MHF3202
Challenge Problem

The Class

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Proposition You are walking up a set of stairs that are spaced in such a way that you can either take one step at a time, or two steps at a time (i.e. you can use every step, or occasionally skip a step). Prove that there are exactly F_n different ways to ascend n steps, where F_n is the nth Fibonacci number.

Proof: We will prove this with strong induction. Define S_n as the number of distinct ways to ascend exactly n steps using only one-step or two-step jumps for $n \in \mathbb{Z},\ n \geq 0$. First, note that the only way to reach the 0th step is to take no jumps at all, since taking a jump would move to a higher step $j > 0$. Since there is exactly one way to do nothing, we conclude $S_0 = 1$. By definition of the Fibonacci sequence used in this proof, $F_0 = 1$, so $F_0 = S_0$ as required. Also note that the only way to reach the 1st step is to take a one-step jump, since taking any more one-step or two-step jumps would move to a higher step $j > 1$, and doing nothing would leave you at step 0. Since there is exactly one way to reach step 1, we conclude $S_1 = 1$. By definition of the Fibonacci sequence used in this proof, $F_1 = 1$, so $F_1 = S_1$ as required. Thus, we have our base cases for $n = 0$ and $n = 1$. For $m \geq 2$, assume $S_k = F_k$ for $0 \leq k \leq m$. We want to show $S_{m+1} = F_{m+1}$. Notice that at the beginning of any sequence, there are two options: either you take a one-step jump or you take a two-step jump. If you take a one-step jump, you arrive at step 1, leaving m steps remaining to reach step $m + 1$. By the inductive hypothesis, there are S_m ways to reach step m. Since each of these ways is a distinct sequence of steps, and adding a one-step jump to the beginning of each sequence creates a unique way to reach $m+1$, the total number of ways to reach step $m+1$ by starting with a one-step jump is precisely S_m. If you take a two-step jump, you arrive at step 2, leaving $m-1$ steps remaining to reach step $m+1$. By the inductive hypothesis, there are S_{m-1} ways to reach step $m-1$. Since each of these ways is a distinct sequence of steps, and adding a two-step jump to the beginning of each sequence creates a unique way to reach $m + 1$, the total number of ways to reach step $m + 1$ by starting with a two-step jump is precisely S_{m-1}. Furthermore, since every way of reaching step $m + 1$ by starting with a one-step jump originates from a distinct way of reaching step m and starts with a one-step jump, and every way of reaching step $m + 1$ by starting with a two-step jump originates from a distinct way of reaching step $m-1$ and starts with a two-step jump, the two sets of sequences are unique from each other, and their intersection is the null set. Thus, there are $S_{m-1} + S_m = S_{m+1}$ ways to get to the $m + 1$st step. By the inductive hypothesis, $S_{m-1} = F_{m-1}$ and $S_m = F_m$, so there are $F_{m-1} + F_m = S_{m+1}$ ways to get to the $m+1$st step. Replacing m with $m+1$ in the definition $F_{m-2} + F_{m-1} = F_m$ yields $F_{m-1} + F_m = F_{m+1}$, and we can say $F_{m+1} = S_{m+1}$. Thus, by induction, $S_n = F_n$ for all $n \geq 0$. 😊