

Converting $1 + e^{i\theta}$ to polar

Claim: Given the complex number $z = 1 + e^{i\theta}$, we claim:

$$z = 1 + e^{i\theta} = 2 \cos \frac{\theta}{2} e^{i\frac{\theta}{2}}$$

Proof:

We will show the claim by showing there exist (r, ϕ) such that $z = re^{i\phi} = 2 \cos \frac{\theta}{2} e^{i\frac{\theta}{2}}$. In other words, z in exponential form with $r = 2 \cos \frac{\theta}{2}$ and angle $\phi = \frac{\theta}{2}$.

$z = (x, y)$, we want r and θ such that $z = (x, y) = (r \cos \theta, r \sin \theta)$.

Find r :

$$r = |z|$$

The modulus of a complex number is the length in Euclidean space. So take the square root of the product of the complex number and its conjugate:

$$\begin{aligned} &= \sqrt{(1 + e^{i\theta})(1 + e^{-i\theta})} \\ &= \sqrt{1 + e^{-i\theta} + e^{i\theta} + e^{i\theta}e^{-i\theta}} \\ &= \sqrt{1 + e^{-i\theta} + e^{i\theta} + 1} \\ &= \sqrt{2 + e^{-i\theta} + e^{i\theta}} \end{aligned}$$

Use Euler's formula for e 's:

$$\begin{aligned} &= \sqrt{2 + \cos \theta + i \sin \theta + \cos \theta - i \sin \theta} \\ &= \sqrt{2 + \cos \theta + \cos \theta + i \sin \theta - i \sin \theta} \\ &= \sqrt{2 + 2 \cos \theta} \\ &= \sqrt{2} \sqrt{1 + \cos \theta} \end{aligned}$$

Use the trigonometric double angle identity $2 \cos^2 \frac{\theta}{2} = 1 + \cos \theta$:

$$\begin{aligned} &= \sqrt{2} \sqrt{2 \cos^2 \frac{\theta}{2}} \\ &= \sqrt{2} \sqrt{2} \sqrt{\cos^2 \frac{\theta}{2}} \end{aligned}$$

$$= 2 \cos \frac{\theta}{2}$$

So $r = 2 \cos \frac{\theta}{2}$.

Find ϕ :