Approaching the Capacity of Sampled Analog Channels

Yuxin Chen†, Yonina C. Eldar†‡, and Andrea J. Goldsmith†
† Department of Electrical Engineering
Stanford University, Stanford, CA 94305
‡Department of Electrical Engineering
Technion, Haifa, Israel 32000

***Abstract*—We explore the capacity of sub-Nyquist sampled analog channels based on modulation and filter bank sampling techniques. In particular, we derive the capacity of sampled analog channels under sampling via modulation banks and filter banks. A connection between these sampling mechanisms and MIMO Gaussian channels is illuminated. For sampling with a single branch of modulation and filtering, we identify the modulation sequence that optimizes capacity. These results illustrate the importance of the sampling technique on the capacity of sampled analog channels for a broad class of nonuniform sampling structures.**

I. Introduction

The capacity of analog waveform channels and the corresponding capacity-achieving water-filling power allocation strategy over frequency are well known [1], and provide much insight and performance targets for practical communication system design. These results implicitly assume sampling above the Nyquist rate at the receiver end. However, hardware and power limitations often preclude sampling at this rate, especially for wideband communication systems. This gives rise to several questions at the intersection of sampling theory and information theory: how much information, in the Shannon sense, can be conveyed through undersampled analog channels, and how sampling structures should be optimized to maximize sampled channel capacity under sub-Nyquist sampling rate constraints.

In the sampling theory literature, various sampling methods have been developed to exploit the structure of bandlimited signals [2]. Examples include filter-bank sampling, first analyzed by Papoulis [3], in which the input signal is sampled through M linear systems. Inspired by recent "compressed sensing" [4] ideas, sub-Nyquist sampling approaches have been developed to exploit the structure of several classes of input signals, such as multiband signals [5] and signals with a finite rate of innovation [6]. Sampling via modulation and filter banks, where the signal is passed through modulation banks and filter banks before sampling, has proven to be very effective for signal reconstruction at sub-Nyquist sampling rates. An example of this sampling mechanism is the modulated wideband converter (MWC) proposed by Mishali *et al.* [5], [7]. With all post-modulation filters chosen to be low-pass filters, MWC performs well in sub-sampling sparse multiband signals with unknown spectral support. In fact, sampling via modulation and filter banks captures most nonuniform sampling techniques used in practice, although it does not include certain techniques such as sampling with a random grid.

Most of the above sampling theoretic work aims at finding optimal sampling and reconstruction mechanisms that achieve either perfect reconstruction of a class of analog signals from noiseless samples, or minimum reconstruction error from noisy samples based on statistical measures (e.g. mean squared error (MSE)). However, they do not consider optimal sampling structures based on the information-theoretic metric of channel capacity. In our previous work [8], we determined optimal transmission strategies and sampling structures for sub-Nyquist filter bank sampling of analog channels. These results indicated that capacity was not always monotonic in sampling rate, and illuminated an intriguing connection between MIMO channel capacity and capacity of undersampled channels. Moreover, the optimal filter designed to maximize capacity was found to be the same as the filter that minimizes the MSE between the original and reconstructed signals, thereby uncovering a new connection between capacity and MSE.

In this paper, we expand on our prior work by considering a broader class of nonuniform sampling mechanisms: that of modulation banks combined with filter banks. In particular, for modulation sequences that are periodic with period T_q, we derive the sampled channel capacity and show its connection to a general MIMO Gaussian channel in the frequency domain. Under single-branch sampling via modulation and filtering, we provide an algorithm to identify the optimal modulation sequence for piece-wise flat channels when T_q is an integer multiple of the sampling period. This single-branch mechanism achieves the same performance as employing an optimal filter bank with each branch sampled at a period T_q. This class of sampling mechanisms with modulation banks, however, only allow for a hardware gain instead of a capacity gain, i.e. the class of modulation-bank sampling does not allow a capacity gain to be harvested compared with the class of filter-bank sampling.

The remainder of the paper is organized as follows. The problem setup is formally stated in Section II. We derive the sampled channel capacity under modulation-bank sampling in Section III, along with an approximate analysis based on MIMO channel capacity. The optimal modulation sequence under single-branch sampling is explored in Section IV.

II. Problem Formulation

Suppose that the transmit signal $x(t)$ is time constrained to the interval $(0,T]$. The channel is modeled as an linear time-invariant (LTI) filter with impulse response $h(t)$ and frequency response $H(f)=\int_{-\infty}^{\infty} h(t)\exp\left(-j2\pi ft\right)\mathrm{d}t$. The

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analog channel output is given by

$$r(t) = h(t) * x(t) + \eta(t),$$

where $\eta(t)$ is stationary zero-mean Gaussian noise with power spectral density $\mathcal{S}_\eta(f)$. We assume throughout the remainder of this paper that perfect channel state information is known at both the transmitter and the receiver.

The analog channel output is passed through a sampling system with sampling rate f_s, which comprises M different branches. Specifically, the received analog signal $r(t)$ in the ith branch is prefiltered by an LTI filter with impulse response $p_i(t)$ and frequency response $P_i(f)$, modulated by a periodic waveform $q_i(t)$ of period T_q, filtered by another LTI filter with frequency response $S_i(f)$, and then sampled uniformly at a rate $\tilde{f}_s := f_s/M = (MT_s)^{-1}$, as illustrated in Fig. 1. Such sampling mechanisms include as special cases many nonuniform sampling methods applied in practice. The first prefilter $P_i(f)$ will be useful in zeroing out out-of-band noise, while the periodic waveforms scramble spectral contents from different aliased frequency sets, thus bringing in more design flexibility that may potentially lead to better exploitation of channel structures. By taking advantage of random modulation sequences to achieve incoherence among different branches, this sampling mechanism has proved useful for sub-sampling analog multiband signals by exploiting spectral sparsity [5]. Note that the filters can introduce arbitrary delay, so that the branches may be sampled at different times.

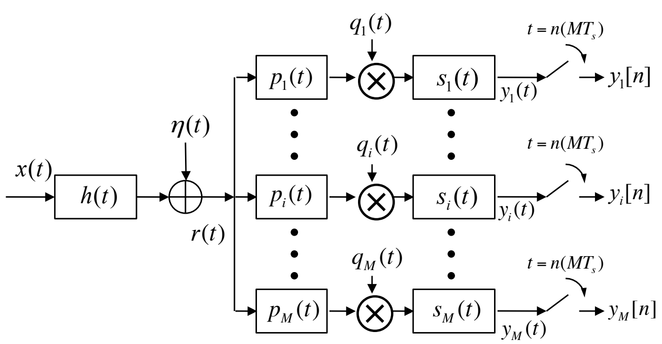

Figure 1. Sampling via modulation banks and filter banks: in each branch, the received signal is prefiltered by an LTI filter with impulse response $p_i(t)$, modulated by a periodic waveform $q_i(t)$, filtered by another LTI filter with impulse response $s_i(t)$, and then sampled at a rate f_s/M.

In the ith branch, the received prefiltered analog signal in the time interval $(0, T]$ prior to sampling can be written as

$$y_i(t) = s_i(t) * (q_i(t) \cdot (p_i(t) * h(t) * x(t) + p_i(t) * \eta(t))), \tag{1}$$

resulting in the digital sequence of samples

$$y_i[n] = y_i(nMT_s) \quad \text{and} \quad \mathbf{y}[n] = [y_1[n], \cdots, y_M[n]]^T. \tag{2}$$

The problem of finding the capacity of the sampled channel can be posed as quantifying the maximum mutual information between the input signal $x(t)$ and the output sequence sampled at a rate f_s as follows

$$C(f_s) = \lim_{T\to\infty} \frac{1}{T} \max_{p(x)} I\left(x[0,T]; \{\mathbf{y}[n]\}_{(0,T]}\right), \tag{3}$$

where $x[0,T]$ denotes the analog input signal over time $[0,T]$, and the maximum is taken over all input distributions $p(x)$ subject to a power constraint $\frac{1}{T}\mathbb{E}(\int_0^T |x(\tau)|^2 \mathrm{d}\tau) \leq P$.

III. Sampled Channel Capacity

A. Capacity Results

In this section, we state in closed form the sampled channel capacity. We first observe that the Fourier transform of each of the periodic modulation sequences $q_i(t)$ is a delta train with a spacing of the inverse period $1/T_q$. Since multiplication in the time domain corresponds to convolution in the spectral domain, the modulation bank scrambles frequency components among different aliased sets. This is reflected in Theorem 1 that characterizes the sampled analog channel capacity.

Assume that $\tilde{T}_s := MT_s = bT_q/a$ where a and b are coprime integers, and that the Fourier transform of $q_i(t)$ is given as $\sum_l c_i^l \delta(f - lf_q)$. Before stating our theorem, we introduce two Fourier symbol matrices $\mathbf{F}^\eta$ and $\mathbf{F}^h$. The $aM \times \infty$-dimensional matrix $\mathbf{F}^\eta$ contains M submatrices with the αth submatrix given by an $a \times \infty$-dimensional matrix $\mathbf{F}_\alpha^\eta \mathbf{F}_\alpha^p$. Here, for any $v \in \mathbb{Z}$, $1 \leq l \leq a$, and $1 \leq \alpha \leq M$, we have

$$(\mathbf{F}_\alpha^\eta)_{l,v} = (\mathbf{F}_\alpha^p)_{v,v} \sum_u \left\{ c_\alpha^u S_\alpha \left(f - uf_q + v\frac{f_q}{b} \right) \exp\left(j2\pi u \frac{bl}{a} \right) \right\}.$$

Also, $\mathbf{F}_\alpha^p$ and $\mathbf{F}^h$ are infinite diagonal matrices such that for all $l \in \mathbb{Z}$:

$$\begin{cases} (\mathbf{F}_\alpha^p)_{l,l} &= P_\alpha\left(f - l\frac{f_q}{b}\right)\sqrt{\mathcal{S}_\eta\left(f - l\frac{f_q}{b}\right)}, \\ \left(\mathbf{F}^h\right)_{l,l} &= \frac{H\left(f - l\frac{f_q}{b}\right)}{\sqrt{\mathcal{S}_\eta\left(f - l\frac{f_q}{b}\right)}}. \end{cases}$$

Theorem 1. *Consider the system shown in Fig. 1. Assume that $h(t)$, $p_i(t)$ and $s_i(t)$ $(1 \leq i \leq M)$ are all continuous, bounded and absolutely Riemann integrable, $\mathbf{F}^\eta$ is right invertible. Additionally, suppose that $h_\eta(t) := \mathcal{F}^{-1}\left(H(f)/\sqrt{\mathcal{S}_\eta(f)}\right)$ satisfies $h_\eta(t) = o\left(t^{-\epsilon}\right)$ for some constant $\epsilon > 1$. The capacity $C(f_s)$ of the sampled channel with a power constraint P is given by*

$$C(f_s) = \int_{-\frac{f_s}{2aM}}^{\frac{f_s}{2aM}} \frac{1}{2} \sum_{i=1}^{aM} \left[\log\left(\nu \cdot \lambda_i(f)\right)\right]^+ \mathrm{d}f, \tag{4}$$

where ν is chosen such that

$$P = \int_{-\frac{f_s}{2aM}}^{\frac{f_s}{2aM}} \sum_{i=1}^{aM} \left[\nu - \lambda_i(f)\right]^+ \mathrm{d}f. \tag{5}$$

Here, $\lambda_i(f)$ denotes the ith largest eigenvalue of $(\mathbf{F}^\eta\mathbf{F}^{\eta*})^{-\frac{1}{2}}\mathbf{F}^\eta\mathbf{F}^h\mathbf{F}^{h*}\mathbf{F}^{\eta*}(\mathbf{F}^\eta\mathbf{F}^{\eta*})^{-\frac{1}{2}}$ at frequency f, and $[x]^+ := \max\{x, 0\}$.

Remark 1. *The right invertibility of $\mathbf{F}^\eta$ ensures that the sampling method is non-degenerate, e.g. the modulation sequence cannot be zero.*

The optimal ν corresponds to a water-filling power allocation strategy based on the singular values of the equivalent channel matrix $\left(\mathbf{F}^{\eta}\mathbf{F}^{\eta *}\right)^{-\frac{1}{2}}\mathbf{F}^{\eta}\mathbf{F}^{h}$, where $\left(\mathbf{F}^{\eta}\mathbf{F}^{\eta *}\right)^{-\frac{1}{2}}$ is due to noise prewhitening and $\mathbf{F}^{\eta}\mathbf{F}^{h}$ is the equivalent channel matrix after modulation and filtering. This result can be interpreted by viewing (4) as the classical MIMO Gaussian channel capacity of the equivalent channel matrix. We note that a closed-form capacity expression may be hard to obtain for general modulated sequences $q_i(t)$. That is because the multiplication operation corresponds to convolution in the frequency domain which does not preserve Toeplitz properties of the original operator associated with the channel filter. However, when $q_i(t)$ is periodic, it can be mapped to a spike train in the frequency domain, which still exhibits block Toeplitz properties, as we describe in more detail later on.

B. Approximate Analysis

Rather than providing a rigorous proof of Theorem 1, we develop here an approximate analysis by relating the modulated aliased channel to a MIMO channel, which allows for a communication based interpretation of the capacity results, similar to the approximate Fourier analysis of Gallager [1] for the capacity of the analog channel. The proof, which is deferred to the full-length version [9], makes use of a discretization argument and asymptotic spectral properties of Toeplitz matrices.

The Fourier transform of the signal prior to modulation in the ith branch at frequency f is given as $P_i(f)\left(H(f)X(f)+N(f)\right)$. Multiplication of this prefiltered signal with the modulation sequence $q_i(t)$ corresponds to convolution in the frequency domain. The modulation sequence $q_i(t)$ is assumed to be periodic with frequency response $\sum_l c_i^l \delta\left(f-lf_q\right)$. Define

$$R(f)=H(f)X(f)+N(f).$$

The channel output is sampled at a rate $\tilde{f}_s=f_s/M$ in the ith branch. We observe that since T_q does not coincide with $\tilde{T}_s:=MT_s$, and that the sampling system is periodic with period $bT_q=a\tilde{T}_s$. Specifically, if we denote by $h(t,\tau)$ the output of the sampling system at time t due to an input at time τ, then $h(t-bT_q,\tau-bT_q)=h(t,\tau)$. We therefore divide all samples in the ith branch into a groups, where the lth ($0\leq l<a$) group contains $\{y_i[l+ka]\mid k\in\mathbb{Z}\}$, as illustrated in Fig. 2(a). Hence, each group is sampled by a uniform grid with rate f_q/b. The sampling system when restricted to the output on each group of the sampling grid can be treated as LTI, thus justifying its equivalent representation in the spectral domain. The equivalent impulse response of the sampling system for the lth group can be given by $\left(\left[p_i(t)\cdot q_i(t+l\tilde{T}_s)\right]*s_i(t)\right)$. Thus, the equivalent Fourier transform of the system output before ideal sampling in the lth group of the ith branch can be written as

$$\begin{aligned}&\tilde{Y}_i^l(f)\\ \overset{\Delta}{=}&S_i(f)\left(P_i(f)R(f)*\sum_u c_i^u\delta\left(f-uf_q\right)\exp\left(j2\pi fl\tilde{T}_s\right)\right)\\ =&S_i(f)\sum_u c_i^u P_i\left(f-uf_q\right)R\left(f-uf_q\right)\exp\left(j2\pi u\frac{bl}{a}\right),\end{aligned}$$

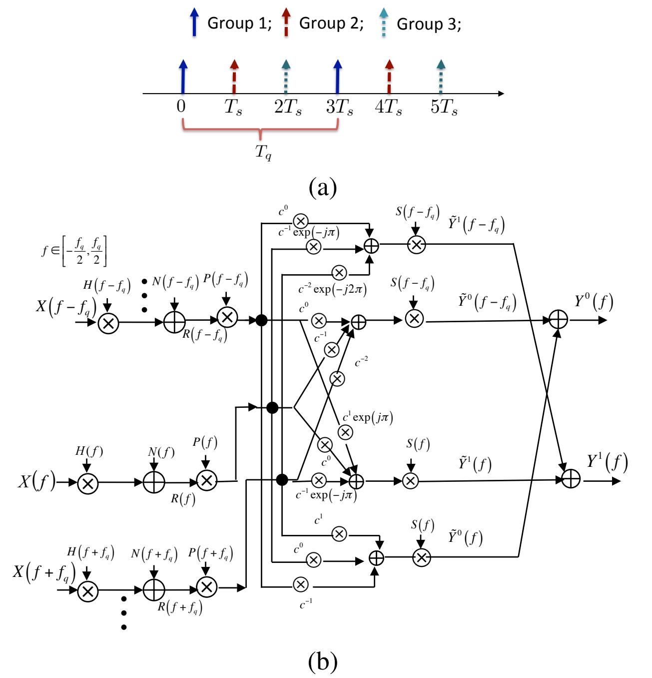

Figure 2. (a) Grouping of sampling grid when $f_s=3f_q$. The sampling grid is divided into 3 groups, where each group forms a uniform grid with rate $f_s/3$; (b) Equivalent MIMO Gaussian channel for sampling via a single branch of modulation and filtering. Here, $f_q=\frac{1}{2}f_s$ and $f\in\left[0,\frac{f_s}{2}\right)$.

which further leads to the sampled sequence in the lth group of the ith branch as

$$\begin{aligned}Y_i^l(f)=&\sum_v \tilde{Y}_i^l\left(f-\frac{vf_q}{b}\right)\\ =&\sum_v A_{l,v}P_i\left(f-v\frac{f_q}{b}\right)H\left(f-v\frac{f_q}{b}\right)X\left(f-v\frac{f_q}{b}\right)\\ &+A_{l,v}P_i\left(f-v\frac{f_q}{b}\right)N\left(f-v\frac{f_q}{b}\right),\end{aligned}$$

where

$$A_{l,v}:=\sum_u c_i^u S_i\left(f-v\frac{f_q}{b}+uf_q\right)\exp\left(j2\pi u\frac{bl}{a}\right).$$

Fig. 2 illustrates this representation for sampling with a single branch of modulation and filtering when $f_s=2f_q$. All the information of the entire sampled data is contained in $\left\{Y_i^l(f)\mid 0\leq l<a,1\leq i\leq M\right\}$, and hence the sampling system can be equivalently represented as a MIMO channel with countable transmit antennas and aM receive antennas.

Define a aliased frequency set with rate $\tilde{f}_s$ as $\{f-l\tilde{f}_s\mid l\in\mathbb{Z}\}$. Due to the convolution operation in the spectral domain, the frequency response of the sampled output at frequency f becomes a linear combination of frequency components $\{X(f)\}$ and $\{N(f)\}$ from several different aliased sets. Here, we introduce the definition of a *modulated aliased frequency set* as a generalization of the aliased set. Specifically, for each

f, its modulated aliased set is the set[1] $\{f - lf_q - k\tilde{f}_s \mid l, k \in \mathbb{Z}\}$. By our assumption that $af_q = b\tilde{f}_s$ with a and b being relatively prime, simple results in number theory imply that

$$\left\{f_0 - lf_q - k\tilde{f}_s \mid l, k \in \mathbb{Z}\right\} = \left\{f_0 - l\frac{f_q}{b} \mid l \in \mathbb{Z}\right\}. \quad (6)$$

In other words, for a given $f_0 \in [-f_q/2b, f_q/2b]$, the sampled output at f_0 depends on the input in the entire modulated aliased set. Since the sampling bandwidth at each branch is $\tilde{f}_s$, all outputs at frequencies $\{f_0 - lf_q/b \mid l \in \mathbb{Z};\ |f_0 - lf_q/b| \leq \frac{\tilde{f}_s}{2}\}$ rely on the inputs in the same modulated aliased set. This can be treated as a Gaussian MIMO channel with a countable number of input antennas at the frequency set $\{f_0 - l\tilde{f}_s/a \mid l \in \mathbb{Z}\}$ and aM groups of receive antennas each associated with one group of sample sequences in one branch. As an example, we illustrate in Fig. 2 the equivalent MIMO Gaussian channel under single-branch sampling via modulation and filtering, when $S(f) = 0$ for all $f \notin [-f_s/2, f_s/2]$.

The effective frequencies of this frequency-selective MIMO Gaussian channel range from $-f_q/2b$ to $f_q/2b$. It can be easily verified that the noise received in the lth group of the ith branch is zero-mean Gaussian with spectral density

$$\sum_{v=-\infty}^{\infty} \left| A_{l,v} P_i \left(f - v\frac{f_q}{b}\right)\right|^2 \mathcal{S}_\eta \left(f - v\frac{f_q}{b}\right), \quad |f| \leq \frac{f_q}{2b}$$

indicating the mutual correlation of noise at different branches. The received noise vector can be whitened by premultiplying by an $M \times M$ whitening matrix $(\mathbf{F}^\eta(f)\mathbf{F}^{\eta*}(f))^{-\frac{1}{2}}$ associated with the modulation and filter banks. After pre-whitening, the equivalent channel matrix becomes

$$(\mathbf{F}^\eta(f)\mathbf{F}^{\eta*}(f))^{-\frac{1}{2}} \mathbf{F}^\eta(f)\mathbf{F}^h(f) = \tilde{\mathbf{F}}^\eta(f)\mathbf{F}^h(f), \quad (7)$$

where $\tilde{\mathbf{F}}^\eta(f) \triangleq (\mathbf{F}^\eta(f)\mathbf{F}^{\eta*}(f))^{-\frac{1}{2}} \mathbf{F}^\eta(f)$. Classical MIMO Gaussian channel capacity results immediately imply that the channel capacity at frequency $f \in [-f_q/2b, f_q/2b]$ can be expressed as

$$\frac{1}{2}\sum_{i=1}^{aM} \left[\log\left(\nu\lambda_i\left(\tilde{\mathbf{F}}^\eta(f)\mathbf{F}^h(f)\mathbf{F}^{h*}(f)\tilde{\mathbf{F}}^{\eta*}(f)\right)\right)\right]^+$$

for some appropriate water level ν. Taking the integral over $[-f_q/2b, f_q/2b]$ and maximizing over all power allocation strategies leads to a universal water level ν and hence our capacity expression.

IV. Examples and Interpretation

We consider whether adding an extra modulation bank provides an implementation gain for the following two special cases.

[1]We note that although each modulated aliased set is countable, it may be a dense set when $f_q/\tilde{f}_s$ is irrational. Under the assumption in Theorem 1, however, the elements in the set have a minimum spacing of f_q/b.

A. $\frac{1}{M}f_s = \frac{1}{a}f_q$ for some integer a

In this case, the modulated aliased set is $\left\{f - k\tilde{f}_s - lf_q \mid k, l \in \mathbb{Z}\right\} = \left\{f - k\tilde{f}_s \mid k \in \mathbb{Z}\right\}$, which is equivalent to the original aliased frequency set. That said, the sampled output $Y(f)$ is still a linear combination of $\left\{H\left(f - k\tilde{f}_s\right)X\left(f - k\tilde{f}_s\right) + N\left(f - k\tilde{f}_s\right) \mid k \in \mathbb{Z}\right\}$. But since linear combinations of these components can be attained by simply adjusting the prefilter response $S(f)$, the modulation bank does not provide any more design degrees of freedom. Therefore, the maximum sampled channel capacity achievable by adding an additional modulation bank is no larger than the one achievable without the modulation sequences.

B. $\frac{1}{M}f_s = bf_q$ for some integer b

In this case, the modulated aliased set is enlarged to $\left\{f - k\tilde{f}_s - lf_q \mid k, l \in \mathbb{Z}\right\} = \{f - lf_q \mid k \in \mathbb{Z}\}$, which may potentially provide implementation gain compared with filter-bank sampling with the same number of branches. We consider the following example. Suppose that the channel contains 3 flat subbands with channel gains as plotted in Fig. 3, and that the noise is of unit spectral density within these 3 subbands and zero otherwise. Here, single-branch sampling via a single filter is employed, where the sampling rate is $f_s = 2$ and the period of the modulation sequence $T_q = 2T_s$. Due to aliasing, Subband 1 and Subband 3 (as illustrated in Fig. 3) are mixed together. The optimal prefilter without modulation would be a band-pass filter with pass band $-1.5 \leq f \leq 0.5$ [8], resulting in a channel consisting of two subbands with respective channel gains 2 and 1.

Now if we employ sampling via modulation followed by a lowpass filter, the channel can be better exploited. Specifically, suppose that the modulation sequence has a period of $2T_s$ and obeys $c^0 = 1$, $c^1 = 100$, $c^2 = 1$, $c^{-2} = 1000$ and $c^i = 0$ for all other i's, and that the cutoff frequency of the low-pass filter is $f_{\text{cutoff}} = 1$. By simple manipulation,

$$\begin{bmatrix} \exp\left(j\pi T_q\left(f - \frac{f_s}{2}\right)\right)Y\left(f - \frac{f_s}{2}\right) \\ \exp\left(j\pi T_q f\right)Y(f) \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 100 & 1001 \\ 100 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2X\left(f - \frac{f_s}{2}\right) + N\left(f - \frac{f_s}{2}\right) \\ X(f) + N(f) \\ 2X\left(f + \frac{f_s}{2}\right) + N\left(f + \frac{f_s}{2}\right) \end{bmatrix}$$

for all $f \in \left[0, \frac{f_s}{2}\right]$. Through noise whitening and eigenvalue decomposition, we can derive a pair of equivalent parallel channels experiencing respective channel gains 2 and 1.99, which outperforms non-modulated sampling via optimal filtering. As illustrated in Fig. 3, $Y(f - f_s/2)$ primarily depends on the frequency component at $f + f_s/2$, while $Y(f)$ primarily depends on the frequency component at $f - f_s/2$: both frequencies have SNR 4. In fact, by increasing c^{-2} and c^1 correspondingly, we can obtain a two-subband channel with respective channel gains both arbitrarily close to 2. That said, single-branch sampling via modulation can achieve the same capacity as applying the optimal filter bank.

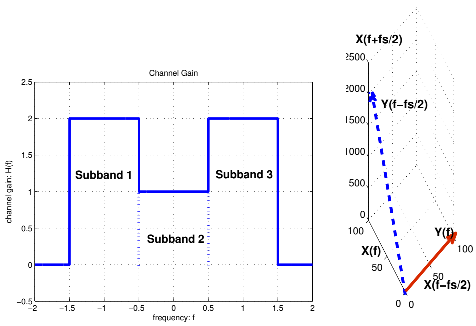

Figure 3. The left plot illustrates the channel gain, where the sampling rate is $f_s = 2$. The right plot illustrates the signal components of the sampled response under sampling via modulation.

More generally, let us consider the following scenario. Suppose that the channel of bandwidth $W = \frac{2L}{K} f_s$ is equally divided into $2L$ subbands each of bandwidth $f_q = f_s/K$ for some integers K and L. The SNR $|H(f)|^2 / \mathcal{S}_\eta(f)$ within each subband is assumed to be flat. For instance, in the presence of white noise, if $f_q \ll B_c$ with B_c being the coherence bandwidth, the channel gain (and hence the SNR) is roughly equal across the subband. Take any $f \in [-f_q/2, f_q/2]$, and run the following algorithm to determine the modulation sequence.

Algorithm 1

1. **Initialize.** Find the K largest elements in $\left\{ \frac{|H(f - l f_q)|^2}{\mathcal{S}_\eta(f - l f_q)} : l \in \mathbb{Z}, -L \le l \le L-1 \right\}$. Denote by $\{l_i : 1 \le i \le K\}$ the index set of these K elements such that $l_1 > l_2 > \cdots > l_K$. Set $i = 1$, $I_{\max} = -L$, $\mathcal{J} = \emptyset$, and $c^i = 0$ for all $i \in \mathbb{Z}$. Let A be a large given number.
2. For $i = 1 : K$
   For $m = I_{\max} : I_{\max} + K - 1$
   if $(m \bmod K) \notin \mathcal{J}$, do
   $\mathcal{J} = \mathcal{J} \cup \{m \bmod K\}$, $I_k = m$,
   $I_{\max} = m + L - 1 - l_i$, $c^{m - l_i} = A^{K+1-i}$ and break;
3. For $i = -L : L-1$
   if $i \in \{I_1, \cdots, I_K\}$, then $S(f + i f_p) = 1$;
   else $S(f + i f_q) = 0$.

The goal of this algorithm is to generate K subbands with high SNR. Due to convolution, the signal in each subband is a linear combination of the frequency components in all frequencies in the modulated aliased set. Adjusting the values of $\left\{ c^i : i \in \mathbb{Z} \right\}$ results in different weights for each component. Here, the signal in each subband being selected through Step 2 will contain one primary component accounting for most of the power of the entire signal. Filtering is further used in Step 3 in order to suppress aliasing. The performance of this algorithm is characterized in the following proposition.

Proposition 1. *Consider the above piecewise flat channel with $2L$ subbands. For a given f_q, the modulation sequence found by Algorithm 1 maximizes capacity when $A \to \infty$.*

In fact, the performance of this algorithm is asymptotically equivalent to the one using sampling via an optimal filter bank with sampling rate f_q at each branch. Single-branch sampling effectively achieves the same performance as multi-branch filter-bank sampling. Hence, this is the preferred approach when building multiple analog filters is more expensive (in terms of power consumption, size, or cost) than a single modulator. We note, however, that for a given overall sampling rate, modulation-bank sampling does not outperform filter-bank sampling in terms of sampled capacity, as stated below.

Proposition 2. *Consider the setup in Theorem 1. For a given overall sampling rate f_s, sampling with M branches of optimal modulation and filter banks does not achieve higher capacity than sampling with an optimal bank of aM filters.*

That said, instead of a capacity improvement, the main advantage of applying modulation bank is a hardware gain, namely, using fewer branches to achieve the same capacity.

V. Concluding Remarks

This paper characterizes the effect upon sampled channel capacity of modulation and filter bank sampling. In particular, we derive the capacity as a function of sampling rate, and identify optimal modulation sequences for single-branch sampling in the presence of piecewise flat channels. We show conditions under which sampling with a single modulator and filter is equivalent to sampling with a bank of filters. With the sampled capacity characterized for most nonuniform sampling methods that are applied in practice, it remains to be seen whether such sampling methods are already optimal in terms of maximizing capacity. An upper bound on sampled capacity under sampling rate constraints for more general nonuniform sampling methods would allow us to evaluate which sampling mechanisms are capacity-achieving for specific channels. Moreover, for channels where there is a gap between achievable rates and the capacity upper bound, these results might provide insight into new sampling methods that might achieve or at least close the gap to capacity. Investigation of capacity under more general nonuniform sampling techniques is a topic of ongoing work.

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