ELE 382: Statistical Signal Processing Fall 2017

Homework 5

Due date: No due date (at the beginning of class)

You are allowed to drop 1 problem without penalty. If you attempt all problems, then the problem with the lowest score will automatically be dropped. In addition, up to 2 bonus points will be awarded to each problem for clean, well-organized, and elegant solutions.

1. Bertsekas 8.6 (*0 points*)

Professor May B. Hard, who has a tendency to give difficult problems in probability quizzes, is concerned about one of the problems she has prepared for an upcoming quiz. She therefore asks her TA to solve the problem and record the solution time. May's prior probability that the problem is difficult is 0.3, and she knows from experience that the conditional PDF of her TA's solution time X, in minutes, is

$$f_{T|\Theta}(x \mid \Theta = 1) = \begin{cases} c_1 e^{-0.04x}, & \text{if } 5 \le x \le 60, \\ 0, & \text{otherwise,} \end{cases}$$

if $\Theta = 1$ (problem is difficult), and is

$$f_{T|\Theta}(x \mid \Theta = 2) = \begin{cases} c_2 e^{-0.16x}, & \text{if } 5 \le x \le 60, \\ 0, & \text{otherwise,} \end{cases}$$

if $\Theta = 2$ (problem is not difficult), where c_1 and c_2 are normalizing constants. She uses the MAP rule to decide whether the problem is difficult.

(a) Given that the TA's solution time was 20 minutes, which hypothesis will she accept and what will be the probability of error?

(b) Not satisfied with the reliability of her decision, May asks four more TAs to solve the problem. The TAs' solution times are conditionally independent and identically distributed with the solution time of the first TA. The recorded solution times are 10, 25, 15, and 35 minutes. On the basis of the five observations, which hypothesis will she now accept, and what will be the probability of error?

2. Bertsekas 8.7 (*0 points*)

We have two boxes, each containing three balls: one black and two white in box 1; two black and one white in box 2. We choose one of the boxes at random, where the probability of choosing box 1 is equal to some given p, and then draw a ball.

(a) Describe the MAP rule for deciding the identity of the box based on whether the drawn ball is black or white.

(b) Assuming that $p = 1/2$, find the probability of an incorrect decision and compare it with the probability of error if no ball had been drawn.

3. Bertsekas 8.8 (*0 points*)

The probability of heads of a given coin is known to be either q_0 (hypothesis H_0) or q_1 (hypothesis H_1). We toss the coin repeatedly and independently, and record the number of heads before a tail is observed for the first time. We assume that $0 < q_0 < q_1 < 1$, and that we are given prior probabilities $\mathbb{P}(H_0)$ and $\mathbb{P}(H_1)$. For parts (a) and (b), we also assume that $\mathbb{P}(H_0) = \mathbb{P}(H_1) = 1/2$.

(a) Calculate the probability that hypothesis H_1 is true, given that there were exactly k heads before the first tail.

(b) Consider the decision rule that decides in favor of hypothesis H_1 if $k \geq k^*$, where $k^*$is some nonnegative integer, and decides in favor of hypothesis H_0 otherwise. Give a formula for the probability of error in terms of k^*, q_0, and q_1. For what value of k^* is the probability of error minimized? Is there another type of decision rule that would lead to an even lower probability of error?

(c) Assume that $q_0 = 0.3$, $q_1 = 0.7$, and $\mathbb{P}(H_1) > 0.7$. How does the optimal choice of k^* (the one that minimizes the probability of error) change as $\mathbb{P}(H_1)$ increases from 0.7 to 1.0?

4. Detection in the presence of fading (*0 points*)

Let the signal and observation, respectively, be

$$X = \begin{cases} -\sqrt{P} & \text{with probability } \frac{1}{2} \\ +\sqrt{P} & \text{with probability } \frac{1}{2}, \end{cases} \qquad \text{and} \qquad Y = AX + Z\,,$$

where $A \geq 0$ and $Z \sim \mathcal{N}(0, N)$, and the random variables $X,\ A$, and Z are independent.

(a) Find the optimal decoding rule $\hat{\Theta}(y)$ for deciding whether $X = -\sqrt{P}$ or $+\sqrt{P}$, i.e., the rule that minimizes the probability of decoding error. Give your answer in terms of intervals of y.

(b) Find an expression for the minimum probability of error in terms of signal power P, noise power N, the pdf $f_A(a)$ of A, and the Q function.

5. Lognormal PDF (*10 points*)

Let $X \sim \mathcal{N}(0, \sigma^2)$. Find the pdf of $Y = e^X$ (known as the lognormal PDF).

6. Nonlinear processing (*0 points*)

Let $X \sim \mathsf{Uniform}[-1, 1]$. Define the random variable

$$Y = \begin{cases} X^2 + 1, & \text{if } |X| \geq 0.5 \\ 0, & \text{otherwise.} \end{cases}$$

Find the CDF of Y, $F_Y(y)$.

7. Iterated expectation (*0 points*) Let Λ and X be random variables with

$$\Lambda \sim f_\Lambda(\lambda) = \begin{cases} \frac{5}{3}\lambda^{\frac{2}{3}} & 0 \leq \lambda \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

and $X \mid \{\Lambda = \lambda\} \sim \mathsf{Exp}(\lambda)$. Find $\mathbb{E}[X]$.

8. Conditioning on a random variable (*0 points*)

Let X and Y be random variables with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1,\, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}[X \mid Y]$.

9. Functions of exponential random variables. (*0 points*) Let X and Y be independent exponentially distributed random variables with the same parameter λ. Define the following three functions of X and Y:

$$U = \max(X, Y), \quad V = \min(X, Y).$$

(a) Find the PDF of U and the PDF of V.

(b)
Find the joint PDF of U and V.

10. Transformation of Gaussians (*0 points*) Let

$$\boldsymbol{X} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right).$$

Find $g_1(\boldsymbol{X})$ and $g_2(\boldsymbol{X})$ such that

$$\boldsymbol{Y} = \begin{bmatrix} g_1(\boldsymbol{X}) \\ g_2(\boldsymbol{X}) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}\right).$$

11. Valid covariance matrix (*0 points*)

For what real values of a and b is the following matrix the covariance matrix of some real-valued random vector?

$$\boldsymbol{K} = \begin{bmatrix} 2 & 1 & b \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

Hint: A symmetric matrix is positive semidefinite if and only if the determinant of every matrix obtained by deleting a set of rows and the corresponding set of columns is nonnegative.

12. Rotation (*0 points*)

Let $\boldsymbol{X}$ be a Gaussian random vector with

$$\mathbb{E}[\boldsymbol{X}] = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad \mathsf{Cov}(\boldsymbol{X}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Find a vector $\boldsymbol{b}$ and orthonormal matrix $\boldsymbol{U}$ such that $\boldsymbol{Y}$ defined by $\boldsymbol{Y} = \boldsymbol{U}^\top(\boldsymbol{X} - \boldsymbol{b})$ is a mean zero Gaussian random vector such that Y_1 and Y_2 are independent.