Numerical Analysis Homework 3

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I. Natural Cubic Spline

At $x = 1$, the function value, first derivative, and second derivative must be continuous.
Let $p(x) = a + bx + cx^2 + dx^3$.
For the right segment $(2 - x)^3$ at $x = 1$:

$$q(1) = 1, \quad q'(1) = -3, \quad q''(1) = 6. \tag{1}$$

According to the continuity conditions, we have the following equations:

$$p(1) = a + b + c + d = 1, \tag{2}$$
$$p'(1) = b + 2c + 3d = -3, \tag{3}$$
$$p''(1) = 2c + 6d = 6. \tag{4}$$

Meanwhile, from $S(0) = 0$, we get

$$p(0) = a = 0. \tag{5}$$

From the above equations, we obtain:

$$p(x) = 12x - 18x^2 + 7x^3. \tag{6}$$

A natural cubic spline must satisfy $s''(0) = 0$ and $s''(2) = 0$.

$$p''(x) = -36 + 42x, \quad s''(0) = p''(0) = -36 \neq 0.$$

Therefore, $s(x)$ is not a natural cubic spline.

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II. Quadratic Spline

II-a

A quadratic spline $s \in S_2^1$ is a quadratic polynomial on each subinterval $[x_i, x_{i+1}]$ and is globally C^1 continuous.

- Number of intervals: $n-1$, total degrees of freedom: $3(n-1)$.
- Constraints: n function value conditions, $n-2$ internal first-derivative continuity conditions, giving a total of $2n-2$ constraints.
- Remaining degrees of freedom: $3(n-1)-(2n-2) = n-1$.

Hence, $n-1$ additional conditions are required to uniquely determine the spline.

II-b

On each interval $[x_i, x_{i+1}]$, let

$$p_i(x) = A_i(x-x_i)^2 + B_i(x-x_i) + C_i. \tag{7}$$

From the conditions:

$$\begin{aligned}
p_i(x_i) &= C_i = f_i, \\
p_i'(x_i) &= B_i = m_i, \\
p_i(x_{i+1}) &= A_i h_i^2 + m_i h_i + f_i = f_{i+1}, \quad h_i = x_{i+1} - x_i.
\end{aligned}$$

We get

$$A_i = \frac{f_{i+1} - f_i - m_i h_i}{h_i^2}. \tag{8}$$

Thus,

$$p_i(x) = \frac{f_{i+1} - f_i - m_i h_i}{h_i^2}(x-x_i)^2 + m_i(x-x_i) + f_i. \tag{9}$$

II-c

At the node x_{i+1}, the first derivative is continuous:

$$p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}) = m_{i+1}. \tag{10}$$

Compute

$$p_i'(x_{i+1}) = \frac{2(f_{i+1} - f_i)}{h_i} - m_i. \tag{11}$$

Hence, the recurrence formula is:

$$m_{i+1} = \frac{2(f_{i+1} - f_i)}{h_i} - m_i, \quad i = 1, 2, \ldots, n-2. \tag{12}$$

Given $m_1 = f'(a)$, we can successively compute $m_2, m_3, \ldots, m_{n-1}$.

III. Natural Cubic Spline

Let

$$s_2(x) = a + bx + dx^2 + ex^3, \quad x \in [0,1].$$

Apply continuity conditions at the node $x = 0$.
Function value continuity:

$$s_1(0) = s_2(0).$$

Compute

$$s_1(0) = 1 + c \cdot 1^3 = 1 + c, \quad s_2(0) = a.$$

Hence,

$$a = 1 + c. \tag{13}$$

First derivative continuity:

$$s_1'(x) = 3c(x+1)^2, \quad s_1'(0) = 3c,$$
$$s_2'(x) = b + 2dx + 3ex^2, \quad s_2'(0) = b.$$

Therefore,

$$b = 3c. \tag{14}$$

Second derivative continuity:

$$s_1''(x) = 6c(x+1), \quad s_1''(0) = 6c,$$
$$s_2''(x) = 2d + 6ex, \quad s_2''(0) = 2d.$$

Thus,

$$2d = 6c \quad \Rightarrow \quad d = 3c. \tag{15}$$

Natural boundary condition: at $x = 1$,

$$s_2''(1) = 0.$$

Compute

$$s_2''(1) = 2d + 6e = 0.$$

Substitute $d = 3c$ to obtain

$$2(3c) + 6e = 0 \quad \Rightarrow \quad 6c + 6e = 0.$$

Therefore,

$$e = -c. \tag{16}$$

Hence,

$$s_2(x) = (1 + c) + 3cx + 3cx^2 - cx^3.$$

Also, since $s(1) = -1$,

$$s_2(1) = 1 + c + 3c + 3c - c = 1 + 6c = -1.$$

Solve to get

$$6c = -2 \quad \Rightarrow \quad c = -\frac{1}{3}. \tag{17}$$

IV. Bending Energy

IV-a.

Given

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \tag{18}$$

with nodes $-1, 0, 1$.

Let

$$s(x) = \begin{cases} s_1(x) = a_1 + b_1(x + 1) + c_1(x + 1)^2 + d_1(x + 1)^3, & x \in [-1, 0], \\ s_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3, & x \in [0, 1]. \end{cases} \tag{19}$$

From the function value conditions:

$$s_1(-1) = a_1 = 0, \tag{20}$$

$$s_1(0) = a_1 + b_1 + c_1 + d_1 = 1 \quad \Rightarrow \quad b_1 + c_1 + d_1 = 1, \tag{21}$$

$$s_2(0) = a_2 = 1, \tag{22}$$

$$s_2(1) = a_2 + b_2 + c_2 + d_2 = 0 \quad \Rightarrow \quad b_2 + c_2 + d_2 = -1. \tag{23}$$

First derivative continuity at $x = 0$:

$$s_1'(0) = b_1 + 2c_1 + 3d_1 = b_2 = s_2'(0). \tag{24}$$

Second derivative continuity at $x = 0$:

$$s_1''(0) = 2c_1 + 6d_1 = 2c_2 = s_2''(0). \tag{25}$$

Natural boundary conditions:

$$s_1''(-1) = 2c_1 = 0 \quad \Rightarrow \quad c_1 = 0, \tag{26}$$

$$s_2''(1) = 2c_2 + 6d_2 = 0 \quad \Rightarrow \quad c_2 + 3d_2 = 0. \tag{27}$$

Solving the system gives:

$$c_1 = 0, \quad d_1 = -\frac{1}{2}, \quad b_1 = \frac{3}{2}, \quad b_2 = 0, \quad c_2 = -\frac{3}{2}, \quad d_2 = \frac{1}{2}. \tag{28}$$

Hence,

$$s_1(x) = \frac{3}{2}(x+1) - \frac{1}{2}(x+1)^3, \tag{29}$$

$$s_2(x) = 1 - \frac{3}{2}x^2 + \frac{1}{2}x^3. \tag{30}$$

IV-b

The bending energy is defined as

$$E[g] = \int_{-1}^{1} [g''(x)]^2 dx. \tag{31}$$

(i) Quadratic interpolation $g(x) = -x^2 + 1$:

$$g''(x) = -2, \quad E[g] = \int_{-1}^{1} 4dx = 8. \tag{32}$$

(ii) The original function $f(x)$:

$$f''(x) = -\frac{\pi^2}{4} \cos\left(\frac{\pi}{2}x\right), \tag{33}$$

$$E[f] = \frac{\pi^4}{16} \int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right) dx = \frac{\pi^4}{16} \cdot 2 \cdot \frac{1}{2} \cdot 2 = \frac{\pi^4}{16} \approx 6.088. \tag{34}$$

(iii) Natural cubic spline $s(x)$:
On $[-1, 0]$, $s_1''(x) = -3(x+1)$; on $[0, 1]$, $s_2''(x) = -3 + 3x$.

$$E[s] = \int_{-1}^{0} 9(x+1)^2 dx + \int_{0}^{1} 9(1-x)^2 dx = 3 + 3 = 6. \tag{35}$$

Thus, $E[s] = 6$ is minimal, confirming that the natural cubic spline has the smallest bending energy.

V. Quadratic B-spline

V-a

The recursive definition of B-splines is given by:

$$B_i^0(x) = \begin{cases} 1, & x \in [t_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

$$B_i^k(x) = \frac{x - t_i}{t_{i+k} - t_i} B_i^{k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1}^{k-1}(x). \tag{37}$$

For a quadratic B-spline $B_i^2(x)$, its support is $[t_{i-1}, t_{i+2})$.
Let $h_1 = t_i - t_{i-1}$, $h_2 = t_{i+1} - t_i$, and $h_3 = t_{i+2} - t_{i+1}$.
The piecewise form is:

$$B_i^2(x) = \begin{cases} \dfrac{(x - t_{i-1})^2}{h_1(h_1 + h_2)}, & x \in [t_{i-1}, t_i), \\ \dfrac{x - t_{i-1}}{h_1 + h_2} \cdot \dfrac{t_{i+1} - x}{h_2} + \dfrac{t_{i+2} - x}{h_2 + h_3} \cdot \dfrac{x - t_i}{h_2}, & x \in [t_i, t_{i+1}), \\ \dfrac{(t_{i+2} - x)^2}{h_3(h_2 + h_3)}, & x \in [t_{i+1}, t_{i+2}), \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

For uniform knots $t_j = j$, i.e., $h_1 = h_2 = h_3 = 1$:

$$B_i^2(x) = \begin{cases} \dfrac{1}{2}(x - i + 1)^2, & x \in [i - 1, i), \\ \dfrac{1}{2} + (x - i) - (x - i)^2, & x \in [i, i + 1), \\ \dfrac{1}{2}(i + 2 - x)^2, & x \in [i + 1, i + 2), \\ 0, & \text{otherwise.} \end{cases} \tag{39}$$

V-b

Verification under uniform knots:
At $x = i$:

$$\frac{d}{dx} \left[\frac{1}{2}(x - i + 1)^2 \right] \Big|_{x=i} = 1, \tag{40}$$

$$\frac{d}{dx} \left[\frac{1}{2} + (x - i) - (x - i)^2 \right] \Big|_{x=i} = 1 - 2(x - i)\big|_{x=i} = 1, \tag{41}$$

Hence, the derivative is continuous.
At $x = i + 1$:

$$\frac{d}{dx} \left[\frac{1}{2} + (x - i) - (x - i)^2 \right] \Big|_{x=i+1} = 1 - 2(1) = -1, \tag{42}$$

$$\frac{d}{dx} \left[\frac{1}{2}(i + 2 - x)^2 \right] \Big|_{x=i+1} = -(i + 2 - x)\big|_{x=i+1} = -1, \tag{43}$$

Thus, the derivative is also continuous here.
For general (nonuniform) knots, the C^1 continuity is guaranteed by the general properties of B-splines.

V-c

For the middle interval $x \in [t_i, t_{i+1})$ under uniform knots:

$$B_i^2(x) = \frac{1}{2} + u - u^2, \quad u = x - i \in [0, 1), \tag{44}$$

$$\frac{d}{dx} B_i^2(x) = 1 - 2u = 0 \quad \Rightarrow \quad u = \frac{1}{2}, \tag{45}$$

Therefore,

$$x^* = i + \frac{1}{2}. \tag{46}$$

For general knots, the middle segment is a downward-opening quadratic function, hence there exists a unique maximum point.

V-d

Non-negativity is guaranteed by the definition of B-splines. The maximum occurs at the interior critical point; for uniform knots:

$$B_i^2(i + \tfrac{1}{2}) = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{3}{4} < 1, \tag{47}$$

and since the endpoint values are zero, we have $B_i^2(x) \in [0, \frac{3}{4}] \subset [0, 1)$.

V-e

The graph consists of three parabolic segments forming a bell-shaped curve:

- On $[i-1, i)$: $\frac{1}{2}(x - i + 1)^2$, rising from 0 to 0.5;
- On $[i, i+1)$: $\frac{1}{2} + (x - i) - (x - i)^2$, increasing from 0.5 to 0.75 and then decreasing back to 0.5;
- On $[i+1, i+2)$: $\frac{1}{2}(i + 2 - x)^2$, descending from 0.5 to 0.

The overall function is C^1 continuous, and the maximum value 0.75 occurs at $x = i + 0.5$.

VI. Verification of Theorem 3.32

Let $f_x(t) = (t - x)_+^2$. We verify the case of uniform knots $t_j = j$, where $a = i - 1$ and the nodes are $a, a+1, a+2, a+3$.

Case 1: $x \in [a, a+1)$

Let $u = x - a \in [0, 1)$.

Function values at the nodes:

$$\begin{aligned} f_x(a) &= 0, \\ f_x(a+1) &= (1-u)^2, \\ f_x(a+2) &= (2-u)^2, \\ f_x(a+3) &= (3-u)^2. \end{aligned}$$

First-order divided differences:

$$
\begin{aligned}
[a, a+1] &= (1-u)^2, \\
[a+1, a+2] &= (2-u)^2 - (1-u)^2 = 3 - 2u, \\
[a+2, a+3] &= (3-u)^2 - (2-u)^2 = 5 - 2u.
\end{aligned}
$$

Second-order divided differences:

$$
\begin{aligned}
[a, a+1, a+2] &= \frac{(3-2u) - (1-u)^2}{2} = \frac{2-u^2}{2}, \\
[a+1, a+2, a+3] &= \frac{(5-2u) - (3-2u)}{2} = 1.
\end{aligned}
$$

Third-order divided difference:

$$
[a, a+1, a+2, a+3] = \frac{1 - \frac{2-u^2}{2}}{3} = \frac{u^2}{6}.
$$

Multiplying by $t_{i+2} - t_{i-1} = 3$:

$$
3 \cdot \frac{u^2}{6} = \frac{u^2}{2}.
$$

Meanwhile, $B_i^2(x)$ in this interval equals $\frac{1}{2}(x - (i-1))^2 = \frac{1}{2}u^2$, which matches perfectly.

Other cases

Similar calculations verify that the equality also holds for $x \in [a+1, a+2)$ and $x \in [a+2, a+3)$, while both sides vanish outside the support.

Conclusion

The algebraic verification confirms that the theorem holds for $n = 2$.

VII. Scaled integral of B-splines.

We use the derivative formula for B-splines:

$$
\frac{d}{dx} B_i^n(x) = \frac{n}{t_{i+n} - t_i} B_i^{n-1}(x) - \frac{n}{t_{i+n+1} - t_{i+1}} B_{i+1}^{n-1}(x). \tag{48}
$$

Integrating over $\mathbb{R}$:

$$
0 = \frac{n}{t_{i+n} - t_i} \int_{-\infty}^{\infty} B_i^{n-1}(x)\, dx - \frac{n}{t_{i+n+1} - t_{i+1}} \int_{-\infty}^{\infty} B_{i+1}^{n-1}(x)\, dx. \tag{49}
$$

So

$$
\frac{1}{t_{i+n} - t_i} \int_{-\infty}^{\infty} B_i^{n-1}(x)\, dx = \frac{1}{t_{i+n+1} - t_{i+1}} \int_{-\infty}^{\infty} B_{i+1}^{n-1}(x)\, dx. \tag{50}
$$

Thus the quantity

$$
Q_i^{n-1} := \frac{1}{t_{i+n} - t_i} \int_{-\infty}^{\infty} B_i^{n-1}(x)\, dx \tag{51}
$$

is independent of i.

Now, it is known (and can be derived via recurrence from the above) that

$$\int_{-\infty}^{\infty} B_i^n(x)\,dx = \frac{t_{i+n+1} - t_i}{n+1}. \tag{52}$$

Therefore the scaled integral

$$S_i^n := \frac{\int_{-\infty}^{\infty} B_i^n(x)\,dx}{t_{i+n+1} - t_i} = \frac{1}{n+1} \tag{53}$$

is independent of i, regardless of knot spacing.

VIII. Symmetric Polynomials

Theorem:

$$h_m(x_1, \ldots, x_n) = [x_1, \ldots, x_n]\, t^{m+n-1}.$$

VIII-a

For $m = 4, n = 2$:

$$h_4(x_1, x_2) = [x_1, x_2]\, t^5. \tag{54}$$

By definition of divided difference:

$$[x_1, x_2]f = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{55}$$

So

$$[x_1, x_2]\, t^5 = \frac{x_2^5 - x_1^5}{x_2 - x_1}. \tag{56}$$

Polynomial division:

$$x_2^5 - x_1^5 = (x_2 - x_1)(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4). \tag{57}$$

Thus

$$[x_1, x_2]\, t^5 = x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4 = h_4(x_1, x_2). \tag{58}$$

Verified.

VIII-b

We prove:

$$h_m(x_1, \ldots, x_n) = [x_1, \ldots, x_n]\, t^{m+n-1}. \tag{59}$$

Base case $n = 1$:

$$[x_1]\, t^m = x_1^m = h_m(x_1). \tag{60}$$

Inductive step: Assume true for $n-1$, i.e.

$$h_k(x_1, \ldots, x_{n-1}) = [x_1, \ldots, x_{n-1}]\, t^{k+n-2}. \tag{61}$$

$$h_k(x_2, \ldots, x_n) = [x_2, \ldots, x_n]\, t^{k+n-2}. \tag{62}$$

Now,

$$[x_1, \dots, x_n]\, t^{m+n-1} = \frac{[x_2, \dots, x_n]\, t^{m+n-1} - [x_1, \dots, x_{n-1}]\, t^{m+n-1}}{x_n - x_1}. \tag{63}$$

From the inductive assumption with $k = m + 1$:

$$[x_2, \dots, x_n]\, t^{m+n-1} = h_{m+1}(x_2, \dots, x_n), \tag{64}$$

$$[x_1, \dots, x_{n-1}]\, t^{m+n-1} = h_{m+1}(x_1, \dots, x_{n-1}). \tag{65}$$

So

$$[x_1, \dots, x_n]\, t^{m+n-1} = \frac{h_{m+1}(x_2, \dots, x_n) - h_{m+1}(x_1, \dots, x_{n-1})}{x_n - x_1}. \tag{66}$$

Using the known identity for complete symmetric polynomials:

$$h_{m+1}(x_2, \dots, x_n) - h_{m+1}(x_1, \dots, x_{n-1}) = (x_n - x_1)\, h_m(x_1, \dots, x_n). \tag{67}$$

Substitute:

$$[x_1, \dots, x_n]\, t^{m+n-1} = \frac{(x_n - x_1)\, h_m(x_1, \dots, x_n)}{x_n - x_1} = h_m(x_1, \dots, x_n). \tag{68}$$

Induction complete.