2.14.1 One simple circuit

Part 1

Claim: Let B be a unitary defined as:

$$B = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

We claim B is unitary

Proof:

We must show $BB^\dagger = B^\dagger B = I$

The adjoint of of B is:

$$B^\dagger = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

Showing $\mathbf{BB^\dagger = I}$:

$$BB^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} (1-(-1)) & (-i+i) \\ (i-i) & 1(-1(-1)+1) \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= I$$

Showing $\mathbf{B^\dagger B = I}$:

$$B^\dagger B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (1-(-1)) & (-i+i) \\ (i-i) & 1(-1(-1)+1) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore B is unitary

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Part 2

Claim:

$$P_{\frac{\pi}{2}} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Proof:

$$BPH = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$$