Problem 1

$\min \; Z(x_1, x_2) = 4(x_1-2)^2 + 3(x_2-4)^2$

s.t. $g_1(x_1, x_2) = -x_1 - x_2 + 5 \le 0$

$g_2(x_1, x_2) = -x_1 + 1 \le 0$

$g_3(x_1, x_2) = -x_2 + 2 \le 0$

$\nabla Z(x_1, x_2) = \begin{bmatrix} 8(x_1-2) \\ 6(x_2-4) \end{bmatrix}, \quad \nabla g_1(x_1, x_2) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\nabla g_2(x_1, x_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \nabla g_3(x_1, x_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

KKT Conditions

$8(x_1^*-2) - \mu_1^* - \mu_2^* = 0$ — (1)

$6(x_2^*-4) - \mu_1^* - \mu_3^* = 0$ — (2)

$\mu_1^*(-x_1^* - x_2^* + 5) = 0$ — (3)

$\mu_2^*(-x_1^* + 1) = 0$ — (4)

$\mu_3^*(-x_2^* + 2) = 0$ — (5)

$-x_1^* - x_2^* + 5 \le 0$ — (6)

$-x_1^* + 1 \le 0$ — (7)

$-x_2^* + 2 \le 0$ — (8)

$\mu_1, \mu_2, \mu_3 \ge 0$ — (9)

If (7) and (8) are binding at the same time, (9) can't be satisfied.

Case 1: (6) is binding, and (7) and (8) aren't binding ($\mu_1^* > 0, \mu_2^* = \mu_3^* = 0$)

$\left.\begin{array}{l} 8(x_1^*-2) - \mu_1^* = 0 \\ 6(x_2^*-4) - \mu_1^* = 0 \end{array}\right\} \Rightarrow 4x_1^* - 3x_2^* = -4$

$x_1^* + x_2^* = 5 \Rightarrow 4x_1^* + 4x_2^* = 20$

$(x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = \left(\frac{11}{7}, \frac{24}{7}, \frac{-24}{7}, 0, 0\right)$ (Infeasible !!)

Case 2: Binding → (6) and (7) ; Not binding → (8) ($\mu_1^*, \mu_2^* > 0, \mu_3^* = 0$)

$8(x_1^*-2) - \mu_1^* - \mu_2^* = 0$

$6(x_2^*-4) - \mu_1^* = 0$ $\Rightarrow (x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (1, 4, 0, -8, 0)$

$x_1^* = 1, \quad x_1^* + x_2^* = 5$ Infeasible !!

Case 3: Binding → (6) and (8) ; Not binding → (7) ($\mu_1^*, \mu_3^* > 0, \mu_2^* = 0$)

$8(x_1^*-2) - \mu_1^* = 0$

$6(x_2^*-4) - \mu_1^* - \mu_3^* = 0 \Rightarrow (x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (3, 2, 8, 0, -20)$

$x_2^* = 2, \quad x_1^* + x_2^* = 5$ Infeasible !!

Case 4: Binding → (7) ; Not binding → (6) and (8) ($\mu_1^* = \mu_3^* = 0, \mu_2^* > 0$)

$8(x_1^*-2) - \mu_2^* = 0$

$6(x_2^*-4) = 0 \quad \Rightarrow (x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (1, 4, 0, -8, 0)$

$x_1^* = 1$ Infeasible !!

Case 5: Binding → (8) ; Not binding → (6) and (7) $(\mu_1^* = \mu_2^* = 0, \mu_3^* > 0)$

$8(x_1^*-2) = 0$

$6(x_2^*-4) - \mu_3^* = 0 \quad \Rightarrow (x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (2, 2, 0, 0, -12)$

$x_2^* = 2$ Infeasible

Case 6: $\mu_1^* = \mu_2^* = \mu_3^* = 0$

$8(x_1^*-2) = 0$

$6(x_2^*-4) = 0 \quad \Rightarrow (x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (2, 4, 0, 0, 0)$

It is a KKT point, and $z(x_1^*, x_2^*) = 0$

$\nabla^2 z(x_1, x_2) = \begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix}$, which is definitely positive.

⇒ This solution is unique !

Problem 2

$\min\ z(x) = 5x_1^2 - 3x_1x_2 + 2x_2^2 + 7$

s.t. $g_1(x) = 5 - 2x_1 - x_2 = 0$

$g_2(x) = -x_1 \le 0, \quad g_3(x) = -x_2 \le 0$

$\nabla z(x) = \begin{bmatrix} 10x_1 - 3x_2 \\ -3x_1 + 4x_2 \end{bmatrix}, \quad \nabla g_1(x) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \nabla g_2(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \nabla g_3(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

KKT Conditions

$10x_1 - 3x_2 - 2\mu_1 - \mu_2 = 0$

$-3x_1 + 4x_2 - \mu_1 - \mu_3 = 0$

$\mu_1(5 - 2x_1 - x_2) = 0$

$\mu_2(-x_1) = 0$

$\mu_3(-x_2) = 0$

$5 - 2x_1 - x_2 = 0$

$-x_1 \le 0$

$-x_2 \le 0$

$\mu_1, \mu_2, \mu_3 \ge 0$

∵ $g_1(x)$ is an equation

∴ $g_1(x)$ must be binding

Then, $\mu_1 > 0$, so we only need to check if $g_2(x)$ and $g_3(x)$ are binding or not.

Moreover, $g_2(x)$ and $g_3(x)$ can't be binding at the same time.

Case 1: Binding → $g_2(x)$, Not binding → $g_3(x)$ $(\mu_1, \mu_2 > 0, \mu_3 = 0)$

$10x_1^* - 3x_2^* - 2\mu_1^* - \mu_2^* = 0$

$-3x_1^* + 4x_2^* - \mu_1^* = 0$

$2x_1^* + x_2^* = 5$

$x_1^* = 0$

$\Rightarrow (x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (0, 5, 20, -55, 0)$

Infeasible!!

Case 2 : Binding $\rightarrow g_3(x)$ , Not binding $\rightarrow g_2(x)$ $(\mu_1, \mu_3 > 0, \mu_2 = 0)$

$10x_1^* - 3x_2^* - 2\mu_1^* = 0$

$-3x_1^* + 4x_2^* - \mu_1^* - \mu_3^* = 0$

$2x_1^* + x_2^* = 5$

$x_2^* = 0$

$\Rightarrow (x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (\frac{5}{2}, 0, \frac{25}{2}, 0, -20)$

Infeasible!!

Case 3 : $\mu_1 > 0$ , $\mu_2 = \mu_3 = 0$

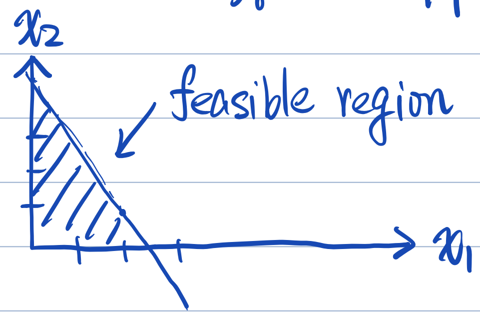
$10x_1^* - 3x_2^* - 2\mu_1^* = 0$

$-3x_1^* + 4x_2^* - \mu_1^* = 0$ $\Rightarrow 16x_1^* - 11x_2^* = 0$

$2x_1^* + x_2^* = 5$ $\Rightarrow 16x_1^* + 8x_2^* = 40$

$(x_1^*, x_2^*, \mu_1^*, \mu_2^*, \mu_3^*) = (\frac{55}{38}, \frac{40}{19}, \frac{85}{38}, 0, 0) \Rightarrow$ a KKT point

$z(x^*) = 5 \cdot (\frac{55}{38})^2 - 3 \cdot \frac{40}{19} \cdot \frac{55}{38} + 2 \cdot (\frac{40}{19})^2 + 7 = \frac{1307}{76}$

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Problem 3

min $z(x) = 2x_1 + x_2$ $\quad \nabla z(x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

s.t. $g_1(x) = -3x_1 - x_2 + 1 \leq 0$ $\quad \nabla g_1(x) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

$g_2(x) = -x_1 - 4x_2 + 2 \leq 0$ $\quad \nabla g_2(x) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

$g_3(x) = x_1 - x_2 + 1 \leq 0$ $\quad \nabla g_3(x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\nabla_x(L, \mu) = \begin{bmatrix} 2 - 3\mu_1 - \mu_2 + \mu_3 \\ 1 - \mu_1 - 4\mu_2 - \mu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note 1 : Since there are only two decision variable , only two of constraints can be binding.

Note 2 : there are at least 2 constraints to be binding to reach an optimum solution , so there are 3 possible cases.

Case 1 : $\mu_1, \mu_2 > 0$ ; $\mu_3 = 0$

$(x_1^*, x_2^*, \mu_1^*, \mu_2^*) = (\frac{2}{11}, \frac{5}{11}, \frac{7}{11}, \frac{1}{11}) \Rightarrow$ a KKT point

$z(x^*) = 2 \cdot \frac{2}{11} + \frac{5}{11} = \frac{9}{11}$

Case 2: $\mu_1, \mu_3 \geq 0$ ; $\mu_2 = 0$

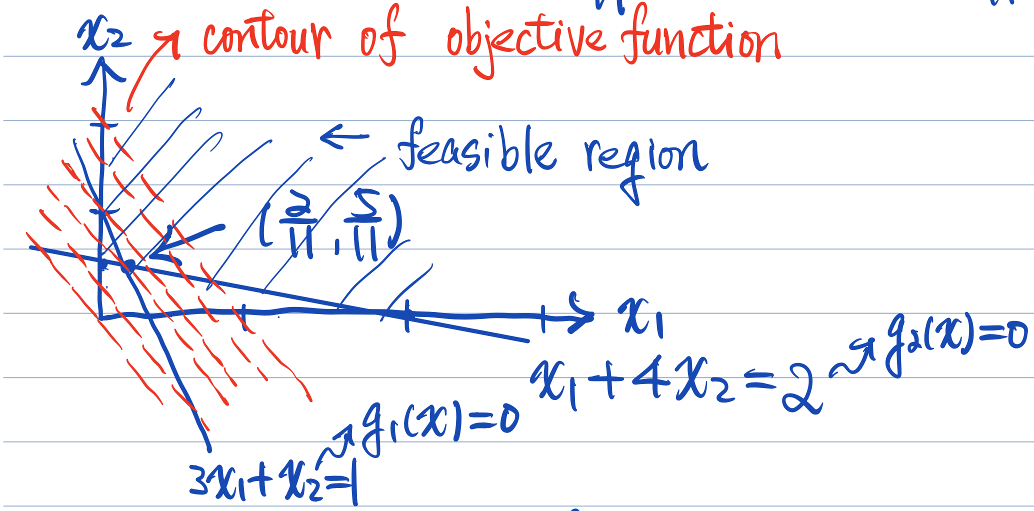
$(x_1^*, x_2^*, \mu_1, \mu_2, \mu_3) = (0, 1, \frac{3}{4}, 0, \frac{1}{4}) \Rightarrow$ a KKT point

$z(x^*) = 1$

Case 3: $\mu_2, \mu_3 \geq 0$ ; $\mu_1 = 0$

$(x_1^*, x_2^*, \mu_1, \mu_2, \mu_3) = (\frac{-2}{5}, \frac{3}{5}, 0, \frac{3}{5}, \frac{-7}{5}) \Rightarrow$ infeasible

The minimum value is $\frac{9}{11}$ when $x^* = (\frac{2}{11}, \frac{5}{11})$




Note 3: Since objective function and all constraints are linear, this problem is linear programming. The optimum must happen at an intersection of any two constraints.