Chapter 5 Vectors

No: Date: 10/2/2023

5.1 Vectors in 3 dimensions

Notation: $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$

$\underline{i}, \underline{j}, \underline{k}$

$\vec{i}, \vec{j}, \vec{k}$

$\langle 1, 1, 1 \rangle$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Let $\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

magnitude, $|\underset{\sim}{u}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$

unit vector, $\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|} = \frac{2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}}{\sqrt{14}}$

$= \frac{2}{\sqrt{14}}\underset{\sim}{i} + \frac{1}{\sqrt{14}}\underset{\sim}{j} - \frac{3}{\sqrt{14}}\underset{\sim}{k}$

direction cosines: $\cos\alpha = \frac{2}{\sqrt{14}}$, $\cos\beta = \frac{1}{\sqrt{14}}$, $\cos\gamma = \frac{-3}{\sqrt{14}}$

direction angles: $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$, $\beta = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$, $\gamma = \cos^{-1}\left(-\frac{3}{\sqrt{14}}\right)$

Parallel

$(\theta = 0° @ 180°)$

① $\underline{a} = k\underline{b}$, $k \in \mathbb{R}$

$\langle 2, 1, 2 \rangle = k\langle 6, 3, 6 \rangle$

$2 = 6k$, $1 = 3k$, $2 = 6k$

$k = \frac{1}{3}$, $k = \frac{1}{3}$, $k = \frac{1}{3}$

$\therefore$ Parallel exist

$\langle 3, 1, 1 \rangle = k\langle 6, 2, 3 \rangle$

$3 = 6k$, $2k = 1$, $3k = 1$

$k = \frac{1}{3}$, $k = \frac{1}{2}$, $k = \frac{1}{3}$

$\therefore$ Parallel does not exist

② $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|$

$\underline{a} \cdot \underline{b} = -|\underline{a}||\underline{b}|$

③ $\underline{a} \times \underline{b} = \underline{0}$

Perpendicular

$(\theta = 90°)$

① $\underline{a} \cdot \underline{b} = 0$

② If $\underset{\sim}{w} = \underset{\sim}{u} \times \underset{\sim}{v}$, then $\underset{\sim}{w} \perp \underset{\sim}{u}$ & $\underset{\sim}{w} \perp \underset{\sim}{v}$

③ $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|$

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5.2 Dot product

$a \cdot b = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$

@ $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

3. $\underset{\sim}{p} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$

$= \langle 3, 4, -3 \rangle$

$\underset{\sim}{q} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$

$= \langle 5, 3, -4 \rangle$

$\underset{\sim}{r} = -5\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$= \langle -5, 2, -3 \rangle$

(a) $\underset{\sim}{p} \cdot \underset{\sim}{q}$

$= \begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix}$

$= 3(5) + 4(3) + (-3)(-4)$

$= 39$

(b) $\underset{\sim}{q} \cdot (\underset{\sim}{p} - \underset{\sim}{r})$

$= \underset{\sim}{q} \cdot \underset{\sim}{p} - \underset{\sim}{q} \cdot \underset{\sim}{r}$

$= \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix}\begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix}$

$= 39 - [-25 + 6 + 12]$

$= 46$

(c) $(\underset{\sim}{p} + 3\underset{\sim}{q}) \cdot 3\underset{\sim}{r}$

$= \left[\begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix} + 3\begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix}\right] \cdot 3\begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix}$

$= \left[\begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 15 \\ 9 \\ -12 \end{pmatrix}\right] \cdot \begin{pmatrix} -15 \\ 6 \\ -9 \end{pmatrix}$

$= \begin{pmatrix} 18 \\ 13 \\ -15 \end{pmatrix} \cdot \begin{pmatrix} -15 \\ 6 \\ -9 \end{pmatrix}$

$= 18(-15) + 13(6) + (-15)(-9) = -57$

4(a) θ

$p = 2i + 4j + k \qquad q = i + j + k$

$\underset{\sim}{p} \cdot \underset{\sim}{q} = 2(1) + 4(1) + 1(1)$

$= 7$

$|\underset{\sim}{p}| = \sqrt{(2)^2 + (4)^2 + (1)^2}$

$= \sqrt{21}$

$|\underset{\sim}{q}| = \sqrt{(1)^2 + (1)^2 + (1)^2}$

$= \sqrt{3}$

$\theta = \cos^{-1}\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$

$\theta = \cos^{-1}\left(\frac{7}{\sqrt{21}\sqrt{3}}\right)$

$= 28.13^\circ$

θ (b)

$p = 2i + 8j - k \qquad q = i - j - 3k$

$\underset{\sim}{p} \cdot \underset{\sim}{q} = 2(1) + 8(-1) - 1(-3)$

$= -3$

$|\underset{\sim}{p}| = \sqrt{(2)^2 + (8)^2 + (-1)^2}$

$= \sqrt{69}$

$|\underset{\sim}{q}| = \sqrt{(1)^2 + (-1)^2 + (-3)^2}$

$= \sqrt{11}$

$\theta = \cos^{-1}\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$

$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{69}\sqrt{11}}\right)$

$= -\frac{3}{\sqrt{759}}$

5(a) $\underset{\sim}{a} = h\underset{\sim}{b}$

$\underset{\sim}{u} = h\underset{\sim}{v}$ if u and v are parallel

$2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} = h(-6\underset{\sim}{i} + 3\underset{\sim}{j} + \lambda\underset{\sim}{k})$

$2 = -6h \qquad -1 = 3h$

$h = -\frac{1}{3} \qquad h = -\frac{1}{3}$

$3 = \lambda h$

$3 = \lambda(-\frac{1}{3})$

$\lambda = -9$

5.3 vector Product

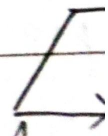
$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underset{\sim}{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underset{\sim}{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underset{\sim}{k}$$

@

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \, \hat{\underset{\sim}{u}} \rightarrow \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{|\underset{\sim}{a} \times \underset{\sim}{b}|}$$

9. $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ $\quad$ $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$ $\quad$ 10(a)



(a) $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -3 & 1 \\ 1 & -4 & 5 \end{vmatrix}$

$$= \begin{vmatrix} -3 & 1 \\ -4 & 5 \end{vmatrix} \underset{\sim}{i} - \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} \underset{\sim}{j} + \begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix} \underset{\sim}{k}$$

$$= [-15 - (-4)]\underset{\sim}{i} - [10 - 1]\underset{\sim}{j} + [-8 - (-3)]\underset{\sim}{k}$$

$$= -11\underset{\sim}{i} + 9\underset{\sim}{j} + 5\underset{\sim}{k}$$

$\vec{OD}$

=

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(b) area

(b) $\underset{\sim}{b} \times \underset{\sim}{a} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -4 & 5 \\ 2 & -3 & 1 \end{vmatrix}$

$$= \begin{vmatrix} -4 & 5 \\ -3 & 1 \end{vmatrix} \underset{\sim}{i} - \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} \underset{\sim}{j} + \begin{vmatrix} 1 & -4 \\ 2 & -3 \end{vmatrix} \underset{\sim}{k}$$

$$= [-4 - (-15)]\underset{\sim}{i} - (1 - 10)\underset{\sim}{j} + [-3 - (-8)]\underset{\sim}{k}$$

$$= 11\underset{\sim}{i} + 9\underset{\sim}{j} + 5\underset{\sim}{k}$$

5.4 Application < line, plane

Equation of line:

$$\underset{\sim}{r} = (2\underset{\sim}{i} + \underset{\sim}{j}) + t(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k})$$

$$\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v}$$

V.E

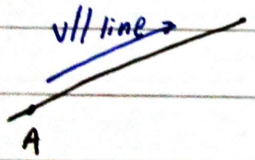
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

P.E

$$x = a_1 + td$$
$$y = a_2 + te$$
$$z = a_3 + tf$$

C.E

$$\frac{x - a_1}{d} = \frac{y - a_2}{e} = \frac{z - a_3}{f}$$

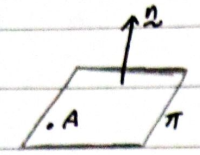


$\underset{\sim}{v}$

- from question, direction vector = v
- 2 points on line $\Rightarrow \underset{\sim}{v} = \overrightarrow{AB}$
- line $\parallel$ (parallel) other line, $\underset{\sim}{v} = \underset{\sim}{v_1}$
- line $\perp$ plane $\underset{\sim}{v} = \underset{\sim}{n}$

Equation of plane :

$\tilde{n} \perp$ plane π



How to get $\tilde{n}$?

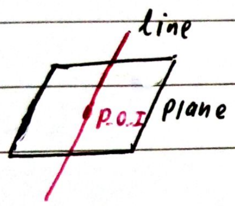
1. from question, normal vector $= \tilde{n}$
2. plane $\perp$ line @ vector $\Rightarrow \tilde{n} = \tilde{v}$
3. plane // other plane $\tilde{n} = \tilde{n}_1$
4. 2 vectors on plane $\Rightarrow \tilde{n} = \tilde{v}_1 \times \tilde{v}_2$
5. 3 points on plane $\Rightarrow \tilde{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

VE $\tilde{r} \cdot \tilde{n} = \tilde{a} \cdot \tilde{n}$

eg: $r \cdot (2\tilde{i} - \tilde{j} + 3\tilde{k}) = 4$

CE $ax + by + cz = d$

eg: $2x - y + 3z = 4$



1. line equation (parametric)

$x = 1 - t = 0 \quad = 0$

$y = 2t \quad = 2$

$z = -t \quad = -1$

2. plane equation (cartesian)

$ax + by + cz = d$

$x + y - z = 5$

3. subs 1 into 2 : to get t $t = 1$

4. subs t into 1

5. write point of intersection (x, y, z)

$(0, 2, -1)$

ANGLE

vector + vector
$\theta = \cos^{-1}\left(\frac{\tilde{v}_1 \cdot \tilde{v}_2}{|\tilde{v}_1||\tilde{v}_2|}\right)$

plane + plane
$\theta = \cos^{-1}\left(\frac{\tilde{n}_1 \cdot \tilde{n}_2}{|\tilde{n}_1||\tilde{n}_2|}\right)$

line + line
$\theta = \cos^{-1}\left(\frac{\tilde{v}_1 \cdot \tilde{v}_2}{|\tilde{v}_1||\tilde{v}_2|}\right)$

normal & line
$\beta = \cos^{-1}\left(\frac{\tilde{n} \cdot \tilde{v}}{|\tilde{n}||\tilde{v}|}\right)$

Plane + line
$\theta = 90° - \beta$