Class
HW

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Question 1: 9.5a

$$
\begin{aligned}
i &= e^{i\pi/2} \\
1-\sqrt{3}i &= 2e^{-i\pi/3} \\
\sqrt{3}+i &= 2e^{i\pi/6} \\
\text{Thus, } i(1-\sqrt{3}3)(\sqrt{3}+1) &= e^{i\pi/2}2e^{-i\pi/3}2e^{i\pi/6} \\
&= 4e^{(i\pi/2)+(-i\pi/3)+(i\pi/6)} \\
&= 4e^{i(\pi/2-\pi/3+\pi/6)} \\
&= 4e^{i(\frac{\pi}{3})} \\
&= 4\left(\cos(\frac{\pi}{3})+\sin(\frac{\pi}{3})i\right) \\
&= 4\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\
&= 2(1+\sqrt{3}i)
\end{aligned}
$$

Question 2: 9.5d

$$
\begin{aligned}
1+\sqrt{3}i &= 2e^{i\frac{\pi}{3}} \\
\text{Thus, } (1+\sqrt{3}i)^{-10} &= \left(2e^{i\frac{\pi}{3}}\right)^{-10} \\
&= 2^{-10}\left(e^{i\frac{\pi}{3}}\right)^{-10} \\
&= 2^{-10}e^{-10i\frac{\pi}{3}} \text{ by de Moivre} \\
&= 2^{-10}e^{i\frac{2\pi}{3}} \\
&= 2^{-10}\left(\cos(\frac{2\pi}{3})+i\sin(\frac{2\pi}{3})\right) \\
&= 2^{-10}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\
&= 2^{-11}\left(-1+\sqrt{3}i\right)
\end{aligned}
$$

Question 3: 9.6

Suppose that $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}, r_1, r_2, \theta_1, \theta_2 \in \mathbb{R}$, and also $-\frac{\pi}{2} < \theta_1, \theta_2 \leq \frac{\pi}{2}$ since if $\mathrm{Re}(z) > 0$, then $-\frac{\pi}{2} < \mathrm{Arg}(z) < \frac{\pi}{2}$.
Then $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)}$.
r_1r_2 can be rewritten as r, since we are only concerned with the principal argument and not the radius.
So, we have $z_1z_2 = re^{i(\theta_1+\theta_2)}$, and $\mathrm{Arg}(z_1z_2) = \theta_1+\theta_2$, since $-\pi < \theta_1+\theta_2 \leq \pi$.
Also, we have that $\mathrm{Arg}(z_1) = \theta_1$, $\mathrm{Arg}(z_2) = \theta_2$, and thus $\mathrm{Arg}(z_1)+\mathrm{Arg}(z_2) = \theta_1+\theta_2$.
Therefore $\mathrm{Arg}(z_1z_2) = \theta_1+\theta_2 = \mathrm{Arg}(z_1)+\mathrm{Arg}(z_2)$.

Question 4: 11.3

$$|-8-8\sqrt{3}i| = \sqrt{64+192} = 16$$
$$\mathrm{Arg}(-8-8\sqrt{3}i) = -\frac{2\pi}{3}$$
$$(-8-8\sqrt{3}i)^{\frac{1}{4}} = \left(16e^{-i\frac{2\pi}{3}}\right)^{\frac{1}{4}}$$
$$\text{There are 4 roots } z_k = 2\exp(\frac{1}{4}i(2\pi k - \frac{2\pi}{3})) \text{ where } k = 0,1,2,3$$
$$z_0 = 2\exp(\frac{1}{4}i(-\frac{2\pi}{3})) = 2\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right) = \sqrt{3} - i$$
$$z_1 = 2\exp(\frac{1}{4}i(2\pi - \frac{2\pi}{3})) = 2\left(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})\right) = 1 - \sqrt{3}i$$
$$z_2 = 2\exp(\frac{1}{4}i(4\pi - \frac{2\pi}{3})) = 2\left(\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})\right) = -\sqrt{3} + i$$
$$z_3 = 2\exp(\frac{1}{4}i(6\pi - \frac{2\pi}{3})) = 2\left(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})\right) = -1 + \sqrt{3}i$$

See attached PDF for image and principal root.

Question 5: 11.4a

$$-1 = e^{i\pi}$$
$$(-1)^{\frac{1}{3}} = \left(e^{i\pi}\right)^{\frac{1}{3}}$$
$$= e^{i\frac{\pi}{3}}$$
$$\text{There are 3 roots } z_k = \exp(\frac{2\pi ik}{3} + \frac{\pi}{3}i) \text{ where } k = 0,1,2$$
$$z_0 = \exp(\frac{\pi}{3}i) = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_1 = \exp(\pi i) = \cos(\pi) + i\sin(\pi) = -1$$
$$z_2 = \exp(\frac{5\pi}{3}i) = \cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3}) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

See attached PDF for image and principal root.

Question 6: 14.1

(a)
$$\text{Domain} = \{z \in \mathbb{C} : z \neq \pm i\}$$

(b)
$$\text{Domain} = \{z \in \mathbb{C} : z \neq 0\}$$

(c)
$$\text{Domain} = \{z \in \mathbb{C} : \mathrm{Re}(z) \neq 0\}$$

(d)
$$\text{Domain} = \{z \in \mathbb{C} : z\bar{z} \neq 1\}$$

Question 7: Exam 1 Fall 2022 Question 2

See attached PDF for image.