Numerical Analysis Homework 2

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I. Linear Interpolation

I-a

First, calculate $p_1(f;x)$ based on the conditions given in the problem:

$$p_1(f;x) = \frac{3-x}{2} \tag{1}$$

$$f(x) - p_1(f;x) = \frac{1}{x} - \frac{3-x}{2} \tag{2}$$

Therefore,

$$f''(\xi(x)) = \left(\frac{1}{x} - \frac{3-x}{2}\right) \cdot \frac{2}{(x-1)(x+2)} = \frac{1}{x} \tag{3}$$

$$\xi(x) = (2x)^{\frac{1}{3}} \tag{4}$$

I-b

Examine the monotonicity of $\xi(x)$ and $f''(\xi(x))$:

$$\xi'(x) = \frac{2^{\frac{1}{3}} x^{-\frac{2}{3}}}{3} > 0, \quad x \in [1,2] \tag{5}$$

$$f'''(\xi(x)) = -x^{-2} < 0, \quad x \in [1,2] \tag{6}$$

Therefore,

$$\min \xi(x) = 2^{\frac{1}{3}}, \quad \max \xi(x) = 4^{\frac{1}{3}}, \quad \max f''(\xi(x)) = 1 \tag{7}$$

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II. Non-negative Polynomials

Since there are n given parameters and a polynomial of degree 2n is required, the following polynomial is a possible reasonable solution:

$$p(x) = \sum_{i=1}^{2n} (x - x_i)^2 \tag{8}$$

III. Proof of Divided Difference Formula

III-a

When $n = 0$,

$$f[t] = e^t, \quad \frac{(e-1)^0}{0!} e^t = e^t \tag{9}$$

When $n = 1$,

$$f[t, t+1] = \frac{e^{t+1} - e^t}{(t+1) - t} = e^t(e-1) = \frac{e-1}{1!} e^t \tag{10}$$

Assume the formula holds for $n = k$, i.e.,

$$f[t, \ldots, t+k] = \frac{(e-1)^k}{k!} e^t \tag{11}$$

Then for $n = k+1$, we have

$$f[t, \ldots, t+k+1] = \frac{f[t+1, \ldots, t+k+1] - f[t, \ldots, t+k]}{(t+k+1) - t} \tag{12}$$

By the induction hypothesis,

$$f[t+1, \ldots, t+k+1] = \frac{(e-1)^k}{k!} e^{t+1}, \quad f[t, \ldots, t+k] = \frac{(e-1)^k}{k!} e^t \tag{13}$$

Substituting, we get

$$f[t, \ldots, t+k+1] = \frac{\frac{(e-1)^k}{k!} e^{t+1} - \frac{(e-1)^k}{k!} e^t}{k+1} \tag{14}$$

$$= \frac{(e-1)^k e^t (e-1)}{k!(k+1)} = \frac{(e-1)^{k+1}}{(k+1)!} e^t \tag{15}$$

Q.E.D.

III-b

From the first equation, set $t = 0$,

$$f[0, 1, \ldots, n] = \frac{(e-1)^n}{n!} \tag{16}$$

From the second equation,

$$f[0,1,\ldots,n] = \frac{e^{\xi}}{n!} \tag{17}$$

Combining the above results,

$$\xi = n\ln(e-1) \approx 0.541n > \frac{n}{2} \tag{18}$$

IV. Newton Formula

IV-a

Newton interpolation formula:

$$p_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) \tag{19}$$

where $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$.
Compute the 0th order divided difference:

$$a_0 = f[x_0] = f(0) = 5 \tag{20}$$

1st order divided differences:

$$f[x_0, x_1] = \frac{f(1) - f(0)}{1-0} = \frac{3-5}{1} = -2 \tag{21}$$

$$a_1 = -2 \tag{22}$$

2nd order divided differences:

$$f[x_1, x_2] = \frac{f(3) - f(1)}{3-1} = \frac{5-3}{2} = 1 \tag{23}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1-(-2)}{3-0} = \frac{3}{3} = 1 \tag{24}$$

$$a_2 = 1 \tag{25}$$

3rd order divided differences:

$$f[x_2, x_3] = \frac{f(4) - f(3)}{4-3} = \frac{12-5}{1} = 7 \tag{26}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{7-1}{4-1} = \frac{6}{3} = 2 \tag{27}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{2-1}{4-0} = \frac{1}{4} = 0.25 \tag{28}$$

$$a_3 = 0.25 \tag{29}$$

Substitute into the interpolation formula, expand and simplify to get the final result:

$$p_3(f;x) = 0.25x^3 - 2.25x + 5 \tag{30}$$

IV-b

Use $p_3(f;x)$ to approximate $f(x)$, find the minimum point within the interval $(1,3)$.

$$p_3'(f;x) = 0.75x^2 - 2.25 = 0 \tag{31}$$

$$x = \sqrt{3} \tag{32}$$

Verification confirms that $x = \sqrt{3}$ is a minimum point.

V. Divided Difference

V-a

Compute the divided differences:

x	0th Order	1st Order	2nd Order	3rd Order	4th Order	5th Order
0	0					
		1				
1	1		6			
		7		15		
1	1		21		42	
		7		99		30
1	1		120		102	
		127		201		
2	128		321			
		448				
2	128					

From the table above, $f[0,1,1,1,2,2] = 30$

V-b

$$f[0,1,1,1,2,2] = \frac{f^{(5)}(\xi)}{5!}, \quad \xi \in (0,2) \tag{33}$$

Substituting the data, solve to get

$$\xi = \sqrt{\frac{10}{7}} \tag{34}$$

VI. Hermite Interpolation

VI-a

Compute the divided differences:

$$\begin{aligned}
f[0,1] &= \frac{2-1}{1-0} = 1 \\
f[1,1] &= f'(1) = -1 \\
f[1,3] &= \frac{0-2}{3-1} = -1 \\
f[3,3] &= f'(3) = 0 \\
f[0,1,1] &= \frac{-1-1}{1-0} = -2 \\
f[1,1,3] &= \frac{-1-(-1)}{3-1} = 0 \\
f[1,3,3] &= \frac{0-(-1)}{3-1} = 0.5 \\
f[0,1,1,3] &= \frac{0-(-2)}{3-0} = \frac{2}{3} \\
f[1,1,3,3] &= \frac{0.5-0}{3-1} = 0.25 \\
f[0,1,1,3,3] &= \frac{0.25-\frac{2}{3}}{3-0} = -\frac{5}{36}
\end{aligned}$$

$$\begin{aligned}
H_4(x) =& 1 + 1\cdot x + (-2)x(x-1) + \frac{2}{3}x(x-1)^2 \\
&+ \left(-\frac{5}{36}\right)x(x-1)^2(x-3)
\end{aligned} \tag{35}$$

$$f(2) \approx H_4(2) = \frac{11}{18} \tag{36}$$

VI-b

Hermite interpolation error formula:

$$f(x) - H_4(x) = \frac{f^{(5)}(\xi)}{5!}(x-0)(x-1)^2(x-3)^2 \tag{37}$$

Therefore, the maximum possible error at $x = 2$ is:

$$|f(2) - H_4(2)| \le \frac{M}{60} \tag{38}$$

VII. Forward and Backward Difference

When $k = 1$:

$$f[x_0, x_1] = \frac{f(x_1 - x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h} \tag{39}$$

Assume the formula holds for $k = m - 1$, i.e.,

$$f[x_0, x_1, \dots x_{m-1}] = \frac{\Delta^{(m-1)} f(x_0)}{h^{m-1}(m-1)!} \tag{40}$$

$$f[x_1, x_2, \dots x_m] = \frac{\Delta^{(m-1)} f(x_1)}{h^{m-1}(m-1)!} \tag{41}$$

Then when $k = m$,

$$f[x_0, x_1, \dots x_m] = \frac{f[x_1, x_2, \dots x_m] - f[x_0, x_1, \dots x_{m-1}]}{x_m - x_0} = \frac{\Delta^{(m)} f(x_0)}{h^m (m)!} \tag{42}$$

$$\Delta^m f(x_0) = h^m m! f[x_0, x_1, \dots x_m] \tag{43}$$

By mathematical induction, the formula is proved.
Similarly, it can be shown that

$$\nabla^k f(x_0) = h^k k! f[x_0, x_{-1}, \dots x_{-k}] \tag{44}$$

VIII. Partial Derivative of Divided Differences

From the recursive definition of divided differences,

$$f[t, x_0, x_1, \dots, x_n] = \frac{f[t, x_1, \dots, x_n] - f[x_0, x_1, \dots, x_n]}{t - x_0} = \frac{g(t) - g(x_0)}{t - x_0}. \tag{45}$$

Taking the limit as $t \to x_0$ on both sides, and since f is differentiable at x_0, the limit exists, yielding

$$\lim_{t \to x_0} f[t, x_0, x_1, \dots, x_n] = \lim_{t \to x_0} \frac{g(t) - g(x_0)}{t - x_0} = g'(x_0). \tag{46}$$

By the definition of divided differences with repeated nodes,

$$f[x_0, x_0, x_1, \dots, x_n] := \lim_{t \to x_0} f[t, x_0, x_1, \dots, x_n]. \tag{47}$$

Therefore,

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \dots, x_n] = g'(x_0) = f[x_0, x_0, x_1, \dots, x_n]. \tag{48}$$

Similarly, due to the symmetry of divided differences with respect to all nodes, for any $k = 0, 1, \dots, n$, we have

$$\frac{\partial}{\partial x_k} f[x_0, x_1, \dots, x_n] = f[x_0, \dots, x_k, x_k, \dots, x_n]. \tag{49}$$

IX. A Min-Max Problem

On the interval $x \in [-1, 1]$, when any polynomial $p(x) = \frac{T_n(x)}{2^{n-1}}$, its maximum absolute value attains the minimum:

$$\min \max_{x\in[-1,1]} |P(x)| = \frac{|a_0|}{2^{n-1}} \tag{50}$$

where

$$T_n(x) = \cos(n \arccos x) \tag{51}$$

Next, map $[a, b]$ to $[-1, 1]$, obtaining the final result:

$$\min \max_{x\in[a,b]} |P(x)| = \frac{|a_0|(b-a)^n}{2^{2n-1}} \tag{52}$$

X. Imitate the Proof of Chebyshev Theorem

Let $T_n(x)$ be the Chebyshev polynomial of degree $n \in \mathbb{N}$, extended to $\mathbb{R}$. For fixed $a > 1$, define

$$\mathcal{P}_n^a := \{p \in \mathcal{P}_n : p(a) = 1\}$$

and

$$\hat{p}_n(x) := \frac{T_n(x)}{T_n(a)}.$$

For $\|f\|_\infty = \max_{x\in[-1,1]} |f(x)|$, we have

$$\forall p \in \mathcal{P}_n^a, \quad \|\hat{p}_n\|_\infty \le \|p\|_\infty.$$

We proceed by contradiction. Suppose there exists some $p \in \mathcal{P}_n^a$ such that

$$\|p\|_\infty < \|\hat{p}_n\|_\infty. \tag{53}$$

Recall that on $[-1, 1]$, $|T_n(x)| \le 1$, and $T_n(a) > 1$ for $a > 1$. Therefore,

$$|\hat{p}_n(x)| \le \frac{1}{T_n(a)} \quad \text{for all } x \in [-1, 1], \tag{54}$$

and

$$\|\hat{p}_n\|_\infty = \frac{1}{T_n(a)} =: M. \tag{55}$$

Let $x_k = \cos(k\pi/n)$ for $k = 0, 1, \ldots, n$ be the extreme points of $T_n(x)$ in $[-1, 1]$, where

$$T_n(x_k) = (-1)^k. \tag{56}$$

Then

$$\hat{p}_n(x_k) = \frac{(-1)^k}{T_n(a)}. \tag{57}$$

From inequality (1), we have $|p(x)| < M$ for all $x \in [-1, 1]$. Define the difference

$$r(x) = \hat{p}_n(x) - p(x). \tag{58}$$

Note that $r(a) = 1 - 1 = 0$.

Now consider the sign of $r(x_k)$:

- For even k: $\hat{p}_n(x_k) = M > 0$ and $p(x_k) < M$, so $r(x_k) > 0$.
- For odd k: $\hat{p}_n(x_k) = -M < 0$ and $p(x_k) > -M$, so $r(x_k) < 0$.

Thus, $r(x)$ alternates in sign at the $n + 1$ points $x_0, x_1, \ldots, x_n$.

By the Intermediate Value Theorem, $r(x)$ has at least n distinct zeros in $(-1, 1)$. Since $r(a) = 0$ with $a > 1$, we have an additional zero outside $[-1, 1]$.

Therefore, $r(x)$ has at least $n+1$ distinct zeros. But $r(x)$ is a polynomial of degree at most n, so this forces $r(x) \equiv 0$, implying $p(x) \equiv \hat{p}_n(x)$, which contradicts assumption (1).

Hence, no such p exists, and we conclude

$$\|\hat{p}_n\|_\infty \leq \|p\|_\infty \quad \text{for all } p \in \mathcal{P}_n^a. \tag{59}$$

XI. Bernstein Polynomials

We prove the identity:

$$B_{n-1,k}(t) = \frac{n-k}{n} B_{n,k}(t) + \frac{k+1}{n} B_{n,k+1}(t) \tag{60}$$

where $B_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$.

Compute the right-hand side:

$$\frac{n-k}{n} B_{n,k}(t) = \frac{n-k}{n} \cdot \binom{n}{k} t^k (1-t)^{n-k} \tag{61}$$

$$\frac{k+1}{n} B_{n,k+1}(t) = \frac{k+1}{n} \cdot \binom{n}{k+1} t^{k+1} (1-t)^{n-k-1} \tag{62}$$

Using the identities:

$$\frac{n-k}{n} \binom{n}{k} = \binom{n-1}{k}, \quad \frac{k+1}{n} \binom{n}{k+1} = \binom{n-1}{k} \tag{63}$$

we obtain:

$$\text{RHS} = \binom{n-1}{k} t^k (1-t)^{n-k} + \binom{n-1}{k} t^{k+1} (1-t)^{n-k-1} \tag{64}$$

Factor out the common term:

$$\text{RHS} = \binom{n-1}{k} t^k (1-t)^{n-k-1} \left[(1-t) + t\right] \tag{65}$$

Since $(1-t) + t = 1$, we have:

$$\text{RHS} = \binom{n-1}{k} t^k (1-t)^{n-1-k} = B_{n-1,k}(t) \tag{66}$$

This completes the proof.

XII. Integral of a Bernstein Basis Polynomial

Using the Beta function:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q > 0 \tag{67}$$

For the integral of $B_{n,k}(t)$:

$$\int_0^1 B_{n,k}(t) dt = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dt \tag{68}$$

Let $p = k+1$, $q = n-k+1$:

$$\int_0^1 t^k (1-t)^{n-k} dt = B(k+1, n-k+1) = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} \tag{69}$$

Using factorial expressions:

$$\Gamma(k+1) = k!, \quad \Gamma(n-k+1) = (n-k)!, \quad \Gamma(n+2) = (n+1)! \tag{70}$$

Thus:

$$\int_0^1 t^k (1-t)^{n-k} dt = \frac{k!(n-k)!}{(n+1)!} \tag{71}$$

Substituting back:

$$\int_0^1 B_{n,k}(t) dt = \binom{n}{k} \cdot \frac{k!(n-k)!}{(n+1)!} = \frac{n!}{k!(n-k)!} \cdot \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1} \tag{72}$$