ELE 382: Statistical Signal Processing	Fall 2017

Homework 1

Due date: Monday, Sep. 25, 2017 (at the beginning of class)

You are allowed to drop 1 problem without penalty. If you attempt all problems, then the problem with the lowest score will automatically be dropped. In addition, up to 2 bonus points will be awarded to each problem for clean, well-organized, and elegant solutions.

1. Rolling dices (Problem 1.4, Bertsekas) (*10 points*)

We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

(a) Find the probability that doubles are rolled.

(b) Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.

(c) Find the probability that at least one die roll is a 6.

(d) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

2. The paradox of induction (Problem 1.26, Bertsekas) (*10 points*)

Consider a statement whose truth is unknown. If we see many examples that are compatible with it, we are tempted to view the statement as more probable. Such reasoning is often referred to as an *inductive inference*. Consider now the statement that "all cows are white." An equivalent statement is that "everything that is not white is not a cow." We then observe several black crows. Our observations are clearly compatible with the statement, but do they make the hypothesis "all cows are white" more likely?

To analyze such a situation, assume that there are two possible states of the world, which we model as complementary events

$$\begin{aligned} A &: \quad \text{all cows are white} \\ A^{c} &: \quad 50\% \text{ of all cows are white} \end{aligned}$$

Let p be the prior probability $\mathbb{P}(A)$ that all cows are white. We make an observation of a cow or a crow, with probability q and $1-q$, respectively, independent of whether event A occurs or not. Assume that $0 < p < 1$, $0 < q < 1$, and that all crows are black.

(a) Given the event $B = \{\text{a black crow was observed}\}$, what is $\mathbb{P}(A \mid B)$?

(b) Given the event $C = \{\text{a white cow was observed}\}$, what is $\mathbb{P}(A \mid C)$?

3. Communication through a noisy channel (Problem 1.31, Bertsekas) (*10 points*)

A source transmits a message (a string of symbols) through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$, respectively, and is received incorrectly with probability ϵ_0 and ϵ_1, respectively. Errors in different symbol transmissions are indepedent.

(a) What is the probability that the kth symbol is received correctly?

(b) What is the probability that the string of symbols 1011 is received correctly?

(c) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. In other words, a 0 (or 1) is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that a 0 is correctly decoded?

(d) For what values of ϵ_0 is there an improvement in the probability of correct decoding of a 0 when the scheme of Part (c) is used?

(e) Suppose that the scheme of part (c) is used. What is the probability that a symbol was 0 given that the received string is 101?

4. Binomial PMF (Problem 2.9, Bertsekas) (*10 points*)

Consider a binomial random variable X with parameters n and p. Let k^* be the largest integer that is less than or equal to $(n+1)p$. Show that the PMF $p_X(k)$ is monotonically nondecreasing with k in the range from 0 to k^*, and is monotonically decreasing with k for $k \geq k^*$.

5. Continuous random variables (Problem 3.8, Bertsekas) (*10 points*)

(a) Consider two continuous random variables Y and Z, and a random variable X that is equal to Y with probability p and to Z with probability $1-p$. What is the PDF of X?

(b) Calculate the CDF of the two-sided exponential random variable X that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0 \\ (1-p)\lambda e^{-\lambda x}, & \text{if } x \geq 0 \end{cases}$$