Kurs Bio144:
Datenanalyse in der Biologie

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Lecture 6: ANCOVA, Short introduction to Linear Algebra
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Overview

- ANCOVA
- ANCOVA as special cases of a linear model
- Introduction to linear Algebra

Note:

ANCOVA = ANalysi of COVAriance (Kovarianzanalyse)

Course material covered today

The lecture material of today is based on the following literature:

- “Getting Started with R” chapter 6.3
- “Lineare regression” chapters 3.A (p. 43-45) and 3.4, 3.5 (p. 39-42)

Recap of ANOVA

- ANOVA is a method to test if the means of two or more groups are different.
- Post-hoc tests and contrasts, including correction for p-values, to understand the differences between the groups.
- Two-way ANOVA for factorial designs, interactions.
- Interpretation of the results.
- Model checking.
- ANOVA is a special case of linear regression with categorical covariates.

Analysis of Covariance (ANCOVA)

An ANCOVA is an analysis of variance (ANOVA), including also at least one continuous covariate.

$\rightarrow$ ANCOVA is a combination of ANOVA and regression, so in principle, there is nothing new to it...

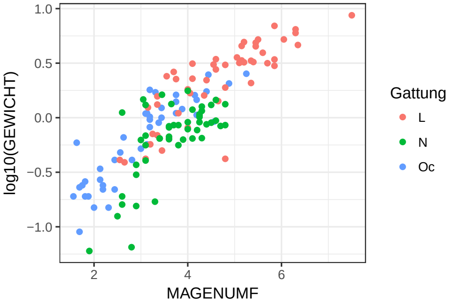
Given a categorical covariate x_i and a continuous covariate z_i. Then the ANCOVA equation (without interactions) is given as

$$y_i = \beta_0 + \beta_1 x_i^{(1)} + ... + \beta_k x_i^{(k)} + \beta_z z_i + e_i \ ,$$

where $x_i^{(k)}$ is the kth dummy variable ($x_i^{(k)}$=1 if ith observation belongs to category k, 0 otherwise).

Note: It is straightforward to add an interaction of x_i with z_i.

Example: Remember the earthworm study from week 3:
"Magenumfang" was used to predict "Gewicht" of the worm, including as covariate also the worm species.



Categorical and continuous covariates were used to predict a continuous outcome (the weight of the worm).

```
> r.lm <- lm(log(GEWICHT) ~  MAGENUMF + Gattung,d.wurm)
>  summary(r.lm)$coef
```

```
              Estimate Std. Error     t value     Pr(>|t|)
(Intercept) -2.5355459 0.22147279 -11.4485663 8.617670e-22
MAGENUMF     0.7118725 0.04528843  15.7186392 1.232126e-32
GattungN    -0.5151344 0.11009219  -4.6791186 6.760621e-06
GattungOc   -0.0907298 0.12791000  -0.7093254 4.793107e-01
```

Remember: What do the *p*-values ($p = 0.48$ and $p < 0.0001$) of the categorical covariate "Gattung" mean?

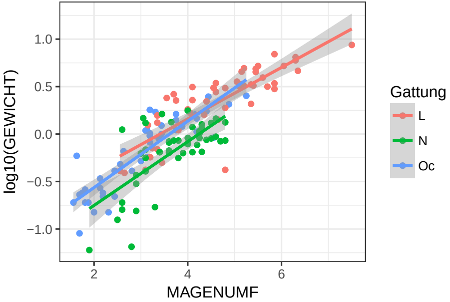
To understand if "Gattung" has an effect, **we need to carry out an *F*-test** (see slides 46/47, week 3). To this end, plot the ANOVA table:

```
> anova(r.lm)
```

```
Analysis of Variance Table

Response: log(GEWICHT)
           Df  Sum Sq Mean Sq F value    Pr(>F)
MAGENUMF    1 104.866 104.866  409.69 < 2.2e-16 ***
Gattung     2   7.177   3.589   14.02 2.842e-06 ***
Residuals 139  35.579   0.256
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Remember that we can also include an interaction term between `MAGENUMF` and `Gattung` to allow for different slopes:



The *F*-test is also needed to check whether the respective interaction term is needed:

```
> r.lm2<- lm(log(GEWICHT) ~  MAGENUMF * Gattung,d.wurm)
> summary(r.lm2)$coef
```

```
                    Estimate Std. Error   t value     Pr(>|t|)
(Intercept)       -2.1274390 0.30859958 -6.893849 1.816840e-10
MAGENUMF           0.6237919 0.06484623  9.619555 4.705940e-17
GattungN          -1.1510597 0.49399105 -2.330123 2.126052e-02
GattungOc         -0.7810625 0.39669094 -1.968945 5.097754e-02
MAGENUMF:GattungN  0.1500462 0.12178262  1.232082 2.200288e-01
MAGENUMF:GattungOc 0.1817710 0.10199782  1.782106 7.694701e-02
```

```
> anova(r.lm2)
```

```
Analysis of Variance Table

Response: log(GEWICHT)
                  Df  Sum Sq Mean Sq  F value    Pr(>F)
MAGENUMF           1 104.866 104.866 414.4743 < 2.2e-16 ***
Gattung            2   7.177   3.589  14.1835 2.521e-06 ***
MAGENUMF:Gattung   2   0.917   0.458   1.8112    0.1673
Residuals        137  34.662   0.253
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

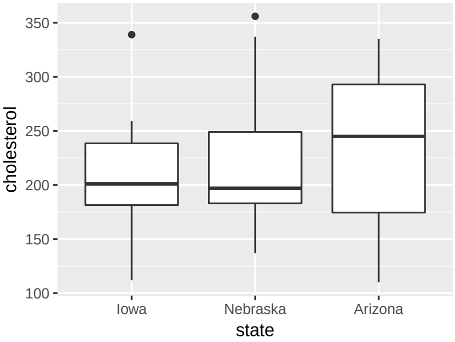
$\rightarrow p = 0.167$, thus interaction is probably not relevant.

A new example: cholesterol levels

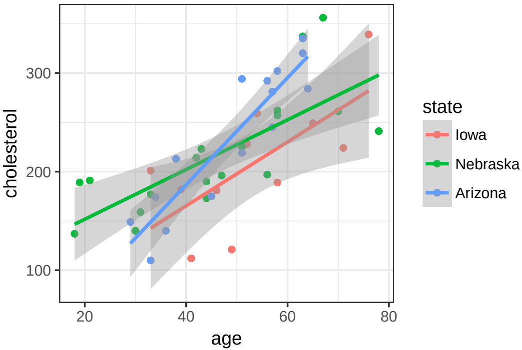
Example: Cholesterol levels [mg/ml] for 30 women from two US states, Iowa and Nebraska, were measured.

Question: Do these levels differ between the states?

Age (years) may be a relevant covariable.



The scatter plot gives an idea about the model that might be useful here:



$\rightarrow$ We include state, age and the interaction of the two.

The model equation is thus given as

$$y_i = \beta_0 + \beta_1 \cdot (state)_i + \beta_2 \cdot (age)_i + \beta_3 \cdot (state)_i \cdot (age)_i + e_i \ , \quad e_i \sim \mathrm{N}(0, \sigma_e^2) \ .$$

```
> r.lm <- lm(cholesterol ~ age*state,data=d.chol)
> anova(r.lm)
```

```
Analysis of Variance Table

Response: cholesterol
          Df Sum Sq Mean Sq F value    Pr(>F)
age        1  96524   96524 61.8961 1.424e-09 ***
state      2  11474    5737  3.6789   0.03438 *
age:state  2  12665    6332  4.0606   0.02501 *
Residuals 39  60819    1559
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Interpretation?

Compare the results from the previous slide to the estimated coefficients:

```
> r.lm <- lm(cholesterol ~ age*state,data=d.chol)
> summary(r.lm)$coef
```

```
                    Estimate Std. Error    t value    Pr(>|t|)
(Intercept)       35.8112138 50.4494150  0.7098440 0.482023781
age                3.2381449  0.9234015  3.5067573 0.001158119
stateNebraska     65.4865523 56.7347108  1.1542590 0.255418196
stateArizona     -65.7063829 66.7677031 -0.9841043 0.331130339
age:stateNebraska -0.7177069  1.0643772 -0.6742975 0.504099737
age:stateArizona   2.1787633  1.2672928  1.7192264 0.093502039
```

Note: The *p*-values for the age coefficient is not the same as in the anova table...

Compare the results for "state" also to the model without interaction:

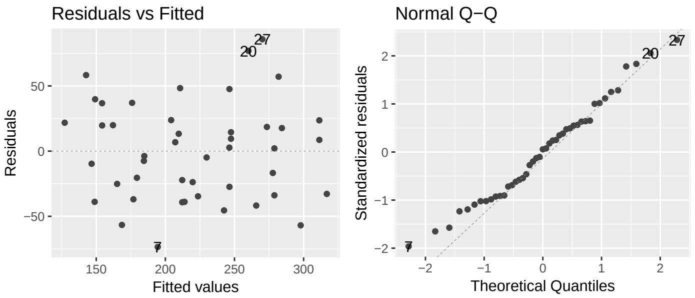
```
> r.lm2 <- lm(cholesterol ~ age ,data=d.chol)
> summary(r.lm2)$coef
```

```
             Estimate Std. Error  t value     Pr(>|t|)
(Intercept) 68.247111  22.830703 2.989269 4.610215e-03
age          3.134982   0.448522 6.989584 1.332242e-08
```

Note: the *p*-value for 'state' is similar to the previous slide (anova table).

Reason: anova tests the models against one another in the **order** specified.

As always, some model checking is necessary:



$\rightarrow$ This seems ok.

A final word on ANCOVA

ANCOVA unifies several concepts that we approached in this course so far:

- Linear regression
- Categorical covariates
- Interactions (of continuous and categorical covariates)
- Analysis of Variance (ANOVA)

As such, it is a **special case of the linear model**.

An introduction to linear Algebra

Who has some knowledge of linear Algebra?

Overview

- The basics about
  - vectors
  - matrices
  - matrix algebra
  - matrix multiplication
- Why is linear Algebra useful?
- What does it have to do with data analysis and statistics?
- Regression equations in matrix notation.

Motivation

Why are vectors, matrices and their algebraic rules useful?

Example 1: The observations for a covariate $\mathbf{x}$ or the response $\mathbf{y}$ for all individuals $1 \leq i \leq n$ can be stored in a vector (vectors and matrices are always given in **bold** letters):

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} , \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{pmatrix} .$$

Example 2: Covariance matrices for multiple variables. Say we have $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. The covariance matrix is then given as

$$\begin{pmatrix} \mathrm{Var}(\mathbf{x}^{(1)}) & \mathrm{Cov}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \\ \mathrm{Cov}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) & \mathrm{Var}(\mathbf{x}^{(2)}) \end{pmatrix} .$$

Example 3: The data (e.g. of some regression model) can be stored in a matrix:

$$\tilde{\mathbf{X}} = \begin{pmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_2^{(1)} & x_2^{(2)} \\ \dots & \dots & \dots \\ 1 & x_n^{(1)} & x_n^{(2)} \end{pmatrix} .$$

This is the so-called design matrix with a vector of 1's in the first column.

Example 4: A linear regression model can be written compactly using matrix multiplication:

$$\mathbf{y} = \tilde{\mathbf{X}} \cdot \tilde{\boldsymbol{\beta}} + \mathbf{e} \ ,$$

with $\tilde{\boldsymbol{\beta}}$ the vector of regression coefficients and $\boldsymbol{\epsilon}$ the vector of residuals.

Why do we discuss this topic in our course?

- Useful for compact notation.
- Enabels you to understand many statistical texts (books, research articles) that remain inaccessible otherwise.
- Useful for efficient coding, e.g. in R, which helps to increase speed and to reduce error rates.
- More advanced concepts often rely on linear algebra, e.g. principal component analysis (PCA) or random effects models.
- Is part of a general education (Allgemeinbildung) ;-)

Matrices

An $n \times m$ Matrix is given as

$$\mathbf{A} = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{array} \right) ,$$

with rows $1 = 1, \dots, n$ and columns $j = 1, \dots, m$.

Quadratic matrix: $n = m$. Example:

$$\left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 6 & 1 & 9 \end{array} \right)$$

Symmetric matrix: $a_{ij} = a_{ji}$. Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} .$$

The diagonal of a quadratic matrix is given by $(a_{11}, a_{22}, \ldots, a_{nn})$.
Example: the diagonal of the above matrix is given as

$$(a_{11}, a_{22}, a_{33}) = (1, 3, 5) .$$

Diagonal matrix: A matrix that has entries $\neq 0$ only on the diagonal.
Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} .$$

Transposing a matrix: Given a matrix $\mathbf{A}$. Exchange the rows by the columns and vice versa. This leads to the transposed matrix $\mathbf{A}^\top$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \quad \Rightarrow \quad \mathbf{A}^\top = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix} .$$

Examples (note also the change of dimensions):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \quad \Rightarrow \quad \mathbf{A}^\top = \begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \quad \Rightarrow \quad \mathbf{A}^\top = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$$

- Transposing a matrix twice leads to the original matrix:

$$(\mathbf{A}^\top)^\top = \mathbf{A} \ .$$

- When a matrix is symmetric, then

$$\mathbf{A}^\top = \mathbf{A} \ .$$

  This is true in particular for diagonal matrices.

Vectors

A vector is nothing else than n numbers written in a column:

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Transposing a vector leads to a *row vector*:

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}^\top = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}$$

Note: By definition (by default), a vector is always a column vector.

Addition and substraction

- Adding and substracting matrices and vectors is only possible when the objects have the same dimension.
- Examples: Elementwise addition (or substraction)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 10 & 10 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

- But this addition is not defined:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{pmatrix} =$$

Multiplication by a scalar

Multiplication with a "number" (scalar) is simple: Multiply each element in a vector or a matrix.

Examples:

$$3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

$$-2 \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \\ 4 \end{pmatrix}$$

Matrix multiplication

The multiplication of two matrices **A** and **B** is defined if

number of columns in **A** = number of rows in **B**.

It is easiest to explain matrix multiplication with an example:

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 1 \cdot 4 & 2 \cdot 1 + 1 \cdot (-2) \\ (-1) \cdot 3 + 0 \cdot 4 & (-1) \cdot 1 + 0 \cdot (-2) \\ 3 \cdot 3 + 1 \cdot 4 & 3 \cdot 1 + 1 \cdot (-2) \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 \\ -3 & -1 \\ 13 & 1 \end{pmatrix}$$

▸ Matrix multiplication app

Matrix multiplication rules I

Attention: Matrix multiplication does not follow the same rules as scalar multiplication!!!

- It can happen that $\mathbf{A} \cdot \mathbf{B}$ can be calculated, but $\mathbf{B} \cdot \mathbf{A}$ is not defined (see example on previous slide).
- In general: $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$, even if both are defined.
- It can happen that $\mathbf{A} \cdot \mathbf{B} = \mathbf{0}$ ($\mathbf{0}$ matrix), although both $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$.
- The Assoziativgesetz holds: $\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$.
- The Distributivgesetz holds:

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\ (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} &= \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \end{aligned}$$

Matrix multiplication rules II

- Transposing inverts the order: $(\mathbf{A} \cdot \mathbf{B})^\top = \mathbf{B}^\top \cdot \mathbf{A}^\top$.
- The product $\mathbf{A} \cdot \mathbf{A}^\top$ is always symmetric.
- All these rules also hold for vectors, which can be interpreted as $n \times 1$ matrices:

$$\mathbf{a} \cdot \mathbf{b}^\top = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_m \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_m \\ \vdots & \vdots & & \vdots \\ a_n b_1 & a_n b_2 & \dots & a_n b_m \end{pmatrix}$$

If $\mathbf{a}$ and $\mathbf{b}$ have the same length:

$$\mathbf{a}^\top \cdot \mathbf{b} = \sum_i a_i b_i$$

Short exercises

Given vectors **a** and **b** and matrix **C**:

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

Calculate, if defined

- $\mathbf{a}^\top \cdot \mathbf{b}$
- $\mathbf{a} \cdot \mathbf{b}^\top$
- $\mathbf{C} \cdot \mathbf{a}$
- $\mathbf{C} \cdot \mathbf{b}$

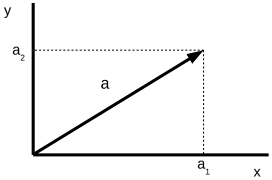
The length of a vector

The length of a vector $\mathbf{a}^\top = (a_1, a_2, \ldots, a_n)$ is defined as $||\mathbf{a}||$ with

$$||\mathbf{a}||^2 = \mathbf{a}^\top \cdot \mathbf{a} = \sum_i a_i^2 \ .$$

This is basically the Pythagoras idea in 2, 3, ... n dimensions.

In 2 dimensions: $||\mathbf{a}||^2 = a_1^2 + a_2^2$:



Identity matrix (Einheitsmatrix)

The identity matrix (of dimension m) is probably the simples matrix that exists. It has 1's on the diagonal and 0's everywhere else:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

Multiplication with the identity matrix leaves a $m \times n$ matrix $\mathbf{A}$ unchanged:

$$\mathbf{A} \cdot \mathbf{I} = \mathbf{A} \ .$$

Inverse matrix

Given a quadratic matrix $\mathbf{A}$ that fulfills

$$\mathbf{B} \cdot \mathbf{A} = \mathbf{I} \ ,$$

then $\mathbf{B}$ is called the inverse of $\mathbf{A}$ (and vice versa). One then writes

$$\mathbf{B} = \mathbf{A}^{-1} \ .$$

Note:

- In that case it also holds that $\mathbf{A} \cdot \mathbf{B} = \mathbf{I}$.
- Therefore: $\mathbf{A} = \mathbf{B}^{-1} \quad \Leftrightarrow \quad \mathbf{B} = \mathbf{A}^{-1}$

- The inverse of $\mathbf{A}$ may not exist. If it exists, $\mathbf{A}$ is regular, otherwise singular.

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

- The inverse of a matrix product is given as

$$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} \ .$$

- It is

$$(\mathbf{A}^\top)^{-1} = (\mathbf{A}^{-1})^\top \ .$$

  Therefore one may also write $\mathbf{A}^{-\top}$.

Linear regression in matrix notation

Linear regression with n data points can be understood as an equation system with n equations.

Remember example 4 from slide 18: We said that a linear regression model can be written compactly using matrix multiplication:

$$\mathbf{y} = \tilde{\mathbf{X}} \cdot \tilde{\boldsymbol{\beta}} + \mathbf{e} \ .$$

Task: Verify this now, using a model with two variables $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ and

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} , \ \tilde{\mathbf{X}} = \begin{pmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_2^{(1)} & x_2^{(2)} \\ \dots & \dots & \dots \\ 1 & x_n^{(1)} & x_n^{(2)} \end{pmatrix} , \ \tilde{\boldsymbol{\beta}} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} , \ \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} .$$

It can be shown (see Stahel 3.4f,g) that the least-squares estimates $\hat{\boldsymbol{\beta}}$ can be calculated as

$$\hat{\boldsymbol{\beta}} = (\tilde{\mathbf{X}}^\top \tilde{\mathbf{X}})^{-1} \cdot \tilde{\mathbf{X}}^\top \cdot \mathbf{y}$$

Does this look complicated?

Let's test this in R ....

Doing linear algebra in R

Let us look at model $\mathbf{y} = \tilde{\mathbf{X}} \cdot \tilde{\boldsymbol{\beta}} + \mathbf{e}$ with coefficients $\beta_0 = 10, \beta_1 = 5, \beta_2 = -2$ and variables

i	$x_i^{(1)}$	$x_i^{(2)}$
1	0	4
2	1	1
3	2	0
4	3	1
5	4	4

Thus the model is given as

$$y_i = 10 + 5x_i^{(1)} - 2x_i^{(2)} + e_i \text{ , for } \quad 1 \leq i \leq n \ .$$

Let us start by generating the "true" response, calculated as $\tilde{\mathbf{X}}\tilde{\boldsymbol{\beta}}$:

```
> x1 <- c(0,1,2,3,4)
> x2 <- c(4,1,0,1,4)
> Xtilde <- matrix(c(rep(1,5),x1,x2),ncol=3)
> Xtilde
```

```
     [,1] [,2] [,3]
[1,]    1    0    4
[2,]    1    1    1
[3,]    1    2    0
[4,]    1    3    1
[5,]    1    4    4
```

```
> t.beta <- c(10,5,-2)
> t.y <- Xtilde%*%t.beta
> t.y
```

```
     [,1]
[1,]    2
[2,]   13
[3,]   20
[4,]   23
[5,]   22
```

Matrix multiplication in R is done by the %*% symbol.

Next, we generate the vector containing the $e_i \sim \mathrm{N}(0, \sigma_e^2)$ with $\sigma_e^2 = 1$:

```
> t.e <- rnorm(5,0,1)
> t.e
```

```
[1]  0.7606833 -0.3257157  0.6830309  0.9070262  0.9342162
```

which we add to the “true” $\mathbf{y} = \tilde{\mathbf{X}}\tilde{\beta}$ values, to obtain the “observed” values:

```
> t.Y <- t.y  + t.e
> t.Y
```

```
          [,1]
[1,]  2.760683
[2,] 12.674284
[3,] 20.683031
[4,] 23.907026
[5,] 22.934216
```

It is now possible to fit the model with `lm`:

```
> r.lm <- lm(t.Y ~ x1 + x2)
> summary(r.lm)$coef
```

```
             Estimate Std. Error   t value    Pr(>|t|)
(Intercept) 10.069826  0.5556231  18.12348 0.003030672
x1           5.157981  0.1866953  27.62780 0.001307540
x2          -1.896970  0.1577864 -12.02239 0.006847617
```

Alternatively, we can use formula

$$\hat{\boldsymbol{\beta}} = (\tilde{\mathbf{X}}^\top \tilde{\mathbf{X}})^{-1} \tilde{\mathbf{X}}^\top \mathbf{y}$$

to find the parameter estimates:

```
> solve(t(Xtilde) %*% Xtilde)  %*%  t(Xtilde) %*% t.Y

          [,1]
[1,] 10.069826
[2,]  5.157981
[3,] -1.896970
```

- `solve()` calculates the inverse (here the inverse of $\tilde{\mathbf{X}}^\top \tilde{\mathbf{X}}$).
- `t()` gives the transposed (here of $\tilde{\mathbf{X}}^\top$).

Task: Do this calculation by yourself and verify for each step that the dimensions of the matrices and the vector are indeed fitting, so that this expression is defined.

Distribution of the β coefficients

See chapter 3.5 in Stahel: Linear algebra is useful to find the actual distribution of the regression coefficients $\boldsymbol{\beta}$.

No need to know this by heart – but please remember:

Finding the joint distribution for $\boldsymbol{\beta}$ is **much harder** without linear algebra.

Appendix

Some R commands for matrix algebra

Reading vectors and a matrices into R:

```
> a <- c(1,2,3)
> a
```

```
[1] 1 2 3
```

```
> A <- matrix(c(1,2,3,4,5,6),byrow=T,nrow=2)
> B <- matrix(c(6,5,4,3,2,1),byrow=T,nrow=2)
> A
```

```
     [,1] [,2] [,3]
[1,]    1    2    3
[2,]    4    5    6
```

```
> B
```

```
     [,1] [,2] [,3]
[1,]    6    5    4
[2,]    3    2    1
```

Adding and subtracting:

```
> A + B
     [,1] [,2] [,3]
[1,]    7    7    7
[2,]    7    7    7
```

```
> A - B
     [,1] [,2] [,3]
[1,]   -5   -3   -1
[2,]    1    3    5
```

However, be careful, R sometims does unreasonable things:

```
> A + a
     [,1] [,2] [,3]
[1,]    2    5    5
[2,]    6    6    9
```

What happened here??

Matrix multiplication:

```
> C <- A %*% t(B)
> C

     [,1] [,2]
[1,]   28   10
[2,]   73   28

> A%*%a

     [,1]
[1,]   14
[2,]   32
```

Matrix inversion (possible for quadratic matrices only):

```
> solve(C)

           [,1]       [,2]
[1,]  0.5185185 -0.1851852
[2,] -1.3518519  0.5185185

> C %*% solve(C)

     [,1] [,2]
[1,]    1    0
[2,]    0    1
```

Why does `solve(A)` or `solve(B)` not work?