MAP2302
Notes

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Contents

Chapter 1

Introduction and Basics

I forgot to take notes on the first 3 sections but if you're using this as review you should probably already know those intuitively by now.

1.1 Basic Definitions and Terminology

1.2 Solutions and Initial Value Problems

1.3 Direction Fields

1.4 Approximation Method of Euler

There are three general approaches to solving DEs:

1. The analytic approach (the major focus of the course).
2. The graphical approach where we get information about solutions by sketching direction/slope fields.
3. The numerical approach. When analytic techniques fail, numerical methods can produce an approximation that is good enough, provided that one understands how to make the method as accurate as possible.

Consider the first-order initial value problem

$$\frac{dy}{dt} = f(t, y), y(0) = y_0 \tag{1.4.1}$$

We can get a numerical solution to this problem by replacing $\frac{dy}{dt}$ with a simple forward difference representation.

$$\frac{y_{i+1} - y_i}{\Delta t} = f(x, y) \tag{1.4.2}$$

Solving for y_{i+1} yields

$$y_{i+1} = y_i + \Delta t \cdot f(t_i, y_i) \tag{1.4.3}$$

It is now possible to march forward in time to obtain a new value of y at a new value of t. We can rename Δt as h and call it the **step size**.

1.4.1 Error Analysis

We can find an exact solution for the DE by a Taylor series expansion:

$$y_{i+1} = y_i + h\frac{dy_i}{dx} + \frac{h^2}{2!}\frac{d^2y_i}{dx^2} + \dots \tag{1.4.4}$$

Manipulating the previous equation we get

$$\frac{y_{i+1} - y_i}{h} = \frac{dy_i}{dx} + \frac{h}{2!}\frac{d^2 y_i}{dx} + \frac{h^2}{3!} d^3 y_i dx^3 + \ldots \tag{1.4.5}$$

Anything after $\frac{dy_i}{dx}$ is considered error due to truncation and is notated as $\mathcal{O}\left(h\right)$. Thus the Euler method is termed a first-order method. Note that the Runge-Kutta method of order 4 is $\mathcal{O}\left(h^4\right)$.

Chapter 2

First-Order Differential Equations

2.1 Seperable DEs

Definition 2.1.1: Separable Differential Equations

A DE is **separable** if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y) \tag{2.1.1}$$

To solve a separable DE, consider these three steps:

1. Check whether the zeroes of $p(y)$ are solutions of the equation so you don't lose a solution
2. Divide both sides by $p(y)$ and multiply both sides by dx to reduce it to the form $h(y)dy = g(x)dx$ where $h(y) = \frac{1}{p(y)}$
3. Integrate both sides to obtain the implicit solution $H(y) = G(x) + C$

Note

Take caution when dividing anywhere so you don't lose a solution; specifically look out for division by 0!

Note

In general, try to apply the initial conditions as soon as possible so you don't have to do extraneous work.

Note

If the integrals are both the same and the initial conditions for y and x are the same, you can reason that the answer is $y = x$ by symmetry.

2.2 Linear Equations

The main premise of this section is finding a situation where you can express the left-hand side of a linear DE as a result of the product rule, and thus integrate neatly.

Definition 2.2.1: First-Order Linear DE

A first-order linear DE has the form

$$a_1(x)\frac{dx}{dy} + a_0(x)y = g(x) \tag{2.2.1}$$

where $a_1(x) \neq (0)$.

There are 3 cases to consider:

1. **The easy case:** $a_0(x) = 0$

   Then, dividing both sides by $a_1(x)$, eq.(2.2.1) can be written as $\frac{dy}{dx} = \frac{b(x)}{a_1(x)}$ and is separable.

   So $y(x) = \int \frac{b(x)}{a_1(x)} dx$.

2. **Product rule:** $a_0(x) = a_1'(x)$

   Then eq.(2.2.1) can be written as $a_1(x)\frac{dy}{dx} + a_1'(x)y = b(x)$, and the left-hand side can be expressed as a result of the product rule:

   $$\frac{d}{dx}[a_1(x)y]$$

   which is the main premise of this chapter. Next, integrate both sides and you're left with

   $$a_1(x)y = \int b(x)dx$$

   So $y(x) = \frac{\int b(x)dx + c}{a_1(x)}$.

3. **The general case:** What if it's linear but not necessarily in an easy form?

   (a) First, write eq.(1.6.1) in the **standard form**:

   $$\frac{dy}{dx} + P(x)y = Q(x) \tag{2.2.2}$$

   where $P(x) = \frac{a_0(x)}{a_1(x)}$ and $Q(x) = \frac{b(x)}{a_1(x)}$

   (b) Multiply both sides of eq.(2.2.2) by the **integrating factor** $\mu(x)$:

   $$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

   (c) Choose $\mu(x) = e^{\int P(x)dx}$ and then the equation from the last step can be written as

   $$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

   We can take the derivative using the product rule to verify that our choice of $\mu(x)$ is valid. After that, we once again reach the main premise of the section and can integrate both sides. So $y(x) = \frac{\int \mu(x)Q(x)dx + C}{\mu(x)}$.

2.3 Exact Equations

If $z = f(x, y)$ is a function having continuous first partial derivatives in a region R of the xy-plane, then its **total differential** is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

The total differential to the **level curve** $f(x, y) = c$ is

$$df = d(c)$$
$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Note

Given $f(x, y) = c$, we can generate a first-order DE by computing the total differential of both sides of the function.

To use this to help solve DEs, we start by rewriting our DE in the following form

$$M(x, y)dx + N(x, y)dy = 0$$

If we can find a function $f(x, y)$ such that

$$\frac{\partial f}{x} = M(x, y)$$
$$\frac{\partial f}{y} = N(x, y)$$

then $df = M(x, y)dx + N(x, y)dy$. In such a case,

$$M(x, y)dx + N(x, y)dy$$

is in **exact form** and

$$M(x, y)dx + N(x, y)dy = 0$$

is an **exact equation**.

If the LHS of a DE is the total differential of a function, we can construct the function by partial integration (if the RHS is 0).

> **Theorem 2.3.1** Test for Exactness
>
> Suppose the first derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then $M(x, y)dx + N(x, y)dy = 0$ is an **exact differential equation** in R if the condition
>
> $$\frac{\partial M}{y}(x, y) = \frac{\partial N}{x}(x, y)$$
>
> holds for all (x, y) in R.

2.3.1 Solving Exact Equations

$M(x, y)dx + N(x, y)dy = 0$

1. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. If exact, write $\frac{\partial f}{\partial x} = M(x, y)$ and then

$$f(x, y) = \int M(x, y)dx + g(y) \tag{2.3.1}$$

3. Determine $g(y)$:
    - take the partial derivative with respect to y of both sides of eq.(2.3.1)

    $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\int M(x, y)dx + g(y)]$$

    - substitute $N(x, y) = \frac{\partial f}{\partial y}$ to determine $g'(y)$
    - find $g(y)$ through integration

    $$g(y) = \int g'(y)dy$$

4. Substitute $g(y)$ into eq.(2.3.1) to get $f(x, y)$
5. Solution:

$$f(x, y) = \int M(x, y)dx + g(y) = c$$

You can also flip M with N and adjust steps 2-4.

2.4 Special Integrating Factors

Definition 2.4.1

If

$$M(x,y)dx + N(x,y)dy = 0 \tag{2.4.1}$$

is not exact, but multiplying the equation by some factor $\mu(x,y)$ makes it exact, then $\mu(x,y)$ is called an **integrating factor** for eq.(2.4.1).

Our goal is to find $\mu(x,y)$ so that

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact. That is,

$$\frac{\partial}{\partial y}[\mu(x,y)M(x,y)] = \frac{\partial}{\partial x}[\mu(x,y)N(x,y)]$$

which expands to

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$
$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu$$

Now there are two cases.

Case 1: If μ is only dependent upon x or is a constant, then $\frac{\partial \mu}{\partial y} = 0$ and then

$$-N\frac{\partial \mu}{\partial x} = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu$$

which when manipulated can be written as

$$\mu(x) = \exp\,[\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}dx]$$

Case 2: If μ is only dependent upon y or is a constant, then $\frac{\partial \mu}{\partial x} = 0$

$$M\frac{\partial \mu}{\partial y} = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu$$

which when manipulated can be written as

$$\mu(y) = \exp\,[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}dx]$$

Theorem 2.4.1 Integrating Factors

Assuming the given equation is not exact, then we have 2 cases to check.

1. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ depends only on x or is a constant, then

   $$\mu(x) = \exp\,[\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}dx]$$

   is an integrating factor of eq.(2.4.1).

2. If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ depends only on y or is a constant, then

   $$\mu(y) = \exp\,[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}dy]$$

is an integrating factor of eq.(2.4.1).

2.5 Substitutions

Suppose we have an equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

that is not separable, linear or exact. We may be able to transform it into something we have a method to solve. The general procedure to substitute is mostly common sense:

1. Identify the type of equation and apply substitution
2. Rewrite the original equation with the new variables
3. Solve the transformed equation
4. Express the solution in terms of original variables

2.5.1 Homogeneous Equations

Definition 2.5.1: Homogeneous Equations

If the right side of

$$\frac{dy}{dx} = f(x,y)$$

can be expressed as a function of $\frac{y}{x}$ and $\frac{x}{y}$ alone, then the equation is **homogeneous**.

Substituting $v = \frac{y}{x}$ will reduce a homogeneous equation to a **separable** first-order DE.

1. Substitute $v = \dfrac{y}{x}$ making our DE

$$\frac{dy}{dx} = G(v) \tag{2.5.1}$$

2. Our variables are now v and x, so we need to make $\frac{dy}{dx}$ in terms of those two.

$$v = \frac{y}{x}$$
$$y = vx$$
$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

You may also use the following instead:

$$dy = xdv + vdx$$

3. Substituting this back into eq.(2.5.1), we can then separate and integrate
4. Put the equation back in terms of x and y

2.5.2 Bernoulli Equations

Definition 2.5.2: Bernoulli Equations

A first-order equation that can be written as

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{2.5.2}$$

where $P(x)$ and $Q(x)$ are continuous on an interval and n is a **real** number is called a **Bernoulli equation**.

If $n = 0$ or $n = 1$, then the equation is linear and can be solved accordingly. Otherwise, we transform it into a linear equation:

1. Divide eq.(2.5.2) by y^n, giving
$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$
2. Substitute $v = y^{1-n}$, making it a linear equation
3. Differentiate $v = y^{1-n}$
$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{1-n}y^n\frac{dv}{dx}$$
4. Substitute this back into eq.(2.5.2), making the equation **linear** in v and possible to solve
$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

2.5.3 Z-Substitution

If you have an equation of the form
$$\frac{dy}{dx} = G(ax + by + c)$$
then the substitution $z = ax + by + c$ makes the equation **separable**.

2.6 First-Order Applications

A geometric problem of interest in engineering is that of finding a family of curves (**orthogonal trajectories**) that intersects a given family of curves orthogonally at each point. Some applications are as follows:

- **Electromagnetic Theory**: Given lines of force of electric field, find the equipotential flux
- **Heat Conduction**: Given lines of constant temperature (isotherms), find lines of heat flow
- **Fluid Mechanics**: Given fluid flow streamlines, find the associated equipotential lines along which the fluid potential is constant

The slope of a curve orthogonal (perpendicular) to a given curve is the negative reciprocal of the original curve's slope. Curves orthogonal to the family $F(x, y) = k$ satisfy the DE
$$\frac{\partial F}{\partial y}dx = \frac{\partial F}{\partial x}dy$$

2.6.1 Exponential Growth and Decay

Suppose the rate of change of a population of organisms is proportional to that quantity:
$$\frac{dP}{dt} = kP$$
where k is the constant of proportionality and P is the population. This is a separable equation that gives us
$$P(t) = Ce^{kt}$$

We have exponential growth when $k > 0$ and exponential decay when $k < 0$. This has applications in bank interest, radioactive decay, detecting art forgeries, and population growth.

2.6.2 Logistic Models

The logistic differential equation has two positive parameters: r, a growth parameter, and K, the carrying capacity. The equation is:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

This is a separable equation with the solution

$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}}$$

where P_0 is the initial population.

2.6.3 Newton's Law of Cooling

$$\frac{dT}{dt} = k(T_e - T)$$

where T is the temperature of an object at time t, T_e is the ambient temperature, and k is a constant governing the rate of cooling. This is a separable equation that can be solved to become

$$T - T_e = Ce^{-kt}$$

which, after applying initial conditions, becomes

$$T(t) = T_e + (T_0 - T_e)e^{-kt}$$

Chapter 3

Linear Second-Order Differential Equations

3.1 The Mass-Spring Oscillator

A damped mass-spring oscillator consists of a mass m attached to a spring fixed at one end. The differential equation that governs the motion of this oscillator can be derived from $F = ma$ and is given by

$$my'' + by' + ky = F_{\text{ext}} \tag{3.1.1}$$

where y is displacement from equilibrium, m is the mass of the spring, k is the spring constant/stiffness from Hooke's law, b is a damping coefficient (friction, opposing the motion of the spring), and F_{ext} is the sum of all externally applied forces.

1. Spring motion when $b = 0$, $F_{\text{ext}} = 0$ simplifies eq.(2.7.1) to

$$my'' + ky = 0$$

   which resembles a sine or cosine graph.

2. Spring motion where $b \neq 0$, $F_{\text{ext}} = 0$ will oscillations that will eventually die out. Systems with low damping are called underdamped systems and will continue to oscillate for a while, in contrast to systems with high damping which are called overdamped systems. Overdamping occurs when the frictional force exceeds the spring force, i.e. $b^2 - 4mk > 0$ which stops any oscillations. In an underdamped system, $b^2 - 4mk < 0$ which is not enough to suppress all oscillations.

3.1.1 External Forces

Typically external forces are sinusoidal with fixed amplitude, and the resulting solution curves will eventually synchronize with these forces and oscillate at the same frequency, though the initial behavior can be erratic. Examples of such systems are sound system speakers, electronic amplifier circuits, and ocean tides (affected by the moon.) It is important to observe that some systems are incredibly sensitive to the resonant frequency: this is the explanation for how perfectly tuned notes can shatter crystal and wind-driven vibrations can destroy bridges. This is a major point to consider when engineers are designing systems, whether you want to avoid such resonant responses or not (as with radio signals).

Example 3.1.1 (Find the synchronous response of the mass-spring oscillator with $F_{\text{ext}} = 5\sin(\omega t)$, $m = 1$, $b = 2$, $k = 4$, and $\omega = 3$.)

$$\begin{aligned} my'' + by' + ky &= F_{\text{ext}} \\ y'' + 2y' + 4y &= 5\sin(3t) \end{aligned}$$

Let

$$
\begin{aligned}
y &= A\cos(3t) + B\sin(3t) \\
y' &= -3A\sin(3t) + 3B\cos(3t) \\
y'' &= -9A\cos(3t) - 9B(\sin_3 t)
\end{aligned}
$$

Plug back into the ODE and solve for A and B

$$
\begin{aligned}
y'' + 2y' + 4y &= -9A\cos(3t) - 9B\sin(3t) - 6A\sin(3t) + 6B\cos(3t) + 4A\cos(3t) + 4B\cos(3t) \\
&= 5\sin(3t) + 0\cos(3t)
\end{aligned}
$$

We can look at the terms attached to the cosines:

$$
\begin{aligned}
-9A + 6B + 4A = 0 \\
B = \frac{5A}{6}
\end{aligned}
$$

Now we look at the terms attached to the sines:

$$
\begin{aligned}
-9B - 6A + 4B &= 5 \\
-5B - 6A &= 5 \\
-\frac{25}{6}A - \frac{36}{6}A &= 5 \\
A &= -\frac{25}{61}
\end{aligned}
$$

Subbing in for A and B, our final answer is:

$$
y(t) = -\frac{25}{61} \cdot \frac{5}{6}\cos(3t) - \frac{25}{61}\sin(3t)
$$

3.2 Homogenous Second-Order Linear Equations

Consider a second-order linear DE:

$$
a(x)y'' + b(x)y' + c(x)y = f(x) \tag{3.2.1}
$$

If $f(x) = 0$, then

$$
a(x)y'' + b(x)y' + c(x)y = 0 \tag{3.2.2}
$$

is the associated **homogenous** DE of eq.(3.2.1).

Note

$y = 0$ is the trivial solution to eq.(3.2.2).

Note

A linear combination of any linearly independent solutions y_1 and y_2

$$
y(t) = c_1 y_1(t) = c_2 y_2(t)
$$

is also a solution.

Definition 3.2.1: Linear Dependence

A pair of functions $y_1(t)$ and $y_2(t)$ are **linearly independent** on an interval I iff neither one is a constant multiple of the other on all of I. They are **linearly dependent** if they are a constant multiple of each other.

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the homogenous DE $a(x)y'' + b(x)y' + c(x)y = 0$ then

$$y_h = c_1 y_1 + c_2 y_2$$

is its **general solution**. In this chapter we focus mainly on second-order DEs with **constant coefficients**.

Now how do we find 2 solutions to eq.(3.2.1)?

Example 3.2.1 (What if we try e^{rt}?)

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

Insert into the DE:

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$e^{rt}(ar^2 + br + c) = 0$$

e^{rt} never equals 0, so the **auxiliary** or **characteristic** equation $ar^2 + br + c$ must equal 0, so we have two roots:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Essentially, we treat the DE like a polynomial. We have three cases to check based on the determinant $b^2 - 4ac$.

1. $b^2 - 4ac > 0$, where we use two **distinct** real roots:
   We either manually factor or use the quadratic equation to get the roots

   $$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
   $$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

   which gives us two solutions

   $$y_1 = e^{r_1 t}$$
   $$y_2 = e^{r_2 t}$$

   and the general solution

   $$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

   (This also correlates to overdamping in the mass-spring oscillator)

2. $b^2 - 4ac = 0$, where we use one **repeated** real root:

   $$r_1 = r_2 = r = -\frac{b}{2a}$$

   which gives us two solutions

   $$y_1 = e^{rt}$$
   $$y_2 = te^{rt}$$

   and the general solution

   $$y_h(t) = c_1 e^{rt} + c_2 te^{rt}$$

   (This correlates to a critically damped mass-spring oscillator)

Example 3.2.2 (Find a general solution to $y'' + 2y' + y = 0$)

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ r &= -1, -1 \\ y_1 &= e^{-t} \\ y_2 &= te^{-t} \\ y_h(t) &= c_1 e^{-t} + c_2 t e^{-t} \end{aligned}$$

3. $b^2 - 4ac < 0$, where we use two **imaginary** roots:

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

where $i = \sqrt{-1}$, $\alpha = \dfrac{b}{2a}$, and $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$. Previously, our solutions were of the form e^{rt}, but if our roots are imaginary, we get the form

$$e^{a+i\beta t} = e^{\alpha t} e^{i\beta t}$$

which is complex; using Euler's formula

$$e^{i\pi} = \cos\theta + i\sin\theta$$

we get the complex solution

$$z(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t) \tag{3.2.3}$$

Lemma 3.2.1 Let $z(t) = u(t) + iv(t)$ be a solution to the DE $ay'' + by' + c = 0$, where $u(t)$ and $v(t)$ are real-valued functions. Then $u(t)$ and $v(t)$ are both real-valued solutions to $ay'' + by' + c = 0$.

We take the first two derivatives

$$\begin{aligned} y &= u + iv \\ y' &= u' + iv' \\ y'' &= u'' + iv'' \end{aligned}$$

Plugging this back into the DE

$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = 0$$

Rearranging for into the real and imaginary parts

$$(au'' + bu' + cu) + i(av'' + bv' + cv) = 0$$

A complex number equals 0 iff both the real part and the imaginary part are zero. Then both $u(t)$ and $v(t)$ are real-valued solutions to the DE. 😄

This lemma allows us to write eq.(3.2.3) as the real general solution

$$y_h(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

or

$$y_h(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

(This correlates to underdamping in the mass-spring oscillator)

Example 3.2.3 (Find a general solution to the IVP $y'' + 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 2$)

$$
\begin{aligned}
r^2 + 2r + 2 = 0 \\
r &= \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i \\
\alpha &= -1 \\
\beta &= 1 \\
y_h(t) &= e^{-t}(c_1 \cos(t) + c_2 \sin(t))
\end{aligned}
$$

Use initial conditions

$$
\begin{aligned}
y_h(0) &= e^0(c_1 \cos(0) + c_2 \sin(0)) = 0 = c_1 \\
y_h(t) &= e^{-t} c_2 \sin(t) \\
y_h'(t) &= c_2 e^{-t} \cos(t) - c_2 e^{-t} \sin(t) \\
y_h'(0) &= c_2 e^0 \cos(0) - c_2 e^0 \sin(0) = 2 = c_2
\end{aligned}
$$

Plug the constants back in to get the general solution

$$y_h(t) = 2e^{-t} \sin(t)$$

This method works for any higher-order homogeneous linear equations with constant coefficients. It takes a similar form to eq.(3.2.1) but adds another solution $y_a(t)$ for each added order. In fact, you will find a combination of the 3 cases mentioned above, as there are obviously more than 2 roots.

Example 3.2.4 ($y''' + y'' - 6y' + 4y = 0$)

$$r^3 + r^2 - 6r + 4 = 0$$

Guess a root

$$r = \frac{\pm\text{factors of the constant}}{\text{factors of the leading coefficient}} = \frac{\pm 4}{1}$$

$r = 1$ is a root, which means $(r-1)$ is a factor, and dividing by that gives us

$$\frac{r^3 + r^2 - 6r + 4}{r-1} = r^2 + 2r - 4$$

Find the roots of the result

$$
\begin{aligned}
r &= \frac{-2 \pm 2\sqrt{5}}{2} \\
r_1 &= 1 \\
r_2 &= -1 + \sqrt{5} \\
r_3 &= -1 - \sqrt{5}
\end{aligned}
$$

Our general solution is

$$y_h(t) = c_1 e^t + c_2 e^{(-1+\sqrt{5})t} + c_3 e^{(-1-\sqrt{5})t}$$

3.3 Nonhomogeneous Equations

We now look at **non-homogeneous** linear equations with constant coefficients of the form

$$ay'' + by' + cy = f(t) \tag{3.3.1}$$

where $f(t) \neq 0$

3.3.1 Method of Undetermined Coefficients

1. Use the form of f to **guess** a form for a particular solution
2. Plug the guessed form back into the DE
3. Determine the coefficients that make your guessed solution an actual equation

Note

Note that this method only works when

1. The DE is linear
2. The coefficients are constants
3. The non-homogeneous term $f(t)$ is one of the types:
   (a) Constants
   (b) Polynomials
   (c) Exponential function
   (d) Sine or Cosine functions
   (e) A linear combination of the above (ex. $P(t)e^{\alpha t}\sin(\beta t)$, where α and β are real numbers)

Example 3.3.1 (Find a particular solution to $y'' + 3y' + 2y = 3t$)

What kind of function y would generate $3t$ if we add the first and second derivatives? Try a linear function:

$$\begin{aligned} y &= At + B \\ y' &= A \\ y'' &= 0 \\ y'' + 3y' + 2y &= \\ 0 + 3A + 2A + 2B &= 3t \end{aligned}$$

This gives us a system of 2 equations based on the coefficients of t:

$$\begin{aligned} t^1 &: 2A = 3 \implies A = \frac{3}{2} \\ t^0 &: 3A + 2B = 0 \implies B = \frac{-3}{2}A = \frac{-9}{4} \end{aligned}$$

Plugging this back in to the guessed function we get the particular solution

$$y_p(t) = \frac{3}{2}t - \frac{9}{4}$$

Many of the guesses depending on the form of $f(t)$ are what you would intuitively expect if you take the derivative of your guess. We may use the following guesses for different forms:

1. If $f(t) = Ct^m$ then our guess for a solution may be

$$y_p = A_m t^m + \ldots + A_1 t + A_0$$

2. If $f(t) = Ce^{\alpha t}$ then
$$y_p = Ae^{\alpha t}$$
3. If $f(t) = C\sin(\beta t)$ or $f(t) = C\cos(\beta t)$ then
$$y_p = A\cos(\beta t) + B\sin(\beta t)$$
(refer back to section 3.1 on mass-spring oscillators)
4. If $f(t)$ is a combination of the different forms, then the guess is usually a combination. For example, if $f(t) = Ct^m e^{\alpha t}$ then
$$y_p = (A_m t^m + \ldots + A_0)e^{\alpha t}$$

Guess and check doesn't always work, though:

Example 3.3.2 ($y'' + y' = 5$)

$$\begin{aligned} y_p &= A \\ y_p' &= 0 \\ y_p'' &= 0 \end{aligned}$$

Insert back into the DE

$$0 + 0 = 5$$

Contradiction!

This is because the suggested function $y_p = A$ is also a solution to the homogeneous equation
$$y'' + y' = 0$$
which has a solution which already includes $y_p = A$. In fact, if any particular solution y_p contains terms that duplicate terms in the solution of the associated homogeneous DE, then y_p must be multiplied by some t^s, where s is the **smallest positive integer that eliminates that duplication**. Let's try ex.(3.3.2) again:

Example 3.3.3 ($y'' + y' = 5$)
Since we know that our guess A is already represented in the solution to the homogeneous equation, let's try multiplying by t:

$$\begin{aligned} y_p &= At \\ y_p' &= A \\ y_p'' &= 0 \end{aligned}$$

Inserting back into the DE

$$\begin{aligned} A &= 5 \\ y_p(t) &= 5t \end{aligned}$$

To summarize everything, we first find the general solution to the associated homogeneous case, and then that allows us to find what we need to guess. We can then use the following tables to easily find the forms to guess:

$f(t)$	$y_p(t)$
$e^{\alpha t}$	$At^s e^{\alpha t}$
$P(t)$	$t^s Q(t)$
$P(t)e^{\alpha t}$	$t^s e^{\alpha t} Q(t)$

where

- $s = 0$ if α is not a root of the characteristic equation
- $s = 1$ if α is not a root of the characteristic equation
- $s = 2$ if α is repeated root of the characteristic equation

$f(t)$	$y_p(t)$
$e^{\alpha t}\begin{cases}\cos(\beta t)\\ \sin(\beta t)\end{cases}$	$t^s e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$
$P(t)\begin{cases}\cos(\beta t)\\ \sin(\beta t)\end{cases}$	$t^s[Q_1(t)\cos(\beta t) + Q_2(t)\sin(\beta t)]$
$P(t)e^{\alpha t}\begin{cases}\cos(\beta t)\\ \sin(\beta t)\end{cases}$	$t^s e^{\alpha t}[Q_1(t)\cos(\beta t) + Q_2(t)\sin(\beta t)]$

where

- $s = 0$ if $\alpha + i\beta$ is not a root of the characteristic equation
- $s = 1$ if $\alpha + i\beta$ is a root of the characteristic equation

Example 3.3.4 (Find a particular solution to $y'' - 6y' + 9y = e^{3t}$)

We first check the homogeneous case to see what we need to find, which gives us

$$y_h = c_1 e^{3t} + c_2 t e^{3t}$$

Now we guess

$$\begin{aligned} y_p &= At^2 e^{3t} \\ y_p' &= (2At + 3At^2)e^{3t} \\ y_p'' &= (2A + 12At + 9At^2)e^{3t} \end{aligned}$$

Substituting back into the DE

$$\begin{aligned} (2At + 3At^2)e^{3t} - 6(2A + 12At + 9At^2)e^{3t} + 9At^2 e^{3t} &= e^{3t} \\ t^2 :\quad 9A - 18A + 9A &= 0 \\ t^1 :\quad 12A - 12A &= 0 \\ t^0 :\quad 2A &= 1 \\ A &= \frac{1}{2} \end{aligned}$$

So the particular solution is

$$y_p(t) = \frac{1}{2}t^2 e^{3t}$$

3.4 The Superposition Principle

What do we do when we have more than one forcing function?

Theorem 3.4.1 The Superposition Principle

Let y_1 be a solution to the DE $ay'' + by' + cy = f_1(t)$, and y_2 be a solution to the DE $ay'' + by' + cy = f_2(t)$. Then, for any constants c_1 and c_2, the function $c_1 y_1 + c_2 y_2$ is a solution to the DE $ay'' + by' + cy = c_1 f_1(t) + c_2 f_2(t)$.

Example 3.4.1 (Given that $y_1 = \frac{3}{2}t - \frac{9}{4}$ is a solution to

$$y'' + 3y' + 2y = 3t$$

and $y_2 = \frac{1}{2}e^{3t}$ is a solution to

$$y'' + 3y' + 2y = 10e^{3t}$$

find a solution to $y'' + 3y + 2y = -9t + 20e^{3t}$.)

Notice that $-9t = -3 \cdot 3t$ and $20e^{3t} = 2 \cdot 10e^{3t}$. Then our particular solution is

$$\begin{aligned} y_p(t) &= c_1 y_1 + c_2 y_2 \\ &= c_1(\frac{3}{2}t - \frac{9}{4}) + c_2(\frac{1}{2}e^{3t}) \end{aligned}$$

Plugging in what we scaled the forcing functions by

$$= -3(\frac{3}{2}t - \frac{9}{4}) + 2(\frac{1}{2}e^{3t})$$

Note

If you want to find the general solution to a nonhomogeneous DE, just add the particular solution to the general solution of the homogeneous version. More formally, if y_p is a particular solution to $ay'' + by' + cy = f(t)$ and $y_h = c_1 y_1 + c_2 y_2$ is a general solution to $ay'' + by' + cy = 0$, then by the superposition principle

$$y_g(t) = y_h(t) + y_p(t)$$

is a **general solution** to $ay'' + by' + cy = f(t)$.

Note

As a general tip, if the forcing functions are both being multiplied by some e^{rt}, you can factor that out and treat the whole right side as only needing one particular solution as opposed to two. In fact, try to look for **any** patterns to make your life easier.

Example 3.4.2 (Write down the particular solution to the DE

$$y'' + 2y' + 2y = 5e^{-t}\sin(t) + 5t^3 e^{-t}\cos(t)$$

)

The root of the right side is $\alpha + i\beta = -1 + i$, which is found in the roots of the characteristic equation $r^2 + 2r + 2 = 0$, which means we need to throw in an extra t. The final form would be

$$y_p(t) = te^{-t}[(A_3t^3 + A_2t^2 + A_1t + A_0)\cos(t) + (B_3t^3 + B_2t^2 + B_1t + B_0)\sin(t)]$$

3.5 Variation of Parameters

3.5.1 Cramer's Rule

Cramer's rule gives us a formula for solving the n unknowns in a system of n linear equations by calculating determinants. When n is small, it provides a simple procedure for solving the system. Let's consider the case of $n = 2$:

Definition 3.5.1

Suppose we want to solve the 2 unknowns in the following system:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a nonzero determinant. Cramer's rule gives the solutions

$$x_1 = \frac{\det(A_1)}{\det(A)}$$
$$x_2 = \frac{\det(A_2)}{\det(A)}$$

where A_i is the matrix obtained from A by replacing the ith column of A with the column vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

i.e.

$$A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$
$$A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{bmatrix}$$

Example 3.5.1 (Solve the system

$$2x + y = 1$$
$$3x - 2y = 5$$

)

Let $W = \det \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$, then $W_1 = \det \begin{bmatrix} 1 & 1 \\ 5 & -2 \end{bmatrix}$ and $W_2 = \det \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and thus

$$x = \frac{W_1}{W} = \frac{-7}{-7} = 1$$
$$y = \frac{W_2}{W} = \frac{7}{-7} = -1$$

3.5.2 Method of Undetermined Coefficients

The method of undetermined coefficients is a simple procedure for determining a particular solution when the equation has constant coefficients and the non-homogeneous term is of a special type. Here, we present a more general method, called variation of parameters for finding a particular solution.

Definition 3.5.2: Wronskian

For any 2 differentiable functions y_1 and y_2, the function

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is called the **Wronskian** of y_1 and y_2.

Theorem 3.5.1 Test for Linear Independence

If $y_1(t)$ and $y_2(t)$ are any two solutions to a homogeneous linear equation on an interval I, and if the determinant of the Wronskian is

$$W[y_1, y_2](t) = 0$$

at any point t on I, then y_1 and y_2 are **linearly dependent** on I, otherwise they are **linearly independent**.

Example 3.5.2 (Test $y_1 = \cos(t)$ and $y_2 = \sin(t)$)

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1 \neq 0$$

The two functions are linearly independent on $(-\infty, \infty)$.

Consider the nonhomogeneous linear second-order DE:

$$a(t)y'' + b(t)y' + c(t)y = f(t) \tag{3.5.1}$$

and let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions for the corresponding homogeneous solutions for the corresponding homogeneous equation (For the rest of the section, we denote functions $a(t)$, $b(t)$, and $c(t)$ by a, b, and c respectively for the rest of the section). A general solution to this homogeneous equation is given by

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

Now let us introduce the variation of parameters, by finding functions v_1 and v_2 such that

$$y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$$

is a particular solution of eq.(3.5.1). Take the derivative

$$y_p' = (v_1'y_1 + v_1y_1') + (v_2'y_2 + v_2y_2')$$

and then assume

$$v_1'y_1 + v_2'y_2 = 0 \tag{3.5.2}$$

then

$$y_p' = v_1y_1' + v_2y_2'$$

and

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$$

Substituting back into eq.(3.5.1), we get

$$\begin{aligned} f &= ay_p'' + by_p' + cy_p \\ &= a(v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'') + b(v_1y_1' + v_2y_2') + c(v_1y_1 + v_2y_2) \\ &= a(v_1'y_1' + v_2'y_2') + v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) \\ &= a(v_1'y_1' + v_2'y_2') \end{aligned}$$

since the terms connected to v_1 and v_2 add up to 0, since they are solutions to the homogeneous case. Now, we have

$$\frac{f}{a} = v_1'y_1' + v_2'y_2'$$

Along with our assumption in eq.(3.5.2), we now have two equations to solve for v_1 and v_2 :

$$\begin{cases} y_1v_1' + y_2v_2' = 0 \\ y_1'v_1' + y_2'v_2' = f(t)/a \end{cases}$$

and we can use Cramer's rule to solve v_1' and v_2':

$$v_1' = \frac{W_1}{W}$$
$$v_2' = \frac{W_2}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$
$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(t)/a & y_2' \end{vmatrix}$$
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(t)/a \end{vmatrix}$$

Note

To make it easier on yourself, always put your DE in standard form (where the coefficient of y'' is 1)!

We integrate to find v_1 and v_2, therefore giving us

$$v_1 = \int \frac{W_1}{W} dt$$
$$v_2 = \int \frac{W_2}{W} dt$$

To give a quick summary in order to find a particular solution to $ay'' + by' + cy = f(t)$:

1. Find two linearly independent solutions $y_1(t)$, $y_2(t)$ to the associated homogeneous equation $ay''+by'+cy = 0$
2. Compute the Wronskian W, W_1, and W_2
3. Determine $v_1(t)$ and $v_2(t)$ by integrating $v_1'(t) = \frac{W_1}{W}$ and $v_2'(t) = \frac{W_2}{W}$

   Note

   We do not need to introduce any constants when computing the definite integrals of v_1' and v_2'.

4. A particular solution of eq.(3.5.1) is

   $$y_p = v_1y_1 + v_2y_2$$

   and a general solution is

   $$y_g = y_h + y_p = c_1y_1 + c_2y_2 + v_1y_1 + v_2y_2$$

   Note

   If necessary, combine like terms in the general solution!

Example 3.5.3 (Find a general solution to $y'' - 2y' + y = e^t \ln(t)$, $t > 0$)

Factoring the characteristic equation $r^2 - 2r + 1$, we get that $r = 1$ is a double root, giving us the homogeneous solution

$$
\begin{aligned}
y_h(t) &= c_1e^t + c_2te^t \\
y_1 &= e^t \\
y_2 &= te^t \\
W &= \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t} \\
W_1 &= \begin{vmatrix} 0 & te^t \\ e^t \ln(t) & e^t + te^t \end{vmatrix} = -te^{2t}\ln(t) \\
W_2 &= \begin{vmatrix} e^t & 0 \\ e^t & e^t \ln(t) \end{vmatrix} = e^{2t}\ln(t) \\
v_1 &= \int \frac{W_1}{W}dt = \int \frac{-te^{2t}\ln(t)}{e^{2t}}dt = -\int t\ln(t)dt = -\frac{t^2}{2}\ln(t) + \frac{t^2}{4} \\
v_2 &= \int \frac{W_2}{W}dt = \int \frac{e^{2t}\ln(t)}{e^{2t}}dt = \int \ln(t)dt = t\ln(t) + t \\
y_p(t) &= v_1y_1 + v_2y_2 = (-\frac{t^2}{2}\ln(t) + \frac{t^2}{4})e^t + (t\ln(t) + t)te^t \\
&= \frac{t^2}{2}e^t\ln(t) - \frac{3}{4}t^2e^t \\
y_g(t) &= y_p(t) + y_h(t) = c_1e^t + c_2te^t + \frac{t^2}{2}e^t\ln(t) - \frac{3}{4}t^2e^t
\end{aligned}
$$

3.6 Cauchy-Euler Equation and Reduction of Order

We have seen that a nonhomogeneous constant-coefficient differential equation $ay'' + by' + cy = f(t)$ has solutions valid over all intervals where $f(t)$ is continuous (ensuring that the integrals containing $f(t)$ exist and are differentiable). However, what happens when the DE has a point where it's undefined or discontinuous? When we have equations with variable coefficients of the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

the most we can expect is that there are solutions valid over the intervals where the four functions $a_2(t)$, $a_1(t)$, $a_0(t)$, and $f(t)$ are continuous; $a_2(t)$ must be nonzero over the function as well.

Theorem 3.6.1 Existence and Uniqueness of Solutions Theorem

Suppose $p(t)$, $q(t)$, and $g(t)$ are **continuous** on an interval (a, b) that **contains** the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$
\begin{aligned}
y''(t) + p(t)y'(t) + q(t)y(t) &= g(t) \\
y(t_0) &= Y_0 \\
y'(t_0) &= Y_1
\end{aligned}
$$

Example 3.6.1 (Determine the largest interval for which the theorem of existence and uniqueness ensures the existence and uniqueness of a solution to the initial value problem:

$$
\begin{aligned}
(t-3)y'' + y' + \sqrt{t}y &= \ln(t) \\
y(1) &= 3 \\
y'(1) &= -5
\end{aligned}
$$

)

We know that $t_0 = 1$. Let us now put the DE in standard form:

$$y'' + \frac{1}{t-3}y' + \frac{\sqrt{t}}{t-3}y = \frac{\ln(t)}{t-3}$$

Analyzing where the all the functions are continuous and taking the most restrictive case, we know that the interval must be either $(0, 3)$ or $(3, \infty)$; only the former contains the initial condition, so the final answer is $(0, 3)$.

3.6.1 Cauchy-Euler Equations

Definition 3.6.1: Cauchy-Euler Equation

A linear equation of the form

$$a_n t^n \frac{d^n y}{dt^n} + a_{n-1} t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 t \frac{dy}{dt} + a_0 y = f(t)$$

where a_n, a_{n-1}, $\ldots$, a_0 are constants, is known as a **Cauchy-Euler equation**. Its characteristic is that the degree k of the each of the coefficients t^k matches the order k of the differentiation $d^k y/dt^k$.

Note

Sometimes a Cauchy-Euler equation can be disguised and needs to be multiplied or divided by some factor of t.

The **characteristic equation** for the Cauchy-Euler equation is

$$ar^2 + (b-a)r + c = 0$$

and thus $y = t^r$ is a solution of the DE whenever r is a solution of the characteristic equation. There are 3 different cases depending on the roots of r.

1. 2 distinct roots where $r_1 \neq r_2$:
   The general solution is
   $$y_h(t) = c_1 t^{r_1} + c_2 t^{r_2}$$
2. one repeated real root:
   The general solution is
   $$y_h(t) = c_1 t^r + c_2 t^r \ln(t)$$
   (this result arises from the reduction of order, which will be covered later)
3. two conjugate complex roots where $r = \alpha \pm i\beta$, $\beta \neq 0$:
   The general solution is
   $$y_h(t) = c_1 t^\alpha \cos(\beta \ln(t)) + c_2 t^\alpha \sin(\beta \ln(t))$$

Example 3.6.2 (Find two linearly independent solutions to the equation

$$t^2 y'' + 3ty' - 3y = 0$$

)

$$
\begin{aligned}
ar^2 + (b-a)r + c &= 0 \\
r^2 + 2r - 3 &= 0 \\
(r+3)(r-1) &= 0 \\
r &= -3, 1 \\
y_h(t) &= c_1 t + c_2 t^{-3} \\
y_1 &= t \\
y_2 &= t^{-3}
\end{aligned}
$$

For a non-homogeneous Cauchy-Euler equation, we again find the 2 linearly independent solutions y_1 and y_2, and then use **variation of parameters** as outlined in the previous section to solve for v_1 and v_2.

Note

Again, remember to put it in standard form for variation of parameters!

Example 3.6.3 (Find a general solution.

$$t^2 y'' - 3ty' + 3y = 2t^4 e^t \quad (t > 0)$$

)

$$
\begin{aligned}
r^2 - 4r + 3 &= 0 \\
r &= 1, 3 \\
y_h(t) &= c_1 t + c_2 t^3 \\
y_1 &= t \\
y_2 &= t^3
\end{aligned}
$$

Putting the DE in standard form

$$
\begin{aligned}
y'' - \frac{3}{t}y' + \frac{3}{t^2}y &= 2t^2 e^t \quad (t > 0) \\
W &= \begin{vmatrix} t & t^3 \\ 1 & 3t^3 \end{vmatrix} = 2t^3 \\
W_1 &= \begin{vmatrix} 0 & t^3 \\ 2t^2 e^t & 3t^2 \end{vmatrix} = -2t^5 e^t \\
W_2 &= \begin{vmatrix} t & 0 \\ 1 & t_2^2 e^t \end{vmatrix} = 2t^3 e^t \\
v_1 &= \int -t^2 e^t dt = -e^t(t^2 - 2t + 2) \\
v_2 &= \int e^t dt = e^t \\
y_p &= v_1 y_1 + v_2 y_2 = -e^t(t^2 - 2t + 2)t + e^t t^3 \\
&= 2t^2 e^t - 2te^t \\
y_g &= c_1 t + c_2 t^3 + 2te^t(t-1)
\end{aligned}
$$

3.6.2 Reduction of Order

Theorem 3.6.2 Reduction of Order

Let $y_1(t)$ be a solution (not identically zero) to the equation $y'' + P(t)y' + Q(t)y = 0$ on an interval I. Then

$$y_2(t) = y_1(t)\int \frac{e^{-\int P(t)dt}}{y_1^2(t)}dt$$

is a second linearly independent solution.

Thm.(3.6.2):

$$y'' + P(t)y' + Q(t)y = 0$$

Let $y_2 = vy_1$

$$\begin{aligned}
y_2' &= v'y_1 + vy_1' \\
y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \\
(v''y_1 + 2v'y_1' + vy_1'') + P(t)v'y_1 + vy_1' + Q(t)vy_1 &= 0 \\
v''y_1 + (2y_1' + Py_1)v' + (y_1'' + Py_1' + Qy_1)v &= 0
\end{aligned}$$

The terms collected under v are equal to 0, so thus

$$v''y_1 + (2y_1' + Py_1)v' = 0$$

Let $y = y_1$, and $w = v'$

$$yw' + (2y' + Py)w = 0$$

This is a separable DE

$$\begin{aligned}
\int \frac{dw}{w} &= -\int \frac{2y' + Py}{y}dt \\
e^{\ln|w|} &= e^{-\int (\frac{2y'}{y} + P)dt} \\
w &= e^{-2\int \frac{dy}{y}}e^{-\int Pdt} \\
w &= y^{-2}e^{-\int Pdt} \\
v' &= y^{-2}e^{-\int Pdt} \\
v &= \int \frac{e^{-\int Pdt}}{y_1^2}dt = \frac{y_2}{y_1} \\
y_2(t) &= y_1(t)\int \frac{e^{-\int P(t)dt}}{[y_1(t)]^2}dt
\end{aligned}$$

😄

Chapter 4

Laplace Transforms

4.1 Definition of the Laplace Transform

Definition 4.1.1: Laplace Transform

Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by

$$\mathscr{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

The domain of $F(s)$ is values of s for which the integral exists. The Laplace transform of f is denoted by $F(s)$ or $\mathscr{L}\{f\}$.

Note

1. F is an improper integral
$$\int_0^\infty e^{-st} f(t) dt = \lim_{N \to \infty} \int_0^N e^{-st} f(t) dt$$
2. In this course, we assume that s and f are sufficiently restricted to guarantee the convergence of the appropriate Laplace transform

Theorem 4.1.1 Linearity of the Laplace Transform

Let f_1 and f_2 be functions whose Laplace transforms exist for $s > a$ and let c_1 and c_2 be any constants. Then for $s > a$,

$$\mathscr{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathscr{L}\{f_1\} + c_2 \mathscr{L}\{f_2\}$$

4.1.1 Existence of the Transform

Definition 4.1.2

A function $f(t)$ is **piecewise continuous** on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

Definition 4.1.3

A function $f(t)$ is **piecewise continuous** on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Note

Infinite discontinuities are not okay! Jumps and holes are.

Definition 4.1.4

A function $f(t)$ is of **exponential order** a if there exist positive constants T and M such that

$$|f(t)| \leq Me^{at} \quad \text{for all } t \geq T$$

Theorem 4.1.2 Sufficient Conditions for Existence of the Transform

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order a, then $\mathscr{L}\{f(t)\}$ exists for $s > a$.

Note

These conditions are **sufficient** but not necessary for the existence of a Laplace transform.

Theorem 4.1.3 Behavior of $F(s)$ as $s \to \infty$

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order a, and $F(s) = \mathscr{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

4.2 Properties of the Laplace Transform

Theorem 4.2.1 Translation in s

If the Laplace transform $\mathscr{L}\{f(t)\} = F(s)$ exists for $s > a$, then

$$\mathscr{L}\{e^{at} f(t)\} = F(s - a)$$

Thm. 4.2.1:

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$
$$\mathscr{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at}$$
$$= \int_0^\infty e^{-(s-a)t} f(t)dt = F(s - a)$$

since changing s without a difference in t has no effect on the end result. 😊

Note

It is sometimes useful to use the symbolism

$$\mathscr{L}\{e^{at} f(t)\} = \mathscr{L}\{f(t)\}|_{s\to s-a}$$

where we say that we "send s to $s - a$ ".

Theorem 4.2.2 Transforms of Derivative

Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order a. Then, for $s > a$,

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$$

Thm. 4.2.2:

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

Using integration by parts,

$$\begin{aligned}\int_0^\infty e^{-st}f'(t)dt &= \lim_{N\to\infty} e^{-st}f(t)|_0^N + \int_0^\infty se^{-st}f(t)dt \\ &= \lim_{N\to\infty}\frac{f(N)}{e^{sN}} - \frac{f(0)}{e^0} + s\int_0^\infty e^{-st}f(t)dt \\ &= s\mathscr{L}\{f(t)\} - f(0)\end{aligned}$$

😆

Theorem 4.2.3 Transforms of Higher Order Derivatives

Let $f(t)$, $f'(t)$,...,$f^{(n-1)}(t)$ be continuous on $[0,\infty)$ and $f^{(n)}(t)$ be piecewise continuous on $[0,\infty)$, with all these functions of exponential order a. Then for $s > a$,

$$\mathscr{L}\{f^{(n)}(t)\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

Theorem 4.2.4 Derivative of a Transform

Let $F(s) = \mathscr{L}\{f(t)\}$. Assume $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order a. Then for $s > a$,

$$\mathscr{L}\{t^n f(t)\} = (-1)^n\frac{d^nF}{ds^n}$$

Thm. 4.2.4:

$$\frac{dF}{ds} = \frac{d}{ds}[\int_0^\infty e^{-st}f(t)dt]$$

We then apply the Leibnitz rule of switching the order of integration and differentiation (which is okay when the integrand is continuous):

$$\begin{aligned}\frac{dF}{ds} &= \int_0^\infty \frac{d}{ds}[e^{-st}]f(t)dt \\ &= -\int_0^\infty te^{-st}f(t)dt \\ &= -\mathscr{L}\{tf(t)\} \\ \mathscr{L}\{t^1f(t)\} &= -1\cdot\frac{dF}{ds}\end{aligned}$$

By induction,

$$\mathscr{L}\{t^n f(t)\} = (-1)^n\frac{d^nF}{ds^n}$$

😆

4.3 Inverse Laplace Transform

Definition 4.3.1: Inverse Laplace Transform

Given a function $F(s)$, if there is a function $f(t)$ that is **continuous** on $[0,\infty)$ and satisfies

$$\mathscr{L}\{f(t)\} = F(s)$$

then $f(t)$ is the **inverse Laplace transform** of $F(s)$ and we use the notation

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

Theorem 4.3.1 Linearity of the Inverse Transform

$\mathscr{L}^{-1}$ is a linear operator. That is, if $\mathscr{L}^{-1}\{F_1\}$ and $\mathscr{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$, then

$$\mathscr{L}^{-1}\{c_1F_1 + c_2F_2\} = c_1\mathscr{L}^{-1}\{F_1\} + c_2\mathscr{L}^{-1}\{F_2\}$$

for any constants c_1 and c_2.

4.3.1 Partial Fractions

Partial fractions are extremely useful in finding inverse Laplace transforms. Given the fraction $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with the degree of P being less than the degree of Q, we know that $Q(s)$ can always be factored into linear and/or irreducible quadratic factors only, in theory. There are 3 basic cases:

1. **Nonrepeated linear factors:**
   If $Q(s) = (s - r_1)(s - r_2)\ldots(s - r_n)$ where r_i are distinct real numbers, the partial fraction expansion has the form
   $$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \ldots + \frac{A_n}{s - r_n}$$
   where A_i are real numbers.

2. **Repeated linear factors:**
   If $Q(s)$ contains the linear factor $(s - r)^m$ where m is the highest power of $s - r$ that divides $Q(s)$, then the portion of the partial fraction expansion of $\frac{P(s)}{Q(s)}$ that corresponds to that factor is
   $$\frac{A_1}{s - r} + \frac{A_2}{(s - r)^2} + \ldots + \frac{A_m}{(s - r)^m}$$
   where A_i are real numbers.

3. **Quadratic factors:**
   If $Q(s)$ contains the quadratic factor $[(s - \alpha)^2 + \beta^2]^m$ where m is the highest power of $(s - \alpha)^2 + \beta^2$ that divides $Q(s)$, then the portion of the partial fraction expansion of $\frac{P(s)}{Q(s)}$ that corresponds to that factor is
   $$\frac{C_1s + D_1}{(s - \alpha)^2 + \beta^2} + \frac{C_2s + D_2}{[(s - \alpha)^2 + \beta^2]^2} + \ldots + \frac{C_ms + D_m}{[(s - \alpha)^2 + \beta^2]^m}$$

Note

We need to express $C_is + D_i$ in the form of $A_i(s - \alpha) + \beta B_1$ for the inverse Laplace transforms. The above partial fraction expansion is equivalent to the form

$$\frac{A_1(s - \alpha) + B_1\beta}{(s - \alpha)^2 + \beta^2} + \ldots + \frac{A_m(s - \alpha) + B_m\beta}{[(s - \alpha)^2 + \beta^2]^m}$$

4.4 Solving Initial Value Problems with Laplace Transforms

Consider a linear IVP with constant coefficients.

1. Take the Laplace transform of both sides of the differential equation
2. Use the properties of the Laplace transform and the initial conditions to reduce it to an algebraic equation in $Y(s)$
3. Solve $Y(s)$
4. Determine the inverse Laplace transform and the solution is $y(t) = \mathscr{L}^{-1}\{Y(s)\}$

Note

You will need

$$\mathscr{L}\{f^{(n)}(t)\} = s^n\mathscr{L}\{f^{(n)}(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

and especially when solving a second order IVP, where we use

$$\mathscr{L}\{y'(t)\} = sY(s) - y(0)$$

and

$$\mathscr{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

Example 4.4.1 (Solve $y' - 3y = e^{2t}$, $y(0) = 1$)

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0) = sY(s) = 1 \\
\mathscr{L}\{3y\} &= 3Y(s) \\
\mathscr{L}\{e^{2t}\} &= \frac{1}{s-2} \\
sY(s) - 1 - 3Y(s) &= \frac{1}{s-2} \\
Y(s)(s-3) &= 1 + \frac{1}{s-2} \\
Y(s) &= \frac{s-1}{(s-2)(s-3)}
\end{aligned}$$

Using PFD

$$\begin{aligned}
Y(s) &= \frac{2}{s-3} - \frac{1}{s-2} \\
\mathscr{L}\{Y(s)\} &= 2e^{3t} - e^{2t} = y(t)
\end{aligned}$$

Note

If the initial conditions given are not where $t = 0$, then use the substitution $y(t) = w(t - n)$ where n is where the initial conditions are.

If the linear equation has coefficients that are polynomials of t, then we apply the property

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

If $n = 1$ and $f(t) = y'(t)$, then

$$\begin{aligned}
\mathscr{L}\{ty'(t)\} &= -\frac{d}{ds}[\mathscr{L}\{y'(t)\}] \\
&= -\frac{d}{ds}(sY(s) - y(0)) \\
&= -sY'(s) - Y(s)
\end{aligned}$$

and if $n = 1$ and $f(t) = y''(t)$, then

$$\mathscr{L}\{ty''(t)\} = -s^2Y'(s) - 2sY(s) + y(0)$$

which means that the DEQ is now a linear differential equation in $Y(s)$ which we can solve easily.

Note

You may need the theorem

$$\lim_{s\to\infty} \mathscr{L}\{f(t)\} = 0$$

4.5 Transformations of Discontinuous Functions

Definition 4.5.1: Unit Step Function

The unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases}$$

Note

$$\begin{aligned} u(t-a) &= \begin{cases} 0, & t-a < 0 \\ 1, & t-a > 0 \end{cases} \\ &= \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} \\ M \cdot u(t-a) &= \begin{cases} 0, & t < a \\ M, & t > a \end{cases} \end{aligned}$$

Note

The unit step function can be used to write piecewise defined functions in a compact form. If our function is defined as

$$f(t) = \begin{cases} g(t), & t < a \\ h(t), & t > a \end{cases}$$

then we can use the unit step function to rewrite it as

$$f(t) = g(t) - g(t)u(t-a) + h(t)u(t-a)$$

Example 4.5.1 (Rewrite

$$g(t) = \begin{cases} 3, & t < 2 \\ -1, & 2 < t < 5 \\ t, & 5 < t < 8 \\ \frac{t^2}{10}, & t > 8 \end{cases}$$

)

$$g(t) = 3 - 3u(t-2) + (-1)u(t-2) - (-1)u(t-5) + tu(t-5) - tu(t-8) + \frac{t^2}{10}u(t-8)$$

Combine like terms

$$g(t) = 3 - 4u(t-2) + (1+t)u(t-5) + (\frac{t^2}{10} - t)u(t-8)$$

If $a \geq 0$, the Laplace transform of $u(t-a)$ is

$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s} \quad \text{for } s > 0$$

Theorem 4.5.1 Translation in t

Let $F(s) = \mathscr{L}\{f(t)\}$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

and, conversely, an inverse Laplace transform of $e^{-as}F(s)$ is

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

Note

If $f(t) = 1$ for all t,

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) = \frac{e^{-as}}{s}$$

Note

Let $g(t) = f(t-a)$, then $f(t) = g(t+a)$. Therefore,

$$\begin{aligned} \mathscr{L}\{f(t-a)u(t-a)\} &= e^{-as}\mathscr{L}\{f(t)\} \\ \implies \mathscr{L}\{g(t)u(t-a)\} &= e^{-as}\mathscr{L}\{g(t+a)\} \end{aligned}$$

Example 4.5.2 (Determine the Laplace transform: $\mathscr{L}\{tu(t-2)\}$)

$$\begin{aligned} g(t) &= t \\ a &= 2 \\ g(t+a) &= g(t+2) = t+2 \\ \mathscr{L}\{tu(t-2)\} &= e^{-2s}\mathscr{L}\{g(t+2)\} = e^{-2s}\mathscr{L}\{t+2\} \\ &= e^{-2s}(\frac{1}{s^2} + \frac{2}{s}) \end{aligned}$$

Example 4.5.3 (Determine the inverse Laplace transform: $\mathscr{L}^{-1}\{\frac{e^{-3s}}{s^2}\}$)

$$\begin{aligned} a &= 3 \\ F(s) &= \frac{1}{s^2} \\ \mathscr{L}^{-1}\{F(s)\} &= \mathscr{L}^{-1}\{\frac{1}{s^2}\} = t \end{aligned}$$

Recall

$$\begin{aligned} \mathscr{L}^{-1}\{e^{-as}F(s)\} &= f(t-a)u(t-a) \\ \mathscr{L}^{-1}\{\frac{1}{s^2} \cdot e^{-3s}\} &= f(t-3)u(t-3) \\ &= t|_{t\to t-3}u(t-3) = (t-3)u(t-3) \end{aligned}$$

4.6 Convolutions

In many case, we are required to determine the inverse Laplace transform of a product of two functions.

Note

$$\mathscr{L}^{-1}\{\frac{1}{s^2+1}G(s)\} \neq \mathscr{L}^{-1}\{\frac{1}{s^2+1}\}\mathscr{L}^{-1}\{G(s)\}$$

Definition 4.6.1: Convolution

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The **convolution** of $f(t)$ and $g(t)$ is defined by

$$(f * g)(t) = \int_0^t f(t-v)g(v)dv$$

Example 4.6.1 (Determine $(t * 1)(t)$)

$$\begin{aligned} t * 1 &= \int_0^t f(t-v)g(v)dv \\ &= \int_0^t (t-v)(1)dv \\ &= (vt - \frac{v^2}{2})|_{v=0}^{v=t} \\ &= \frac{t^2}{2} \end{aligned}$$

Note

Convolution is different from multiplication

$$(t * 1)(t) = \frac{t^2}{2} \neq t$$

Theorem 4.6.1 Properties of Convolution

Let $f(t)$, $g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

1. $f * g = g * f$
2. $f * (g + h) = (f * g) + (f * h)$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0$

Theorem 4.6.2 Convolution Theorem

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α. If $F(s) = \mathscr{L}\{f(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$, then

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

or, equivalently,

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Thm 4.6.2:

$$\mathscr{L}\{f * g\} = \int_0^\infty e^{-st}[\int_0^t f(t-v)g(v)dv]dt$$

Now we introduce the unit step function

$$= \int_0^\infty e^{-st}[\int_0^t u(t-v)f(t-v)g(v)dv]dt$$

Reverse the order of integration

$$\begin{aligned}
&= \int_0^\infty g(v) \int_0^\infty e^{-st} u(t-v) f(t-v) dt dv \\
&= \int_0^\infty g(v) e^{-sv} F(s) dv \\
&= F(s) \int_0^\infty g(v) e^{-sv} dv \\
&= F(s)G(s)
\end{aligned}$$

😄

Note

When $g(t)$ and $\mathscr{L}\{g(t)\} = G(s) = \dfrac{1}{s}$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathscr{L}\{\int_0^t f(v)dv\} = \frac{F(s)}{s}$$

and the inverse form of the above formula

$$\int_0^t f(v)dv = \mathscr{L}^{-1}\{\frac{F(s)}{s}\}$$

can be used in lieu of partial fractions when s^n is a factor of the denominator and $f(t) = \mathscr{L}^{-1}\{F(s)\}$ is easy to integrate.

Example 4.6.2 (Determine $\mathscr{L}^{-1}\{\frac{1}{s(s^2+1)}\}$)

$$\begin{aligned}
\mathscr{L}^{-1}\{\frac{1}{s(s^2+1)}\} &= \mathscr{L}^{-1}\{\frac{1}{s} \cdot \frac{1}{s^2+1}\} \\
&= 1 * \sin(t) \\
&= \int_0^t 1 \cdot \sin(v) dv \\
&= -\cos(v)|_{v=0}^{v=t} \\
&= 1 - \cos(t)
\end{aligned}$$

Example 4.6.3 (Use the convolution theorem to solve the **integral equation**

$$y(t) = 4t + \int_0^t y(t-v)\sin(v)dv$$

)

$$
\begin{aligned}
y(t) &= 4t + y(t) * \sin(t) \\
Y(s) &= \frac{4}{s^2} + Y(s) \cdot \frac{1}{s^2+1} \\
Y(s)[1 - \frac{1}{s^2+1}] &= \frac{4}{s^2} \\
Y(s)[\frac{s^2}{s^2+1}] &= \frac{4}{s^2} \\
Y(s) &= \frac{4(s^2+1)}{s^4} = \frac{4}{s^2} + \frac{4}{s^4} \\
y(t) &= 4t + \frac{2}{3}t^3
\end{aligned}
$$

Laplace transforms are also useful in solving **integro-differential equations**, which are equations that involve a derivative as well as an integral of the unknown function.

Example 4.6.4 (Solve $y'(t) + y(t) + \int_0^t y(v)dv = 1$, $y(0) = 0$)

$$
\begin{aligned}
y'(t) + y(t) + 1 * y(t) &= 1 \\
sY(s) - y(0) + Y(s) + \frac{1}{2}Y(s) &= \frac{1}{s} \\
Y(s) &= \frac{1}{s}(\frac{1}{s+1+\frac{1}{s}}) \\
&= \frac{1}{s^2+s+1} \\
&= \frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\
y(t) &= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t)
\end{aligned}
$$

4.6.1 Impulse Response Function

Suppose we have an equation of the form

$$ay'' + by' + cy = g(t), \quad y(0) = y'(0) = 0$$

Then $g(t)$ is the input of the system and $y(t)$ is the output. We would like to have a convenient way to study the behavior of the system for different inputs. Let $Y(s) = \mathscr{L}\{y(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$. Then we define the **transfer function**

$$H(s) = \frac{Y(s)}{G(s)} = \frac{1}{as^2 + bs + c}$$

We can now easily study the behavior of the system given different inputs by simply multiplying by the transfer function. The function $h(t) = \mathscr{L}^{-1}\{H(s)\}$ is called the **impulse response function** because it describes the solution when a mass-spring system is struck by a hammer.

Solutions Using the Impulse Response Function

Let I be an interval containing the origin and let $g(t)$ be continuous on I. To solve the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0,\ y'(0) = y_1$$

1. Find the unique solution $y_h(t)$ to the homogeneous IVP

$$ay'' + by' + cy = 0,\ y(0) = y_0,\ y'(0) = y_1$$

2. Determine the impulse response function

$$h(t) = \mathscr{L}^{-1}\{\frac{1}{as^2 + bs + c}\}$$

3. The solution is given by

$$y(t) = y_h(t) + h * g = y_h(t) + \int_0^t h(t-v)g(v)dv$$

4.7 Dirac Delta Function

Definition 4.7.1: Dirac Delta Function

The Dirac delta function, $\delta(t)$, is a "function" that is zero everywhere but ∞ at $t = 0$.

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

We can shift the argument to get

$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ \infty, & t = a \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

The Laplace transform of the Dirac delta function is given by

$$\int_{-\infty}^{\infty} e^{-st}\delta(t-a)dt = e^{-st}|_{t=a} = e^{-as}$$

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}, \text{ for } a \geq 1$$

$$\mathscr{L}\{\delta(t)\} = e^{-0s} = 1$$

$$\mathscr{L}\{1\} = \delta(t)$$

The Dirac delta function has a relationship with the unit step function:

$$\int_{-\infty}^{t} \delta(x-a)dx = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} = u(t-a)$$

Differentiating both sides with respect to t yields

$$\delta(t-a) = u'(t-a)$$

Thus the Dirac delta "function" is the derivative of the unit step function.

Example 4.7.1 (Evaluate $\int_{-\infty}^{\infty} e^{3t}\delta(t)dt$)

$$\int_{-\infty}^{\infty} e^{3t}\delta(t)dt = e^{3t}|_{t=0} = e^0 = 1$$

Example 4.7.2 ($\mathscr{L}\{t^3\delta(t-3)\}$)

There are two ways to solve this:

1. By the definition

$$\int_{-\infty}^{\infty} e^{-st} t^3 \delta(t-3) dt = e^{-st} t^3 |_{t=3} = 27e^{-3s}$$

2. Recall

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$$

Thus

$$\begin{aligned}
\mathscr{L}\{t^3 \delta(t-3)\} &= (-1)^3 \frac{d^3}{ds^3} [\mathscr{L}\{\delta(t-3)\}] \\
&= -\frac{d^3}{ds^3} (e^{-3s}) \\
&= 27e^{-3s}
\end{aligned}$$

Example 4.7.3 ($\mathscr{L}\{e^t \delta(t-3)\}$)

Recall the translation property:

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a)$$

Thus

$$\mathscr{L}\{e^t \delta(t-3)\} = e^{-3s}|_{s \to s-1} = e^{3(1-s)}$$

In addition, it is useful to know that the impulse response function $h(t)$ can be considered as the response of the system to the impulse $\delta(t)$.

Chapter 5

Series Solutions of Differential Equations

5.1 The Taylor Polynomial Approximation

Definition 5.1.1: Taylor Polynomials

The Taylor polynomial of degree n centered at x_0 approximating $f(x)$ with n derivatives at x_0 is given by

$$p_n(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Example 5.1.1 (Solve $y' = y^2,\ y(0) = 2$)

$$\begin{aligned}
p_3(x) &= \frac{y(0)}{0!} + \frac{y'(0)x}{1!} + \frac{y''(0)x^2}{2!} + \frac{y'''(0)x^3}{3!} \\
y(0) &= 2 \\
y' &= y^2 \implies y'(0) = (y(0))^2 = 2^2 = 4
\end{aligned}$$

To find $y''(x)$, differentiate both sides of the DE implicit w.r.t. x

$$\begin{aligned}
\frac{d}{dx}y' &= \frac{d}{dx}y^2 \\
y''(x) &= 2y \cdot y' \\
y''(0) &= 2y(0) \cdot y'(0) = 2(2)(4) = 16 \\
\frac{d}{dx}y''(x) &= \frac{d}{dx}2yy' \\
y''' &= 2(y' \cdot y' + y \cdot y'') \\
y'''(0) &= 2((y'(0))^2 + y(0)y''(0)) = 96 \\
p_3(x) &= 2 + 4x + 8x^2 + 16x^3
\end{aligned}$$

5.2 Power Series and Analytic Functions

Recall **power series**, which are a certain type of Taylor polynomial. A power series about a point x_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

where a_n is a coefficient, x_0 is the center, and n is always a non-negative exponent. We say that a power series **converges** at the point $x = c$ if

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(c-x_0)^n$$

exists as a finite number. If the limit does not exist, the power series **diverges** at $x = c$.

Theorem 5.2.1

There is a number ρ ($0 \le \rho \le \infty$) called the **radius of convergence** of a power series such that the series converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$. If the series converges for all values of x, then $\rho = \infty$.

Theorem 5.2.2 Ratio Test

If, for any large n, the coefficients of a_n are nonzero and satisfy $\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = L$ where $0 \le L < \infty$, then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $\rho = L$.

Example 5.2.1 (Determine the convergence of

$$\sum_{n=0}^{\infty} \frac{(n+2)!}{n!}(x+2)^n$$

)

$$\sum_{n=0}^{\infty} \frac{(n+2)!}{n!}(x+2)^n = \sum_{n=0}^{\infty} (n+2)(n+1)(x+2)^n$$

Using the ratio test

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_n \text{ coefficient}}{a_{n+1} \text{ coefficient}}\right| &= \lim_{n\to\infty} \left|\frac{(n+2)(n+1)}{(n+3)(n+2)}\right| = 1 = \rho \\ |x+2| &< 1 \\ -1 < x+2 &< 1 \\ -3 < x &< -1 \end{aligned}$$

Checking the endpoints

$$x = -3$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)(x+2)^n = \sum_{n=0}^{\infty} (n+2)(n+1)(-1)^n$$

$$\lim_{n\to\infty} (n+2)(n+1)(-1)^n \text{ DNE}$$

This diverges

$$x = -1$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)(x+2)^n = \sum_{n=0}^{\infty}(n+2)(n+1)(1)^n$$

$$\lim_{n\to\infty}(n+2)(n+1) = \infty$$

This diverges, and so our final interval of convergence is

$$(-3, 1)$$

Note

When shifting summation indices, whatever you do under $\sum$, you must do the opposite of in the function.

Example 5.2.2

$$\sum_{n=1}^{\infty}(n+1)x^{n-1} = \sum_{n=0}^{\infty}(n+2)x^n$$

Theorem 5.2.3

If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for all x in some open interval, then each coefficient $a_n = 0$.

Note

This will be important, as when you solve for the coefficient a with the highest subscript, this will give you your **recurrence relation**, which will help solve DEs.

Theorem 5.2.4

If $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a positive radius of convergence R, then f is differentiable in $|x-x_0| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} \quad \text{for } |x-x_0| < R$$

Note

It is important to shift the starting index n to avoid negative exponents on the x if necessary.

Furthermore, f is integrable in $|x-x_0| < R$ and

$$\int f(x)dx = C + \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1} \quad \text{for } |x-x_0| < R$$

Example 5.2.3 (Find a power series for $\ln(1+x)$)

$$\ln(1+x) = \int \frac{dx}{1+x}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\ln(1+x) = \int \sum_{n=0}^{\infty}(-1)^n x^n dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C$$

Evaluate C when $x = 0$

$$\ln(1+0) = \ln(1) = 0 = \sum_{n=0}^{\infty} \frac{(-1)^n(0)}{n+1} + C$$

$$C = 0$$

Our final answer is

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

Definition 5.2.1: Analytic Functions

A function is **analytic** at x_0 if it can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with a positive (or an infinite) radius of convergence.

Note

If a function does not have derivatives of all orders at x_0, then it cannot be analytic at x_0.

5.3 Power Series Solutions to Linear DEQs

Consider the DEQ

$$a_2(x)y'' + a_1(x)y' + a_0 y = 0 \tag{5.3.1}$$

and it's standard form

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$.

Definition 5.3.1

A point x_0 is called an **ordinary point** of equation 5.3.1 if both $P(x)$ and $Q(x)$ are analytic at x_0. If x_0 is not an ordinary point, it is called a **singular point** of the equation.

5.3.1 Power Series Method About an Ordinary Point

Assuming that y is analytic at x_0, to solve equation 5.3.1,

1. Let

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

2. Find y' and y'', and substitute them into the DEQ
3. Find the recurrence relation and solve for a_n
4. Apply any given initial conditions if applicable

Example 5.3.1 (Find a power series solution about $x = 0$ to $y'' + y = 0$.)

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n &= 0 \end{aligned}$$

The coefficients of x^n must be 0 for all powers of x; solving for a_{n+2} gives us our recurrence relation

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

Notice that we can treat evens and odds separately, since there is no relation between them in the recurrence relation

For the evens

$$\begin{aligned} n &= 0 \quad a_2 = \frac{-a_0}{2(1)} = \frac{-a_0}{2!} \\ n &= 2 \quad a_4 = \frac{-a_2}{4(3)} = \frac{a_0}{4!} \\ n &= 4 \quad a_6 = \frac{-a_4}{6(5)} = \frac{-a_0}{6!} \end{aligned}$$

This pattern suggests that if $n = 2k$, then $a_{2k} = \frac{(-1)^k}{(2k)!} a_0$

For the odds

$$\begin{aligned} n &= 1 \quad a_3 = \frac{-a_1}{3(2)} = \frac{-a_1}{3!} \\ n &= 3 \quad a_5 = \frac{-a_3}{5(4)} = \frac{a_1}{5!} \\ n &= 5 \quad a_7 = \frac{-a_5}{7(6)} = \frac{-a_1}{7!} \end{aligned}$$

This pattern suggests that if $n = 2k+1$, then $a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1$

Now we can construct our solution

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \dots \\
&= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 - \frac{a_0}{6!} x^6 + \dots \\
&= a_0 [1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots] + a_1 [x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots] \\
&= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
&= a_0 \cos(x) + a_1 \sin(x)
\end{aligned}$$

Theorem 5.3.1 Existence of Analytic Solutions

Suppose x_0 is an ordinary point for the equation

$$y'' + P(x) y' + Q(x) y = 0$$

Then the equation has two linearly independent analytic solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Moreover, a power series solution converges on at least on some interval $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point of the equation in the **complex plane**.

Note

When finding series expansions about points not at $x = 0$, then it is important to adjust any instances of x in the DE, as the series expansion of y is now

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$