The Ontario Teachers' Pension Fund

Case 1

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Part B of this case is based closely on cases prepared by David Hsieh

Purpose and Overview:

This assignment is designed to introduce you to concepts of portfolio allocation when there are many assets. Investors with access to risk-free leverage face a tradeoff between expected mean returns and expected portfolio standard deviation. You will investigate the implications of this tradeoff for asset allocation decisions in Part A of the assignment.

In Part B of the assignment you will analyze a real world asset allocation example based on the outputs of a similar analysis, the case of the Ontario Teachers' Pension Planning Board (OTPPB). The OTPPB was created in 1990 to manage the Ontario Teachers' Pension Fund. The pension fund is the largest fund in Canada; at the end of 2004, it had net assets of over C$81 billion. The board is responsible for the retirement of 158,000 elementary and secondary school teachers, 97,000 retired teachers and their survivors, and 88,000 former teachers with entitlements in the plan. This example will introduce you to the process of interpreting an investor's objectives and constraints and choosing a portfolio allocation that satisfies those objectives. The OTPPB is subject to both investment constraints (no short sales) and financial constraints (pension liabilities) and you will address how to approach the investment problem in this more realistic setting.

Directions:

You should submit no more than 9 slides total between parts A and B, and your group may be asked to present these slides in class to summarize your conclusions. The recommended allocation of these 9 slides across the parts of the case is 3 for Part A, 3 for Part B1, and 3 for Part B2.

Directions (Part A):

Download the R data file "A.RData" from the course website/Sakai. This file has one worksheet titled "Nominal". You will conduct standard mean-variance analysis, and you will have the option to do so using the equations provided in the notes. Since Part B will involve constrained optimizations, it is highly recommended that you use this opportunity to familiarize yourself with the matrix equation approach.

It is strongly recommended that you read and understand the notes at the end of this document before proceeding.

The analysis for this portion of the assignment should be conducted entirely in nominal terms. Since investors consume out of real returns, we would prefer to conduct this analysis in real terms, but for this dataset there is no real risk-free asset. The nominal analysis should be viewed as a simplification.

You do not need to submit the R file that results from the following directions. The goal of these directions is to help you analyze the data you have been given and organize your outputs in R. The questions and items for presentation will ask you to understand, interpret, and present some of the outputs of the directions, so it is strongly recommended that you complete the analysis detailed in the directions. You may choose to use a tool other than R to complete this portion of the assignment.

1. On the “Nominal” sheet calculate the mean and standard deviation of monthly returns for each asset in nominal terms. The risky assets include equity, bonds, gold, and commodities, while the T-Bill is our proxy for a risk-free asset.
2. Calculate the correlation between each of the risky assets’ returns in nominal terms.
3. Use the statistics you have already estimated to calculate the variance covariance (VCV) matrix for the risky assets’ returns in nominal terms.
4. Assume that the next month's return behaves like a random draw from a multivariate normal distribution with the characteristics of past nominal returns. Use the items you have already calculated to determine the allocation of the Maximal Sharpe Ratio (MSR) portfolio (also called the mean-variance efficient (MVE) portfolio of risky assets or the tangency portfolio) and the desired allocations of mean-variance investors with risk aversion of 1.3, 2.8, 6.5, 10.5, and 16.9 assuming the mean return on the T-Bill is a good proxy for the risk-free rate. Also calculate the allocation of the Global Minimum Variance (GMV) portfolio.
5. Calculate the expected Sharpe Ratio of each of the 7 portfolios then create a plot that labels where each falls in expected return (vertical axis) expected standard deviation space (horizontal axis). Your presentation will include this plot.

Questions and Requests for Comment (Part A):

Please use PowerPoint to communicate your responses to these required questions in a concise and organized manner suitable for presentation. Certain questions will ask you to present the results of some of the directions in the previous section as part of the PowerPoint. Your response will be submitted in PDF format, so embedded workbooks or formulas will not be considered in determining your grade. Remember that your grade is heavily weighted towards the logic of your interpretation of the results. You should submit no more than 3 slides for this portion of the assignment, and your group may be asked to present these slides in class to summarize your conclusions.

1. **Present (3 slides):** Include the plot produced in Direction 5, clearly labeled. Include a table of the GMV, MSR, and mean variance optimized portfolios' allocations and describe the differences in these allocations in words (Hint: something about the risky asset weights is constant for everything but the GMV). Explain conceptually how a mean variance investor with access to a risk-free asset and short sales determines and adjusts their portfolio allocation based on risk aversion (Hint: what portfolio of risky assets do they buy and how much of the risk free asset do they hold as risk aversion changes?).

Part B (Ontario Teachers' Pension Fund):

This portion of the case is based closely on the Ontario Teachers' case study as prepared by David Hsieh.

History:
The Ontario Teachers' Pension Fund was created in 1917. Up until 1989, the pension fund was restricted to invest in non-marketable bonds issued by the Government of Ontario. In 1989, it was determined that the pension fund was underfunded by nearly C$ 8 billion. The Province of Ontario, as the plan sponsor, agreed to make up the shortfall over a 40 year period, through a series of special funding payments.

In 1990, the OTPPB was created as an independent board with authority to invest all assets and administer the pension plan. The OTPPB set a long term goal of earning 5% above inflation. In 1989, the pension fund held only bonds issued by the Province of Ontario. When the OTPPB was formed, it implemented the advice of a pension fund consultant, and began shifting assets into equities.

In 1992, the Ontario Teachers' Federation became a co-sponsor of the pension fund, and shared equal responsibility with the Province of Ontario for any future shortfall.

Pension Plans in General:
There are two types of pension plans: defined contribution and defined benefit. The difference is the how the benefits are calculated. In defined contribution plans, benefits are a function of the investment results of the plan. In defined benefit plans, benefits are a function of worker tenure and salary without reference to the plan's investment results.

Defined contribution plans pool the contributions made by the employer and/or the employees in an account specific to the employee and provide a set of possible investments for the employee to choose from. At retirement, the employee withdraws funds from the plan to spend, where the available amount is based on the value of the combined contributions and the return outcomes of the employee investment elections. 401(k) plans are defined contribution plans.

Defined benefit plans also pool the contributions made by the employer and/or employee, but in a single fund whose distributions pay all employee benefits, generally managed by the employer or its designee. At retirement, the employee receives benefits based on a pre-defined formula, usually involving the length of employment and the pre-retirement annual salary. In some cases, benefits are also indexed to inflation. Given that retirement benefits are not a function of the investment process, it is possible that a pension plan's assets are below the present value of its liabilities. If that happens, the plan sponsor is usually responsible for making up the difference, although bankruptcy proceedings at private sponsors can mitigate this responsibility.

The Ontario Teachers' Pension Plan
The Ontario Teachers' plan is a defined benefit plan. Benefits for the plan participants are calculated using the following formula:

2% × average of "best 5-year salary" × number of years of service.

This amount is indexed to the Canadian Consumer Price Index (CPI).

Currently, teachers contribute 8.9% of their annual salary, matched by their employers. Table 1 provides the contributions and benefits paid during the period from 1990 until 2004.

Investment and Performance: 1990-2004

Since the creation of the OTPPB, the pension fund has increased its allocation to equities. Starting from 0% in equities in 1989, the asset mix reached 44% in equities in 1993, and 75% in 1997. At the end of 2004, approximately 49% of the plan's assets were in equities, principally shares in public and private companies and equity-return derivative contracts, and the remainder were in fixed income securities, largely government bonds and debentures. (This assumes the inflation-sensitive part of the portfolio is invested in inflation-indexed bonds.) Table 2 provides the net investments, performance, and net assets during the period from 1990 to 2004. The surplus is the difference between the actuarial value of the net assets and the accrued benefit benefits.

Directions:

The OTTPB has hired your team to advise them on the asset allocation in their portfolio. More specifically, you are asked to answer two main questions:

1. Is the current asset allocation consistent with its long-term goal of providing the promised distributions without increasing teacher contribution rates?

2. Given the liabilities, is this long-term goal reasonable given the available investment options and their expected return characteristics?

More detailed information related to these three questions is given in Parts B1 and B2 below. R files with relevant data can be downloaded from the course website/Sakai.

Part B1 (3 Slides): The Efficient Frontier and Current Asset Allocation

The pension fund invests in three main asset classes: equity, fixed income, and inflation sensitive assets. For the purposes of this case (in both Parts B1 and B2), you should assume that inflation-sensitive assets have similar characteristics as fixed income securities.

Consider the assumed long run real (above inflation) mean, standard deviations, and correlations for six asset classes (short-term bonds, medium-term bonds, long-term bonds, Canadian equity, US equity, and non-North American equity) in the R data file B1.RData. Use these means, standard deviations, and correlations to find the mean-standard deviation frontier for these assets classes, assuming no short sales and no access to a risk-free asset.

Present (~3 Slides): Present the estimated portfolio frontier (mean, standard deviation, and Sharpe Ratio of portfolio returns) for the levels of mean return included in the simulation tables below using either a table or an appropriately scaled and labeled plot. Comment on the sensitivity of the frontier to the assumed means, standard deviations, and correlations (i.e. change all means and re-estimate at least once, change all standard deviations and re-estimate at least once, etc.). Discuss the weights in equity and non-equity investments. The OTPPB currently targets a 5% real return. Discuss whether the current asset allocation is consistent with this goal.

Part B2 (3 Slides): The Asset Allocation and Long Term Goal

After your team has determined the mean-standard frontier of the six asset classes, you need to advise the OTPPB which portfolio it should pick. To do so, your team is given the following information for a typical teacher in the retirement system:

1. He or she works from age 25 until age 64, then retires from age 65 until death at age 85. The starting salary is $50,000, which is expected to rise at 1% above inflation.
2. While working, a teacher contributes to the retirement fund at the rate of 8.9%which is matched by the employer.
3. At retirement, the retirement benefits are calculated as 2% of the average best 5-year salary times the number of years of service (the formula was given above).
4. The benefits are indexed to the rate of inflation.

A prototype spreadsheet is given in the R data file B2.RData “payment” sheet. Note that the real risk-free rate is assumed to be 2% for discounting purposes but investors do not have access to a real risk-free asset. You will analyze how OTPPB should invest the retirement funds to minimize the likelihood of an increase of the contribution rate, which is needed if the pension fund does not have sufficient assets to pay future benefits.

You are provided with the outputs of a simulation of many return paths in the tables below. In this simulation, we record the distribution of Terminal Wealth at age 85 and Net Pension Assets at several ages, 55, 64 (retirement), and 75 under the assumed contribution and distribution plan. Negative Terminal Wealth should be interpreted as the cumulative shortfalls at age 85 after distributions have been netted out of wealth. Negative Net Pension Assets represent the case where the present value of distributions (discounted at the risk-free rate) exceeds the current portfolio value plus the present value of contributions.

You are presented with three return scenarios (specific sample paths of the simulation) in the B2.RData “r” sheet for several levels of expected mean portfolio return (each scenario in “r” is a 61 by 4 matrix. It records the sample paths of the simulation for 4 levels of expected mean portfolio return: 4%, 5%, 6%, 7%). Calculate the running total of net pension assets (Portfolio Value + NPV of Contributions – NPV of Benefits) for each level of mean return for each of the three scenarios. If, while making distributions, the portfolio value of the fund becomes negative you should accumulate the negative portfolio values at the risk free rate of 2% as though the fund had received a risk free loan from the government to fund the shortfalls. Create a table or set of tables summarizing these totals at ages 55, 64, 75, and 85 for each of the three scenarios.

Using these tables of simulated outcome distributions, identify about how often you would expect return outcomes at least as bad as each of the three scenarios to occur based on the outcomes for the 7% expected return portfolio. In other words, find the entry closest to the number you computed for the value of net pension assets at 85 in scenario 1 for the 7% mean return portfolio in the table for the simulation outputs at age 85 in the row for the 7% mean return. Record the percentile of the corresponding column. For example, the terminal wealth in Scenario 1 for the 7% return portfolio is a deficit of about \$358,000, very close to the value of the 25^{th} percentile of the distribution reported in the age 85 table in the final row, so you would record 25%. Repeat this process for each age (55, 64, 75, and 85) and each scenario (1, 2, and 3). Add these percentiles to your table of the net pension assets in each scenario at each age for the 7% mean return portfolios.

Present (~2 Slides): Currently, the OTPPB targets returns 5% above inflation. Is this a reasonable target given their long run goal of providing the promised distributions without increasing contribution rates? Use the simulation outputs to recommend and justify a choice of return target to the OTPPB. Present the tables produced by your calculations of the results of each of the three return scenarios and use them to explain and support your recommendation.

Present (~1 Slide): The remaining questions should be addressed in your presentation but are intentionally open ended; the goal is to think critically about the methodology of the simulation as a way to approach determining the OTPPB's return target. You should not need any additional quantitative analysis to answer these questions. What assumptions about the demographics of the teachers do you need to make to draw these conclusions based on the simulation we conducted (Hint: we simulated outcomes for one teacher with a non-random retirement and death age and a constant asset allocation)? How are your conclusions about the return targets likely vary with the inputs to the analysis (Hint: consider at least two of the following: average age of the teachers at retirement and death, mean and standard deviation of asset returns, covariance of asset returns, salary growth rates, or contribution rates)?

Special Note: In both Parts B1 and B2, it is important to perform all your analysis in real (i.e. inflation adjusted) returns, since the inputs in the two R data files are given in real terms.

Table 1: Changes in Net Assets (in C$ billions)

	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
Income															
Investment Income	1.03	3.84	2.15	5.91	0.53	5.66	7.44	7.25	5.14	10.12	6.21	(1.74)	(1.41)	11.42	10.80
Contributions															
Members/transfers	0.53	0.60	0.66	0.69	0.73	0.64	0.62	0.59	0.61	0.63	0.62	0.64	0.68	0.71	0.75
Province of Ontario	0.59	0.66	0.74	0.71	0.70	0.67	0.67	0.65	0.65	0.66	0.66	0.68	0.70	0.72	0.75
Special payments	0.19	0.27	0.44	0.00	0.00	0.00	0.15	0.46	0.49	0.13	0.00	0.00	0.00	0.00	0.00
Total Income	2.34	5.37	3.99	7.31	1.96	6.97	8.88	8.95	6.89	11.54	7.49	(0.42)	(0.03)	12.85	12.30
Expenditures															
Benefits paid	0.76	0.83	0.92	1.00	1.13	1.26	1.52	1.80	2.10	2.28	2.54	3.08	3.08	3.20	3.43
Expenses	0.02	0.03	0.04	0.04	0.05	0.06	0.07	0.09	0.10	0.12	0.13	0.16	0.13	0.19	0.22
Total Expenditures	0.78	0.86	0.96	1.04	1.18	1.32	1.59	1.89	2.20	2.40	2.67	3.24	3.21	3.39	3.65
Increases (Decreases)															
in Net Assets	1.56	4.51	3.03	6.27	0.78	5.65	7.29	7.06	4.69	9.14	4.82	(3.66)	(3.24)	9.46	8.65

Source: various annual reports of the Ontario Teachers' Pension Plan.

Table 2: Net Investments, Performance, and Net Assets (in C$ billions unless otherwise indicated)

	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
Investments															
Fixed income	18.12	19.21	19.46	17.57	11.41	12.51	10.62	10.28	11.48	17.30	13.32	7.09	13.96	19.38	13.91
Equities - Canadian	0.52	2.62	3.22	7.51	9.94	12.22	17.37	19.43	17.61	19.89	17.74	17.06	13.43	15.19	16.80
Equities - Foreign	0.81	1.80	3.25	7.04	10.71	12.29	16.01	19.96	24.02	21.76	23.14	24.28	18.19	19.13	23.09
Inflation-sensitive	0.00	0.18	0.45	1.16	1.34	1.99	2.34	3.29	5.00	8.15	17.85	19.69	19.86	20.73	27.92
Total	19.45	23.81	26.38	33.28	33.40	39.01	46.34	52.96	58.11	67.10	72.05	68.12	65.44	74.43	81.72
Performance															
Rate of return	5.6%	19.6%	8.9%	21.7%	1.7%	16.9%	19.0%	15.6%	9.9%	17.4%	9.3%	-2.3%	-2.0%	18.0%	14.7%
Benchmark	n/a	15.4%	8.0%	20.5%	-0.3%	17.2%	18.1%	15.6%	11.9%	17.6%	5.3%	-5.3%	-4.8%	13.5%	10.6%
Long-term goal	9.5%	8.3%	6.6%	6.2%	4.7%	6.2%	6.7%	5.2%	5.5%	7.1%	7.7%	5.2%	8.9%	7.0%	6.8%
Actuarial Value of Net Assets	19.46	23.81	26.39	30.77	34.23	38.24	43.02	48.91	54.38	59.99	68.78	72.43	75.86	79.16	82.79
Accrued Pension Benefits	23.01	27.47	29.81	34.00	36.85	38.74	41.83	44.46	48.64	52.11	58.56	65.43	73.67	83.12	96.73
Surplus	(3.55)	(3.66)	(3.42)	(3.23)	(2.62)	(0.50)	1.19	4.45	5.74	7.88	10.22	7.00	2.19	(3.96)	(13.94)

Source: various annual reports of the Ontario Teachers' Pension Plan.

Simulation of Terminal Wealth (85)

Percentile of Terminal Wealth

Mean Portfolio Return	99.0%	95.0%	75.0%	50.0%	25.0%	5.0%	1.0%	Percent Negative
1.5%	(214,758)	(348,529)	(508,662)	(597,476)	(671,838)	(756,986)	(808,029)	99.9%
3.0%	1,116,307	605,162	22,725	(253,608)	(445,313)	(633,537)	(732,102)	73.5%
3.5%	2,107,599	1,266,825	348,960	(72,646)	(355,048)	(604,404)	(725,198)	55.4%
4.0%	3,620,990	2,196,491	781,363	154,699	(254,659)	(578,181)	(724,307)	41.1%
4.5%	5,864,665	3,517,297	1,330,141	427,731	(142,175)	(560,644)	(731,977)	31.6%
5.0%	9,505,287	5,497,452	2,017,649	713,677	(53,760)	(577,614)	(764,922)	27.0%
5.5%	15,418,583	8,354,994	2,842,746	980,982	(10,486)	(621,775)	(816,706)	25.3%
6.0%	24,630,091	12,265,502	3,784,552	1,206,168	(12,131)	(682,104)	(873,056)	25.3%
6.5%	38,839,529	17,655,841	4,753,779	1,348,021	(82,558)	(752,087)	(932,989)	27.1%
7.0%	66,359,402	25,262,375	5,176,671	1,021,785	(357,227)	(888,660)	(1,025,249)	34.3%

Simulation of Net Pension Asset at 75

Percentile of Net Pension Asset

Mean Portfolio Return	99.0%	95.0%	75.0%	50.0%	25.0%	5.0%	1.0%	Percent Negative
1.5%	(178,310)	(295,335)	(450,882)	(538,661)	(613,023)	(698,171)	(749,213)	99.9%
3.0%	757,923	413,623	(6,882)	(220,879)	(392,033)	(574,722)	(673,287)	75.5%
3.5%	1,369,459	834,748	219,512	(81,264)	(311,047)	(545,588)	(666,383)	58.2%
4.0%	2,208,633	1,386,592	492,954	78,264	(224,510)	(519,389)	(665,491)	44.1%
4.5%	3,383,511	2,111,486	821,407	255,394	(138,963)	(502,204)	(673,162)	34.3%
5.0%	5,183,174	3,146,529	1,215,907	433,185	(76,947)	(518,819)	(706,106)	29.5%
5.5%	7,931,583	4,563,725	1,675,483	598,133	(47,643)	(562,960)	(757,891)	27.5%
6.0%	11,984,534	6,468,801	2,187,721	741,695	(54,052)	(623,288)	(814,241)	27.2%
6.5%	17,841,386	9,000,886	2,712,634	844,564	(98,700)	(693,272)	(874,174)	28.3%
7.0%	28,879,347	12,676,579	3,021,842	692,762	(307,946)	(829,845)	(966,434)	34.7%

Simulation of Net Pension Asset at Retirement (64)

Mean Portfolio Return	Percentile of Net Pension Asset 99.0%	95.0%	75.0%	50.0%	25.0%	5.0%	1.0%	Percent Negative
1.5%	(149,025)	(239,616)	(348,411)	(414,345)	(471,056)	(542,027)	(584,391)	100.0%
3.0%	369,747	154,537	(85,753)	(222,085)	(331,875)	(460,942)	(533,139)	85.2%
3.5%	674,066	364,719	33,972	(145,802)	(286,398)	(445,934)	(532,230)	71.4%
4.0%	1,063,517	625,061	172,612	(63,552)	(241,083)	(434,818)	(536,022)	57.8%
4.5%	1,571,768	950,338	331,164	25,195	(197,717)	(429,673)	(545,946)	47.3%
5.0%	2,317,597	1,386,846	516,495	113,438	(168,456)	(443,575)	(573,642)	40.4%
5.5%	3,391,645	1,967,511	725,697	195,290	(155,631)	(474,214)	(613,943)	36.8%
6.0%	4,872,041	2,712,842	949,308	267,252	(156,773)	(513,589)	(658,739)	35.1%
6.5%	6,916,490	3,664,392	1,181,910	317,019	(179,252)	(562,820)	(706,568)	35.0%
7.0%	10,815,288	5,131,235	1,368,566	267,002	(288,424)	(662,103)	(786,572)	39.5%

Simulation of Net Pension Asset at 55

Mean Portfolio Return	Percentile of Net Pension Asset 99.0%	95.0%	75.0%	50.0%	25.0%	5.0%	1.0%	Percent Negative
1.5%	(167,806)	(217,111)	(284,613)	(323,747)	(358,412)	(402,733)	(431,257)	100.0%
3.0%	86,839	(16,361)	(151,365)	(225,404)	(287,323)	(362,702)	(408,293)	95.9%
3.5%	227,550	85,856	(93,342)	(188,472)	(265,837)	(357,366)	(410,620)	88.9%
4.0%	402,365	208,027	(28,495)	(149,102)	(245,065)	(354,404)	(415,449)	79.1%
4.5%	623,887	353,453	43,645	(108,665)	(225,893)	(354,614)	(423,526)	69.1%
5.0%	936,498	549,728	126,788	(69,351)	(214,120)	(364,671)	(441,821)	60.8%
5.5%	1,369,157	804,374	218,451	(33,503)	(210,817)	(382,899)	(467,071)	54.3%
6.0%	1,943,608	1,117,923	317,545	(2,019)	(214,724)	(406,505)	(494,649)	50.3%
6.5%	2,707,716	1,510,095	418,879	20,544	(226,620)	(435,156)	(524,881)	48.0%
7.0%	4,099,734	2,115,391	506,415	3,225	(278,793)	(492,706)	(574,451)	49.8%

Notes:

Portfolio Math with Many Assets:

Notation: **w** a vector of portfolio weights, **E(R)** a vector of their expected returns, **Ω** their variance-covariance matrix (VCV), and r_f the risk-free rate.

Expected Return $E(r_p) = \mathbf{w'E(R)}$

Expected Variance $\sigma^2_p = \mathbf{w'\,\Omega\,w}$

Sharpe Ratio $= (E(r_p)-r_f)/\sigma_p$

Remember that for the purposes of this assignment you will use historical means or sample standard deviations of real returns to estimate each of the inputs into these equations.

Unconstrained Allocation Problem Solutions with Many Assets:

Notation: **w** a vector of portfolio weights exclusive of the risk-free asset, **1** a vector of ones, **E(R)** a vector of their expected returns, **Ω** their variance-covariance matrix (VCV) of risky asset returns, A the mean-variance investor's risk aversion (a scalar number), and r_f the risk-free rate (a scalar number).

We consider here the Maximal Sharpe Ratio (MSR) (Mean Variance Efficient (MVE), Tangency) Portfolio, the Global Minimum Variance (GMV) Portfolio, and portfolios for mean-variance investors with a given risk aversion, A.

The MSR Allocation: $\mathbf{w_{MSR} = \Omega^{-1}(E(R)}-r_f\mathbf{1})/(\mathbf{1'\Omega^{-1}(E(R)}-r_f\mathbf{1}))$

The GMV Allocation: $\mathbf{w_{GMV} = \Omega^{-1}1/(1'\Omega^{-1}1)}$

Allocation for a given Risk Aversion A: $\mathbf{w_A = \Omega^{-1}(E(R)}-r_f\mathbf{1})/A$

Note that each of these equations is the solution to a particular optimization problem, maximizing Sharpe Ratio (MSR), minimizing variance (GMV), or maximizing the following objective:

$U = \mathbf{w_A'E(R)} + (1-\mathbf{1'\,w_A})r_f - A/2\,(\mathbf{w_A'\,\Omega\,w_A})$

for the risk aversion A mean variance investor.

All of these optimizations are subject to the unit cost portfolio constraint that $\mathbf{w'1}+w_{rf} = 1$.