MAA4402
Notes

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April 23, 2025

Contents

Chapter 1

Complex Numbers

1.1 Lecture 1

Definition 1.1.1: Complex Numbers

$$z = (x, y) = x + iy \text{ where } x, y \in \mathbb{R}$$

Definition 1.1.2: Complex Addition

$$\text{Let } z_1 = (x_1, y_1), z_2 = (x_2, y_2), \text{ where } x_1, x_2, y_1, y_2 \in \mathbb{R}$$
$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) := (x_1 + x_2) + i(y_1 + y_2)$$

Definition 1.1.3: Complex Multiplication

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

1.1.1 Properties of Complex Numbers

Definition 1.1.4: Set of Complex Numbers

Let $\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\} = \{x + iy : x, y \in \mathbb{R}\}$ be the set of complex numbers

Let $z, z_1, z_2, z_3 \in \mathbb{C}$. Then

1. $$z_1 + z_2 = z_2 + z_1$$
2. $$z_1 z_2 = z_2 z_1$$
3. $$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$
4. $$z + 0 = z \text{ where } 0 = (0, 0) = 0 + 0i$$
5. $$z * 1 = z \text{ where } 1 = (1, 0) = 1 + 0i$$

6.
$$z + (-z) = 0 \text{ where } z = (x, y) = x + iy \text{ and } -z = (-x, -y) = (-x) + i(-y)$$

7.
$$\mathbb{C} \text{ is closed under addition and multiplication}$$

8.
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

9.
$$z_1(z_2 z_3) = (z_1 z_2) z_3$$

1.1.2 Real and Imaginary Parts

Definition 1.1.5: Real and Imaginary Parts

Let $z = x + iy$ where $x, y \in \mathbb{R}$

$$x = \Re(z) = \mathrm{Re}(z)$$
$$y = \Im(z) = \mathrm{Im}(z)$$

Example 1.1.1

$\Re(2 + 3i) = 2$ and $\Im(2 + 3i) = 3$

1.1.3 Inverses

Theorem 1.1.1

Let $z = x + iy, x, y \in \mathbb{R}$. Then $\overline{z} = x - iy$. Then there exists a $w \in \mathbb{C}$ such that $z * w = 1$. Namely, $w = \frac{\overline{z}}{z * \overline{z}} = \frac{1}{x^2+y^2}\overline{z} = \frac{x}{x^2+y^2} + i(\frac{-y}{x^2+y^2})$

Note

Since $z = (x, y) \neq (0, 0), x^2 + y^2 \neq 0$

Note

$$w = \frac{1}{z} = z^{-1}$$

Note

$$\frac{z_1}{z_2} = z_1(z_2^{-1}) \text{ if } z_2 \neq 0$$

1.1.4 Absolute Value

Definition 1.1.6: Absolute Value

Let $z = x + iy, x, y \in \mathbb{R}$. The absolute value of z is the distance between z and 0.

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z * \overline{z}}$$

and

$$|z|^2 = z * \overline{z}$$

1.2 Lecture 2

1.2.1 Properties of Conjugates and Absolute Values

Theorem 1.2.1 Properties of Conjugates and Absolute Values

Let $z, z_1, z_2 \in \mathbb{C}$. Then

1. $$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

2. $$\overline{z_1 z_2} = \overline{z_1} * \overline{z_2}$$

3. $$\text{If } z_2 \neq 0, \text{ then } \overline{(\frac{z_1}{z_2})} = (\frac{\overline{z_1}}{\overline{z_2}})$$

4. $$\Re(z) = \frac{1}{2}(z + \overline{z}) \text{ and } \Im(z) = \frac{1}{2i}(z - \overline{z})$$

5. $$|z_1 z_2| = |z_1||z_2|$$

6. $$\Re(z) \leq |\Re(z)| \leq |z|$$

7. $$\Im(z) \leq |\Im(z)| \leq |z|$$

8. $$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof:

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \text{ by (1)} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \text{ since } \overline{z_1\overline{z_2}} = \overline{z_1} * \overline{\overline{z_2}} = \overline{z_1} z_2 \\
&= |z_1|^2 + 2\Re(z_1\overline{z_2}) + |z_2|^2 \text{ since } z + \overline{z} = 2\Re(z) \text{ by (4)} \\
&\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \text{ by (6)} \\
&= |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2 \text{ by (2)} \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2 \\
\text{So } |z_1 + z_2|^2 &\leq (|z_1| + |z_2|)^2 \\
\text{And } |z_1 + z_2| &\leq |z_1| + |z_2| \\
\text{Since } |z_1 + z_2|, |z_1|, |z_2| &\geq 0
\end{aligned}$$

😊

1.3 Lecture 3

1.3.1 Distance Between Points

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. Then the distance between z_1, z_2 is the distance between (x_1, y_1) and (x_2, y_2). This is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |(x_1 - x_2) + i(y_1 - y_2)| = |z_1 - z_2|$$

1.3.2 Equation of a Circle

Let $z_0 \in \mathbb{C}, R > 0$. Then let $S = \{z \in \mathbb{C} : |z - z_0| = R\}$ which is a circle cantered around z_0 with radius R.

1.3.3 Polar Form of a Complex Number

For $\theta \in \mathbb{R}, e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

$$|e^{i\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$$

Let $z \in \mathbb{C}, z \neq 0, z = x + iy, x, y \in \mathbb{R}$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = |z| > 0 \\ x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

The polar form of z is

$$\begin{aligned} z &= r\cos(\theta) + ir\sin(\theta) \\ &= r(\cos(\theta) + i\sin(\theta)) \\ &= re^{i\theta} \end{aligned}$$

Given $z \neq 0, z \in \mathbb{C}, r = |z|$ is unique.

$$\frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \tan(\theta) \text{ if } x \neq 0$$

θ is not unique.

Note

$$\theta \neq tan^{-1}(\frac{y}{x})$$

Each value of θ is called an argument of z and denoted by $\arg(z)$. The principal value of $\arg(z)$ is denoted by $\text{Arg}(z)$ and is the unique angle θ that satisfies $-\pi < \theta \leq \pi$.

Note

$\arg(z)$ is a set of numbers.

Example 1.3.1 (Let $z = -\sqrt{3} + i$. Find the polar form of $z, |z|, \arg(z), \text{Arg}(z)$.)

$$
\begin{aligned}
|z| &= |-\sqrt{3}+i| = \sqrt{3+1} = \sqrt{4} = 2 \\
\tan(\theta) &= \frac{y}{x} = -\frac{1}{\sqrt{3}} \\
\tan(\frac{\pi}{6}) &= \frac{1}{\sqrt{3}} \\
\theta &= (\pi - \frac{\pi}{6}) + 2\pi n \\
\arg(z) &= \frac{5\pi}{6} + 2\pi n \text{ where } n \in \mathbb{Z} \\
\operatorname{Arg}(z) &= \frac{5\pi}{6}
\end{aligned}
$$

So the polar form of $z = 2e^{i\frac{5\pi}{6}}$.

Note

$\operatorname{Arg}(z)$ is not defined at 0.

Theorem 1.3.1

Let $\theta, \phi \in \mathbb{R}$. Then $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$.

Proof:

$$
\begin{aligned}
e^{i(\theta+\phi)} &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\
&= (\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) + i(\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)) \\
&= (\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi)) \\
&= e^{i\theta}e^{i\phi}
\end{aligned}
$$

😄

Note

If $z = x + iy, x, y \in \mathbb{R}$, then

$$e^z = e^{x+iy} := e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

Theorem 1.3.2

Let $z_1, z_2 \in \mathbb{C}, z_1, z_2 \neq 0$. Then $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

1.4 Lecture 4

Theorem 1.4.1 De Moivre's Theorem

Let $\theta \in \mathbb{R}$. Then $(e^{i\theta})^n = e^{in\theta}$ for $n = 1, 2, 3, \ldots$ Following from this, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.

Proof: The result is clearly true for $n = 1$.

Assume $k \geq 1$ is a fixed integer and the result is true for $n = k$, i.e. $(e^{i\theta})^k = e^{ik\theta}$.

$$\begin{aligned}
(e^{i\theta})^{k+1}(e^{i\theta})^k e^{i\theta} &= e^{ik\theta}e^{i\theta} \text{ by the induction hypothesis} \\
&= e^{i(k\theta+\theta)} \\
&= e^{i(k+1)\theta} \text{ and the result is true for } n = k+1 \\
\text{Hence, } (e^{i\theta})^n &= e^{in\theta} \text{for all integers } n \geq 1 \text{ by mathematical induction}
\end{aligned}$$

😊

Example 1.4.1 (Find $(1+i)^{100}$)

$$\begin{aligned}
1+i &= \sqrt{2}e^{\frac{\pi i}{4}} \\
(1+i)^{100} &= (\sqrt{2}e^{\frac{\pi i}{4}})^{100} \\
&= (\sqrt{2})^{100}(e^{\frac{\pi i}{4}})^{100} \\
&= 2^{50}e^{\frac{\pi i 100}{4}} \text{ by DeMoivre's theorem} \\
&= 2^{50}e^{25\pi i} \\
&= 2^{50}e^{\pi i} \\
&= -2^{50}
\end{aligned}$$

Proposition 1.4.1 Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ and $r_1, r_2 > 0$ and $\theta_1, \theta_2 \in \mathbb{R}$.
$z_1 = z_2 \iff r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi n$ for some $n \in \mathbb{Z}$

1.4.1 Solving for Powers of Complex Numbers

We now try to find solutions of $z^n = 1$.

Example 1.4.2 (Solving $z^3 = 1$)

Let $z = re^{i\theta}, r > 0$ and $\theta \in \mathbb{R}$.

$$\begin{aligned}
z^3 = 1 &\iff r^3 e^{3i\theta} = 1e^{i\theta} \\
&\iff r^3 = 1, 3\theta = 0 + 2\pi n (n \in \mathbb{Z}) \\
&\iff r = 1, \theta = \frac{2\pi n}{3}(n \in \mathbb{Z})
\end{aligned}$$

For $n \in \mathbb{Z}, n = 3q + k(q \in \mathbb{Z}, k = 0, 1, 2)$

$$\begin{aligned}
\frac{2\pi n}{3} &= \frac{2\pi}{3}(3q+k) \\
&= 2\pi q + \frac{2\pi k}{3}(q \in \mathbb{Z}, k = 0, 1, 2) \\
\text{So } z^3 = 1 &\iff z = e^{\frac{2\pi k i}{3}}(k = 0, 1, 2)
\end{aligned}$$

The solutions are

$$\begin{aligned}
z &= z_0 = 1 \\
\text{and } z &= z_1 = e^{\frac{2\pi i}{3}} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) \\
&= -\frac{1}{2} + \frac{\sqrt{3}i}{2} \\
\text{and } z &= z_2 = e^{\frac{4\pi i}{3}} = \cos(\frac{4\pi}{3} + i\sin(\frac{4\pi}{3})) \\
&= -\frac{1}{2} - \frac{\sqrt{3}i}{2}
\end{aligned}$$

Theorem 1.4.2

Let n be a positive integer. The equation $z^n = 1$ has n complex solutions

$$z = e^{\frac{2\pi ik}{n}} \text{ where } k = 0, 1, 2, \ldots, n-1$$

where the points on the unit circle are vertices of an n-gon, and the points are called **roots of unity**.

Theorem 1.4.3

Let $w \neq 0, w \in \mathbb{C}$. Let $w = \rho e^{i\phi}(\rho > 0, \phi \in \mathbb{R})$. The equation $z^n = w$ has n complex solutions

$$z = \rho^{\frac{1}{n}} e^{\frac{i\phi}{n} + \frac{2\pi ik}{n}}, k = 0, 1, 2, \ldots, n-1$$

Proof: Let $\tau = \rho^{\frac{1}{n}} \exp(\frac{i\phi}{n}) = p^{\frac{1}{n}} e^{\frac{i\rho}{n}}$
Then

$$\begin{aligned}
\tau^n &= (\rho^{\frac{1}{n}} e^{\frac{i\phi}{n}})^n \\
&= \rho e^{i\phi} \\
&= w
\end{aligned}$$

$\therefore z = \tau$ is a solution

$$\begin{aligned}
z^n = w &\iff (\frac{z}{\tau})^n = \frac{z^n}{\tau^n} = \frac{w}{w} = 1 \\
&\iff \frac{z}{\tau} = \exp(\frac{2\pi ik}{n}), k = 0, 1, \ldots, n-1 \\
&\iff z = \tau\exp(\frac{2\pi ik}{n}) = \rho^{\frac{1}{n}} \exp(\frac{i\phi}{n}) \exp(\frac{2\pi ik}{n}), k = 0, 1, \ldots, n-1 \\
&\iff z = \rho^{\frac{1}{n}} \exp(\frac{i\phi}{n} + \frac{2\pi ik}{n}), k = 0, 1, \ldots, n-1
\end{aligned}$$

😊

Example 1.4.3 (Solve $z^4 = -16$)

$$-16 = 16e^{i\pi} \text{ where } \rho = 16, \phi = \pi$$

There are 4 solutions:

$$z = 2\exp(\frac{\pi}{4}(1 + 2k)), k = 0, 1, 2, 3$$

The solutions are

$$
\begin{aligned}
z_k &= 2\exp(\frac{\pi i}{4}(2k+1)), k = 0, 1, 2, 3 \\
z_0 &= 2(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = \sqrt{2}(1+i) \\
z_1 &= \sqrt{2}(-1+i) \\
z_2 &= \sqrt{2}(-1-i) \\
z_3 &= \sqrt{2}(1-i)
\end{aligned}
$$

Chapter 2

Analytic Functions

2.1 Lecture 6

2.1.1 Functions and Mappings

We will consider functions

$$f : S \mapsto \mathbb{C}, \text{ where } S \subseteq \mathbb{C}$$

where S is the domain of f.

Example 2.1.1 (Find the domain of each function.)

1. $f(z) = \frac{1}{z}$

   Domain $= \{z \in \mathbb{C} : z \neq 0\}$

2. $\mathrm{Arg}(z)$

   Domain $= \{z \in \mathbb{C} : z \neq 0\}$

   Note

   Let $z \neq 0$. Then $z = \rho e^{i\phi}$ for some $\rho > 0, -\pi < \phi \leq \pi$.

   $$\mathrm{Arg}(z) = \phi$$

3. $\mathrm{Arc}(\frac{1}{z)}$

   Domain $= \{z \in \mathbb{C} : z \neq 0\}$

Any complex function $f : S \mapsto \mathbb{C}$ can be written as $f(z) = u(x, y) + iv(x, y)$ where $u, v \in \mathbb{R}$ and $z = x + iy, x, y \in \mathbb{R}$.

Example 2.1.2

Let $\mathbb{C}* = \mathbb{C}\backslash\{0\}$. $f : \mathbb{C}* \mapsto \mathbb{C}$ by $f(z) = \frac{1}{z}$ can be written as $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = u+iv$ where $u = \frac{x}{x^2+y^2}, v = \frac{-y}{x^2+y^2}$.

Note

Let $A \subset S, f : S \mapsto \mathbb{C}$. The image of A under the map f is

$$f(A) = \{f(a) : a \in A\}$$

2.2 Lecture 7

2.2.1 Formal Definition of Limit

Recall that $\lim_{x \to a} f(x) = L$ if $f(x)$ can be made arbitrarily close to L by choosing x close enough to to a (as long as $x \neq a$). Algebraically, this means that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ provided $0 < |x - a| < \delta$.

Now we examine the complex version of this:

$$\lim_{z \to z_0} f(z) = w_0 \tag{2.2.1}$$

Suppose $f(z)$ is defined on a deleted neighborhood of z_0.

Note

A deleted/punctured neighborhood

$$D'(z_0, R_0) = \{z \in \mathbb{C} : 0 < |z - z_0 < |R_0\}$$

We say the limit of $f(z)$ as z approaches z_0 is w_0 and we write this as equation 2.2.1 if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ provided $0 < |z - z_0| < \delta$.

Example 2.2.1 (Prove $\lim_{z \to i}(2z + 1) = 1 + 2i$ using the formal definition.)

Proof Template: Suppose ϵ is any positive real number.
Choose $\delta =$ something.
Then suppose $0 < |z - z_0| < \delta \ldots |f(z) - w_0| < \epsilon$.
Hence $\lim_{z \to z_0} f(z) = w_0$. 😊

Scratch: $f(z) = 2z+1, z_0 = i, w_0 = 1+2i$. Let $\epsilon > 0$. We want $\delta > 0$ so if $0 < |z-z_0| < \epsilon$ i.e. $0 < |z-i| < \epsilon$ then $|f(z) - w_0| = |2z + 1 - (1 + 2i)| < \epsilon$. $|(2z + 1) - (1 + 2i)| = |2z - 2i| = 2|z - i| < \epsilon$. Take $\delta = \frac{\epsilon}{2}$

Proof: Suppose ϵ is any positive real number.
Choose $\delta = \frac{\epsilon}{2}$.
Then suppose $0 < |z - i| < \delta = \frac{\epsilon}{2}$.
We have $|(2z + 1) - (1 + 2i)| = |2z - 2i| = 2|z - i| < 2 * \frac{\epsilon}{2} = \epsilon$, which is $|(2z + 1) - (1 + 2i)| < \epsilon$.
Hence $\lim_{z \to i}(2z + 1) = 1 + 2i$. 😊

2.3 Lecture 8

2.3.1 Properties of Limits

Theorem 2.3.1 Theorem on Limits

Suppose f is defined on a punctured neighborhood of z_0.
Let $f(z) = u(x, y) + iv(x, y), z = x + iy, x, y \in \mathbb{R}$.
Let $z_0 = x_0 + iy_0, x_0, y_0 \in \mathbb{R}$ and $w_0 = u_0 + iv_0, u_0, v_0 \in \mathbb{R}$.
Then

$$\lim_{z \to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

and we can apply rules we learned in calculus.

Example 2.3.1 (Find $\lim_{z\to 1+i}\frac{1}{z}$)

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$
$$\lim_{(x,y)\to(1,1)} \frac{x}{x^2+y^2} = \frac{1}{2}$$
$$\lim_{(x,y)\to(1,1)} \frac{-y}{x^2+y^2} = -\frac{1}{2}$$
$$\lim_{z\to(1+i)} \frac{1}{z} = \frac{1}{2} - \frac{i}{2}$$

2.4 Lecture 9

2.4.1 More Properties of Limits

Theorem 2.4.1

Suppose $f(z)$ and $g(z)$ are defined on a deleted neighborhood of z_0, and

$$\lim_{z\to z_0} f(z) = w_0 \text{ and } \lim_{z\to z_0} g(z) = w_1$$

Then

1. $$\lim_{z\to z_0} f(z)+g(z) = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z) = w_0 + w_1$$
2. $$\lim_{z\to z_0} f(z)g(z) = w_0 w_1$$
3. $$\lim_{z\to z_0} \gamma f(z) = \gamma w_0 \text{ where } \gamma \text{ is a complex constant}$$
4. $$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1} \text{ if } w \neq 0$$
5. $$\lim_{z\to z_0} z^n = z_0^n \text{ if } n \text{ is a positive integer}$$

   Note

   Also true if n is a negative integer and $z_0 \neq 0$.
6. $$\lim_{z\to z_0} P(z) = P(z_0) \text{ if } P(z) \text{ is a polynomial}$$
7. $$\lim_{z\to z_0} R(z) = \frac{P(z_0)}{Q(z_0)} \text{ if } Q(z_0) \neq 0, P(z), Q(z) \text{ are polynomials, and } R(z) = \frac{P(z)}{Q(z)} \text{ is a rational function}$$

2.4.2 Point of Infinity

What are limits to infinity? Suppose $z_0 \in \mathbb{C}$ and $f(z)$ is defined on a deleted neighborhood of z_0 s. We say $\lim_{z \to z_0} f(z) = \infty$ if for every $M > 0$ there is a $\delta > 0$ such that $|f(z)| > M$ for $0 < |z - z_0| < \delta$.

Note

$$|f(z)| > M \iff |\frac{1}{f(z)} - 0| = |\frac{1}{f(z)}| = \frac{1}{|f(z)|} < \frac{1}{M} \iff |\frac{1}{f(z)}| < \frac{1}{M}$$

Theorem 2.4.2

1. $$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$$
2. $$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f(\frac{1}{z}) = w_0$$
3. $$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$$

Example 2.4.1 (Show $\lim_{z \to \infty} \frac{z}{(z-1)(z-2)} = \infty$)

$\lim_{z \to 1} \frac{(z-1)(z-2)}{z} = 0$. Therefore $\lim_{z \to 1} \frac{z}{(z-1)(z-2)} = \infty$.

2.4.3 Definition of Derivative

Suppose $f(z)$ is defined on an open neighborhood of z_0

$$D(z_0, r_0) = \{z : |z - z_0| < r_0\} \text{ for some } r_0 > 0$$

f is differentiable at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, in which case we write

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

where $z - z_0 = \Delta z, z = z_0 + \Delta z$

2.5 Lecture 10

2.5.1 Differentiability

Note

Let n be a positive integer.

$$f(z) = z^n \text{ is differentiable at all } z$$

$$f'(z) = nz^{n-1} \text{ for all } z$$

Example 2.5.1 (Determine the differentiability of $f(z) = \overline{z}$.)
Consider the limit

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}\end{aligned}$$

We show that the limit $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist. Consider the limit along the real axis:

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Now consider the limit along the imaginary axis:

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{\Delta y} = \lim_{\Delta y \to 0} \frac{-i}{i} = -1$$

These limits are not the same, and thus $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist. Hence, the function $f(z) = \overline{z}$ is differentiable nowhere, even though it is continuous.

Theorem 2.5.1

$f(z) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x = +iy_0$ iff $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Example 2.5.2

$f(z) = \overline{z} = x - iy$ is continuous everywhere since $u(x,y) = x$ and $v(x,y) = -y$ are continuous everywhere.

Note

If f is differentiable at z_0, then f is continuous at z_0, but not necessarily the other way around.

2.5.2 Necessary Conditions for Differentiability

Theorem 2.5.2

Suppose $f(z)$ is defined on an open neighborhood of z_0. Suppose $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy, x, y \in \mathbb{R}$.
If f is differentiable at $z_0 = x_0 + iy_0$, then $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist at the point (x_0, y_0).
Also,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}\end{aligned}$$

at (x_0, y_0).

Note

These are called the **Cauchy-Riemann equations**.

Furthermore,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

2.6 Lecture 11

Example 2.6.1

Let $f(z) = |z|^2 = x^2 + y^2$ where $z = x + iy$.
Let $u = x^2 + y^2, v = 0$.
Then $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial y} = 0$.
So the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist for all (x, y).
Plugging these into the Cauchy-Riemann equations, we get

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \iff \begin{cases} 2x = 0 \\ 0 = -2y \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

So the Cauchy-Riemann equations hold only for $(x, y) = (0, 0)$.
What does this theorem imply? It implies that $f(z)$ is **not** differentiable at any point $z \neq 0$.

Proof: Suppose that f is differentiable at $z_0 = x_0 + iy_0$.

Then

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z0)}{\Delta z}$$

exists.

Also,

$$\lim_{\substack{\Delta \to 0 \\ \text{(along the real axis)}}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{2.6.1}$$

$$= \lim_{\substack{\Delta \to 0 \\ \text{(along the imaginary axis)}}} \frac{f(z_0 + \Delta z) - f(z_0}{\Delta z} \tag{2.6.2}$$

and these 2 limits exist.

Along the real axis, $\Delta z = \Delta x$. Let $\Delta x \to 0$. Then

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta x} \\ &= \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\left(\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right) \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \end{aligned}$$

So equation (2.6.1) is equal to $\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$.

Along the imaginary axis, $\Delta z = i\Delta y$. Let $\Delta y \to 0$. Then

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{i\Delta y} \\ &= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\left(\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}\right) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \end{aligned}$$

So equation (2.6.2) is equal to $\frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$.

Then

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

So

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ at } (x_0, y_0)$$

😄

2.7 Lecture 12

2.7.1 Sufficient Conditions for Differentiability

Theorem 2.7.1

Suppose $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy, x, y, u, v \in \mathbb{R}$) is defined on an open neighborhood $D(z_0, r)$ of $z_0 = x_0 + iy_0$ for some $r > 0$.
Suppose the derivatives part $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist at each point in $D(z_0, r)$ and are continuous at (x_0, y_0).
Then if the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ hold at (x_0, y_0), then f is differentiable at z_0 and

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

Note

Remember that you can have a function where the Cauchy-Riemann equations hold at a point but the derivative at that point does not exist.

2.7.2 Differentiability Rules

Assume $f(z)$ and $g(z)$ are differentiable on some domain.

1. $$\frac{d}{dz}(f(z) + g(z) = f'(z) + g'(z)$$

2. $$\frac{d}{dz}f(z)g(z) = f(z)g'(z) + f'(z)g(z)$$

3. $$\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$$

4. $$\frac{d}{dz}f(g(z)) = f'(g(z))g'(z) \text{ provided that the range of } g \subseteq \text{the domain of } f$$

2.7.3 Cauchy-Riemann Equations in Polar Coordinates

Theorem 2.7.2

Let $S = \{re^{i\theta} : r_1 < r < r_2, \theta_1 < \theta < \theta_2\}$ where $0 < \theta_2 - \theta_1 \leq 2\pi$.
Suppose $f(z) = u(r, \theta) + iv(r, \theta), z = re^{i\theta} \in S$.
Suppose the partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial \theta}$ exist and are continuous on S.
If the Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

hold at $(r_0, \theta 0)$, then f is differentiable at $z_0 = r_0 e^{i\theta_0}$ and

$$f'(z_0) = e^{-i\theta_0}\left(\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right).$$

Example 2.7.1

Define $g(z) = \sqrt{r}e^{\frac{i\theta}{2}}, z = re^{i\theta}$ for $r > 0$ and $-\pi < \theta < \pi$.

$$\begin{aligned} g(z) = \sqrt{r}e^{\frac{i\theta}{2}} &= \sqrt{r}\left(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right) \\ &= \sqrt{r}\cos(\frac{\theta}{2}) + i\sqrt{r}\sin(\frac{\theta}{2}) \end{aligned}$$

Let $u = \sqrt{r}\cos(\frac{\theta}{2}), v = \sqrt{r}\sin(\frac{\theta}{2})$. The partial derivatives

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{2\sqrt{r}}\cos(\frac{\theta}{2}) \\ \frac{\partial v}{\partial r} &= \frac{1}{2\sqrt{r}}\sin(\frac{\theta}{2}) \\ \frac{\partial u}{\partial \theta} &= -\frac{\sqrt{r}}{2}\sin(\frac{\theta}{2}) \\ \frac{\partial v}{\partial \theta} &= \frac{\sqrt{r}}{2}\cos(\frac{\theta}{2}) \end{aligned}$$

exist and are continuous for $r > 0$. The Cauchy-Riemann equations hold for $r > 0$ and $-\pi < \theta < \pi$ since

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{2\sqrt{r}}\cos(\frac{\theta}{2}) = \frac{1}{r}\left(\frac{\sqrt{r}}{2}\cos(\frac{\theta}{2})\right) = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= \frac{1}{2\sqrt{r}}\sin(\frac{\theta}{2}) = -\frac{1}{r}\left(-\frac{\sqrt{r}}{2}\sin(\frac{\theta}{2})\right) = -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}$$

Hence $g(z)$ is differentiable on D and

$$\begin{aligned} g'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) &= e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos(\frac{\theta}{2}) + i\frac{1}{2\sqrt{r}}\sin(\frac{\theta}{2})\right) \\ &= \frac{1}{2\sqrt{r}e^{i\theta}}\left(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right) \\ &= \frac{1}{2\sqrt{r}e^{i\theta}}e^{\frac{i\theta}{2}} \\ &= \frac{1}{2\sqrt{r}e^{\frac{i\theta}{2}}} \\ &= \frac{1}{2g(z)} \text{ for } z \in D \end{aligned}$$

Note

$g(z) = \sqrt{r}e^{\frac{i\theta}{2}}$ where $z = re^{i\theta}$ is the **principal square root** $\sqrt{z}$ of z.

2.8 Lecture 13

2.8.1 Analytic Functions

Definition 2.8.1: Analyticity at a Point

$f(z)$ is analytic at a point z_0 if f is differentiable at each point in some open neighborhood $D(z_0, r)$ of z_0 for some $r > 0$.

Definition 2.8.2: Openness

A set $S \subseteq \mathbb{C}$ is open if for every $z_0 \in S$, there is a $\delta_0 > 0$ such that

$$z_0 \in D(z_0, \delta_0) \subseteq S.$$

Definition 2.8.3: Closedness

$F \subseteq \mathbb{C}$ is closed if $F^c = \{z \in \mathbb{C} : z \notin F\}$ is open.

Note

A set can be also be neither open nor closed.

Definition 2.8.4: Analyticity on an Open Set

$f(z)$ is analytic on an open set if it is analytic at each point of the set.
This is equivalent to saying that f is differentiable at each point of S (assuming the set is open).

Definition 2.8.5: Entirety

f is entire if $f(z)$ is analytic (which implies differentiable) at each point in $\mathbb{C}$.

Note

Any polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ (each $a_j \in \mathbb{C}$) is entire.

Definition 2.8.6: Isolated Singularity

z_0 is an isolated singularity of $f(z)$ if f is **not** differentiable at z_0 but is differentiable on some deleted neighborhood $D'(z_0, r_0)$ of z_0.

Theorem 2.8.1

Let D, E be open.
Suppose $f : D \mapsto \mathbb{C}$ is analytic on D, and $g : E \mapsto \mathbb{C}$ is analytic on E.
Then $g \circ f : D \mapsto \mathbb{C}$ is analytic provided $f(D) \subseteq E$.
Further, $\frac{d}{dz} g \circ f(z) = \frac{d}{dz} g(f(z)) = g(f(z)f'(z)$ for $z \in D$.

Note

We need to make sure $f(z)$ is in the domain of g where g is analytic.

Example 2.8.1

Recall the principal square root function $g(z) = \sqrt{r}e^{\frac{i\theta}{2}}, r > 0, -\pi < \theta < \pi, z = re^{i\theta}$.
g is analytic on $D = \{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$ and we know $g'(z) = \frac{1}{2g(z)}$ for $z \in D$.
Let $G(z) = g(2z - 2 + i)$.

Show G is analytic on the half-plane $H = \{z : \text{Re}(z) > 1\}$ and $G'(z) = \frac{1}{g(2z-2+i)}, z \in H$.
Let $h(z) = 2z - 2 + i$. $h(z)$ is entire because it's a polynomial.
$G = g \circ h$, i.e. $G(z) = g(h(z)) = g(2z - 2 + i)$. We need to make sure that if $z \in H$, $h(z)$ is in D, which is the domain where g is analytic.
So suppose $z \in H$. $z = x + iy, x > 1$.
$h(z) = 2z - 2 + i = 2x + iy - 2 + i = 2(x - 1) + i(2y + 1)$.
$x > 1$ so $\text{Re}(h(z)) > 0$ and $-\frac{\pi}{2} < \text{Arg}(h(z)) < \frac{\pi}{2}$ and $h(z) \in D$.
So G is analytic on H and $G'(z) = g'(h(z))h'(z) = \frac{1}{2g(h(z))} * 2 = \frac{1}{g(2z-2+i)}$ for $z \in H$.

2.9 Lecture 14

2.9.1 Constant Functions

Definition 2.9.1: Connectedness

An open set $S \subseteq \mathbb{C}$ is connected if any points $z_1, z_2 \in S$ can be connected by polygonal lines joined end-to-end and wholly contained in S.

Definition 2.9.2

A non-empty open and connected subset of $\mathbb{C}$ is called a domain.

Theorem 2.9.1

Suppose $S \subseteq \mathbb{C}$ is a domain.
If $f : S \mapsto \mathbb{C}$ is analytic on S, and $f'(z) = 0$ for all $z \in S$, then f is constant on S.
That is, $f(z) = k$ for all $z \in S$ and some complex constant k.

Idea of Proof: Suppose $S \subseteq \mathbb{C}$ is a domain, $f(z) = u(x, y) + iv(x, y)$ is analytic on D, and $f'(z) = 0$ for all $z \in S$.
We know $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0$ for all $(x, y) \in S$.
Basically, taking any straight lines to get between two points (since S is connected) will ensure that the derivative of $u(x, y)$ and $v(x, y)$ for any of those lines is 0.
That means $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0$, and then $f(z)$ is constant. 😊

2.10 Lecture 15

2.10.1 Harmonic Functions

Definition 2.10.1

Let $D \subseteq \mathbb{R}^2$ be a domain (open and connected). A function $f : D \mapsto \mathbb{R}$ is harmonic if the second-order partial derivatives of $f(x, y)$ are continuous and satisfy the Laplace equation $f_{xx} + f_{yy} = 0$ for all $(x, y) \in D$.

Example 2.10.1 ($h(x,y) = x^3 - 3xy^2$)

$$
\begin{aligned}
h_x &= 3x^2 - 3y^2 \\
h_y &= -6xy \\
h_{xx} &= 6x \\
h_{xy} &= -6y \\
h_{yy} &= -6x \\
h_{yx} &= -6y
\end{aligned}
$$

and the second-order partial derivatives are continuous for all (x,y).
$h_{xx} + h_{yy} = 6x - 6x = 0$ for all (x,y).
Hence $h(x,y)$ is harmonic on $\mathbb{R}^2$.

Theorem 2.10.1

Let D be a domain in $\mathbb{C}$ and suppose $f : D \mapsto \mathbb{C}$ is analytic on D.
Let $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy \in D(x,y,u,v \in \mathbb{R})$.
Then the functions $u(x,y), v(x,y)$ are harmonic on D.

Proof: Suppose D is a domain and $f = u(x,y) + iv(x,y)$ is analytic on D.
Then u and v satisfy the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ for $(x,y) \in D$.
By a later theorem, all partial derivatives of $u(x,y)$ and $v(x,y)$ to all orders exist and are continuous (since f is analytic on D).
$u_{xx} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}) = -u_{yy}$ and $u_{xx} + u_{yy} = 0$ and $u(x,y)$ is harmonic on D.
$v(x,y)$ was left as an exercise. 😄

Example 2.10.2

$$
\begin{aligned}
f(z) = z^3 = (x+iy)^3 &\text{ is entire.} \\
&= (x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3) \\
&= x^3 + 3x^2yi - 3xy^2 - iy^3 \\
&= (x^3 - 3xy^2) + i(3x^2y - y^3).
\end{aligned}
$$

The theorem implies $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$ are harmonic on $\mathbb{R}^2$.

Chapter 3

Elementary Functions

3.1 Lecture 15

3.1.1 The Exponential Function

Definition 3.1.1

$\exp(z) = e^z := e^x e^{iy} = e^x(\cos(y) + i\sin(y)), z = x + iy, x, y \in \mathbb{R}$

Properties:

1. $\exp(z)$ is entire and $\frac{d}{dx}e^z = e^z$
2. $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for $z_1, z_2 \in \mathbb{C}$
3. $\exp(z) \neq 0$

   Proof: Let $z = x + iy, x, y \in \mathbb{R}$.

   $$\begin{aligned} \exp(z) &= e^x e^{iy} \\ |\exp(z)| &= |e^x e^{iy}| = |e^x||e^{iy}| \\ &= e^x(1) = e^x > 0. \end{aligned}$$

   So $\exp(z) \neq 0$. ☺

4. $\exp(z_1 - z_2) = \frac{\exp(z_1)}{\exp(z_2)}$
5. $\exp(z + 2\pi i) = \exp(z)$ for $z \in \mathbb{C}$ ($\exp(z)$ has period $2\pi i$)
6. The range of $\exp(z) = \{e^z : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}$ (set of non-zero complex numbers)

   Proof: Let $w \neq 0$. So $w = \rho e^{i\phi}$ for some $\rho > 0, \phi \in \mathbb{R}$.
   We want to solve $e^z = w$.

   $$\begin{aligned} e^z &= w \\ \iff e^x e^{iy} &= \rho e^{i\phi} \\ \iff e^x &= \rho \text{ and } y = \rho + 2\pi n, n \in \mathbb{Z} \\ \iff x &= \ln(\rho), y = \phi + 2\pi n, n \in \mathbb{Z} \\ \iff z &= \ln(\rho) + i(\phi + 2\pi n), n \in \mathbb{Z} \end{aligned}$$

   That is, $z = \ln|w| + i\arg(w)$.
   So e^z can be any complex number except 0. ☺

3.2 Lecture 16

3.2.1 The Logarithmic Function

Recall, $e^x = y \iff x = \ln(y)$ assuming $y > 0$.

Definition 3.2.1

The complex exponential is $\exp(z) = w \iff z = \ln|w| + i\arg(w)$ assuming $w \neq 0$, which is called "the complex log of w." It is a multi-valued function.
So $\exp(z) = w \iff z = \log(w)$ assuming $w \neq 0$.

Example 3.2.1 (Solve $e^z = -2$.)

$$\begin{aligned} z &= \log(-2) = \ln|-2| + i\arg(-2) \\ &= \ln(2) + i(\pi + 2\pi n), n \in \mathbb{Z}. \end{aligned}$$

Example 3.2.2 (Find $\exp(\ln(2) + \frac{\pi i}{4})$.)

$$\begin{aligned} \exp(\ln(2 + \frac{\pi i}{4})) &= e^{\ln(2)} e^{\frac{\pi i}{4}} \\ &= 2(\cos(\pi/4) + i\sin(\pi/4)) \\ &= 2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \\ &= \sqrt{2} + i\sqrt{2} \\ &= \sqrt{2}(1+i). \end{aligned}$$

Definition 3.2.2: Principal Log

Let $z \neq 0, z \in \mathbb{C}$.
The principal log of z is written $\mathrm{Log}(z)$ is defined by

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$$

for $z \neq 0$.

Note

Remember that $-\pi < \mathrm{Arg}(z) \leq \pi$.

Example 3.2.3

$$\begin{aligned}
\operatorname{Log}(1) &= \ln|1| + i\operatorname{Arg}(1) \\
&= \ln(1) + 0 = 0 \\
\operatorname{Log}(i) &= \ln|i| + i\operatorname{Arg}(i) \\
&= \ln(1) + i\frac{\pi}{2} = i\frac{\pi}{2} \\
\operatorname{Log}(-1) &= \ln|-1| + i\operatorname{Arg}(-1) \\
&= \ln(1) + i\pi = i\pi \\
\operatorname{Log}(-1-i) &= \ln|-1-i| + i\operatorname{Arg}(-1-i) \\
&= \ln(\sqrt{2}) + i(-\frac{3\pi}{4}) \\
&= \frac{1}{2}\ln(2) - \frac{3\pi i}{4}
\end{aligned}$$

Note

$\operatorname{Log}(z)$ is **not** continuous on the negative real axis.

$$\begin{aligned}
\lim_{\substack{z\to 1 \\ \text{(on the upper half of a circle)}}} &= \lim_{\theta\to\pi^-} \operatorname{Log}(e^{i\theta}) \\
&= \lim_{\theta\to\pi^-} \ln(|e^{i\theta}|) + i\operatorname{Arg}(e^{i\theta}) \\
&= \lim_{\theta\to\pi^-} \ln(1) + i0 \\
&= i\pi \\
\lim_{\substack{z\to 1 \\ \text{(on the lower half of a circle)}}} &= \lim_{\theta\to\pi^+} \operatorname{Log}(e^{i\theta}) \\
&= \lim_{\theta\to\pi^+} \operatorname{Log}(e^{i\theta}) \\
&= \lim_{\theta\to\pi^+} \ln(|e^{i\theta}) + i\operatorname{Arg}(e^{i\theta}) \\
&= \lim_{\theta\to\pi^+} \ln(1) + i(\theta - 2\pi) \\
&= -\pi i
\end{aligned}$$

The two limits are not equal so $\lim_{z\to -1}\operatorname{Log}(z)$ does not exist. So $\operatorname{Log}(z)$ is defined on the negative real axis but is not continuous there.

Theorem 3.2.1

Let $D = \{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$.
$\operatorname{Log}(z)$ is analytic on D and $\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z}$ for $z \in D$.

Proof: Let $z \in D, z = re^{i\theta}, r > 0, -\pi < \theta < \pi$.
Then $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z) = \ln(r) + i\theta$.
So let $u = \ln(r), v = \theta$.
The partial derivatives

$$\begin{aligned}
u_r &= \frac{1}{r} \\
v_r &= 0 \\
u_\theta &= 0 \\
v_\theta &= 1
\end{aligned}$$

exist and are continuous for $r > 0$.
$u_r = \frac{1}{r} = \frac{1}{r} * 1 = \frac{1}{r} u_\theta$ and $v_r = 0 = -\frac{1}{r} * 0 = -\frac{1}{r} u_\theta$.
So the Cauchy-Riemann equations hold for $r > 0$ and $-\pi < \theta < \pi$.
So $\text{Log}(z)$ is differentiable and analytic on D, and

$$\frac{d}{dz} \text{Log}(z) = e^{-i\theta} (u_r + iv_r) = e^{-i\theta} \left(\frac{1}{r} + i0 \right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

for $z \in D$. 😊

3.3 Lecture 17

3.3.1 Branches of Logarithms

Branches help you to work on the negative real axis.

Definition 3.3.1: Branches

Let $\alpha \in \mathbb{R}$.
Let $D_\alpha = \{re^{i\theta} : r > 0, \alpha < \theta < \alpha + 2\pi\}$. This is called a **branch cut**.
For $z \in D_\alpha$, $\log_\alpha(z) = \ln(r) + i\theta$ where $z = re^{i\theta}, r > 0, \alpha < \theta < \alpha + 2\pi$.

Note

$\log_\alpha(z)$ does not mean $\log(z)$ to base α. We are just looking at it in the domain of the branch.

This function is a **branch** of $\log(z)$.

Note

$\text{Log}(z) = \log_\alpha(z)$, where $\alpha = -\pi$.

Theorem 3.3.1

Let α be a real number.
Let $D_\alpha = \{re^{i\theta} : r > 0, \alpha < \theta < \alpha + 2\pi\}$.
$z \in D_\alpha$, $\log_\alpha(z) = \ln(r) + i\theta$ where $z = re^{i\theta}, r > 0, \alpha < \theta < \alpha + 2\pi$.
This function is analytic on D_α, and its derivative is

$$\frac{d}{dz} \log_\alpha(z) = \frac{1}{z}.$$

Example 3.3.1 ($\alpha = \frac{\pi}{2}$, so $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$)

$$\begin{aligned}
|-i| &= 1 \\
\arg(-1) &= -\frac{\pi}{2} + 2n\pi \text{ for } n \in \mathbb{Z} \\
\log_{\frac{\pi}{2}}(-i) &= \ln|-i| + i(-\frac{\pi}{2} + 2\pi) \\
&= 0 + i\frac{3\pi}{2} \\
\log_{\frac{\pi}{2}}(1) &= \ln|1| + i(0 + 2\pi) = 2\pi.
\end{aligned}$$

3.3.2 Properties of Logarithms

1. $\exp(\log(z)) = z$ for $z \neq 0$
2. $\log(\exp(z)) = z + 2\pi i n \quad (n \in \mathbb{Z})$ for all z
3. $\operatorname{Log}(\exp(z)) = z$ if $-\pi < \operatorname{Im}(z) \leq \pi$

   Proof: Let $z = x + iy, x, y \in \mathbb{R}$.
   $\exp(z) = e^x e^{iy}$

$$\begin{aligned}\operatorname{Log}(\exp(z)) &= \ln(e^x) + i\operatorname{Arg}(e^x e^{iy})\\ &= x + iy \text{ since } -\pi < y \leq \pi\end{aligned}$$

☺

4. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ for $z_1, z_2 \neq 0$

Note

In general, it is not necessarily true that $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$ for $z_1, z_2 \neq 0$

3.4 Lecture 18

3.4.1 Complex Trigonometric Functions

Definition 3.4.1

$$\begin{aligned}\cos(z) &:= \frac{1}{2}(\exp(iz) + \exp(-iz))\\ \sin(z) &:= \frac{1}{2i}(\exp(iz) - \exp(-iz))\end{aligned}$$

for $z \in \mathbb{C}$.

3.4.2 Properties of Complex Trig Functions

1. $\sin(z)$ and $\cos(z)$ are entire and $\frac{d}{dz}\sin(z) = \cos(z)$ and $\frac{d}{dz}\cos(z) = -\sin(z)$

   Proof: $\exp(z), \exp(iz), \exp(-iz)$ are entire.
   Hence, $\cos(z)$ and $\sin(z)$ are entire.

$$\begin{aligned}\frac{d}{dz}\sin(z) &= \frac{d}{dz}(\frac{1}{2i}(\exp(iz) - \exp(-iz)))\\ &= \frac{d}{dz}(\frac{1}{2i}(i\exp(iz) + i\exp(-iz)))\\ &= \frac{1}{2}(\exp(iz) + \exp(-iz))\\ &= \cos(z)\end{aligned}$$

   $\frac{d}{dz}\cos(z)$ is similar. ☺

2. $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$
   $\cos(z_1 + z_2) = \cos(z_1)\cos(2) - \sin(z_1)\sin(z_2)$

3. $\sin(z+\pi/2) = \cos(z)$
   $\cos(z+\pi/2) = -\sin(z)$

4. $\sin^2(z) + \cos^2(z) = 1$

   Proof:

$$
\begin{aligned}
(\sin(z))^2 + (\cos(z))^2 &= \left(\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)^2 + \left(\frac{1}{2}\left(e^{iz} + ie^{-iz}\right)^2\right)\right) \\
&= -\frac{1}{4}\left(e^{2iz} - 2 + e^{-2iz}\right) + \frac{1}{4}\left(e^{2iz} + 2 + e^{-2iz}\right) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}
$$

😃

5. $|\sin(z)|^2 = (\sin(x))^2 + (\sinh(y))^2$

   Proof: We use the fact that $\overline{\exp(z)} = \exp(\overline{z})$.
   $\sin(z) = \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)$

$$
\begin{aligned}
|\sin(z)|^2 &= \sin(z)\overline{\sin(z)} \\
&= \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)\overline{\frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)} \\
&= \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)\left(\frac{1}{-2i}\right)\left(\exp(-i\overline{z}) - \exp(i\overline{z})\right) \\
&= \frac{1}{4}\left(\exp(i(z-\overline{z})) - \exp(i(z+\overline{z})) - \exp(-i(z+\overline{z})) + \exp(-i(z-\overline{z}))\right) \\
\text{We know that } z+\overline{z} &= 2x,\ z-\overline{z} = 2iy \\
\text{So } |\sin(z)|^2 &= \frac{1}{4}\left(\left(e^{-2y} + e^{2y}\right) - \left(e^{2xi} + e^{-2xi}\right)\right) \\
(\sin(x))^2 + (\sinh(y))^2 &= \left(\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right)^2 + \left(\frac{1}{2}\left(e^{y} - e^{-y}\right)^2\right) \\
&= -\frac{1}{4}\left(e^{2-ix} - 2 + e^{-2xi}\right) + \frac{1}{4}\left(e^{2y} - 2 + 2^{-2y}\right) \\
&= \frac{1}{4}\left(\left(e^{2y} + e^{-2y}\right) - \left(e^{2xi} + -2xi\right)\right)
\end{aligned}
$$

   Hence $|\sin(z)|^2 = (\sin(x)^2 + (\sinh(y))^2$.

   Note

   It is not true that $|\sin(z)| \leq 1$ for all $z \in \mathbb{C}$ because $|\sin(z)|^2$ goes to $+\infty$ if y goes to ∞ since $\sinh(y)$ goes to ∞ as y goes to ∞.

😃

   $|\cos(z)|^2 = (\cos(x))^2 + (\sinh(y))^2$

Note

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$
$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

6. $(\cosh(t))^2 - (\sinh(t))^2 = 1$
   So the points $(x, y) = (\cosh(t), \sinh(t))$ are a hyperbola $y^2 - x^2 = 1$.
7. $\sin(z) = 0 \iff z = n\pi, n \in \mathbb{Z}$
   $\cos(z = 0) \iff z = n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$

Example 3.4.1 (Solve $\sin(z) = 2$.)

$$\begin{aligned}
&\sin(z) = 2 \\
\iff &\frac{1}{2i}(e^{iz} - e^{-iz}) = 2 \\
\iff &\frac{1}{2i}(w - \frac{1}{z}) = 2 \text{ where } w = e^{iz} \text{ (noting that } w \neq 0) \\
\iff &w - \frac{1}{w} = 4i \\
\iff &w^2 - 1 = 4iw \\
\iff &w^2 - 4iw - 1 = 0 \\
\iff &w = \frac{4i \pm \sqrt{-12}}{2} = 2i \pm \sqrt{-3} \\
\iff &w = i(2 \pm \sqrt{3}) \\
\iff &\exp(iz) = i(2 \pm \sqrt{3}) \\
\iff &iz = \log(i(2 \pm \sqrt{3})) \\
&= \ln|i(2 \pm \sqrt{3}| + i\arg(i(2 \pm \sqrt{3})) \\
&= \ln(2 \pm \sqrt{3}) + i\left(\frac{\pi}{2} + 2\pi n\right) \text{ for } n \in \mathbb{Z} \\
\iff &z = (\frac{\pi}{2} + 2\pi n) - i\ln(2 \pm \sqrt{3}) \text{ for } n \in \mathbb{Z}
\end{aligned}$$

Note

$0 < \sqrt{3} \approx 1.7 < 2, 2 - \sqrt{3} > 0$

3.4.3 Complex Power Function

Definition 3.4.2

$$z^c := e^{c\log(z)}$$

Note

This is function has multiple values.

The principal value of z^c is $\mathrm{PV}(z^c) := \exp(c\,\mathrm{Log}(z))$ for $z \neq 0, c \in \mathbb{C}$.

Example 3.4.2 (Find i^i and $\mathrm{PV}(i^i)$.)

$$\begin{aligned} i^i &= \exp(i\log(i)) \\ &= \exp(i(\ln|i| + i\arg(z))) \\ &= \exp(i(0 + i(\frac{\pi}{2} + 2\pi n))) \text{ for } n \in \mathbb{Z} \\ &= e^{-\left(\frac{\pi}{2}+2\pi n\right)} \text{ for } n \in \mathbb{Z} \\ \mathrm{PV}(i^i) &= \exp(i\,\mathrm{Log}(i)) = \exp(i(0 + i\frac{\pi}{2})) = e^{-\frac{\pi}{2}} \end{aligned}$$

3.4.4 Branches of the Complex Power Function

Assume c is a complex constant.

Let $\alpha \in \mathbb{R}$.
$D_\alpha = \{re^{i\theta} : r > 0, \alpha < \theta < \alpha + 2\pi\}$
The branch of $\log(z) = \log_\alpha(z)$.
The branch of z^c is $f_\alpha(z) = \exp(c\log_\alpha(z))$.
$f'_\alpha(z) = \exp(c\log_\alpha(z))(c \cdot \frac{1}{z}) = c\frac{\exp(c\log_\alpha(z))}{\exp(\log_\alpha(z))} = c\exp((c-1)\log_\alpha(z)) = cz^{c-1}$.

Chapter 4

Integrals

4.1 Lecture 19

4.1.1 Basic Integrals

Definition 4.1.1

Let $w(t) = u(t) + iv(t)$ where $u(t), v(t)$ are real-valued.
$w'(t) := u'(t) + iv'(t)$

$$\int_a^b w(t)dt = \int_a^b (u(t) + iv(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$$

assuming t is real.

Example 4.1.1 (Find $\frac{d}{dt}e^{it}$ and $\int_0^{\frac{\pi}{4}} e^{it}dt$.)

$$\begin{aligned}
e^{it} &= \cos(t) + i\sin(t) \\
\frac{d}{dt}e^{it} &= -\sin(t) + i\cos(t) \\
&= i(\cos(t) + i\sin(t)) \\
&= ie^{it} \\
\int_0^{\frac{\pi}{4}} e^{it}dt &= \int_0^{\frac{\pi}{4}} (\cos(t) + i\sin(t))dt \\
&= \int_0^{\frac{\pi}{4}} \cos(t)dt + i\int_0^{\frac{\pi}{4}} \sin(t)dt \\
&= \sin(t)|_0^{\pi/4} + i(-\cos(t))|_0^{\pi/4} \\
&= \frac{\sqrt{2}}{2} + i(1 - \frac{\sqrt{2}}{2})
\end{aligned}$$

4.2 Lecture 20

4.2.1 Anti-Derivatives

Example 4.2.1

$$
\begin{aligned}
e^{it} &= \cos(t) + i\sin(t) \\
W(t) &= \frac{e^{it}}{i} \\
&= -ie^{it} = \sin(t) - i\cos(t) \\
W'(t) &= \cos(t) + i\sin(t) = e^{it} = w(t) \\
\text{So } \int_0^{\pi/4} e^{it} &= W(\pi/4) - W(0) \\
&= -i(e^{\pi/4} - e^0) = -i(\sqrt{2}/2 + i\sqrt{2}/2 - 1) \\
&= \sqrt{2}/2 + i(1 - \sqrt{2}/2)
\end{aligned}
$$

Theorem 4.2.1

Suppose $D \subseteq \mathbb{C}$ is a domain, $f : D \to \mathbb{C}$ is analytic, and $(a, b) \subset D$.
Let $w(t) = f(t)$ for $a < t < b$.
Then $w'(t) = f'(z)$ where $z = t$.

Example 4.2.2

$f(z) = \exp(iz)$ is entire
$f'(z) = i\exp(iz)$
$\frac{d}{dt}e^{it} = i\exp(it) = ie^{it}$

4.2.2 Curves and Contours

Definition 4.2.1: Complex Curve

A curve in $\mathbb{R}^2$ is given by continuous functions $x = x(t),\ y = y(t),\ a \leq t \leq b$.
The complex version is defined as $z(t) = x(t) + iy(t),\ a \leq t \leq b$.

Definition 4.2.2: Smooth Curve

A curve given by $(x(t), y(t)),\ a \leq t \leq b$ is smooth if $x'(t),\ y'(t)$ are continuous on (a, b) and $(x'(t), y'(t)) \neq (0, 0)$ for $a < t < b$.
The complex version is where $z'(t) = x'(t) + iy'(t) \neq 0$ for $a < t < b$.

4.2.3 Arc Length

Definition 4.2.3: Arc Length

The arc length of a smooth curve is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b |z'(t)| dt$$

since $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |z(t)|$

4.2.4 Contour Integral

Definition 4.2.4: Contour Integral

Let C be a smooth curve parameterized by $z(t) = x(t) + iy(t),\ a \leq t \leq b$.
Suppose $f(z)$ is continuous on C.
The contour integral

$$\int_C f(z)dz := \int_a^b f(z(t))z'(t)dt$$

where $dz = z'(t)dt$

Example 4.2.3

Find $\int_C \overline{z}dz$.

$$\begin{aligned} z &= t(2+i),\ 0 \leq t \leq 1 \\ &= 2t + ti \\ z' &= 2 + i \\ dz &= (2+i)dt \\ \overline{z} &= t(2-i) \\ \int_C \overline{z}dz &= \int_0^1 t(2-i)(2+i)dt \\ &= \int_0^1 5tdt \\ &= \frac{5}{2} \end{aligned}$$

4.3 Lecture 21

4.3.1 Parameterizations

We now look at parameterizations of specific shapes.

For a full circle:
$|z - z_0| = r_0$
$z - z_0 = r_0 e^{it}$
$z = z_0 + r_0 e^{it},\ 0 \leq t \leq 2\pi$
For an arc of a circle:
$z = z_0 + r_0 e^{it},\ \alpha \leq t \leq \beta$
For a line segment from z_1 to z_2 :

$z = z_1 + t(z_2 - z_1),\ 0 \le t \le 1$

Example 4.3.1 (Parameterize the line segment from $-1+i$ to 3.)

$$\begin{aligned} z &= (-1+i) + t(3 - (-1+i)) \\ &= (-1+i) + t(4-i) \\ &= (-1+4t) + i(1-t),\ 0 \le t \le 1 \end{aligned}$$

For a curve that is a graph of a function $y = f(x)$:
$a \le x \le b$ and $y = f(x)$
The curve C is parameterized by $z = t + if(t),\ a \le t \le b$

4.4 Lecture 22

4.4.1 Properties of Integrals

Provided that the integrals exist,

1.
$$\text{Re}(\int_a^b f(t)dt) = \int_a^b \text{Re}(f(t))dt$$
$$\text{Im}(\int_a^b f(t)dt) = \int_a^b \text{Im}(f(t))dt$$

2.
$$\int_a^b \omega_0 f(t)dt = \omega_0 \int_a^b f(t)dt$$
if ω_0 is a complex constant

3.
$$|\int_a^b f(t)dt| \le \int_a^b |f(t)|dt$$

Proof: **Case 1:** $f(t)$ is real-valued.
Then $-|f(t)| \le f(t) \le |f(t)|$.
Thus $-\int_a^b |f(t)|dt \le \int_a^b f(t)dt \le \int_a^b |f(t)|dt$.
So $|\int_a^b f(t)dt| \le \int_a^b |f(t)|dt$.

Case 2: $\int_a^b f(t)dt = 0$.
$|\int_a^b f(t)dt| = 0 \le \int_a^b |f(t)|dt$ since $|f(t)| \ge 0|$ for $a \le t \le 1$.

Case 3: $\int_a^b f(t)dt = re^{i\theta}$ where $r > 0,\ \theta \in \mathbb{R}$.
$|\int_a^b f(t)dt| = |re^{i\theta}| = r = e^{-i\theta}\int_a^b f(t)dt = \int_a^b e^{-i\theta}f(t)dt$ (since $e^{-i\theta}$ is a constant).
$r = \text{Re}(r)$ (since $r \in \mathbb{R}$).
Thus, $r = \text{Re}(\int_a^b e^{-i\theta}f(t)dt) = \int_a^b \text{Re}(e^{-i\theta}f(t)dt) \le \int_a^b |e^{-i\theta}f(t)|dt$ since $\text{Re}(x) \le |x|$.
Then, $\int_a^b |e^{-i\theta}f(t)|dt = \int_a^b |f(t)|dt$ since $|e^{-i\theta}f(t)| = |e^{-i\theta}||f(t)| = |f(t)|$.
So $|\int_a^b f(t)dt| = r \le \int_a^b |f(t)|dt$.

😊

4.4.2 Properties of Contour Integrals

Provided that the integrals exist,

1. $$\int_C (f(z)+g(z))dz = \int_C f(z)dt + \int_C g(z)dz$$

2. $$\int_C \alpha f(z)dz = \alpha \int_C f(z)dz$$

   if α is a complex constant.

3. $$\int_{-C} f(z)dz = -\int_C f(z)dz$$

   where $-C$ is where you plug in $-t$ into the parameterization of z, i.e. you use $z(-t)$ for $-b \leq t \leq -a$

4. Suppose $|f(z)| \leq M$ for $z \in \mathbb{C}$.

   Then

   $$|\int_C f(z)dz| \leq LM$$

   where L is the length of C.

   Proof: Assume $L =$ Length of $C = \int_a^b |z'(t)|dt$ where C is parameterized by $z(t),\ a \leq t \leq b$.
   Assume $|f(z)| \leq M$ for $z \in C$, for M is a constant.
   $|\int_C f(z)dz| = |\int_a^b f(z(t))z'(t)dt| \leq \int_a^b |f(z(t))z'(t)|dt$ by property 3 of section 4.4.1 (the previous section).
   For $a \leq t \leq b,\ z(t) \in C$.
   $|f(z(t))| \leq M$, and $|f(z(t))z'(t)| = |f(z(t))||z'(t)| \leq M|z'(t)|$ for $a \leq t \leq b$.
   $|\int_C f(z)dz| \leq |\int_a^b |f(z(t))z'(t)|dt \leq \int_a^b M|z'(t)|dt = M\int_a^b |z'(t)|dt = ML$. 😊

Example 4.4.1

Previously in example 4.2.3, we showed that the $\int_C \overline{z}dz = \frac{5}{2}$.
Now we find $\int_C \overline{z}dz$ for the curve that goes from the origin to i, and from i to $1+i$.
We break this up into 2 curves:
$C_a:\ z = 0 + t(i-0) = it,\ 0 \leq t \leq 1$
$\overline{z} = -it$
$dz = idt$

$$\begin{aligned}\int_{C_a} \overline{z}dz &= \int_0^1 (-it)idt \\ &= \int_0^1 tdt \\ &= \frac{1}{2}\end{aligned}$$

$C_b:\ z = i + t,\ 0 \leq t \leq 2$
$\overline{z} = t - i$
$z = i + t$

$$\begin{aligned}\int_{C_b} \overline{z}dz &= \int_0^2 (t-i)dt \\ &= 2 - 2i\end{aligned}$$

So

$$\int_C \overline{z}dz = \int_{C_a} \overline{z}dz + \int_{C_b} \overline{z}dz = \frac{5}{2} - 2i$$

Note

Even though examples 4.2.3 and 4.4.1 start and finish at the same points, the contour of integral of $\overline{z}$ over each of the curves are not equal. They will be equal if the function is analytic, but that will be proved later.

4.5 Lecture 23

Example 4.5.1 (Show that $|\int_C \frac{\overline{z}+1}{z^3-2}dz| \leq \pi/2$ where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ in the first quadrant.)

Let L be the length of C which is $\frac{1}{4} \cdot 4\pi = \pi$.
For $z \in C,\ |\frac{\overline{z}+1}{z^3-2}| = |\overline{z}+1| \cdot \frac{1}{|z^3-2|} \leq 3 \cdot \frac{1}{6} = \frac{1}{2} = M$ since $|\overline{z}+1| \leq 3$ and $|z^3 - 2| \geq 6$.
So $|\int_C \left(\frac{\overline{z}+1}{z^3-2}\right) dz| \leq LM = \pi \cdot \frac{1}{2} = \pi/2$.

We also proved property 3 of section 4.4.1 and property 4 of section 4.4.2.

4.5.1 Anti-Derivatives of Contour Integrals

Suppose D is a domain and $f : D \to \mathbb{C}$ is continuous.
$F : D \to \mathbb{C}$ is an anti-derivative of f if F is analytic on D and $F'(z) = f(z)$ for $z \in D$.

Example 4.5.2

$F(z) = \frac{1}{3}z^3$ is clearly an anti-derivative of $f(z) = z^2$.

Note

$F'(z) = f(z)$ for $z \in \mathbb{C},\ F$ is entire and f is continuous.

Theorem 4.5.1

Suppose $D \subseteq \mathbb{C}$ is a domain, $F : D \to \mathbb{C}$ is analytic, C is a contour, $C \subset D$ parameterized by $\gamma(t),\ a \leq t \leq b$.

1. Then
$$\frac{d}{dt}F(\gamma(t)) = F(\gamma(t))\gamma'(t)$$
2. If $f : D \to \mathbb{C}$ is continuous and F is an anti-derivative of f, then
$$\int_C f(t)dz = F(z_2) - F(z_1)$$

Proof:

$$\begin{aligned}
\int_C f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b F'(\gamma(t))\gamma'(t)dt \\
&= \int_a^b \frac{d}{dt}F(\gamma(t))dt \\
&= F(\gamma(t))|_{t=a}^{t=b} \\
&= F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1)
\end{aligned}$$

😊

4.6 Lecture 24

Note

Part 2 of the previous theorem (4.5.1) is true for a general contour (i.e. a piecewise smooth contour that you break up into parts).

Example 4.6.1 (Find $\int_C z^2 dz$ where C is given as some funny curve that is really difficult to parameterize, where $z_1 = 1$ and $z_2 = 2i$.)

$f(z) = z^2$ is continuous and has an entire analytic anti-derivative $F(z) = \frac{z^3}{3}$.
So

$$\begin{aligned}\int_C &= F(2i) - F(1) \\ &= \frac{(2i)^3}{3} - \frac{1}{3} \\ &= -\frac{8i}{3} - \frac{1}{3} \\ &= -\frac{1}{3}(1 + 8i).\end{aligned}$$

Example 4.6.2 (Find $\int_C \sin(z) dz$ where C is given another random funny curve that is really difficult to parameterize but is also in a loop (i.e. a closed contour) starting from $z = 1$.)

$f(z) = \sin(z)$ is continuous and has an entire anti-derivative $F(z) = -\cos(z)$.
So

$$\begin{aligned}\int_C \sin(z) dz &= F(1) - F(1) \\ &= 0.\end{aligned}$$

Theorem 4.6.1

Suppose $f(z)$ is continuous and has an analytic anti-derivative on a domain D.
If C is a closed contour in D, then

$$\int_C f(z) dz = 0.$$

Theorem 4.6.2

Suppose $D \subseteq \mathbb{C}$ is a domain, and $f : D \to \mathbb{C}$ is continuous.
The following are equivalent:

1. f has an analytic anti-derivative F on D.
2. $\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$ for any two contours $C_1, C_2 \in D$ that start and finish at the same points, i.e. f is **independent of path**.
3. $\int_C f(z) dz = 0$ for any closed contour $C \subset D$.

It is "clear" that (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3). It is also "clear" that (2) $\Longleftrightarrow$ (3).

Idea of Proof of (2) $\Longrightarrow$ (1) (find proof in book): Assume (2).
Let $z_0 \in D$ be fixed and let $z \in D$.
Let C_z be any contour for z_0 to z.
Define $F(z) + \int_{C_z} f(w) dw$.
It can be shown that F is differentiable in D and $F'(z) = f(z)$. 😄

Theorem 4.6.3 Cauchy's Integral Theorem

Let C be a simple (doesn't cross itself) closed contour with positive orientation.
Let R be the closed region bounded by C ($R = C \cup$ (Whatever's inside C). R is not a domain.

Note

For some domain D, $R \subset D$ and f is analytic on D.

Suppose $f(z)$ is continuous on R and analytic at each point of R, and suppose f' is continuous.
Then $\int_C f(z)dz = 0$.
Notice that these are the conditions that work for Green's Theorem.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy, x, y \in \mathbb{R}$.

$$\begin{aligned}
\int_C f(z)dz &= \int_a^b f(x(t) + iy(t))(x'(t) + iy'(t))dt \\
&= \int_a^b (u(x(t), y(t)) + iv(x(t), y(t)))(x'(t) + iy'(t)) \\
&= \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt \\
&= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \text{ by Green's Theorem} \\
&= \iint_R 0dxdy + i\iint_R 0dxdy \text{ since } f \text{ is analytic so that } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\
&= 0
\end{aligned}$$

☺

4.7 Lecture 25

We proved theorem 4.6.3.

Goursat proved that the condition that f' is continuous is **not** necessary.

Theorem 4.7.1 Cauchy-Goursat Theorem

Suppose C is a simple closed contour.
Suppose f is analytic at each point of the curve C and at each point inside C (i.e. analytic on every point in the region R).

Note

This implies that f is analytic on a region slightly bigger than R.

Then

$$\int_C f(z)dz = 0.$$

4.8 Lecture 26

4.8.1 Simply and Multiply Connected Domains

Definition 4.8.1: Simple Connected Domain

A domain D is simply connected if any closed contour $C \subset D$ encloses only points in D (i.e. it has no "holes").

Definition 4.8.2: Multiply Connected Domains

Any domain that is **not** simply connected is multiply connected.

Theorem 4.8.1

Suppose f is analytic on a simply connected domain D.
Then $\int_C f(z)dz = 0$ for all closed contours $C \subset D$.

Proof: Follows by the Cauchy-Goursat Theorem. 😊

Corollary 4.8.1

Any analytic function on a simply connected domain D as an anti-derivative (analytic on D).

Corollary 4.8.2

Any entire function has an anti-derivative (which is also entire).

Theorem 4.8.2 The Cauchy-Goursat Theorem for Multiply Connected Domains

Suppose R is a closed region bounded by finitely many disjoint simple closed contours $C_1, C_2, \ldots, C_n$ oriented so that any point in on the interior of R is on the "left."
Let $B = \bigcup_{j=1}^{n} C_j$.
If f is analytic on R then

$$\int_B f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz = 0.$$

4.8.2 Deformation of Path

Suppose we are given a contour C_1 and we want to know $\int_{C_1} f(z)dz$. We can then look at a contour C_2 within C_1 that contains the "bad parts/singularities" of C_1 to make the contour easier to work with.

More formally, if C_1, C_2 are simple closed contours (with positive orientation) and C_2 is inside C_1, and $f(z)$ is analytic on the closed region bounded by C_1 and C_2, by the Cauchy-Goursat Theorem for multiply connected domains,

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0.$$

So

$$\int_{C_1} f(z)dz = -\int_{-C_2} f(z)dz = \int_{C_2} f(z)dz.$$

4.8.3 Cauchy's Integral Formula

Let C be a simple closed contour with positive orientation.

Suppose f is analytic on C and at each point inside C.

Let z_0 be any point inside C.

Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

Note

$f(z_0)$ for z_0 inside C is completely determined by the values of $f(z)$ for $z \in C$.

Lemma 4.8.1

Let C be the simple closed circle $|z - z_0| = R$ with positive orientation where $z_0 \in \mathbb{C}$ and $R > 0$. Then

$$\int_C \frac{1}{z - z_0} dz = 2\pi i.$$

Proof: Let $C : z = z_0 + Re^{it}, 0 \leq t \leq 2\pi$ so that $dz = Rie^{it}dt$ and $z - z_0 = Re^{it}$.

$$\begin{aligned} \int_C \frac{dz}{z - z_0} &= \int_0^{2\pi} \frac{Rie^{it}}{Re^{it}} dt \\ &= \int_0^{2\pi} idt \\ &= i \int_0^{2\pi} 1dt \\ &= 2\pi i \end{aligned}$$

☺

4.9 Lecture 27

We proved lemma 4.8.1.

Now we show that this applies for any $f(z)$.

Let C be a simple closed contour with positive orientation.

Let z_0 be a point inside C.

If $f(z)$ is analytic on C and at each point inside C, then

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = f(z_0).$$

Proof: Suppose z_0 is inside C and f is analytic on C and inside C.

Let C_R be the simple circle $|z - z_0| = R$ with positive orientation and let C_R be inside C since the inside of C is a domain.

$f(z)$ and $\frac{f(z)}{z - z_0}$ is analytic on the closed region bounded by C and C_R.

By the principle of deformation of paths,

$$\begin{aligned}
\int_C \frac{f(z)}{z-z_0}dz &= \int_{C_R} \frac{f(z)}{z-z_0}dz \\
&= \int_{C_R} \frac{f(z)-f(z_0)+f(z_0)dz}{z-z_0} \\
&= \int_{C_R} \frac{f(z)-f(z_0)dz}{z-z_0} + \int_{C_R} \frac{f(z_0)dz}{z-z_0} \\
&= \int_{C_R} \frac{f(z)-f(z_0)dz}{z-z_0} + f(z_0)\int_{C_R} \frac{1}{z-z_0}dz \\
&= \int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}dz + f(z_0)2\pi i \\
\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0} &= \frac{1}{2\pi i}\int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}dz + f(z_0) \\
\left|\left(\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz\right) - f(z_0)\right| &= \left|\frac{1}{2\pi i}\int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}dz\right|
\end{aligned}$$

Let M_R be the maximum of $|f(z)-f(z_0)|$ when $|z-z_0| \leq R$.

Since f is analytic at z_0 and continuous at z_0,

$$\lim_{R\to 0} M_R = 0.$$

So for $z \in C_R,\ |z-z_0| = R$,

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \frac{|f(z)-f(z_0)|}{R} \leq \frac{M_R}{R}.$$

Let L be the length of C_R which is $2\pi R$.

$$0 \leq \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz - f(z_0)\right| \leq \frac{1}{2\pi}(2\pi R)\frac{M_R}{R} = M_R.$$

Hence $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz = f(z_0)$. ☺

Example 4.9.1

Take the square centered at the origin with side lengths of 4 with a positive orientation.
Find $\int_C \frac{\cos(z)}{z(z^2+8)}dz$.
$z(z^2+8) = 0$ for $z = 0,\ z^2 = -8$, i.e. $z = 0, \pm i\sqrt{8}$.
Here $z_0 = 0,\ f(z) = \frac{\cos(z)}{z^2+8}$ and $f(z)$ is analytic for $z \neq \pm 2\sqrt{2}i$ which lie outside C.
So

$$\begin{aligned}
\int_C \frac{\cos(z)}{z(z^2+8)}dz &= \int_C \frac{f(z)}{z-0}dz \\
&= 2\pi i f(0) \text{ by the Cauchy Integral Formula} \\
&= 2\pi(\frac{1}{8}) \\
&= \frac{\pi i}{4}
\end{aligned}$$

Theorem 4.9.1 Cauchy Integral Formula for Derivatives

Suppose C is a simple closed contour with positive orientation and z is a point inside C and f is analytic on C and inside C.

So by the Cauchy Integral Formula,

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \\
f'(z) &= \frac{1}{2\pi i}\frac{d}{dz}\int_C \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_C \frac{\partial}{\partial z}\left(\frac{f(w)}{w-z}\right) dw \text{ which can be proved} \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw^2 \\
f'(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz \text{ provided } z_0 \text{ is inside } C
\end{aligned}
$$

It turns out that all derivatives $f^{(n)}z_0$ exist and

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Example 4.9.2

Find $\int_C \frac{\sinh(z)}{z^4}dz$ where C is a square with side lengths of 2 centered at the origin with positive orientation.
$f(z) = \sinh(z) = \frac{1}{2}(e^z - e^{-z})$ is entire.
Here $z_0 = 0$ is inside C and $n = 3$.
By the Cauchy Integral Formula for Derivatives,

$$\int_C \frac{\sinh(z)}{z^4} = 2\pi i\frac{f^{(3)}(0)}{3!} = \frac{2\pi i}{6} = \frac{\pi i}{3}$$

since $f^{(3)}(z) = \cosh(z)$.

4.10 Lecture 28

4.10.1 Maximum Modulus Principle

Lemma 4.10.1

Let f be analytic on an open disk $D(z_0, r)$ where $z_0 \in \mathbb{C},\ r > 0$.
Suppose that $|f(z)| \le |f(z_0)|$ for all $z \in D$.
Then $f(z)$ is constant on D, i.e. $f(z) = f(z_0)$ for all $z \in D$.

Proof: Let $z_1 \in D$ with $z_1 \ne z_0$.
Let $\rho = |z_0 - z_1|$, and let C_ρ be the positively oriented circle with center z_0 and radius ρ.
$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-z_0}dz$ by the Cauchy Integral Formula
Parameterize C_ρ by $z = z_0 + \rho e^{it}, 0 \le t \le 2\pi$.
Then $dz = \rho i e^{it}dt$.

We get

$$
\begin{aligned}
f(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{it}}{\rho e^{it}}\rho i e^{it} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{it})dt \\
|f(z_0)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+\rho e^{it})dt\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|dt \\
\text{Since } z_0+\rho e^{it} &\in D \text{ for } 0 \leq t \leq 2\pi \\
\text{we get } |f(z_0+\rho e^{it})| &\leq |f(z_0)|. \\
\text{Hence } \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|dt &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|dt \\
&= |f(z_0)|. \\
\text{So } |f(z_0)| &= \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|dt \\
|f(z_0)| &= \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|dt \\
0 &= \frac{1}{2\pi}\int_0^{2\pi} \left(|f(z_0)| - |f(z_0+\rho e^{it})|\right)|dt
\end{aligned}
$$

Since $|f(z_0)| - |f(z_0+\rho e^{it})|$ is a continuous function on $[0, 2\pi]$, it must be 0 everywhere.
So $|f(z_0)| = |f(z_0+\rho e^{it})|$ for $0 \leq t \leq 2\pi$.
Hence $|f(z_1)| = |f(z_0)|$ for all $z_1 \in D$.
Thus $f(z)$ is constant on D. 😄

The Maximum Modulus Principle states the following:
Let f be analytic on a domain D.
Assume f is not constant.
Then $|f(z)|$ has no maximum on D.
This means there is no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$.

4.11 Lecture 29

Example 4.11.1 ($f(z) = z$)
Let $D(0,1) = D$.
Then $|z| < 1$ and $|f(z)| = |z| < 1$.
There is no point z_0 in D such that $|f(z)| = |z| \leq |z_0|$.

Idea of Proof: Suppose by way of contradiction that there is such a point z_0 such that $|f(z)| \leq |f(z_0)|$ holds.
Let $w \in D,\ w \neq z_0$.
Then there is a polygonal path from z_0 to w, and you can apply the previous lemma to every part of that path.
Continuing this way, we eventually show that $f(w) = f(z_0)$.
This means f is constant on D, which is a contradiction.
So if f is analytic and non-constant on D then $|f(z)| \leq |f(z_0)|$ cannot happen. 😄

Corollary 4.11.1

Let R be a closed bounded region in $\mathbb{C}$ such that $R = D \cup \text{boundary of } D$ where D is a domain.
Suppose f is analytic on D and continuous on R.
$|f(z)|$ reaches its maximum on the boundary an nowhere inside provided f is not constant.

4.11.1 Cauchy's Inequality

Let $R > 0$ and suppose f is analytic on the closed disk $\overline{D}(z_0, R) = \{z : |z - z_0| \leq R\}$.
Let C be the boundary of this disk $C = \{z : |z - z_0| = R\}$.
Suppose M_R is the max of $|f(z)|$ on C.
Then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}.$$

Proof: By the previous corollary, $|f(z)| \leq M_R$ for $z \in C$ and $|z - z_0| < R$.
By the Cauchy Integral formula for derivatives,

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

since z_0 is inside C.

$$\begin{aligned} \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| &\leq \frac{M_R}{|z - z_0|^{n+1}} \text{ for } z \in C \\ &= \frac{M_R}{R^{n+1}} \\ \text{Length of } C = L &= 2\pi R \\ \left| \frac{f^{(n)}(z_0)}{n!} \right| &= \frac{1}{2\pi} \left| \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \\ &\leq \frac{1}{2\pi} 2\pi R \cdot \frac{M_R}{R^{n+1}} \\ &= \frac{M_R}{R^n} \end{aligned}$$

😄

4.11.2 Liouville's Theorem

Theorem 4.11.1

Any bounded entire function is constant.

Proof: Let f be entire and bounded.
Then there is a constant M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.
Let z_0 be any point in $\mathbb{C}$ and let R be any positive real number.
Then $|f(z)| \leq M$ for z on the circle $|z - z_0| = R$.
By Cauchy's Inequality, $0 \leq |f'(z)| \leq \frac{M_R}{R^1}$ (M does **not** depend on R).
Letting $R \to \infty$ we get $|f'(z_0)| = 0$ and $f'(z_0) = 0$.
But z_0 was any complex number, so

$$f'(z) = 0$$

for all $z \in \mathbb{C}$.
Therefore f is a constant function, since $\mathbb{C}$ is a domain. 😄

4.11.3 Fundamental Theorem of Algebra

Theorem 4.11.2

Any complex polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ where $(a_n \neq 0)$ and $n \geq 1$ has a complex root.
That is, $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$.
The proof of this relies on Liouville's Theorem.

Proof: We prove this by way of contradiction.
Consider $f(z) = \frac{1}{P(z)}$.
If P does **not** have any zeroes, then f is entire.
Observe that

$$\begin{aligned}\lim_{z\to\infty} f(z) &= \lim_{z\to\infty} \frac{1}{a_0 + a_1 z + \ldots + a_n z^n} \\ &= \lim_{z\to\infty} \frac{1}{z^n}\left(\frac{1}{(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n})}\right) \\ &= 0.\end{aligned}$$

By definition of a limit, there exists $k > 0$ such that $|f(z)| \leq 1$ for any z such that $|z| \geq k$.
Since f is continuous, we know that there exists $M > 0$ such that the max value of $|f(z)| \leq M$ for $z \in \overline{D}(0, k)$.
So the max of $|f(z)| \leq \max\{1, M\}$ for $z \in \mathbb{C}$.
So f is a bounded entire function.
By Liouville's Theorem, f is constant, which is a contradiction since $P(x)$ is not constant and thus f is not constant.
So $P(x)$ must have a complex root. 😄

Corollary 4.11.2

Any complex polynomial can be completely factored.
$P(z) = a_n(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)$ for some complex constants $\alpha_1, \alpha_2, \ldots \alpha_n$ which are not necessarily distinct (i.e. multiplicity).

Chapter 5

Series

5.1 Lecture 30

We proved the FTA from the previous lecture.

5.1.1 Convergence of Sequences

Definition 5.1.1

An infinite sequence $z_1, z_2, \ldots, z_n, \ldots$ of complex numbers (also written $\{z_n\}_{n=1}^{\infty}$) converges to a complex number w if

$$\lim_{n \to \infty} |z_n - w| = 0.$$

If this is the case we say that the sequence has a limit point w and we write $\lim_{n \to \infty} z_n = w$.

Example 5.1.1

Let $z_n = 1 + \frac{e^{in}}{n}, n = 1, 2, \ldots$
We claim that $\lim_{n \to \infty} z_n = 1$.

$$\begin{aligned}
|z_n - w| &= |1 + \frac{e^{in}}{n} - 1| \\
&= |\frac{e^{in}}{n}| \\
&= \frac{|e^{in}|}{n} \\
&= \frac{1}{n}
\end{aligned}$$

5.1.2 Convergence of Series

We want to make sense of infinite sums:

$$\sum_{n=1}^{\infty} z_n.$$

We define finite sums as

$$S_m := \sum_{n=1}^{m} z_n.$$

which is a sequence! More specifically, $\{S_m\}_{m=1}^{\infty}$ is the sequence of partial sums.

Definition 5.1.2

A series of complex numbers $\sum_n z^n$ is said to be convergent if there exists $w \in \mathbb{C}$ such that

$$\lim_{m\to\infty} S_m = w.$$

Example 5.1.2 (Geometric Series)

Consider

$$\sum_{n=0}^{\infty} z^n, \ |z| < 1.$$

Then we have

$$\begin{aligned} S_m &= \sum_{n=0}^{m} z^n \\ &= 1 + z + \ldots + z^m \\ zS_m &= z + z^2 + \ldots + z^{m+1} \\ S_m - zS_m &= 1 - z^{m+1} \\ &= S_m(1-z) \\ \implies S_m &= \frac{1-z^{m+1}}{1-z}. \\ \text{Also, observe that since } |z| < 1, \ &\lim_{k\to\infty} z^k = 0. \\ \text{So } \lim_{m\to\infty} S_m &= \frac{1}{1-z}. \end{aligned}$$

So

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Definition 5.1.3: Power Series

Let $z_0 \in \mathbb{C}$.
A power series in $z - z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where $a_0, a_1, \ldots a_n, \ldots \in \mathbb{C}$.

Example 5.1.3

The geometric series is a power series with $z_0 = 0$.

Example 5.1.4

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

for $|z| < \infty$.

5.2 Lecture 31

5.2.1 Taylor Series

Theorem 5.2.1

Let $z_0 \in \mathbb{C}, r_0 > 0$.
Suppose that f is analytic on the disk $D(z_0, r_0)$.
Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges for every $z \in D$, and

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z)$$

for all $|z - z_0| < r_0$.

Example 5.2.1 (MacLaurin Expansions, i.e. where $z_0 = 0$.)

1. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \; |z| < 1$
2. $\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}(-1)^n z^n, \; |z| < 1$

Note

More generally,

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}, \; |w| < 1$$

for "some stuff" w.

3. $\sum_{n=0}^{\infty}(z^2)^n = \sum_{n=0}^{\infty} w^n = \frac{1}{1-w}(\text{for } |w| < 1) = \frac{1}{1-z^2}(\text{for } |z| < 1)$
   Note that $|w| = |z^2| = |z|^2 < 1 \implies |z| < 1$

5.2.2 Laurent Series

A Laurent expansion is a Taylor expansion that also allows for negative integers as exponents.

Theorem 5.2.2

Suppose f is analytic on the annulus (a ring) $r_2 < |z - z_0| < r_1$.
Let C be a simple closed contour with positive orientation around z_0.
Then f has an expansion

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \; z \text{ s.t. } r_2 < |z - z_0| < r_1 \\ &= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \\ \text{where } a_n &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \; b_n = a_{-n} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz. \end{aligned}$$

The expansion is called the Laurent expansion and it is **unique**.
The second sum is called the **principal part**.
In particular, $b_1 = a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz$ is the **residue** of f at z_0 denoted $\text{Res}_{z_0} f$.

Example 5.2.2 (Find the Laurent expansion about $z_0 = 0$ for $f(z) = \frac{1}{z(z-1)}$.)

First we look at $0 < |z| < 1$.

$$\begin{aligned}
f(z) &= \frac{1}{z}\frac{1}{z-1} \\
&= \frac{1}{z}\left(-\frac{1}{1-z}\right) \\
&= \frac{1}{z}\left(-\sum_{n=0}^{\infty} z^n\right) \\
&= -\sum_{n=0}^{\infty} z^{n-1} \\
&= -\frac{1}{z} + (\sum_{n=0}^{\infty} -z^n).
\end{aligned}$$

Now we consider another annulus $1 < |z| < \infty \implies |\frac{1}{z}| = \frac{1}{|z|} < 1$.

$$\begin{aligned}
f(z) &= \frac{1}{z}\frac{1}{z-1} \\
&= \frac{1}{z}\frac{\frac{1}{z}}{1-\frac{1}{z}} \\
&= \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} \\
&= \frac{1}{z^2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^2\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{z^{n+2}} \\
&= \sum_{n=2}^{\infty}\frac{1}{z^n} \\
&\implies a_n = 0,\ b_1 = 0,\ b_n = 1 \text{ for } n \geq 2.
\end{aligned}$$

Chapter 6

Residues and Poles

6.1 Lecture 32

6.1.1 Isolated Singular Points

Definition 6.1.1

We say that $z_0 \in \mathbb{C}$ is a singular point (or singularity) of the function f when:

1. f is not analytic at z_0
2. Each neighborhood of z_0 contains points at which f is analytic

Definition 6.1.2

We say the singularity of z_0 of f is **isolated** when there exists $r > 0$ such that f is analytic on the deleted/punctured neighborhood $D'(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$.

Example 6.1.1 ($f(z) = \frac{e^z}{z^2+1}$ on $\mathbb{C} \setminus \{\pm i\}$)

$\pm i$ are singular points of f (and also the only ones).

Note

It wouldn't matter how you defined f at $\pm i$ because it's not differentiable.

They are also **isolated**; in fact, f is complex differentiable on $\mathbb{C} \setminus \{\pm i\}$.

Example 6.1.2 ($f(z) = \frac{1}{\sin(\frac{1}{z})}$)

Singular points arise at $z = 0$ and where $\sin(\frac{1}{z}) = 0$, i.e. $z = \frac{1}{n\pi}, n = \pm 1, \pm 2, \ldots$
The points $z = \frac{1}{n\pi}$ are isolated.
$z = 0$ is not because each $D'(0, r)$ contains $\frac{1}{n\pi}$ when $n \gg 0$.

Let z_0 be an isolated singular point of f and assume that f is analytic on $D' = D'(z_0, r)$.
D' is a special kind of annulus and thus has a Laurent expansion.

$$
\begin{aligned}
f(z) &= \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \\
&= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}
\end{aligned}
$$

where C is any simple closed positively oriented contour in D' surrounding z_0.

Example 6.1.3 (Calculate $I = \int_C \frac{e^z}{(z-1)^2} dz$ where C is the counterclockwise circle around $z = 1$.)

Let $f(z) = \frac{e^z}{(z-1)^2}$.
Then I is $\int_C f(z)dz = 2\pi i \operatorname{Res}_{z=1} f$ where C is in $D'(1, r)$ with $r > 1$.
Now we find an expansion of the form $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-1)^n$ valid in D', and then $I = 2\pi i a_{-1}$.

$$\begin{aligned}
\frac{e^z}{(z-1)^2} &= \frac{e \cdot e^{z-1}}{(z-1)^2} \\
&= \frac{e}{(z-1)^2} \cdot \left(1 + (z-1) + \frac{(z-1)^2}{2} + \dots\right) \\
&= e\left((z-1)^{-2} + (z-1)^{-1} + \frac{1}{2} + \frac{1}{6}(z-1) + \dots\right) \\
\text{The residue is } b_1 = a_{-1} &= \text{the coefficient of } (z-1)^{-1} \\
&= e \\
\text{So } \int_C \frac{e^z}{(z-1)^2} dz &= 2\pi i e.
\end{aligned}$$

6.2 Lecture 33

6.2.1 Cauchy's Residue Theorem

Theorem 6.2.1 Cauchy's Residue Theorem

Suppose C is a simple closed contour with positive orientation.
Suppose $f(z)$ is analytic on C and inside C except for finitely many singularities $z_1, z_2, \dots, z_n$ inside the contour C.
Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).$$

Proof: Let $C_1, C_2, \dots, C_n$ be disjoint circles centered at $z_1, z_2, \dots z_n$ inside C with negative orientation.
Let $B = C \cup C_1 \cup \dots \cup C_n$.
By the Cauchy-Goursat theorem for multiply connected domains,

$$\begin{aligned}
0 &= \int_B f(z)dz \\
&= \int_C f(z)dz + \int_{C_1} f(z)dz + \dots \int_{C_n} f(z)dz \\
\int_C f(z)dz &= -\sum_{k=1}^{n} \int_{C_k} f(z)dz \\
&= \sum_{k=1}^{n} \int_{-C_k} f(z)dz \\
&= 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).
\end{aligned}$$

😊

Example 6.2.1

Find $\int_C \frac{e^z}{(z^2-1)}dz$ where C is the simple circle $|z|=2$ with positive orientation.
$f(z)=\frac{e^z}{(z^2-1)}=\frac{e^z}{(z-1)(z+1)}$ has singularities at $z=\pm 1$.
So $f(z)$ is analytic on C and inside C except for singularities at $z=\pm 1$ which lie inside C.
By Cauchy's Residue Theorem,

$$\begin{aligned}
\int_C f(z) &= \int_C \frac{e^z}{z^2-1}dz \\
&= 2\pi i(\text{Res}_{z=1} f(z) + \text{Res}_{z=-1} f(z) \\
&\text{We want a series in powers of } z-1. \\
f(z) &= \frac{e^z}{(z-1)(z+1)} \\
&= \frac{1}{z-1}\cdot\frac{e^z}{z+1} \\
&\frac{e^z}{z+1} \text{ is analytic for } z\neq -1 \text{ so the Taylor series near } z=1 \text{ exists.} \\
g(z) &= \frac{e^z}{z+1} \text{ is analytic for } |z-1|<2 \\
&= \sum_{n=0}^{\infty}\frac{g^{(n)}(1)}{n!}(z-1)^n \\
&= g(1)+\frac{g'(1)}{1!}(z-1)+\dots \\
&= \frac{e}{2}+\dots \text{ for } |z-1|<2 \\
f(z) &= \frac{1}{z-1}(g(z)) \\
&= \frac{e}{2}(z-1)^{-1}+g'(1)(z-1)^0+\dots \text{ for } 0<|z-1|<2 \\
\text{So } \text{Res}_{z=1} f(z) &= \frac{e}{2} \\
&\frac{e^z}{z-1} \text{ is analytic for } z\neq 1 \text{ so the Taylor series near } z=-1 \text{ exists.} \\
f(z) &= \frac{1}{z+1}\cdot\frac{e^z}{z-1} \\
&= \frac{1}{(z+1)}(h(z)) \\
&= \frac{1}{z+1}\left(\frac{e^{-1}}{-2}+\frac{h'(-1)}{1!}(z+1)^0+\dots\right) \\
\text{Res}_{z=-1} f(z) &= -\frac{1}{2e} \\
\text{So } \int_C f(z)dz &= 2\pi i(\frac{e}{2}-\frac{1}{2e})
\end{aligned}$$

Theorem 6.2.2

Suppose $\phi(z)$ is analytic at $z=z_0$.
Let $f(z)=\frac{\phi(z)}{(z-z_0)^m}$ where m is a positive integer.
Then $\text{Res}_{z=z_0} f(z)=\frac{\phi^{(m-1)}(z_0)}{(m-1)!}$

Proof: Since ϕ is analytic at $z = z_0$,

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \text{ (for } z \text{ near } z_0\text{) by Taylor's Theorem}$$

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

$$= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \text{ which is the Laurent Series near } z_0$$

$$\text{Res}_{z=z_0} f(z) = \text{Coefficient of } (z-z_0)^{-1} \text{ in the Laurent Series of } f(z) \text{ near } z = z_0$$

$$\text{We want } n = m = 1\text{, so } n = m - 1$$

$$\text{So } \text{Res}_{z=z_0} f(z) = \sum_{n=0}^{\infty} \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

😊

6.3 Lecture 34

We proved theorem 6.2.1 and theorem 6.2.2.

6.3.1 Principal Part

Suppose $f(z)$ has an isolated singularity at $z = z_0$.

By Laurent's theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$$

for $0 < |z - z_0| < r_0$ for some $r_0 > 0$.

Like we described earlier, the second sum is the principal part of $f(z)$ near $z = z_0$.

6.3.2 Types of Singularities

There are three types of (isolated) singularities:

1. **Removable Singularity**

   The principal part is 0.

   Example 6.3.1

   $$f(z) = \frac{\sin(z)}{z}$$

   $$= \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots) \qquad = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \ldots \text{ for } z \neq 0 \text{ which is the Laurent Series.}$$

   The principal part is 0, and so $z = 0$ is a removable singularity.
   Let $g(z) = 1 - 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \ldots$ which is entire.
   So $f(z) = g(z)$ for $z \neq 0$.

2. **Pole of Order m**

   The principal part has finitely many terms

   $$\frac{b_1}{(z-z_0)^1} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m} \text{ where } b_m \neq 0.$$

In this case, the singularity $z = z_0$ is called a pole of order m.

Example 6.3.2

$$\begin{aligned} f(z) &= \frac{1}{z^6(\cos(z) - 1)} \\ &= -\frac{1}{2!} \cdot \frac{1}{z^4} + \frac{1}{4!} \cdot \frac{1}{z^2} - \frac{1}{6!} + \frac{1}{8!} z^2 + \dots \text{ for } z \neq 0. \end{aligned}$$

The principal part near $z = 0$ is $-\frac{1}{2!} \cdot \frac{1}{z^4} + \frac{1}{4!} \cdot \frac{1}{z^2}$ so that $z = 0$ is a pole of order 4.

Note

$$\operatorname{Res}_{z=0} f(z) = \text{coefficient of } z^{-1} = 0.$$

3. **Essential Singularity**
   The principal part has infinitely many non-zero terms.

Example 6.3.3

$f(z) = \exp(\frac{1}{z})$ has a singularity at $z = 0$.

$$\begin{aligned} \exp(z) &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \text{ for all } z \\ f(z) &= \exp(\frac{1}{z}) \\ &= 1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \dots \text{ for all } z \neq 0. \end{aligned}$$

The principal part has infinitely many non-zero terms, so $z = 0$ is an essential singularity.

Example 6.3.4

Let C be a contour around 0 with positive orientation.
Find $\int_C \exp(\frac{1}{z}) dz$.

$$\begin{aligned} \int_C \exp(\frac{1}{z}) dz &= 2\pi i \operatorname{Res}_{z=0} f(z) \text{ by Cauchy's Residue Theorem} \\ &= 2\pi i \cdot 1 \\ &= 2\pi i \end{aligned}$$

6.4 Lecture 35

6.4.1 Residue at Infinity Theorem

Theorem 6.4.1

If a function f is analytic at all points of $\mathbb{C}$ except for a finite number of singularities that lie inside a simple closed contour C with positive orientation.
Then

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0}(\frac{1}{z^2} f(\frac{1}{z})).$$

Proof: Choose $R > 0$ large enough so that C is inside C_R (center 0 and radius R with positive orientation).

So all the singularities $z_1, z_2, \ldots z_n$ are inside C which is inside C_R.
So $f(z)$ is analytic on the closed region bounded by C and C_R, hence $\int_c f(z)dz = \int_{C_R} f(z)dz$ by the Deformation of Paths Theorem.

$$
\begin{aligned}
C_R: \; z(t) &= Re^{it}, \; 0 \leq t \leq 2\pi. \\
dz &= Rie^{it}dt. \\
\int_{C_R} f(z)dz &= \int_0^{2\pi} f(Re^{it})Rie^{it}dt \\
\widetilde{C}_R: \; z(t) &= \frac{1}{R}e^{-it}, \; 0 \leq t \leq 2\pi \\
dz &= -\frac{i}{R}e^{-it}dt \\
\text{Let } g(z) &= \frac{1}{z^2}f(\frac{1}{z}) \\
\int_{-\tilde{C}_R} g(z)dz &= \operatorname{Res}_{z=0} g(z) \text{ since the only singularity of } g \text{ is at } z=0
\end{aligned}
$$

since if $|z| < \frac{1}{R}$, $|\frac{1}{z}| > R$, and $f(\frac{1}{z})$ is analytic.

$$
\begin{aligned}
\text{But } \int_{-\tilde{C}_R} g(z)dz &= -\int_{\tilde{C}_R} g(z)dz \\
&= -\int_0^{2\pi} g\left(\frac{1}{R}e^{-it}\right)\frac{-i}{R}e^{-it}dt \\
&= -\int_0^{2\pi} \left(R^2 e^{2it}\right) f(Re^{it}) \left(\frac{-ie^{-it}}{R}\right) dt \\
&= \int_0^{2\pi} f(Re^{it})iRe^{it}dt \\
&= \int_{C_R} f(z)dz \\
\text{Hence } \int_C f(z)dz &= \int_{C_R} f(z)dz \\
&= \int_{-\tilde{C}_R} g(z)dz \\
&= 2\pi i \operatorname{Res}_{z=0} g(z) \\
\text{So } \frac{1}{2\pi i}\int_C f(z)dz &= \operatorname{Res}_{z=0}(\frac{1}{z^2}f(\frac{1}{z})).
\end{aligned}
$$

😊

Example 6.4.1

Let C be the positively oriented circle $|z| = 2$.
Find $\int_C \frac{4z-5}{z(z-1)}dz$.

1. **First Method**

   $f(z) = \frac{4z-5}{z(z-1)}$ is analytic except for singularities at $z = 0, 1$ which lie inside C.

   By Cauchy's Residue Theorem, $\int_C f(z)dz = 2\pi i \left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z)\right)$

$$
\begin{aligned}
\text{For } z &= 0 \\
f(z) &= \frac{4z-5}{z(z-1)} \\
&= \frac{\phi(z)}{z} \text{ where } \phi(z) = \frac{4z-5}{z-1} \text{ which is analytic at } z=0 \\
\operatorname{Res}_{z=0} f(z) &= \phi(0) \\
&= 5 \\
\text{For } z &= 1 \\
f(z) &= \frac{4z-5}{z(z-1)} \\
&= \frac{\psi(z)}{z-1} \text{ where } \psi(z) = \frac{4z-5}{z} \text{ which is analytic at } z=1 \\
\operatorname{Res}_{z=1} f(z) &= \psi(1) \\
&= -1 \\
&= \\
\text{So } \int_C f(z)dz &= 2\pi i(5-1) \\
&= 8\pi i
\end{aligned}
$$

2. **Second Method**

   $f(z) = \frac{4z-5}{z(z-1)}$ is analytic on all of $\mathbb{C}$ except for singularities at $z = 0, 1$ which both lie inside C.

   So by the Residue at Infinity Theorem,

$$
\begin{aligned}
\int_C f(z)dz &= 2\pi i \operatorname{Res}_{z=0} \left(\frac{1}{z^2} f\left(1/z\right) \right) \\
\text{Let } g(z) &= \frac{1}{z^2} f\left(\frac{1}{z}\right) \\
&= \frac{1}{z^2} \left(\frac{4(\frac{1}{z}) - 5}{(\frac{1}{z})(\frac{1}{z} - 1)} \right) \\
&= \frac{\frac{4}{z} - 5}{(1-z)} \cdot \frac{z}{z} \\
&= \frac{4-5z}{z(1-z)} \\
&= \frac{\phi(z)}{z} \text{ where } \phi(z) = \frac{4-5z}{1-z} \text{ which is analytic at } z=0 \\
\operatorname{Res}_{z=0} \left(\frac{1}{z^2} f\left(\frac{1}{z}\right) \right) &= \operatorname{Res}_{z=0} g(z) \\
&= \frac{\phi(0)}{1} \\
&= 4 \\
\int_C f(z)dz &= 2\pi i(4) \\
&= 8\pi i
\end{aligned}
$$

6.5 Lecture 36

We proved theorem 6.4.1.

6.5.1 Zeroes of Analytic Functions

$f(z)$ has a zero of order m at z_0 if $f(z)$ is analytic at $z = z_0$ and $0 = f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0)$ and $f^{(m)}(z_0) \neq 0$.

Since f is analytic at $z = 0$, it has a Taylor Series

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \text{ for } |z-z_0| < R_0, \text{ some } R_0 > 0 \\
&= \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \\
&= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \ldots \\
&= (z-z_0)^m \left(\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}}{(m+1)!}(z-z_0)^1 + \ldots \right) \\
&= g(z) \text{ which is analytic at } z = z_0 \text{ and} \\
g(z_0) &= \frac{f^{(m)}(z_0)}{m!} \neq 0
\end{aligned}$$

Theorem 6.5.1

Let f be analytic at z_0.
The following are equivalent:

1. f has a zero of order m at z_0
2. $f(z) = (z-z_0)^m g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$

Theorem 6.5.2

Suppose $f : D \to \mathbb{C}$ is analytic and D is a domain.
Let $Z = Z(f) = \{z \in D : f(z) = 0\}$ be the set of zeroes.
Suppose $z_0 \in Z(f)$ and $\{z_n\}_{n=1}^{\infty} \subset Z(f)$ and $\lim_{n\to\infty} z_n = z_0$ and each $z_n \neq z_0$.
Then $f(z) = 0$ for all $z \in D$.
First we claim that $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2, \ldots$

Proof: We know $f(z_0) = 0$.
Suppose by way of contradiction that $f^{(n)}(z_0) \neq 0$ for some $n \geq 1$.
Let n_0 be the smallest such n.
So $f(z_0) = f'(z_0) = \ldots = f^{(n_0-1)}(z_0) = 0$ and $f^{(n_0)}(z_0) \neq 0$.
So $f(z) = (z-z_0)^{n_0} g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.
Here $n_0 \geq 1$.
$0 = f(z_n) = (z_n - z_0)^{n_0} g(z_n)$ which implies $g(z_n) = 0$ for $n \geq 1$ since $z_n \neq z_0$ for $n \geq 1$.
But g is analytic at z_0 which implies g is continuous at z_0.
So $\lim_{n\to\infty} g(z_n) = g(z_0)$ since $\lim_{n\to\infty} z_n = z_0$.
So $0 = g(z_0)$ which is a contradiction.
Hence our first claim is true. 😊

This means $f^{(n)}(z_0) = 0$ for all $n \geq 0$.
Let $E = \{z \in D : f^{(n)}(z) = 0\}$ for all $n \geq 0$.
So $z_0 \in E$ and E is non-empty.

$E = \bigcap_{n \geq 1} E_n$ where $E_n = \{z \in D : f^{(n)}(z) = 0\}$.
Each E_n is closed since $f^{(n)}$ is continuous.
So E is closed.
Our second claim is that E is an open subset of D.

Proof: Let $w_0 \in E$.
So $f^{(n)}(w_0) = 0$ for all $n \geq 0$.
But f is analytic at w_0 so $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w_0)}{n!}(z - w_0)^n$ for $z \in D(w_0, r_0)$ for some $r_0 > 0$.
So $f(z) = 0$ for all $z \in D(w_0, r_0)$.
Therefore $D(w_0, r_0) \subset Z(f)$ and $f^{(n)}(z_0) = 0$ for all $n \geq 0,\ z \in D(w_0, r_0)$.
So $D(w_0, r_0) \subset E$ and E is open.
Therefore E is a non-empty open and closed subset of D.
Hence $E = D$ since D is open and connected by a theorem from Advanced Calculus.
It follows that $f(z) = 0$ for all $z \in D$. ☺

6.6 Lecture 37

We proved theorem 6.5.2.

Corollary 6.6.1

Suppose $\mathbb{R} \subset D$ is a domain in $\mathbb{C}$ and $f : D \to \mathbb{C}$ is analytic.
If $f(x) = 0$ for all $x \in \mathbb{R}$, then $f(z) = 0$ for all $z \in D$.

Proof: Suppose $f(x) = 0$ for all $x \in \mathbb{R}$.
Let $z_n = \frac{1}{n} \to 0 = z_0$ as $n \to \infty$ and $f(z_n) = f(0) = 0$.
The previous theorem implies $f(z) = 0$ for all $z \in D$. ☺

Example 6.6.1

$f(z) = \sin^2(z) + \cos^2(z) - 1$ is entire.
We know $f(x) = \sin_2(x) + \cos_2(x) - 1 = 0$ for all $x \in \mathbb{R}$.
So by the previous corollary, $f(z) = 0$ for all $z \in \mathbb{C}$ and $\sin^2(z) + \cos^2(z) = 1$ for all $z \in \mathbb{C}$.

Theorem 6.6.1

Suppose f and g are analytic on a domain D and $f(z) = g(z)$ on some open disk $D(z_0, r_0) \subset D$.
Then $f(z) = g(z)$ for all $z \in D$.

Proof: Let $h(z) = f(z) - g(z)$.
Then h is analytic on D and $h(z) = 0$ for all $z \in D(z_0, r_0) \subset D$.
Let $z_n = z_0 + \frac{1}{n}$ for a large enough n.
Then z_n converges to z_0.
Since $z_n \in D(z_0, r_0)$ and each $z_n \neq z_0$ and $h(z_n) = h(z_0) = 0$, the previous theorem implies $h(z_0) = 0$ for all $z \in D$ and $f(z) = g(z)$ for $z \in D$. ☺

6.6.1 Analytic Continuation

Suppose f is analytic on a domain D.
Suppose g_1 and g_2 are analytic on a different domain D'.
Let $D \subset D'$.
Suppose $f(z) = g_1(z) = g_2(z)$ for $z \in D$.
Then $g_1(z) = g_2(z)$ for all $z \in D$.

6.7 Lecture 38

Example 6.7.1

Let $f(z) = \sum_{n=0}^{\infty} z^n$.
Then f is analytic on the disc $D(0,1)$.
So $f(z)$ is defined for $|z| < 1$.
Let $g(z) = \frac{1}{1-z}$ for $z \neq 1$.
Then g is analytic on $\mathbb{C} \setminus \{1\}$.
So $f(z) = g(z)$ for $z \in D(0,1)$.
Then $g(z)$ is the analytic continuation of $f(z)$ to $\mathbb{C} \setminus \{1\}$.

6.7.1 Riemann Zeta Function

Define $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ where $s = \sigma + it,\ \sigma, t \in \mathbb{R}$.

$$\begin{aligned} n^s &= \exp(s \ln(n)) \\ |n^s| &= |\exp((\sigma + it)\ln(n))| \\ &= |\exp(2\ln(n))\exp(it\ln(n))| \\ &= \exp(\sigma \ln(n)) \\ &= n^\sigma \end{aligned}$$

The series converges for $\mathrm{Re}(s) = \sigma > 1$.

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{m} \frac{1}{n^s} \\ &= \frac{m}{m^s} + \int_1^m \frac{\lfloor u \rfloor}{u^{s+1}} du \\ \text{As } n \to \infty,\ \zeta(s) &= s\int_1^\infty \frac{\lfloor u \rfloor}{u^{s+1}} du \text{ for } \mathrm{Re}(s) = \sigma > 1 \\ &= s\int_1^\infty \frac{u}{u^{s+1}} du - s\int_0^\infty \frac{u - \lfloor u \rfloor}{u^{s+1}} du \\ &= \frac{s}{s-1} - s\int_1^\infty \frac{u - \lfloor u \rfloor}{u^{s+1}} du \text{ converges for } \sigma > 0. \end{aligned}$$

So $\zeta(s)$ can be analytically continued to

$$\frac{s}{s-1} + s\int_1^\infty \frac{u - \lfloor u \rfloor}{s^{s+1}} du$$

where the second part of the sum is analytic for $\mathrm{Re}(s) = \sigma > 0$.

So there is a pole of order 1 at $s = 1$.

The **Riemann Hypothesis** states that if $s = \sigma + it,\ 0 < \sigma < 1$ and $\zeta(s) = 0$, then $\sigma = \frac{1}{2}$.

Corollary 6.7.1

The number of primes less than or equal to x is $\mathrm{Li}(x) + O(\sqrt{x}\ln(x))$ where $\mathrm{Li}(x) = \int_0^\infty \frac{1}{\ln(t)} dt \simeq \frac{x}{\ln(x)}$.