

1.11.3 Many Computational Paths

Part A

Claim: Suppose a quantum computational process, follows n different paths with each path contributing a probability amplitude of $z_k = \frac{1}{n}e^{ik\psi}$, where $\psi \in (0, 2\pi), k \in 1, \dots, n-1$. We claim that the total probability of realizing the final output configuration is

$$P = \frac{1}{n^2} \frac{\sin^2(n\frac{\psi}{2})}{\sin^2(\frac{\psi}{2})}$$

Proof: Let z_t be the total complex amplitude of the final output configuration. That is:

$$z_t = \sum_{k=0}^{n-1} z_k$$

Plug in the value of Z_k as given above:

$$z_t = \frac{1}{n} \sum_{k=0}^{n-1} e^{ik\psi} = \frac{1}{n} \sum_{k=0}^{n-1} z^k$$

Using the identity $1 + z + z^2 + \dots + z^n = \frac{1-z^{n+1}}{1-z}$:

$$z_t = \frac{1}{n} \frac{1 - z^n}{1 - z}$$

Therefore probability is:

$$\begin{aligned} P = |z|^2 &= \left| \frac{1}{n} \frac{1 - z^n}{1 - z} \right|^2 \\ &= \frac{1}{n^2} \left| \frac{1 - e^{i\psi n}}{1 - e^{i\psi}} \right|^2 \\ &= \frac{1}{n} \frac{(1 - e^{i\psi n})(1 - e^{-i\psi n})}{(1 - e^{i\psi})(1 - e^{-i\psi})} \\ &= \frac{1}{n^2} \left(\frac{2 - e^{i\psi n} - e^{-i\psi n}}{2 - e^{-i\psi} - e^{i\psi}} \right) \\ &= \frac{1}{n^2} \left(\frac{2 - e^{i\psi n} - e^{-i\psi n}}{2 - e^{-i\psi} - e^{i\psi}} \right) \\ &= \frac{1}{n^2} \left(\frac{2 - [e^{i\psi n} + e^{-i\psi n}]}{2 - [e^{-i\psi} + e^{i\psi}]} \right) \end{aligned}$$

Simplify denominator:

$$2 - [e^{-i\psi} + e^{i\psi}]$$

Use $e^{i\psi} = \cos(\psi) + i \sin(\psi)$

$$\begin{aligned} &= 2 - [\cos(-\psi) + i \sin(-\psi) + \cos(\psi) + i \sin(\psi)] \\ &= 2 - [\cos(\psi) - i \sin(\psi) + \cos(\psi) + i \sin(\psi)] \\ &= 2 - [\cos(\psi) + \cos(\psi)] \\ &= 2 - 2 \cos(\psi) \end{aligned}$$

Simplify numerator:

$$\begin{aligned} &2 - [e^{-in\psi} + e^{in\psi}] = \\ &= 2 - [\cos(-n\psi) + i \sin(-n\psi) + \cos(n\psi) + i \sin(n\psi)] \\ &= 2 - [\cos(n\psi) + \cos(n\psi) + i \sin(n\psi) - i \sin(n\psi)] \\ &= 2 - 2 \cos(n\psi) \end{aligned}$$

Plug back in to get:

$$P = |z|^2 = \frac{1}{n^2} \frac{2 - 2 \cos(n\psi)}{2 - 2 \cos(\psi)}$$

Note: See trig identities for modulus of complex number

Use the trig identity: $\cos(\theta) = 1 - 2 \sin^2\left(\frac{\theta}{2}\right)$

$$\begin{aligned} P = |z|^2 &= \frac{1}{n^2} \frac{2 - 2[1 - 2 \sin^2\left(\frac{n\psi}{2}\right)]}{2 - 2[1 - 2 \sin^2\left(\frac{\psi}{2}\right)]} \\ &= \frac{1}{n^2} \frac{4 \sin^2\left(\frac{n\psi}{2}\right)}{4 \sin^2\left(\frac{\psi}{2}\right)} \\ &= \frac{1}{n^2} \frac{\sin^2\left(\frac{n\psi}{2}\right)}{\sin^2\left(\frac{\psi}{2}\right)} \end{aligned}$$

Therefore the total probability from all paths is:

$$P = \frac{1}{n^2} \frac{\sin^2(n\frac{\psi}{2})}{\sin^2(\frac{\psi}{2})}$$

Part B

When the relative angle is zero we have $\psi = 0$, the probability is $P = 1$ as shown below in the graph.

Part C

We can see the probability reaches zero at $\psi = \pi$

Plot:

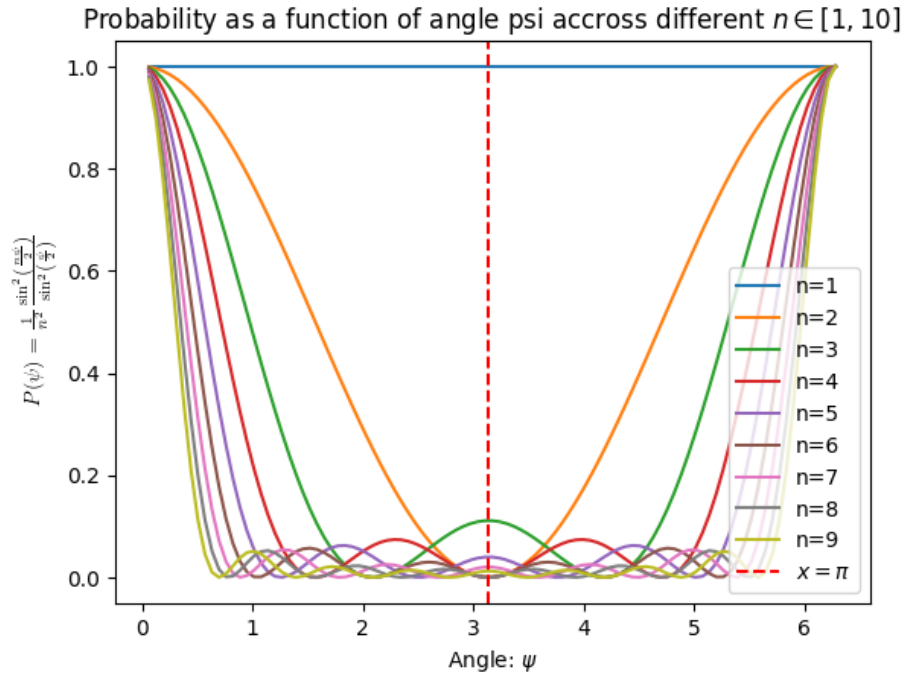


Figure 1: Plot of probability