1.11.3 Many Computational Paths

Part A

Claim: Suppose a quantum computational process, follows n different paths with each path contributing a probability amplitude of $z_k = \frac{1}{n}e^{ik\psi}$, where $\psi \in (0, 2\pi), k \in 1, \ldots, n-1$. We claim that the total probability of realizing the final output configuration is

$$P = \frac{1}{n^2}\frac{\sin^2(n\frac{\psi}{2})}{\sin^2(\frac{\psi}{2})}$$

Proof: Let z_t be the total complex amplitude of the final output configuration. That is:

$$z_t = \sum_{k=0}^{n-1} z_k$$

Plug in the value of Z_k as given above:

$$z_t = \frac{1}{n}\sum_{k=0}^{n-1} e^{ik\psi} = \frac{1}{n}\sum_{k=0}^{n-1} z^k$$

Using the identity $1 + z + z^2 + \ldots + z^n = \frac{1-z^{n+1}}{1-z}$:

$$z_t = \frac{1}{n}\frac{1-z^n}{1-z}$$

Therefore probability is:

$$P = |z|^2 = |\frac{1}{n}\frac{1-z^n}{1-z}|^2$$

$$= \frac{1}{n^2}|\frac{1-e^{i\psi n}}{1-e^{i\psi}}|^2$$

$$= \frac{1}{n}\frac{(1-e^{i\psi n})(1-e^{-i\psi n})}{(1-e^{i\psi})(1-e^{-i\psi})}$$

$$= \frac{1}{n^2}(\frac{2-e^{i\psi n}-e^{-i\psi n}}{2-e^{-i\psi}-e^{i\psi}})$$

$$= \frac{1}{n^2}(\frac{2-e^{i\psi n}-e^{-i\psi n}}{2-e^{-i\psi}-e^{i\psi}})$$

$$= \frac{1}{n^2}(\frac{2-[e^{i\psi n}+e^{-i\psi n}]}{2-[e^{-i\psi}+e^{i\psi}]})$$

Simplify denominator:

$$2 - [e^{-i\psi} + e^{i\psi}]$$

Use $e^{i\psi} = \cos(\psi) + i\sin(\psi)$

$$= 2 - [\cos(-\psi) + i\sin(-\psi) + \cos(\psi) + i\sin(\psi)]$$

$$= 2 - [\cos(\psi) - i\sin(\psi) + \cos(\psi) + i\sin(\psi)]$$

$$= 2 - [\cos(\psi) + \cos(\psi)]$$

$$= 2 - 2\cos(\psi)$$

Simplify numerator:

$$2 - [e^{-in\psi} + e^{in\psi}] =$$

$$= 2 - [\cos(-n\psi) + i\sin(-n\psi) + \cos(n\psi) + i\sin(n\psi)]$$

$$= 2 - [\cos(n\psi) + \cos(n\psi) + i\sin(n\psi) - i\sin(n\psi)]$$

$$= 2 - 2\cos(n\psi)$$

Plug back in to get:

$$P = |z|^2 = \frac{1}{n^2}\frac{2 - 2\cos(n\psi)}{2 - 2\cos(\psi)}$$

Note: See trig identities for modulus of complex number

Use the trig identity: $\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

$$P = |z|^2 = \frac{1}{n^2}\frac{2 - 2[1 - 2\sin^2\left(\frac{n\psi}{2}\right)]}{2 - 2[1 - 2\sin^2\left(\frac{\psi}{2}\right)]}$$

$$= \frac{1}{n^2}\frac{4\sin^2\left(\frac{n\psi}{2}\right)}{4\sin^2\left(\frac{\psi}{2}\right)}$$

$$= \frac{1}{n^2}\frac{\sin^2\left(\frac{n\psi}{2}\right)}{\sin^2\left(\frac{\psi}{2}\right)}$$

Therefore the total probability from all paths is:

$$P = \frac{1}{n^2}\frac{\sin^2(n\frac{\psi}{2})}{\sin^2(\frac{\psi}{2})}$$

Part B

When the relative angle is zero we have $\psi = 0$, the probability is $P = 1$ as shown below in the graph.

Part C

We can see the probability reaches zero at $\psi = \pi$

Plot:

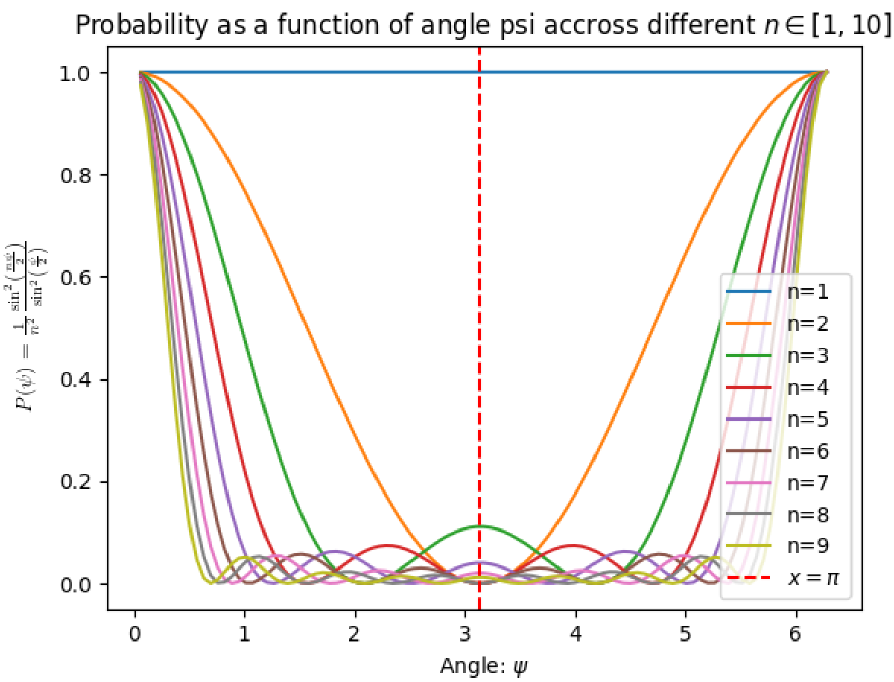

Figure 1: Plot of probability