MAA4402
HW1

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Question 1: 2.2

(a) Suppose $z \in \mathbb{C}$, then $z = x + iy$, where $x, y \in \mathbb{R}$. Then

$$\begin{aligned} \operatorname{Re}(iz) &= \operatorname{Re}(ix + i^2 y) \\ &= \operatorname{Re}(ix - y) \\ &= -y \\ &= -(y) \\ &= -\operatorname{Im}(x + iy) \\ &= -\operatorname{Im}(z) \end{aligned}$$

(b) Suppose $z \in \mathbb{C}$, then $z = x + iy$, where $x, y \in \mathbb{R}$. Then

$$\begin{aligned} \operatorname{Im}(iz) &= \operatorname{Im}(ix + i^2 y) \\ &= \operatorname{Im}(ix - y) \\ &= x \\ &= \operatorname{Re}(x + iy) \\ &= \operatorname{Re}(z) \end{aligned}$$

Question 2: 2.5

Proof: Let $z_1, z_2 \in \mathbb{C}$, with $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) \text{ by the definition of a complex number} \\ &= (x_1 x_2 - y_1 y_2, y_1 x_2 + y_2 x_1) \text{ by the definition of complex multiplication} \\ &= (x_2 x_1 - y_2 y_1, y_1 x_2 + y_2 x_1) \text{ by the commutative property of multiplication of real numbers} \\ &= (x_2 x_1 - y_2 y_1, y_2 x_1 + y_1 x_2) \text{ by the commutative property of addition of real numbers} \\ &= (x_2, y_2, x_1, y_1) \text{ by the definition of complex multiplication} \\ &= z_2 z_1 \text{ by the definition of a complex number} \end{aligned}$$

$\therefore z_1 z_2 = z_2 z_1$ showing that the multiplication of complex numbers is commutative ☺

Question 3: 2.6a

Proof: Let $z_1, z_2, z_3 \in \mathbb{C}$, with $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, z_3 = x_3 + iy_3$, where $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. Then

$$\begin{aligned} (z_1 + z_2) + z_3 &= (x_1 + iy_1 + x_2 + iy_2) + x_3 + iy_3 \text{ by the definition of a complex number} \\ &= [(x_1 + x_2) + i(y_1 + y_2)] + x_3 + iy_3 \text{ by the definition of complex addition} \\ &= ((x_1 + x_2) + x_3) + i((y_1 + y_2) + y_3) \text{ by the definition of complex addition} \\ &= (x_1 + (x_2 + x_3)) + i(y_1 + (y_2 + y_3)) \text{ by the associativity of addition in } \mathbb{R} \\ &= x_1 + iy_1 + [(x_2 + x_3) + i(y_2 + y_3)] \text{ by the definition of complex addition} \\ &= x_1 + iy_1 + (x_2 + iy_2 + x_3 + iy_3) \text{ by the definition of complex addition} \\ &= z_1 + (z_2 + z_3) \text{ by the definition of a complex number} \end{aligned}$$

$\therefore (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ showing that addition of complex numbers is associative ☺

Question 4: 3.1

Proof: Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$.
Then

$$\begin{aligned}
(z_1 z_2)(z_3 z_4) &= z_1(z_2(z_3 z_4)) \text{ by associativity of complex multiplication} \\
&= z_1((z_2 z_3) z_4) \text{ by associativity of complex multiplication} \\
&= z_1((z_3 z_2) z_4) \text{ by commutativity of complex multiplication} \\
&= z_1(z_3(z_2 z_4)) \text{ by associativity of complex multiplication} \\
&= (z_1 z_3)(z_2 z_4) \text{ by associativity of complex multiplication}
\end{aligned}$$

$\therefore (z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4)$ ☺

Question 5: 5.2

1. $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$

   Proof: Let $z \in \mathbb{C}, z = x + iy$ where $x, y \in \mathbb{R}$.
   Then

$$\begin{aligned}
\operatorname{Re}(z) &= \operatorname{Re}(x+iy) = x \leq |x| \\
&= \sqrt{x^2} \leq \sqrt{x^2+y^2} \text{ since } 0 \leq x^2 \leq x^2 + y^2 \\
\text{but } \sqrt{x^2+y^2} &= |z|
\end{aligned}$$

   $\therefore \operatorname{Re}(z) \leq |\operatorname{Re}(z)| \leq z$ ☺

2. $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$

   Proof: Let $z \in \mathbb{C}, z = x + iy$ where $x, y \in \mathbb{R}$.
   Then

$$\begin{aligned}
\operatorname{Im}(z) &= \operatorname{Im}(x+iy) = y \leq |y| \\
&= \sqrt{y^2} \leq \sqrt{x^2+y^2} \text{ since } 0 \leq y^2 \leq x^2 + y^2 \\
\text{but } \sqrt{x^2+y^2} &= |z|
\end{aligned}$$

   $\therefore \operatorname{Im}(z) \leq |\operatorname{Im}(z)| \leq z$ ☺

Question 6: 5.5

See attached scan.