MHF3202
HW5

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Question 1

Lemma 0.0.1 Let $n \in \mathbb{N}$. The relation $\equiv_n$ on the set $\mathbb{Z}$ is reflexive, symmetric, and transitive.

Proof: Proved in book. ☺

Proof: Since the prime numbers $P \subset \mathbb{N}$, any relation $\equiv_p$ on $\mathbb{Z}$ is reflexive, symmetric, and transitive by lemma 0.0.1. ☺

Question 2

Proof: We seek to disprove that $\sim_k$ is not reflexive, symmetric, or transitive.
That is, we want to show that there exists some $k \in \mathbb{R}$ such that $\sim_k$ is none of those properties.
To that end, fix $k = 0$.
Then, $(a, b) \sim_0 (c, d)$ if $c = a^2 - b^2$ and $d = 2ab$.
To show that $\sim_0$ is not reflexive, take $(1, 1) \in \mathbb{R}^2$.
Then $c = 1^2 - 1^2 = 0 \neq 1$ and $d = 2(1)(1) = 2 \neq 1$, so $(1, 1) \not\sim_0 (1, 1)$.
Thus $\sim_0$ is not reflexive.
To show that $\sim_0$ is not symmetric, observe that $(1, 1) \sim_0 (0, 2)$.
Now, observe that $c = 0^2 - 2^2 = -4 \neq 1$ and $d = 2(0)(2) = 0 \neq 1$, so $(0, 2) \not\sim_0 (1, 1)$.
Thus $\sim_0$ is not symmetric.
To show that $\sim_0$ is not transitive, consider that $(1, 1) \sim_0 (2, 0)$ and $(2, 0) \sim_0 (-4, 0)$.
However, we can see that $c = 0^2 - 0^2 = 0 \neq -4$ and $d = 2(1)(1) = 2 \neq 0$, so $(1, 1) \not\sim_0 (-4, 0)$.
Thus $\sim_0$ is not transitive.
Since $0 \in \mathbb{R}$, we have that $\sim_k$ is not reflexive, symmetric, or transitive for any k. ☺

Question 3

Proof: Consider the relation $R = \{(1, 1), (2, 2), (3, 3), (5, 5), (8, 8)\}$.
Observe that for every $a \in A$, we have (a, a), i.e. aRa (even though that is not the same for the relation from B to A), so R is reflexive.
Observe that if aRa, then it is trivial to flip it to find that aRa, and so it is symmetric.
Finally, observe that it transitivity is trivial by the fact that no element is related to any other element except itself.
So equality is reflexive, symmetric, and transitive from A to B. ☺

Question 4

Lemma 0.0.2 Let $f : A \to B$ be a function from A to B, and suppose $|A| = \infty$ and $|B| < \infty$. Then f is not bijective.

Proof: Suppose f, A, B are as given.
Then $|A| = \infty > |B|$, and so f is not injective by the pigeonhole principle.
Since f is not injective, it cannot be bijective. ☺

Proof: We seek to disprove that such a bijection exists.
To that end, let A and B be the sets as given.
Observe that $B = \{0\}$, since if $x > 8$ or $x < 8$, then $48x - 3x^2 - 192 < 0$ and taking the square root of a negative number is not defined in the reals, and if $x = 8$, then $48x - 3x^2 = 0 \in \mathbb{Q}$.
Observe that $A = \{a \in \mathbb{Z} : a \geq -4\}$, since $y = (x - 1)^2 - 4,\ x \in \mathbb{R}$ attains all real numbers greater than or equal to -4.
By lemma 0.0.2, since $|A| = \infty$ and $|B| = 0 < \infty$, there cannot be any function f that is a bijection. ☺

Question 5

Proof: We seek to prove that such a bijection exists.
To that end, let A and B be the sets as given.
Observe that A is the set of all of all odd integers, and that B is the set of all even integers.

Note

Not sure if I needed to prove that but I'm on a time crunch so I'll leave it for now.

Consider the function $h(x) = x - 1$.
To prove that h is injective, suppose by way of contraposition that $h(x) = h(y)$ for some $x, y \in \mathbb{Z}$.
Then $x - 1 = y - 1$, and adding 1 to both sides, we get $x = y$.
So h is injective.
To prove that h is surjective, consider some element $b \in B$.
Since b is even, it can be written as $2k$ for some $k \in \mathbb{Z}$ by definition of even numbers.
Then, there is some number $a = 2k+1 \in \mathbb{Z}$ by addition of integers such that $h(a) = a-1 = 2k+1-1 = 2k$.
Since a is an odd integer by definition of odd numbers, $a \in A$.
So h is surjective, and thus bijective. ☺

Question 6

Corollary 0.0.1 Let $f : A \to B$ be a function from A to B, and suppose $|A| = \infty$ and $|B| < \infty$. Then there exists $b \in B$ such that there exists some $X \subseteq A$ such that $f(X) = b$ and $|X| = \infty$.

Proof: Suppose by way of contradiction that no such X exists.
Let $N = \max_{b \in B} |\{x : f(x) = b\}| < \infty$.
Then $|A| \leq |B| \cdot N < \infty$, which is a contradiction. ☺

Proof: Fix $k \in \mathbb{N}$.
Let $\{S_{k,i}\}_{i=0}^{N}$ be the set of all k-length strings of numerals.
Observe, there are at most 10^k different substrings, so $N = 10^k < \infty$.
Furthermore, since $e \in \mathbb{R} \setminus \mathbb{Q}$, it has an infinite decimal expansion, which we can express in terms of $S_{k,i}$ as follows:

$$2.S_{k,i_1} S_{k,i_2} S_{k,i_3} \ldots$$

Now, define $f : A \to B$ where $A = \mathbb{N}$ and $B = \{S_{k,i}\}_{i=0}^{N=10^k}$ where $f(n) = S_{k,i_n}$.
By corollary 0.0.1, there is an S_{k,i_0} such that there are infinitely many $n \in \mathbb{N}$ such that $f(n) = S_{k,i_0}$.
Thus the k-length S_{k,i_0} appears infinitely many times, and our claim is proven. ☺

Question 7

Proof: We seek to prove that f is bijective.
To show that f is injective, suppose by way of contraposition that there is some $x, y \in \mathbb{R}$ such that $f(x) = f(y)$.
Then, $x^3 = y^3$, and taking the cube root on both sides, we get $x = y$.
Thus, f is injective.
To show that f is surjective, take an arbitrary $b \in \mathbb{R}$.
We seek some $a \in \mathbb{R}$ such that $f(a) = b$.
We have $a^3 = b$, and taking the cube root of both sides, we have $a = \sqrt[3]{b}$.
Since $a \in \mathbb{R}$, we have that f is surjective and thus bijective.

Note

It feels scary throwing around the cube root so easily, but I'm taking you at your word about what you said in class.

☺

Question 8

Proof: We seek to prove that f is bijective.
To show that it is injective, suppose we have some $(a, b), (c, d) \in \{0, 2\} \times \mathbb{N}$ such that $(a, b) \neq (c, d)$, i.e. $a \neq c$ or $b \neq d$.

Case 1: $a \neq c$

If $a \neq c$, then either $a = 0$ or $c = 0$.

If $a = 0$, then $f(a, b) = b - ab = b - 0 = b > 0$, since $b \in \mathbb{N}$.

Then, $c = 2$ and $f(c, d) = d - cd = d - 2d = -d < 0$, since $d \in \mathbb{N}$.

Thus, $f(a, b) \neq f(c, d)$.

Similarly, if $a = 2$, then $c = 0$ and we get the same result.

Note that this does not depend on if $b = d$ or $b \neq d$.

Case 2: $b \neq d$

If $b \neq d$ and $a = c$, then either $a = c = 0$ or $a = c = 2$.

If $a = c = 0$, then $f(a, b) = b \neq d = f(c, d)$.

If $a = c = 2$, then $f(a, b) = -b \neq -d = f(c, d)$, since $b, d \in \mathbb{N}$.

Since our cases are exhaustive, we have that if any $(a, b) \neq (c, d)$, then $f(a, b) \neq f(c, d)$, and f is injective.
To show that it is surjective, take an arbitrary $e \in \mathbb{Z} \setminus \{0\}$.
We seek some $(m, n) \in \{0, 2\} \times \mathbb{N}$ for which $f(m, n) = e$.

Case 1: $e > 0$

If $e > 0$, we can take $m = 0$ which gives us $f(0, n) = n - 0n = n$.

Observe that since $e > 0$, $e \in \mathbb{N}$.

Since $n \in \mathbb{N}$, we can take $n = e$, and so we have (m, n) such that $f(m, n) = e$.

Case 2: $e < 0$

If $e < 0$, we can take $m = 2$ which gives us $f(2, n) = n - 2n = -n$.

Observe that since $e < 0$, $-e > 0 \in \mathbb{N}$.

Since $n \in \mathbb{N}$, we can take $n = -e$, and so we have (m, n) such that $f(m, n) = -n = e$.

Since our cases are exhaustive, for any $e \in \mathbb{Z} \setminus \{0\}$, we have an (m, n) such that $f(m, n) = e$, so f is surjective.
Since f is injective and surjective, it is bijective. 😊

Question 9

Proof: We claim that $A = \mathbb{Z}$ is the largest set such that $\sim$ is an equivalence relation on A.
Observe, $f(x) = x^2 - 6x + 9 = (x - 3)^2$ by factoring.
Since $x \in \mathbb{Z}$, $(x - 3)^2 \in \mathbb{Z}$ by multiplication and addition of integers.
So, the codomain of f can actually be restricted to the perfect squares, i.e. $D = \{m : n \in \mathbb{Z}, m = n^2\}$.
Thus, we redefine $f : \mathbb{Z} \to D$, $f(x) = x^2 - 6x + 9$.
Now, we prove that f is surjective.
To that end, take any $y \in D$.
We have that $\sqrt{y} \in \mathbb{Z}$ because it is a perfect square.
So, we can take $x = \sqrt{y} + 3 \in \mathbb{Z}$, and $f(x) = y$.
So f is surjective.
To prove that $\sim$ is reflexive, take any integer $a \in \mathbb{Z}$.
Notice that $a^2 \in D$, and since f is surjective there is some c such that $f(c) = a^2 = a \cdot a$.

By definition, $a \sim a$, and $\sim$ is reflexive.
To prove that $\sim$ is symmetric, suppose that $a \sim b$ for some $a, b \in \mathbb{Z}$.
By definition, $a \cdot b = f(c)$ for some c.
Since multiplication is commutative, we can write $b \cdot a = f(c)$.
Thus $b \sim a$ and $\sim$ is transitive.
To prove that $\sim$ is transitive, suppose that $a \sim b$ and $b \sim d$ for some $a, b, d \in \mathbb{Z}$.
By definition, $a \cdot b = f(c)$ for some $c \in \mathbb{Z}$, and $b \cdot d = f(e)$ for some $e \in \mathbb{Z}$.
Observe that $f(c) = j^2$ and $f(e) = k^2$ for some $j, k \in \mathbb{Z}$ respectively.
Then we can write $a \cdot b = j^2$ and $b \cdot d = k^2$.
If $a = b$ or $b = d$, then $a \cdot d = k^2$ or $a \cdot d = j^2$ respectively and transitivity is trivial.

Note

I know this isn't exhaustive. This frustrates me so much but I'm not sure how to formalize the fact that $a \cdot d$ is a square in the other case. Like, I have the concept in my head; if they're perfect squares, then their product is a perfect square. If not, then since they're related, their product is a perfect square, and thus some exponents in their prime factorization are odd and become even when multiplied by another with an odd exponent and become perfect squares.

Thus, $\sim$ is transitive.
So $\sim$ is an equivalence relation on $A = \mathbb{Z}$. 😄

Outside of the proof, I would go even further and say that the set of perfect squares is an equivalence class and the negative perfect squares are another equivalence class, since multiplying perfect squares gives another perfect square (as proved earlier in the class I think).
Actually, there are equivalence classes for numbers whose products have a prime factorization such that all of them have the same parity for each exponent, if that makes sense, like 2, 8, 18, 128, etc.
Is that a partition? I don't wanna think about that right now.