Bayesian Inference

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Bayesian Inference: Main Idea

data

unknow variable

- Maximum likelihood estimation (MLE):

$$\hat{\theta} = \arg\max_{\theta} p(D \mid \theta).$$

Parameter θ is unknown but deterministic (frequentist view).

- Bayesian inference:
  - θ is viewed as a random variable (even when it is actually deterministic).
  - Use Bayes' rule to calculate the posterior distribution:

$$\boxed{p(\theta \mid D) = \frac{p(D \mid \theta)p(\theta)}{p(D)}.}$$

posterior

$P(\theta \mid D)$: posterior

$P(D \mid \theta)$: likelihood

$P(\theta)$: Prior

$$P(\theta \mid D) = \frac{P(D \mid \theta)\, P(\theta)}{\int P(D \mid \theta) P(\theta)\, d\theta}$$

$$P(D) = \int P(D \mid \theta) P(\theta)\, d\theta$$

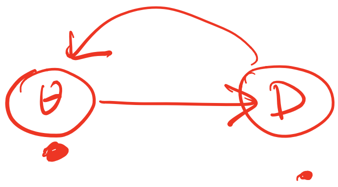
$$\propto P(D \mid \theta) P(\theta)$$

Bayes' Rule.

$$P(\theta, D) = P(\theta)\, P(D \mid \theta)$$

(Chain Rule)

$$= P(D)\, P(\theta \mid D)$$



$$\frac{P(\theta)P(D \mid \theta)}{P(D)} = \frac{P(D)\, P(\theta \mid D)}{P(D)}$$

$$\Rightarrow$$

$$= P(\theta \mid D)$$

Example: Did the Sun Just Explode?

- We have a device that detects if the sun explodes with high accuracy:

$$p(x = \theta \mid \theta) = 1 - \alpha,$$
$$p(x = 1 - \theta \mid \theta) = \alpha.$$

α: error

Known, fixed

$\alpha = 0.0001$

$\theta \in \{0, 1\}$: if the sun explode; $x \in \{0, 1\}$: if the alarm fires.

- If the alarm fires ($x = 1$). Should we believe sun has exploded or not?

① MLE:

$$\hat{\theta} = \arg\max_{\theta \in \{0,1\}} P(x=1 \mid \theta) = \begin{cases} \alpha & \theta = 0 \\ 1-\alpha & \theta = 1 \end{cases}$$

$= 1$

② Bayesian Inference:

Step 1: prior $P(\theta) = \begin{cases} 10^{-100k} \triangleq \beta, & \theta = 1 \\ \approx 1-\beta & \theta = 0 \end{cases}$

Step 2: posterior:

$$P(\theta \mid x=1) = \frac{P(x=1 \mid \theta)\, P(\theta)}{P(x=1)} \propto P(x=1 \mid \theta)\, P(\theta)$$

$$= \begin{cases} (1-\alpha)\beta & \theta = 1 \\ \alpha(1-\beta) & \theta = 0 \end{cases}$$

$\theta \overset{?}{=} 0/1$?

$$\underbrace{P(\theta=1 \mid x=1)}_{(1-\alpha)\beta} \lesseqgtr \underbrace{P(\theta=0 \mid x=1)}_{\alpha(1-\beta)}$$

$\Rightarrow$ predict $\theta = 1$ if $P(\theta=1 \mid x=1) < P(\theta=0 \mid x=1)$

$\Rightarrow (1-\alpha)\beta < \alpha(1-\beta)$

$\Rightarrow \dfrac{\beta}{1-\beta} < \dfrac{\alpha}{1-\alpha}$ ✗

predict $\theta = 0$

$\beta \approx 0 \qquad \alpha = 0.0001.$