Numerical Analysis Homework 1

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Due time: 2025 年 9 月 21 日

I. Bisection Method and Interval

I-a

$w_n = 2^{1-n}$

I-b

$\sup L = 2^{-n}$

II. Prove Step Count for Guaranteed Relative Error in Bisection

Proof: When the bisection method is at step n, according to the conclusion of Problem I, the supremum of the distance between the root and the midpoint of the interval is $\frac{b_0-a_0}{2^{1+n}}$, and the relative error is $\frac{b_0-a_0}{r \cdot 2^{1+n}}$ (where r is the exact solution). Considering the worst-case scenario, where r takes the smallest possible value a_0 to maximize the relative error, the following condition must be satisfied:

$$\frac{b_0 - a_0}{a_0 \cdot 2^{1+n}} \leq \epsilon \tag{1}$$

Taking the logarithm of both sides of equation (1) and simplifying, we finally obtain:

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1 \tag{2}$$

III. Prove Step Count for Guaranteed Relative Error in Bisection

From the given information:

$$p(x)' = 12x^2 - 4x \tag{3}$$

The iteration formula for Newton's method is:

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = x_n - \frac{4x_n^3 - 2x_n^2 + 3}{12x_n^2 - 4x_n} \tag{4}$$

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The following table shows the iteration results calculated using a calculator:

表 1: Newton's Method Iteration Results ($x_0 = -1$)

Iteration Step	x_n	$p(x_n)$	$p'(x_n)$	x_{n+1}
0	-1.0	-3.0	16.0	-0.8125
1	-0.8125	-0.4658203125	11.171875	-0.770797
2	-0.770797	-0.01984	10.212724	-0.768854
3	-0.768854	0.00003	10.16906	-0.768857
4	-0.768857	≈ 0		

IV. Linear Convergence with Constant Factor

From the given information:

$$e_n = x_n - \alpha, \quad f(\alpha) = 0 \tag{5}$$

Substituting into the variant of Newton's iteration formula yields:

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_0)} \tag{6}$$

Performing a Taylor expansion of $f(x)$ around α gives:

$$f(x_n) = f(\alpha + e_n) = f'(\alpha)e_n + \frac{f''(\alpha)}{2}e_n^2 + o(e_n^3) \tag{7}$$

Substituting (7) into (6) and ignoring higher-order terms yields:

$$e_{n+1} = e_n\left(1 - \frac{f'(\alpha)}{f'(x_0)}\right) - \frac{f''(\alpha)}{2f'(x_0)}e_n^2 + o(e_n^3) \approx e_n\left(1 - \frac{f'(\alpha)}{f'(x_0)}\right) \tag{8}$$

Comparing (8) with $e_{n+1} = Ce_n^s$, we find:

$$s = 1, \quad C = 1 - \frac{f'(\alpha)}{f'(x_0)} \tag{9}$$

V. Convergence Analysis

Calculating the fixed point of $x_{n+1} = \arctan x_n$:

$$x = \arctan x \tag{10}$$

There is only one solution $x = 0$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$. To analyze the convergence of the fixed point, consider the function:

$$g(x) = x - \arctan x, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \tag{11}$$

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0 \tag{12}$$

$g(x)$ is a monotonically increasing odd function within its domain. For the iteration sequence:

$$x_{n+1} = \arctan x_n < x_n, \quad x \in (0, \frac{\pi}{2}) \tag{13}$$

$$x_{n+1} = \arctan x_n > x_n, \quad x \in (-\frac{\pi}{2}, 0) \tag{14}$$

It converges to the fixed point $x = 0$.

VI. Value of Continued Fraction

Creating the iteration sequence:

$$x_1 = \frac{1}{p}, \quad x_2 = \frac{1}{p + \frac{1}{p}}, \quad x_3 = \frac{1}{p + \frac{1}{p+\frac{1}{p}}}, \cdots \tag{15}$$

$\Longrightarrow$

$$x_{n+1} = \frac{1}{p + x_n} \tag{16}$$

Calculating the fixed point:

$$x = \frac{1}{p + x} \tag{17}$$

$\Longrightarrow$

$$x = \frac{-p \pm \sqrt{p^2 + 4}}{2} \tag{18}$$

That is, the sequence $\{x_n\}$ converges to (18).

VII. Further Thinking on Problem II

When $a_0 < 0 < b_0$, $\log a_0$ is undefined, making formula (2) invalid. Examining the expression for relative error:

$$\frac{|x_n - r|}{|r|} \leq \epsilon \tag{19}$$

This means n must satisfy:

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log|r|}{\log 2} - 1 \tag{20}$$

In this case, if r is very close to 0, the relative error can be amplified arbitrarily, making the concept of relative error an unsuitable method here.

VIII. Multiple Zero in Newton's Method

VIII-a

When r is a k-fold zero of f, near $x = r$ we have:

$$f(x) \approx \frac{f^{(k)}(r)}{k!}(x - r)^k \tag{21}$$

$$f'(x) \approx \frac{f^{(k)}(r)}{(k-1)!}(x-r)^{(k-1)} \tag{22}$$

Substituting into Newton's iteration formula gives:

$$x_{n+1} \approx x_n - \frac{x_n - r}{k} \tag{23}$$

The absolute error e_n satisfies the following relation:

$$e_{n+1} \approx (1 - \frac{1}{k})e_n \tag{24}$$

From this, it can be seen that when Newton's iteration converges significantly slower (linear convergence instead of quadratic convergence), it may indicate a multiple zero.

VIII-b

Proof. Let r be a k-fold zero of f, then:

$$f(x) = (x-r)^k g(x), \quad g(r) \neq 0 \tag{25}$$

$$f'(x) = (x-r)^{(k-1)}[kg(x) + (x-r)g'(x)] \tag{26}$$

Substituting into the modified Newton's method iteration formula given in the problem yields:

$$x_{n+1} = x_n - k\frac{(x_n - r)^k g(x_n)}{(x_n - r)^{k-1}\left[kg(x_n) + (x_n - r)g'(x_n)\right]} \tag{27}$$

$$= x_n - k\frac{(x_n - r)g(x_n)}{kg(x_n) + (x_n - r)g'(x_n)} \tag{28}$$

Let $e_n = x_n - r$, then:

$$x_{n+1} = r + e_n - k\frac{e_n g(x_n)}{kg(x_n) + e_n g'(x_n)} \tag{29}$$

$$= r + e_n^2\left[\frac{g'(x_n)}{kg(x_n) + e_n g'(x_n)}\right] \tag{30}$$

Therefore, the error recurrence relation is:

$$e_{n+1} = x_{n+1} - r = e_n^2\left[\frac{g'(x_n)}{kg(x_n) + e_n g'(x_n)}\right] \tag{31}$$

When $x_n \to r$, $e_n \to 0$, and $g(x_n) \to g(r) \neq 0$, $g'(x_n) \to g'(r)$, hence:

$$\frac{g'(x_n)}{kg(x_n) + e_n g'(x_n)} \to \frac{g'(r)}{kg(r)} = C \quad \text{(constant)} \tag{32}$$

So:

$$e_{n+1} \approx Ce_n^2 \tag{33}$$

This shows that the modified iteration has quadratic convergence.