MAA4402
HW3

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Question 1: Fall 2022 Exam 1 Question 3

a Let $z_0 \in \mathbb{C}$ and suppose $f : D'(z_0, r) \mapsto \mathbb{C}$ for some $r > 0$. Let $w_0 \in \mathbb{C}$. We say that the limit of $f(z)$ as z approaches z_0 is w_0 and write

$$\lim_{z \to z_0} f(z) = w_0$$

if for any real positive ϵ, there is a real positive δ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

b Suppose that ϵ is any real positive number.

Choose $\epsilon = \delta$.

Then suppose $0 < |z - z_0| < \epsilon$.

We have

$$\begin{aligned} |\overline{z} - \overline{z_0}| &= |\overline{z - z_0}| \\ &= |z - z_0| \\ &\leq \epsilon = \delta \end{aligned}$$

Hence

$$\lim_{z \to z_0} (\overline{z}) = (\overline{z_0})$$

c Let

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

where $z = x + iy, x, y \in \mathbb{R}$

(a) i. Along the real axis, $z = x$ and $\overline{z} = x$, so

$$\lim_{\substack{z \to 0 \\ \text{(along the real axis)}}} \left(\frac{z}{\overline{z}}\right)^2 = \lim_{x \to 0} \left(\frac{x}{x}\right)^2 = 1$$

ii. Along the imaginary axis, $z = iy$ and $\overline{z} = -iy$, so

$$\lim_{\substack{z \to 0 \\ \text{(along the imaginary axis)}}} \left(\frac{z}{\overline{z}}\right)^2 = \lim_{y \to 0} \left(\frac{iy}{-iy}\right)^2 = \left(\frac{i}{-i}\right)^2 = 1$$

Hence

$$\lim_{\substack{z \to 0 \\ \text{(along the real axis)}}} f(z) = \lim_{\substack{z \to 0 \\ \text{(along the imaginary axis)}}} f(z)$$

(b) Along $y = x$, $z = x + ix = x(1 + i)$, so

$$\lim_{\substack{z \to 0 \\ \text{(along } y=x\text{)}}} \left(\frac{z}{\overline{z}}\right)^2 = \lim_{x \to 0} \left(\frac{x(1+i)}{x(1-i)}\right)^2 = (\frac{1+i}{1-i})^2 = i^2 = -1$$

This limit does not equal the limits found in the previous parts, thus the limit

$$\lim_{z \to 0} \left(\frac{z}{\overline{z}}\right)^2$$

does not exist.

Question 2: 18.10

a Let $f(z) = \frac{4z^2}{(z-1)^2}$ for $z \neq 1$. Then $f(\frac{1}{z}) = \frac{4(\frac{1}{z})^2}{((\frac{1}{z})-1)^2}$ for $z \neq 1$

$$f(\frac{1}{z}) = \frac{4(\frac{1}{z})^2}{((\frac{1}{z})-1)^2} = \frac{4(\frac{1}{z})^2z^2}{((\frac{1}{z})-1)^2z^2} = \frac{4}{(1-z)^2}$$
$$\lim_{z\to 0} f(\frac{1}{z}) = \frac{4}{(1-0)^2} = 4$$

and

$$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$$

b Let $f(z) = \frac{1}{(z-1)^3}$ for $z \neq$. Then $\frac{1}{f(z)} = (z-1)^3$

$$\lim_{z\to 1} \frac{1}{f(z)} = \lim_{z\to 1}(z-1)^3 = (1-1)^3 = 0$$

and

$$\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$$

c Let $f(z) = \frac{z^2+1}{z-1}$ for $z \neq 1$. Then $f(\frac{1}{z}) = \frac{\frac{1}{z^2}+1}{\frac{1}{z}-1}$ for $z \neq 1$ and

$$\begin{aligned}\frac{1}{f(\frac{1}{z})} &= \frac{\frac{1}{z}-1}{\frac{1}{z^2}+1} \\ &= \frac{(\frac{1}{z}-1)z^2}{(\frac{1}{z^2}+1)z^2} \\ &= \frac{z-z^2}{1+z^2} \\ &= \lim_{z\to 0}\frac{1}{f(\frac{1}{z})} = \frac{0-0}{1+0} \\ &= 0\end{aligned}$$

and

$$\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$$

Question 3: 20.8

a Consider the limit

$$\begin{aligned}\lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} &= \lim_{\Delta z\to 0} \frac{\mathrm{Re}(z+\Delta z)-\mathrm{Re}(z)}{\Delta z} \\ &= \frac{\mathrm{Re}(z)-\mathrm{Re}(z)+\mathrm{Re}(\Delta z)}{\Delta z} \\ &= \frac{\mathrm{Re}(\Delta z)}{\Delta z}\end{aligned}$$

(a) Along the real axis, $\Delta z = \Delta x$ and $\text{Re}(\Delta z) = \Delta x$, so

$$\begin{aligned}\lim_{\substack{\Delta z \to 0 \\ \text{(along the real axis)}}} \frac{\text{Re}(\Delta z)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} \\ &= 1\end{aligned}$$

(b) Along the imaginary axis, $\Delta z = \Delta iy$ and $\text{Re}(\Delta z) = 0$, so

$$\begin{aligned}\lim_{\substack{\Delta z \to 0 \\ \text{(along the imaginary axis)}}} \frac{\text{Re}(\Delta z)}{\Delta z} &= \lim_{\Delta y \to 0} \frac{0}{\Delta y} \\ &= 0\end{aligned}$$

So

$$\lim_{\substack{\Delta z \to 0 \\ \text{(along the real axis)}}} \frac{\text{Re}(\Delta z)}{\Delta z} = 1 \neq 0 = \lim_{\substack{\Delta z \to 0 \\ \text{(along the imaginary axis)}}} \frac{\text{Re}(\Delta z)}{\Delta z}$$

and $\lim_{\Delta z \to 0} \frac{\text{Re}(\Delta z)}{\Delta z}$ does not exist. Hence, the function $f(z) = \text{Re}(z)$ is not differentiable anywhere

b Consider the limit

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\text{Im}(z + \Delta z) - \text{Im}(z)}{\Delta z} \\ &= \frac{\text{Im}(z) - \text{Im}(z) + \text{Im}(\Delta z)}{\Delta z} \\ &= \frac{\text{Im}(\Delta z)}{\Delta z}\end{aligned}$$

(a) Along the real axis, $\Delta z = \Delta x$ and $\text{Im}(\Delta z) = 0$, so

$$\begin{aligned}\lim_{\substack{\Delta z \to 0 \\ \text{(along the real axis)}}} \frac{\text{Im}(\Delta z)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{0}{\Delta x} \\ &= 0\end{aligned}$$

(b) Along the imaginary axis, $\Delta z = \Delta i \Delta y$ and $\text{Im}(\Delta z) = iy$, so

$$\begin{aligned}\lim_{\substack{\Delta z \to 0 \\ \text{(along the imaginary axis)}}} \frac{\text{Im}(\Delta z)}{\Delta z} &= \lim_{\Delta y \to 0} \frac{i\Delta y}{\Delta y} \\ &= i\end{aligned}$$

So

$$\lim_{\substack{\Delta z \to 0 \\ \text{(along the real axis)}}} \frac{\text{Im}(\Delta z)}{\Delta z} = 0 \neq i = \lim_{\substack{\Delta z \to 0 \\ \text{(along the imaginary axis)}}} \frac{\text{Im}(\Delta z)}{\Delta z}$$

and $\lim_{\Delta z \to 0} \frac{\text{Im}(\Delta z)}{\Delta z}$ does not exist. Hence, the function $f(z) = \text{Im}(z)$ is not differentiable anywhere

Question 4: 24.1

c $f(z) = 2x + ixy^2 = u(x, y) + iv(x, y)$ where $u = 2x$ and $v = xy^2$.

$$\frac{\partial u}{\partial x} = 2 \neq 2xy = \frac{\partial v}{\partial y}$$

so $f'(z)$ does not exist at any point.

d

$$\begin{aligned} f(z) &= e^x e^{-iy} \\ &= e^x \left(\cos(-y) + i \sin(-y)\right) \\ &= u(x, y) + iv(x, y) \\ u &= e^x \cos(-y) \\ v &= e^x \sin(-y) \end{aligned}$$

$$\frac{\partial u}{\partial x} = e^x \cos(-y) \neq -e^x \cos(-y) = \frac{\partial v}{\partial y}$$

so $f'(z)$ does not exist at any point.