第四章 频域变换基础

一、 傅里叶级数和变换介绍

- **1768年生于法国**
- **1807年提出“任何周期信号都可用正弦函数级数表示”**
- **1829年狄里赫利第一个给出收敛条件**
- **拉格朗日反对发表**
- **1822年首次发表在“热的分析理论”一书中**

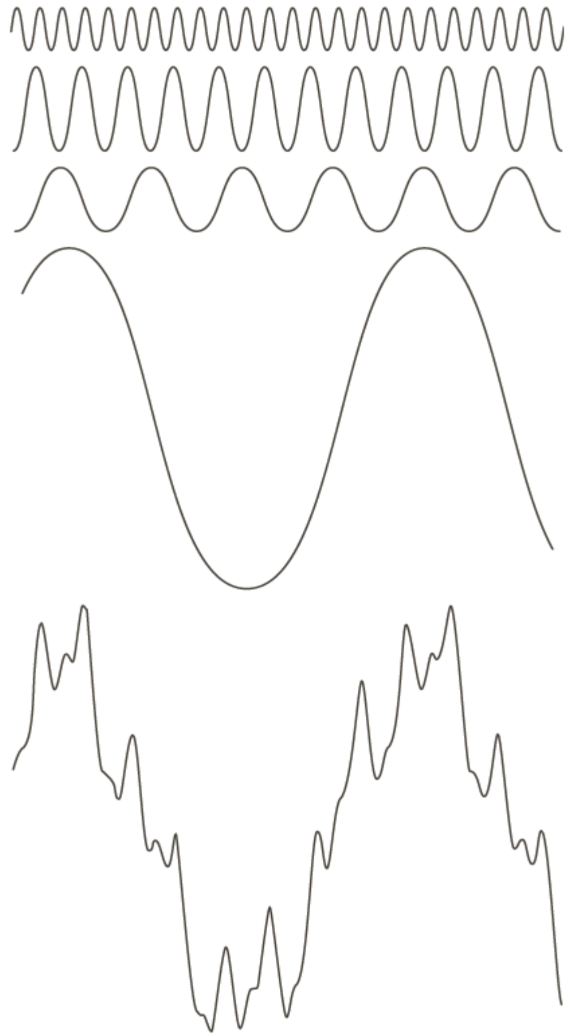


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

二、复数

定义：　$C=R+jI$　　$C=|C|e^{j\theta}$

共轭：

$C^*=R-jI$

欧拉公式：

$e^{j\theta} = \cos\theta + j\sin\theta$

极坐标：

$C=|C|\ (\cos\theta+j\sin\theta)$

$|C|=\sqrt{R^2 + I^2}$

$\theta=\mathrm{arctg}(I/R)$

复函数：

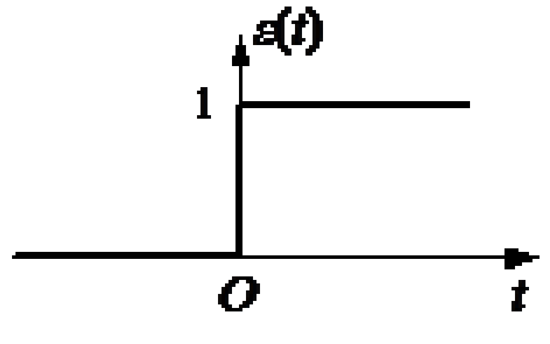
$F(u)=R(u)+jI(u)$

$F^*(u)=R(u)-jI(u)$

三、阶跃和冲激信号

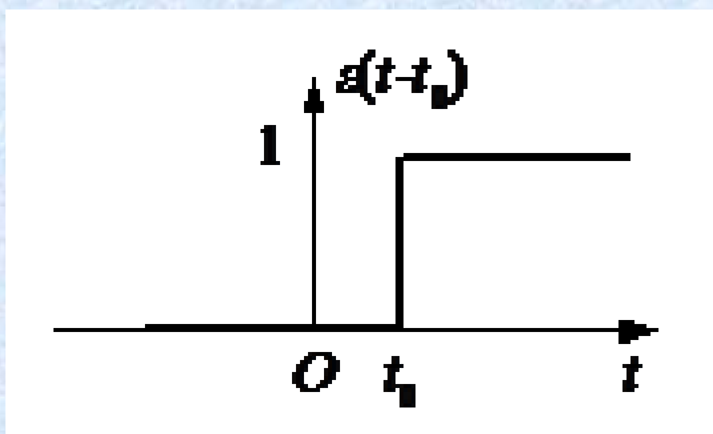
单位阶跃信号

$$\varepsilon(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$$



延迟 时间的阶跃信号

$$\varepsilon(t-t_0)=\begin{cases}0 & t<t_0\\ 1 & t>t_0\end{cases}$$



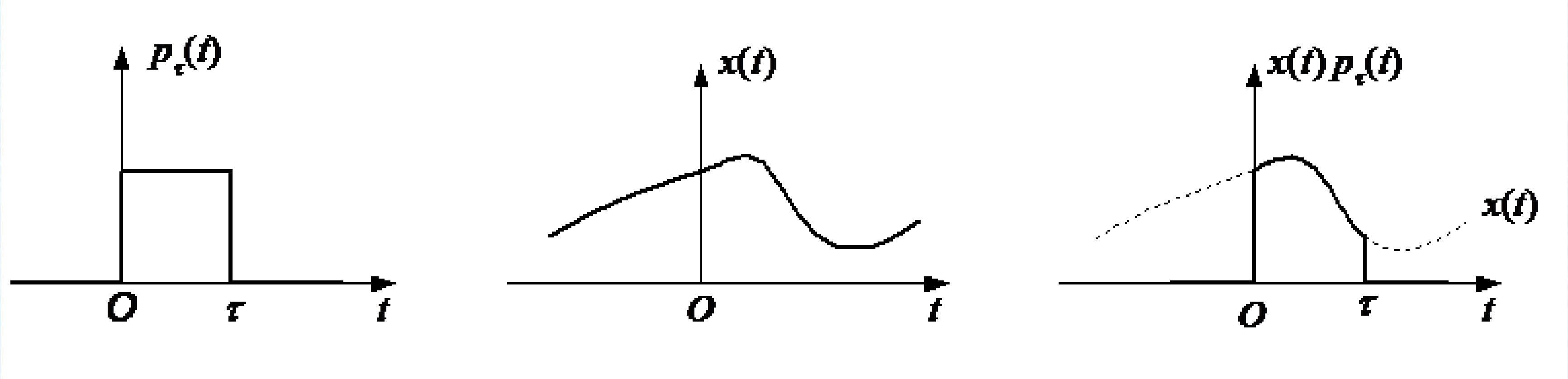
当给常规函数$x(t)$乘以$\varepsilon(t)$后，$x(t)\varepsilon(t)$截取了$t>0$时的$x(t)$，$x(t)\varepsilon(t)$在$t<0$时为零，在$t>0$时为$x(t)$

零时刻起始的宽度为 τ 的矩形脉冲 $p_\tau(t)$

阶跃函数表示

$$p_\tau(t)=\varepsilon(t)-\varepsilon(t-\tau)$$

给任一常规函数 $x(t)$ 乘以 $p_\tau(t)$ 可以截取脉冲范围内的 $x(t)$



信号的截取

阶跃函数可把分段光滑函数用一个表达式表示

$$x(t)=\begin{cases} x_1(t) & t<0 \\ x_2(t) & 0<t<t_1 \\ x_3(t) & t>t_1 \end{cases}$$

$$\begin{aligned} x(t) &= x_1(t)[1-\varepsilon(t)]+x_2(t)[\varepsilon(t)-\varepsilon(t-t_1)]+x_3(t)\varepsilon(t-t_1) \\ &= x_1(t)+[x_2(t)-x_1(t)]\varepsilon(t)+[x_3(t)-x_2(t)]\varepsilon(t-t_1) \end{aligned}$$

任一函数 $x(t)$ 与阶跃函数 $\varepsilon(t)$ 乘积的积分

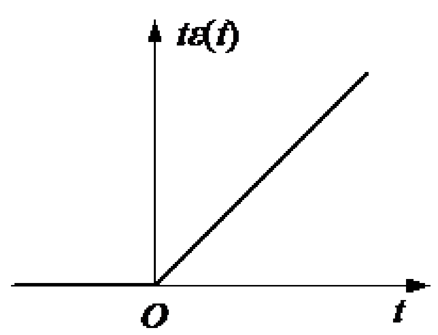
$$\int_{-\infty}^{t} x(\tau)\varepsilon(\tau)\mathrm{d}\tau = \begin{cases} 0 & t<0 \\ \int_{0}^{t} x(\tau)\mathrm{d}\tau & t>0 \end{cases}$$

$$= \left[\int_{0}^{t} x(\tau)\mathrm{d}\tau\right]\varepsilon(t)$$

单位阶跃函数的积分

$$\int_{-\infty}^{t} \varepsilon(\tau)\mathrm{d}\tau = \begin{cases} 0 & t<0 \\ t & t>0 \end{cases}$$

$$= t\varepsilon(t)$$

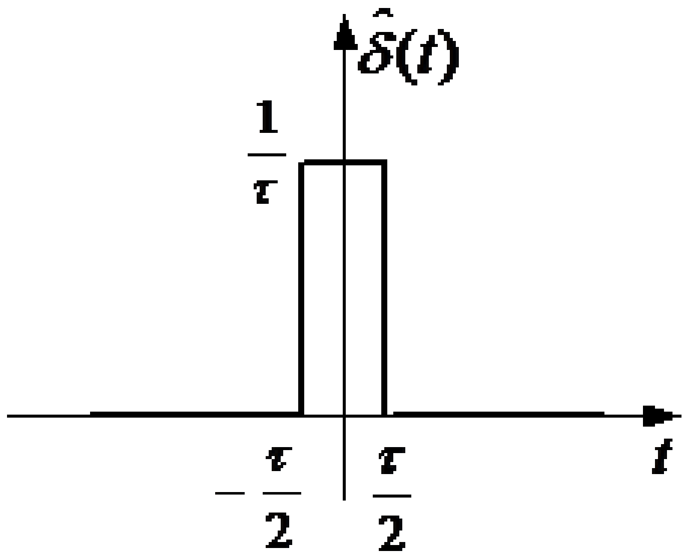


冲激信号

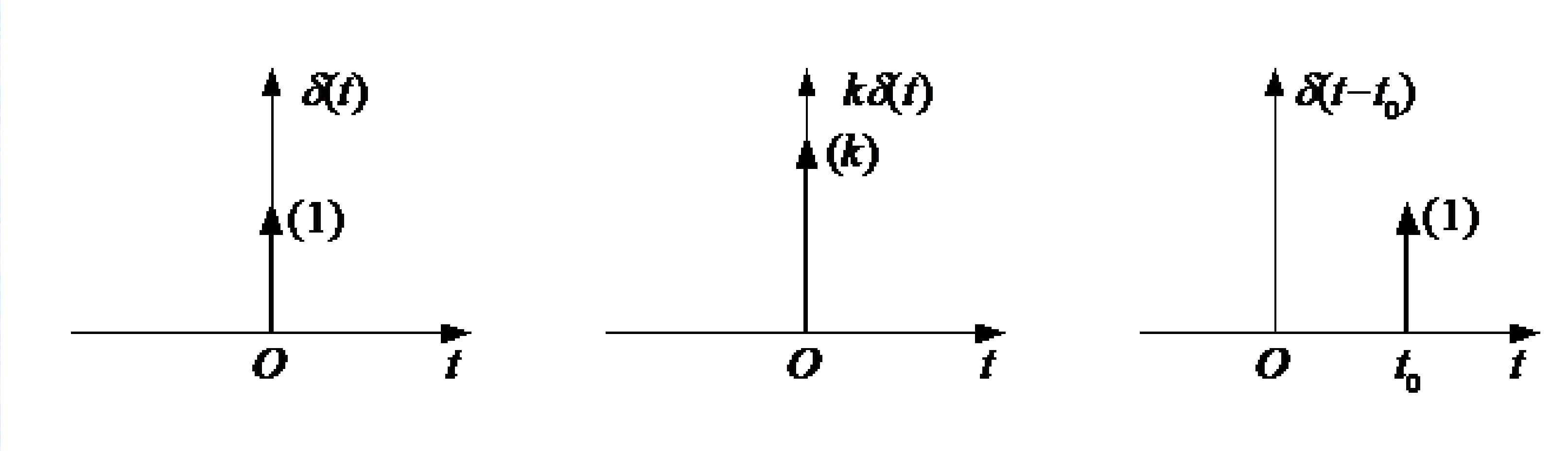
$$\begin{cases} \delta(t) = 0 & t < 0,\quad t > 0 \\ \delta(t) = \infty & t = 0 \\ \int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 & \end{cases}$$

下图所示宽度为τ、高度为$\frac{1}{\tau}$的脉冲$\hat{\delta}(t)$不论τ为何值，其面积总为1，当$\tau\to 0$时，$\hat{\delta}(t)$具有$\delta(t)$的所有特征

$$\delta(t)=\lim_{\tau\to 0}\hat{\delta}(t)$$



冲激函数与时间轴所围面积称为冲激函数强度，单位冲激函数的强度为1，而冲激函数 $k\delta(t)$ 的强度为k。延迟t_0 时刻的单位冲激函数为 $\delta(t-t_0)$



冲激函数性质

- 冲激函数的积分为阶跃函数

$$\int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

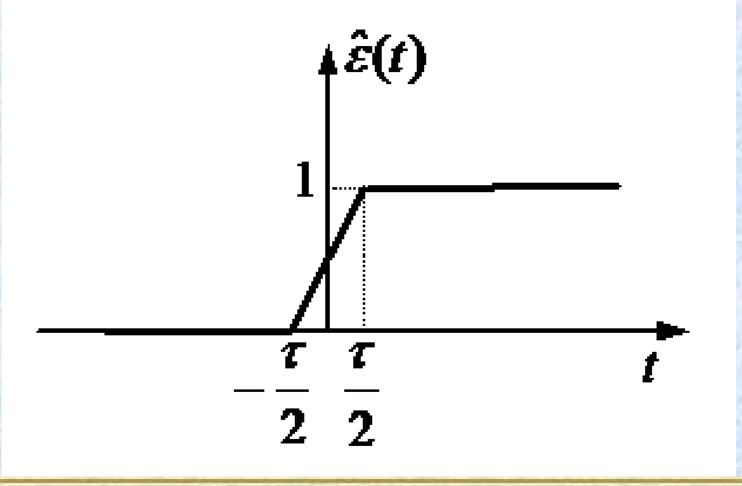
$$\int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau = \varepsilon(t)$$

由微积分知识，上式的逆关系应该为

$$\delta(t) = \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}$$

注意：从严格的常规函数微积分角度考虑，$\varepsilon(t)$在t=0处不存在导数。可看作是$\hat{\varepsilon}(t)$在 τ →0时极限

$$\frac{\mathrm{d}\hat{\varepsilon}(t)}{\mathrm{d}t}=\hat{\delta}(t)$$



- 冲激函数的抽样性质

冲激函数具有抽取常规函数X(t) 在t=t_0处样本的作用

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

上式说明冲激函数与常规函数乘积的积分为冲激出现时刻该函数的值，称为冲激函数的抽样性质

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

- 尺度变换性质

$$\delta(at)=\frac{1}{|a|}\delta(t)$$

其中a为不等于0的实常数

在一些文献中冲激函数 也常狄拉克(Dirac)函数

单位离散冲激

$$\delta(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0\end{cases}$$

$$\sum_{-\infty}^{\infty}\delta(x)=1$$

单位离散冲激抽样

$$\sum_{-\infty}^{\infty}f(x)\delta(x-x_0)=f(x_0)$$

单位离散冲激串

$$S_{\Delta T}(t)=\sum_{-\infty}^{\infty}\delta(t-n\Delta T)$$

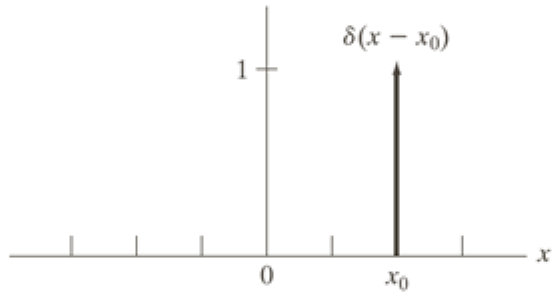


FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

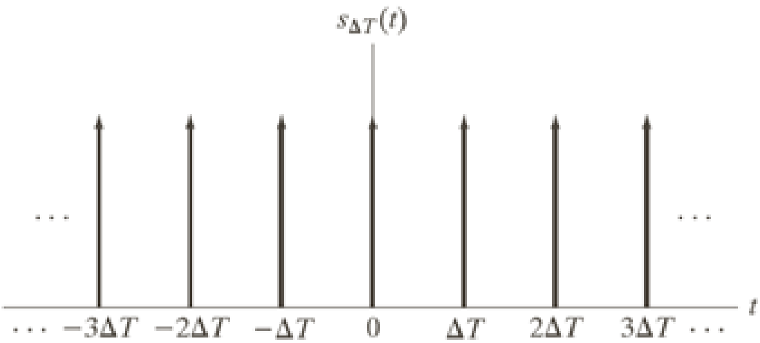


FIGURE 4.3 An impulse train.

三、连续傅里叶变换

傅里叶级数

周期信号可展开成正交函数线性组合的无穷级数：

. 三角函数式的 傅立里叶级数 $\{\cos n\omega_1 t, \sin n\omega_1 t\}$

. 复指数函数式的傅里叶级数 $\{ e^{jn\omega_1 t} \}$

- Fourier展开

$L^2[-l, l]$空间的概念

基本函数族 $\left\{1, \cos\frac{n\pi}{l}x, \sin\frac{n\pi}{l}x\right\}_{n=1}^{\infty}$

函数 $f(x)$ 的Fourier展开式

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(\xi)d\xi \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(\xi)\cos\frac{n\pi}{l}\xi d\xi \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(\xi)\sin\frac{n\pi}{l}\xi d\xi$$

完备性的概念

Dirichlet定理-Fourier展开收敛定理

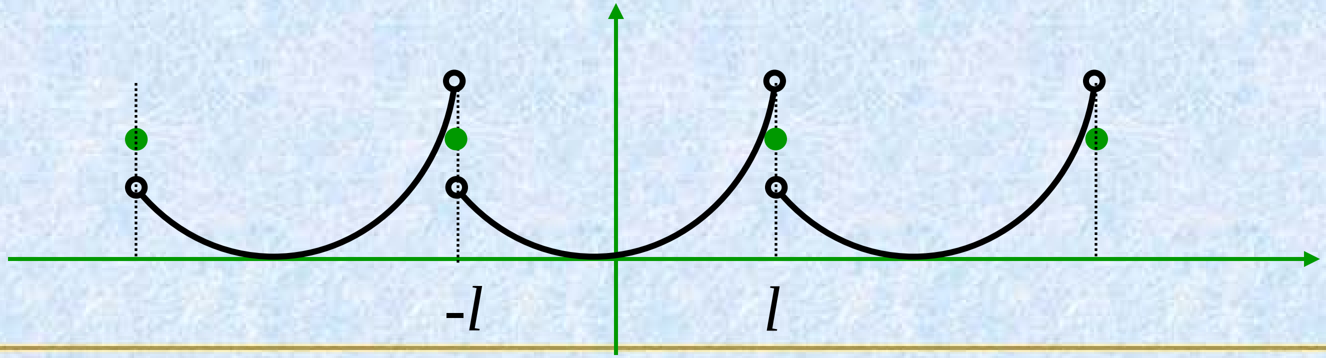
若 $f(x)$ 满足：

(1)处处连续，或在每个周期内只有有限个第一类间断点；

(2) 在每个周期内只有有限个极值点，

则

$$\text{函数} f(x) \text{的} Fourier \text{展开} = \begin{cases} f(x) & \text{在连续点} x \\ \frac{1}{2}[f(x-0)+f(x+0)] & \text{在间断点} x \end{cases}$$



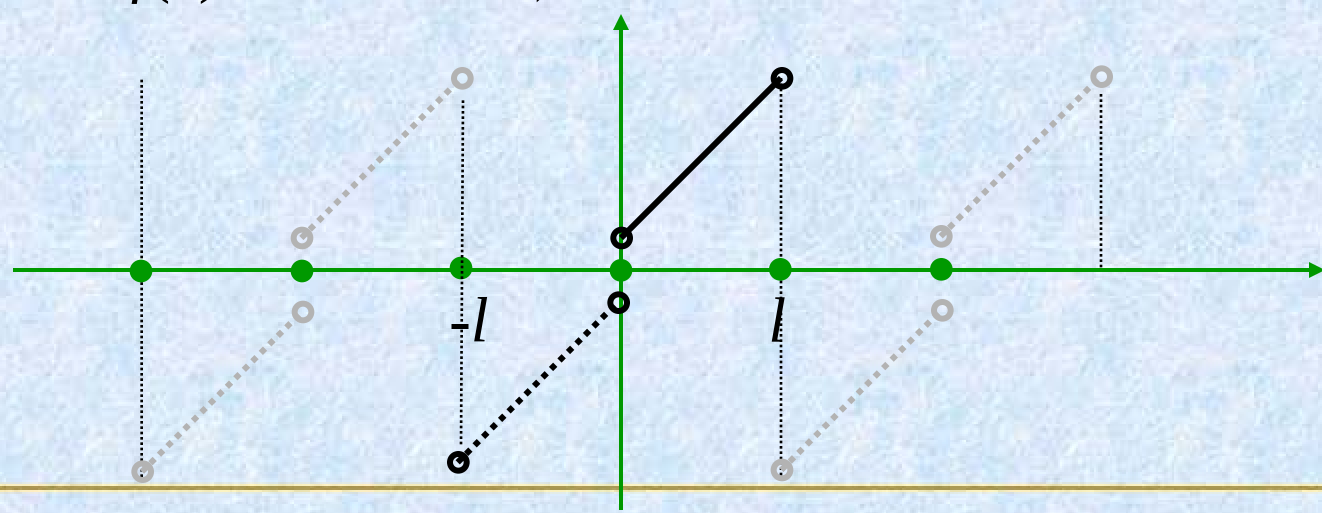
正弦级数和余弦级数

若函数 $f(x)$ 是奇函数，则Fourier展开成正弦级数

若函数 $f(x)$ 是偶函数，则Fourier展开成余弦级数

例子

例1：设 $f(x) = x+1$, $x \in (0, l)$, 试将其展开成正弦级数.



复形式的Fourier级数

基本函数族 $\left\{ e^{i\frac{n\pi}{l}x} \right\}_{n=-\infty}^{+\infty}$

函数 $f(x)$ 的Fourier展开式

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{l}x} \qquad c_n = \frac{1}{l}\int_{-l}^{l} f(\xi) e^{-i\frac{n\pi}{l}\xi} d\xi$$

复形式的Fourier积分定理

$$f(x)=\int_{-\infty}^{+\infty}F(\omega)e^{i\omega x}d\omega$$

其中

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\xi)e^{-i\omega\xi}d\xi$$

F(◆) 被称为Fourier变换

Fourier积分定理被称为反演公式

☞$^{-1}$ ☞ = I

注意与书上公式的比较

若令 $\mu=\frac{\omega}{2\pi}$ **可得书上公式**

书上的例子

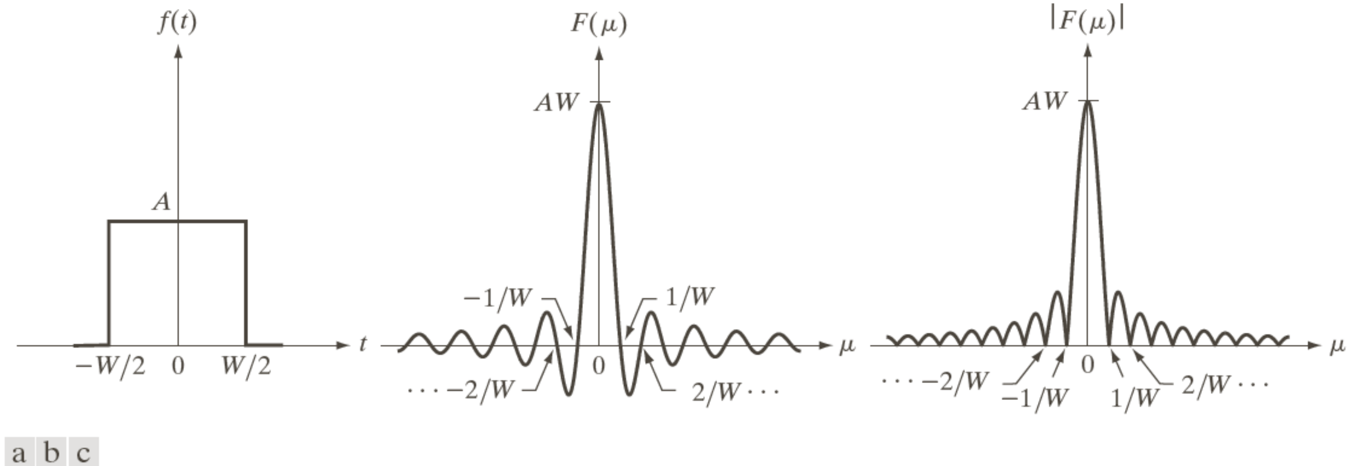


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

离散信号采样

$$f^{\sim}(t) = f(t)S_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$f(t)$ 被采样的连续信号

$S_{\Delta t}(t)$ 是冲激串

$$S_{\Delta T}(t) = \sum_{-\infty}^{\infty} \delta(t - n\Delta T)$$

冲激的傅里叶变换

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu t_0} = \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)$$

冲激串的傅里叶变换

傅里叶展开 $S_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j2n\pi t/\Delta T}$

$$\Downarrow$$

$$F\{e^{j2n\pi t/\Delta T}\} = \delta(\mu - n/\Delta T)$$

$$\Downarrow$$

$$S(\mu) = F\{S_{\Delta T}(t)\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - n/\Delta T)$$

结论：

冲激串的傅里叶变换还是冲激串，周期为 $1/\Delta T$，变换后的频域周期和空域周期成反比

四、连续函数的卷积

一维卷积

$$g(x) = f(x) * h(x) = \int_{-\infty}^{+\infty} f(\xi) h(x-\xi) d\xi$$

二维卷积

$$g(x,y) = \iint_{-\infty}^{+\infty} f(\xi,\eta) h(x-\xi, y-\eta) d\xi \, d\eta$$

卷积与频域的关系

$$f(t)h(t) \leftrightarrow F(\mu) * H(\mu)$$

$$f(t) * h(t) \leftrightarrow F(\mu)H(\mu)$$

卷积的图解说明

（1）变量代换：将函数 $x(t)$、 $h(t)$ 的自变量 t 置换为 τ，得到 $x(\tau)$、$h(\tau)$。

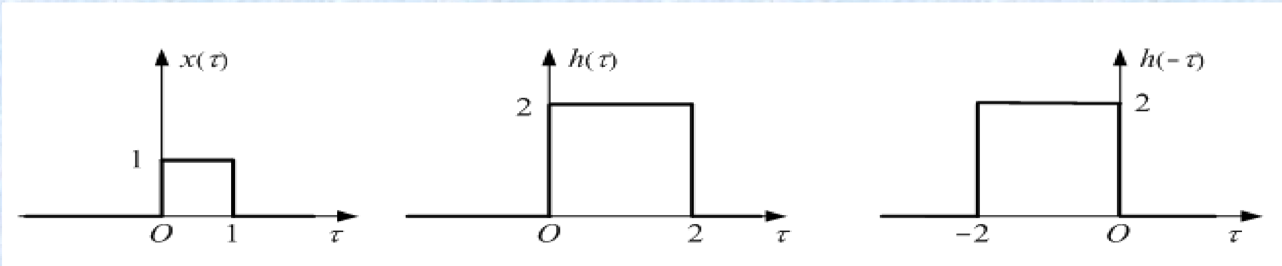
（2）反转：把 $h(\tau)$ 反转可得 $h(-\tau)$。

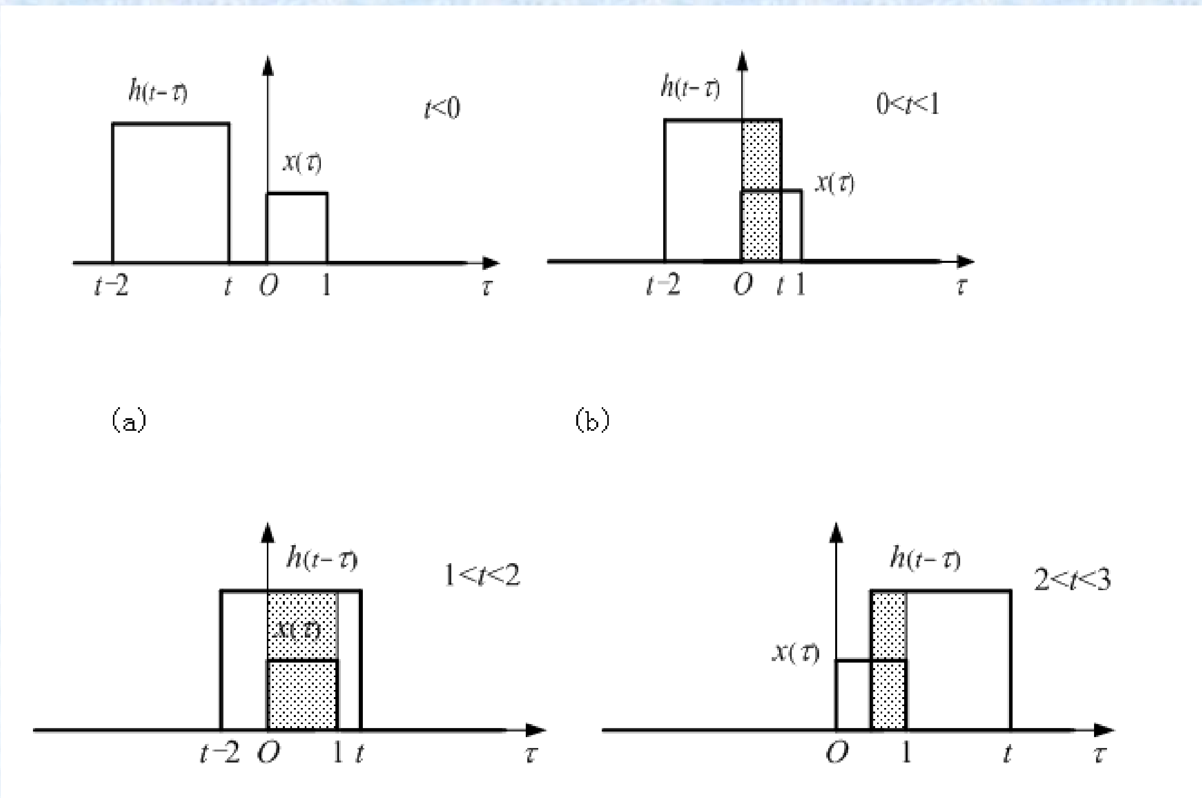
（3） 移位：将 $h(-\tau)$ 沿 τ 轴移位 t 时间，得 $h(t-\tau)$ 的波形。如果 $t<0$，则将 $h(-\tau)$ 向左移；如果 $t>0$，将 $h(-\tau)$ 向右移。

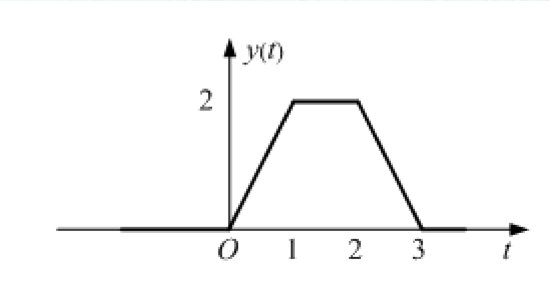
（4）相乘：函数 $x(t)$ 与 $h(t-\tau)$ 相乘，两波形重叠部分相乘有值，不重叠部分乘积为零。

（5）积分： $x(t)$ 与 $h(t-\tau)$ 乘积曲线下的面积即为 t 时刻的卷积值。在求积分时，积分的上、下限不仅依赖于信号本身，还依赖于卷积过程中信号波形的相对位置。

例2.2-3 已知 $x(t)=\varepsilon(t)-\varepsilon(t-1)$， $h(t)=2[\varepsilon(t)-\varepsilon(t-2)]$。求 $y(t)=x(t)*h(t)$。







演示

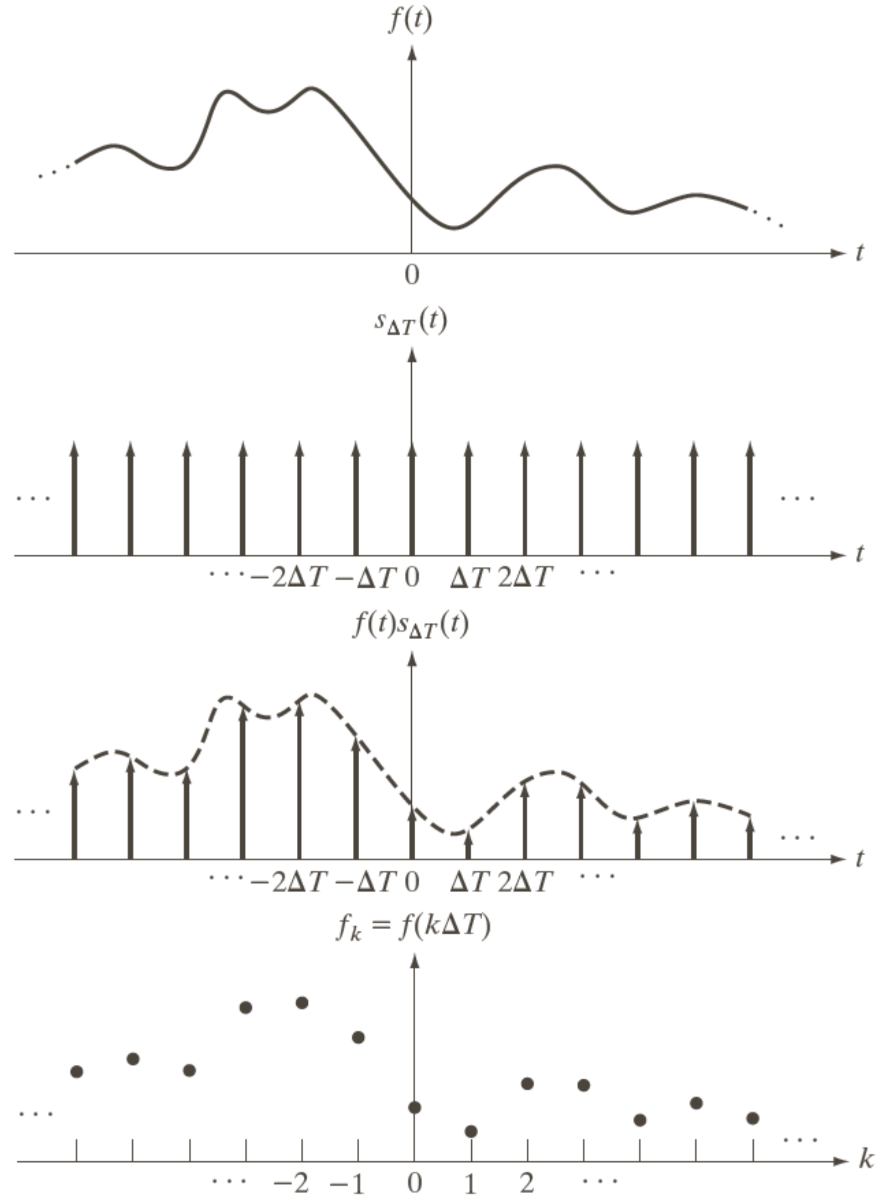
五、信号取样与恢复

取样公式：

$$f^{\sim}(t) = f(t)S_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

其中$f^{\sim}(t)$ 表示取样后的函数，任意取样值f_k值为：

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) = f(k\Delta T)$$



a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

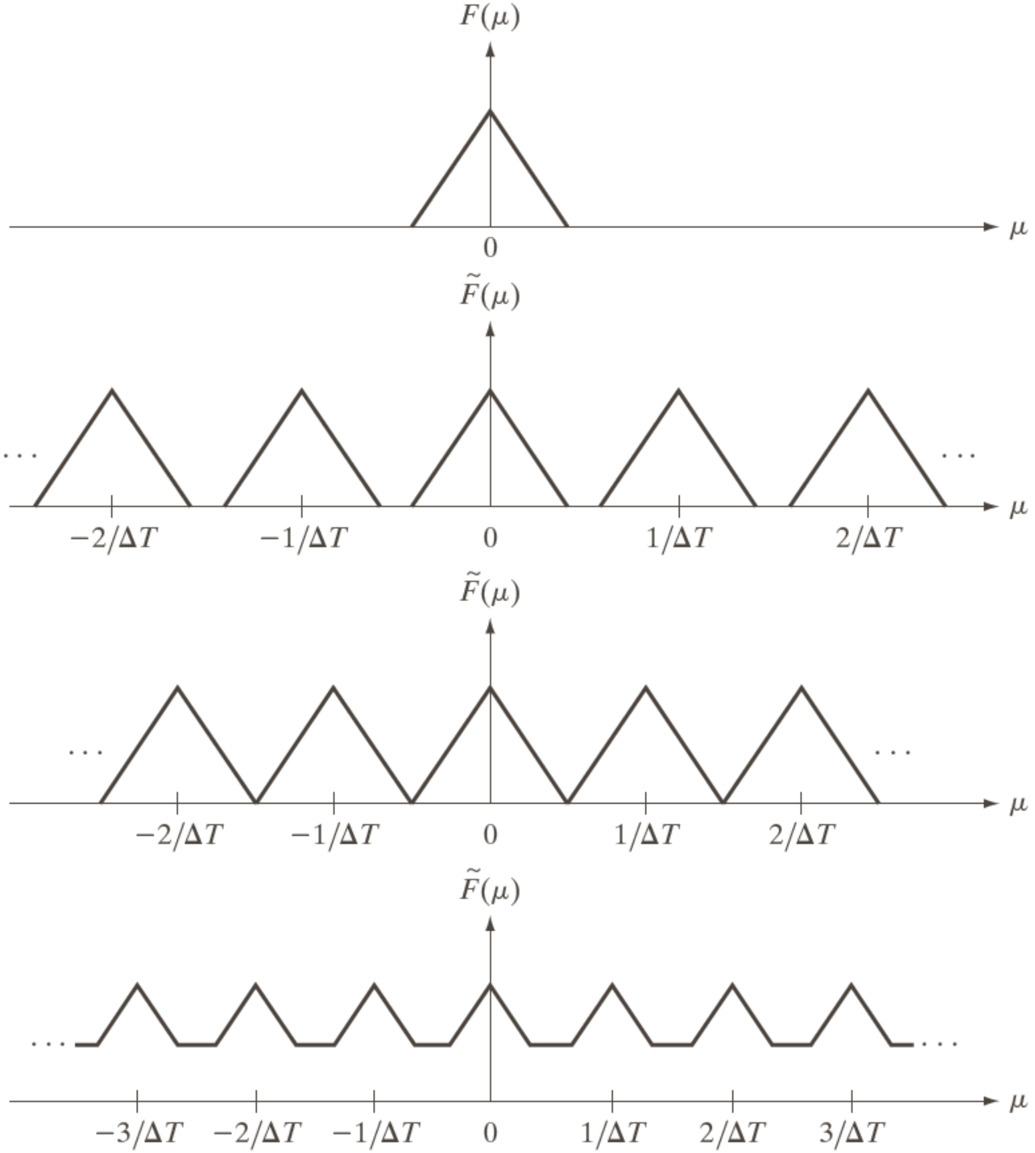
取样函数的傅里叶变换

$$F^{\sim}(\mu) = F(\mu) * S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F(\mu - n/\Delta T)$$

P133

从该式可得到何种结论？

取样函数的傅里叶变换是F(μ) 的一个拷贝的无限周期序列



a
b
c
d

FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

采样定理

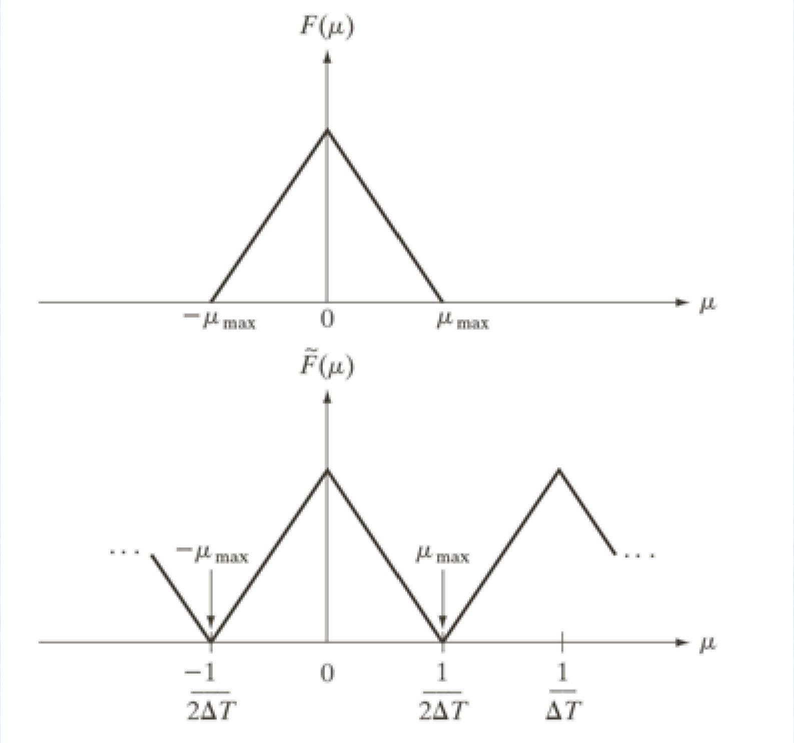
带限函数 $\left[-\mu_{\max}, \mu_{\max}\right]$

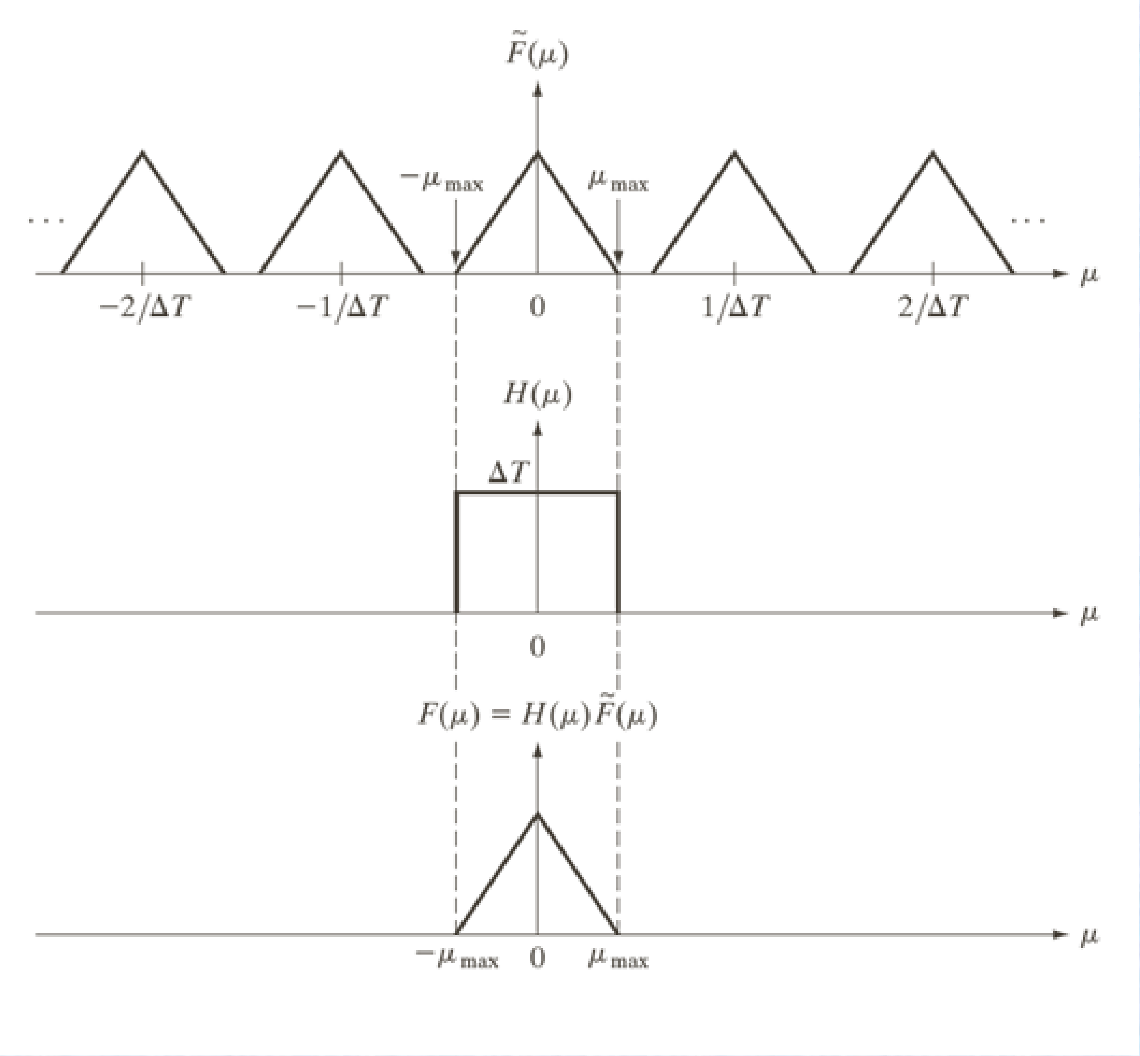
如果以超过函数最高频率两倍的取样率来获取样本，连续带限函数可以从它的样本集中来恢复。

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

如果恰好以最高频率两倍取样会怎样？

如何F~(μ) 从获得f(t)?




$\tilde{F}(\mu)$
$-\mu_{max}$
μ_{max}
...
...
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
μ
$H(\mu)$
ΔT
0
μ
$F(\mu) = H(\mu)\tilde{F}(\mu)$
$-\mu_{max}$
0
μ_{max}
μ

$$F(\mu) = \tilde{F}(\mu)H(\mu)$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t}$$

低通滤波器　　　重建滤波器

$$H(\mu) = \begin{cases} \Delta T & \mu \in [-\mu_{max}, \mu_{max}] \\ 0 & \text{其它} \end{cases}$$

由取样重建函数

$$
\begin{aligned}
f(t) &= h(t)\star\tilde{f}(t) \\
&= \int_{-\infty}^{\infty} h(z)\tilde{f}(t-z)dz \\
&= \int_{-\infty}^{\infty} \frac{\sin(\pi z/\Delta T)}{(\pi z/\Delta T)} \sum_{n=-\infty}^{\infty} f(t-z)\delta(t-n\Delta T-z)dz \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(\pi z/\Delta T)}{(\pi z/\Delta T)} f(t-z)\delta(t-n\Delta T-z)dz \\
&= \sum_{n=-\infty}^{\infty} f(n\Delta T)\frac{\sin[\pi(t-n\Delta T)/\Delta T]}{[\pi(t-n\Delta T)/\Delta T]} \\
&= \sum_{n=-\infty}^{\infty} f(n\Delta T)\operatorname{sinc}[(t-n\Delta T)/\Delta T].
\end{aligned}
$$

从上式可以得出什么结论？

六、离散傅里叶变换

直接对采样函数进行取样

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t} = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

令 $\mu = m/M\Delta T$ 得到对应的M个频域离散集合

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \quad m=0,1,2 \dots M-1$$

反变换

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \quad n=0,1,2 \dots M-1$$

周期性

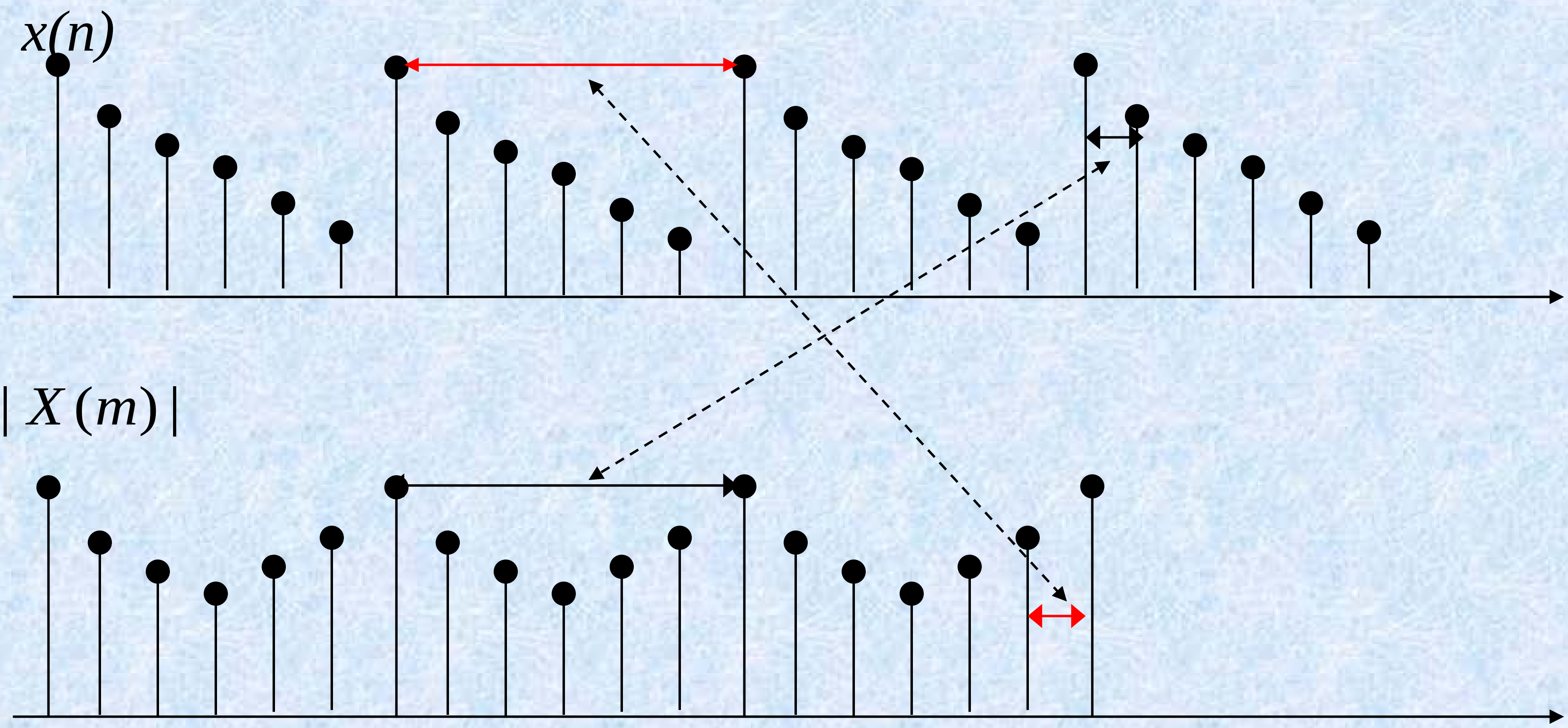
域	连续性	周期性
时域	离散	周期
频域	离散	周期

时域的离散造成频域的延拓（周期性）。因而频域的离散也会造成时域的延拓（周期性）。

- 在离散情况下对于信号恢复能否用空域中的卷积替代频域中的乘积？

f(t)*h(t) 直接代替？

取样和频率间隔的关系



$T = M\Delta T$

$\Delta\mu = \frac{1}{M\Delta T} = \frac{1}{T}$

$\Omega = M\Delta\mu = \frac{1}{\Delta T}$

七、混淆

考虑欠采样情况

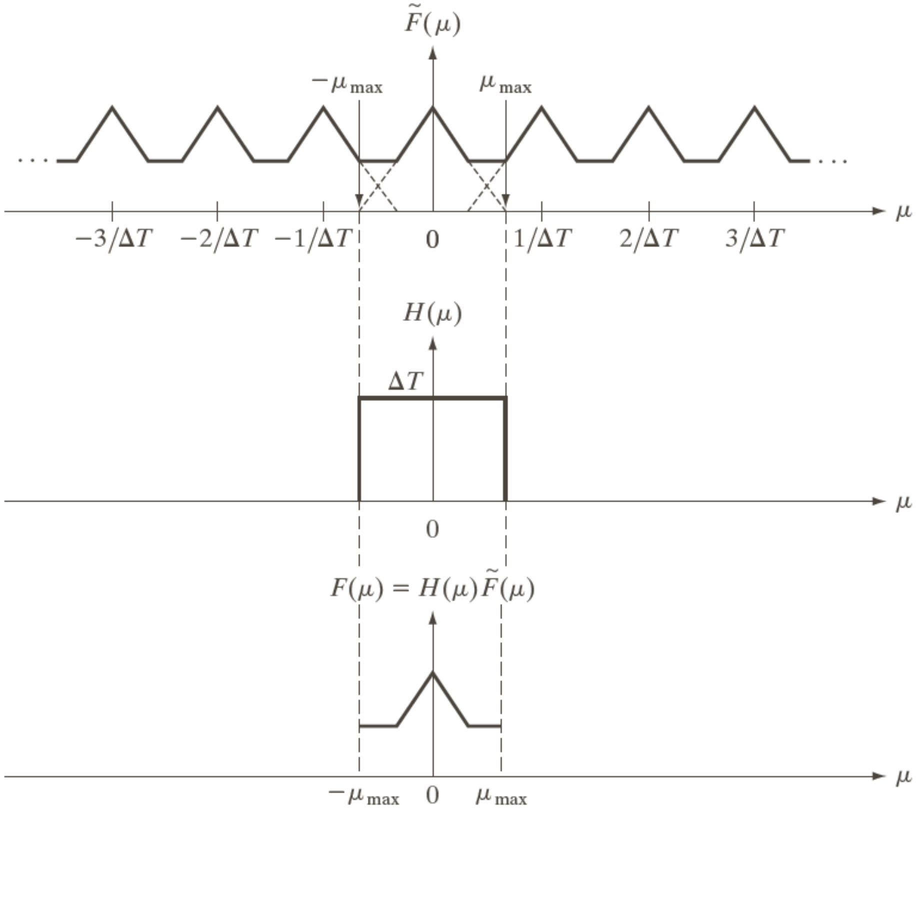


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

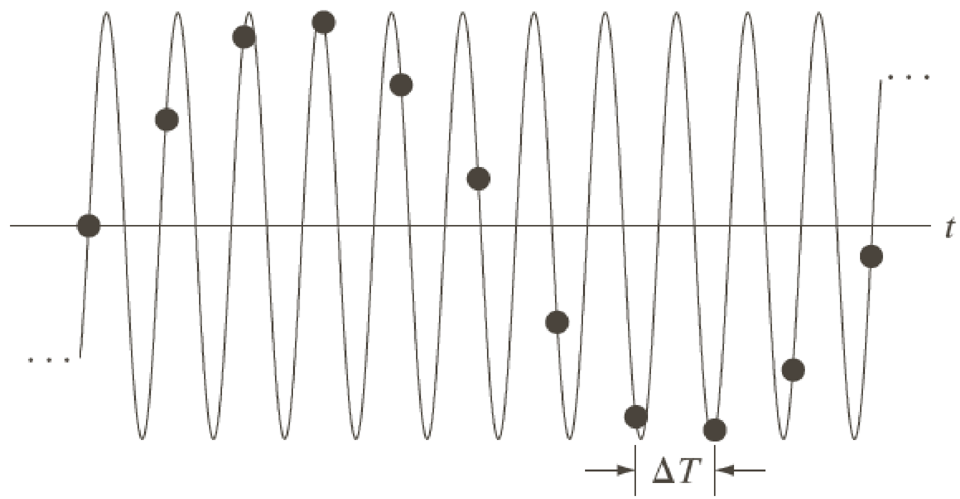
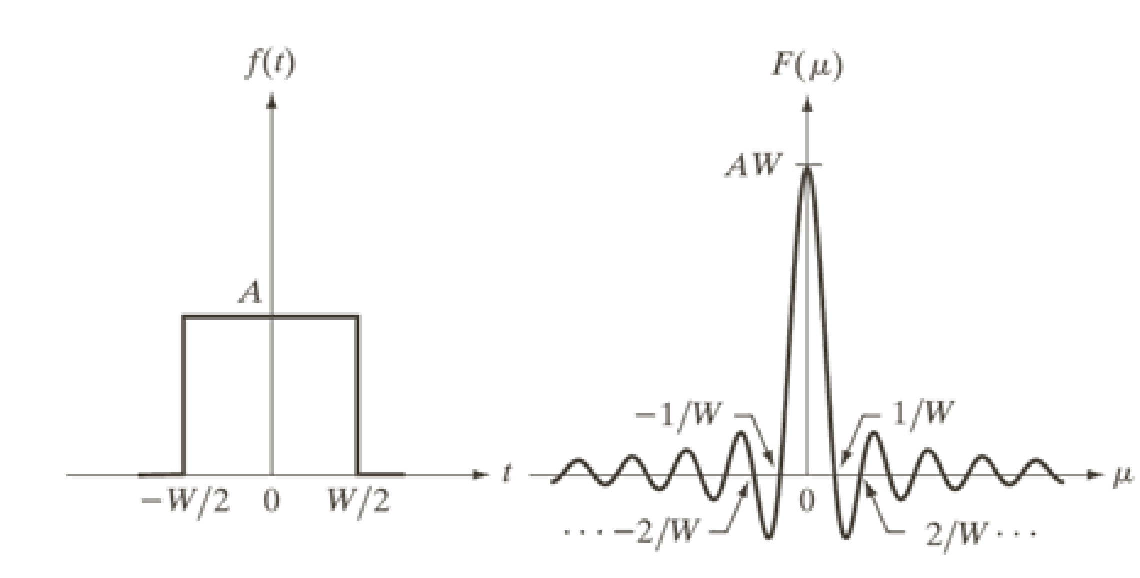


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

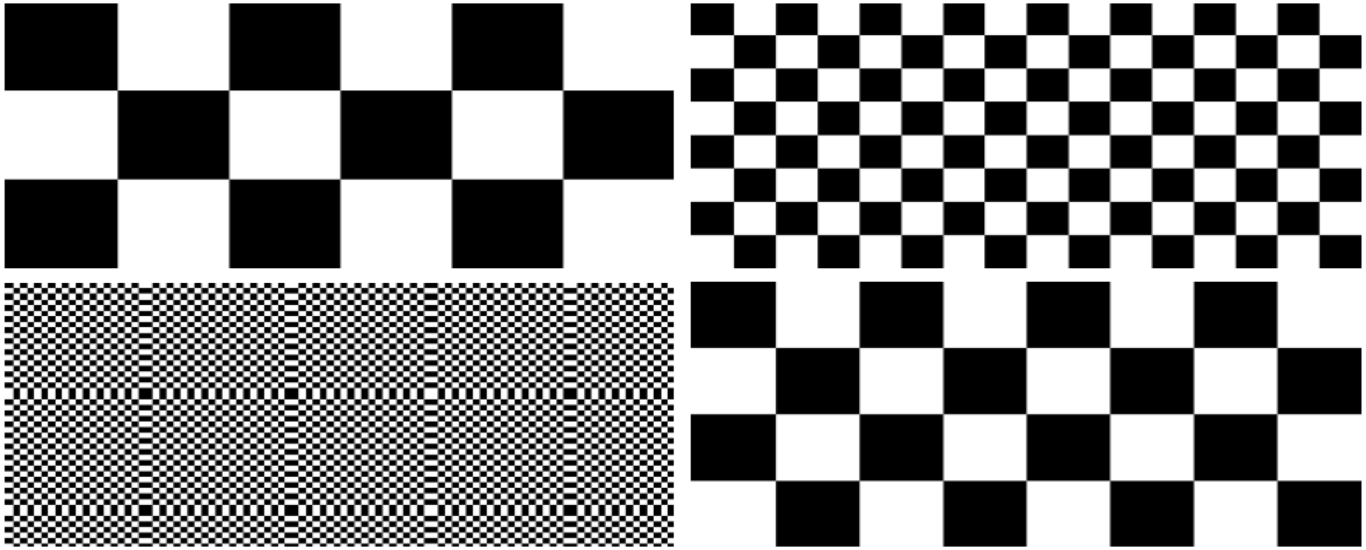
混淆无法完全避免？为什么？

$$h(t)=\begin{cases}1, & 0\le t\le T\\ 0, & \text{其它}\end{cases}$$



图像的 混淆

何为图像的混淆？空间和时间上的考虑



a b
c d

FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

镜头散焦处理

图像插值和重取样

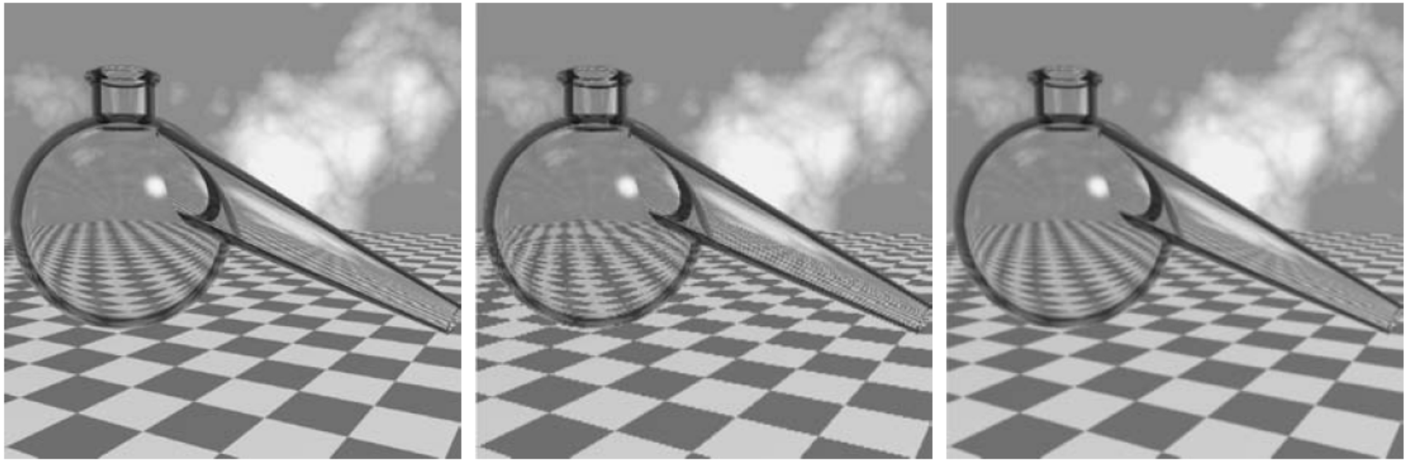
放大　过采样
缩小　欠采样

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3 × 3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

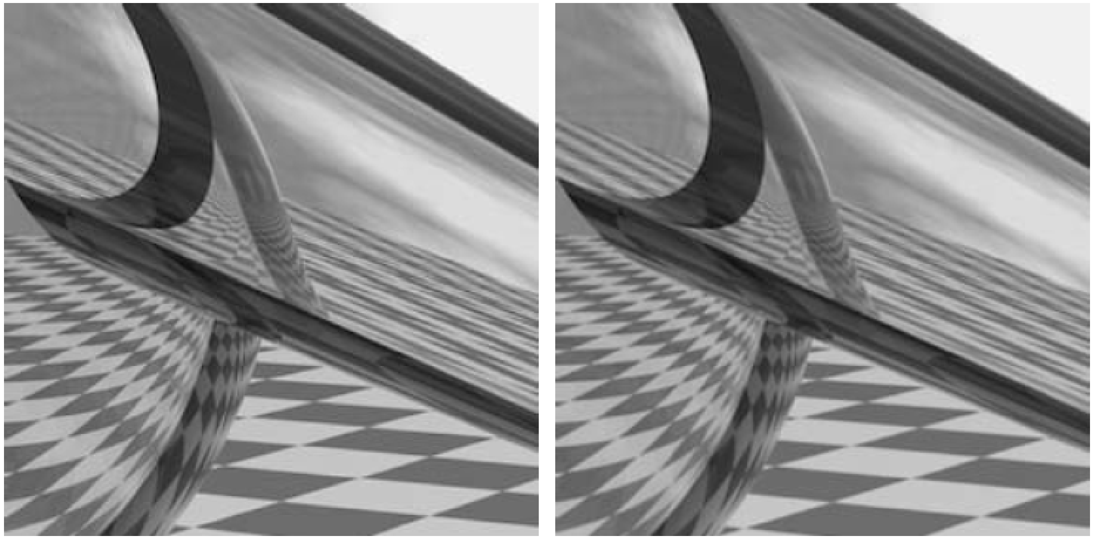
抗混淆：　平滑处理　超取样

混淆对边缘的影响



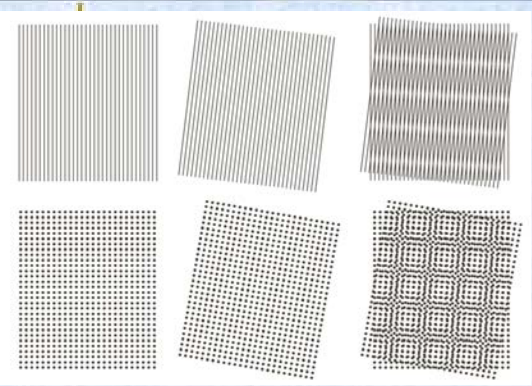
a b c

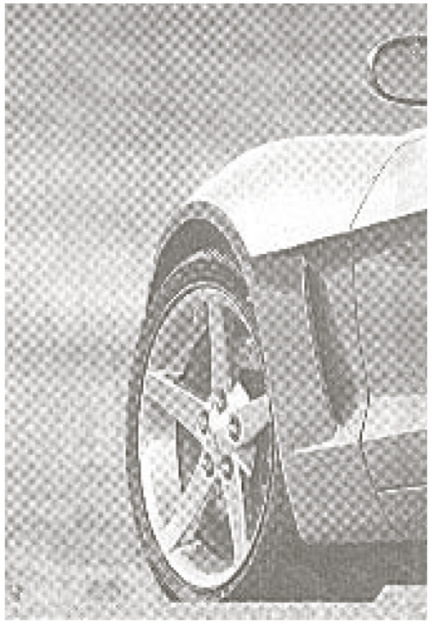
FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

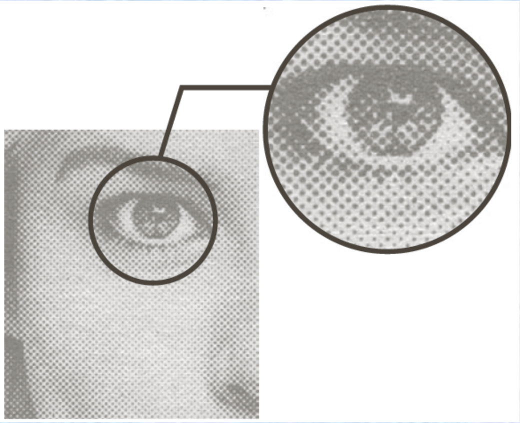


a b

FIGURE 4.19 Image zooming. (a) A 1024×1024 digital image generated by pixel replication from a 256×256 image extracted from the middle of Fig. 4.18(a). (b) Image generated using bi-linear interpolation, showing a significant reduction in jaggies.







摩尔波纹

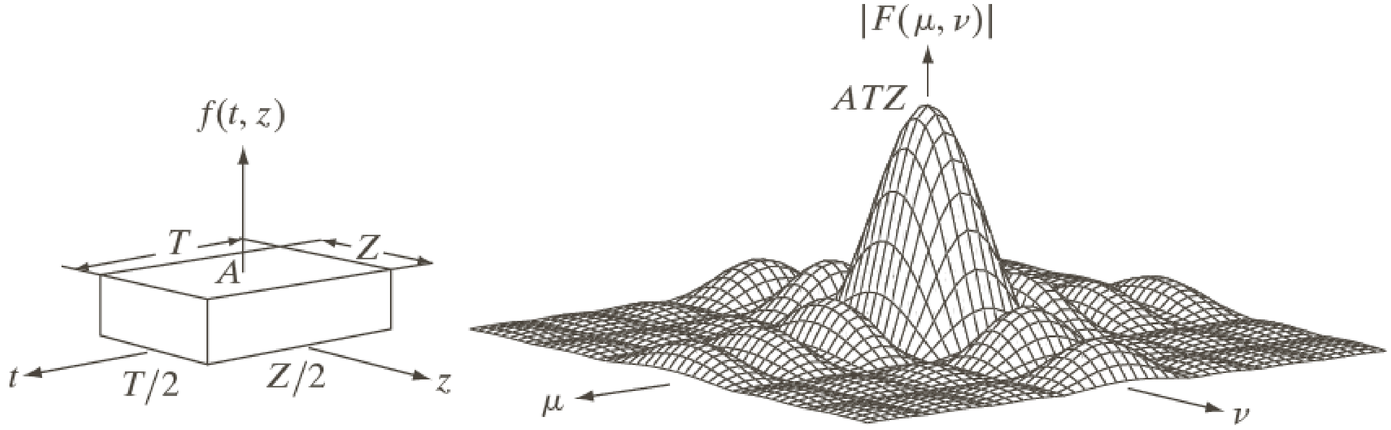
八、二维傅里叶变换

连续正变换 $F(\mu, v) = \iint_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t+vz)} dtdz$

连续反变换 $f(t, z) = \iint_{-\infty}^{\infty} F(\mu, v)\, e^{j2\pi(\mu t+vz)} d\mu dv$

离散正变换 $F(\mu, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\mu x/M+vy/N)}$

离散反变换 $f(x, y) = \frac{1}{MN} \sum_{\mu=0}^{M-1} \sum_{v=0}^{N-1} f(\mu, v) e^{j2\pi(\mu x/M+vy/N)}$

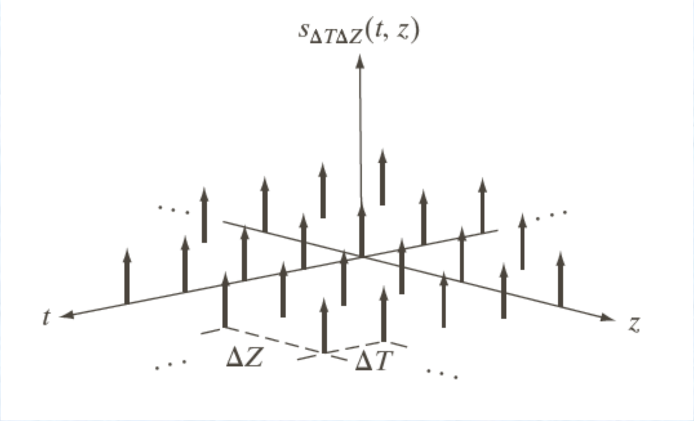


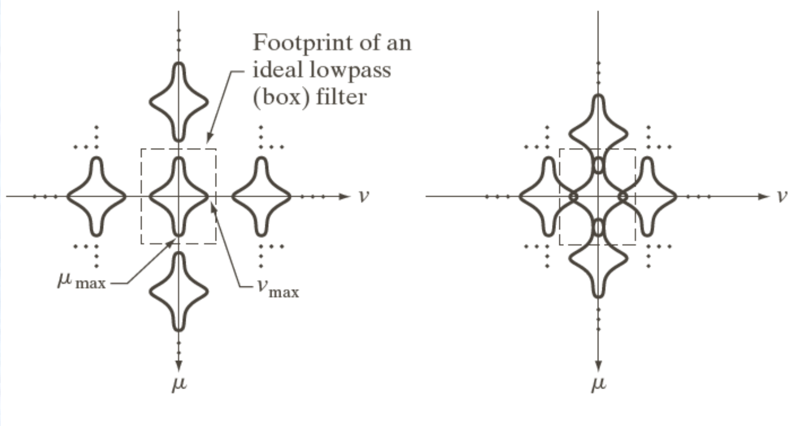
a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

二维带限函数 $F(\mu, v) = 0, |\mu| \geq \mu_{max}$ 且 $|v| \geq v_{max}$

二维取样定理 $\frac{1}{\Delta T} > 2\mu_{max}$ 且 $\frac{1}{\Delta Z} > 2v_{max}$





二维离散傅里叶变换性质

空间和频率间隔的关系

$$\Delta\mu = \frac{1}{M\Delta T} \quad , \quad \Delta v = \frac{1}{N\Delta Z}$$

平移和旋转

$$f(x,y)e^{j2\pi(\mu_0 x/M + v_0 y/N)} \leftrightarrow F(\mu - \mu_0, v - v_0)$$

$$f(x - x_0, y - y_0) \leftrightarrow F(\mu, v)e^{-j2\pi(x_0\mu/M + y_0 v/N)}$$

$$f(r, \theta + \theta_0) \leftrightarrow F(\omega, \varphi + \varphi_0)$$

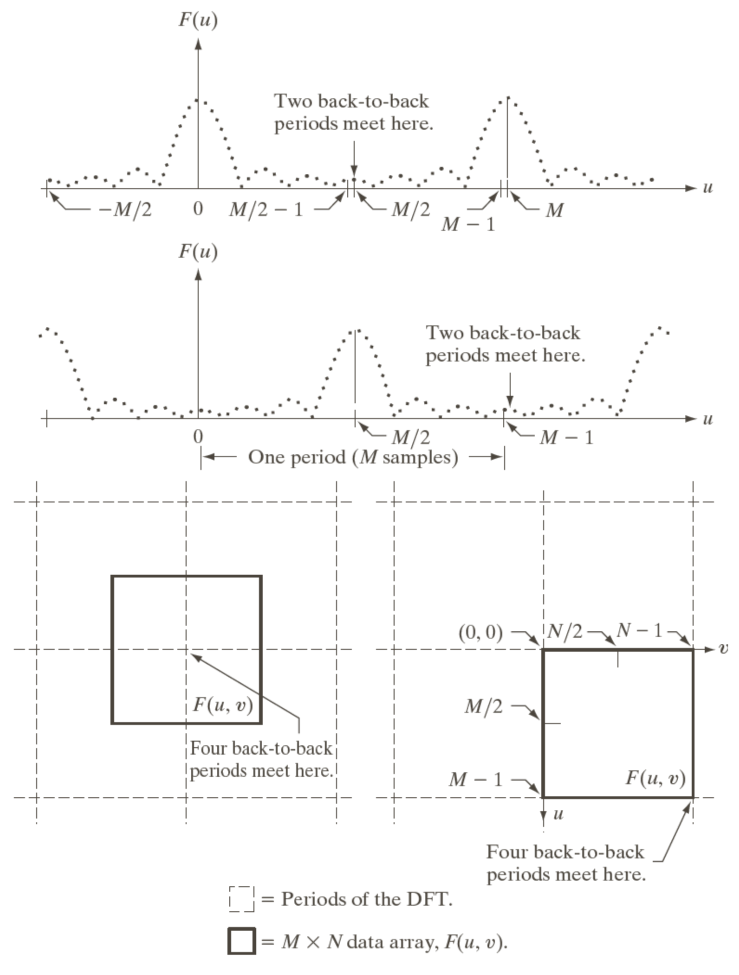
周期性

$$f(x,y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$$

$$F(u,v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$$

如何利用周期性和平移性来移动F(0,0)?

$$f(x,y)(-1)^{x+y} \leftrightarrow F(u - \frac{M}{2}, v - \frac{N}{2})$$


F(u)
Two back-to-back
periods meet here.
u
−M/2
0
M/2 − 1
M/2
M − 1
M
F(u)
Two back-to-back
periods meet here.
u
0
M/2
M − 1
One period (M samples)
F(u, v)
Four back-to-back
periods meet here.
(0, 0)
N/2
N − 1
v
M/2
M − 1
F(u, v)
u
Four back-to-back
periods meet here.
= Periods of the DFT.
= M × N data array, F(u, v).

对称性

任意函数可表示为一个奇函数和一个偶函数之和

奇函数：反对称　　偶函数：对称

离散情况下我们考虑的的是中点对称

$$W_e(x,y) = W_e(M-x, N-y)$$

$$W_o(x,y) = -W_o(M-x, N-y)$$

结论：奇离散函数的所有样本和为0

一个离散奇函数和一个离散偶函数乘积之和的累加为0

请参考书上的例子

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u, v)$ and $I(u, v)$ are the real and imaginary parts of $F(u, v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

二维DFT及其反变换的一些对称性质。书上例4.11是对上述性质的验证

九、傅里叶谱和相角

二维傅里叶变换复形式

$$F(u,v)=\left|F(u,v)\right|\bullet e^{j\theta(u,v)}$$

谱

$$\left|F(u,v)\right|=\sqrt{R^2(u,v)+I^2(u,v)}$$

相角

$$\theta(u,v)=tg^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

功率谱

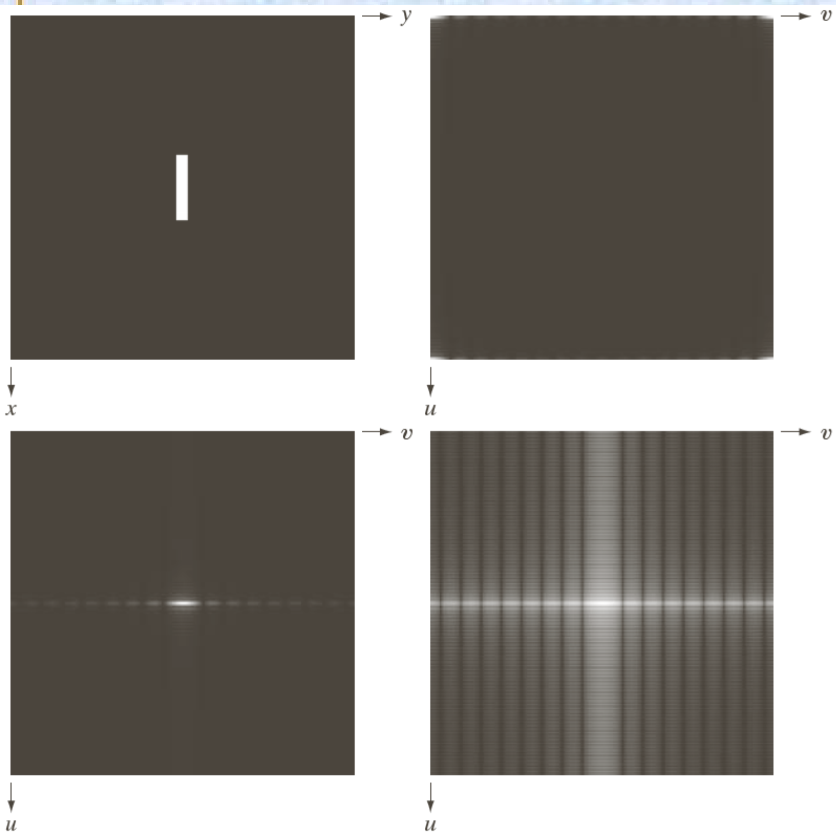
$$P(u,v)=\left|F(u,v)\right|^2=R(u,v)^2+I(u,v)^2$$

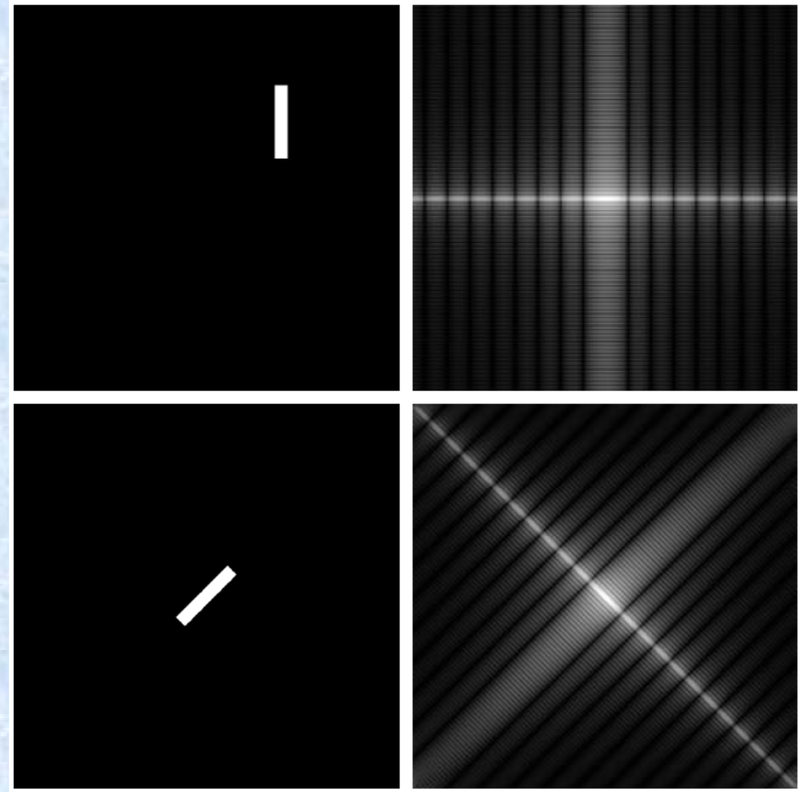
谱：偶对称　　　　　　相角：奇对称

为什么？

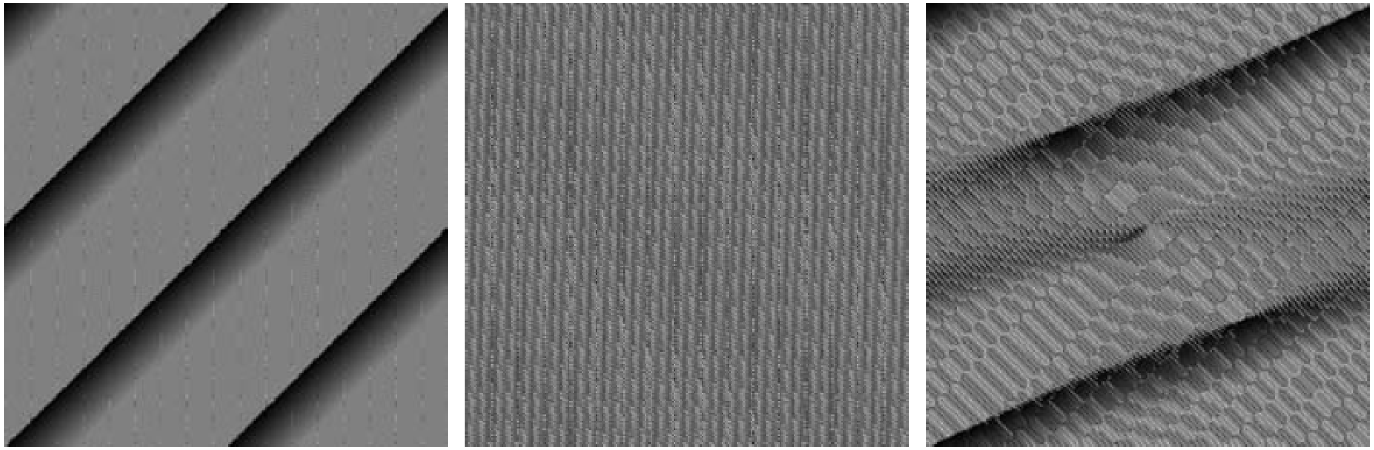
直流分量：$F(0,0)$处的值，即$(f(x,y)$的 平均值的MN倍

$$F(0,0)=MN\bar{f}(x,y)$$





谱的例子



a b c

FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

相角的例子

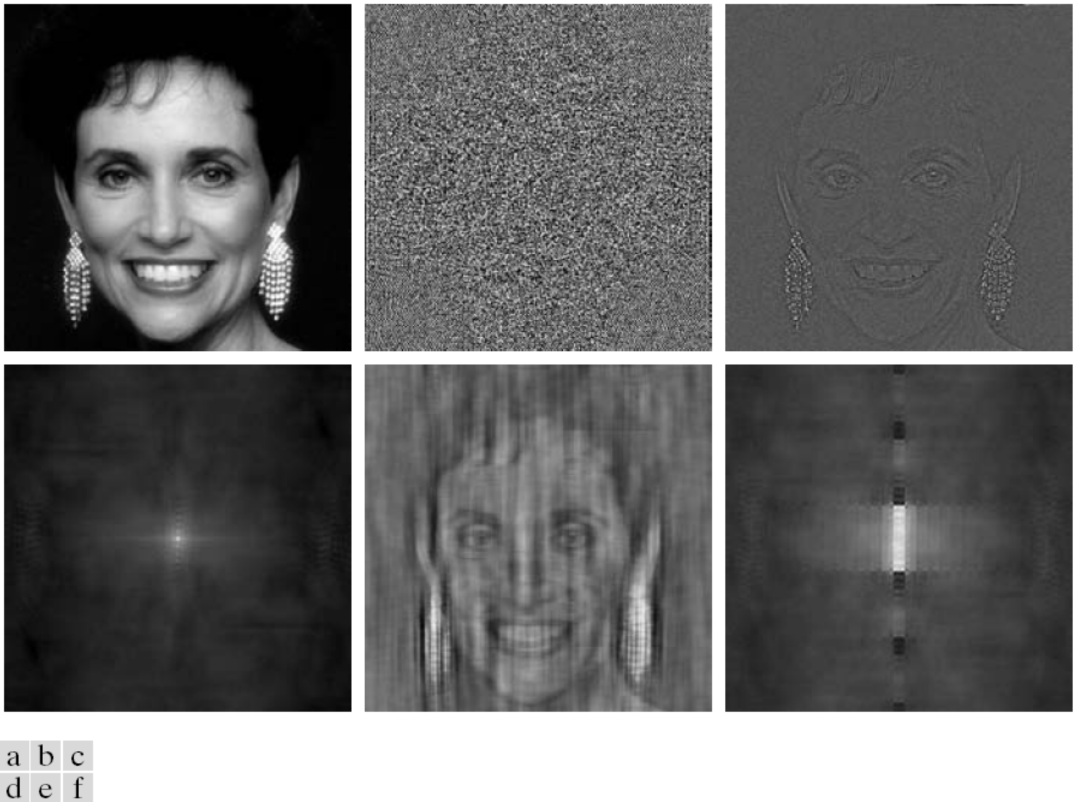


FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

利用不同的谱和相角对图像进行重建

可以总结两条结论：

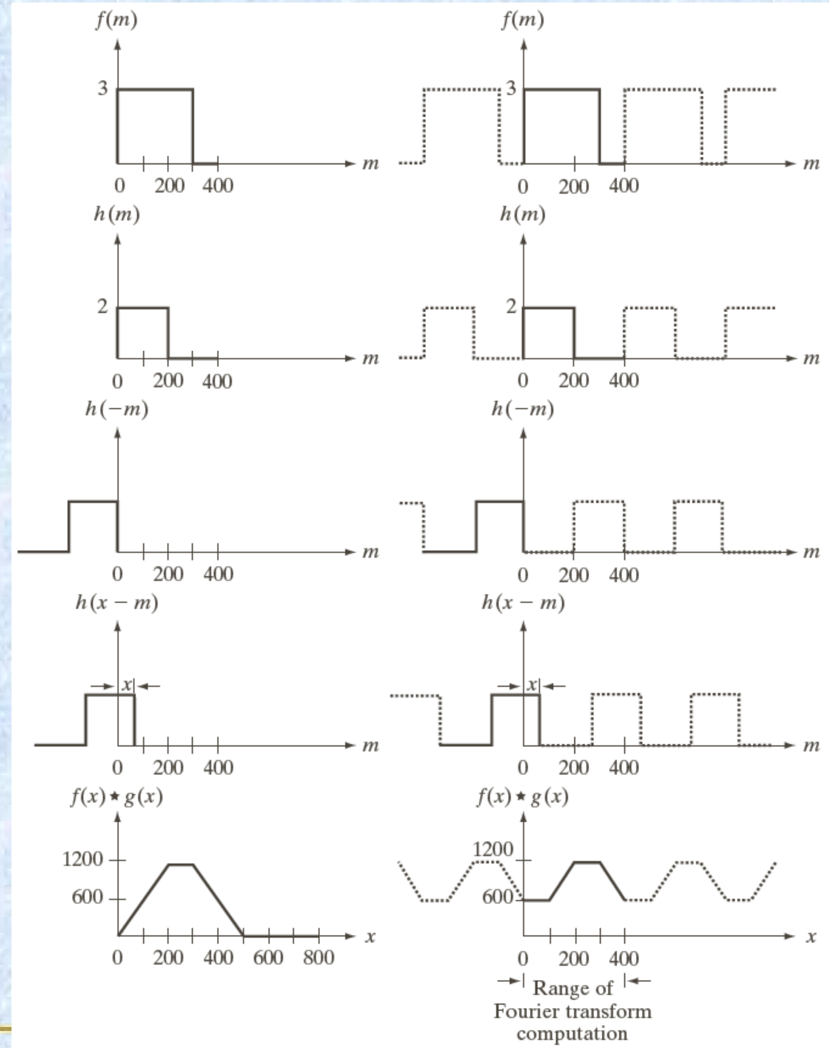
（1）幅度谱决定了一幅图像中含有的各种频率分量的多少

（2）相位谱决定了每一种频率分量在图像中的位置。

只要每一种频率分量保持在图像中的正确位置，那么图像的完整性就能得到很好的保持，这也就是为什么在信号或图像处理中通常只对幅度谱进行处理的原

十、卷积的缠绕问题

为什么会出现缠绕？



a f
b g
c h
d i
e j

FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

解决方案:

$$P \geq A + B - 1$$

补0到两个函数中，其中A和B分别是两个函数样本数。

推广到二维:

$$f_p(x,y) = \begin{cases} f(x,y), & 0 \leq x \leq A-1 \text{ 和 } 0 \leq y \leq B-1 \\ 0, & A \leq x \leq P \text{ 和 } B \leq y \leq Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y), & 0 \leq x \leq C-1 \text{ 和 } 0 \leq y \leq D-1 \\ 0, & C \leq x \leq P \text{ 和 } D \leq y \leq Q \end{cases}$$

其中

$$P \geq A + C - 1 \qquad Q \geq B + D - 1$$

填充解决了缠绕现象，但也带来了新的问题：
频率泄露

为什么？如何消除？建议使用高斯函数

十一、快速傅里叶变换

4点序列{2，3，3，2 DFT的计算复杂度

$$X[m]=\sum_{k=0}^{N-1}x[k]W_N^{km},\quad m=0,1,\cdots N-1$$

$$X[0]=2W_N^0+3W_N^0+3W_N^0+2W_N^0=10$$

$$X[1]=2W_N^0+3W_N^1+3W_N^2+2W_N^3=-1-j$$

$$X[2]=2W_N^0+3W_N^2+3W_N^4+2W_N^6=0$$

$$X[3]=2W_N^0+3W_N^3+3W_N^6+2W_N^9=-1+j$$

复数加法 **$N(N-1)$** 复数乘法 **N^2**

如何提高DFT的运算效率?

解决问题的思路

1. 将长序列DFT分解为短序列的DFT

2. 利用 W_N^{km} 的周期性、对称性、可约性。

W_N^{km} 的性质

1)周期性

$$W_N^{(k+N)m} = W_N^{k(m+N)} = W_N^{km}$$

2) 对称性

$$W_N^{mk+\frac{N}{2}} = -W_N^{mk} \qquad \left(W_N^{km}\right)^* = W_N^{-mk}$$

3)可约性

$$W_N^{mk} = W_{nN}^{nmk}$$

$$W_N^{mk} = W_{N/n}^{mk/n}, \quad N/n\text{为整数}$$

方法

将时域序列逐次分解为一组子序列，利用W_N^{km}的特性，由子序列的DFT来实现整个序列的DFT。

基2时间抽取(Decimation in time)FFT算法

$$x[k] \to \begin{cases} x[2r] \\ x[2r+1] \end{cases} \quad r = 0,1,\cdots\frac{N}{2}-1$$

基2频率抽取(Decimation in frequency)FFT算法

$$X[m] \to \begin{cases} X[2m] \\ X[2m+1] \end{cases}$$

下面介绍DIF—FFT算法。

设序列x(n)的长度为N，且满足

$N = 2^M$，M 为自然数

按n的奇偶把x(n)分解为两个N/2点的子序列

$$x_1(r) = x(2r), \quad r = 0,1,\cdots\frac{N}{2}-1$$

$$x_2(r) = x(2r+1), \quad r = 0,1,\cdots\frac{N}{2}-1$$

则x(n)的DFT为

$$
\begin{aligned}
X(k) &= \sum_{n=} x(n)W_N^{kn} + \sum_{n=} x(n)W_N^{kn} \\
&= \sum_{r=0}^{N/2-1} x(2r)W_N^{2kr} + \sum_{r=0}^{N/2-1} x(2r+1)W_N^{k(2r+1)} \\
&= \sum_{r=0}^{N/2-1} W_N^{2kr} x_1(r) + W_N^{k} \sum_{r=0}^{N/2-1} x_2(r)W_N^{2kr}
\end{aligned}
$$

由于

$$W_N^{2kr} = W_{N/2}^{kr}$$

所以

$$X(k) = \sum_{r=0}^{N/2-1} x_1(r)W_{N/2}^{kr} + W_N^k \sum_{r=0}^{N/2-1} x_2(r)W_{N/2}^{kr} = X_1(k) + W_N^k X_2(k)$$

其中$X_1(k)$和$X_2(k)$分别为$x_1(r)$和$x_2(r)$的N/2点DFT，即

$$X_1(k)=\sum_{r=0}^{N/2-1} x_1(r)W_{N/2}^{kr}=DFT[x_1(r)]$$

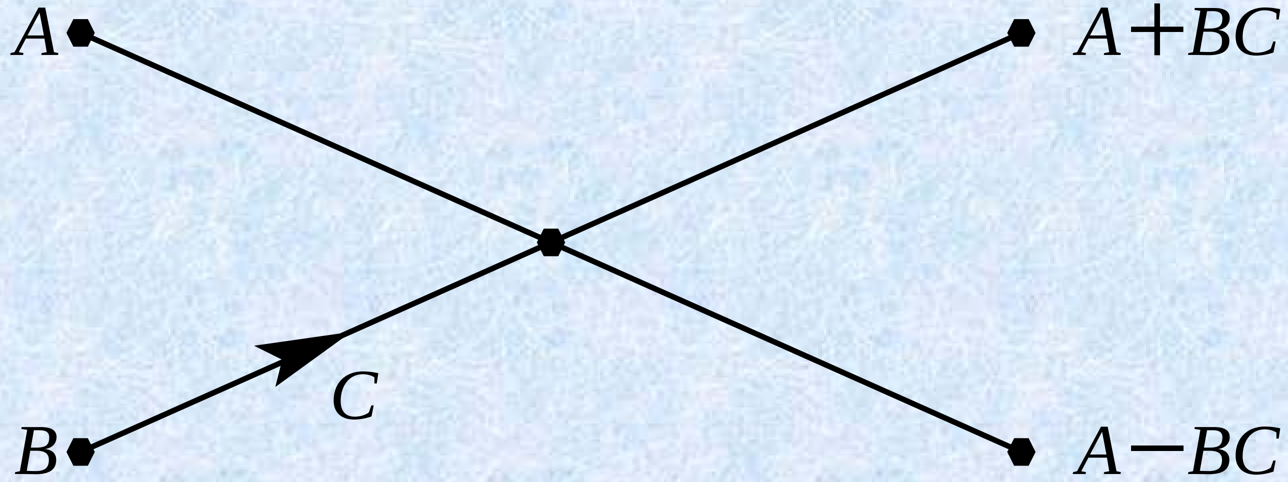
$$X_2(k)=\sum_{r=0}^{N/2-1} x_2(r)W_{N/2}^{kr}=DFT[x_2(r)]$$

由于$X_1(k)$和$X_2(k)$均以N/2为周期，且

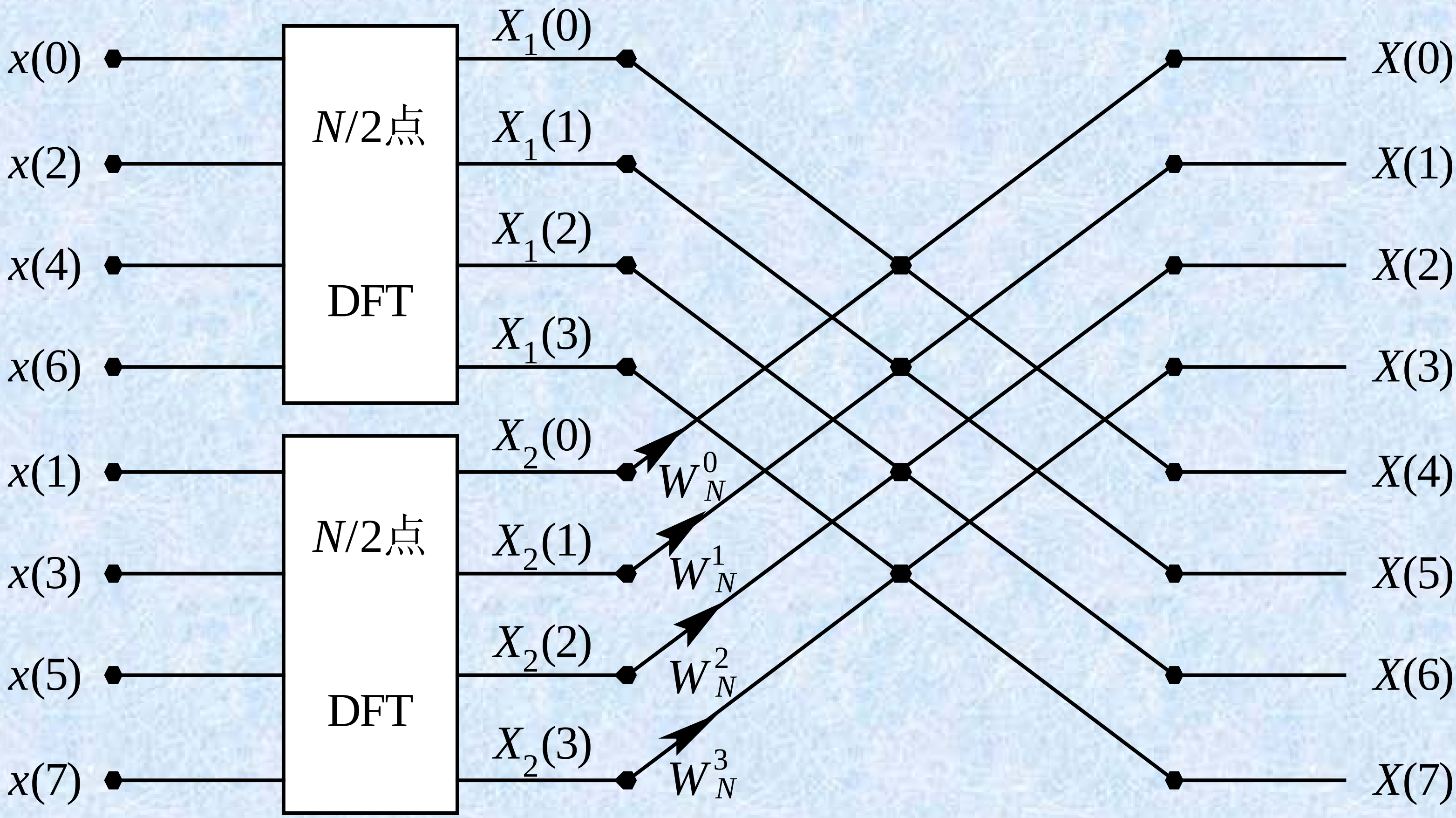
$W_N^{k+\frac{N}{2}}=-W_N^k$ ，所以X(k)又可表示为

$$X(k)=X_1(k)+W_N^k X_2(k) \quad k=0,1,\cdots\frac{N}{2}-1$$

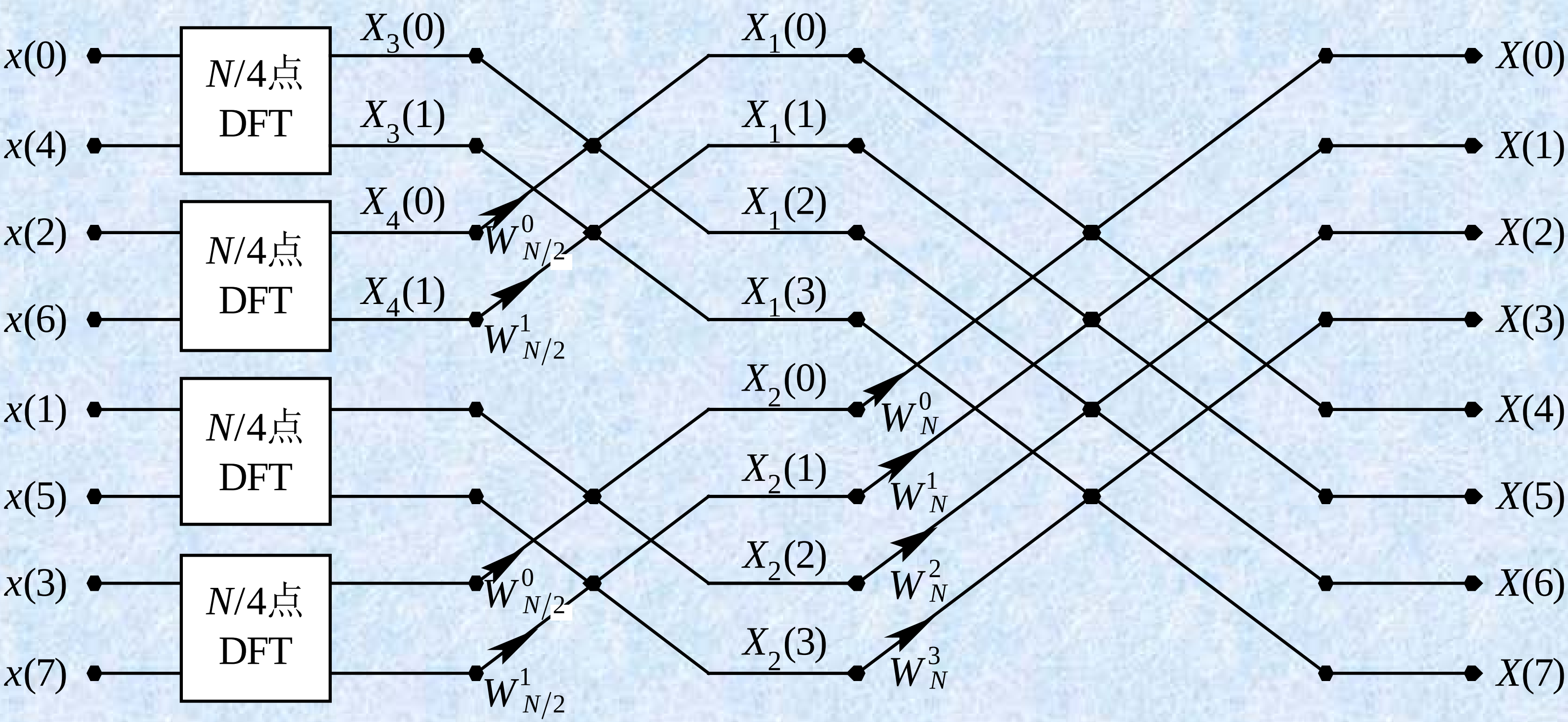
$$X(k+\frac{N}{2})=X_1(k)-W_N^k X_2(k) \quad k=0,1,\cdots\frac{N}{2}-1$$



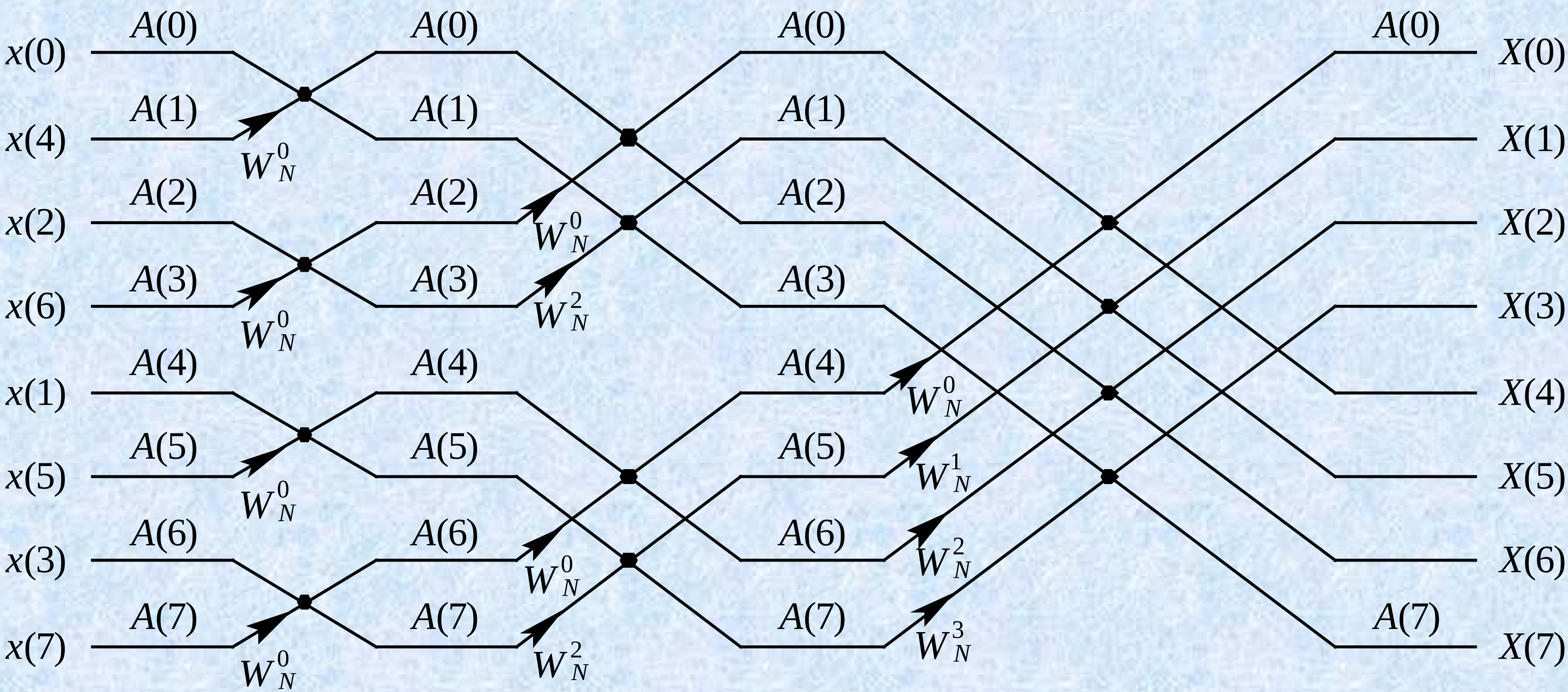
蝶形运算符号



N点DFT的一次时域抽取分解图(N=8)



N点DFT的第二次时域抽取分解图(N=8)



N点DIT—FFT运算流图(N=8)

DIT—FFT算法与直接计算DFT运算量的比较

每一级运算都需要N/2次复数乘和N次复数加(每个蝶形需要两次复数加法)。所以，M级运算总共需要的复数乘次数为

$$C_M(2) = \frac{N}{2} \cdot M = \frac{N}{2} \log_2 N$$

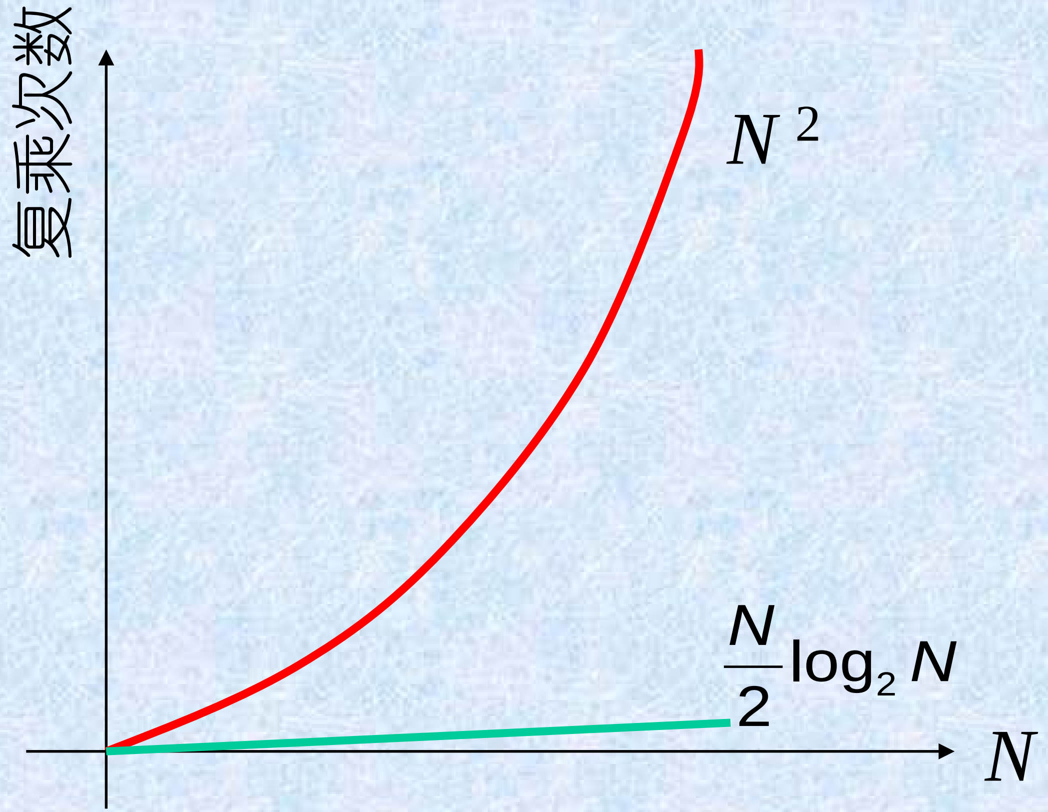
复数加次数为

$$C_A(2) = N \cdot M = N \log_2 N$$

例如，$N=2^{10}=1024$时

$$\frac{N^2}{(N/2)\log_2 N} = \frac{1048576}{5120} = 204.8$$

复乘次数 $\frac{N}{2}\log_2 N$



FFT算法与直接计算DFT所需乘法次数的比较曲线

十二、总结

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi(ux/M+vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u, v)\, e^{j2\pi(ux/M+vy/N)}$
3) Polar representation	$F(u, v) = \lvert F(u, v)\rvert e^{j\phi(u,v)}$
4) Spectrum	$\lvert F(u, v)\rvert = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$ $R = \text{Real}(F); \quad I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
6) Power spectrum	$P(u, v) = \lvert F(u, v)\rvert^2$
7) Average value	$\overline{f}(x, y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

(*Continued*)

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u,v) = F(u+k_1M, v) = F(u, v+k_2N)$ $= F(u+k_1M, v+k_2N)$ $f(x,y) = f(x+k_1M, y) = f(x, y+k_2N)$ $= f(x+k_1M, y+k_2N)$
9) Convolution	$f(x,y) \bigstar h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$
10) Correlation	$f(x,y) \star h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m, y+n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

(Continued)

	Name	DFT Pairs
7)	Correlation theorem†	$f(x,y) \star h(x,y) \Leftrightarrow F^*(u,v)\,H(u,v)$ $f^*(x,y)h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$
8)	Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$
9)	Rectangle	$\text{rect}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10)	Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $j\dfrac{1}{2}\left[\delta(u+Mu_0, v+Nv_0) - \delta(u-Mu_0, v-Nv_0)\right]$
11)	Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $\dfrac{1}{2}\left[\delta(u+Mu_0, v+Nv_0) + \delta(u-Mu_0, v-Nv_0)\right]$
	The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12)	*Differentiation* (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)	$\left(\dfrac{\partial}{\partial t}\right)^m \left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m (j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu)$; $\dfrac{\partial^n f(t,z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13)	*Gaussian*	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

†Assumes that the functions have been extended by zero padding. Convolution and correlation are associative, commutative, and distributive.