MHF3202
Exam 2

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Question 1: Q1

Let $n \in \mathbb{N}$, and let $a_n = (5n)^2$.
Then $a_n = (5n)^2 = 25n^2 = 9n^2 + 16n^2 = (3n)^2 + (4n)^2 = b_n^2 + c_n^2$ where $b_n = 3n,\ c_n = 4n$, and $b_n, c_n \in \mathbb{N}$ since multiplying two positive numbers results in another positive number.
So a_n can be written as the sum of two perfect squares, as desired.

Question 2: Q3

Proof: We prove this by way of induction.
Base Case: For $n = 2$, observe that $F_0 + F_1 + F_2 = 0 + 1 + 1 = 2 = 3 - 1 = F_4 - 1$.
So we have our base case.
Inductive Hypothesis: Assume that for some fixed $j > 1 \in \mathbb{N},\ \sum_{k=0}^{j} F_k = F_{j+2} - 1$.
Inductive Step: By the inductive hypothesis, $\sum_{k=0}^{j} F_k = F_{j+2} - 1$.
Adding F_{j+1} to both sides, we get $\sum_{k=0}^{j} F_k + F_{j+1} = F_{j+1} + F_{j+2} - 1$.
However, $F_{j+1} + F_{j+2} = F_{j+3}$ by definition of a Fibonacci number.
Substituting and also reindexing the summation, we get $\sum_{k=0}^{j+1} F_k = F_{j+3} - 1$ as desired.
So for all $j > 1,\ \sum_{k=0}^{j+1} F_k = F_{j+3} - 1$.
Therefore, by induction, our original claim is proved. 😊

Question 3: Q5

Lemma 0.0.1 Let $a, b, c \in \mathbb{Z}$. If $a \mid b$, then $a \mid bc$.

Let $a, b, c \in \mathbb{Z}$, and let $a \mid b$.
Then, $b = ad$ for some $d \in \mathbb{Z}$ by definition of divisibility.
Observe, $bc = adc = ae$ where $e = dc \in \mathbb{Z}$ by multiplication of integers.
So $a \mid bc$ by definition of divisibility.

Proof: Let $n \in \mathbb{N}$.
Then let product of three consecutive natural numbers starting from n be $m = n(n+1)(n+2)$.
First, we show that either $2 \mid n$ or $2 \mid n+1$.
If n is even, then $n = 2a$ for some $a \in \mathbb{N}$ by definition of even numbers and since $n > 0$.
Thus, $2 \mid n$ by definition of divisibility.
If n is odd, then $n = 2b + 1$ for some $b \in \mathbb{N}$ since $n > 0$.
Then $n + 1 = 2b + 2 = 2c$ for $c = b + 1 \in \mathbb{N}$ (since $b > 0$) and $2 \mid n + 1$ by definition of divisibility.
So either $2 \mid n$ or $2 \mid n + 1$.
Because of this, $2 \mid m$ by lemma 0.0.1 by replacing c in the lemma with the product of the two factors that are not either n or $n + 1$.
Now, we show that since n, $n + 1$, and $n + 2$ are 3 consecutive numbers, at least one of them must be divisible by 3.
If $3 \nmid n + 2$, then

$$n + 2 = 3d + e \tag{0.0.1}$$

where $d \in \mathbb{N}$ and e is either 1 or 2 by the division algorithm.
If $e = 2$, then subtracting 2 from both sides of equation 0.0.1 results in $n = 3d$, and $3 \mid n$ by definition of divisibility.
Similarly, if $e = 1$, then subtracting 1 from both sides of equation results in $n + 1 = 3d$, and $3 \mid n + 1$ by definition of divisibility.
So either $3 \mid n$, $3 \mid n + 1$, or $3 \mid n + 2$.
Because of this, $3 \mid m$ by lemma 0.0.1 by replacing c in the lemma with the product of the two factors which are not divisible by 3.
Since $2 \mid m$, then $m = 2k$ for some $k \in \mathbb{N}$ and m is even by definition of even numbers.
Then since $3 \mid m$, $m = 3j$ for some $j \in \mathbb{N}$.

Because m is even, then j must be even since 3 is odd and given an odd natural number, only by multiplying by an even natural number can you get an even natural number as a result.
Since j is even, we can write $m = 3(2f)$ for some $f \in \mathbb{N}$ by definition of even numbers.
So $m = 6f$, and $6 \mid m$ by definition of divisibility. 😊

Question 4: Q6

Let $a, b \in \mathbb{Z}$.
Suppose by way of contradiction that $a^2 + 4b - 2 = 0$.
Then $a^2 + 4b = 2$.
If a is odd, then a^2 is odd, and since $4b = (2)2b$ is an even number, we have a contradiction, since the sum of an odd and an even integer is odd, and 2 is not odd.
If a is even, then $a = 2c$ for some $c \in \mathbb{Z}$ by definition of even number.
By substitution, $(2c)^2 + 4b = 2$ and $4c^2 + 4b = 2$.
We can write this as $2 = 4(c^2 + b) = 4d$ for $d = c^2 + b \in \mathbb{Z}$ by the sum and multiplication of integers.
By definition of divisibility, $4 \mid 2$, which is also a contradiction.
So we have shown that for any arbitrary choice of a and b that $a^2 + 4b - 2 \neq 0$.

Question 5: Q7

We seek to disprove the original claim by direct proof, i.e. we want to show that there exists $k \in \mathbb{Z}$ such that $A_k \neq B_k$.
Consider $k = 0$.
Observe that $A_0 = \{x \in \mathbb{Z} : |x| \leq 18\} = \{-18, -17, -16, \ldots, 16, 17, 18\}$.
Also observe that $B_0 = \{y \in \mathbb{Z} : y = -2x^2 + 12x - 9, 0 \leq x \leq 6\} = \{-9, 1, 7, 9\}$.
By inspection, we can see that $18 \in A_0$ but $18 \notin B_0$, so $A_0 \neq B_0$.
Thus our original claim is disproven.

Question 6: Q9

1. The proof does not give δ as given in the hypothesis; it should have given the upper bound of 10 because then you might run into problems with δ being too big.

2. In case 9.1 the proof does not explain its choice of $\frac{1}{2}\delta$. While this doesn't necessarily invalidate the proof, it should either be explained or given as δ instead for clarity.

3. In case 9.1 the proof does not clearly explain why the set $\{(a, x) : c < x : d\}$ has infinite cardinality. We thus cannot see that S_δ has infinite cardinality; that is, unless we show that the proposed set has infinite cardinality by virtue of x being a real number.

4. Case 9.2 does not give a correct base case as the "base case" is essentially assuming the inductive hypothesis which invalidates induction. We should instead give a base $n = 0$ and find a way to iterate from there.

5. At the end of case 9.2 the proof assumes that you can take $S = D_n$. However, D_N is not necessarily a square, and so we cannot use it to concretely prove the claim. Also, it sets the equality for S instead of S_δ which is not defined. Instead, the proof should have guaranteed in some way that D_N was a square, which you can't do by assuming that if the area of a shape is a perfect square then the shape itself is a square (which the proof does). Also, you would need to set $S_\delta = D_N$ instead of S.