1. (20pts) The following ordinary difference table is for $f(x) = x + \frac{sin(x)}{3}$. Use it to find

(a) $f'(0.72)$ from a cubic polynomial.

(b) $f'(1.33)$ from a quadratic.

(c) $f'(0.50)$ from a fourth-degree polynomial.

In each part, choose the best starting i-value.

$s = \frac{x - x_i}{h}$

$h = 0.2.$

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	0.30	0.3985	0.2613	-0.0064	-0.0022	0.0003
1	0.50	0.6598	0.2549	-0.0086	-0.0018	0.0004
2	0.70	0.9147	0.2464	-0.0104	-0.0014	0.0005
3	0.90	1.1611	0.2360	-0.0118	-0.0010	
4	1.10	1.3971	0.2241	-0.0128		
5	1.30	1.6212	0.2113			
6	1.50	1.8325				

$$f(x) = f_i + s\Delta f_i + \frac{s(s-1)}{2!}\Delta^2 f_i + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_i + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f_i \dots$$

$$f'(x) = \frac{1}{h}\Big(\Delta f_i + \frac{s+(s-1)}{2!}\Delta^2 f_i + \frac{s(s-1)+s(s-2)+(s-1)(s-2)}{3!}\Delta^3 f_i$$

$$+ \frac{s(s-1)(s-2)+\dots+(s-1)(s-2)(s-3)}{4!}\Delta^4 f_i + \dots\Big)$$

(a.) $0.72 \rightarrow$ choose x_1, x_2, x_3, x_4 $\quad s = \frac{0.72-0.5}{0.2} = 1.1$

$$\therefore f'(0.72) \approx \frac{1}{0.2}\Big(0.2549 + \frac{1.1+0.1}{2}\cdot(-0.0086) + \frac{1.1\times 0.1 + 1.1\times(-0.9) + 0.1\times(-0.9)}{3!}(-0.0018)\Big)$$

$$= 1.250155$$

(b.) $1.33 \rightarrow$ choose x_4, x_5, x_6 . $\quad s = \frac{1.33-1.1}{0.2} = 1.15$

$$f'(1.33) \approx \frac{1}{0.2}\Big(0.2241 + \frac{1.15+0.15}{2}\cdot(-0.0128)\Big) = 1.0789$$

(c) $0.5 \rightarrow$ choose $x_2 \sim x_6$. $\quad s = 0.$

$$f'(0.5) = \frac{1}{0.2}\Big(0.2549 + \frac{-1}{2}(-0.0086) + \frac{2}{6}(-0.0018) + \frac{-6}{24}(0.0004)\Big)$$

$$= 1.2925$$

2. (20pts) Use the method of undetermined coefficients to obtain the formulas for $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$ at x_0 using five evenly spaced points from x_2 to x_{-2}, together with their error terms.

$$f''(x_0) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0 f_0 + C_1 f_1 + C_2 f_2.$$

let $P(u) = f(u + x_0)$

- for $P(u) = 1$. $f_{-2} = \cdots = f_2 = 1$.

  $f''(x_0) = C_{-2} + C_{-1} + C_0 + C_1 + C_2 = P''(0) = 0.$

- for $P(u) = u$,

  $f_{-2} = P(-2h) = -2h$, $f_{-1} = P(-h) = -h$, $f_0 = P(0) = 0$. $f_1 = P(h) = h$. $f_2 = P(2h) = 2h$.

  $f''(x_0) = C_{-2}(-2h) + C_{-1}(-h) + C_0 \cdot 0 + C_1(h) + C_2(2h) = P''(0) = 0$

- for $P(u) = u^2$.

  $f_{-2} = P(-2h) = 4h^2$. $f_{-1} = P(-h) = h^2$. $f_0 = P(0) = 0$. $f_1 = P(h) = h^2$. $f_2 = P(2h) = 4h^2$

  $\therefore f''(x_0) = C_{-2}(4h^2) + C_{-1}(h^2) + C_0 \cdot 0 + C_1(h^2) + C_2(4h^2) = P''(0) = 0$.

- for $P(u) = u^3$

  $f_{-2} = P(-2h) = -8h^3$, $f_{-1} = P(-h) = -h^3$. $f_0 = P(0) = 0$. $f_1 = P(h) = h^3$. $f_2 = P(2h) = 8h^3$

  $\therefore f''(x_0) = C_{-2}(-8h^3) + C_{-1}(-h^3) + C_0 \cdot 0 + C_1(h^3) + C_2(8h^3) = P''(0) = 0$.

- for $P(u) = u^4$

  $f_{-2} = P(-2h) = 16h^4$. $f_{-1} = P(-h) = h^4$. $f_0 = P(0) = 0$. $f_1 = P(h) = h^4$, $f_2 = P(2h) = 16h^4$

  $\therefore f''(x_0) = C_{-2}(16h^4) + C_{-1}(h^4) + C_0 \cdot 0 + C_1 \cdot (h^4) + C_2(16h^4) = P''(0) = 0$.

$$\therefore \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1/12 \\ 4/3 \\ -5/2 \\ 4/3 \\ -1/12 \end{bmatrix} \cdot \frac{1}{h^2}$$

$$\therefore f''(x_0) = \frac{-f_{-2} + 16f_{-1} - 30f_0 + 16f_1 - f_2}{12h^2}$$

similarly, for $f'''(x_0)$. change RHS to: ($\because P'''(0) = 6$ when $P(u) = u^3$)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -1 \\ 1/2 \end{bmatrix} \cdot \frac{1}{h^3}$$

$$\therefore f'''(x_0) = \frac{-f_{-2} + f_{-1} - f_1 + f_2}{2h^3}$$

similarly, for $f^{(4)}(x_0)$. change RHS to : ($\because$ $p^{(4)}(0) = 24$ when $p(u) = u^4$)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{bmatrix} \Rightarrow \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \end{bmatrix} \cdot \frac{1}{h^4}$$

$$\therefore f^{(4)}(x_0) = \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4}$$

3. (20pts) Simpson's $\frac{1}{3}$ rule, although based on passing a quadratic through three evenly spaced points, actually gives the exact answer if $f(x)$ is a cubic. The implication is that the area under any cubic between $x = a$ and $x = b$ is identical to the area of a parabola that matches the cubic at $x = a$, $x = b$, and $x = \frac{a+b}{2}$. Prove this.

by Simpson's $\frac{1}{3}$ rule, integrating a function $f(x)$ over the interval $[a,b]$ is given by: $\int_a^b f(x)\,dx = \frac{b-a}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$,

and now we know $f(x)$ is a cubic polynomial, let $f(x) = C_3x^3 + C_2x^2 + C_1x + C_0$

$$\int_a^b f(x)\,dx = \int_a^b C_3x^3 + C_2x^2 + C_1x + C_0 = \left.\frac{C_3}{4}x^4 + \frac{C_2}{3}x^3 + \frac{C_1}{2}x^2 + C_0x\right|_a^b$$

$$= \frac{C_3}{4}(b^4 - a^4) + \frac{C_2}{3}(b^3 - a^3) + \frac{C_1}{2}(b^2 - a^2) + C_0(b-a)$$

$$\frac{b-a}{6}\left(C_3\left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right) + C_2\left(a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right) + C_1\left(a + 4\left(\frac{a+b}{2}\right) + b\right) + C_0 \cdot 6\right)$$

$$\Rightarrow C_3 : \frac{(b-a)(2a^3 + a^3 + 3a^2b + 3ab^2 + b^3 + 2b^3)}{12} = \frac{3(b-a)(a^3 + a^2b + ab^2 + b^3)}{12} = \frac{b^4 - a^4}{4}$$

$$C_2 : \frac{(b-a)(a^2 + a^2 + 2ab + b^2 + b^2)}{6} = \frac{2(b-a)(a^2 + ab + b^2)}{6} = \frac{b^3 - a^3}{3}$$

$$C_1 : \frac{(b-a)(3a + 3b)}{6} = \frac{b^2 - a^2}{2}$$

$$C_0 : \frac{b-a}{6} \cdot 6 = b - a.$$

So. the approximate integration is exact the answer.

4. (20pts) Compute the integral of $f(x) = \dfrac{sin(x)}{x}$ between $x = 0$ and $x = 1$ using Simpson's $\frac{1}{3}$ rule with $h = 0.5$ and then with $h = 0.25$. (Remember that the limit of $\dfrac{sin(x)}{x}$ at $x = 0$ is 1.) From these two results, extrapolate to get a better result. What is the order of the error after the extrapolation? Compare your answer with the true answer.

$h = 0.5$: $\displaystyle\int_0^1 f(x)\,dx = \frac{h}{3}\left(f(0) + 4f(0.5) + f(1)\right)$

$= \frac{1}{6}\left(1 + 4 \times 0.9589 + 0.8415\right)$

$= 0.94614588$

$h = 0.25$: $\displaystyle\int_0^1 f(x)\,dx = \int_0^{0.5} f(x)\,dx + \int_{0.5}^1 f(x)\,dx$

$= \frac{h}{3}\left(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right)$

$= \frac{1}{12}\left(1 + 4 \cdot 0.9896 + 2 \cdot 0.9589 + 4 \cdot 0.9089 + 0.8415\right)$

$= 0.94608693$

order of the error after extrapolation : 5

① : $0.94614588 + \alpha \cdot (0.5)^4$

② : $0.94608693 + \alpha\,(0.25)^4$

16.

$\frac{16}{15}\left(② - ①/16\right) = (16 \cdot ② - ①)/15 = ② + (② - ①)/15$

$= 0.94608693 + (-0.0000039) = 0.94608303$

correct answer: 0.94608307

5. (20pts) Evaluate the following integral, and compare your answers to the analytical solution. Use $h = 0.1$ in both directions in parts (a) and (b),

(a) Using the trapezoidal rule in both directions.

(b) Using Simpson's $\frac{1}{3}$ rule in both directions.

(c) Using Gaussian quadrature, three-term formulas in both directions.

$$\int_{-0.2}^{1.4} \int_{0.4}^{2.6} e^x sin(2y)\, dy dx$$

$\displaystyle\int_{-0.2}^{1.4}\int_{0.4}^{2.6} e^x \sin 2y\,dy\,dx = \int_{0.2}^{1.4} e^x\,dx \cdot \int_{0.4}^{2.6} \sin(2y)\,dy$

I ran the provided code and got following answer.

```
❯ python p5.py
Trapezoidal Rule: 0.3683399550766352
Simpson's 1/3 Rule: 0.3692685194703215
Gaussian Quadrature: 0.37237771648186674
```

Correct answer : 0.369265016

6. (20pts) Please use Monte Carlo Integration to compute the double integral of $f(x,y) = (x-1)^2 + \frac{y^2}{16}$ where $R = [-2,3] \times [-1,2]$.

$$\iint_R f(x,y)\,dy\,dx$$

I ran the provided code and got following answer.

```
❯ python p6.py
Estimated integral using Monte Carlo integration with N = 100: 31.724277989408137
❯ python p6.py
Estimated integral using Monte Carlo integration with N = 10000: 35.7150202938199
❯ python p6.py
Estimated integral using Monte Carlo integration with N = 1000000: 35.89583278912296
```

Correct answer : 35.9375