MHF3202
Exam 1

Oliver Deng

February 9, 2025

Question 1

1.
$$E = \{x : x = 2n, n \in \mathbb{Z}\}$$
where E is the set of all even real numbers.

2.
$$P = \{x : x = 2n, n \in \mathbb{Z}, x \text{ is a prime number}\}$$
where P is the set of all even prime numbers.

3.
$$F = \{(x, y) : x \in \mathbb{R}, y = \sin(x^2) - 3\}$$
where F is the graph of the function $f(x) = \sin(x^2) - 3$.

4.
$$\mathcal{P}(X) = \{A : A \subseteq X, X = \{x : x = 17n, n \in \mathbb{Z}\}\}$$
where $\mathcal{P}(X)$ is the power set of all real numbers that are evenly divisible by 17.

Question 2

For any number n bigger than N, the power set of A is the set of all strict subsets X of A, where the cardinality of X is less than n.
This statement is false, since X can also be the set A itself.

Question 3

$(\neg P) \wedge (Q \implies R)$ and $(P \wedge R) \vee (\neg Q)$ are not equivalent because their respective truth tables Table 1 and Table 2 have different values.

Table 1: Truth table of $(\neg P) \wedge (Q \implies R)$

P	Q	R	$\neg P$	$Q \implies R$	$(\neg P) \wedge (Q \implies R)$
T	T	T	F	T	F
T	T	F	F	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Table 2: Truth table of $(P \wedge R) \vee (\neg Q)$

P	Q	R	$(P \wedge R)$	$\neg Q$	$(P \wedge R) \vee (\neg Q)$
T	T	T	T	F	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	T	T

Question 4

The original supposition is that for all numbers x in a subset A of the real numbers, there is a number y in a subset B such that multiplying x and y results in a number that is **not** an integer.
It does not necessarily follow that there exists an x in that subset A such that every element y in B, when multiplied by x, is not an integer.
Consider the sets $A = \{\pi, 5\}$ and $B = \{2, 3\}$. For all x in A, both elements of B guarantee that for all $x, xy \in \mathbb{Z}$ is false, since both $\pi * 2$ or $\pi * 3$ are not integers.
However, we can say that there exists an x in A such that for all elements y in B, $xy \in \mathbb{Z}$. Take 5 from A. We can easily see that for both (all) elements in B, $xy \in \mathbb{Z}$, since $5 * 2 = 10$ and $5 * 3 = 15$, and both of those numbers are integers.
Thus, the claim in question does not follow from the original claim.

Question 5

$$\forall x \in \mathbb{R}, \exists N \in \mathbb{N} \text{ s.t. } \forall y (y > N), \frac{1}{y} < x$$

Question 6

1. $$\overline{A} = \{x : (x \notin \mathbb{Q}) \text{ or } (\forall y \in \mathbb{N}, 3xy \notin \mathbb{N})\}$$

2. $$A \cap B = \{x \in \mathbb{Z} : 2x \in N\}$$

3. A does not have a least element because given a rational number, you can always find a smaller one, and you can also find a bigger natural number that satisfies the second condition, since the natural numbers do not have a largest element.

   A also does not a largest element, because the rational numbers include the positive integers, which satisfy both conditions, and also do not have a largest element.

Question 7

1. No

2. The power set of A and the union of all B_i are not equal, since the union of all B_i do not consider two sets: the empty set and A itself.

   The empty set is not considered, since the empty set's cardinality is 0, but B_i only considers sets with cardinality from 1 to N. However, the empty set is a part of A's power set.

   A itself is not considered, because B_i only considers proper subsets of A, but A is not a proper subset of itself. However, A is a part of its power set.

   Therefore,

   $$\mathcal{P}(A) \setminus \bigcup_{i=1}^{N} B_i = \{A, \varnothing\}$$

Question 8

1. We can say that it is true.

2. ***Proof:*** Suppose that $P \implies Q$ is false.

   Then we know by Table 3 that P is true and Q is false, since that is the only case where implication is false.

   Also, since P is true, then $(\neg R \vee P)$ is true, and then the whole right-hand side of the implication in the claim is true.

   Then, again by Table 3, we know that in any case where the right-hand side of an implication is true, the implication must also be true.

   Hence, we know that $(R \vee S) \vee (R \wedge P) \implies ((\neg P) \wedge Q) \vee ((\neg R) \vee P)$ is true. ☺

Table 3: Truth table for implication

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Question 9

Let $D(x)$ be the statement that "x is a dog."
Let $C(x)$ be the statement that "x is a cat."
Let $O(x)$ be the statement that "x is an octopus."
Let $L(x)$ be the statement that "x can learn most tricks from their keepers."
Let $T(x)$ be the statement that "x can trick its keepers about what it has learned."

1. $$(\forall x(D(x) \implies L(x))) \wedge (\exists y(C(y) \wedge L(y))) \wedge (\forall z(T(z) \implies O(z)))$$

2. $$(\exists x(D(x) \wedge \neg L(x))) \vee (\forall y(C(y) \implies \neg L(y))) \vee (\exists z(T(z) \wedge \neg O(z)))$$

3. Either there exists an x such that it's a dog and cannot learn most tricks from its keepers, or for all y, y being a cat implies that it cannot learn most tricks from its keeper, or there exists a z such that it's an octopus and it cannot trick its keepers about what it has learned.

Question 10

1. The truth or falseness of the claim is unknowable.

2. ***Proof:*** Suppose that $(P \implies Q) \veebar (Q \implies R)$ is true.
   Then we have 2 cases according to the truth table given:

   (a) Case 1: $P \implies Q$ is true and $Q \implies R$ is false.
   Then, by Table 3, we know that Q is true and that R is false.
   If Q is true, then P can be either true or false in order to satisfy that $P \implies Q$ is true.
   Now, we can construct Table 4 with those 2 cases to show that $(P \implies R) \iff (Q \iff \neg P)$ is true.

   (b) Case 2: $P \implies Q$ is false and $Q \implies R$ is true.
   Then, by Table 3, we know that P is true and that Q is false.
   If Q is false, then R can be either true or false in order to satisfy that $Q \implies R$ is true.
   Now, we can construct Table 5 with those 2 cases to show that $(P \implies R) \iff (Q \iff \neg P)$ is either true or false, which is the same as saying that it is unknown.

   Since there is one case that is unknowable, we can see that $(P \implies R) \iff (Q \iff \neg P)$ is unknowable. ☺

Table 4: Truth table for case 1

P	Q	R	$\neg P$	$Q \iff \neg P$	$P \implies R$	$(Q \iff \neg P) \iff (P \implies R)$
T	T	F	F	F	F	T
F	T	F	T	T	T	T

Table 5: Truth table for case 2

P	Q	R	$\neg P$	$Q \iff \neg P$	$P \implies R$	$(Q \iff \neg P) \iff (P \implies R)$
T	F	T	F	T	T	T
T	F	F	F	T	F	F