Solution for "Quantum Computation and Quantum Information: 10th Anniversary Edition" by Nielsen and Chuang

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Contents

Chapter 2

Introduction to quantum mechanics

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2.2

$$\begin{aligned}
A\left|0\right\rangle &= A_{11}\left|0\right\rangle + A_{21}\left|1\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 0,\ A_{21} = 1 \\
A\left|1\right\rangle &= A_{12}\left|0\right\rangle + A_{22}\left|1\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 1,\ A_{22} = 0 \\
\therefore A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

input: $\{\left|0\right\rangle, \left|1\right\rangle\}$, output: $\{\left|1\right\rangle, \left|0\right\rangle\}$

$$\begin{aligned}
A\left|0\right\rangle &= A_{11}\left|1\right\rangle + A_{21}\left|0\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 1,\ A_{21} = 0 \\
A\left|1\right\rangle &= A_{12}\left|1\right\rangle + A_{22}\left|0\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 0,\ A_{22} = 1 \\
A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

2.3

From eq (2.12)

$$A\left|v_i\right\rangle = \sum_j A_{ji}\left|w_j\right\rangle$$

$$B\left|w_j\right\rangle = \sum_k B_{kj}\left|x_k\right\rangle$$

Thus

$$\begin{aligned}
BA\left|v_i\right\rangle &= B\left(\sum_j A_{ji}\left|w_j\right\rangle\right)\\
&= \sum_j A_{ji}B\left|w_j\right\rangle\\
&= \sum_{j,k} A_{ji}B_{kj}\left|x_k\right\rangle\\
&= \sum_k\left(\sum_j B_{kj}A_{ji}\right)\left|x_k\right\rangle\\
&= \sum_k (BA)_{ki}\left|x_k\right\rangle\\
\therefore (BA)_{ki} &= \sum_j B_{kj}A_{ji}
\end{aligned}$$

2.4

$$\begin{aligned}
I\left|v_j\right\rangle = \sum_i I_{ij}\left|v_i\right\rangle &= \left|v_j\right\rangle,\ \forall j.\\
&\Rightarrow I_{ij} = \delta_{ij}
\end{aligned}$$

2.5

Defined inner product on $\mathcal{C}^n$ is

$$((y_1,\cdots,y_n),(z_1,\cdots,z_n)) = \sum_i y_i^* z_i.$$

Verify (1) of eq (2.13).

$$\begin{aligned}
\left((y_1,\cdots,y_n),\sum_i\lambda_i(z_{i1},\cdots,z_{in})\right) &= \sum_i y_i^*\left(\sum_j\lambda_j z_{ji}\right)\\
&= \sum_{i,j} y_i^*\lambda_j z_{ji}\\
&= \sum_j\lambda_j\left(\sum_i y_i^* z_{ji}\right)\\
&= \sum_j\lambda_j\left((y_1,\cdots,y_n),(z_{j1},\cdots,z_{jn})\right)\\
&= \sum_i\lambda_i\left((y_1,\cdots,y_n),(z_{i1},\cdots,z_{in})\right).
\end{aligned}$$

Verify (2) of eq (2.13),

$$((y_1, \cdots, y_n), (z_1, \cdots, z_n))^* = \left(\sum_i y_i^* z_i\right)^* \tag{2.1}$$
$$= \left(\sum_i y_i z_i^*\right) \tag{2.2}$$
$$= \left(\sum_i z_i^* y_i\right) \tag{2.3}$$
$$= ((z_1, \cdots, z_n), (y_1, \cdots, y_n)) \tag{2.4}$$

.

Verify (3) of eq (2.13),

$$\begin{aligned}((y_1, \cdots, y_n), (y_1, \cdots, y_n)) &= \sum_i y_i^* y_i \\ &= \sum_i |y_i|^2\end{aligned}$$

Since $|y_i|^2 \geq 0$ for all i. Thus $\sum_i |y_i|^2 = ((y_1, \cdots, y_n), (y_1, \cdots, y_n)) \geq 0$.

From now on, I will show the following statement,

$$((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0 \text{ iff } (y_1, \cdots, y_n) = 0.$$

($\Leftarrow$) This is obvious.
($\Rightarrow$) Suppose $((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0$. Then $\sum_i |y_i|^2 = 0$.

Since $|y_i|^2 \geq 0$ for all i, if $\sum_i |y_i|^2 = 0$, then $|y_i|^2 = 0$ for all i. Therefore $|y_i|^2 = 0 \Leftrightarrow y_i = 0$ for all i. Thus,

$$(y_1, \cdots, y_n) = 0.$$

2.6

$$\begin{aligned}\left(\sum_i \lambda_i \left|w_i\right\rangle, \ \left|v\right\rangle\right) &= \left(\left|v\right\rangle, \ \sum_i \lambda_i \left|w_i\right\rangle\right)^* \\ &= \left[\sum_i \lambda_i \left(\left|v\right\rangle, \ \left|w_i\right\rangle\right)\right]^* \ (\because \text{linearlity in the 2nd arg.}) \\ &= \sum_i \lambda_i^* \left(\left|v\right\rangle, \ \left|w_i\right\rangle\right)^* \\ &= \sum_i \lambda_i^* (\left|w_i\right\rangle, \ \left|v\right\rangle)\end{aligned}$$

2.7

$$
\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0 \\
\frac{|w\rangle}{\| |w\rangle \|} &= \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\frac{|v\rangle}{\| |v\rangle \|} &= \frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

2.8

If $k = 1$,

$$
\begin{aligned}
|v_2\rangle &= \frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \\
\langle v_1|v_2\rangle &= \langle v_1| \left(\frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \right) \\
&= \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle \langle v_1|v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \\
&= 0.
\end{aligned}
$$

Suppose $\{v_1, \cdots v_n\}$ $(n \leq d-1)$ is a orthonormal basis. Then

$$
\begin{aligned}
\langle v_j|v_{n+1}\rangle &= \langle v_j| \left(\frac{|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \right) \quad (j \leq n) \\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \langle v_j|v_i\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \delta_{ij}}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\
&= \frac{\langle v_j|w_{n+1}\rangle - \langle v_j|w_{n+1}\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\
&= 0.
\end{aligned}
$$

Thus Gram-Schmidt procedure produces an orthonormal basis.

2.9

$$
\begin{aligned}
\sigma_0 &= I = |0\rangle \langle 0| + |1\rangle \langle 1| \\
\sigma_1 &= X = |0\rangle \langle 1| + |1\rangle \langle 0| \\
\sigma_2 &= Y = -i\, |0\rangle \langle 1| + i\, |1\rangle \langle 0| \\
\sigma_3 &= Z = |0\rangle \langle 0| - |1\rangle \langle 1|
\end{aligned}
$$

2.10

$$
\begin{aligned}
|v_j\rangle \langle v_k| &= I_V |v_j\rangle \langle v_k| I_V \\
&= \left(\sum_p |v_p\rangle \langle v_p|\right) |v_j\rangle \langle v_k| \left(\sum_q |v_q\rangle \langle v_q|\right) \\
&= \sum_{p,q} |v_p\rangle \langle v_p|v_j\rangle \langle v_k|v_q\rangle \langle v_q| \\
&= \sum_{p,q} \delta_{pj}\delta_{kq} |v_p\rangle \langle v_q|
\end{aligned}
$$

Thus

$$
(|v_j\rangle \langle v_k|)_{pq} = \delta_{pj}\delta_{kq}
$$

2.11

$$
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \det(X - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda \pm 1
$$

If $\lambda = -1$,

$$
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Thus

$$
|\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}
$$

If $\lambda = 1$

$$
|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

$$
X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|\lambda = -1\rangle, \ |\lambda = 1\rangle\}
$$

2.12

$$
\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I\right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1
$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$
|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

Because $|\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c \, |\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

2.13

Suppose $|\psi\rangle \, , \; |\phi\rangle$ are arbitrary vectors in V.

$$\begin{aligned} (|\psi\rangle \, , \; (|w\rangle \langle v|) \, |\phi\rangle)^* &= \left((|w\rangle \langle v|)^\dagger \, |\psi\rangle \, , \; |\phi\rangle \right)^* \\ &= \left(|\phi\rangle \, , \; (|w\rangle \langle v|)^\dagger \, |\psi\rangle \right) \\ &= \langle \phi| \, (|w\rangle \langle v|)^\dagger \, |\psi\rangle \, . \end{aligned}$$

On the other hand,

$$\begin{aligned} (|\psi\rangle \, , \; (|w\rangle \langle v|) \, |\phi\rangle)^* &= (\langle \psi|w\rangle \langle v|\phi\rangle)^* \\ &= \langle \phi|v\rangle \langle w|\psi\rangle \, . \end{aligned}$$

Thus

$$\begin{aligned} \langle \phi| \, (|w\rangle \langle v|)^\dagger \, |\psi\rangle = \langle \phi|v\rangle \langle w|\psi\rangle \; \text{ for arbitrary vectors } \; |\psi\rangle \, , \; |\phi\rangle \\ \therefore (|w\rangle \langle v|)^\dagger = |v\rangle \langle w| \end{aligned}$$

2.14

$$\begin{aligned} ((a_i A_i)^\dagger \, |\phi\rangle \, , \; |\psi\rangle) &= (|\phi\rangle \, , \; a_i A_i \, |\psi\rangle) \\ &= a_i (|\phi\rangle \, , \; A_i \, |\psi\rangle) \\ &= a_i (A_i^\dagger \, |\phi\rangle \, , \; |\psi\rangle) \\ &= (a_i^* A_i^\dagger \, |\phi\rangle \, , \; |\psi\rangle) \\ \therefore (a_i A_i)^\dagger = a_i^* A_i^\dagger & \end{aligned}$$

2.15

$$\begin{aligned} ((A^\dagger)^\dagger \, |\psi\rangle \, , \; |\phi\rangle) &= (|\psi\rangle \, , \; A^\dagger \, |\phi\rangle) \\ &= (A^\dagger \, |\phi\rangle \, , \; |\psi\rangle)^* \\ &= (|\phi\rangle \, , \; A \, |\psi\rangle)^* \\ &= (A \, |\psi\rangle \, , \; |\phi\rangle) \\ \therefore (A^\dagger)^\dagger = A & \end{aligned}$$

2.16

$$
\begin{aligned}
P &= \sum_i |i\rangle \langle i| \, . \\
P^2 &= \left(\sum_i |i\rangle \langle i|\right) \left(\sum_j |j\rangle \langle j|\right) \\
&= \sum_{i,j} |i\rangle \langle i|j\rangle \langle j| \\
&= \sum_i |i\rangle \langle j| \, \delta_{ij} \\
&= \sum_i |i\rangle \langle i| \\
&= P
\end{aligned}
$$

2.18

Suppose $|v\rangle$ is a eigenvector with corresponding eigenvalue λ.

$$
\begin{aligned}
U |v\rangle &= \lambda |v\rangle \, . \\
1 &= \langle v|v\rangle \\
&= \langle v| I |v\rangle \\
&= \langle v| U^\dagger U |v\rangle \\
&= \lambda\lambda^* \langle v|v\rangle \\
&= \|\lambda\|^2 \\
\therefore \lambda &= e^{i\theta}
\end{aligned}
$$

2.19

$$
X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
$$

2.20

$$
\begin{aligned}
U &\equiv \sum_i |w_i\rangle \langle v_i| \\
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|UU^\dagger A U U^\dagger|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \langle v_p|v_q\rangle \langle w_q|A|w_r\rangle \langle v_r|v_s\rangle \langle w_s|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s|v_j\rangle \\
&= \sum_{p,r} \langle v_i|w_p\rangle \langle w_r|v_j\rangle A''_{pr}
\end{aligned}
$$

2.21

Suppose M be Hermitian. Then $M = M^\dagger$.

$$\begin{aligned}M &= IMI\\ &= (P+Q)M(P+Q)\\ &= PMP + QMP + PMQ + QMQ\end{aligned}$$

Now $PMP = \lambda P$, $QMP = 0$, $PMQ = PM^\dagger Q = (QMP)^* = 0$. Thus $M = PMP + QMQ$. Next prove QMQ is normal.

$$\begin{aligned}QMQ(QMQ)^\dagger &= QMQQM^\dagger Q\\ &= QM^\dagger QQMQ \quad (M = M^\dagger)\\ &= (QM^\dagger Q)QMQ\end{aligned}$$

Therefore QMQ is normal. By induction, QMQ is diagonal ... (following is same as Box 2.2)

2.22

Suppose A is a Hermitian operator and $|v_i\rangle$ are eigenvectors of A with eigenvalues λ_i. Then

$$\langle v_i|A|v_j\rangle = \lambda_j \langle v_i|v_j\rangle .$$

On the other hand,

$$\langle v_i|A|v_j\rangle = \langle v_i|A^\dagger|v_j\rangle = \langle v_j|A|v_i\rangle^* = \lambda_i^* \langle v_j|v_i\rangle^* = \lambda_i^* \langle v_i|v_j\rangle = \lambda_i \langle v_i|v_j\rangle$$

Thus

$$(\lambda_i - \lambda_j) \langle v_i|v_j\rangle = 0.$$

If $\lambda_i \neq \lambda_j$, then $\langle v_i|v_j\rangle = 0$.

2.23

Suppose P is projector and $|\lambda\rangle$ are eigenvectors of P with eigenvalues λ. Then $P^2 = P$.

$$P|\lambda\rangle = \lambda|\lambda\rangle \text{ and } P|\lambda\rangle = P^2|\lambda\rangle = \lambda P|\lambda\rangle = \lambda^2|\lambda\rangle .$$

Therefore

$$\begin{aligned}\lambda &= \lambda^2\\ \lambda(\lambda-1) &= 0\\ \lambda &= 0 \text{ or } 1.\end{aligned}$$

2.24

Def of positive $\langle v|A|v\rangle \geq 0$ for all $|v\rangle$.

Suppose A is a positive operator. A can be decomposed as follows.

$$\begin{aligned}A &= \frac{A+A^\dagger}{2} + i\frac{A-A^\dagger}{2i}\\ &= B + iC \quad \text{where } B = \frac{A+A^\dagger}{2}, \quad C = \frac{A-A^\dagger}{2i}.\end{aligned}$$

Now operators B and C are Hermitian.

$$\begin{aligned}\langle v|A|v\rangle &= \langle v|B+iC|v\rangle \\ &= \langle v|B|v\rangle + i\langle v|C|v\rangle \\ &= \alpha + i\beta \quad \text{where } \alpha = \langle v|B|v\rangle\,,\ \beta = \langle v|C|v\rangle\,.\end{aligned}$$

Since B and C are Hermitian, $\alpha,\ \beta \in \mathbb{R}$. From def of positive operator, β should be vanished. Hence $\beta = \langle v|C|v\rangle$ for all $|v\rangle$, i.e. $C = 0$.

Therefore $A = B$. Since B is Hermitian, positive operator A is also Hermitian.

2.25

$$\langle\psi|A^\dagger A|\psi\rangle = \|A\,|\psi\rangle\|^2 \geq 0 \text{ for all } |\psi\rangle\,.$$

Thus $A^\dagger A$ is positive.

2.26

$$\begin{aligned}|\psi\rangle^{\otimes 2} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}|\psi\rangle^{\otimes 3} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \\ &= \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \\ &= \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\end{aligned}$$

2.27

$$\begin{aligned}X \otimes Z &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \\ &= \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

In general, tensor product is not commutable.

2.28

$$
\begin{aligned}
(A \otimes B)^* &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* \\
&= \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
&= A^* \otimes B^*.
\end{aligned}
$$

$$
\begin{aligned}
(A \otimes B)^T &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{1m}^TB^T \\ \vdots & \ddots & \vdots \\ A_{n1}^TB^T & \cdots & A_{nm}^TB^T \end{bmatrix} \\
&= A^T \otimes B^T.
\end{aligned}
$$

$$
\begin{aligned}
(A \otimes B)^\dagger &= ((A \otimes B)^*)^T \\
&= (A^* \otimes B^*)^T \\
&= (A^*)^T \otimes (B^*)^T \\
&= A^\dagger \otimes B^\dagger.
\end{aligned}
$$

2.29
Suppose U_1 and U_2 are unitary operators. Then

$$\begin{aligned}(U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger &= U_1 U_1^\dagger \otimes U_2 U_2^\dagger \\ &= I \otimes I.\end{aligned}$$

Similarly,

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = I \otimes I.$$

2.30
Suppose A and B are Hermitian operators. Then

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B. \tag{2.5}$$

Thus $A \otimes B$ is Hermitian.

2.31
Suppose A and B are positive operators. Then

$$\langle\psi| \otimes \langle\phi| (A \otimes B) |\psi\rangle \otimes |\phi\rangle = \langle\psi|A|\psi\rangle \langle\phi|B|\phi\rangle .$$

Since A and B are positive operators, $\langle\psi|A|\psi\rangle \geq 0$ and $\langle\phi|B|\phi\rangle \geq 0$ for all $|\psi\rangle, |\phi\rangle$. Then $\langle\psi|A|\psi\rangle \langle\phi|B|\phi\rangle \geq 0$. Thus $A \otimes B$ is positive if A and B are positive.

2.32
Suppose P_1 and P_2 are projectors. Then

$$\begin{aligned}(P_1 \otimes P_2)^2 &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2.\end{aligned}$$

Thus $P_1 \otimes P_2$. is also projector.

2.33

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.6}$$

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

2.34
Suppose $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

$$\begin{aligned}\det(A - \lambda I) &= (4 - \lambda)^2 - 3^2 \\ &= \lambda^2 - 8\lambda + 7 \\ &= (\lambda - 1)(\lambda - 7)\end{aligned}$$

Eigenvalues of A are $\lambda = 1,\ 7$. Corresponding eigenvectors are $|\lambda = 1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, $|\lambda = 7\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$.

Thus

$$A = |\lambda = 1\rangle\langle\lambda = 1| + 7\,|\lambda = 7\rangle\langle\lambda = 7|\,.$$

$$\begin{aligned}\sqrt{A} &= |\lambda = 1\rangle\langle\lambda = 1| + \sqrt{7}\,|\lambda = 7\rangle\langle\lambda = 7| \\ &= \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{\sqrt{7}}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}1+\sqrt{7} & -1+\sqrt{7}\\ -1+\sqrt{7} & 1+\sqrt{7}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\log(A) &= \log(1)\,|\lambda = 1\rangle\langle\lambda = 1| + \log(7)\,|\lambda = 7\rangle\langle\lambda = 7| \\ &= \frac{\log(7)}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\end{aligned}$$

2.35

$$\begin{aligned}\vec{v}\cdot\vec{\sigma} &= \sum_{i=1}^{3} v_i\sigma_i \\ &= v_1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + v_2\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + v_3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \\ &= \begin{bmatrix}v_3 & v_1 - iv_2\\ v_1 + iv_2 & -v_3\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\det(\vec{v}\cdot\vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\ &= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)\end{aligned}$$

Eigenvalues are $\lambda = \pm 1$. Let $|\lambda_{\pm 1}\rangle$ be eigenvectors with eigenvalues ± 1.

Since $\vec{v}\cdot\vec{\sigma}$ is Hermitian, $\vec{v}\cdot\vec{\sigma}$ is diagonalizable. Then

$$\vec{v}\cdot\vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|$$

Thus

$$\begin{aligned}\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right) &= e^{i\theta}\,|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\ &= (\cos\theta + i\sin\theta)\,|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\ &= \cos\theta(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + i\sin\theta(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\ &= \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}.\end{aligned}$$

$\because$ Since $\vec{v} \cdot \vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthogonal. Thus

$$|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}| = I.$$

2.36

$$\begin{aligned}
\mathrm{Tr}(\sigma_1) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0 \\
\mathrm{Tr}(\sigma_2) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = 0 \\
\mathrm{Tr}(\sigma_3) &= \mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1 - 1 = 0
\end{aligned}$$

2.37

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_{i,j} \langle i|A|j\rangle \langle j|B|i\rangle \\
&= \sum_{i,j} \langle j|B|i\rangle \langle i|A|j\rangle \\
&= \sum_j \langle j|BA|j\rangle \\
&= \mathrm{Tr}(BA)
\end{aligned}$$

2.38

$$\begin{aligned}
\mathrm{Tr}(A+B) &= \sum_i \langle i|A+B|i\rangle \\
&= \sum_i (\langle i|A|i\rangle + \langle i|B|i\rangle) \\
&= \sum_i \langle i|A|i\rangle + \sum_i \langle i|B|i\rangle \\
&= \mathrm{Tr}(A) + \mathrm{Tr}(B).
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}(zA) &= \sum_i \langle i|zA|i\rangle \\
&= \sum_i z \langle i|A|i\rangle \\
&= z \sum_i \langle i|A|i\rangle \\
&= z\,\mathrm{Tr}(A).
\end{aligned}$$

2.39

(1)

$(A, B) \equiv \mathrm{Tr}(A^\dagger B)$.

(i)

$$\begin{aligned}
\left(A, \sum_i \lambda_i B_i\right) &= \mathrm{Tr}\left[A^\dagger \left(\sum_i \lambda_i B_i\right)\right] \\
&= \mathrm{Tr}(A^\dagger \lambda_1 B_1) + \cdots + \mathrm{Tr}(A^\dagger \lambda_n B_n) \quad (\because \text{Execise 2.38}) \\
&= \lambda_1 \mathrm{Tr}(A^\dagger B_1) + \cdots + \lambda_n \mathrm{Tr}(A^\dagger B_n) \\
&= \sum_i \lambda_i \mathrm{Tr}(A^\dagger B_i)
\end{aligned}$$

(ii)

$$\begin{aligned}
(A, B)^* &= \left(\mathrm{Tr}(A^\dagger B)\right)^* \\
&= \left(\sum_{i,j} \langle i|A^\dagger|j\rangle \langle j|B|i\rangle\right)^* \\
&= \sum_{i,j} \langle i|A^\dagger|j\rangle^* \langle j|B|i\rangle^* \\
&= \sum_{i,j} \langle j|B|i\rangle^* \langle i|A^\dagger|j\rangle^* \\
&= \sum_{i,j} \langle i|B^\dagger|j\rangle \langle j|A|i\rangle \\
&= \sum_{i} \langle i|B^\dagger A|i\rangle \\
&= \mathrm{Tr}(B^\dagger A) \\
&= (B, A).
\end{aligned}$$

(iii)

$$\begin{aligned}
(A, A) &= \mathrm{Tr}(A^\dagger A) \\
&= \sum_i \langle i|A^\dagger A|i\rangle
\end{aligned}$$

Since $A^\dagger A$ is positive, $\langle i|A^\dagger A|i\rangle \geq 0$ for all $|i\rangle$.

Let a_i be i-th column of A. If $\langle i|A^\dagger A|i\rangle = 0$, then

$$\langle i|A^\dagger A|i\rangle = a_i^\dagger a_i = \|a_i\|^2 = 0 \text{ iff } a_i = \mathbf{0}.$$

Therefore $(A, A) = 0$ iff $A = \mathbf{0}$.

(2)

(3)

2.40

$$\begin{aligned}
[X,Y] &= XY - YX \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \\
&= 2iZ
\end{aligned}$$

$$\begin{aligned}
[Y,Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\
&= 2iX
\end{aligned}$$

$$\begin{aligned}
[Z,X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 2i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 2iY
\end{aligned}$$

2.41

$$\begin{aligned}
\{\sigma_1, \sigma_2\} &= \sigma_1\sigma_2 + \sigma_2\sigma_1 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\{\sigma_2, \sigma_3\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\{\sigma_3, \sigma_1\} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\sigma_0^2 &= I^2 = I \\
\sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I \\
\sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^2 = I \\
\sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I
\end{aligned}
$$

2.42

$$
\frac{[A, B] + \{A, B\}}{2} = \frac{AB - BA + AB + BA}{2} = AB
$$

2.43

From eq (2.75) and eq (2.76), $\{\sigma_j, \sigma_k\} = 2\delta_{jk} I$. From eq (2.77),

$$
\begin{aligned}
\sigma_j \sigma_k &= \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} \\
&= \frac{2i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l + 2\delta_{jk} I}{2} \\
&= \delta_{jk} I + i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l
\end{aligned}
$$

2.44

By assumption, $[A, B] = 0$ and $\{A, B\} = 0$, then $AB = 0$. Since A is invertible, multiply by A^{-1} from left, then

$$
\begin{aligned}
A^{-1} AB &= 0 \\
IB &= 0 \\
B &= 0.
\end{aligned}
$$

2.45

$$
\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= \left[B^\dagger, A^\dagger \right]
\end{aligned}
$$

2.46

$$
\begin{aligned}
[A, B] &= AB - BA \\
&= -(BA - AB) \\
&= -[B, A]
\end{aligned}
$$

2.47

$$\begin{aligned}(i[A,B])^\dagger &= -i[A,B]^\dagger\\ &= -i\left[B^\dagger, A^\dagger\right]\\ &= -i[B,A]\\ &= i[A,B]\end{aligned}$$

2.48

(Positive)
Since P is positive, it is diagonalizable. Then $P = \sum_i \lambda_i |i\rangle\langle i|$, $(\lambda_i \geq 0)$.

$$J = \sqrt{P^\dagger P} = \sqrt{PP} = \sqrt{P^2} = \sum_i \sqrt{\lambda_i^2}\, |i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i| = P.$$

Therefore polar decomposition of P is $P = UP$ for all P. Thus $U = I$, then $P = P$.

(Unitary)
Suppose unitary U is decomposed by $U = WJ$ where W is unitary and J is positive, $J = \sqrt{U^\dagger U}$.

$$J = \sqrt{U^\dagger U} = \sqrt{I} = I$$

Since unitary operators are invertible, $W = UJ^{-1} = UI^{-1} = UI = U$. Thus polar decomposition of U is $U = U$.

(Hermitian)
Suppose $H = UJ$.

$$J = \sqrt{H^\dagger H} = \sqrt{HH} = \sqrt{H^2}.$$

Thus $H = U\sqrt{H^2}$.

In general, $H \neq \sqrt{H^2}$.
From spectral decomposition, $H = \sum_i \lambda_i |i\rangle\langle i|$, $\lambda_i \in \mathbb{R}$.

$$\sqrt{H^2} = \sqrt{\sum_i \lambda_i^2 |i\rangle\langle i|} = \sum_i \sqrt{\lambda_i^2}\, |i\rangle\langle i| = \sum_i |\lambda_i|\, |i\rangle\langle i| \neq H$$

2.49
Normal matrix is diagonalizable, $A = \sum_i \lambda_i |i\rangle\langle i|$.

$$J = \sqrt{A^\dagger A} = \sum_i |\lambda_i|\, |i\rangle\langle i|.$$

$$U = \sum_i |e_i\rangle\langle i|$$

$$A = UJ = \sum_i |\lambda_i|\, |e_i\rangle\langle i|.$$

2.50

Define $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $A^\dagger A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation of $A^\dagger A$ is $\det(A^\dagger A - \lambda I) = \lambda^2 - 3\lambda + 1$. Eigenvalues of $A^\dagger A$ are $\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$ and associated eigenvectors are $|\lambda_\pm\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}$.

$$A^\dagger A = \lambda_+ |\lambda_+\rangle\langle\lambda_+| + \lambda_- |\lambda_-\rangle\langle\lambda_-| .$$

$$\begin{aligned} J = \sqrt{A^\dagger A} &= \sqrt{\lambda_+} |\lambda_+\rangle\langle\lambda_+| + \sqrt{\lambda_-} |\lambda_-\rangle\langle\lambda_-| \\ &= \sqrt{\frac{3+\sqrt{5}}{2}} \cdot \frac{5-\sqrt{5}}{40} \begin{bmatrix} 4 & 2\sqrt{5}-2 \\ 2\sqrt{5}-2 & 6-2\sqrt{5} \end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{2}} \cdot \frac{5+\sqrt{5}}{40} \begin{bmatrix} 4 & -2\sqrt{5}-2 \\ -2\sqrt{5}-2 & 6+2\sqrt{5} \end{bmatrix} \end{aligned}$$

$$J^{-1} = \frac{1}{\sqrt{\lambda_+}} |\lambda_+\rangle\langle\lambda_+| + \frac{1}{\sqrt{\lambda_-}} |\lambda_-\rangle\langle\lambda_-| .$$

$$U = AJ^{-1}$$

I'm tired.

2.51

$$H^\dagger H = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

2.52

$$H^\dagger = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$$

Thus

$$H^2 = I.$$

2.53

$$\begin{aligned} \det(H - \lambda I) &= \left(\frac{1}{\sqrt{2}} - \lambda \right) \left(-\frac{1}{\sqrt{2}} - \lambda \right) - \frac{1}{2} \\ &= \lambda^2 - \frac{1}{2} - \frac{1}{2} \\ &= \lambda^2 - 1 \end{aligned}$$

Eigenvalues are $\lambda_{\pm} = \pm 1$ and associated eigenvectors are $|\lambda_{\pm}\rangle = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 \pm \sqrt{2} \end{bmatrix}$.

2.54
Since $[A, B] = 0$, A and B are simultaneously diagonalize, $A = \sum_i a_i |i\rangle\langle i|$, $B = \sum_i b_i |i\rangle\langle i|$.

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_i \exp(a_i)\,|i\rangle\langle i|\right)\left(\sum_i \exp(b_i)\,|i\rangle\langle i|\right) \\
&= \sum_{i,j} \exp(a_i + b_j)\,|i\rangle\,\langle i|j\rangle\,\langle j| \\
&= \sum_{i,j} \exp(a_i + b_j)\,|i\rangle\langle j|\,\delta_{i,j} \\
&= \sum_i \exp(a_i + b_i)\,|i\rangle\langle i| \\
&= \exp(A + B)
\end{aligned}$$

2.55

$$H = \sum_E E\,|E\rangle\langle E|$$

$$\begin{aligned}
U(t_2 - t_1)U^\dagger(t_2 - t_1) &= \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right)\exp\left(\frac{iH(t_2 - t_1)}{\hbar}\right) \\
&= \sum_{E,E'} \left(\exp\left(-\frac{iE(t_2 - t_1)}{\hbar}\right)|E\rangle\langle E|\right)\left(\exp\left(-\frac{iE'(t_2 - t_1)}{\hbar}\right)|E'\rangle\langle E'|\right) \\
&= \sum_{E,E'} \left(\exp\left(-\frac{i(E - E')(t_2 - t_1)}{\hbar}\right)|E\rangle\langle E'|\,\delta_{E,E'}\right) \\
&= \sum_E \exp(0)\,|E\rangle\langle E| \\
&= \sum_E |E\rangle\langle E| \\
&= I
\end{aligned}$$

Similarly, $U^\dagger(t_2 - t_1)U(t_2 - t_1) = I$.

2.56

$U = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \quad (|\lambda_i| = 1)$.

$$\log(U) = \sum_j \log(\lambda_j)\,|\lambda_j\rangle\langle\lambda_j| = \sum_j i\theta_j\,|\lambda_j\rangle\langle\lambda_j| \text{ where } \theta_j = \arg(\lambda_j)$$

$$K = -i\log(U) = \sum_j \theta_j\,|\lambda_j\rangle\langle\lambda_j|\,.$$

$$K^\dagger = (-i\log U)^\dagger = \left(\sum_j \theta_j\,|\lambda_j\rangle\langle\lambda_j|\right)^\dagger = \sum_j \theta_j^*\,|\lambda_j\rangle\langle\lambda_j| = \sum_j \theta_j\,|\lambda_j\rangle\langle\lambda_j| = K$$

2.57

$$|\phi\rangle \equiv \frac{L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

$$\langle\phi|M_m^\dagger M_m|\phi\rangle = \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle}$$

$$\frac{M_m |\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \cdot \frac{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}}$$

2.58

$$\begin{aligned}
\langle M\rangle &= \langle\psi|M|\psi\rangle = \langle\psi|m|\psi\rangle = m\langle\psi|\psi\rangle = m \\
\langle M^2\rangle &= \langle\psi|M^2|\psi\rangle = \langle\psi|m^2|\psi\rangle = m^2\langle\psi|\psi\rangle = m^2 \\
\text{deviation} &= \langle M^2\rangle - \langle M\rangle^2 = m^2 - m^2 = 0.
\end{aligned}$$

2.59

$$\begin{aligned}
\langle X\rangle &= \langle 0|X|0\rangle = \langle 0|1\rangle = 0 \\
\langle X^2\rangle &= \langle 0|X^2|0\rangle = \langle 0|X|1\rangle = \langle 0|0\rangle = 1 \\
\text{standard deviation} &= \sqrt{\langle X^2\rangle - \langle X\rangle^2} = 1
\end{aligned}$$

2.60

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma} &= \sum_{i=1}^{3} v_i\sigma_i \\
&= v_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\det(\vec{v}\cdot\vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)
\end{aligned}$$

Eigenvalues are $\lambda = \pm 1$.

(i) if $\lambda = 1$

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma} - \lambda I &= \vec{v}\cdot\vec{\sigma} - I \\
&= \begin{bmatrix} v_3 - 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - 1 \end{bmatrix}
\end{aligned}$$

Eigenvector is $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix}$.

$$
\begin{aligned}
|\lambda_1\rangle\langle\lambda_1| &= \frac{1+v_3}{2} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1-v_3}{v_1+iv_2} \end{bmatrix} \\
&= \frac{1+v_3}{2} \begin{bmatrix} 1 & \frac{v_1-iv_2}{1+v_3} \\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\
&= \frac{1}{2} \left(I + \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
&= \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})
\end{aligned}
$$

(ii) If $\lambda = -1$.

$$
\begin{aligned}
\vec{v}\cdot\vec{\sigma} - \lambda I = \vec{v}\cdot\vec{\sigma} + I \\
&= \begin{bmatrix} v_3+1 & v_1-iv_2 \\ v_1+iv_2 & -v_3+1 \end{bmatrix}
\end{aligned}
$$

Eigenvalue is $|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$.

$$
\begin{aligned}
|\lambda_{-1}\rangle\langle\lambda_{-1}| &= \frac{1-v_3}{2} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1+v_3}{v_1+iv_2} \end{bmatrix} \\
&= \frac{1-v_3}{2} \begin{bmatrix} 1 & -\frac{v_1-iv_2}{1-v_3} \\ -\frac{v_1+iv_2}{1-v_3} & \frac{1+v_3}{1-v_3} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1-v_3 & -(v_1-iv_2) \\ -(v_1+iv_2) & 1+v_3 \end{bmatrix} \\
&= \frac{1}{2} \left(I - \begin{bmatrix} v_3 & v_1-iv_2 \\ (v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
&= \frac{1}{2}(I - \vec{v}\cdot\vec{\sigma}).
\end{aligned}
$$

2.61

$$
\begin{aligned}
\langle\lambda_1|0\rangle\langle 0|\lambda_1\rangle &= \langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle \\
&= \langle 0|\frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})|0\rangle \\
&= \frac{1}{2}(1+v_3)
\end{aligned}
$$

Post-measurement state is

$$
\begin{aligned}
\frac{|\lambda_1\rangle\langle\lambda_1|0\rangle}{\sqrt{\langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle}} &= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \cdot \frac{1}{2}\begin{bmatrix} 1+v_3 \\ v_1+iv_2 \end{bmatrix} \\
&= \sqrt{\frac{1}{2}(1+v_3)}\begin{bmatrix} 1 \\ \frac{v_1+iv_2}{1+v_3} \end{bmatrix} \\
&= \sqrt{\frac{1+v_3}{2}}\begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \\
&= |\lambda_1\rangle .
\end{aligned}
$$

2.62

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