MAA4402
HW4

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Question 1: 24.2b

Let $f(z) = e^{-x}e^{-iy} = e^{-x}\left(\cos(-y) + i\sin(-y)\right)$, where $z = x + iy, x, y \in \mathbb{R}$.
Let $u(x,y) = e^{-x}\cos(-y) = e^{-x}\cos(y)$ and $v = e^{-x}\sin(-y) = -e^{-x}\sin(y)$.
Then $u_x = -e^{-x}\cos(-y) = -e^{-x}\cos(y) = v_y$ and $v_x = e^{-x}\sin(y) = -u_y$.
Since $u_x = v_y$ and $v_x = -u_y$ everywhere and since these derivatives are continuous everywhere, the conditions in the theorem are satisfied at all points in the complex plane. Thus $f'(z)$ exists everywhere and

$$f'(z) = u_x + iv_x = e^{-x}\cos(y) - ie^{-x}\sin(y).$$

For $f''(z)$, take $f'(z)$.
Let $U(x,y) = e^{-x}\cos(y)$ and $V = -e^{-x}\sin(y)$.
Then $U_x = -e^{-x}\cos(y) = V_y$ and $V_x = e^{-x}\sin(y) = -U_y$.
Since $U_x = V_y$ and $V_x = -U_y$ everywhere and since these derivatives are continuous everywhere, the conditions in the theorem are satisfied at all points in the complex plane. Thus $f''(z)$ exists everywhere and

$$f''(z) = U_x + iV_x = -e^{-x}\cos(y) + ie^{-x}\sin(y).$$

Question 2: 24.3c

Let $z = x + iy, x, y \in \mathbb{R}$.
Then $f(z) = z\,\mathrm{Im}(z) = (x + iy)(y) = xy + iy^2$.
Let $u(x,y) = xy$ an d let $v(x,y) = y^2$.
Then

$$\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases} \iff \begin{cases} y = 2y \\ 0 = -x \end{cases} \iff \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Since $u_x = v_y$ and $v_x = -u_y$ only at $z = 0$ and are continuous there, $f'(z)$ exists at $z = 0$ and $f'(0) = u(x,y) + iv(x,y) = y + 0 = 0$.

Question 3: 24.4b

Let $f(z) = e^{-\theta}\cos(\ln(r)) + ie^{-\theta}\sin(\ln(r))$.
Let $u(r,\theta) = e^{-\theta}\cos(\ln(r))$ and $v(r,\theta) = e^{-\theta}\sin(\ln(r))$.
Then $u_r = -e^{-\theta}\sin(\ln(r))\frac{1}{r} = \frac{1}{r}v_\theta$ and $v_r = e^{-\theta}\cos(\ln(r))\frac{1}{r} = -\frac{1}{r}u_\theta$.
Since $u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$ when $r > 0, 0 < \theta < 2\pi$ and are continuous there, $f'(z)$ exists in that domain and

$$\begin{aligned} f'(z) &= e^{-i\theta}\left(-e^{-\theta}\sin(\ln(r))\frac{1}{r} + ie^{-\theta}\cos(\ln(r))\frac{1}{r}\right) \\ &= i\frac{1}{z}\left(e^{-\theta}\cos(\ln(r)) + ie^{-\theta}\sin(\ln(r))\right) \\ &= i\frac{f(z)}{z}. \end{aligned}$$

Question 4: 24.5

$$\begin{aligned}
u_x &= u_r \cos(\theta) - u_\theta \frac{\sin(\theta)}{r} \\
u_y &= u_r \sin(\theta) + u_\theta \frac{\cos(\theta)}{r} \\
u_x \cos(\theta) &= u_r \cos^2(\theta) - u_\theta \sin(\theta)\cos(\theta)\frac{1}{r} \\
u_y \sin(\theta) &= u_r \sin^2(\theta) + u_\theta \cos(\theta)\sin(\theta)\frac{1}{r} \\
u_x \cos(\theta) + u_y \sin(\theta) &= u_r(\cos^2(\theta) + \sin^2(\theta)) + u_\theta \cos(\theta)\sin(\theta)\frac{1}{r} - u_\theta \cos(\theta)\sin(\theta)\frac{1}{r} \\
&= u_r
\end{aligned}$$

Now we have u_x and u_y in terms of u_r and u_θ. Now if the equations in (6) are satisfied,

$$\begin{aligned}
u_x &= u_r \cos(\theta) - u\theta \frac{\sin(\theta)}{r} = v_y \\
u_r &= \left(u_x + u_\theta \frac{\sin(\theta)}{r}\right) \frac{1}{\cos(\theta)}
\end{aligned}$$

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Question 5: 24.6

Suppose that $f'(z) = u_x + iv_x$.
Then

$$u_x + iv_x = u_x \left(\sin^2(\theta) + \cos^2(\theta)\right) + iv_x \left(\sin^2(\theta) + \cos^2(\theta)\right)$$

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Question 6: 26.5

Let $z = x + iy, x, y \in \mathbb{R}$ and $h(z) = 2x - 1 + i$
Then $\mathrm{Re}(2z - 2 + i) = \mathrm{Re}(2x + 2iy - 2 + i) = 2x + 2 = x + 1$ which, when $x > 1$ means that $h(z) > 2$.
So G is analytic in the half-plane $x > 1$.

Question 7: 26.6

Let $g(z) = \ln(r) + i\theta$, and let $D = \{z : r > 0, 0 < \theta < 2\pi\}$.
Suppose $z \in D$, where $z = x + iy, x, y \in \mathbb{R}$.