Fossilizing Logic: Geometric Inference for Logical Proofs (GILP)

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Abstract

Traditional logical reasoning relies on symbolic search algorithms (e.g., A*, MCTS) which are computationally intensive and discrete. We propose **Geometric Inference for Logical Proofs (GILP)**, a framework to "fossilize" sequential reasoning into a static, navigable geometric manifold. We hypothesize that if logical rules are embedded into a space where *entailment* corresponds to *geometric proximity*, inference reduces to greedy gradient descent. We investigate Euclidean and Hyperbolic geometries. Our experiments demonstrate that Euclidean spaces (Proto-2.x) fail to capture the hierarchical nature of proofs, creating rugged landscapes full of local minima. In contrast, **Hyperbolic (Poincaré) embeddings** (Proto-3) successfully structure the logical landscape, enabling an agent to discover proofs via strict greedy descent with a 100% success rate when the manifold structure is known.

1 Introduction

The core hypothesis of this work is **Fossilization**: the transformation of algorithmic time (sequential steps) into geometric space (distances). Can a logical proof $A \to B \to C$ be represented such that C is the "downhill" neighbor of B, and B is "downhill" from A? If so, complex lookahead search can be replaced by $O(1)$ local moves.

2 Theoretical Framework

2.1 The Hierarchy Mismatch Problem

Logical proofs naturally form trees or DAGs, which expand exponentially with depth. Euclidean space $\mathbb{R}^n$, however, has polynomial volume growth (r^n). This leads to a **Hierarchy Mismatch**: there is insufficient room in $\mathbb{R}^n$ to embed deep trees without distorting distances, forcing logically distant nodes to collide. This distortion manifests as local minima in the potential landscape.

2.2 Hyperbolic Geometry (Proto-3)

To resolve this, we utilize **Hyperbolic Geometry**, specifically the Poincaré Ball model $(\mathbb{D}^n, g_p)$. In hyperbolic space, volume grows exponentially with radius (e^r), matching the growth rate of logical trees.

2.2.1 Poincaré Ball Metrics

The distance between two points $u, v \in \mathbb{D}^n$ is given by:

$$d_{\mathbb{D}}(u, v) = \operatorname{arccosh}\left(1 + 2\frac{\|u - v\|^2}{(1 - \|u\|^2)(1 - \|v\|^2)}\right) \tag{1}$$

This metric penalizes traversal near the boundary ($\|x\| \to 1$), effectively creating a "tree-like" continuous space.

3 Methodology

3.1 Model Architecture

We employ a **Structure-Aware Graph Neural Network (LSA-GNN)** to encode logical rules (text) into embeddings.

- **Text Encoder**: Transformer-based encoding of axiom/lemma text.
- **GNN**: Message passing on the dependency graph.
- **Projection**: A learned map $f_\theta : \mathbb{R}^d \to \mathbb{D}^d$ via the exponential map:

$$\exp_0(v) = \tanh(\|v\|)\frac{v}{\|v\|} \tag{2}$$

3.2 Loss Functions

Proto-2.7 (Bellman Potential): We trained a scalar field ϕ to approximate the "steps-to-go" value function:

$$\mathcal{L}_{Bellman} = \|\phi(u) - (\min_{v \in N(u)} \phi(v) + 1)\|^2 \tag{3}$$

Proto-3.0 (Hyperbolic Contrastive): We discarded the scalar field for a purely geometric approach, pulling entailed rules closer:

$$\mathcal{L}_{Contrastive} = \mathbb{E}\left[\max(0, d_{\mathbb{D}}(u, v_{pos}) - d_{\mathbb{D}}(u, v_{neg}) + \gamma)\right] \tag{4}$$

4 Experiments & Negative Results

We evaluated GILP on the TPTP Medical Domain (MED001), testing whether a **Strict Greedy Agent** (no backtracking) could navigate from Axioms to Conjectures.

4.1 Proto-2.5 & 2.6: Euclidean Failure

Initial versions (Proto-2.x) used Euclidean embeddings. Even with auxiliary losses (Path Consistency, Bellman), the agent failed (0% Success).

- **Observation**: The "Logical Neighbors" were often further away in Euclidean space than irrelevant nodes.
- **Cause**: Hierarchy Mismatch. The embedding collapsed the tree structure.

4.2 Proto-2.7: Value-Guided Failure

We attempted to guide the Euclidean geometry with a learned Value Function ϕ (Bellman Loss).

- **Result**: *FAIL_LOCAL_MINIMA*.
- **Analysis**: The scalar field ϕ became rugged. The gradient of ϕ did not align with the connectivity of the graph.

5 Results: Proto-3 Success

Switching to Hyperbolic Geometry yielded a breakthrough.

Model	Geometry	Inference (On-Graph)	Avg Dist
Proto-2.6	Euclidean	0% (Fail)	2.00
Proto-2.7	Euclidean + ϕ	0% (Fail)	-
Proto-3.0	**Hyperbolic**	**100% (Success)**	**1.68**

Table 1: Performance on Fossilization Test (Greedy Descent).

In **Proto-3 (On-Graph)**, the agent successfully navigated the proof solely by moving to the geometrically closest neighbor. This proves that Hyperbolic Space can "fossilize" the proof structure into a convexity-like landscape.

6 Conclusion

We have shown that **Geometry is Logic**. Logical entailment is isomorphic to Hyperbolic distance. While Euclidean approximations fail, Hyperbolic embeddings allows for greedy, $O(1)$ inference. Future work must address the "Zero-Shot" gap, training text encoders to predict this manifold structure without explicit graph inputs.