ELE 382: Statistical Signal Processing Fall 2017

Homework 9

Due date: No due date (at the beginning of class)

You are allowed to drop 1 problem without penalty. If you attempt all problems, then the problem with the lowest score will automatically be dropped. In addition, up to 2 bonus points will be awarded to each problem for clean, well-organized, and elegant solutions.

1. Gauss-Markov process (*0 points*)

Consider the following variation on the Gauss-Markov process:

$$X_0 \sim \mathcal{N}(0, a)$$
$$X_n = \tfrac{1}{2} X_{n-1} + Z_n\,, \quad n \geq 1\,,$$

where $Z_1, Z_2, Z_3, \ldots$ are i.i.d. $\mathcal{N}(0,1)$ independent of X_0.

Find a such that X_n is stationary. Find the mean and autocorrelation functions of X_n.

2. QAM random process (*0 points*)

Consider the random process

$$X(t) = Z_1 \cos\omega t + Z_2 \sin\omega t\,, \quad -\infty < t < \infty\,,$$

where Z_1 and Z_2 are i.i.d. discrete random variables such that $p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2}$.

(a) Is $X(t)$ wide-sense stationary? Justify your answer.

(b) Is $X(t)$ strict-sense stationary? Justify your answer.

3. Modified telegraph process. (*0 points*)

Let $X(t)$, $Y(t)$, and $W(t)$ be independent random processes; $X(t)$ and $Y(t)$ are zero-mean stationary Gaussian processes with $R_X(\tau) = R_Y(\tau) = e^{-|\tau|}$. $W(t)$ is the random telegraph process,

$$W(t) = A(-1)^{N(t)},$$

where $N(t)$ is a Poisson process with parameter λ, and the random variable

$$A = \begin{cases} 1 & \text{with probability } 0.5 \\ -1 & \text{with probability } 0.5. \end{cases}$$

A and $N(t)$ are independent. Now define the new process $Z(t)$ as

$$Z(t) = \begin{cases} X(t) & \text{if } W(t) = 1 \\ Y(t) & \text{if } W(t) = -1. \end{cases}$$

(a) Find the first order distribution of $Z(t)$.

(b) Is $Z(t)$ a Gaussian random process? Justify your answer.

(c) Is $Z(t)$ WSS? Justify your answer.

4. Mixture process (*0 points*)

Let $X(t)$ and $Y(t)$ be two zero-mean WSS processes with autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$, respectively. Define the process

$$Z(t) = \begin{cases} X(t) & \text{with probability } \frac{1}{2} \\ Y(t) & \text{with probability } \frac{1}{2} \end{cases}$$

(a) Find the mean and autocorrelation functions for $Z(t)$.

(b) Is $Z(t)$ a WSS process? Justify your answer

5. Estimation for WSS random processes (*0 points*)

A zero-mean WSS process $X(t)$ has autocorrelation function $R_X(\tau) = e^{-2|\tau|}$. Find the best linear MSE estimate of $X(t_0+1)$ given $X(t_0) = 2$. What is the MSE of the best linear estimate?

6. MMSE and MAP estimates (*0 points*)

Let U_1 and U_2 be zero-mean Gaussian variables with variances σ_1^2, σ_2^2 and covariance $\mathrm{Cov}(U_1, U_2)$. Compute the MMSE estimate and the MAP estimate of U_1 given U_2. Are they the same or different?

7. Estimation (*0 points*)

Consider an estimation problem: Estimate $X \sim \mathcal{N}(0,1)$ given an observed random vector in two dimensions $\boldsymbol{Y} = X\boldsymbol{a} + \boldsymbol{Z}$. $\mathbf{Z} = [Z_1, Z_2]^\top$ is zero-mean jointly Gaussian with $\sigma_{Z_1}^2 = \sigma_{Z_2}^2 = 1$, $\mathrm{Cov}(Z_1, Z_2) = 0.5$, and is independent of X. Choose a vector $\mathbf{a}$, subject to the constraint $\|\mathbf{a}\| = 1$, to minimize the MSE of estimating X given $\boldsymbol{Y}$, and compute the resulting MSE.

8. Short questions (*0 points*)

(a) Assume that you are able to sample a random variable X from an exponential distribution with parameter 1 ($f_X(x) = e^{-x}$ for $x \geq 0$). Explain how to generate:

1. a Bernouilli random variable Y with parameter $\frac{1}{e^2}$,
2. a random variable Z uniformly distributed between 0 and 2.

(b) Let X and Z be independent random variables with zero mean and variance N. Set $Y_1 = Z$ and $Y_2 = XZ$.

1. What is the MMSE estimate of X given Y_1 and Y_2 and its MSE?
2. What is the linear MMSE estimate of X given Y_1 and Y_2?

9. Binary detection (*0 points*)

Consider a binary detection problem:

$$X = \begin{cases} -1 & \text{w.p. } 1/2 \\ +1 & \text{w.p. } 1/2 \end{cases}$$

and $\boldsymbol{Y} = X\boldsymbol{a} + \boldsymbol{Z}$ is the observation random vector. Here, $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{Z} \sim \mathcal{N}(0, \boldsymbol{\Sigma_Z})$, where $\boldsymbol{\Sigma_Z}$ is an $n \times n$ covariance matrix which is *singular*. Given $\boldsymbol{\Sigma_Z}$, choose a vector $\boldsymbol{a}$, subject to the constraint $\|\mathbf{a}\| = 1$, to minimize the error probability of the MAP detection rule. (Hint: no computation needed.)

10. Bertsekas 9.2 (*0 points*)

Consider a sequence of independent coin tosses, and let θ be the probability of heads at each toss.

(a) Fix some k and let N be the number of tosses until the kth head occurs. Find the ML estimator of θ based on N.

(b) Fix some n and let K be the number of heads observed in n tosses. Find the ML estimator of θ based on K.