x, y, and z are functions of t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

2.5.2 Multivariable Case

If $z = f(x, y)$ is a differentiable function of x and y, and both $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of two variables s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Note

It's easier to remember the chain rule if you use a tree diagram.

Note

When a function is only dependent on one function, then the regular $\frac{dy}{dz}$ notation is used, but when it's dependent on more than one, then the partial $\frac{\partial z}{\partial x}$ notation is used.

2.5.3 Implicit Differentiation

If $F(x, y) = 0$ defines a differentiable function $y = f(x)$ implicitly, then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$
$$\frac{\partial F}{\partial x} = -\frac{F_x}{F_y}$$

Note

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad \text{provided } F_y \neq 0$$

Example 2.5.2 (Let $x^3 + y^3 = \tan(xy)$. Find $\frac{dy}{dx}$.)

$$\begin{aligned} F(x, y) &= x^3 + y^3 - \tan(xy) \\ F_x &= 3x^2 - y\sec^2(xy) \\ F_y &= 3y^2 - x\sec^2(xy) \\ \frac{dy}{dx} &= -\frac{3x^2 - y\sec^2(xy)}{3y^2 - x\sec^2(xy)} \end{aligned}$$

2.6 Directional Derivatives and Gradients

We know that f_x gives the rate of change in the x direction, and similarly with f_y and y, and now we want to find the rate of change in the direction of any arbitrary vector $\hat{u} = \langle u_1, u_2\rangle$.

Definition 2.6.1: Directional Derivative

The **directional derivative** of f at (a, b) in the direction of unit vector $\hat{u} = \langle u_1, u_2\rangle$ is

$$D_{\hat{u}} f(a, b) = \lim_{h\to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a, b)}{h}$$

Note

If $\hat{u} = \hat{i} = \langle 1, 0\rangle$, then $D_{\hat{i}} f = \frac{\partial f}{\partial x}$ and if $\hat{u} = \hat{j} = \langle 0, 1\rangle$ then $D_{\hat{j}} f = \frac{\partial f}{\partial y}$

Geometrically, if the vertical plane that passes through the point $(a, b, f(a, b))$ in the direction $\hat{u}$ intersect the surface $z = f(x, y)$ in a curve C, then the rate of change of f in the direction $\hat{u}$ is the slope of the tangent line at $(a, b, f(a, b))$ of curve C in the direction of $\hat{u}$.

Unfortunately, the definition of the directional derivative does not provide a user-friendly means of computing the derivative. This is remedied with the next theorem:

Theorem 2.6.1 Directional Derivatives

$$D_{\hat{u}} f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2$$

Note

When the direction is given by a vector whose length is not 1, we must scale it to be a unit vector first

Example 2.6.1 (Find the directional derivative of $f(x, y) = y^2 \ln(x)$ at the point $(1, 4)$ in the direction of $\langle 1, -1\rangle$.)

First we must scale the directional vector

$$\hat{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$$

Then we find the directional derivative

$$\begin{aligned} f_x &= \frac{y^2}{x} \\ f_x(1,4) &= 14 \\ f_y &= 2y\ln(x) \\ f_y(1,4) &= 0 \\ D_{\hat{u}}f(1,4) &= 16(\frac{1}{\sqrt{2}}) \end{aligned}$$

We can also write any unit vector $\hat{u} = \langle \cos(\theta), \sin(\theta)\rangle$, where θ is the angle the unit vector $\hat{u}$ makes with the positive x-axis. Then

$$D_{\hat{u}}f(a, b) = f_x(a, b)\cos(\theta) + f_y(a, b)\sin(\theta)$$

2.6.1 Gradient Vectors

We can also write the computation formula for the direction derivative as a dot product.

$$D_{\hat{u}}f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2 = \langle f_x(a, b), f_y(a, b)\rangle \cdot \langle u_1, u_2\rangle$$

Definition 2.6.2: Gradient

The **gradient** of f is the vector

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

This is quite easy to extend to multiple dimensions. For example, in 3 dimensions,

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

The formula

$$D_{\hat{u}}f(x, y) = \nabla f(x, y) \cdot \hat{u}$$

not only tells us how to compute the directional derivative, but also tells us in what direction $\hat{u}$ the directional derivative is the greatest and the smallest. Given a differentiable function $f(x, y)$, fix a point (a, b).

$$\begin{aligned} D_{\hat{u}}f(x, y) &= \nabla f(x, y) \cdot \hat{u} \\ &= |\nabla f(a, b)||\hat{u}|\cos(\theta) \quad \text{where } \theta \text{ is the angle between the gradient and } \hat{u} \\ &= |\nabla f(a, b)|\cos(\theta) \end{aligned}$$

Therefore,

$$-|\nabla f(a, b)| \le D_{\hat{u}}f(a, b) \le |\nabla f(a, b)|$$

Theorem 2.6.2 Let f be differentiable at the point $P(a, b)$ with $\nabla f(a, b) \neq \vec{0}$.

- f has its maximum rate of increase at P in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a,b)|$.

- f has its maximum rate of decrease at P in the direction of the gradient $-\nabla f(a,b)$. The rate of change in this direction is $-|\nabla f(a,b)|$.
- $D_{\hat{u}} f(a,b) = 0$ in any direction orthogonal to $\nabla f(a,b)$.

2.7 Gradients and Tangent Planes

We observe that

1. The function does not change at (a,b) in any direction **orthogonal** to the gradient $\nabla f(a,b)$.
2. The curve $f(x,y) = k$ is a level curve, on which function values are constant.

We can combine these two observations:

Theorem 2.7.1 Let f be a differentiable function at (a,b), and let $\nabla f(a,b) \neq 0$.
The line tangent to the level curve of f at $(a,b) \perp \nabla f(a,b)$.

2.7.1 Tangent Planes

A surface in $\mathbb{R}^3$ may be defined in at least two different ways:

- **Explicitly** in the form $z = f(x,y)$
- **Implicitly** in the form $F(x,y,z) = k$

We already know the equation for a tangent plane for an explicit function, but now we define the tangent plane for an implicit function:

Theorem 2.7.2 For a surface $F(x,y,z) = k$, the equation of the tangent plane at (a,b,c) is

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0 \tag{2.7.1}$$

which comes from the fact that

$$\nabla F(a,b,c) \cdot \langle x-a, y-b, z-c \rangle = 0$$

Note

A surface $z = f(x,y)$ can be rewritten as $F(x,y,z) = f(x,y) - z = 0$ which lets us rewrite eq.(2.7.1) as

$$\frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b) + f(a,b) = z$$

which is a result we showed in section 2.4.

Definition 2.7.1: Normal Line

Let $P(a,b,c)$ be a point on the surface $F(x,y,z) = 0$ with $\nabla F(a,b,c) \neq 0$. The line through the point P having the direction $\nabla F(a,b,c)$ is called the **normal line** to the surface at P.

Example 2.7.1 (Find equations of the tangent plane and normal line to the surface

$$f(x,y,z) = x^2 + y^2 + z - 9 = 0$$

at the point $(1,2,4)$.)

Recall

$$F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0$$
$$F_x=2x$$
$$F_y=2y$$
$$F_z=1$$

Plugging in initial conditions

$$2(x-1)+4(y-2)+(z-4)=0$$

Simplifying, we get our tangent plane

$$2x+4y+z-14=0$$

Our normal line would be represented by the equations

$$x=2t+1$$
$$y=4t+2$$
$$z=t+4$$

2.8 Maximum and Minimum Values

Definition 2.8.1: Local Extremum

A function $f(x,y)$ has a **local maximum value** at (a,b) if $f(x,y)\le f(a,b)$ for all domain points (x,y) near (a,b).
A function $f(x,y)$ has a **local minimum value** at (a,b) if $f(x,y)\ge f(a,b)$ for all domain points (x,y) near (a,b).

Definition 2.8.2: Critical Point

An interior point (a,b) in the domain of f is a **critical point** of f if either

1. $f_x(a,b)=f_y(a,b)=0$, or $\nabla f=\vec{0}$
2. f_x or f_y does not exist at (a,b)

Example 2.8.1 (Find all the critical points of

$$f(x,y)=x^4-2x^2+y^2-4y+6$$

)

$$f_x=4x^3-4x=0$$
$$4x(x-1)(x+1)=0$$
$$x=0,1,-1$$
$$f_y=2y-4=0$$
$$y=2$$

Our critical points are $(0,2)$, $(1,2)$, and $(-1,2)$.

Theorem 2.8.1
If f has a local max or min value at (a,b) and f_x and f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$

Definition 2.8.3: Saddle Point

A function f has a **saddle point** at a critical point (a,b) if $f_x(a,b) = f_y(a,b) = 0$ but f does not have a local extremum at (a,b).

2.8.1 Second Derivative Test

If we want to classify critical points, we need to take the second derivative. Suppose $f(x,y)$ has continuous partial derivatives for all points near (a,b) and $f_x(a,b) = f_y(a,b) = 0$. We define the **discriminant** to be

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

1. If $D > 0$ and $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$, then $f(a,b)$ is a local max
2. If $D > 0$ and $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$, then $f(a,b)$ is a local min
3. If $D < 0$, then f has a saddle point at (a,b)
4. If $D = 0$, then the test is inconclusive

2.8.2 Absolute Extrema

Definition 2.8.4: Absolute Extrema

f has an **absolute maximum** at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) in the domain of f.
f has an **absolute minimum** at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) in the domain of f.

Theorem 2.8.2 Extreme Value Theorem
Let $f(x,y)$ be a continuous function on a closed, bounded set D. Then

1. f has both a max and min value on D
2. The extreme values occur either at critical points in the interior of D or at points on the boundary of D

Note
Make sure to check ALL extreme endpoints, like corners, max x and y values, etc.

2.9 Lagrange Multipliers

We want to optimize $f(x,y)$ subject to a constraint $g(x,y) = c$. Basically, we want to find the maximum value k for a level curve, and this will happen when they are tangent. Two curves have a common perpendicular line if they are tangent at that point. We know that ∇f is perpendicular to its level curves and ∇g is perpendicular to the constraint curve.

Theorem 2.9.1 Lagrange's Theorem
Let f and g have continuous first partial derivatives such that f has an extremum at a point (a,b) on the smooth constraint curve $g(x,y) = c$. If $\nabla g(a,b) \neq 0$, then there is a real number λ such that

$$\nabla f(a,b) = \lambda \nabla g(a,b)$$

where the scalar λ is called a **Lagrange multiplier**.

2.9.1 Method of Lagrange Multipliers

Assume that the level curve $g(x,y)=c$ is bounded and $\nabla g(x,y) \neq \vec{0}$ on the curve $g(x,y)=c$. To find the max and min values of $f(x,y)$ subject to the constraint $g(x,y)=c$:

1. Find the values of x, y, and λ by solving the system of equations
$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = c \end{cases}$$
2. Evaluate f at the points (x,y) found in the first step; the largest and smallest values are the max and min values

Example 2.9.1 (Find the max and min values of
$$f(x,y) = x^2 - y^2 - 1$$
subject to
$$x^2 + y^2 = 1$$
)

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ \nabla f &= \langle 2x, -2y \rangle \\ \nabla g &= \langle 2x, 2y \rangle \\ &\begin{cases} 2x = \lambda 2x \\ 2y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases} \\ 2x - \lambda 2x &= 0 \\ 2x(1-\lambda) &= 0 \\ x &= 0 \\ \lambda &= 0 \end{aligned}$$

If $x = 0$
$$\begin{aligned} 0^2 + y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

If $\lambda = 1$
$$\begin{aligned} -2y &= 2y \\ y &= 0 \\ x^2 + 0^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Our points are $(0,1)$, $(0,-1)$, $(1,0)$, and $(-1,0)$
$$\begin{aligned} f(0,1) &= -2 \\ f(0,-1) &= -2 \\ f(1,0) &= 0 \\ f(-1,0) &= 0 \end{aligned}$$

Our max is 0 and our min is -2

Note

This method doesn't work well for constraint regions like a square or a triangle because there isn't one equation to represent the constraint region.

Note

Remember that Lagrange multipliers find values on the curve $g(x, y) = c$; if we want to find max/min values within the constraint, then we need to find critical points on the inside.

2.9.2 More Constraints

If we have another constraint, additional Lagrange multipliers are used. For example, if we have two constraints, we solve the system of equations

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = c_1 \\ h(x,y,z) = c_2 \end{cases}$$

Chapter 3

Integration

3.1 Double Integrals

3.1.1 Double Integrals Over Rectangles

Consider a continuous function $f(x, y) \geq 0$ on a rectangular region $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d]$. The volume under the surface can be approximated by

$$V \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i, y_j)\Delta x \Delta y$$

and provided that the limit exists, the actual volume is

$$V = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i, y_j)\Delta x \Delta y$$

Definition 3.1.1: Double Integral

The **double integral** of $f(x, y)$ over a rectangle R is defined as

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy$$

and if the limit exists, we say that f is **integrable** over R. Also, the right side of the equality is an **iterated integral** and we work our way from the inside out.

Example 3.1.1 (

$$\int_1^9 \int_1^4 \frac{1}{\sqrt{xy}}dxdy$$

)

$$
\begin{aligned}
\int_1^9 \int_1^4 \frac{1}{\sqrt{xy}} dxdy &= \int_1^9 (\int_1^4 x^{-\frac{1}{2}} \frac{1}{\sqrt{y}} dx) dy \\
&= \int_1^9 (2x^{\frac{1}{2}} \frac{1}{\sqrt{y}} |_1^4) dy \\
&= \int_1^9 (\frac{4}{\sqrt{y}} - \frac{2}{\sqrt{y}}) dy \\
&= 2 \int_1^9 y^{-\frac{1}{2}} dy \\
&= 4y^{\frac{1}{2}} |_1^9 \\
&= 8
\end{aligned}
$$

Theorem 3.1.1 Fubini's Theorem

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

If $f(x,y) = g(x)h(y)$ on $R = [a,b] \times [c,d]$, then

$$\iint_R g(x)h(y)dA = \int_c^d \int_a^b g(x)h(y)dxdy = \int_c^d h(y)dy \int_a^b g(x)dx$$

3.1.2 Double Integrals Over General Regions

Consider a function $f(x,y)$ and a general plane region D. We enclose this region D in a rectangle R and define a new function $F(x,y)$ on R by

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \text{ in } D \\ 0 & (x,y) \text{ not in } D \end{cases}$$

If F is integrable over R, then we define the **double integral of f over D** by

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA$$

This is not actually how we calculate double integrals in practice, though. Instead, we classify D into two types:

1. $D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

2. $D = \{(x,y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$

3.1.3 Changing the Order of Integration

Sometimes changing the order of integration in the iterated integral may simplify the evaluation process. It is useful to use the limits in the original integral to first sketch the region D.

3.1.4 Area of Regions

We can find the area of a region by use of single integration, but if we change the way we think about area, we can actually use a double integral to do the same thing. Notice that the area of the region D is $A(D) = \iint 1dA$ (If you look closely enough, you're essentially setting up a regular integral but in terms of a double integral).

Example 3.1.2 (Set up a double integral that represents the area of the region bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$.)

Splitting up the region into two parts, we get

$$\iint_D 1dydx = \int_{-2}^{0} \int_{x^2}^{4x+12} dydx + \int_{0}^{3} \int_{x^2}^{-x+12} dydx$$

3.1.5 Volume of Regions Between Two Surfaces

We can find the volume between two continuous surfaces $z_1 = f(x, y)$ and $z_2 = g(x, y)$ with $g(x, y) \leq f(x, y)$ on a region D in the xy-plane.

$$V = \iint_D (z_1 - z_2)dA = \iint_D (f(x, y) - g(x, y))dA$$

3.2 Change of Coordinates

3.2.1 Polar Coordinates

Polar to rectangular:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Rectangular to polar:

$$r = \sqrt{x^2 + y^2} \qquad \tan(\theta) = \frac{y}{x}$$

3.2.2 Cylindrical Coordinates

This is essentially polar coordinates but with z, represented by the ordered triple (r, θ, z).

Cylindrical to rectangular:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

Rectangular to cylindrical:

$$r = \sqrt{x^2 + y^2} \qquad \tan(\theta) = \frac{y}{x} \qquad z = z$$

3.2.3 Spherical Coordinates

In $\mathbb{R}^3$, a point P is represented by the ordered triple (ρ, θ, ϕ) where

- ρ is the distance between P and the origin $(p \geq 0)$
- θ is the same angle used to describe the same location in cylindrical units $(0 \leq \theta \leq 2\pi)$
- ϕ is the angle formed between the positive z-axis and the line segment from the origin to the point $(0 \leq \phi \leq \pi)$

Spherical to rectangular:

$$x = \rho\sin(\phi)\cos(\theta) \qquad y = \rho\sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi)$$

Rectangular to spherical

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \tan(\theta) = \frac{y}{x} \qquad \cos(\phi) = \frac{z}{\rho}$$

3.3 Change of Variables in Multiple Integrals

3.3.1 Transformations in the Plane

Transformations from a region S in the uv-plane to the region R in the xy-plane are done by equations of the form

$$x = g(u,v) \qquad y = h(u,v)$$

which allows you to integrate over a rectangle, which is a lot nicer. Now, we must ask how the integral of $f(x,y)$ over R is related to the integral of $f(g(u,v),h(u,v))$ over S.

3.3.2 Change of Variables for Double Integrals

If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),h(u,v))|J(u,v)|dudv$$

where

$$J(u,v) = |\frac{\partial(x,y)}{\partial(u,v)}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the **Jacobian** of the transformation.

Note

We can change the variables from (x,y) to (u,v) in a double integral by simply replacing

$$dA = |J(u,v)|dudv$$

when we use the substitution $x = g(u,v)$ and $y = h(u,v)$ and then change the limits of integration accordingly.

To save on time, we can calculate the Jacobian by calculating $|\frac{\partial(u,v)}{\partial(x,y)}|$ and taking the reciprocal. That is,

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1}$$

Note

The Jacobian switching from rectangular to polar coordinates is always r.

3.4 Double Integrals in Polar Coordinates

3.4.1 Rectangular Polar Regions

If f is continuous in the polar region $R = \{(r,\theta)|a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

Note

$$dA = rdrd\theta$$

Example 3.4.1 (Compute $\iint_R 2xydA$, where $R = \{(r,\theta)|2 \leq r \leq 4, 0 \leq \theta \leq \frac{\pi}{2}\}$)

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_2^4 2r\cos(\theta)r\sin(\theta)rdrd\theta &= \int_0^{\frac{\pi}{2}} 2\cos(\theta)\sin(\theta)d\theta\int_2^4 r^3dr\\ &= \int_0^{\frac{\pi}{2}}\sin(2\theta)d\theta\int_2^4 r^3dr\\ &= -\frac{1}{2}\cos(2\theta)|_0^{\frac{\pi}{2}}\frac{r^4}{4}|_2^4\\ &= 60\end{aligned}$$

3.4.2 General Polar Regions

If f is continuous in the polar region $D = \{(r,\theta)|0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, then

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos(\theta), r\sin(\theta)rdrd\theta$$

3.4.3 Polar Areas and Volumes

As in rectangular coordinates, if a solid is bounded by the surface $z = f(r,\theta)$ over a polar region

$$D = \{(r,\theta)|0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$$

then the volume of the solid is

$$V = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r,\theta)rdrd\theta$$

and the area of the polar region D is

$$A = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} rdrd\theta$$

3.5 Triple Integrals

3.5.1 Rectangular Coordinates

Assume we have a box $B = \{(x,y,z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$. We can define the triple integral of $f(x,y,z)$ over B as

$$\iiint_B f(x,y,z)dV = \lim_{l\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i,y_j,z_k)\Delta i\Delta j\Delta k$$

Theorem 3.5.1 Fubini's Theorem for Triple Integrals

If f is continuous on $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)dV = \int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz$$

In fact, there are 6 different ways to set up an triple integral (they are not listed out for the sake of your eyes).

Now we expand the definition of the triple integral to compute a triple integral over a more general bounded region E in $\mathbb{R}^3$. A solid region E is type 1 if E lies between the graphs of two continuous functions of x and y :

$$E = \{(x,y,z)|(x,y)\in D, u_1(x,y) \leq z \leq u_2(x,y)\}$$

where D is the projection of E onto the xy-plane. Then

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} dz\right) dA$$

We can use this concept to evaluate when the region is a type 2 (y is between two functions of x and z) or type 3 (x is between two functions of y and z) region.

Volume of Solids by Triple Integrals

The volume of a solid E is defined as $V(E) = \iiint_E 1dV$.

3.5.2 Change of Variables for Triple Integrals

We now extend the concept of change of variables to triple integrals. Consider the transformation $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$. The **Jacobian** of the transformation is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

and

$$\iiint_R f(x,y,z)dV = \iiint_S f(g(u,v,w), h(u,v,w), k(u,v,w))|J(u,v,w)|dudvdw$$

Note

$$dV = |J(u,v,w)|dudvdw$$

The Jacobian for cylindrical coordinates is

$$dV = rdzdrd\theta$$

3.5.3 Cylindrical Coordinates

If f is continuous over the solid region

$$E = \{(r,\theta,z)|(r,\theta) \in D, u_1(r,\theta) \leq z \leq u_2(r,\theta)\}$$

where D is the projection of E onto the $r\theta$-plane, then

$$\iiint_E f(r,\theta,z)dV = \iint_D \left(\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)rdz\right) dA$$

In particular, if $D = \{(r,\theta)|g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta\}$, then we have

$$\iiint_E f(r,\theta,z)dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)rdzdrd\theta$$

Example 3.5.1 (Evaluate $\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^2 \frac{1}{1+x^2+y^2} dzdydx$)

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}dzdydx = \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{2}\frac{r}{1+r^2}dzdrd\theta$$
$$= \int_{0}^{\pi}d\theta\int_{0}^{3}\frac{r}{1+r^2}\int_{0}^{2}dz$$
$$= 2\pi\int_{0}^{10}\frac{du}{2u}$$
$$= \pi\ln|u||_{1}^{10}$$
$$= \pi\ln(10)$$

3.5.4 Spherical Coordinates

If f is continuous over the solid region

$$E = \{(\rho,\theta,\phi)|g(\theta,\phi)\leq\rho\leq h(\theta,\phi),\alpha\leq\alpha\leq\beta,a\leq\phi\leq b\}$$

then f is integrable over E and

$$\iiint_E f(\rho,\theta,\phi)dV = \int_{\alpha}^{\beta}\int_{a}^{b}\int_{g(\theta,\phi)}^{h(\theta,\phi)} f(\rho,\theta,\phi)\rho^2\sin(\phi)d\rho d\phi d\theta$$

Note

$$dV = \rho^2\sin(\phi)d\rho d\phi d\theta$$

Example 3.5.2 (Evaluate $\iiint_E 2zdV$, where E is the solid bounded by the hemisphere $z = \sqrt{4-x^2-y^2}$ and the xy-plane.)

$$\iiint_E 2zdV = \int_{0}^{2\pi}\int_{0}^{\frac{\pi}{2}}\int_{0}^{2} 2\rho\cos(\phi)\rho^2\sin(\phi)d\rho d\phi d\theta$$
$$= \int_{0}^{2\pi}d\theta\int_{0}^{\frac{\pi}{2}}2\cos(\phi)\sin(\phi)d\phi\int_{0}^{2}\rho^3 d\rho$$
$$= 2\pi\int_{0}^{\frac{\pi}{2}}\sin(2\phi)d\phi[\frac{\rho^4}{4}]|_{0}^{2}$$
$$= 8\pi[-\frac{1}{2}\cos(2\phi)]|_{0}^{\frac{\pi}{2}}$$
$$= 8\pi$$

Chapter 4

Vector Calculus

4.1 Vector Fields

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space.

Definition 4.1.1: Vector Fields

A **vector field** on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\vec{F}(x, y)$, where D is a subset of $\mathbb{R}^2$. The vector field is written as

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$$

where P and Q are the component functions of $\vec{F}$. The vector field $\vec{F}$ is continuous or differentiable on D if P and Q are continuous and differentiable on D. Similarly, we can do this for any higher dimension.

Note

A scalar field is a function $f : \mathbb{R}^3 \to \mathbb{R}$ vs. a vector field which is a function $f : \mathbb{R}^3 \to \mathbb{R}^3$.

4.1.1 Gradient Fields

One very important category of vector field is that of **gradient fields** or **conservative vector fields**. As the name implies, the field is generated by taking the gradient of a scalar-valued function.

Definition 4.1.2: Conservative Vector Fields

A vector field $\vec{F}$ is called **conservative** if there exists a differentiable function such that

$$\vec{F} = \nabla f$$

The function f is called the **potential function** for $\vec{F}$.

Note

Not every vector field is conservative!

4.1.2 Cross-Partial Property of Conservative Vector Fields

Theorem 4.1.1

Let P, Q, and R have continuous first partial derivatives. If the vector field $\vec{F} = \langle P, Q, R \rangle$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

Note

If $\vec{F} = \langle P, Q \rangle$, then only the first equality must hold.

The converse of the theorem is not necessarily true. It can only help you determine if a field is **not** conservative. However, we can add a component to allow us to tell if a vector field is conservative based on the equalities:

Theorem 4.1.2

Let P, Q, and R have continuous first partial derivatives on an **open simply connected region** D. The vector field $\vec{F} = \langle P, Q, R \rangle$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

on D.

Note

$\mathbb{R}^2$ and $\mathbb{R}^3$ are open simply connected.

Note

Any constant vector field is conservative over open simply connected region D.

To find the potential function f, integrate one component and take the derivative with respect to another component, which allows you to find the difference and then integrate that, plugging that back in to the original integration. Repeat for any other components.

4.2 Line Integrals

4.2.1 Scalar Line Integrals in the Plane

We introduce a new type of integral called a **line integral** $\int_C f(x,y)ds$ for which we integrate over a piecewise smooth curve C.

Definition 4.2.1

If f is defined on a smooth curve C, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \sum_{i=1}^{\infty} f(x_i, y_i)\Delta s$$

if this limit exists.

Theorem 4.2.1

If the curve C is given by $\vec{r} = \langle x(t), y(t) \rangle$ and $a \leq t \leq b$, then

$$\int_C f(x,y)ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|dt$$

where $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = |\vec{r}\,'(t)|dt$

Geometrically, if $f(x,y) \geq 0$, then $\int_C f(x,y)ds$ is the are under the surface $z = f(x,y)$ and above the curve C. We can see that the line integral does not depend on the parameterization $\vec{r}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same.

Now if C is a **piecewise smooth curve** (i.e. C is the union of a finite number of smooth curves), then

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \ldots + \int_{C_n} f(x,y)ds$$

4.2.2 Scalar Line Integrals in Space

It is easy to extend the previous section's results to a smooth curve in $\mathbb{R}^3$. If the curve is parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $a \leq t \leq b$, then the line integral of a continuous real-valued function $f(x,y,z)$ along C is given by

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))|\vec{r}\,'(t)|dt$$

4.2.3 Vector Line Integrals

Definition 4.2.2

Let $\vec{F}$ be a continuous vector field defined on a smooth curve C. The **line integral** of $\vec{F}$ on C is given by

$$\int_C \vec{F} \cdot \vec{T} ds$$

which is a measure of how much the vector field points in the direction of C.

Note

If the curve C is given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $a \leq t \leq b$, then

$$\begin{aligned}\int_C \vec{F} \cdot \vec{T} ds &= \int_a^b \vec{F} \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}|\vec{r}\,'(t)|dt \\ &= \int_a^b \vec{F} \cdot \vec{r}\,'(t)dt\end{aligned}$$

Let $\vec{F} = \langle P, Q, R \rangle$. For each of the component functions of $\vec{r}(t)$, we can write

$$dx = x'(t)dt \quad dy = y'(t)dt \quad dz = z'(t)dt$$

Therefore,

$$\int_C \vec{F} \cdot \vec{T} ds = \int_a^b (Px'(t) + Qy'(t) + Rz'(t))dt = \int_C Pdx + \int_C Qdy + \int_C Rdz$$

Note

$\int_C f(x,y)ds$ remains unchanged if C is given a new parameterization, but

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

Note

You will often see line integrals of a vector field expressed as

$$\int_C \vec{F} \cdot d\vec{r}$$

where $d\vec{r} = \langle dx, dy, dz \rangle$, instead of the usual $\int_a^b \vec{F} \cdot \vec{r}\,'(t)dt$.

4.3 The Fundamental Theorem of Line Integrals

Theorem 4.3.1 Fundamental Theorem for Line Integrals

Let C be a smooth curve with para motorization $\vec{r}(t)$ and $a \leq t \leq b$. Let f be a function of two or three variables whose first-order partial derivatives exist and are continuous on C (a potential function for ∇f). Then

$$\int_C \nabla f \cdot d\vec{r} = f|_{\vec{r}(a)}^{\vec{r}(b)} = f(\vec{r}(a)) - f(\vec{r}(b))$$

Note

If $\vec{F}$ is a conservative vector field, there are potential functions f such that $\vec{F} = \nabla f$. Therefore,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Example 4.3.1 (Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle 2xy + z, x^2, x \rangle$ and C is a path from $(1, -1, 2)$ to $(2, 2, 3)$.)

Our first step is to determine if $\vec{F}$ is conservative

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = 1 = \frac{\partial P}{\partial z}$$

$\vec{F}$ is conservative, and so we now find the potential function

$$f(x,y,z) = x^2y + zx$$

Now we plug in the endpoints

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= (x^2y + 2x)|_{(1,-1,2)}^{(2,2,3)} \\ &= 2^2 * 2 + 3 * 2 - (1(-1) + 2) = 13\end{aligned}$$

4.3.1 Curves

Definition 4.3.1: Closed Curve

A curve is **closed** if it intersects itself at its endpoints.

Definition 4.3.2: Simple Curve

A curve is **simple** if it intersects itself **only** at its endpoints.

4.3.2 Circulation of a Vector Field

Let $\vec{F}$ be a continuous vector field and C be a smooth, **closed** oriented curve in $\mathbb{R}^3$. The line integral $\int_C \vec{F} \cdot \vec{T} ds$ sums the tangential components of $\vec{F}$ along C. This integral is called the **circulation** of $\vec{F}$ along C. Since this integral is evaluated on a closed curve it is often represented symbolically by

$$\oint_C \vec{F} \cdot \vec{T}$$

Note

If $\vec{F}$ is a conservative vector field and C is a closed curve, then $\oint_C \vec{F} d\vec{r} = 0$

4.3.3 Independence of Path

Definition 4.3.3: Independence of Path

Let $\vec{F}$ be a continuous vector field with domain D. The vector field $\vec{F}$ is **independent of path** if

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

for any path C_1 and C_2 in D with the same initial and terminal points.

Theorem 4.3.2

If $\vec{F}$ is continuous on an open connected region, then the vector field $\vec{F}$ is independent of path if and only if $\vec{F}$ is conservative.

4.3.4 Regions

Definition 4.3.4: Open Region

A region D is **open** if it does not contain any of its boundary points.

Definition 4.3.5: Connected Region

An open region D is **connected** if any two points in D can be connected by a continuous curve lying entirely in D.

Definition 4.3.6: Simply Connected Region

An open connected region D is **simply connected** if every simple closed curve in D can be contracted to a point in D without ever leaving D.

Note

In two dimensions, a region is simply connected if it is connected and has no holes.

4.3.5 Equivalent Conditions for Line Integrals

Let $\vec{F}$ be continuous on an **open simply connected** region D. The following statements are equivalent.

- $\vec{F}$ is conservative on D
- $\int_C \vec{F} \cdot d\vec{r}$ is independent of path C in D
- $\oint \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D
- Cross-partial property holds

Note

The simply connectedness matters

4.4 Green's Theorem

Green's theorem relates a line integral around a simply closed plane curve C and a double integral over the region D enclosed by the curve. The boundary curve of D is denoted by ∂D and it is **positively oriented** if we walk along the curve in the direction of orientation with D always on the left.

Theorem 4.4.1 Green's Theorem

Let C be a piecewise smooth simple closed curve (positively oriented) in the plane and let D be the region bounded by $C = \delta D$. Let $\vec{F} = \langle P, Q \rangle$ be a vector field whose component functions have continuous partial derivatives on D. Then

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D} Pdx + Qdy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$$

Note

- Green's Theorem can be used only for a 2D vector field $\vec{F}$.
- Green's Theorem implies
$$\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA = \iint_D 0dA = 0$$
since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ is the cross-partial property.
- Green's Theorem holds for nonsimply connected regions.
$$\iint_D Q_x - P_y dA = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy$$

Example 4.4.1 (Evaluate $\oint_C (2y^3 - 3x^2)dx + (2x^3 + 5y^2)dy$, where C is the boundary (oriented clockwise) of the region determined by $y = \sqrt{4 - x^2}$ and $y = 0$.)

$$
\begin{aligned}
P &= 2y^3 - 3x^2 \\
Q &= 2x^3 + 5y^2 \\
\iint_D Q_x - P_y dA &= \iint_D (6x^2 - 6y^2) dA \\
&= \int_0^{\pi} \int_0^2 (6r^2\cos^2(\theta) - 6r^2\sin^2(\theta)) r dr d\theta \\
&= \int_0^{\pi} \cos(2\theta) d\theta \int_0^2 6r^3 dr \\
&= 0
\end{aligned}
$$

4.4.1 Calculating Area Using Green's Theorem

Theorem 4.4.2

If D is a closed plane region bounded by a piecewise smooth simple closed curve C, oriented counterclockwise. Then the area of the region D is

$$A = \int_C x dy = -\int_C y dx = \frac{1}{2}\int_C x dy - y dx$$

Recall that the area of the region D is $A = \iint_D 1 dA$. We wish to choose $\vec{F}$ such that P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Many vector fields satisfy this, and the most common of these are

1. $$\vec{F} = \langle 0, x \rangle$$

2. $$\vec{F} = \langle -y, x \rangle$$

3. $$\vec{F} = \langle -\frac{y}{2}, \frac{y}{2} \rangle$$

Then

$$A = \iint_D 1 dA = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \oint_C P dx + Q dy$$

gives us Thm. 4.4.2.

4.5 Parametric Surfaces

Definition 4.5.1: Parametric Surface

Let x, y, and z be functions of u and v that are continuous on a domain D. The set of points (x, y, z) given by

$$\vec{r}(u, v) = \langle f(u, v), g(u, v), h(u, v) \rangle$$

is called a **parametric surface**.

Example 4.5.1 (Parameterize the plane through the point $(2, 4, 7)$ and parallel to the vectors $\langle 1, 0, 3 \rangle$ and $\langle 2, 4, 8 \rangle$.)

$$
\begin{aligned}
\vec{r}(u,v) &= \langle 2,4,7\rangle + u\langle 1,0,3\rangle + v\langle 2,7,8\rangle \\
&= \langle 2+u+2v, 4+7v, 7+3u+8v\rangle
\end{aligned}
$$

4.5.1 Normal Vectors and Tangent Planes

Let S be a parametric surface given by

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

over an open region D such that x, y, and z have continuous partial derivatives on D. The **partial derivatives of $\vec{r}$** are defined as

$$
\begin{aligned}
\vec{r}_u &= \frac{\partial \vec{r}}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle \\
\vec{r}_v &= \frac{\partial \vec{r}}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle
\end{aligned}
$$

which are the rates of change of $\vec{r}$ along the direction of u or v. Each of these partial derivatives can be interpreted geometrically in terms of tangent vectors and they lie in the tangent plane to the surface at a given point. If the normal vector $\vec{r}_u \times \vec{r}_v$ is **not zero** in D, then the surface S is called **smooth** and S has a tangent plane (through the given point and perpendicular to the normal vector).

4.5.2 Area of a Parametric Surface

Let S be a smooth parametric surface defined over an open region D. If each point on the surface S corresponds to exactly one point in the domain D, then the **surface area** of S is

$$\iint_S dS = \iint_S |\vec{r}_u \times \vec{r}_v| dA$$

Note

For a surface S explicitly given by $z = k(x,y)$, we can parameterize the surface using

$$\vec{r}(x,y) = \langle x, y, k(x,y)\rangle$$

So

$$
\begin{aligned}
\vec{r}_x &= \langle 1, 0, \frac{\partial k}{\partial x}\rangle \\
\vec{r}_y &= \langle 0, 1, \frac{\partial k}{\partial y}\rangle \\
\vec{r}_x \times \vec{r}_y &= \langle -\frac{\partial k}{\partial x}, -\frac{\partial k}{\partial y}, 1\rangle \\
\text{and } |\vec{r}_x \times \vec{r}_y| &= \sqrt{(\frac{\partial k}{\partial x})^2 + (\frac{\partial k}{\partial y})^2 + 1}
\end{aligned}
$$

This implies that the surface area of S is

$$
\begin{aligned}
S &= \iint_D |\vec{r}_x \times \vec{r}_y| dA \\
&= \iint_D \sqrt{(\frac{\partial k}{\partial x})^2 + (\frac{\partial k}{\partial y})^2 + 1 dA}
\end{aligned}
$$

4.6 Surface Integrals

4.6.1 Scalar Surface Integrals

$$\iint_S f(x,y,z)dS = \iint_D f(x(u,v), y(u,v), z(u,v))|\vec{r}_u \times \vec{r}_v|dA$$

Theorem 4.6.1

Let f be a continuous real function on a surface $S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ with domain D. Assume that the tangent vectors $\vec{r}_u$ and $\vec{r}_v$ are continuous on D and that the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$ is nonzero on D. Then the **surface integral** of f over S is

$$\begin{aligned}\iint_S f(x,y,z)dS &= \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|dA \\ &= \iint_D f(\vec{r}(u,v))|\vec{n}|dA\end{aligned}$$

Note

For $f(x,y,z) = 1$, the above integral gives the surface area of S.

Note

Recall for a surface $z = k(x,y)$ that

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{k_x^2 + k_y^2 + 1}$$

and therefore

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,k(x,y))\sqrt{k_x^2 + k_y^2 + 1}dA$$

4.6.2 Orientation of a Surface

When we defined vector line integrals, the curve of integration needed an orientation. Similarly, when we define a surface integral of a vector field, we need the notion of an **oriented surface**. Unit normal vectors are used to induce an orientation to a surface S in space. If S is a smooth surface given by $\vec{r}(u,v)$, we can identify two unit vectors $\hat{n}$ and $-\hat{n}$ at any point on S, where

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

For a **closed surface** S such as a sphere, we say that S has a **positive** orientation if we choose the unit normal vector **pointing outward** from the solid. For a surface which is **not closed**, the surface is oriented in the statement of the problem by specifying which unit normal vector is to be selected.

4.6.3 Surface Integral of a Vector Field

One of the principal applications involving the vector form of a surface integral relates to the **flux integral**. It is the rate at which something passes through a surface S if $\vec{F}$ is the velocity field of the movement (in other words, it is how much a vector field points in the direction of the normal vector). To calculate the flux we integrate the component of $\vec{F}$ parallel to $\hat{n}$ at each point on the surface over S.

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n}dS$$

Note

As was the case for line integrals, we can introduce some new notation and write $\hat{n}dS = d\vec{S}$ so that $\iint_S \vec{F} \cdot \hat{n}dS = \iint_S \vec{F} \cdot d\vec{S}$.

Definition 4.6.1: Flux

Let $\vec{F}$ be a continuous vector field defined on an oriented surface S with unit normal vector $\hat{n}$. Then the **surface integral** $\vec{F}$ over S, or the **flux** of $\vec{F}$ across S, is given by

$$\iint_S \vec{F} \cdot \hat{n} dS = \iint_S \vec{F} \cdot d\vec{S}$$

Note

If S is given parametrically by $\vec{r}(u, v)$, then

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \hat{n} dS \\ &= \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| dA\end{aligned}$$

where D is the parameter domain. Thus we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

Note

If S is given explicitly as $z = k(x, y)$, then

$$\iint_S \vec{F} \cdot \hat{n} dS = \iint_D \vec{F} \cdot \langle -k_x, -k_y, 1 \rangle dA$$

4.7 Stokes' Theorem

4.7.1 Curl

Definition 4.7.1: Curl

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ where the component functions are differentiable in $\mathbb{R}^3$ is

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$$

If $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is known as **irrotational**. Also note that the curl of a vector field is another vector field.

The curl of a vector field $\vec{F}$ is the tendency for $\vec{F}$ to rotate about a given point. Suppose we place a paddle wheel into a vector field. The paddle wheel rotates the fastest when the axis of the wheel points in the direction of $\nabla \times \vec{F}$ at the given point.

Theorem 4.7.1

If $f(x, y, z)$ is a scalar-valued function which has continuous second partial derivatives, then

$$\text{curl}(\nabla f) = \vec{0}$$

Essentially, if $\vec{F}$ is conservative, then $\text{curl}(\vec{F}) = \vec{0}$ since $\vec{F} = \nabla f$. In addition, the converse is true as long as the domain is imply connected. That is, if $\text{curl}(\vec{F}) = \vec{0}$, then F is conservative.

4.7.2 3D Version of Green's Theorem

Recall Green's theorem from section 4.4. Let us modify $\vec{F} = \langle P, Q \rangle$ by considering it to be three-dimensional: $\vec{F} = \langle P(x, y), Q(x, y), 0 \rangle$. Then

$$\nabla \times \vec{F} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$$

and Green's theorem can be expressed in vector form as

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \text{curl}(\vec{F}) \cdot \hat{k} dA$$

Here D is a region in the xy-plane and $\hat{k}$ is a unit normal vector to D (or the x-axis) at every point. If D is instead an orientable surface in space, there is an obvious way to alter this equation, and it turns out to still be true:

4.7.3 Positive Orientation of a Boundary Curve

Suppose S is a smooth surface oriented by its unit normal vector $\hat{n}$. The orientation of S induces the **positive orientation** of the boundary curve ∂S. If we walk along the boundary curve in the direction of its orientation with our "head" pointing in the direction of $\hat{n}$, then the surface is always on our left.

4.7.4 Stokes' Theorem

Theorem 4.7.2 Stokes' Theorem

Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve ∂S with positive orientation. Assume that $\vec{F} = \langle P, Q, R \rangle$ is a vector field whose components have continuous partial derivatives on S. Then

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$$

where $\hat{n}$ is the unit normal vector to S determined by the orientation of S.

Note

If S is a closed surface (like a sphere), then

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS = 0$$

Note

When a problem says to verify Stokes' Theorem, that means to prove that the result of taking the line integral of the curve around the surface is the same as taking the surface integral of the curl.

4.8 Divergence Theorem

Definition 4.8.1: Divergence

The **divergence** of a vector field $\vec{F} = \langle P, Q, R \rangle$ where the component functions are differentiable is

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

If $\text{div}(\vec{F}) = 0$, then $\vec{F}$ is called **incompressible** or **source free**. Note that the divergence of a vector field is a scalar-valued function.

For a physical interpretation of divergence, we can think of the divergence of a vector field at a point as the net flux of $\vec{F}$ per unit volume through a small sphere surrounding the point. If you have a point and the vector fields are going outward, you have a positive divergence. If they're going inward, it's an negative divergence.

4.8.1 Flux Form of Green's Theorem

Theorem 4.8.1

Let D be a region for which Green's Theorem holds and ∂D be the positive oriented boundary of D. If $\vec{F} = \langle P, Q \rangle$ is a twice continuous differentiable vector field on D, then

$$\oint_{\partial D} \vec{F} \cdot \hat{n} ds = \iint_D \nabla \cdot \vec{F} dA$$

where $\hat{n}$ is the outward unit normal vector of ∂D.

Thm. 4.8.1:

$$\begin{aligned}
\oint_{\partial D} \vec{F} \cdot \hat{n} ds &= \oint_{\partial D} \langle P, Q \rangle \cdot \hat{n} ds \\
&= \oint_{\partial D} \langle -Q, P \rangle \cdot \hat{T} ds \\
&= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} dA \\
&= \iint_D \nabla \cdot \vec{F} dA
\end{aligned}$$

😄

This is also called the 2D Divergence Theorem. With minor changes this turns into another equation: the Divergence Theorem. It gives the relationship between the divergence of a vector field within a volume and the flux of the vector field across the surface of the enclosed volume.

4.8.2 The Divergence Theorem

Theorem 4.8.2 The Divergence Theorem

Suppose E is a bounded, closed region in space that has a piecewise smooth boundary $S = \partial E$ oriented outward. Let $\vec{F}$ be a vector field whose component functions have continuous first partial derivatives in E. Then

$$\oiint_{\partial E} \vec{F} \cdot \hat{n} dS = \iiint_E \text{div } \vec{F} dV$$

where $\hat{n}$ is the outward unit normal vector on S.

Theorem 4.8.3

If $\vec{F} = \langle P, Q, R \rangle$ is a vector field and P, Q, and R have continuous second partial derivatives, then

$$\text{div}(\text{curl } \vec{F}) = 0$$

Thm. 4.8.3:

$$\text{div}\langle R_y - Q_z, P_z - R_y, Q_x - P_y \rangle = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

😄

Theorem 4.8.4

div $\vec{G} = 0 \implies \vec{G} =$ curl $\vec{F}$ for some $\vec{F}$, provided that the domain is simply connected.

Note

The flux of a constant vector field across the surface of a solid is 0.

$$\begin{aligned}\oiint \vec{F} \cdot \hat{n} ds &= \iiint \text{div } \vec{F} dV \\ &= \iiint 0 dV \\ &= 0\end{aligned}$$