Mathematical Tools

Topics:

- Approximation
- Convex Optimization
- Integration
- Symbolic Mathematics

Approximation

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

In [2]:

```
def f(x):
    return np.sin(x) + 0.5 * x
```

In [3]:

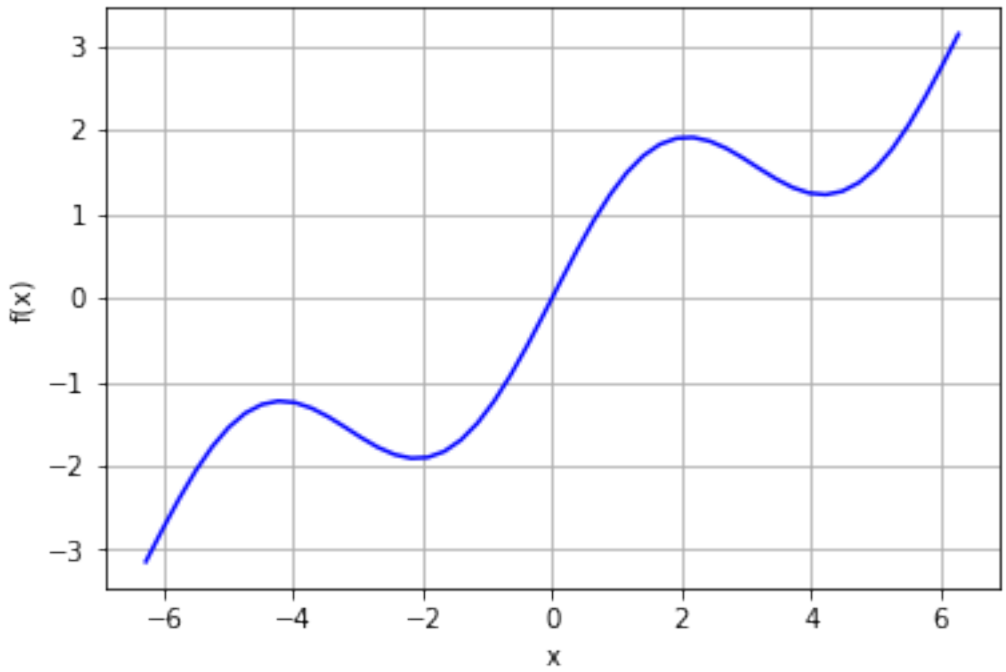
```
x = np.linspace(-2 * np.pi, 2 * np.pi, 50)
```

In [4]:

```
plt.plot(x, f(x), 'b')
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[4]:

```
Text(0,0.5,'f(x)')
```



Regression

$$\min_{\alpha_1,\ldots,\alpha_D} \frac{1}{I} \sum_{d=1}^{I} \left(y_i - \sum_{d=1}^{D} \alpha_d \cdot b_d\left(x_i\right) \right)^2$$

Use Monomial Basis Functions

In [5]:

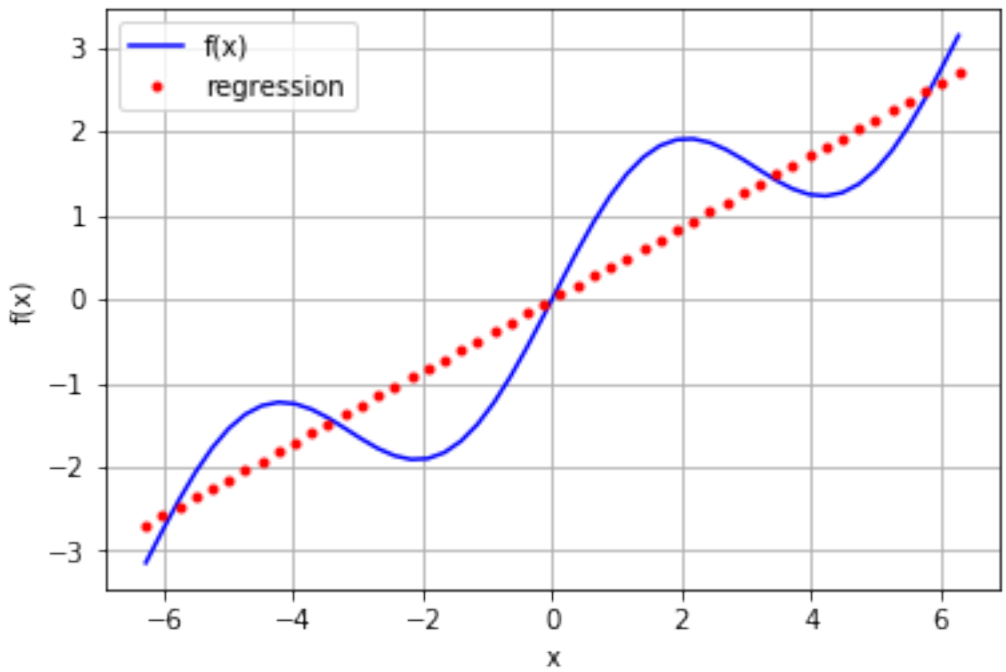
```
reg = np.polyfit(x, f(x), deg=1)
ry = np.polyval(reg, x)
```

In [6]:

```
plt.plot(x, f(x), 'b', label='f(x)')
plt.plot(x, ry, 'r.', label='regression')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[6]:

```
Text(0,0.5,'f(x)')
```



In [7]:

```
np.sum((f(x) - ry) ** 2) / len(x)
```

Out[7]:

```
0.4206477371868664
```

Use Individual Basis Functions

In [8]:

```
matrix = np.zeros((4, len(x)))
matrix[3, :] = x ** 3
matrix[2, :] = x ** 2
matrix[1, :] = x
matrix[0, :] = 1
```

In [9]:

```
reg = np.linalg.lstsq(matrix.T, f(x))[0]
```

```
/Users/chuang/Applications/miniconda3/envs/PyQuant/lib/python3.5/s
ite-packages/ipykernel_launcher.py:1: FutureWarning: `rcond` param
eter will change to the default of machine precision times ``max(M
, N)`` where M and N are the input matrix dimensions.
To use the future default and silence this warning we advise to pa
ss `rcond=None`, to keep using the old, explicitly pass `rcond=-1`
.
  """Entry point for launching an IPython kernel.
```

In [10]:

```
reg
```

Out[10]:

```
array([ 1.52685368e-14,  5.62777448e-01, -1.11022302e-15, -5.43553
615e-03])
```

In [11]:

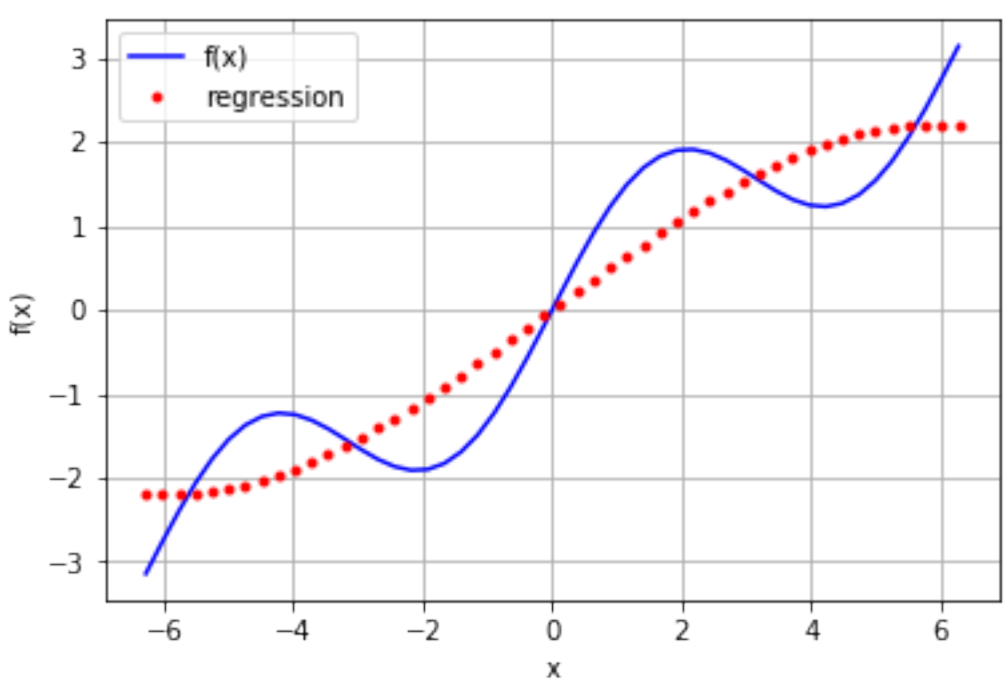
```
ry = np.dot(reg, matrix)
```

In [12]:

```
plt.plot(x, f(x), 'b', label='f(x)')
plt.plot(x, ry, 'r.', label='regression')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[12]:

```
Text(0,0.5,'f(x)')
```



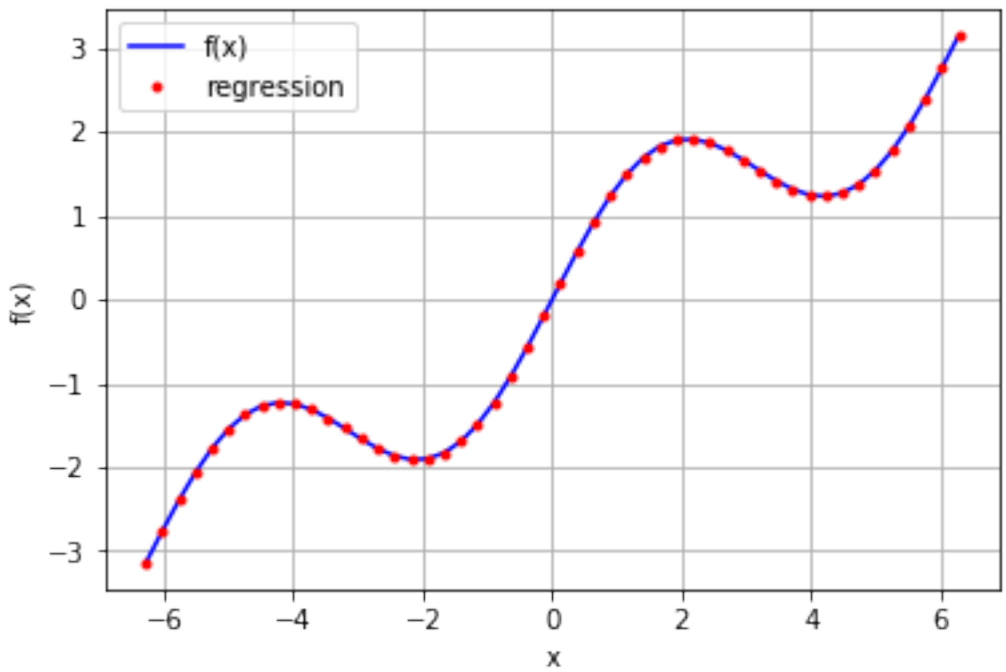
In [13]:

```
matrix[3, :] = np.sin(x)
reg = np.linalg.lstsq(matrix.T, f(x))[0]
ry = np.dot(reg, matrix)
plt.plot(x, f(x), 'b', label='f(x)')
plt.plot(x, ry, 'r.', label='regression')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

```
/Users/chuang/Applications/miniconda3/envs/PyQuant/lib/python3.5/s
ite-packages/ipykernel_launcher.py:2: FutureWarning: `rcond` param
eter will change to the default of machine precision times ``max(M
, N)`` where M and N are the input matrix dimensions.
To use the future default and silence this warning we advise to pa
ss `rcond=None`, to keep using the old, explicitly pass `rcond=-1`
.
```

Out[13]:

```
Text(0,0.5,'f(x)')
```



In [14]:

```
np.allclose(f(x), ry)
```

Out[14]:

```
True
```

In [15]:

```
np.sum((f(x) - ry) ** 2) / len(x)
```

Out[15]:

```
1.8541312760604798e-31
```

In [16]:

```
reg
```

Out[16]:

```
array([9.26243218e-17, 5.00000000e-01, 0.00000000e+00, 1.00000000e
+00])
```

Noisy Data

In [17]:

```
xn = np.linspace(-2 * np.pi, 2 * np.pi, 50)
xn = xn + 0.15 * np.random.standard_normal(len(xn))
yn = f(xn) + 0.25 * np.random.standard_normal(len(xn))
```

In [18]:

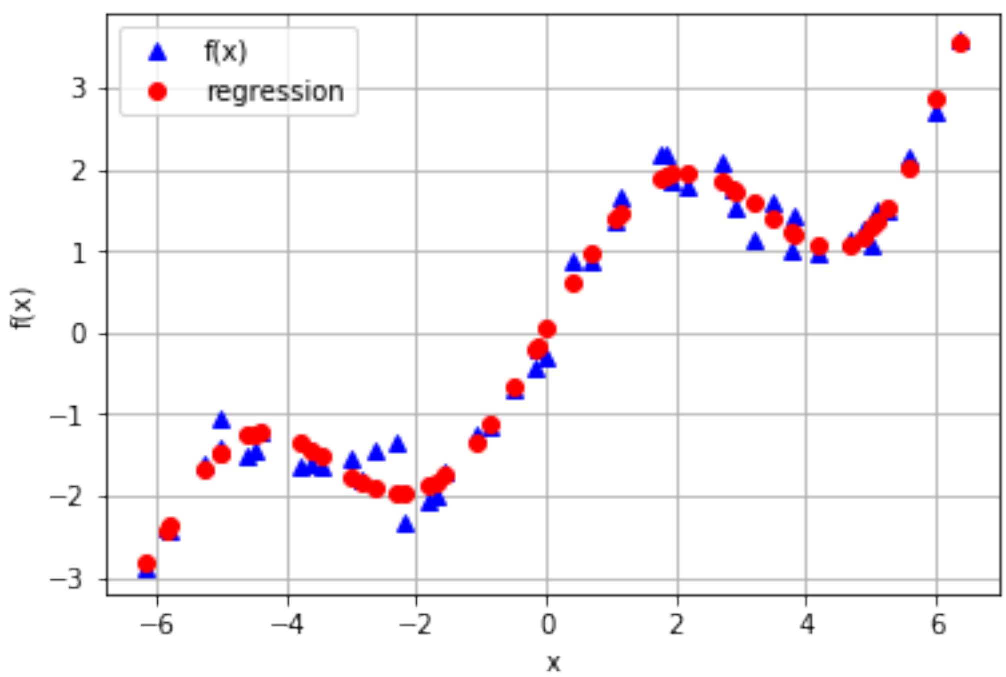
```
reg = np.polyfit(xn, yn, 7)
ry = np.polyval(reg, xn)
```

In [19]:

```
plt.plot(xn, yn, 'b^', label='f(x)')
plt.plot(xn, ry, 'ro', label='regression')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[19]:

```
Text(0,0.5,'f(x)')
```



Unsorted Data

In [20]:

```
xu = np.random.rand(50) * 4 * np.pi - 2 * np.pi
yu = f(xu)
```

In [21]:

```
print(xu[:10].round(2))
print(yu[:10].round(2))
```

```
[ 0.44  4.04 -4.19 -2.21 -5.72  1.24 -5.16  2.4  -4.54  2.85]
[ 0.64  1.24 -1.23 -1.91 -2.33  1.56 -1.67  1.88 -1.29  1.71]
```

In [22]:

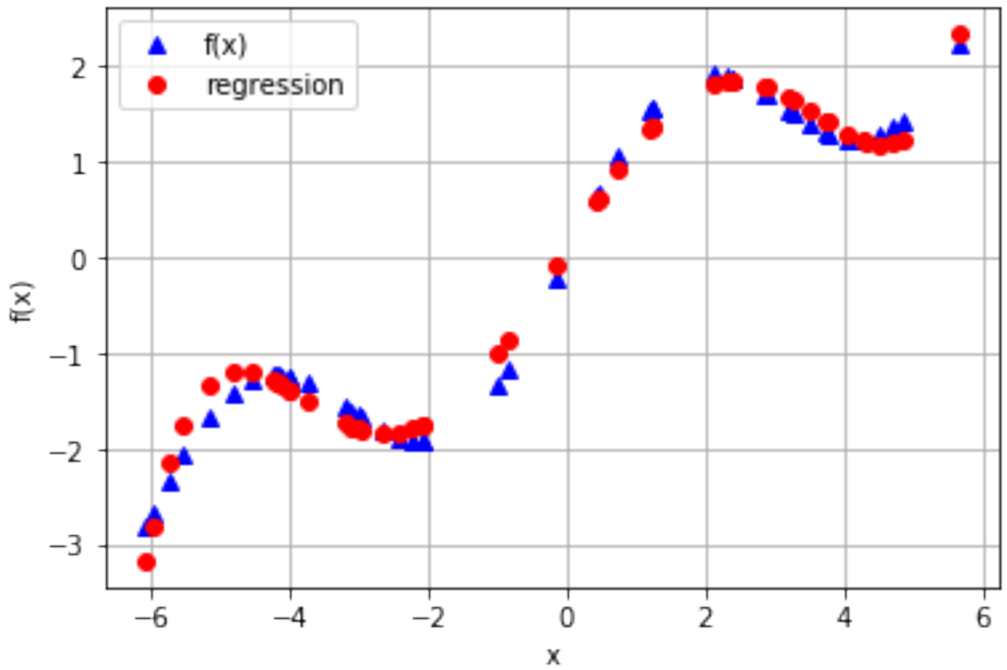
```
reg = np.polyfit(xu, yu, 5)
ry = np.polyval(reg, xu)
```

In [23]:

```
plt.plot(xu, yu, 'b^', label='f(x)')
plt.plot(xu, ry, 'ro', label='regression')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[23]:

```
Text(0,0.5,'f(x)')
```



Multiple Dimensions

In [24]:

```
def fm(x, y):
    return np.sin(x) + 0.25 * x + np.sqrt(y) + 0.05 * y ** 2
```

In [25]:

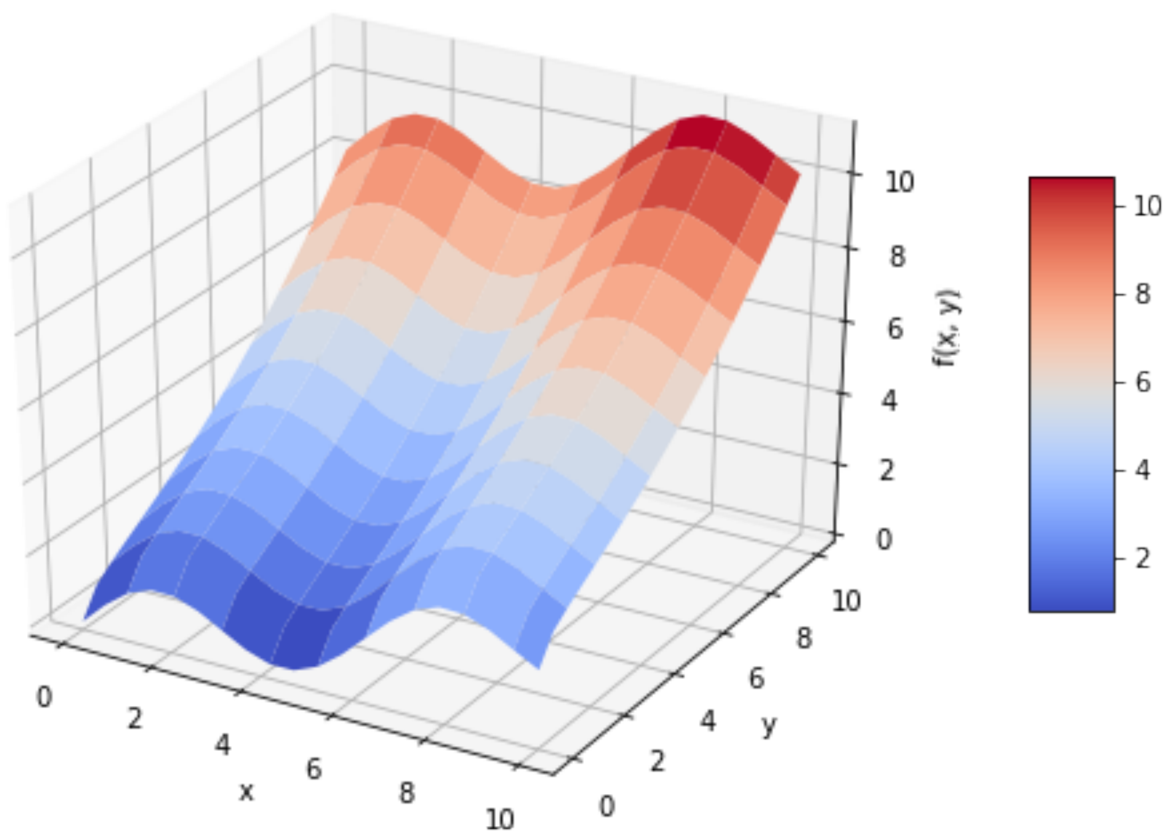
```
x = np.linspace(0, 10, 20)
y = np.linspace(0, 10, 20)
X, Y = np.meshgrid(x, y)
Z = fm(X, Y)
x = X.flatten()
y = Y.flatten()
```

In [26]:

```
from mpl_toolkits.mplot3d import Axes3D
import matplotlib as mpl

fig = plt.figure(figsize=(9, 6))
ax = fig.gca(projection='3d')
surf = ax.plot_surface(X, Y, Z, rstride=2, cstride=2,
                       cmap=mpl.cm.coolwarm,
                       lw=0.5, antialiased=True)
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('f(x, y)')
fig.colorbar(surf, shrink=0.5, aspect=5)
```

Out[26]:

```
<matplotlib.colorbar.Colorbar at 0x112773390>
```



In [27]:

```
matrix = np.zeros((len(x), 7))
matrix[:, 6] = np.sqrt(y)
matrix[:, 5] = np.sin(x)
matrix[:, 4] = y ** 2
matrix[:, 3] = x ** 2
matrix[:, 2] = y
matrix[:, 1] = x
matrix[:, 0] = 1
```

In [28]:

```
import statsmodels.api as sm
model = sm.OLS(fm(x, y), matrix).fit()
```

```
/Users/chuang/Applications/miniconda3/envs/PyQuant/lib/python3.5/i
mportlib/_bootstrap.py:222: RuntimeWarning: numpy.dtype size chang
ed, may indicate binary incompatibility. Expected 96, got 88
  return f(*args, **kwds)
/Users/chuang/Applications/miniconda3/envs/PyQuant/lib/python3.5/i
mportlib/_bootstrap.py:222: RuntimeWarning: numpy.dtype size chang
ed, may indicate binary incompatibility. Expected 96, got 88
  return f(*args, **kwds)
/Users/chuang/Applications/miniconda3/envs/PyQuant/lib/python3.5/i
mportlib/_bootstrap.py:222: RuntimeWarning: numpy.dtype size chang
ed, may indicate binary incompatibility. Expected 96, got 88
  return f(*args, **kwds)
```

In [29]:

```
model.summary()
```

Out[29]:

OLS Regression Results

Dep. Variable:	y	**R-squared:**	1.000
Model:	OLS	**Adj. R-squared:**	1.000
Method:	Least Squares	**F-statistic:**	1.953e+30
Date:	Sun, 19 Aug 2018	**Prob (F-statistic):**	0.00
Time:	15:17:03	**Log-Likelihood:**	12175.
No. Observations:	400	**AIC:**	-2.434e+04
Df Residuals:	393	**BIC:**	-2.431e+04
Df Model:	6		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	-4.163e-15	3.77e-15	-1.104	0.270	-1.16e-14	3.25e-15
x1	0.2500	9.79e-16	2.55e+14	0.000	0.250	0.250
x2	-4.247e-15	3.19e-15	-1.330	0.184	-1.05e-14	2.03e-15
x3	-3.4e-16	9.41e-17	-3.613	0.000	-5.25e-16	-1.55e-16
x4	0.0500	1.66e-16	3.01e+14	0.000	0.050	0.050
x5	1.0000	1.17e-15	8.53e+14	0.000	1.000	1.000
x6	1.0000	6.25e-15	1.6e+14	0.000	1.000	1.000

Omnibus:	31.276	**Durbin-Watson:**	0.154
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	34.656
Skew:	0.692	**Prob(JB):**	2.98e-08
Kurtosis:	2.593	**Cond. No.**	587.

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

In [30]:

```
a = model.params
def reg_func(a, x, y):
    f6 = a[6] * np.sqrt(y)
    f5 = a[5] * np.sin(x)
    f4 = a[4] * y ** 2
    f3 = a[3] * x ** 2
    f2 = a[2] * y
    f1 = a[1] * x
    f0 = a[0] * 1
    return f6 + f5 + f4 + f3 + f2 + f1 + f0
```

In [31]:

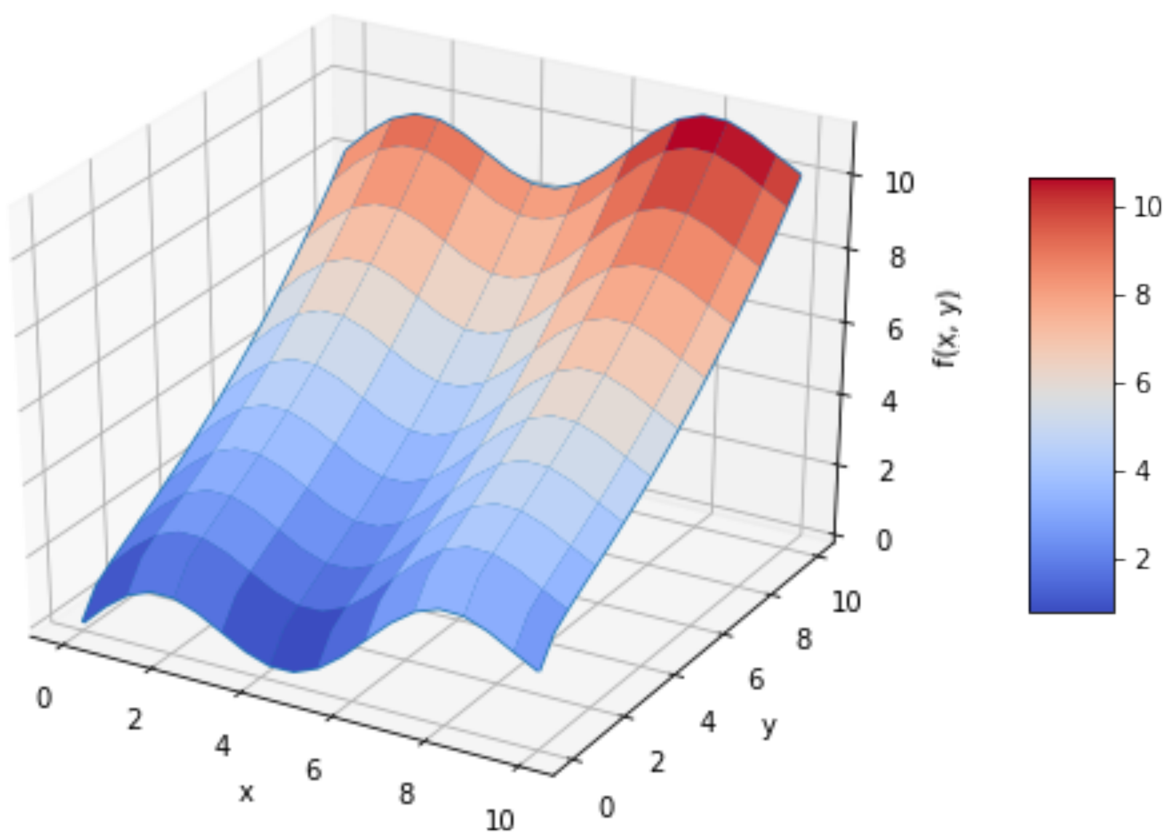
```
RZ = reg_func(a, X, Y)
```

In [32]:

```
fig = plt.figure(figsize=(9, 6))
ax = fig.gca(projection='3d')
surf1 = ax.plot_surface(X, Y, Z, rstride=2, cstride=2,
                        cmap=mpl.cm.coolwarm,
                        lw=0.5, antialiased=True)
surf2 = ax.plot_wireframe(X, Y, RZ, rstride=2, cstride=2,
                          label='regression')
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('f(x, y)')
fig.colorbar(surf, shrink=0.5, aspect=5)
```

Out[32]:

```
<matplotlib.colorbar.Colorbar at 0x1c166c5080>
```



Interpolation

In [33]:

```
import scipy.interpolate as spi
```

In [34]:

```
x = np.linspace(-2 * np.pi, 2 * np.pi, 25)
```

In [35]:

```
def f(x):
    return np.sin(x) + 0.5 * x
```

In [36]:

```
ipo = spi.splrep(x, f(x), k=1)
```

In [37]:

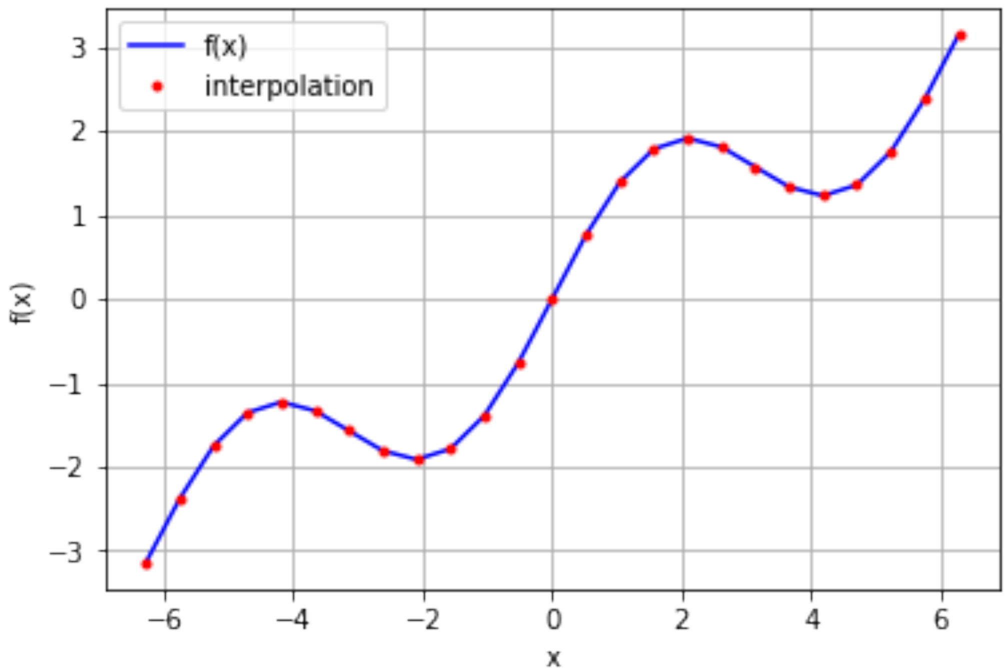
```
iy = spi.splev(x, ipo)
```

In [38]:

```
plt.plot(x, f(x), 'b', label='f(x)')
plt.plot(x, iy, 'r.', label='interpolation')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[38]:

```
Text(0,0.5,'f(x)')
```



In [39]:

```
np.allclose(f(x), iy)
```

Out[39]:

```
True
```

In [40]:

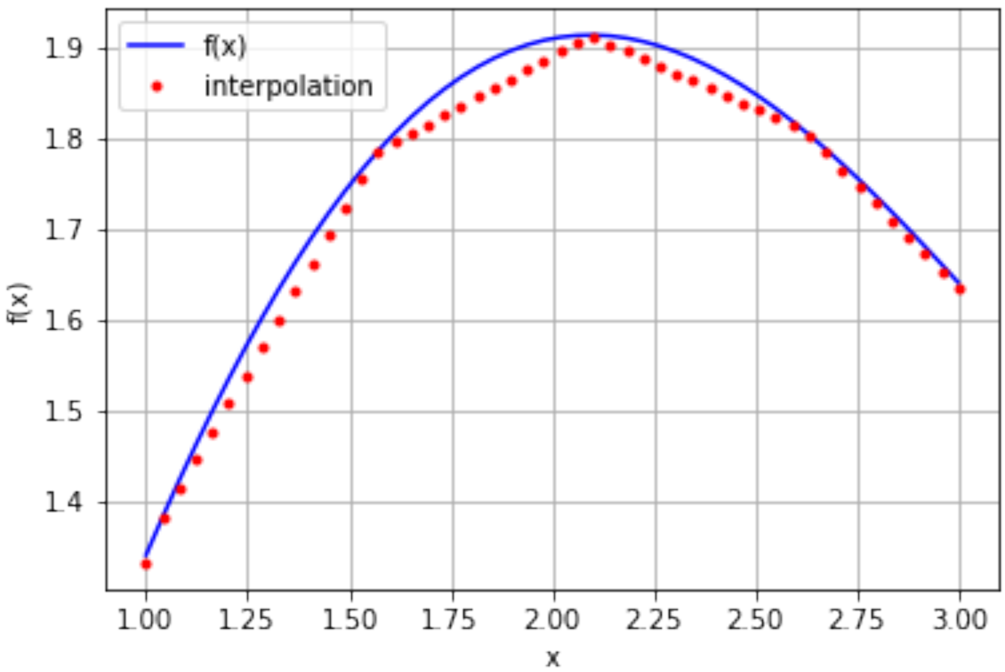
```
xd = np.linspace(1., 3., 50)
iyd = spi.splev(xd, ipo)
```

In [41]:

```
plt.plot(xd, f(xd), 'b', label='f(x)')
plt.plot(xd, iyd, 'r.', label='interpolation')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[41]:

```
Text(0,0.5,'f(x)')
```



In [42]:

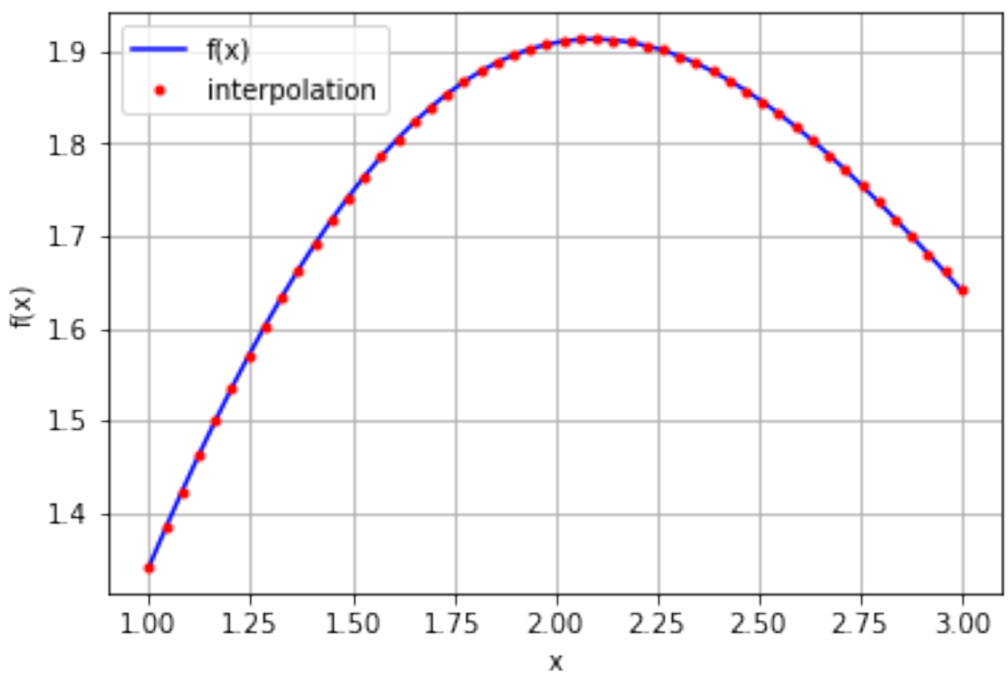
```
ipo = spi.splrep(x, f(x), k=3)
iyd = spi.splev(xd, ipo)
```

In [43]:

```
plt.plot(xd, f(xd), 'b', label='f(x)')
plt.plot(xd, iyd, 'r.', label='interpolation')
plt.legend(loc=0)
plt.grid(True)
plt.xlabel('x')
plt.ylabel('f(x)')
```

Out[43]:

```
Text(0,0.5,'f(x)')
```



In [44]:

```
np.allclose(f(xd), iyd)
```

Out[44]:

```
False
```

In [45]:

```
np.sum((f(xd) - iyd) ** 2) / len(xd)
```

Out[45]:

```
1.1349319851436892e-08
```

Convex Opitimization

In [46]:

```
def fm(x, y):
    return np.sin(x) + 0.05 * x ** 2 + np.sin(y) + 0.05 * y ** 2
```

In [47]:

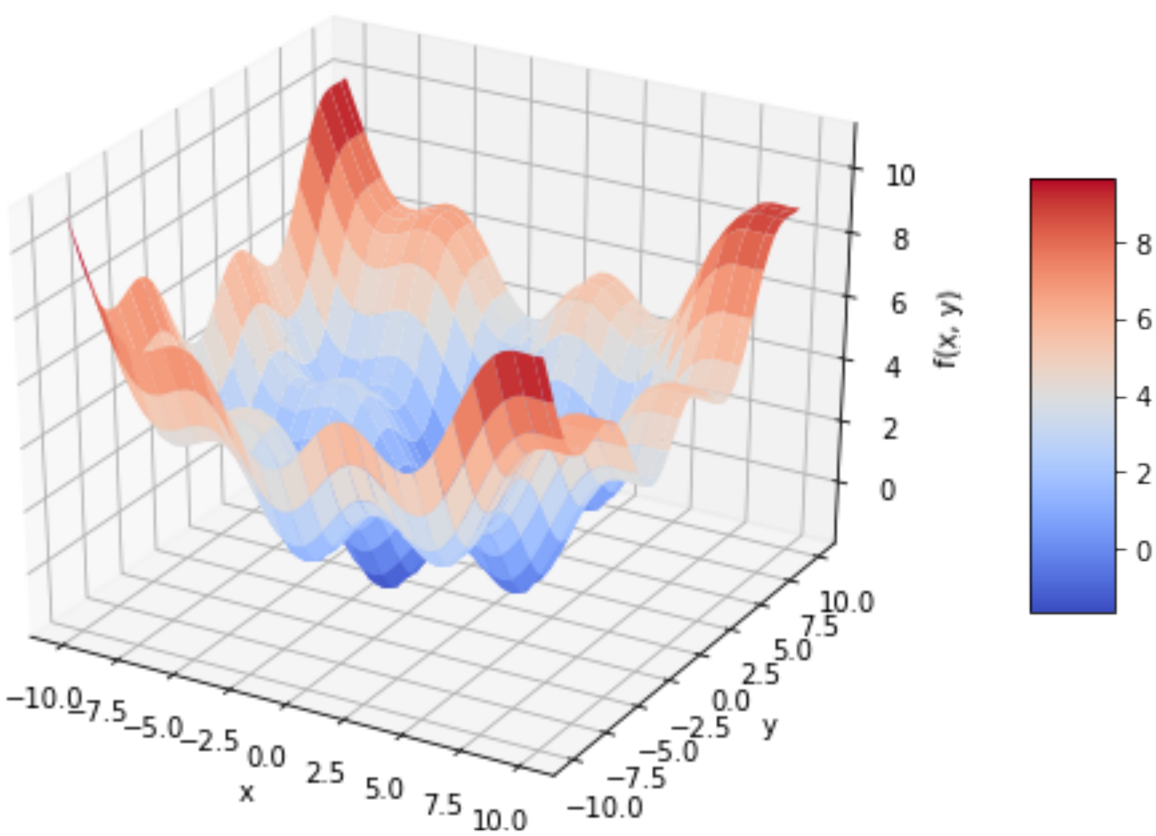
```
x = np.linspace(-10, 10, 50)
y = np.linspace(-10, 10, 50)
X, Y = np.meshgrid(x, y)
Z = fm(X, Y)
```

In [48]:

```
fig = plt.figure(figsize=(9, 6))
ax = fig.gca(projection='3d')
surf = ax.plot_surface(X, Y, Z, rstride=2, cstride=2,
                      cmap=mpl.cm.coolwarm,
                      lw=0.5, antialiased=True)
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('f(x, y)')
fig.colorbar(surf, shrink=0.5, aspect=5)
```

Out[48]:

```
<matplotlib.colorbar.Colorbar at 0x1c16aa44a8>
```



In [49]:

```
import scipy.optimize as spo
```

Global Optimization

In [50]:

```
def fo(grid):
    x, y = grid
    z = np.sin(x) + 0.05 * x ** 2 + np.sin(y) + 0.05 * y ** 2
    if output == True:
        print(x, y, z)
    return z
```

In [51]:

```
output = True
grid = (slice(-10, 10.1, 5), slice(-10, 10.1, 5))
spo.brute(fo, grid, finish=None)
```

```
-10.0 -10.0 11.088042221778739
-10.0 -10.0 11.088042221778739
-10.0 -5.0 7.752945385552508
-10.0 0.0 5.5440211108893696
-10.0 5.0 5.835096836226231
-10.0 10.0 10.0
-5.0 -10.0 7.752945385552509
-5.0 -5.0 4.417848549326277
-5.0 0.0 2.2089242746631386
-5.0 5.0 2.5
-5.0 10.0 6.664903163773769
0.0 -10.0 5.5440211108893696
0.0 -5.0 2.2089242746631386
0.0 0.0 0.0
0.0 5.0 0.29107572533686155
0.0 10.0 4.4559788891106304
5.0 -10.0 5.835096836226231
5.0 -5.0 2.5
5.0 0.0 0.29107572533686155
5.0 5.0 0.5821514506737231
5.0 10.0 4.747054614447491
10.0 -10.0 10.0
10.0 -5.0 6.664903163773769
10.0 0.0 4.4559788891106304
10.0 5.0 4.747054614447492
10.0 10.0 8.911957778221261
```

Out[51]:

```
array([0., 0.])
```

In [52]:

```
output = False
grid = (slice(-10, 10.1, 0.1), slice(-10, 10.1, 0.1))
opt1 = spo.brute(fo, grid, finish=None)

opt1
```

Out[52]:

```
array([-1.4, -1.4])
```

In [53]:

```
fm(opt1[0], opt1[1])
```

Out[53]:

```
-1.7748994599769203
```

Local Optimization

In [54]:

```
output = True
opt2 = spo.fmin(fo, opt1, xtol=0.001, ftol=0.001, maxiter=15, maxfun=20)

opt2
```

```
-1.4000000000000004 -1.4000000000000004 -1.7748994599769203
-1.4700000000000004 -1.4000000000000004 -1.774329079766041
-1.4000000000000004 -1.4700000000000004 -1.7743290797660412
-1.3300000000000003 -1.4700000000000006 -1.7695827276986251
-1.4350000000000005 -1.4175000000000004 -1.7756403866946224
-1.4350000000000005 -1.3475000000000004 -1.7722175692069706
-1.4087500000000004 -1.4393750000000005 -1.7754569915832503
-1.4437500000000005 -1.4568750000000006 -1.7751135039067365
-1.4328125000000007 -1.4426562500000006 -1.7755861787931349
-1.4590625000000008 -1.4207812500000006 -1.7751589553124218
-1.4213281250000005 -1.4347265625000005 -1.7756764959744498
-1.4235156250000003 -1.4095703125000005 -1.7755407435528803
-1.4304882812500006 -1.4343847656250006 -1.775695506287223
-1.4168164062500006 -1.4516113281250007 -1.7753471595488823
-1.4304541015625005 -1.4260278320312505 -1.7757197974424463
-1.4396142578125009 -1.4256860351562504 -1.775644433943365
-1.4258996582031256 -1.4324664306640629 -1.7757110023056175
-1.4258654785156253 -1.4241094970703128 -1.775717648378783
-1.4304199218750004 -1.4176708984375002 -1.7756679961090878
-1.4270297241210943 -1.4287675476074222 -1.7757246992239009
Warning: Maximum number of function evaluations has been exceeded.
```

Out[54]:

```
array([-1.42702972, -1.42876755])
```

In [55]:

```
fm(opt2[0], opt2[1])
```

Out[55]:

```
-1.7757246992239009
```

In [56]:

```
# Basin Hopping

output = False
spo.fmin(fo, (2.0, 2.0), maxiter=250)
```

```
Optimization terminated successfully.
         Current function value: 0.015826
         Iterations: 46
         Function evaluations: 86
```

Out[56]:

```
array([4.2710728 , 4.27106945])
```

Constrained Optimization

Consider the utility optimization problem of an investor who can invest in two risky securities. Both securities cost $q_a = q_b = 10$ today. After one year, they have a payoff of 15 USD and 5 USD, respectively, in state u, and of 5 USD and 12 USD, respectively, in state d. Both states are equally likely. Denote the vector payoffs for the two securities by r_a and r_b, respectively.

The investor has a budget of $w_0 = 100$ USD to invest and derives utility from future wealth according to the utility function $u(w) = \sqrt{w}$, where w is the wealth (USD amount) available.

In [57]:

```
def Eu(grid):
    s, b = grid
    return -(0.5 * np.sqrt(s * 15 + b * 5) + 0.5 * np.sqrt(s * 5 + b * 12))

def constraint(grid):
    s, b = grid
    return 100 - s * 10 - b * 10

cons = ({'type': 'ineq', 'fun': constraint})
bnds = ((0, 1000), (0, 1000))
```

In [58]:

```
result = spo.minimize(Eu, [5, 5], method='SLSQP', bounds=bnds, constraints=con
s)
```

In [59]:

```
result
```

Out[59]:

```
     fun: -9.700883611487832
     jac: array([-0.48508096, -0.48489535])
 message: 'Optimization terminated successfully.'
    nfev: 21
     nit: 5
    njev: 5
  status: 0
 success: True
       x: array([8.02547122, 1.97452878])
```

In [60]:

```
result['x']
```

Out[60]:

```
array([8.02547122, 1.97452878])
```

In [61]:

```
np.dot(result['x'], [10, 10]) # The budget constraint is binding!
```

Out[61]:

```
99.99999999999999
```

Integration

Integration is an important mathematical tools when it comes to valuation and option pricing. This stems from the fact that risk-neutral values of derivatives can be expressed in general as the discounted *expectation* of their payoff under the risk-neutral (martingale) measure.

In [62]:

```
import scipy.integrate as sci
```

In [63]:

```
def f(x):
    return np.sin(x) + 0.5 * x
```

In [64]:

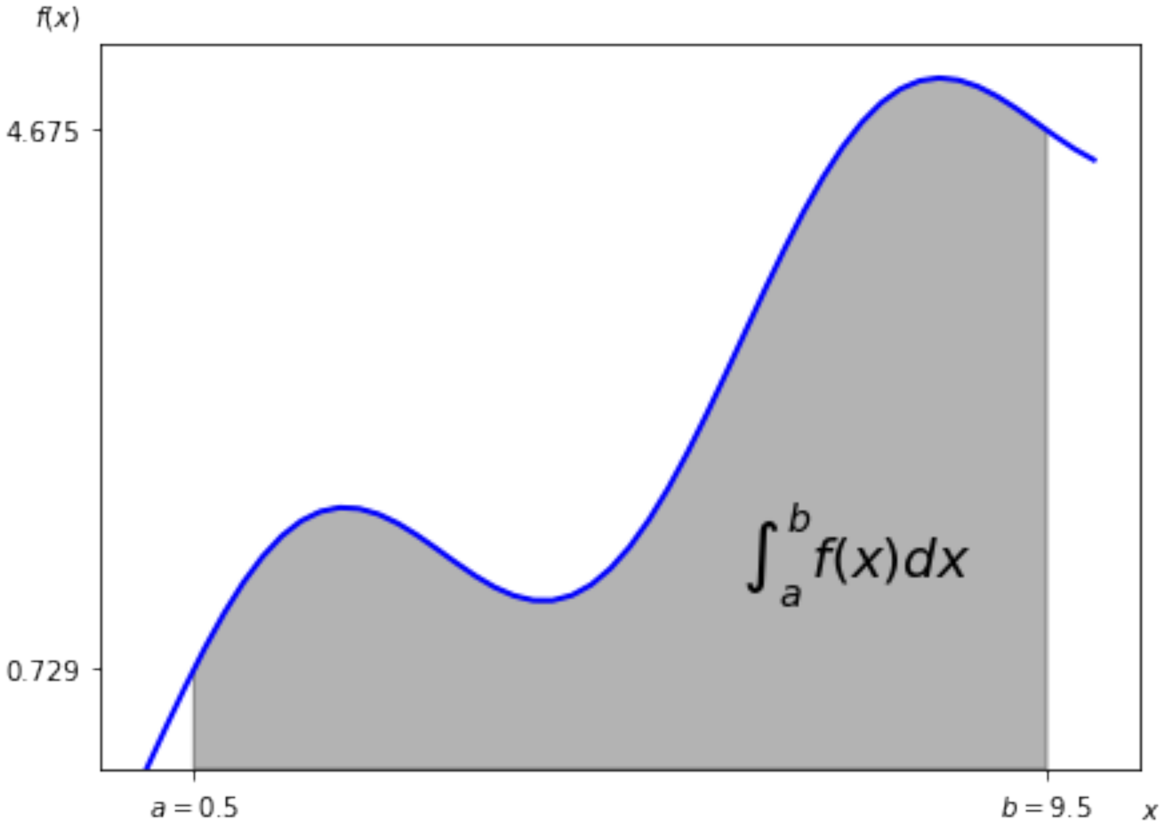
```
a = 0.5
b = 9.5
x = np.linspace(0, 10)
y = f(x)
```

In [65]:

```
from matplotlib.patches import Polygon

fig, ax = plt.subplots(figsize=(7, 5))
plt.plot(x, y, 'b', lw=2)
plt.ylim(ymin=0)

Ix = np.linspace(a, b)
Iy = f(Ix)
verts = [(a, 0)] + list(zip(Ix, Iy)) + [(b, 0)]
poly = Polygon(verts, facecolor='0.7', edgecolor='0.5')
ax.add_patch(poly)

plt.text(0.75 * (a + b), 1.4, r"$\int_a^b f(x)dx$",
        horizontalalignment='center', fontsize=20)
plt.figtext(0.9, 0.075, '$x$')
plt.figtext(0.075, 0.9, '$f(x)$')

ax.set_xticks((a, b))
ax.set_xticklabels(('$a=0.5$', '$b=9.5$'))
ax.set_yticks((f(a), f(b)))
```

Out[65]:

```
[<matplotlib.axis.YTick at 0x1c16b37f28>,
 <matplotlib.axis.YTick at 0x1c16b37860>]
```



Numerical Integration

In [66]:

```
sci.fixed_quad(f, a, b)[0] # fixed Gaussian quadrature
```

Out[66]:

```
24.366995967084602
```

In [67]:

```
sci.quad(f, a, b)[0] # adaptive quadrature
```

Out[67]:

```
24.374754718086752
```

In [68]:

```
sci.romberg(f, a, b) # romberg integration
```

Out[68]:

```
24.374754718086713
```

In [69]:

```
xi = np.linspace(0.5, 9.5, 25)
```

In [70]:

```
sci.trapz(f(xi), xi) # trapezoidal rule
```

Out[70]:

```
24.352733271544516
```

In [71]:

```
sci.simps(f(xi), xi) # Simpson's rule
```

Out[71]:

```
24.37496418455075
```

Integration by Simulation

In [72]:

```
for i in range(1, 20):
    np.random.seed(1000)
    x = np.random.random(i * 10) * (b - a) + a
    print(np.sum(f(x)) / len(x) * (b - a))
```

```
24.804762279331463
26.522918898332378
26.265547519223976
26.02770339943824
24.99954181440844
23.881810141621663
23.527912274843253
23.507857658961207
23.67236746066989
23.679410416062886
24.424401707879305
24.239005346819056
24.115396924962802
24.424191987566726
23.924933080533783
24.19484212027875
24.117348378249833
24.100690929662274
23.76905109847816
```

Symbolic Computation

In [73]:

```
import sympy as sy
```

Basics

In [74]:

```
x = sy.Symbol('x')
y = sy.Symbol('y')
```

In [75]:

```
type(x)
```

Out[75]:

```
sympy.core.symbol.Symbol
```

In [76]:

```
sy.sqrt(x)
```

Out[76]:

```
sqrt(x)
```

In [77]:

```
3 + sy.sqrt(x) - 4 ** 2
```

Out[77]:

```
sqrt(x) - 13
```

In [78]:

```
f = x ** 2 + 0.5 * x ** 2 + 3 / 2
```

In [79]:

```
sy.simplify(f)
```

Out[79]:

```
1.5*x**2 + 1.5
```

In [80]:

```
sy.init_printing(pretty_print=False, use_unicode=False)
```

In [81]:

```
print(sy.pretty(f))
```

```
     2
1.5*x  + 1.5
```

In [82]:

```
print(sy.pretty(sy.sqrt(x) + 0.5))
```

```
  ___
\/ x  + 0.5
```

In [83]:

```
pi_str = str(sy.N(sy.pi, 100))
pi_str[:40]
```

Out[83]:

```
'3.14159265358979323846264338327950288419'
```

In [84]:

```
pi_str[-40:]
```

Out[84]:

```
'4592307816406286208998628034825342117068'
```

In [85]:

```
pi_str.find('15926')
```

Out[85]:

```
4
```

Equations

In [86]:

```
sy.solve(x ** 2 - 1)
```

Out[86]:

```
[-1, 1]
```

In [87]:

```
sy.solve(x ** 3 + 0.5 * x ** 2 - 1)
```

Out[87]:

```
[0.858094329496553, -0.679047164748276 - 0.839206763026694*I, -0.6
79047164748276 + 0.839206763026694*I]
```

In [88]:

```
sy.solve(x ** 2 + y ** 2)
```

Out[88]:

```
[{x: -I*y}, {x: I*y}]
```

Integration

In [89]:

```
a, b = sy.symbols('a b')
```

In [90]:

```
print(sy.pretty(sy.Integral(sy.sin(x) + 0.5 * x, (x, a, b))))
```

```
  b
  /
 |
 |  (0.5*x + sin(x)) dx
 |
/
a
```

In [91]:

```
int_func = sy.integrate(sy.sin(x) + 0.5 * x, x)
```

In [92]:

```
print(sy.pretty(int_func))
```

```
      2
0.25*x  - cos(x)
```

In [93]:

```
Fb = int_func.subs(x, 9.5).evalf()
Fa = int_func.subs(x, 0.5).evalf()
```

In [94]:

```
Fb - Fa
```

Out[94]:

```
24.3747547180867
```

In [95]:

```
int_func_limits = sy.integrate(sy.sin(x) + 0.5 * x, (x, a, b))
print(sy.pretty(int_func_limits))
```

```
        2         2
- 0.25*a  + 0.25*b  + cos(a) - cos(b)
```

In [96]:

```
int_func_limits.subs({a : 0.5, b : 9.5}).evalf()
```

Out[96]:

```
24.3747547180868
```

In [97]:

```
sy.integrate(sy.sin(x) + 0.5 * x, (x, 0.5, 9.5))
```

Out[97]:

```
24.3747547180867
```

Differentiation

In [98]:

```
int_func.diff()
```

Out[98]:

```
0.5*x + sin(x)
```

In [99]:

```
f = sy.sin(x) + 0.05 * x ** 2 + sy.sin(y) + 0.05 * y ** 2
```

In [100]:

```
del_x = sy.diff(f, x)
del_x
```

Out[100]:

```
0.1*x + cos(x)
```

In [101]:

```
del_y = sy.diff(f, y)
del_y
```

Out[101]:

```
0.1*y + cos(y)
```

In [102]:

```
xo = sy.nsolve(del_x, -1.5)
xo
```

Out[102]:

```
-1.42755177876459
```

In [103]:

```
yo = sy.nsolve(del_y, -1.5)
yo
```

Out[103]:

```
-1.42755177876459
```

In [104]:

```
f.subs({x : xo, y : yo}).evalf()
```

Out[104]:

```
-1.77572565314742
```

In [105]:

```
# uneducated guess might trap the algorithm in a local minimum instead of a gl
obal one

xo = sy.nsolve(del_x, 1.5)
yo = sy.nsolve(del_y, 1.5)
f.subs({x : xo, y : yo}).evalf()
```

Out[105]:

```
2.27423381055640
```

In [ ]:

```

```