Image Compression

Course: Machine Learning (INF267)

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1 Project Description

The aim of the project is the implementation of the Expectation - Maximization algorithm in python. This algorithm will be used for the segmentation and compression of a colour image.

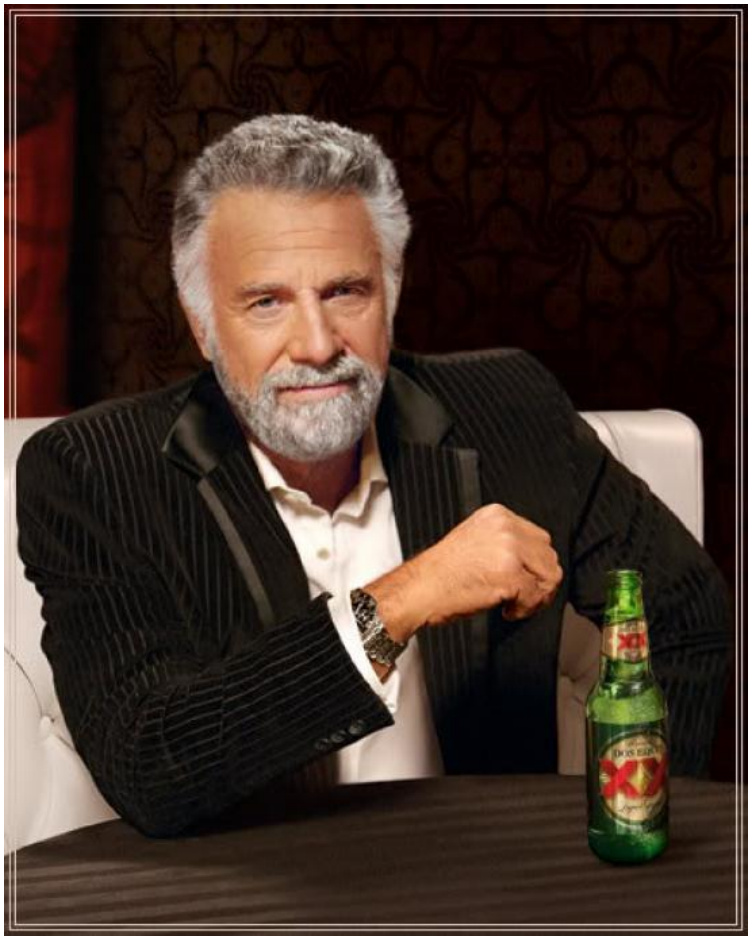
```
import random
import numpy as np
from PIL import Image as img
from numpy import linalg as LA
from matplotlib import pyplot as plt
%matplotlib inline
```

2 Training Image

The size of the image is $550 \times 690 \times 3$ and it is loaded as a $N \times D$ array, where each row n_i refers to each pixel of the image and each column d_j contains the RGB values of the specific pixel. In total we have 379,500 RGB pixels.

```
im = img.open("./images/im.jpg", "r")
image = list(im.getdata())
image = np.asarray(image)
n, d = image.shape
```

3 View Image



4 Normalize the data set

Pixels' values are integers that range from 0 (black) to 255 (white). So, we divide each feature by the maximum value, in order to normalize our data in the range [0, 1]. In that way we avoid overflows and we are able to handle all the variables in the same scale.

```
# Normalize the pixels' values
image = image.astype(float)/255
```

5 Description of EM Algorithm

The logic behind the above algorithm is that data are generated by a statistical process. More specifically we assume that data are described by a mixture of distributions, each of them corresponds to a cluster and the parameters of each distribution (π_k, μ_k, Σ_k) describes the correspoding cluster. So, the aim is to find the parameters of the distributions/clusters that best fit the data (pixels) and assign each pixel to its nearest one, getting its values.

Algorithm 1 Expectation - Maximization Algorithm

1: Select an initial set of model parameters $\{\pi_k, \mu_k, \Sigma_k\}_{k=1}^{K}$
2: **while** not converge **do**:
3: **Expectation Step:** For each pixel, calculate the probability that
4: pixel belongs to each distribution.
5: **Maximization Step:** Given the probabilities from the expectation,
6: step find the new estimates of the parameters that maximize the log
7: likelihood.

In our case, we assume that our statistical model will be described by a multivariate mixture of Gaussians. Our data (pixels) are 3-dimensional vectors, each of those corresponds to an RGB value. Therefore, we expect each cluster to be described by a 3-dimensional mean value and a 3-dimensional covariance, one for each channel. So, the equation of the multivariate normal distribution for a specific cluster k is defined as:

$$P(x|k) = \mathcal{N}(x|\mu_k, \Sigma_k) = \frac{1}{(2\pi)^{\frac{D}{2}}|\Sigma_k|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu_k)^T\Sigma_k^{-1}(x-\mu_k)}$$

Another hypothesis we make, is that the pixels' values (RGB) are independent, which means an isotropic covariance matrix, i.e. a matrix with all values equal to zero, except from those of the diagonal that are equal. In this case the representation of the covariance matrix is:

$$\Sigma_k = \begin{bmatrix} \sigma_k^2 & 0 & 0 \\ 0 & \sigma_k^2 & 0 \\ 0 & 0 & \sigma_k^2 \end{bmatrix} = \sigma_k^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \Sigma_k = \sigma_k^2 I,$$

where I is the identity matrix and σ^2 is the variance of the channels.

So, the above equation of the normal distribution is simplified as:

$$P(x|k) = \mathcal{N}(x|\mu_k, \Sigma_k) = \frac{1}{(2\pi)^{\frac{D}{2}}|\Sigma_k|^{\frac{1}{2}}}\, e^{-\frac{1}{2}(x-\mu_k)^T\Sigma_k^{-1}(x-\mu_k)} \quad \Longleftrightarrow$$

$$P(x|k) = \mathcal{N}(x|\mu_k, \sigma_k^2) = \frac{1}{(2\pi)^{\frac{D}{2}}|\sigma_k^2 I|^{\frac{1}{2}}}\, e^{-\frac{1}{2}(x-\mu_k)^T\left((\sigma_k I)^2\right)^{-1}(x-\mu_k)} \quad \Longleftrightarrow$$

$$P(x|k) = \mathcal{N}(x|\mu_k, \sigma_k^2) = \frac{1}{(2\pi)^{\frac{D}{2}}(\sigma_k^{2D}|I|)^{\frac{1}{2}}}\, e^{-\frac{1}{2\sigma_k^2}(x-\mu_k)^T(x-\mu_k)} \quad \Longleftrightarrow$$

$$P(x|k) = \mathcal{N}(x|\mu_k, \sigma_k^2) = \frac{1}{(2\pi)^{\frac{D}{2}}(\sigma_k^2)^{\frac{D}{2}}}\, e^{-\frac{1}{2\sigma_k^2}\|x-\mu_k\|^2} \quad \Longleftrightarrow$$

$$P(x|k) = \mathcal{N}(x|\mu_k, \sigma_k^2) = \frac{1}{(2\pi\sigma_k^2)^{\frac{D}{2}}}\, e^{-\frac{1}{2\sigma_k^2}\left((x_r-\mu_{kr})^2+(x_g-\mu_{kg})^2+(x_b-\mu_{kb})^2\right)} \quad \Longleftrightarrow$$

$$P(x|k) = \mathcal{N}(x|\mu_k, \sigma_k^2) = \frac{1}{\sqrt{2\pi\sigma_k^2}^D}\, e^{-\frac{1}{2\sigma_k^2}(x_r-\mu_{kr})^2} \quad e^{-\frac{1}{2\sigma_k^2}(x_g-\mu_{kg})^2} \quad e^{-\frac{1}{2\sigma_k^2}(x_b-\mu_{kb})^2} \quad \Longleftrightarrow$$

$$P(x|k) = \mathcal{N}(x|\mu_k, \sigma_k^2) = \prod_{d=1}^{D}\frac{1}{\sqrt{2\pi\sigma_k^2}}\, e^{-\frac{1}{2\sigma_k^2}(x_d-\mu_{kd})^2} \quad \Longleftrightarrow$$

$$P(x|k) = \prod_{d=1}^{D} P(x_d|\mu_{kd}, \sigma_k^2)$$

6 Initialize Parameters

The first step of the Expectation - Maximization algorithm is to initialize each cluster with a mean value, a variance and a priori probability that it occurs. The result is a $K \times D$ mean value matrix, where K is the number of clusters and D is the number of the correspoding mean RGB values and a $K \times 1$ variance matrix, while the variance of each cluster is the same for all the channels. As for the priori probabilities of the clusters, we initialize their values uniformly in a $1 \times K$ matrix.

```
1
2  def init_parameters(image, k):
3
4      """ Initialize all the necessary variables for the EM algorithm.
5
6      :param image: The N x D matrix with the pixels' values of the initial
           image
7      :param k: The number of categories/clusters in which we want to segment
           the image
8      :return:
9          mu: The K x D matrix with the initial centroids/means of each cluster
10         sigma: The K x 1 matrix with the initial variance of each cluster
11         pi: The 1 x K matrix with the initial priori probability of each
             cluster
12
13     """
14
15     # choose the starting centroids/means randomly as pixels of the image
16     mu = random.sample(list(image), k)
17     mu = np.asarray(mu)
18
19     # initialize the variance matrices for each gaussian
20     sigma = np.random.rand(k,1)
21
22     # initialize the priori probabilities for each gaussian
23     pi = [1/k] * k
24     pi = np.asarray(pi)
25
26     return mu, sigma, pi
```

7 Probability Density Function

The Probability Density Function computes the probability that a random variable x has been generated from a normal distribution with mean $_k$ and variance σ_k^2. In this case, the function below computes the probability density of a specific cluster k and only for one channel d (RGB value) of every pixel. (We want to check how close is the red value of a pixel to the mean red value of a distribution etc.)

Therefore, the equation is defined as:

$$p(x_d|k) = \mathcal{N}(x_d|\mu_{kd}, \sigma_k^2) = \frac{1}{\sqrt{2\pi{\sigma_k}^2}} e^{-\frac{1}{2{\sigma_k}^2}(x_d - kd)^2}$$

```
def density_function (image_j, mu_j, sigma_j):

    """ Compute probability density for each pixel's value and each category.

    :param image_j: The N x 1 matrix containing the pixels' values of a
        specific rgb channel
    :param mu_j: The mean value of one category/distribution and the
        correspoding rgb channel
    :param sigma_j: The variance of one category/distribution
    :return prob: The N x 1 matrix with the probabilities that each value of
        image_j has been generated from a normal distribution with mean mu_j
        and variance sigma_j.

    """

    prob = ( 1 / np.sqrt( 2 * np.pi * sigma_j  ) ) * \
            np.exp( - (image_j - mu_j) ** 2 /( 2 * sigma_j ) )

    return prob
```

8 Expectation Step

Through Expectation step, the algorithm finds the posteriori probabilities that each pixel belongs to each cluster. **As for the probability density function, it should be noted that it first calculates individually the probabilities of every rgb channel and then computes the product of them for every pixel, while rgb values are independent.** In that way, it gets the total probabilities for every pixel and cluster (which consists of rgb channels) and is able to calculate the posteriori probabilities.

The total equation of the Expectation procedure is:

$$p(k|x) = \frac{p(x|k)p(k)}{p(x)} \quad \Longrightarrow$$

$$\gamma(z_k) = \frac{\pi_k \prod_{d=1}^{D} \mathcal{N}(x_d|\mu_{kd}, \sigma_k^2)}{\sum_{k=1}^{K} \pi_k \prod_{d=1}^{D} \mathcal{N}(x_d|\mu_{kd}, \sigma_k^2)}, \quad k = 1, ..., K$$

```
def expectation(image, mu, sigma, pi):

    """ Implementation of Expectation step of the algorithm.

    :param image: The N x D matrix with the pixels' values of the initial
          image
    :param mu: The K x D matrix with the current centroids/means of each
          cluster
    :param sigma: The K x 1 matrix with the current variance of each
          cluster
    :param pi: The 1 x K matrix with the current priori probability of each
          cluster
    :return:
        prob: The N x K matrix with the product of the likelihood and the
          priori probability of each pixel and cluster
        r: The N x K matrix with the posteriori probabilities

    """

    r = np.zeros((n,k))
    prob = np.zeros((n,k))

    for i in range(k):
        red = np.zeros((n,1))
        green = np.zeros((n,1))
        blue = np.zeros((n,1))


```

```
        # compute for each rgb channel of the image the density function
        # for the current category k
        red = density_function(image[:,0], mu[i,0], sigma[i])
        green = density_function(image[:,1], mu[i,1], sigma[i])
        blue = density_function(image[:,2], mu[i,2], sigma[i])

        # compute the total probability that each pixel belongs to the
        # current category k
        density = red * green * blue
        prob[:,i] = pi[i] * density

    # compute the posteriori probabilities
    for i in range(len(prob)):
        total = np.sum(prob[i])
        r[i] = prob[i]/total

    return prob, r

```

9 Maximization Step

After computing the cluster membership probabilities for all the pixels, we compute new estimates for μ_k, σ_k and π_k, in order to best fit them. This is the maximization step. Here, the new estimate for the mean of a distribution is just a weighted average of the points, where the weights are the probabilities that the points belong to that distribution. These probabilities has occured through the expectation step.

The following procedure happens for every cluster ($\times K$) and every rgb channel ($\times D$):

$$\mu_{kd}^{new} = \frac{\sum_{n=1}^{N} \gamma(z_{nk}) x_{nd}}{\sum_{n=1}^{N} \gamma(z_{nk})}, \quad k = 1, ..., K \quad \& \quad d = 1, ..., D$$

As for the variance, the new estimations are:

$$\Sigma_k^{new} = \frac{\sum_{n=1}^{N} \gamma(z_{nk})(x_n - m_k^{new})(x_n - m_k^{new})^T}{D \sum_{n=1}^{N} \gamma(z_{nk})}, \quad k = 1, ..., K \implies$$

$$\Sigma_k^{new} = \frac{\sum_{n=1}^{N} \gamma(z_{nk}) \ \|x_n - \mu_k^{new}\|^2}{D \sum_{n=1}^{N} \gamma(z_{nk})}, \quad k = 1, ..., K \implies$$

$$\Sigma_k^{new} = \frac{\sum_{n=1}^{N} \gamma(z_{nk}) \left((x_{nr} - \mu_{kr}^{new})^2 + (x_{ng} - \mu_{kg}^{new})^2 + (x_{nb} - \mu_{kb}^{new})^2 \right)}{D \sum_{n=1}^{N} \gamma(z_{nk})}, \quad k = 1, ..., K \implies$$

$$\Sigma_k^{new} = \frac{\sum_{n=1}^{N} \sum_{d=1}^{D} \gamma(z_{nk})(x_{nd} - \mu_{kd}^{new})^2}{D \sum_{n=1}^{N} \gamma(z_{nk})}, \quad k = 1, ..., K \quad \Longrightarrow$$

Lastly, the new estimations for priori probabilities are defined as:

$$\pi_k^{new} = \frac{\sum_{n=1}^{N} \gamma(z_{nk})}{N}, \quad k = 1, ..., K$$

```
def maximization(image, r):

    """ Update all the necessary variables.

    :param image: The N x D matrix with the pixels' values of the initial
          image
    :param r: The N x K matrix with the posteriori probabilities
    :return:
        mu: The K x D matrix with the updated centroids/means of each
          category
        sigma: The K x 1 matrix with the updated variance of each category
        pi: The 1 x K matrix with the updated priori probabilities of each
          category

    """

    # update mean values of each category/cluster
    temp = np.dot(r.T, image)
    total = np.sum(r, axis = 0, keepdims = True)
    mu = temp / total.T

    # update variance of each category/cluster
    for i in range(k):
        temp = np.power( (image - mu[i]), 2 )
        temp = r[:,i] * np.sum(temp, axis = 1)
        temp = np.sum(temp, axis = 0)

        sigma[i] = temp / (d * total[0,i])

    # update priori probabilities of each category/cluster
    pi = total[0] / n

    return mu, sigma, pi

```

10 Reconstruction Error

The function below computes the mean error of the reconstructed image, as the distance between its pixels' values and the values of the pixels of the initial image. For the distance calculation, $L2$ Norm is used.

So, the reconstruction error is defined as:

$$error = \frac{1}{N} \sum_{n=1}^{N} \|x_{true,n} - x_{r,n}\|^2$$

```
def error_function(image, new_image):

    """ Computes the reconstruction error of the new_image.

    :param image: The N x D matrix with the pixels' values of the initial
           image
    :param new_image: The N x D matrix with the pixels' values of the
           reconstructed image
    :return error: The error (distance) between the pixels' values of the two
           images

    """

    error = 1/n * sum(LA.norm(image - new_image, axis = 1)**2)

    return error

```

11 Convergence Function

The function below implements the terminal criterion. As a metric we use the **log-likelihood**, that we want to maximize. As far as its value increases, the algorithm continues with new calculations, othewise it terminates.

```
def convergence(old_likelihood, new_likelihood, tol):

    """ Check convergence of the algorithm.

    :param old_likelihood: The log-likelihood of the previous iteration
    :param new_likelihood: The log-likelihood of the current iteration
    :param tol: The tolerance value
    :return:
        old_likelihood: The updated log-likelihood for next iterations
        match: The boolean variable describing the convergence

    """

    if np.abs(new_likelihood - old_likelihood) < tol:
        match = True
    else:
        old_likelihood = new_likelihood
        match = False

    return old_likelihood, match

```

12 Expectation - Maximization Algorithm

The function below is called for each predefined number of clusters. Each time it runs the expectation and maximization step, until the convergence to the best values that describe the compressed image.

```
def EM_algorithm(image, mu, sigma, pi, error):

    """ The implementation of the Expectation - Maximization algorithm.

    :param image: The N x D matrix with the pixels' values of the initial image
    :param mu: The K x D matrix with the initial centroids/means of each
          cluster
    :param sigma: The K x 1 matrix with the initial variance of each cluster
    :param pi: The 1 x K matrix with the initial priori probability of each
          cluster
    :param error: The matrix with the reconstruction error of the images
          for each chosen number of categories/clusters


```

```
    :return:
        new_image: The N x D matrix with the pixels' values of the
          reconstructed image
        error: The recostruction error of the new_image
        mu: The K x D matrix with the final centroids/means of each category
        sigma: The K x 1 matrix with the final variance of each category
        pi: The 1 x K matrix with the final priori probabilities of each
          category

    """

    new_image = np.zeros((n,k))
    old_likelihood = 0
    match = False
    tol = 1e-2

    # while the terminal criterion is not satisfied; continue the iterations
    while not match:
        # run the expectation step
        prob, r = expectation(image, mu, sigma, pi)
        # run the maximization step
        mu, sigma, pi = maximization(image, r)

        # compute log-likelihood
        total = np.sum(prob, axis = 1)
        new_likelihood = np.sum(np.log(total))
        # check for convergence
        old_likelihood, match = convergence(old_likelihood, new_likelihood, tol
    )

    # update the pixels' values with the mean values of the most probable
    # category they belong to
    new_image = mu[np.argmax(r, axis = 1)]
    # compute the error (distance) between the initial image and the new one
    error.append(error_function(image, new_image))

    return new_image, error, mu, sigma, pi

```

13 Plot Reconstructed Image

```
    def plot_image(new_image, k):

    """ Plot the reconstructed image.

    :param new_image: The N x D matrix with the pixels' values of the
           reconstructed image
    :param k: The number of categories/clusters

    """

    imag = np.reshape(new_image, (690, 550, 3))
    plt.imsave('./output/' + str(k) + '_Categories.jpg', imag)

    fig = plt.figure()
    a = fig.add_subplot(1, 2, 1)
    imag = np.reshape(image, (690, 550, 3))
    imgplot = plt.imshow(imag)
    a.set_title('Original Image')

    a = fig.add_subplot(1, 2, 2)
    imag = np.reshape(new_image, (690, 550, 3))
    imgplot = plt.imshow(imag)
    a.set_title('K= ' +str(k) )
    plt.show()

```

14 Plot Error

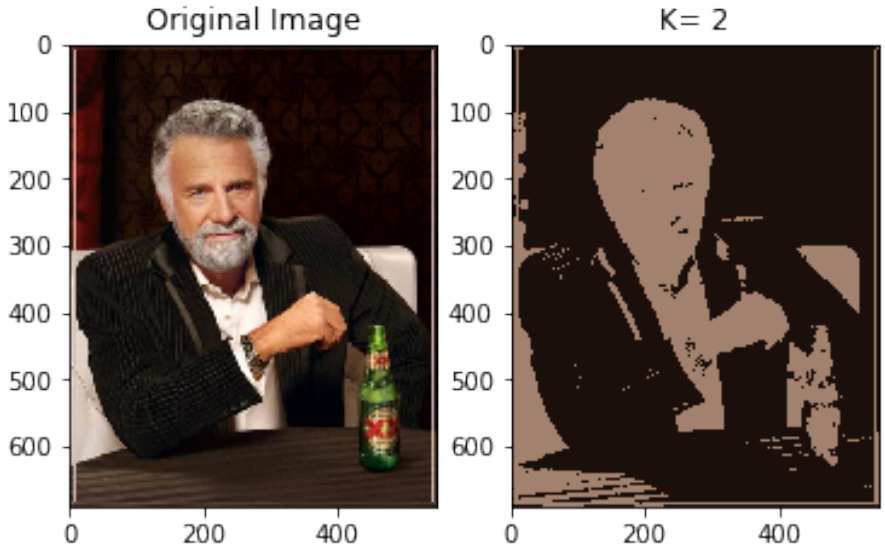
```
    def plot_error(error):

    """ Plots the error line for different number of clusters.

    :param error: The 7 x 1 matrix with the final error of each predefined
         number of clusters

    """

    x = range( 1, len(error)+1 )
    y = error
    plt.plot( x, y )
    plt.ylabel( 'error' )
    plt.xlabel( 'clusters' )
    plt.title( "Error by increasing the number of clusters: " )
    plt.xticks( x )
    plt.show()

```

15 Run Expectation - Maximization Algorithm

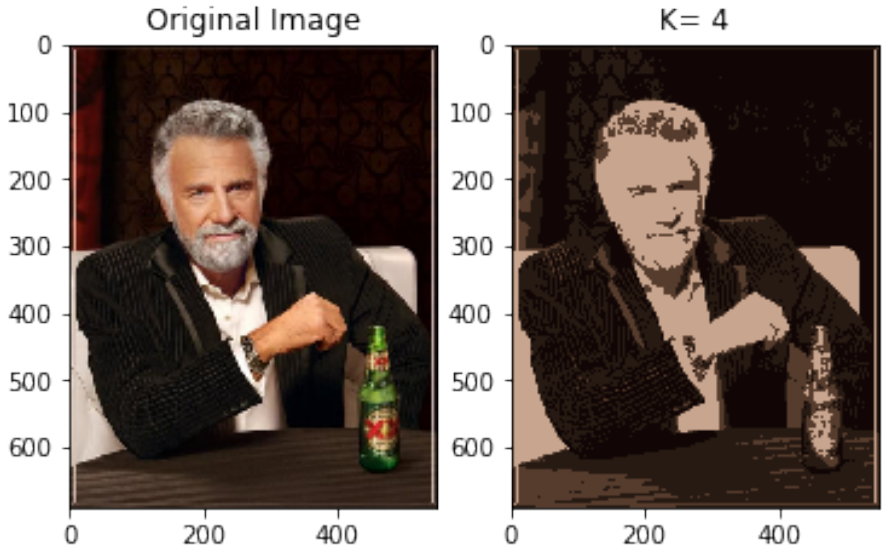
The procedure below implements the Expectation - Maximization algorithm for different number of distributions/clusters (k). Each time, it generates a new compressed image based on k different colours and calculates the reconstruction error from the initial one.

```
# number of categories/clusters
k = 2
error = []

# reconstruct the initial image for different number of categories/clusters
# (e.g. 2,4,8,16,32,64,128)
while k <= 128:
    # initialize parameters
    mu, sigma, pi = (init_parameters(image, k))
    # run Expectation - Maximization algorithm
    new_image, error, mu, sigma, pi = EM_algorithm(image, mu, sigma, pi, error)
    # plot reconstructed image
    plot_image(new_image, k)

    e = error[int(np.log2(k)-1)]
    print("The reconstruction error for " + str(k) + " clusters is: " + str(e)
    k *= 2

# plot the error function
plot_error(error)
```

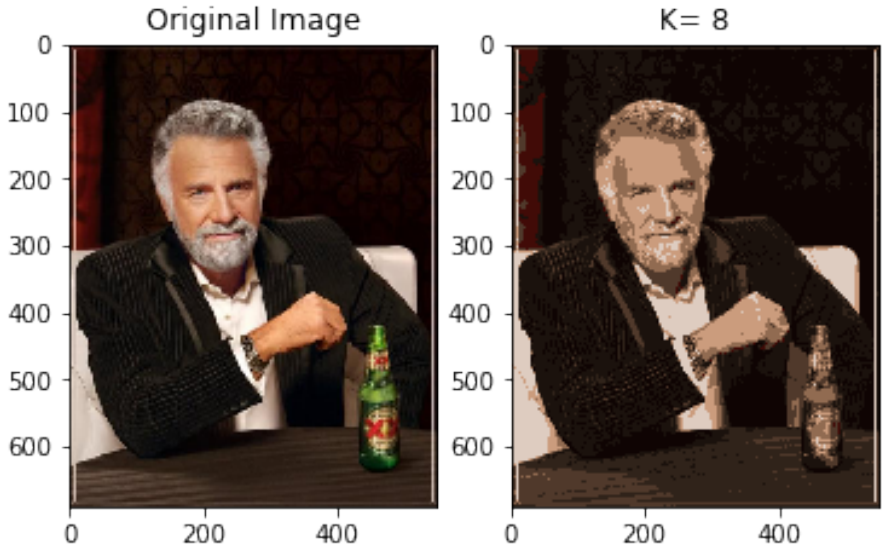
16 Output



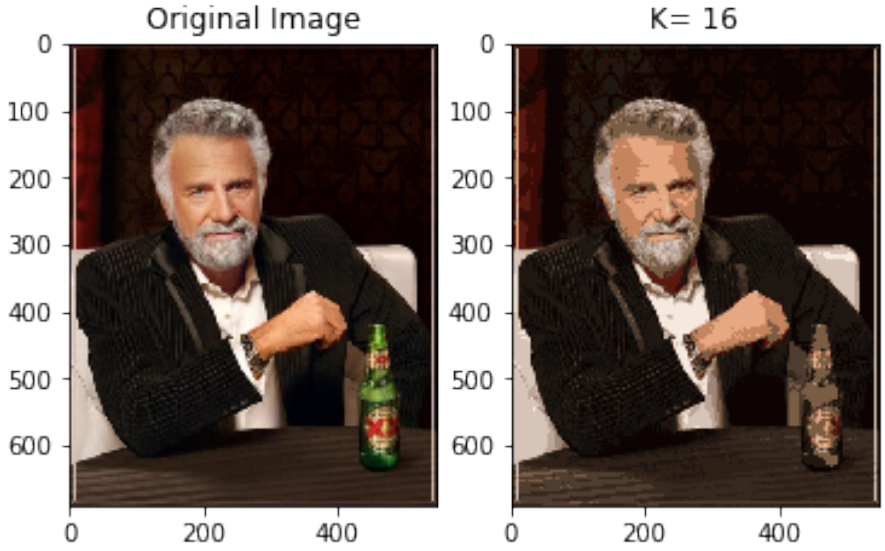
The reconstruction error for 2 clusters is: 0.04874145718980165



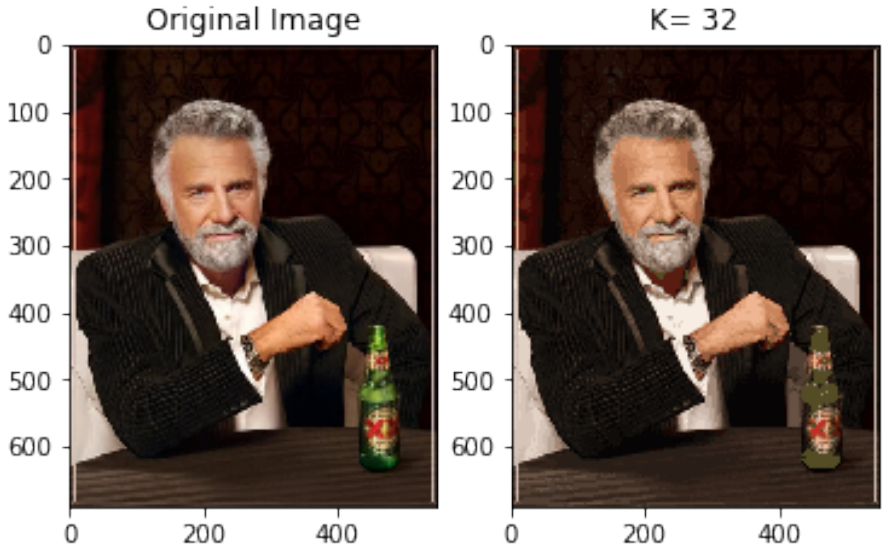
The reconstruction error for 4 clusters is: 0.01680458914182042



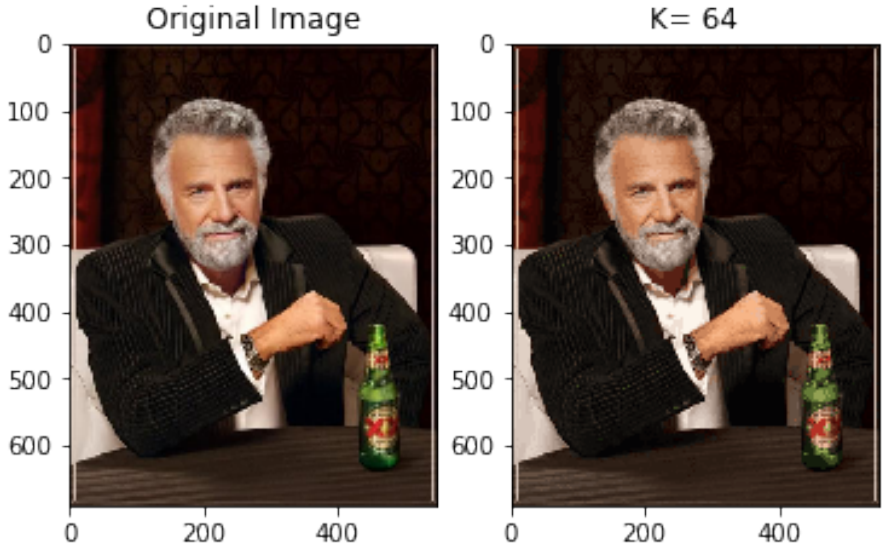
The reconstruction error for 8 clusters is: 0.006135087897959467



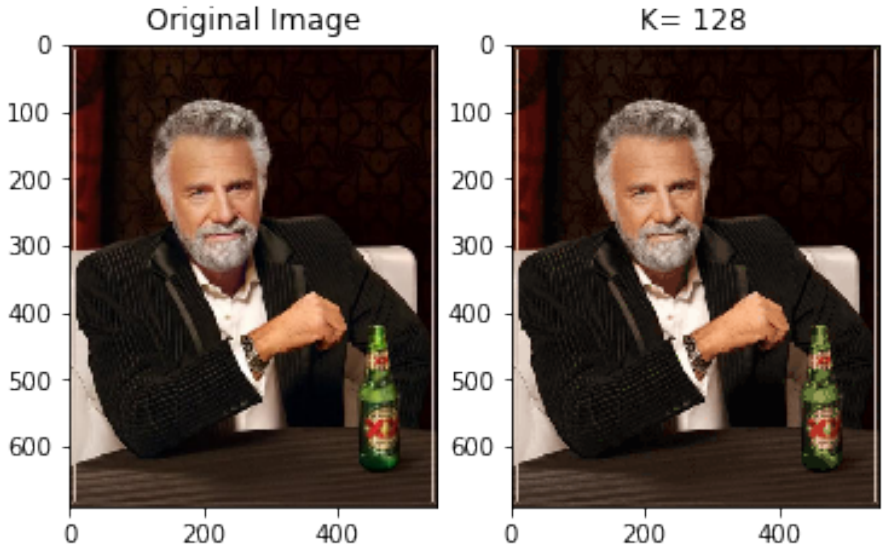
The reconstruction error for 16 clusters is: 0.0033341249012054817



The reconstruction error for 32 clusters is: 0.0021857102234665313



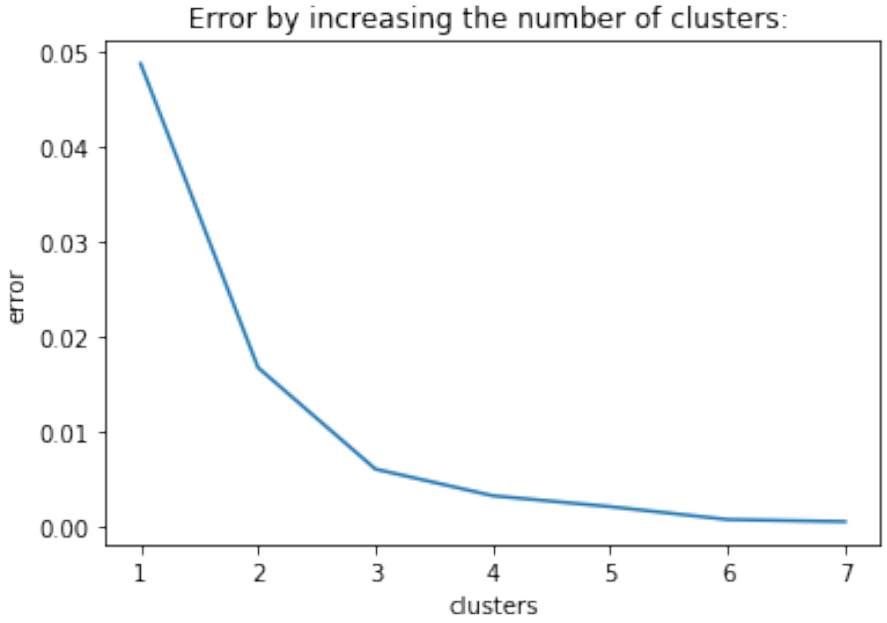
The reconstruction error for 64 clusters is: 0.0008342308137593823



The reconstruction error for 128 clusters is: 0.0006146841027920311

17 Cost Curve

Clusters/Categories	Iterations	Reconstruction Error
2	23	0.04882768790822941
4	107	0.016801744183826944
8	154	0.007940573068342224
16	452	0.0038803141642229227
32	548	0.0019798988582252985
64	1693	0.0009322353167936464
128	2013	0.000520232704578564



As we can see, up to 32 number of clusters the cost curve of the reconstructed image rapidly decreases. This means that the quality of the image is markedly improved and approaches the initial one even more. After that, the cost remains almost the same. So, the compression of the image using more than 32 clusters does not offer so much improvement in the quality of the reconstructed image concerning its bigger size.

References

[1] C. M. Bishop, *Pattern Recognition and Machine Learning (Information Science and Statistics)*. Springer, 2006.