How HPC from Claim 2 Can Be Simplified

In this note, I show how the result of Claim 2 can be simplified to reveal a clear interference term for each input and output.

$$
\begin{aligned}
sin(2A) &= 2\,sin(A)\,cos(A) \\
cos(2A) &= cos^2(A) - sin^2(A) \\
&= 1 - 2sin^2(A) \\
&= 2cos^2(A) - 1 \\
\tan(2A) &= \frac{2\tan(A)}{1 - tan^2(A)}
\end{aligned}
$$

Figure 1: Useful Trig Identities

Claim: From Claim 2, we have

$$HPH = \frac{1}{2}\begin{bmatrix} 1 + e^{i\psi} & 1 - e^{i\psi} \\ 1 - e^{i\psi} & 1 + e^{i\psi} \end{bmatrix}$$

We will show HPH can be simplified as:

$$HPH = e^{i\frac{\psi}{2}}\begin{bmatrix} \cos\frac{\psi}{2} & -i\sin\frac{\psi}{2} \\ -i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}$$

Proof: To show the identity, we demonstrate the following:

1. $U_{BL} = U_{TR} = 1 - e^{i\psi} = -i2e^{i\frac{\psi}{2}}\sin\frac{\psi}{2}$
2. $U_{TL} = U_{BR} = 1 + e^{i\psi} = 2e^{i\frac{\psi}{2}}\cos\frac{\psi}{2}$

Where $e^{i\frac{\psi}{2}}$ is the global phase factor.

Showing 1:

$$-2i\sin\frac{\psi}{2}e^{i\frac{\psi}{2}}$$

Using Euler's formula (wiki):

$$= -2i \sin\frac{\psi}{2}[\cos\frac{\psi}{2} + i\sin\frac{\psi}{2}]$$

$$= -i[2\sin\frac{\psi}{2}\cos\frac{\psi}{2} + 2i\sin^2\frac{\psi}{2}]$$

Using double angle identity: $\sin(2A) = 2\cos(A)\sin(A)$

$$= -i[\sin\psi + 2i\sin^2\frac{\psi}{2}]$$

$$= 2\sin^2\frac{\psi}{2} - i\sin\psi$$

Using double angle identity: $2\sin^2(A) = 1 - \cos(2A)$

$$= 1 - \cos\psi - i\sin\psi$$

$$= 1 - [\cos\psi + i\sin\psi]$$

Using Euler's formula:

$$= 1 - e^{i\psi}$$

Therefore, we have shown $1 - e^{i\psi} = -ie^{i\frac{\psi}{2}}\sin\frac{\psi}{2}$

Showing 2

TODO: finish this part