MHF3202
HW4

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Question 1

Proof: We prove this by way of induction.
Base Case: Observe for $n = 0,\ p(x) = a_0x^0 = a_0(1) = a_0$, so $p(x) \in \mathbb{Z}$ as desired.
Inductive Hypothesis: Assume for $n = k,\ 0 \le k \le n$ that $x^n \in \mathbb{Z}$ and $p(x) \in \mathbb{Z}$.
Inductive Step: Consider a polynomial $q(x)$ of degree $n+1$ written as $q(x) = a_{n+1}x^{n+1} + p(x)$, where $a_{n+1} \in \mathbb{Z}$.
Notice that $a_{n+1}x^{n+1} = a_{n+1}x^n x$.
Since $x^n \in \mathbb{Z}$ by the inductive hypothesis, $a_{n+1}x^n x \in \mathbb{Z}$ by multiplication of integers.
Since $p(x) \in \mathbb{Z}$ by the inductive hypothesis, $q(x) \in \mathbb{Z}$ by addition of integers.
So for $n+1,\ q(x) \in \mathbb{Z}$.
Therefore, by induction, $p(x) \in \mathbb{Z}$ if $x \in \mathbb{Z}$. ☺

Question 2

Note
I tried proving this as a corollary but it proved too hard for my 11PM brain :(

Lemma 0.0.1 The sum and product of rational numbers is rational.

Proof: Suppose $x, y \in \mathbb{Q}$ where $x = \frac{p}{q}, y = \frac{a}{b}$ for $p, q, a, b \in \mathbb{Z},\ q, b \neq 0$.
Then $x + y = \frac{p}{q} + \frac{a}{b} = \frac{pb+aq}{qb}$ where $pb, aq, qb \in \mathbb{Z}$ by multiplication of integers and $qb \neq 0$ since $q \neq 0$ and $b \neq 0$.
Then by definition of rational numbers, $x + y \in \mathbb{Q}$.
Next, we want to show that $x \cdot y \in \mathbb{Q}$.
Multiplying, we get $xy = \left(\frac{p}{q}\right) \cdot \left(\frac{a}{b}\right) = \frac{pa}{qb}$ where $pa, qb \in \mathbb{Z}$ by multiplication of integers and $qb \neq 0$ since $q \neq 0$ and $b \neq 0$.
Then, by definition of rational numbers, $x \cdot y \in \mathbb{Q}$. ☺

Proof: We prove this by way of induction.
Base Case: Observe for $n = 0,\ p(x) = a_0x^0 = a_0(1) = a_0$, so $p(x) \in \mathbb{Q}$ as desired.
Inductive Hypothesis: Assume for $n = k,\ 0 \le k \le n$ that $x^n \in \mathbb{Q}$ and $p(x) \in \mathbb{Q}$.
Inductive Step: Consider a polynomial $q(x)$ of degree $n+1$ written as $q(x) = a_{n+1}x^{n+1} + p(x)$, where $a_{n+1} \in \mathbb{Q}$.
Notice that $a_{n+1}x^{n+1} = a_{n+1}x^n x$.
Since $x^n \in \mathbb{Q}$ by the inductive hypothesis, $a_{n+1}x^n x \in \mathbb{Q}$ by lemma 0.0.1.
Since $p(x) \in \mathbb{Q}$ by the inductive hypothesis, $q(x) \in \mathbb{Q}$ by lemma 0.0.1.
So for $n+1,\ q(x) \in \mathbb{Q}$.
Therefore, by induction, $p(x) \in \mathbb{Q}$ if $x \in \mathbb{Q}$. ☺

Question 3

Proof: We seek to disprove this, i.e. prove that there exists some $A, B, C, D \subset \mathbb{R}$ s.t. $(A \triangle B) \cup (C \triangle D) \not\subseteq (A \cup B) \triangle (C \cup D)$ or $(A \triangle B) \cup (C \triangle D) \not\supseteq (A \cup B) \triangle (C \cup D)$.
Let $A = \{1, 2\}, B = \{11, 12\}, C = \{2, 3\}, D = \{12, 13\}$.
Observe, $A \triangle B = \{1, 2, 11, 12\}$ and $C \triangle D = \{2, 3, 12, 13\}$.
Thus $S_1 = (A \triangle B) \cup (C \triangle D) = \{1, 2, 3, 11, 12, 13\}$.
Also, observe $A \cup B = \{1, 2, 11, 12\}$ and $C \cup D = \{2, 3, 12, 13\}$.
Thus $S_2 = (A \cup B) \triangle (C \cup D) = \{1, 3, 11, 3\}$.
Then, take $2 \in S_1$, and notice by inspection that $2 \notin S_2$.
So, $S_1 \not\subseteq S_2$, and our original claim is disproven. ☺

Question 4

Finding a closed form:

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + (n-1) + n \\
\text{Also, } S_n &= n + (n-1) + \ldots + 3 + 2 + 1 \\
\text{So, } 2S_n &= (n+1) + ((n-1)+2) + \ldots + (2 + (n-1)) + (1+n) \\
&= n(n+1) \\
\text{Thus, } S_n &= \frac{n(n+1)}{2}
\end{aligned}$$

Proof: We prove this by way of induction.
Let $S_n = \sum_i^n i$.
Base Case: Observe that for $n = 1$, $S_1 = \frac{1(1+1)}{2} = 1$.
So the formula gives the correct sum for $n = 1$.
Inductive Hypothesis: Suppose that for $k \in \mathbb{N}$, $S_k = \frac{k(k+1)}{2}$.
Inductive Step: By definition, $S_{k+1} = S_k + (k+1)$.
By the inductive hypothesis,

$$\begin{aligned}
S_{k+1} &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k^2+k}{2} + \frac{2k+2}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2} \text{ as desired.}
\end{aligned}$$

So, for all $k \in \mathbb{N}$, $\sum_{i=1}^{k+1} i = S_{k+1}$.
Therefore, by induction, our original claim is proven. ☺

Question 5

Proof: We prove this by way of induction.
Let $A_n = \sum_{i=1}^{n} i^3$ and $B_n = \left(\sum_{i=1}^{n} i\right)^2$, i.e. $B_n = \left(\frac{n(n+1)}{2}\right)^2$ by using the previous problem as a lemma.
Base Case: Consider $n = 1$.
Observe that $A_1 = 1$ and $B_1 = 1$.
Inductive Hypothesis: Assume that for $k \in \mathbb{N}$, $A_k = B_k$.
Inductive Step: By definition, $A_{k+1} = A_k + (k+1)^3$.

By the inductive hypothesis,

$$\begin{aligned}
A_{k+1} &= B_k + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k^3+3k^2+3k+1)}{4} \\
&= \frac{k^4+2k^3+k^2}{4} + \frac{4(k^3+3k^2+3k+1)}{4} \\
&= \frac{k^4+6k^3+13k^2+12k+4}{4} \\
&= \frac{(k^2+2k+1)(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= B_{k+1} \text{ as desired.}
\end{aligned}$$

So $A_{k+1} = B_{k+1}$ for all $n \in \mathbb{N}$.
Therefore, by induction, our original claim is proven. ☺

Question 6

Proof: We prove this by way of induction.
Base Case: Let $x = 1$.
Observe that $x^2 = 1$ and $x^2 - 1 = 1 - 1 = 8(0)$.
So we have that $x^2 \equiv 1 \pmod 8$ for $x = 1$.
Inductive Hypothesis: Assume that $x^2 \equiv 1 \pmod 8$ for $x = k$ where $k = 2m+1,\ m \in \mathbb{N}$.
Inductive Step: By the inductive hypothesis, $x^2 - 1 = 8n,\ n \in \mathbb{N}$.
By substitution, $(2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 8n$.
Adding $8k+8$ to both sides, we get $4k^2 + 12k + 9 - 1 = (2k+3)^2 - 1 = (2k+1+2)^2 - 1 = 8n$.
By definition of congruence of integers, $(2k+1+2)^2 \equiv 1 \pmod 8$.
So, by substitution, we have that $(x+2)^2 \equiv 1 \pmod 8$, which is the next odd number after x.
Therefore, by induction, $x^2 \equiv 8 \pmod 1$ for any odd natural number. ☺

Question 7

Note

This may have been a typo, since we don't have anything for congruence of real numbers. However, this is almost due and I didn't want to bother you so I sort of came up with an example that also uses some kind of extension for some of the things we've done. I'm fine if you take points off.

Proof: We claim that this is false.
Consider $x = \frac{5}{2}, y = \frac{1}{2}$, and $n = 2$.
Notice that $\frac{5}{2} - \frac{1}{2} = 2(1)$.
So, by definition of congruence (of real numbers?) $\frac{5}{2} \equiv \frac{1}{2} \pmod 2$.
However, notice that $(\frac{5}{2})^3 - (\frac{1}{2})^3 = \frac{125}{8} - \frac{1}{8} = \frac{124}{8} = \frac{41}{2}$.
Observe, by the (extended?) division algorithm, $\frac{41}{2} = 10 \cdot 2 + \frac{1}{2}$.
In particular, $r \neq 0$.
So $\left(\frac{5}{2}\right)^3 - \left(\frac{1}{2}\right)^3 \nmid 2$, and so $\left(\frac{5}{2}\right)^3 \not\equiv \left(\frac{1}{2}\right)^3 \pmod 2$. ☺

Question 8

1. This is **not** a valid proof.

2. In step 1, computing the first few Fibonacci numbers does **not** necessarily establish periodicity. Using the first few computed numbers like it does later in the proof does not rigorously show anything, and should only be used to convince yourself if the original claim **appears** true or false.

   In the inductive step, induction is not properly used, i.e. ChatGPT did not establish that $m+1$ works because of m. If it wanted to use something like periodicity (which is still invalid, but for the sake of argument assume it works) it should have used something like direct proof.

   Also, it is unclear what linear recurrence is (or, at the very least, it has not been proven), and should be defined and proven.

Question 9

I'm running out of time for this so I'm just gonna give a quick sketch (I obviously don't expect points for this, I just want to put it down for you to see).

Idea of Proof: Base case: Observe for a 2x2 square that we can cover it with a triomino.
Inductive hypothesis: Assume that for any n, the $2^n \times 2^n$ mutilated chessboard may be covered by trimonioes.
Inductive step: Not sure how to articulate this, but if you split up the chessboard into quarters, you can fill the mutilated corner with a triomino and then you basically need to prove that you can fill the remaining L shape with triominoes, and since that L shape is itself a triomino, it implies that you can double the size again for the $n+1$ case. 😄

Question 10

Not actually sure how to go about this one yet, but I have 5 minutes before midnight so I'll send this and think about it later.