Random processes

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Princeton University, Fall 2018

Outline

- Random processes
- Mean and autocorrelation functions

Random processes

A random process (RP) (or stochastic process) is an infinite indexed collection of random variables $\{X(t) : t \in \mathcal{T}\}$, defined over a common probability space

- The index parameter t is typically time, but can also be others (e.g. a spatial dimension)
- RPs are used to model random experiments that evolve in time
  - Received sequence/waveform at the output of a communication channel
  - Packet arrival times at a node in a communication network
  - Scores of an NBA team in consecutive games
  - Daily price of a stock
  - Winnings or losses of a gambler

Two ways to view a random process

A random process can be viewed as a function $X(t, \omega)$ of two variables, time $t \in \mathcal{T}$ and the outcome of the underlying random experiment $\omega \in \Omega$

- For fixed t, $X(t, \omega)$ is a random variable over Ω
- For fixed ω, $X(t, \omega)$ is a deterministic function of t, called a sample function

Discrete time random process

A random process is said to be *discrete time* if $\mathcal{T}$ is a countably infinite set, e.g.,

- $\mathcal{N} = \{0, 1, 2, \ldots\}$
- $\mathcal{Z} = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$

- In this case the process is denoted by X_n, for $n \in \mathcal{N}$, a countably infinite set, and is simply an infinite sequence of random variables
- A sample function for a discrete time process is also called a sample path

Continuous time random process

A random process is *continuous time* if $\mathcal{T}$ is a continuous set

- Example: sinusoidal signal with random phase

$$X(t) = \alpha \cos(\omega t + \Theta)\,, \quad t \geq 0$$

where $\Theta \sim \mathsf{Unif}[0, 2\pi]$, and α and ω are constants

Specifying a random process

- We can specify the random process by describing the set of sample functions (sequences, paths) and explicitly providing a probability measure over the set of events (subsets of sample functions)
- This way of specifying a random process has very limited applicability, and is suited only for very simple processes

Specifying a random process

- Alternatively, one can specify a random process (directly or indirectly) by specifying all its n-th order cdfs (pdfs, pmfs), i.e., the joint cdf (pdf, pmf) of the samples

$$X(t_1), X(t_2), \ldots, X(t_n)$$

for every order n and for every set of n points $t_1, t_2, \ldots, t_n \in \mathcal{T}$

Important classes of random processes

I.I.D. process: $\{X_n : n \in \mathcal{N}\}$ is an i.i.d. process if the r.v.s X_n are i.i.d.

- Example (Bernoulli process): $\{X_n : n \in \mathcal{N}\}$ i.i.d. $\sim$ Bern(p)
- Example (discrete-time white Gaussian noise): $X_1, \ldots, X_n, \ldots$ i.i.d. Gaussian
- Here we specified the n-th order PMFs (PDFs) of the processes by specifying the first-order PMF (PDF) and stating that the r.v.s are independent

Random walk

- Let $Z_1, Z_2, \ldots, Z_n, \ldots$ be i.i.d., where

$$Z_n = \begin{cases} +1 & \text{with prob. } \frac{1}{2} \\ -1 & \text{with prob. } \frac{1}{2} \end{cases}$$

- The random walk process is defined by

$$X_0 = 0$$
$$X_n = \sum_{i=1}^{n} Z_i\,, \quad n \geq 1$$

- The sample path for a random walk is a sequence of integers, e.g.,

$$0, +1, 0, -1, -2, -3, -4, \ldots$$

or

$$0, +1, +2, +3, +4, +3, +4, +3, +4, \ldots$$

Markov processes

- A discrete-time random process X_n is said to be a Markov process if the process future and past are conditionally independent given its present value

- Mathematically this can be rephrased in several ways. For example, if the r.v.s $\{X_n : n \geq 1\}$ are discrete, then the process is Markov iff

$$p_{X_{n+1}|\boldsymbol{X}_1^n}(x_{n+1}|x_n, \boldsymbol{x}_1^{n-1}) = p_{X_{n+1}|X_n}(x_{n+1}|x_n)$$

  for every n

Markov processes

- I.I.D. processes are Markov

- Random walk is Markov. To see this, observe that

$$
\begin{aligned}
\mathbb{P}\{X_{n+1} = x_{n+1} | \boldsymbol{X}^n = \mathbf{x}^n\} &= \mathbb{P}\{X_n + Z_{n+1} = x_{n+1} | \boldsymbol{X}^n = \boldsymbol{x}^n\} \\
&= \mathbb{P}\{X_n + Z_{n+1} = x_{n+1} | X_n = x_n\} \\
&= \mathbb{P}\{X_{n+1} = x_{n+1} | X_n = x_n\}
\end{aligned}
$$

Independent increment processes

- A discrete-time random process $\{X_n : n \geq 0\}$ is said to be independent increment if the increment random variables

$$X_{n_1},\ X_{n_2} - X_{n_1},\ \ldots,\ X_{n_k} - X_{n_{k-1}}$$

are independent for all sequences of indices such that $n_1 < n_2 < \cdots < n_k$

- Example: Random walk is an independent increment process because

$$X_{n_1} = \sum_{i=1}^{n_1} Z_i\,, \quad X_{n_2} - X_{n_1} = \sum_{i=n_1+1}^{n_2} Z_i\,, \quad \ldots, \quad X_{n_k} - X_{n_{k-1}} = \sum_{i=n_{k-1}+1}^{n_k} Z_i$$

are independent because they are functions of independent random vectors

Independent increment processes

- The independent increment property makes it easy to find the n-th order pmfs of a random walk process from knowledge only of the first-order pmf
- Example: Find $\mathbb{P}\{X_5 = 3,\, X_{10} = 6,\, X_{20} = 10\}$ for random walk process $\{X_n\}$

  Solution: We use the independent increment property as follows

$$\begin{aligned}
&\mathbb{P}\{X_5 = 3,\, X_{10} = 6,\, X_{20} = 10\} \\
&\quad = \mathbb{P}\{X_5 = 3,\, X_{10} - X_5 = 3,\, X_{20} - X_{10} = 4\} \\
&\quad = \mathbb{P}\{X_5 = 3\}\, \mathbb{P}\{X_5 = 3\}\, \mathbb{P}\{X_{10} = 4\} \\
&\quad = \binom{5}{4} 2^{-5} \binom{5}{4} 2^{-5} \binom{10}{7} 2^{-10} = 3000 \cdot 2^{-20}
\end{aligned}$$

Independent increment processes

- In general if a process is independent increment, then it is also Markov. To see this let X_n be an independent increment process and define

$$\boldsymbol{\Delta X}_1^n = [X_1,\, X_2 - X_1,\, \ldots,\, X_n - X_{n-1}]^\top$$

Then

$$\begin{aligned} &p_{X_{n+1}|\boldsymbol{X}_1^n}(x_{n+1} \,|\, \boldsymbol{x}_1^n) \\ &\quad = \mathbb{P}\{X_{n+1} = x_{n+1} \mid \boldsymbol{X}_1^n = \boldsymbol{x}_1^n\} \\ &\quad = \mathbb{P}\{X_{n+1} - X_n + X_n = x_{n+1} \mid \boldsymbol{\Delta X}_1^n = \boldsymbol{\Delta x}_1^n,\, X_n = x_n\} \\ &\quad = \mathbb{P}\{X_{n+1} = x_{n+1} \mid X_n = x_n\} \end{aligned}$$

- The converse is not necessarily true, e.g. I.I.D. processes are Markov but not independent increment

Independent increment processes

- The independent increment property can be extended to continuous-time processes:

  A process $X(t)$, $t \geq 0$, is said to be independent increment if $X(t_1)$, $X(t_2) - X(t_1)$, $\ldots$, $X(t_k) - X(t_{k-1})$ are independent for every $0 \leq t_1 < t_2 < \ldots < t_k$ and every $k \geq 2$

- Markovity can also be extended to continuous-time processes:

  A process $X(t)$ is said to be Markov if $X(t_{k+1})$ and $(X(t_1), \ldots, X(t_{k-1}))$ are conditionally independent given $X(t_k)$ for every $0 \leq t_1 < t_2 < \ldots < t_k < t_{k+1}$ and every $k \geq 3$

Counting processes

A continuous-time random process $N(t)$, $t \geq 0$, is said to be a counting process if $N(0) = 0$ and $N(t) = n$, $n \in \{0, 1, 2, \ldots\}$, is the number of events from 0 to t (hence $N(t_2) \geq N(t_1)$ for every $t_2 > t_1 \geq 0$)

- The events may be:
  - Photon arrivals at an optical detector
  - Packet arrivals at a router
  - Student arrivals at a class

Poisson process

The Poisson process is a *counting process* in which the events are "independent of each other"

More precisely, $N(t)$ is a Poisson process with rate $\lambda > 0$ if

- $N(0) = 0$
- $N(t)$ is independent increment
- $(N(t_2) - N(t_1)) \sim \text{Poisson}(\lambda(t_2 - t_1))$ for all $t_2 > t_1 \geq 0$

Poisson process

To find the kth order PMF, we use the independent increment property

$$\begin{aligned}
&\mathbb{P}\{N(t_1) = n_1, N(t_2) = n_2, \ldots, N(t_k) = n_k\} \\
&\quad = \mathbb{P}\{N(t_1) = n_1, N(t_2) - N(t_1) = n_2 - n_1, \ldots, \\
&\qquad\qquad N(t_k) - N(t_{k-1}) = n_k - n_{k-1}\} \\
&\quad = p_{N(t_1)}(n_1) p_{N(t_2)-N(t_1)}(n_2 - n_1) \ldots p_{N(t_k)-N(t_{k-1})}(n_k - n_{k-1})
\end{aligned}$$

Poisson process

- **Merging**: The sum of independent Poisson process is Poisson.
- **Branching**: Let $N(t)$ be a Poisson process with rate λ. We split $N(t)$ into two counting subprocesses $N_1(t)$ and $N_2(t)$ such that $N(t) = N_1(t) + N_2(t)$ as follows
  - Each event is randomly and independently assigned to process $N_1(t)$ with probability p, otherwise it is assigned to $N_2(t)$
  - Then $N_1(t)$ is a Poisson process with rate $p\lambda$ and $N_2(t)$ is a Poisson process with rate $(1-p)\lambda$

Mean and autocorrelation functions

- For a random vector $\boldsymbol{X}$ the first and second order moments are
  - mean $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{X}]$
  - correlation matrix $R_{\boldsymbol{X}} = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]$

- For a random process $X(t)$ the first and second order moments are
  - mean function: $\mu_X(t) = \mathbb{E}[X(t)]$ for $t \in \mathcal{T}$
  - autocorrelation function: $R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right]$ for $t_1, t_2 \in \mathcal{T}$

Autocovariance function

- The autocovariance function of a random process is defined as

$$C_X(t_1, t_2) = \mathbb{E}\left[\big(X(t_1) - \mathbb{E}[X(t_1)]\big)\big(X(t_2) - \mathbb{E}[X(t_2)]\big)\right]$$

  The autocovariance function can be expressed using the mean and autocorrelation functions as

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Example: I.I.D. process

I.I.D. process $\{X_n : n \geq 0\}$

$$\mu_X(n) = \mathbb{E}[X_1]$$

$$R_X(n_1, n_2) = \mathbb{E}[X_{n_1} X_{n_2}] = \begin{cases} \mathbb{E}[X_1^2] & n_1 = n_2 \\ (\mathbb{E}[X_1])^2 & n_1 \neq n_2 \end{cases}$$

Example: random phase signal process

Random phase signal process: $X(t) = \alpha\cos(\omega t + \Theta)$ with $\Theta \sim \text{Unif}(0, 2\pi)$

$$\mu_X(t) = \mathbb{E}[\alpha\cos(\omega t + \Theta)] = \int_0^{2\pi} \frac{\alpha}{2\pi}\cos(\omega t + \theta)\,\mathrm{d}\theta = 0$$

$$\begin{aligned}
R_X(t_1, t_2) &= \mathbb{E}\left[X(t_1)X(t_2)\right] \\
&= \int_0^{2\pi} \frac{\alpha^2}{2\pi}\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\,\mathrm{d}\theta \\
&\overset{\text{(i)}}{=} \int_0^{2\pi} \frac{\alpha^2}{4\pi}\big[\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))\big]\,\mathrm{d}\theta \\
&= \underbrace{\frac{\alpha^2}{2}\cos(\omega(t_1 - t_2))}_{\text{depends only on time difference}}
\end{aligned}$$

where (i) follows from the trigonometric identity

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta)$$

Example: random walk

Random walk: $X_i = \sum_{i=1}^{n} Z_i$ with $Z_i = \begin{cases} 1, & \text{with prob. } 0.5 \\ -1, & \text{else} \end{cases}$

$$\begin{aligned}
\mu_X(n) &= \mathbb{E}\left[\sum_{i=1}^{n} Z_i\right] = \sum_{i=1}^{n} 0 = 0 \\
R_X(n_1, n_2) &= \mathbb{E}[X_{n_1} X_{n_2}] \\
&= \mathbb{E}\left[X_{n_1}(X_{n_2} - X_{n_1} + X_{n_1})\right] \\
&= \mathbb{E}[X_{n_1}^2] = n_1 \quad \text{assuming } n_2 \geq n_1 \\
&= \min\{n_1, n_2\} \quad \text{in general}
\end{aligned}$$

Example: Poisson process

Poisson process $N(t)$ with rate λ

$$\begin{aligned}
\mu_N(t) &= \lambda t \\
R_N(t_1,t_2) &= \mathbb{E}[N(t_1)N(t_2)] \\
&= \mathbb{E}\left[N(t_1)(N(t_2)-N(t_1)+N(t_1))\right] \\
&= \lambda t_1 \times \lambda(t_2-t_1) + \lambda t_1 + \lambda^2 t_1^2 \quad \text{assuming } t_2 \geq t_1 \\
&= \lambda t_1 + \lambda^2 t_1 t_2 \quad \text{assuming } t_2 \geq t_1 \\
&= \lambda \min\{t_1,t_2\} + \lambda^2 t_1 t_2
\end{aligned}$$

Gaussian random processes

- A Gaussian random process (GRP) is a random process $X(t)$ such that

$$[X(t_1),\, X(t_2),\, \ldots,\, X(t_n)\,]^\top$$

  is a GRV for all $t_1, t_2, \ldots, t_n \in \mathcal{T}$

- Since the joint PDF for a GRV is specified by its mean and covariance matrix, a GRP is specified by its mean $\mu_X(t)$ and autocorrelation $R_X(t_1, t_2)$ functions

Gauss-Markov process

Let Z_n, $n \geq 1$, be an i.i.d. process with $Z_1 \sim \mathcal{N}(0, \sigma^2)$

The Gauss-Markov process is a first-order autoregressive process defined by

$$
\begin{aligned}
X_1 &= Z_1 \\
X_n &= \alpha X_{n-1} + Z_n \,, \quad n > 1 \,,
\end{aligned}
$$

where $|\alpha| < 1$

Gauss-Markov process

This process is a GRP, since $X_1 = Z_1$ and $X_k = \alpha X_{k-1} + Z_k$ where $Z_1, Z_2, \ldots$ are i.i.d. $\mathcal{N}(0, \sigma^2)$,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{n-2} & \alpha^{n-3} & \cdots & 1 & 0 \\ \alpha^{n-1} & \alpha^{n-2} & \cdots & \alpha & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix}$$

is a linear transformation of a GRV and is therefore a GRV

Gauss Markov process

Mean functions:

$$\begin{aligned}\mu_X(n) &= \mathbb{E}[X_n] = \mathbb{E}[\alpha X_{n-1} + Z_n] \\ &= \alpha\,\mathbb{E}[X_{n-1}] + \mathbb{E}[Z_n] = \alpha\,\mathbb{E}[X_{n-1}] = \alpha^{n-1}\,\mathbb{E}[Z_1] = 0\end{aligned}$$

Gauss Markov process

Autocorrelation function: for $n_2 > n_1$ we write

$$X_{n_2} = \alpha^{n_2-n_1} X_{n_1} + \sum_{i=0}^{n_2-n_1-1} \alpha^i Z_{n_2-i}$$

Thus

$$R_X(n_1, n_2) = \mathbb{E}[X_{n_1} X_{n_2}] = \alpha^{n_2-n_1}\, \mathbb{E}[X_{n_1}^2] + 0\,,$$

since X_{n_1} and Z_{n_2-i} are independent, zero mean for $0 \le i \le n_2 - n_1 - 1$

Next, to find $\mathbb{E}[X_{n_1}^2]$, consider

$$\begin{aligned}\mathbb{E}[X_1^2] &= \sigma^2 \\ \mathbb{E}[X_{n_1}^2] &= \mathbb{E}\left[(\alpha X_{n_1-1} + Z_{n_1})^2\right] = \alpha^2\, \mathbb{E}[X_{n_1-1}^2] + \sigma^2\end{aligned}$$

Gauss Markov process

Thus

$$\mathbb{E}[X_{n_1}^2] - \frac{\sigma^2}{1-\alpha^2} = \alpha^2 \left(\mathbb{E}[X_{n_1-1}^2] - \frac{\sigma^2}{1-\alpha^2} \right)$$

$$\implies \quad \mathbb{E}[X_{n_1}^2] - \frac{\sigma^2}{1-\alpha^2} = (\alpha^2)^{n_1-1} \left(\mathbb{E}[X_1^2] - \frac{\sigma^2}{1-\alpha^2} \right)$$

A little algebra gives

$$\mathbb{E}[X_{n_1}^2] = \frac{1-\alpha^{2n_1}}{1-\alpha^2}\sigma^2$$

Finally the autocorrelation function is

$$R_X(n_1, n_2) = \alpha^{|n_2-n_1|} \, \frac{1-\alpha^{2\min\{n_1,n_2\}}}{1-\alpha^2}\sigma^2$$

Reference

[1] "*Lecture notes for Statistical Signal Processing*," A. El Gamal.