Stationary random processes

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Outline

- Stationary random processes
- Power spectral density

Stationarity

Stationarity refers to time invariance of some, or all, of the statistics of a random process, such as mean, autocorrelation, n-th-order distribution

- We define two types of stationarity: strict sense (SSS) and wide sense (WSS)

Strict sense stationarity

A random process $X(t)$ (or X_n) is said to be SSS if all its finite order distributions are time invariant, i.e., the joint cdfs (pdfs, pmfs) of

$$X(t_1), X(t_2), \ldots, X(t_k) \quad \text{and} \quad X(t_1+\tau), X(t_2+\tau), \ldots, X(t_k+\tau)$$

are the same for all k, all $t_1, t_2, \ldots, t_k$, and all time shifts τ

- So for a SSS process, the first-order distribution is independent of t, and the second-order distribution — the distribution of any two samples $X(t_1)$ and $X(t_2)$ — depends only on $\tau = t_2 - t_1$

Strict sense stationarity

Example (random phase signal): $X(t) = \alpha \cos(\omega t + \Theta)$ where $\Theta \in$ Unif$[0, 2\pi]$ is SSS

- Check that both the first order and second order PDF are stationary (exercise)
- If we know that $X(t_1) = x_1$ and $X(t_2) = x_2$, the sample path is totally determined (except when $x_1 = x_2 = 0$, where two paths may be possible), and thus all n-th order pdfs are stationary

Strict sense stationarity

- IID processes are SSS
- Random walks are not SSS (for example, we know that $\mathbb{E}[X_1^2] = 1$ and $\mathbb{E}[X_2^2] = 2$, which is clearly not stationary)
- Poisson processes are not SSS (for example, we know that $\mathbb{E}[N(t)] = \lambda t$, which is clearly not stationary)

Wide sense stationarity

A random process $X(t)$ is said to be wide-sense stationary (WSS) if its mean and autocorrelation functions are time invariant, i.e.,

- $\mathbb{E}[X(t)] = \mu$, independent of t
- $R_X(t_1, t_2)$ is a function only of the time difference $t_2 - t_1$
- $\mathbb{E}[X(t)^2] < \infty$ (technical condition)

Since $R_X(t_1, t_2) = R_X(t_2, t_1)$, for any wide sense stationary process $X(t)$, $R_X(t_1, t_2)$ is a function only of $|t_2 - t_1|$

Wide sense stationarity

Clearly SSS implies WSS. The converse is not necessarily true

Example: Let

$$X(t) = \begin{cases} +\sin t & \text{with prob. } 1/4 \\ -\sin t & \text{with prob. } 1/4 \\ +\cos t & \text{with prob. } 1/4 \\ -\cos t & \text{with prob. } 1/4 \end{cases}$$

- $\mathbb{E}[X(t)] = 0$ and $R_X(t_1, t_2) = \frac{1}{2}\cos(t_2 - t_1)$, thus $X(t)$ is WSS
- But $X(0)$ and $X(\frac{\pi}{4})$ do not even have the same range, so the first order PMF is not stationary. Hence, the process is not SSS

Wide sense stationarity

- For Gaussian random processes, WSS implies SSS, since the process is completely specified by its mean and autocorrelation functions

- Random walk is not WSS, since $R_X(n_1, n_2) = \min\{n_1, n_2\}$ is not time invariant; similarly Poisson process is not WSS

Autocorrelation function of WSS processes

Let $X(t)$ be a WSS process. We often relabel $R_X(t_1, t_2)$ as $R_X(\tau)$ where $\tau = t_1 - t_2$

1. $R_X(\tau)$ is real and even, i.e., $R_X(\tau) = R_X(-\tau)$ for every τ
2. $|R_X(\tau)| \leq R_X(0) = \mathbb{E}[X^2(t)]$, the "average power" of $X(t)$

   Proof: For every t,

$$\begin{aligned}
(R_X(\tau))^2 &= \{\mathbb{E}[X(t)X(t+\tau)]\}^2 \\
&\leq \mathbb{E}[X^2(t)]\,\mathbb{E}[X^2(t+\tau)] \quad \text{(by Cauchy-Schwarz)} \\
&= (R_X(0))^2 \quad \text{(by stationarity)}
\end{aligned}$$

$\square$

Autocorrelation function of WSS processes

3. If $R_X(T) = R_X(0)$ for some $T \neq 0$, then $R_X(\tau)$ is periodic with period T, that is,

$$R_X(\tau) = R_X(\tau + T) \quad \text{for every } \tau$$

Proof: we again use Cauchy-Schwarz inequality: for every τ,

$$\begin{aligned}
&\left[R_X(\tau) - R_X(\tau+T)\right]^2 \\
&\quad = \left[\,\mathbb{E}\left(X(t)(X(t+\tau) - X(t+\tau+T))\right)\,\right]^2 \\
&\quad \leq \mathbb{E}[X^2(t)]\,\mathbb{E}\left[(X(t+\tau) - X(t+\tau+T))^2\right] \\
&\quad = R_X(0)(2R_X(0) - 2R_X(T)) \\
&\quad = R_X(0)(2R_X(0) - 2R_X(0)) = 0
\end{aligned}$$

Thus $R_X(\tau) = R_X(\tau + T)$ for all τ $\square$

Autocorrelation function of WSS processes

The necessary and sufficient conditions for a function to be an autocorrelation function for a WSS process is that it be real, even, and positive semidefinite

By positive semidefinite, we mean that for any n, any $t_1, t_2, \ldots, t_n$ and any real vector $\mathbf{a} = (a_1, \ldots, a_n)$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j R(t_i - t_j) \geq 0$$

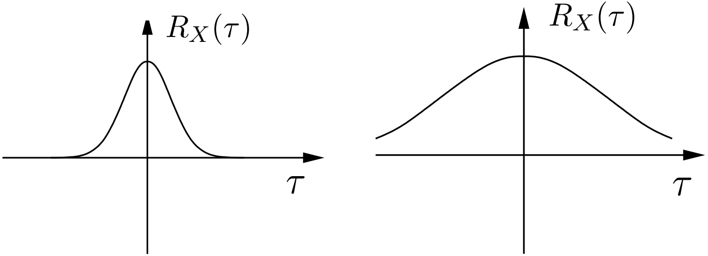
Autocorrelation function of WSS processes

- To see why positive semidefiniteness is necessary, recall that the correlation matrix for a random vector must be positive semidefinite, so if we take a set of n samples from the WSS random process, their correlation matrix must be positive semidefinite

- The positive semidefinite condition may be difficult to verify directly. It turns out, however, to be equivalent to the condition that the Fourier transform of $R_X(\tau)$, which is called the power spectral density $S_X(f)$, is nonnegative for all frequencies f

Interpretation of autocorrelation function

- Let $X(t)$ be WSS with zero mean. If $R_X(\tau)$ drops quickly with τ, this means that samples become uncorrelated quickly as we increase τ. Conversely, if $R_X(\tau)$ drops slowly with τ, samples are highly correlated



Interpretation of autocorrelation function

- So $R_X(\tau)$ is a measure of the rate of change of $X(t)$ with time t, i.e., the frequency response of $X(t)$

- It turns out that this is not just an intuitive interpretation — the Fourier transform of $R_X(\tau)$ (the power spectral density) is in fact the average power density of $X(t)$ over frequency

Power spectral density

- The power spectral density (PSD) of a WSS random process $X(t)$ is the Fourier transform of $R_X(\tau)$:

$$S_X(f) = \mathcal{F}\big(R_X(\tau)\big) = \int R_X(\tau) e^{-i2\pi\tau f} \mathrm{d}\tau$$

- For a discrete time process X_n, the power spectral density is the discrete-time Fourier transform (DTFT) of the sequence $R_X(n)$:

$$S_X(f) = \sum_{n=-\infty}^{\infty} R_X(n) e^{-i2\pi n f}, \quad |f| < \tfrac{1}{2}$$

Power spectral density

- $R_X(\tau)$ (or $R_X(n)$) can be recovered from $S_X(f)$ by taking the inverse Fourier transform or inverse DTFT:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{i2\pi\tau f} \mathrm{d}f$$

$$R_X(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{i2\pi n f} \mathrm{d}f$$

Properties of power spectral density

1. $S_X(f)$ is real and even, since the Fourier transform of the real and even function $R_X(\tau)$ is real and even

2. $\int S_X(f)\mathrm{d}f = R_X(0) = \mathbb{E}[X^2(t)]$, the average power of $X(t)$, i.e., the area under S_X is the average power

Properties of power spectral density

3. $S_X(f)$ is the average power density, i.e., the average power of $X(t)$ in the frequency band $[f_1, f_2]$ is

$$\int_{-f_2}^{-f_1} S_X(f)\,\mathrm{d}f + \int_{f_1}^{f_2} S_X(f)\,\mathrm{d}f = 2\int_{f_1}^{f_2} S_X(f)\,\mathrm{d}f$$

(we will show this soon)

WSS processes and LTI systems

Consider a linear time invariant (LTI) system with impulse response $h(t)$ and transfer function $H(f) = \mathcal{F}(h(t))$. Suppose the input to the system is a WSS process $X(t)$, then the system output is given by

$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)\mathrm{d}\tau$$

WSS processes and LTI systems

(a) Cross-correlation function:

$$\begin{aligned} R_{YX}(\tau) &= \mathbb{E}[Y(t+\tau)X(t)] \\ &= \mathbb{E}[\int_{-\infty}^{\infty} h(s)X(t+\tau-s)X(t)\mathrm{d}s] \\ &= \int_{-\infty}^{\infty} h(s)R_X(\tau-s)\mathrm{d}s \\ &= h(\tau) * R_X(\tau) \end{aligned}$$

(b) Auto-correlation function of system output:

$$\begin{aligned} R_Y(\tau) &= \mathbb{E}[Y(t+\tau)Y(t)] \\ &= \mathbb{E}[Y(t+\tau)\int_{-\infty}^{\infty} h(s)X(t-s)\mathrm{d}s] \\ &= \int_{-\infty}^{\infty} h(s)R_{YX}(\tau+s)\mathrm{d}s \\ &= R_{YX}(\tau) * h(-\tau) = R_X(\tau) * h(\tau) * h(-\tau) \end{aligned}$$

WSS processes and LTI systems

(c) Cross power spectral density

$$\begin{aligned} S_{YX}(f) &= \mathcal{F}(R_{XY}(\tau)) \\ &= S_X(f)H(f) \end{aligned}$$

(d) Power spectral density of system output:

$$\begin{aligned} S_Y(f) &= \mathcal{F}(R_Y(\tau)) \\ &= |H(f)|^2 S_X(f) \end{aligned}$$

Properties of power spectral density

- We now prove Property 3. Let $Y(t) = h(t) * X(t)$ with $h(t)$ representing the ideal band-pass filter (with band $[f_1, f_2] \cap [-f_2, -f_1]$), then

$$\begin{aligned}
\mathbb{E}[Y^2(t)] &= \int_{-\infty}^{\infty} S_Y(f)\mathrm{d}f \\
&= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\mathrm{d}f \\
&= \int_{-f_2}^{f_1} S_X(f)\mathrm{d}f + \int_{f_1}^{f_2} S_X(f)\mathrm{d}f \\
&= 2\int_{f_1}^{f_2} S_X(f)\mathrm{d}f
\end{aligned}$$

- From Property 3, it follows that $S_X(f) \geq 0$

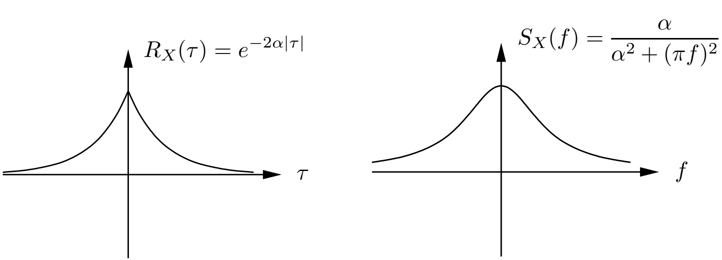
Properties of power spectral density

In general, a function $S(f)$ is a PSD of a WSS process if and only if it is real, even, nonnegative, and

$$\int S(f)\,\mathrm{d}f < \infty$$

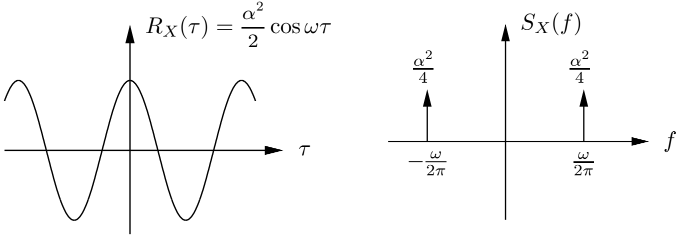
Examples

1.



$R_X(\tau) = e^{-2\alpha|\tau|}$
τ
$S_X(f) = \dfrac{\alpha}{\alpha^2 + (\pi f)^2}$
f

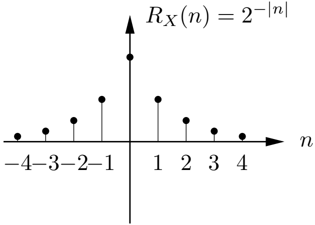
2.

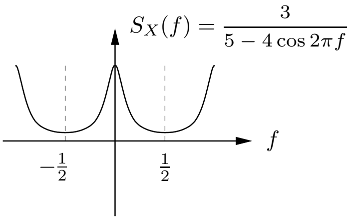


$R_X(\tau) = \dfrac{\alpha^2}{2}\cos\omega\tau$
τ
$S_X(f)$
$\frac{\alpha^2}{4}$
$\frac{\alpha^2}{4}$
f
$-\frac{\omega}{2\pi}$
$\frac{\omega}{2\pi}$

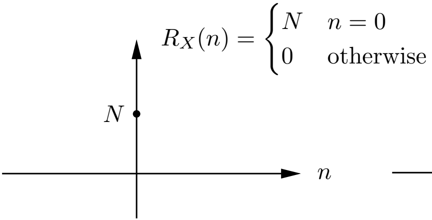
Examples

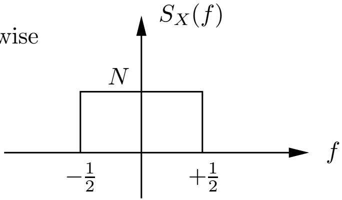
3.





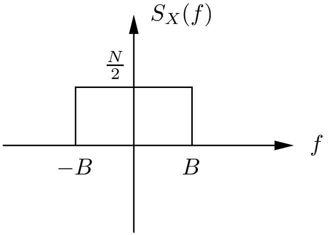
4. Discrete-time white noise process

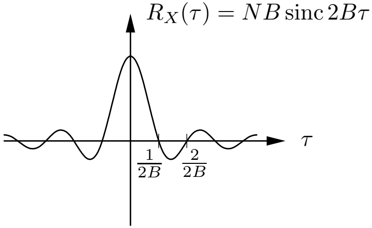




Examples

5. Band-limited white noise process





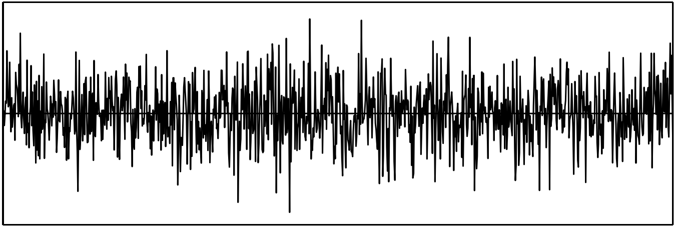
6. White noise process: If we let $B \to \infty$ in the previous example, we obtain a white noise process, which has

$$S_X(f) = \frac{N}{2} \quad \text{for all } f$$

$$R_X(\tau) = \frac{N}{2}\delta(\tau)$$

Examples

If, in addition, $X(t)$ is a GRP, then we obtain the famous white Gaussian noise (WGN) process



Remarks on white noise

- For a white noise process, all samples are uncorrelated
- The process is not physically realizable, since it has infinite power
- However, it plays a similar role in random processes to point mass in physics and delta function in linear systems

Relationship between periodogram and PSD

- Let $x(t)$, $t \geq 0$, be a deterministic signal. Define

$$x_T(t) = \begin{cases} x(t), & 0 \leq t \leq T \\ 0, & \text{otherwise,} \end{cases}$$

  and its Fourier transform

$$X_T(f) = \mathcal{F}[x_T(t)]$$

- The autocorrelation function of $x(t)$ is defined as

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)\ dt$$

Relationship between periodogram and PSD

- The power spectrum of $x(t)$ if defined as

$$S_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2,$$

  assuming the limit exists

- It can be shown that

$$S_x(f) = \mathcal{F}[R_x(\tau)]$$

Relationship between periodogram and PSD

- There is a similar relationship for WSS processes. Let $X(t)$, $t \geq 0$, be a WSS random process
  Define
  $$X_T(t) = \begin{cases} X(t), & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$
  and its Fourier Transform
  $$X_T(f) = \mathcal{F}[X_T(t)]$$

Relationship between periodogram and PSD

- Define the time average autocorrelation function of $X(t)$ as as

$$\langle R_X(\tau)\rangle_T = \frac{1}{T}\int_0^T X_T(t)X_T(t+\tau)dt$$

- Through some simple but technical calculation, we can get

$$R_X(\tau) = \lim_{T\to\infty} \mathbb{E}[\langle R_X(\tau)\rangle_T]$$

- Define the power spectrum of $X_T(t)$ (called periodogram) as

$$\langle S_X(f)\rangle_T = \frac{1}{T}|X_T(f)|^2$$

  Finally, we claim that (without proof)

$$S_X(f) = \lim_{T\to\infty} \mathbb{E}[\langle S_X(f)\rangle_T]$$

Relationship between periodogram and PSD

- Remark: This result underlies a basic method for estimating the PSD: For a given T, compute the periodogram for several sample paths of the random process (i.e., in several independent experiments), and average the results
- Remark: Increasing the number of sample paths over which the averaging is taken reduces the noise in the estimate, while repeating the entire procedure for larger T improves the frequency resolution of the estimate

Reference

[1] "*Lecture notes for Statistical Signal Processing*," A. El Gamal.