Solution for Quantum Computation and Quantum Information by Nielsen and Chuang

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Contents

2 Introduction to quantum mechanics

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2.2

$$\begin{aligned}
&A\left|0\right\rangle = A_{11}\left|0\right\rangle + A_{21}\left|1\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 0,\ A_{21} = 1 \\
&A\left|1\right\rangle = A_{12}\left|0\right\rangle + A_{22}\left|1\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 1,\ A_{22} = 0 \\
&\therefore A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

input: $\{\left|0\right\rangle, \left|1\right\rangle\}$, output: $\{\left|1\right\rangle, \left|0\right\rangle\}$

$$\begin{aligned}
&A\left|0\right\rangle = A_{11}\left|1\right\rangle + A_{21}\left|0\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 1,\ A_{21} = 0 \\
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&A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

2.3

$$\begin{aligned}
A\left|v_i\right\rangle &= \sum_j A_{ji}\left|w_j\right\rangle \\
B\left|w_j\right\rangle &= \sum_k B_{kj}\left|x_k\right\rangle
\end{aligned}$$

Thus

$$\begin{aligned}
BA\left|v_i\right\rangle &= B\left(\sum_j A_{ji}\left|w_j\right\rangle\right) \\
&= \sum_j A_{ji} B\left|w_j\right\rangle \\
&= \sum_{j,k} A_{ji} B_{kj}\left|x_k\right\rangle \\
&= \sum_k \left(\sum_j B_{kj} A_{ji}\right)\left|x_k\right\rangle \\
&= \sum_k (BA)_{ki}\left|x_k\right\rangle \\
\therefore (BA)_{ki} &= \sum_j B_{kj} A_{ji}
\end{aligned}$$

2.4

$$
I\left|v_j\right\rangle = \sum_i I_{ij}\left|v_i\right\rangle = \left|v_j\right\rangle,\ \forall j.
$$
$$
\Rightarrow I_{ij} = \delta_{ij}
$$

2.6

$$
\begin{aligned}
\left(\sum_i \lambda_i \left|w_i\right\rangle,\ \left|v\right\rangle\right) &= \left(\left|v\right\rangle,\ \sum_i \lambda_i \left|w_i\right\rangle\right)^* \\
&= \left[\sum_i \lambda_i \left(\left|v\right\rangle,\ \left|w_i\right\rangle\right)\right]^* (\because \text{linearlity in the 2nd arg.}) \\
&= \sum_i \lambda_i^* \left(\left|v\right\rangle,\ \left|w_i\right\rangle\right)^* \\
&= \sum_i \lambda_i^* (\left|w_i\right\rangle,\ \left|v\right\rangle)
\end{aligned}
$$

2.7

$$
\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix} = 1 - 1 = 0 \\
\frac{\left|w\right\rangle}{\|\left|w\right\rangle\|} &= \frac{\left|w\right\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix} \\
\frac{\left|v\right\rangle}{\|\left|v\right\rangle\|} &= \frac{\left|v\right\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ -1\end{bmatrix}
\end{aligned}
$$

2.8

If $k = 1$,

$$
\begin{aligned}
\left|v_2\right\rangle &= \frac{\left|w_2\right\rangle - \langle v_1|w_2\rangle \left|v_1\right\rangle}{\|\left|w_2\right\rangle - \langle v_1|w_2\rangle \left|v_1\right\rangle\|} \\
\langle v_1|v_2\rangle &= \langle v_1| \left(\frac{\left|w_2\right\rangle - \langle v_1|w_2\rangle \left|v_1\right\rangle}{\|\left|w_2\right\rangle - \langle v_1|w_2\rangle \left|v_1\right\rangle\|}\right) \\
&= \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle \langle v_1|v_1\rangle}{\|\left|w_2\right\rangle - \langle v_1|w_2\rangle \left|v_1\right\rangle\|} \\
&= 0.
\end{aligned}
$$

Suppose $\{v_1, \cdots v_n\}$ $(n \leq d-1)$ is a orthonormal basis. Then

$$\begin{aligned}
\langle v_j | v_{n+1} \rangle &= \langle v_j | \left(\frac{|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i | w_{n+1} \rangle |v_i\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i | w_{n+1} \rangle |v_i\rangle \|} \right) \quad (j \leq n) \\
&= \frac{\langle v_j | w_{n+1} \rangle - \sum_{i=1}^n \langle v_i | w_{n+1} \rangle \langle v_j | v_i \rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i | w_{n+1} \rangle |v_i\rangle \|} \\
&= \frac{\langle v_j | w_{n+1} \rangle - \sum_{i=1}^n \langle v_i | w_{n+1} \rangle \delta_{ij}}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i | w_{n+1} \rangle |v_i\rangle \|} \\
&= \frac{\langle v_j | w_{n+1} \rangle - \langle v_j | w_{n+1} \rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i | w_{n+1} \rangle |v_i\rangle \|} \\
&= 0.
\end{aligned}$$

Thus Gram-Schmidt procedure produces an orthonormal basis.

2.9

$$\begin{aligned}
\sigma_0 &= I = |0\rangle \langle 0| + |1\rangle \langle 1| \\
\sigma_1 &= X = |0\rangle \langle 1| + |1\rangle \langle 0| \\
\sigma_2 &= Y = -i |0\rangle \langle 1| + i |1\rangle \langle 0| \\
\sigma_3 &= Z = |0\rangle \langle 0| - |1\rangle \langle 1|
\end{aligned}$$

2.10

$$\begin{aligned}
|v_j\rangle \langle v_k| &= I_V |v_j\rangle \langle v_k| I_V \\
&= \left(\sum_p |v_p\rangle \langle v_p| \right) |v_j\rangle \langle v_k| \left(\sum_q |v_q\rangle \langle v_q| \right) \\
&= \sum_{p,q} |v_p\rangle \langle v_p | v_j \rangle \langle v_k | v_q \rangle \langle v_q| \\
&= \sum_{p,q} \delta_{pj} \delta_{kq} |v_p\rangle \langle v_q|
\end{aligned}$$

Thus

$$\left(|v_j\rangle \langle v_k| \right)_{pq} = \delta_{pj} \delta_{kq}$$

2.11

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \det(X - \lambda I) = \det \left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) = 0 \Rightarrow \lambda \pm 1$$

If $\lambda = -1$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus

$$|\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If $\lambda = 1$

$$|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|\lambda = -1\rangle\,,\ |\lambda = 1\rangle\}$$

2.12

$$\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I\right) = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because $|\lambda = 1\rangle \langle\lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c\,|\lambda = 1\rangle \langle\lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

2.13

Suppose $|\psi\rangle\,,\ |\phi\rangle$ are arbitrary vectors in V.

$$\begin{aligned}
(|\psi\rangle\,,\ (|w\rangle\langle v|)\,|\phi\rangle)^* &= \Big((|w\rangle\langle v|)^\dagger\,|\psi\rangle\,,\ |\phi\rangle\Big)^* \\
&= \Big(|\phi\rangle\,,\ (|w\rangle\langle v|)^\dagger\,|\psi\rangle\Big) \\
&= \langle\phi|\,(|w\rangle\langle v|)^\dagger\,|\psi\rangle\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(|\psi\rangle\,,\ (|w\rangle\langle v|)\,|\phi\rangle)^* &= (\langle\psi|w\rangle\langle v|\phi\rangle)^* \\
&= \langle\phi|v\rangle\langle w|\psi\rangle\,.
\end{aligned}$$

Thus

$$\langle\phi|\,(|w\rangle\langle v|)^\dagger\,|\psi\rangle = \langle\phi|v\rangle\langle w|\psi\rangle \text{ for arbitrary vectors } |\psi\rangle\,,\ |\phi\rangle$$

$$\therefore (|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$$

2.14

$$
\begin{aligned}
((a_i A_i)^\dagger \left|\phi\right\rangle,\ \left|\psi\right\rangle) &= (\left|\phi\right\rangle,\ a_i A_i \left|\psi\right\rangle) \\
&= a_i(\left|\phi\right\rangle,\ A_i \left|\psi\right\rangle) \\
&= a_i(A_i^\dagger \left|\phi\right\rangle,\ \left|\psi\right\rangle) \\
&= (a_i^* A_i^\dagger \left|\phi\right\rangle,\ \left|\psi\right\rangle) \\
\therefore (a_i A_i)^\dagger = a_i^* A_i^\dagger &
\end{aligned}
$$

2.15

$$
\begin{aligned}
((A^\dagger)^\dagger \left|\psi\right\rangle,\ \left|\phi\right\rangle) &= (\left|\psi\right\rangle,\ A^\dagger \left|\phi\right\rangle) \\
&= (A^\dagger \left|\phi\right\rangle,\ \left|\psi\right\rangle)^* \\
&= (\left|\phi\right\rangle,\ A \left|\psi\right\rangle)^* \\
&= (A \left|\psi\right\rangle,\ \left|\phi\right\rangle) \\
\therefore (A^\dagger)^\dagger = A &
\end{aligned}
$$

2.16

$$
\begin{aligned}
P &= \sum_i \left|i\right\rangle \left\langle i\right| . \\
P^2 &= \left(\sum_i \left|i\right\rangle \left\langle i\right| \right) \left(\sum_j \left|j\right\rangle \left\langle j\right| \right) \\
&= \sum_{i,j} \left|i\right\rangle \left\langle i|j\right\rangle \left\langle j\right| \\
&= \sum_i \left|i\right\rangle \left\langle j\right| \delta_{ij} \\
&= \sum_i \left|i\right\rangle \left\langle i\right| \\
&= P
\end{aligned}
$$

2.18

Suppose $|v\rangle$ is a eigenvector with corresponding eigenvalue λ.

$$\begin{aligned}
U|v\rangle &= \lambda |v\rangle . \\
1 &= \langle v|v\rangle \\
&= \langle v| I |v\rangle \\
&= \langle v| U^\dagger U |v\rangle \\
&= \lambda\lambda^* \langle v|v\rangle \\
&= \|\lambda\|^2 \\
\therefore \lambda &= e^{i\theta}
\end{aligned}$$

2.19

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2.20

$$\begin{aligned}
U &\equiv \sum_i |w_i\rangle \langle v_i| \\
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|UU^\dagger AUU^\dagger|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \langle v_p|v_q\rangle \langle w_q|A|w_r\rangle \langle v_r|v_s\rangle \langle w_s|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s|v_j\rangle \\
&= \sum_{p,r} \langle v_i|w_p\rangle \langle w_r|v_j\rangle A''_{pr}
\end{aligned}$$

2.26