

Solution for "Quantum Computation and Quantum Information:  
10th Anniversary Edition" by Nielsen and Chuang

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## Chapter 2

# Introduction to quantum mechanics

### 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

### 2.2

$$\begin{aligned} A|0\rangle &= A_{11}|0\rangle + A_{21}|1\rangle = |1\rangle \Rightarrow A_{11} = 0, A_{21} = 1 \\ A|1\rangle &= A_{12}|0\rangle + A_{22}|1\rangle = |0\rangle \Rightarrow A_{12} = 1, A_{22} = 0 \\ \therefore A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

input:  $\{|0\rangle, |1\rangle\}$ , output:  $\{|1\rangle, |0\rangle\}$

$$\begin{aligned} A|0\rangle &= A_{11}|1\rangle + A_{21}|0\rangle = |1\rangle \Rightarrow A_{11} = 1, A_{21} = 0 \\ A|1\rangle &= A_{12}|1\rangle + A_{22}|0\rangle = |0\rangle \Rightarrow A_{12} = 0, A_{22} = 1 \\ A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

### 2.3

From eq (2.12)

$$\begin{aligned} A|v_i\rangle &= \sum_j A_{ji}|w_j\rangle \\ B|w_j\rangle &= \sum_k B_{kj}|x_k\rangle \end{aligned}$$

Thus

$$\begin{aligned}
 BA|v_i\rangle &= B\left(\sum_j A_{ji}|w_j\rangle\right) \\
 &= \sum_j A_{ji}B|w_j\rangle \\
 &= \sum_{j,k} A_{ji}B_{kj}|x_k\rangle \\
 &= \sum_k \left(\sum_j B_{kj}A_{ji}\right)|x_k\rangle \\
 &= \sum_k (BA)_{ki}|x_k\rangle \\
 \therefore (BA)_{ki} &= \sum_j B_{kj}A_{ji}
 \end{aligned}$$

## 2.4

$$\begin{aligned}
 I|v_j\rangle &= \sum_i I_{ij}|v_i\rangle = |v_j\rangle, \quad \forall j. \\
 &\Rightarrow I_{ij} = \delta_{ij}
 \end{aligned}$$

## 2.5

Defined inner product on  $\mathcal{C}^n$  is

$$((y_1, \dots, y_n), (z_1, \dots, z_n)) = \sum_i y_i^* z_i.$$

Verify (1) of eq (2.13).

$$\begin{aligned}
 \left((y_1, \dots, y_n), \sum_i \lambda_i (z_{i1}, \dots, z_{in})\right) &= \sum_i y_i^* \left(\sum_j \lambda_j z_{ji}\right) \\
 &= \sum_{i,j} y_i^* \lambda_j z_{ji} \\
 &= \sum_j \lambda_j \left(\sum_i y_i^* z_{ji}\right) \\
 &= \sum_j \lambda_j ((y_1, \dots, y_n), (z_{j1}, \dots, z_{jn})) \\
 &= \sum_i \lambda_i ((y_1, \dots, y_n), (z_{i1}, \dots, z_{in})).
 \end{aligned}$$

Verify (2) of eq (2.13),

$$((y_1, \dots, y_n), (z_1, \dots, z_n))^* = \left( \sum_i y_i^* z_i \right)^* \quad (2.1)$$

$$= \left( \sum_i y_i z_i^* \right) \quad (2.2)$$

$$= \left( \sum_i z_i^* y_i \right) \quad (2.3)$$

$$= ((z_1, \dots, z_n), (y_1, \dots, y_n)) \quad (2.4)$$

Verify (3) of eq (2.13),

$$\begin{aligned} ((y_1, \dots, y_n), (y_1, \dots, y_n)) &= \sum_i y_i^* y_i \\ &= \sum_i |y_i|^2 \end{aligned}$$

Since  $|y_i|^2 \geq 0$  for all  $i$ . Thus  $\sum_i |y_i|^2 = ((y_1, \dots, y_n), (y_1, \dots, y_n)) \geq 0$ .

From now on, I will show the following statement,

$$((y_1, \dots, y_n), (y_1, \dots, y_n)) = 0 \text{ iff } (y_1, \dots, y_n) = 0.$$

( $\Leftarrow$ ) This is obvious.

( $\Rightarrow$ ) Suppose  $((y_1, \dots, y_n), (y_1, \dots, y_n)) = 0$ . Then  $\sum_i |y_i|^2 = 0$ .

Since  $|y_i|^2 \geq 0$  for all  $i$ , if  $\sum_i |y_i|^2 = 0$ , then  $|y_i|^2 = 0$  for all  $i$ . Therefore  $|y_i|^2 = 0 \Leftrightarrow y_i = 0$  for all  $i$ . Thus,

$$(y_1, \dots, y_n) = 0.$$

## 2.6

$$\begin{aligned} \left( \sum_i \lambda_i |w_i\rangle, |v\rangle \right) &= \left( |v\rangle, \sum_i \lambda_i |w_i\rangle \right)^* \\ &= \left[ \sum_i \lambda_i (|v\rangle, |w_i\rangle) \right]^* (\because \text{linearity in the 2nd arg.}) \\ &= \sum_i \lambda_i^* (|v\rangle, |w_i\rangle)^* \\ &= \sum_i \lambda_i^* (|w_i\rangle, |v\rangle) \end{aligned}$$

## 2.7

$$\begin{aligned}\langle w|v\rangle &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0 \\ \frac{|w\rangle}{\| |w\rangle \|} &= \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \frac{|v\rangle}{\| |v\rangle \|} &= \frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

**2.8**

If  $k = 1$ ,

$$\begin{aligned}|v_2\rangle &= \frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \\ \langle v_1|v_2\rangle &= \langle v_1| \left( \frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \right) \\ &= \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle \langle v_1|v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \\ &= 0.\end{aligned}$$

Suppose  $\{v_1, \dots, v_n\}$  ( $n \leq d-1$ ) is an orthonormal basis. Then

$$\begin{aligned}\langle v_j|v_{n+1}\rangle &= \langle v_j| \left( \frac{|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \right) \quad (j \leq n) \\ &= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \langle v_j|v_i\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\ &= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \delta_{ij}}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\ &= \frac{\langle v_j|w_{n+1}\rangle - \langle v_j|w_{n+1}\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\ &= 0.\end{aligned}$$

Thus Gram-Schmidt procedure produces an orthonormal basis.

**2.9**

$$\begin{aligned}\sigma_0 &= I = |0\rangle \langle 0| + |1\rangle \langle 1| \\ \sigma_1 &= X = |0\rangle \langle 1| + |1\rangle \langle 0| \\ \sigma_2 &= Y = -i |0\rangle \langle 1| + i |1\rangle \langle 0| \\ \sigma_3 &= Z = |0\rangle \langle 0| - |1\rangle \langle 1|\end{aligned}$$

**2.10**



$$\begin{aligned}
|v_j\rangle \langle v_k| &= I_V |v_j\rangle \langle v_k| I_V \\
&= \left( \sum_p |v_p\rangle \langle v_p| \right) |v_j\rangle \langle v_k| \left( \sum_q |v_q\rangle \langle v_q| \right) \\
&= \sum_{p,q} |v_p\rangle \langle v_p| v_j\rangle \langle v_k| v_q\rangle \langle v_q| \\
&= \sum_{p,q} \delta_{pj} \delta_{kq} |v_p\rangle \langle v_q|
\end{aligned}$$

Thus

$$(|v_j\rangle \langle v_k|)_{pq} = \delta_{pj} \delta_{kq}$$

## 2.11

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \det(X - \lambda I) = \det \left( \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) = 0 \Rightarrow \lambda = \pm 1$$

If  $\lambda = -1$ ,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus

$$|\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If  $\lambda = 1$

$$|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|\lambda = -1\rangle, |\lambda = 1\rangle\}$$

## 2.12

$$\det \left( \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I \right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1$$

Therefore the eigenvector associated with eigenvalue  $\lambda = 1$  is

$$|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because  $|\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c |\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

**2.13**

Suppose  $|\psi\rangle, |\phi\rangle$  are arbitrary vectors in  $V$ .

$$\begin{aligned} (|\psi\rangle, (|w\rangle\langle v|)|\phi\rangle)^* &= \left((|w\rangle\langle v|)^\dagger|\psi\rangle, |\phi\rangle\right)^* \\ &= \left(|\phi\rangle, (|w\rangle\langle v|)^\dagger|\psi\rangle\right) \\ &= \langle\phi|( |w\rangle\langle v|)^\dagger|\psi\rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} (|\psi\rangle, (|w\rangle\langle v|)|\phi\rangle)^* &= (\langle\psi|w\rangle\langle v|\phi\rangle)^* \\ &= \langle\phi|v\rangle\langle w|\psi\rangle. \end{aligned}$$

Thus

$$\begin{aligned} \langle\phi|( |w\rangle\langle v|)^\dagger|\psi\rangle &= \langle\phi|v\rangle\langle w|\psi\rangle \text{ for arbitrary vectors } |\psi\rangle, |\phi\rangle \\ \therefore (|w\rangle\langle v|)^\dagger &= |v\rangle\langle w| \end{aligned}$$

**2.14**

$$\begin{aligned} ((a_i A_i)^\dagger|\phi\rangle, |\psi\rangle) &= (|\phi\rangle, a_i A_i|\psi\rangle) \\ &= a_i(|\phi\rangle, A_i|\psi\rangle) \\ &= a_i(A_i^\dagger|\phi\rangle, |\psi\rangle) \\ &= (a_i^* A_i^\dagger|\phi\rangle, |\psi\rangle) \\ \therefore (a_i A_i)^\dagger &= a_i^* A_i^\dagger \end{aligned}$$

**2.15**

$$\begin{aligned} ((A^\dagger)^\dagger|\psi\rangle, |\phi\rangle) &= (|\psi\rangle, A^\dagger|\phi\rangle) \\ &= (A^\dagger|\phi\rangle, |\psi\rangle)^* \\ &= (|\phi\rangle, A|\psi\rangle)^* \\ &= (A|\psi\rangle, |\phi\rangle) \\ \therefore (A^\dagger)^\dagger &= A \end{aligned}$$

**2.16**

$$\begin{aligned}
P &= \sum_i |i\rangle \langle i|. \\
P^2 &= \left( \sum_i |i\rangle \langle i| \right) \left( \sum_j |j\rangle \langle j| \right) \\
&= \sum_{i,j} |i\rangle \langle i|j\rangle \langle j| \\
&= \sum_{i,j} |i\rangle \langle j| \delta_{ij} \\
&= \sum_i |i\rangle \langle i| \\
&= P
\end{aligned}$$

### 2.18

Suppose  $|v\rangle$  is a eigenvector with corresponding eigenvalue  $\lambda$ .

$$\begin{aligned}
U|v\rangle &= \lambda|v\rangle. \\
1 &= \langle v|v\rangle \\
&= \langle v|I|v\rangle \\
&= \langle v|U^\dagger U|v\rangle \\
&= \lambda\lambda^* \langle v|v\rangle \\
&= \|\lambda\|^2 \\
\therefore \lambda &= e^{i\theta}
\end{aligned}$$

### 2.19

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

### 2.20

$$\begin{aligned}
U &\equiv \sum_i |w_i\rangle \langle v_i| \\
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|UU^\dagger AUU^\dagger|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \langle v_p|v_q\rangle \langle w_q|A|w_r\rangle \langle v_r|v_s\rangle \langle w_s|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s|v_j\rangle \\
&= \sum_{p,r} \langle v_i|w_p\rangle \langle w_r|v_j\rangle A''_{pr}
\end{aligned}$$

**2.21**

Suppose  $M$  be Hermitian. Then  $M = M^\dagger$ .

$$\begin{aligned} M &= IMI \\ &= (P + Q)M(P + Q) \\ &= PMP + QMP + PMQ + QMQ \end{aligned}$$

Now  $PMP = \lambda P$ ,  $QMP = 0$ ,  $PMQ = PM^\dagger Q = (QMP)^* = 0$ . Thus  $M = PMP + QMQ$ . Next prove  $QMQ$  is normal.

$$\begin{aligned} QMQ(QMQ)^\dagger &= QMQQM^\dagger Q \\ &= QM^\dagger QMQ \quad (M = M^\dagger) \\ &= (QM^\dagger Q)QMQ \end{aligned}$$

Therefore  $QMQ$  is normal. By induction,  $QMQ$  is diagonal ... (following is same as Box 2.2)

**2.22**

Suppose  $A$  is a Hermitian operator and  $|v_i\rangle$  are eigenvectors of  $A$  with eigenvalues  $\lambda_i$ . Then

$$\langle v_i | A | v_j \rangle = \lambda_j \langle v_i | v_j \rangle.$$

On the other hand,

$$\langle v_i | A | v_j \rangle = \langle v_i | A^\dagger | v_j \rangle = \langle v_j | A | v_i \rangle^* = \lambda_i^* \langle v_j | v_i \rangle^* = \lambda_i^* \langle v_i | v_j \rangle = \lambda_i \langle v_i | v_j \rangle$$

Thus

$$(\lambda_i - \lambda_j) \langle v_i | v_j \rangle = 0.$$

If  $\lambda_i \neq \lambda_j$ , then  $\langle v_i | v_j \rangle = 0$ .

**2.23**

Suppose  $P$  is projector and  $|\lambda\rangle$  are eigenvectors of  $P$  with eigenvalues  $\lambda$ . Then  $P^2 = P$ .

$$P|\lambda\rangle = \lambda|\lambda\rangle \text{ and } P|\lambda\rangle = P^2|\lambda\rangle = \lambda P|\lambda\rangle = \lambda^2|\lambda\rangle.$$

Therefore

$$\begin{aligned} \lambda &= \lambda^2 \\ \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \text{ or } 1. \end{aligned}$$

**2.24**

Def of positive  $\langle v | A | v \rangle \geq 0$  for all  $|v\rangle$ .

Suppose  $A$  is a positive operator.  $A$  can be decomposed as follows.

$$\begin{aligned} A &= \frac{A + A^\dagger}{2} + i \frac{A - A^\dagger}{2i} \\ &= B + iC \quad \text{where } B = \frac{A + A^\dagger}{2}, \quad C = \frac{A - A^\dagger}{2i}. \end{aligned}$$

Now operators  $B$  and  $C$  are Hermitian.

$$\begin{aligned} \langle v|A|v \rangle &= \langle v|B + iC|v \rangle \\ &= \langle v|B|v \rangle + i \langle v|C|v \rangle \\ &= \alpha + i\beta \quad \text{where } \alpha = \langle v|B|v \rangle, \quad \beta = \langle v|C|v \rangle. \end{aligned}$$

Since  $B$  and  $C$  are Hermitian,  $\alpha, \beta \in \mathbb{R}$ . From def of positive operator,  $\beta$  should be vanished because  $\langle v|A|v \rangle = 0$  is real. Hence  $\beta = \langle v|C|v \rangle = 0$  for all  $|v\rangle$ , i.e.  $C = 0$ .

Therefore  $A = A^\dagger$ .

Reference: MIT 8.05 Lecture note by Prof. Barton Zwiebach.  
[https://ocw.mit.edu/courses/physics/8-05-quantum-physics-ii-fall-2013/lecture-notes/MIT8\\_05F13\\_Chap\\_03.pdf](https://ocw.mit.edu/courses/physics/8-05-quantum-physics-ii-fall-2013/lecture-notes/MIT8_05F13_Chap_03.pdf)

**Proposition. 2.0.1.** *Let  $T$  be a linear operator in a complex vector space  $V$ . If  $(u, Tv) = 0$  for all  $u, v \in V$ , then  $T = 0$ .*

*Proof.* Suppose  $u = Tv$ . Then  $(Tv, Tv) = 0$  for all  $v$  implies that  $Tv = 0$  for all  $v$ . Therefore  $T = 0$ .  $\square$

**Theorem. 2.0.1.** *If  $(v, Av) = 0$  for all  $v \in V$ , then  $A = 0$ .*

*Proof.* First, we show that  $(u, Tv) = 0$  if  $(v, Av) = 0$ . Then apply proposition 2.0.1. Suppose  $u, v \in V$ . Then  $(u, Tv)$  is decomposed as

$$\begin{aligned} (u, Tv) &= \frac{1}{4} \left[ (u + v, T(u + v)) - (u - v, T(u - v)) + \frac{1}{i}(u + iv, T(u + iv)) \right. \\ &\quad \left. - \frac{1}{i}(u - iv, T(u - iv)) \right]. \end{aligned}$$

If  $(v, Tv) = 0$  for all  $v \in V$ , the right hand side of above eqn vanishes. Thus  $(u, Tv) = 0$  for all  $u, v \in V$ . Then  $T = 0$ .  $\square$

## 2.25

$$\langle \psi|A^\dagger A|\psi \rangle = \|A|\psi \rangle\|^2 \geq 0 \quad \text{for all } |\psi \rangle.$$

Thus  $A^\dagger A$  is positive.

## 2.26

$$\begin{aligned}
|\psi\rangle^{\otimes 2} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
|\psi\rangle^{\otimes 3} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \\
&= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

**2.27**

$$\begin{aligned}
X \otimes Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

In general, tensor product is not commutable.

### 2.28

$$\begin{aligned}
 (A \otimes B)^* &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* \\
 &= \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
 &= A^* \otimes B^*.
 \end{aligned}$$

$$\begin{aligned}
 (A \otimes B)^T &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T \\
 &= \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
 &= \begin{bmatrix} A_{11}B^T & \cdots & A_{1m}^T B^T \\ \vdots & \ddots & \vdots \\ A_{n1}^T B^T & \cdots & A_{nm}^T B^T \end{bmatrix} \\
 &= A^T \otimes B^T.
 \end{aligned}$$

$$\begin{aligned}
 (A \otimes B)^\dagger &= ((A \otimes B)^*)^T \\
 &= (A^* \otimes B^*)^T \\
 &= (A^*)^T \otimes (B^*)^T \\
 &= A^\dagger \otimes B^\dagger.
 \end{aligned}$$

### 2.29

Suppose  $U_1$  and  $U_2$  are unitary operators. Then

$$\begin{aligned}
 (U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger &= U_1 U_1^\dagger \otimes U_2 U_2^\dagger \\
 &= I \otimes I.
 \end{aligned}$$

Similarly,

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = I \otimes I.$$

### 2.30

Suppose  $A$  and  $B$  are Hermitian operators. Then

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B. \quad (2.5)$$

Thus  $A \otimes B$  is Hermitian.

### 2.31

Suppose  $A$  and  $B$  are positive operators. Then

$$\langle \psi | \otimes \langle \phi | (A \otimes B) | \psi \rangle \otimes | \phi \rangle = \langle \psi | A | \psi \rangle \langle \phi | B | \phi \rangle .$$

Since  $A$  and  $B$  are positive operators,  $\langle \psi | A | \psi \rangle \geq 0$  and  $\langle \phi | B | \phi \rangle \geq 0$  for all  $|\psi\rangle, |\phi\rangle$ . Then  $\langle \psi | A | \psi \rangle \langle \phi | B | \phi \rangle \geq 0$ . Thus  $A \otimes B$  is positive if  $A$  and  $B$  are positive.

### 2.32

Suppose  $P_1$  and  $P_2$  are projectors. Then

$$\begin{aligned} (P_1 \otimes P_2)^2 &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2. \end{aligned}$$

Thus  $P_1 \otimes P_2$  is also projector.

### 2.33

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.6}$$

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

### 2.34

$$\text{Suppose } A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}.$$

$$\begin{aligned} \det(A - \lambda I) &= (4 - \lambda)^2 - 3^2 \\ &= \lambda^2 - 8\lambda + 7 \\ &= (\lambda - 1)(\lambda - 7) \end{aligned}$$

Eigenvalues of  $A$  are  $\lambda = 1, 7$ . Corresponding eigenvectors are  $|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,  $|\lambda = 7\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

Thus

$$A = |\lambda = 1\rangle \langle \lambda = 1| + 7 |\lambda = 7\rangle \langle \lambda = 7|.$$



$$\begin{aligned}
\sqrt{A} &= |\lambda = 1\rangle\langle\lambda = 1| + \sqrt{7} |\lambda = 7\rangle\langle\lambda = 7| \\
&= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 + \sqrt{7} & -1 + \sqrt{7} \\ -1 + \sqrt{7} & 1 + \sqrt{7} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\log(A) &= \log(1) |\lambda = 1\rangle\langle\lambda = 1| + \log(7) |\lambda = 7\rangle\langle\lambda = 7| \\
&= \frac{\log(7)}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

### 2.35

$$\begin{aligned}
\vec{v} \cdot \vec{\sigma} &= \sum_{i=1}^3 v_i \sigma_i \\
&= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\det(\vec{v} \cdot \vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)
\end{aligned}$$

Eigenvalues are  $\lambda = \pm 1$ . Let  $|\lambda_{\pm 1}\rangle$  be eigenvectors with eigenvalues  $\pm 1$ .

Since  $\vec{v} \cdot \vec{\sigma}$  is Hermitian,  $\vec{v} \cdot \vec{\sigma}$  is diagonalizable. Then

$$\vec{v} \cdot \vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|$$

Thus

$$\begin{aligned}
\exp(i\theta \vec{v} \cdot \vec{\sigma}) &= e^{i\theta} |\lambda_1\rangle\langle\lambda_1| + e^{-i\theta} |\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= (\cos \theta + i \sin \theta) |\lambda_1\rangle\langle\lambda_1| + (\cos \theta - i \sin \theta) |\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \cos \theta (|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + i \sin \theta (|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\
&= \cos(\theta) I + i \sin(\theta) \vec{v} \cdot \vec{\sigma}.
\end{aligned}$$

$\because$  Since  $\vec{v} \cdot \vec{\sigma}$  is Hermitian,  $|\lambda_1\rangle$  and  $|\lambda_{-1}\rangle$  are orthogonal. Thus

$$|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}| = I.$$

### 2.36

$$\begin{aligned}\mathrm{Tr}(\sigma_1) &= \mathrm{Tr} \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = 0 \\ \mathrm{Tr}(\sigma_2) &= \mathrm{Tr} \left( \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) = 0 \\ \mathrm{Tr}(\sigma_3) &= \mathrm{Tr} \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = 1 - 1 = 0\end{aligned}$$

**2.37**

$$\begin{aligned}\mathrm{Tr}(AB) &= \sum_i \langle i|AB|i\rangle \\ &= \sum_i \langle i|AIB|i\rangle \\ &= \sum_{i,j} \langle i|A|j\rangle \langle j|B|i\rangle \\ &= \sum_{i,j} \langle j|B|i\rangle \langle i|A|j\rangle \\ &= \sum_j \langle j|BA|j\rangle \\ &= \mathrm{Tr}(BA)\end{aligned}$$

**2.38**

$$\begin{aligned}\mathrm{Tr}(A+B) &= \sum_i \langle i|A+B|i\rangle \\ &= \sum_i (\langle i|A|i\rangle + \langle i|B|i\rangle) \\ &= \sum_i \langle i|A|i\rangle + \sum_i \langle i|B|i\rangle \\ &= \mathrm{Tr}(A) + \mathrm{Tr}(B).\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(zA) &= \sum_i \langle i|zA|i\rangle \\ &= \sum_i z \langle i|A|i\rangle \\ &= z \sum_i \langle i|A|i\rangle \\ &= z \mathrm{Tr}(A).\end{aligned}$$

**2.39**

(1)  $(A, B) \equiv \text{Tr}(A^\dagger B)$ .

(i)

$$\begin{aligned}
 \left(A, \sum_i \lambda_i B_i\right) &= \text{Tr} \left[ A^\dagger \left( \sum_i \lambda_i B_i \right) \right] \\
 &= \text{Tr}(A^\dagger \lambda_1 B_1) + \cdots + \text{Tr}(A^\dagger \lambda_n B_n) \quad (\because \text{Exercise 2.38}) \\
 &= \lambda_1 \text{Tr}(A^\dagger B_1) + \cdots + \lambda_n \text{Tr}(A^\dagger B_n) \\
 &= \sum_i \lambda_i \text{Tr}(A^\dagger B_i)
 \end{aligned}$$

(ii)

$$\begin{aligned}
 (A, B)^* &= \left( \text{Tr}(A^\dagger B) \right)^* \\
 &= \left( \sum_{i,j} \langle i|A^\dagger|j\rangle \langle j|B|i\rangle \right)^* \\
 &= \sum_{i,j} \langle i|A^\dagger|j\rangle^* \langle j|B|i\rangle^* \\
 &= \sum_{i,j} \langle j|B|i\rangle^* \langle i|A^\dagger|j\rangle^* \\
 &= \sum_{i,j} \langle i|B^\dagger|j\rangle \langle j|A|i\rangle \\
 &= \sum_i \langle i|B^\dagger A|i\rangle \\
 &= \text{Tr}(B^\dagger A) \\
 &= (B, A).
 \end{aligned}$$

(iii)

$$\begin{aligned}
 (A, A) &= \text{Tr}(A^\dagger A) \\
 &= \sum_i \langle i|A^\dagger A|i\rangle
 \end{aligned}$$

Since  $A^\dagger A$  is positive,  $\langle i|A^\dagger A|i\rangle \geq 0$  for all  $|i\rangle$ .

Let  $a_i$  be  $i$ -th column of  $A$ . If  $\langle i|A^\dagger A|i\rangle = 0$ , then

$$\langle i|A^\dagger A|i\rangle = a_i^\dagger a_i = \|a_i\|^2 = 0 \text{ iff } a_i = \mathbf{0}.$$

Therefore  $(A, A) = 0$  iff  $A = \mathbf{0}$ .

(2)

(3)

$$\begin{aligned}
[X, Y] &= XY - YX \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \\
&= 2iZ
\end{aligned}$$

$$\begin{aligned}
[Y, Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\
&= 2iX
\end{aligned}$$

$$\begin{aligned}
[Z, X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 2i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 2iY
\end{aligned}$$

## 2.41

$$\begin{aligned}
\{\sigma_1, \sigma_2\} &= \sigma_1\sigma_2 + \sigma_2\sigma_1 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\{\sigma_2, \sigma_3\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\{\sigma_3, \sigma_1\} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sigma_0^2 &= I^2 = I \\
\sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I \\
\sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^2 = I \\
\sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I
\end{aligned}$$

**2.42**

$$\frac{[A, B] + \{A, B\}}{2} = \frac{AB - BA + AB + BA}{2} = AB$$

**2.43**

From eq (2.75) and eq (2.76),  $\{\sigma_j, \sigma_k\} = 2\delta_{jk}I$ . From eq (2.77),

$$\begin{aligned}
\sigma_j \sigma_k &= \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} \\
&= \frac{2i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l + 2\delta_{jk}I}{2} \\
&= \delta_{jk}I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l
\end{aligned}$$

**2.44**

By assumption,  $[A, B] = 0$  and  $\{A, B\} = 0$ , then  $AB = 0$ . Since  $A$  is invertible, multiply by  $A^{-1}$  from left, then

$$\begin{aligned}
A^{-1}AB &= 0 \\
IB &= 0 \\
B &= 0.
\end{aligned}$$

**2.45**

$$\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= [B^\dagger, A^\dagger]
\end{aligned}$$

**2.46**

$$\begin{aligned}
[A, B] &= AB - BA \\
&= -(BA - AB) \\
&= -[B, A]
\end{aligned}$$

**2.47**

$$\begin{aligned}
(i[A, B])^\dagger &= -i[A, B]^\dagger \\
&= -i[B^\dagger, A^\dagger] \\
&= -i[B, A] \\
&= i[A, B]
\end{aligned}$$

**2.48**

(Positive )

Since  $P$  is positive, it is diagonalizable. Then  $P = \sum_i \lambda_i |i\rangle\langle i|$ , ( $\lambda_i \geq 0$ ).

$$J = \sqrt{P^\dagger P} = \sqrt{PP} = \sqrt{P^2} = \sum_i \sqrt{\lambda_i^2} |i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i| = P.$$

Therefore polar decomposition of  $P$  is  $P = UP$  for all  $P$ . Thus  $U = I$ , then  $P = P$ .

(Unitary)

Suppose unitary  $U$  is decomposed by  $U = WJ$  where  $W$  is unitary and  $J$  is positive,  $J = \sqrt{U^\dagger U}$ .

$$J = \sqrt{U^\dagger U} = \sqrt{I} = I$$

Since unitary operators are invertible,  $W = UJ^{-1} = UI^{-1} = UI = U$ . Thus polar decomposition of  $U$  is  $U = U$ .

(Hermitian)

Suppose  $H = UJ$ .

$$J = \sqrt{H^\dagger H} = \sqrt{HH} = \sqrt{H^2}.$$

Thus  $H = U\sqrt{H^2}$ .

In general,  $H \neq \sqrt{H^2}$ .

From spectral decomposition,  $H = \sum_i \lambda_i |i\rangle\langle i|$ ,  $\lambda_i \in \mathbb{R}$ .

$$\sqrt{H^2} = \sqrt{\sum_i \lambda_i^2 |i\rangle\langle i|} = \sum_i \sqrt{\lambda_i^2} |i\rangle\langle i| = \sum_i |\lambda_i| |i\rangle\langle i| \neq H$$

**2.49**

Normal matrix is diagonalizable,  $A = \sum_i \lambda_i |i\rangle\langle i|$ .

$$J = \sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle\langle i|.$$

$$U = \sum_i |e_i\rangle\langle i|$$

$$A = UJ = \sum_i |\lambda_i| |e_i\rangle\langle i|.$$

**2.50**

Define  $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ .  $A^\dagger A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ .

Characteristic equation of  $A^\dagger A$  is  $\det(A^\dagger A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0$ . Eigenvalues of  $A^\dagger A$  are  $\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}$  and associated eigenvectors are  $|\lambda_{\pm}\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}$ .

$$A^\dagger A = \lambda_+ |\lambda_+\rangle\langle\lambda_+| + \lambda_- |\lambda_-\rangle\langle\lambda_-|.$$

$$\begin{aligned} J = \sqrt{A^\dagger A} &= \sqrt{\lambda_+} |\lambda_+\rangle\langle\lambda_+| + \sqrt{\lambda_-} |\lambda_-\rangle\langle\lambda_-| \\ &= \sqrt{\frac{3+\sqrt{5}}{2}} \cdot \frac{5-\sqrt{5}}{40} \begin{bmatrix} 4 & 2\sqrt{5}-2 \\ 2\sqrt{5}-2 & 6-2\sqrt{5} \end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{2}} \cdot \frac{5+\sqrt{5}}{40} \begin{bmatrix} 4 & -2\sqrt{5}-2 \\ -2\sqrt{5}-2 & 6+2\sqrt{5} \end{bmatrix} \end{aligned}$$

$$J^{-1} = \frac{1}{\sqrt{\lambda_+}} |\lambda_+\rangle\langle\lambda_+| + \frac{1}{\sqrt{\lambda_-}} |\lambda_-\rangle\langle\lambda_-|.$$

$$U = AJ^{-1}$$

I'm tired.

**2.51**

$$H^\dagger H = \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

**2.52**

$$H^\dagger = \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$$

Thus

$$H^2 = I.$$

**2.53**

$$\begin{aligned} \det(H - \lambda I) &= \left( \frac{1}{\sqrt{2}} - \lambda \right) \left( -\frac{1}{\sqrt{2}} - \lambda \right) - \frac{1}{2} \\ &= \lambda^2 - \frac{1}{2} - \frac{1}{2} \\ &= \lambda^2 - 1 \end{aligned}$$

Eigenvalues are  $\lambda_{\pm} = \pm 1$  and associated eigenvectors are  $|\lambda_{\pm}\rangle = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 \pm \sqrt{2} \end{bmatrix}$ .

**2.54**

Since  $[A, B] = 0$ ,  $A$  and  $B$  are simultaneously diagonalize,  $A = \sum_i a_i |i\rangle\langle i|$ ,  $B = \sum_i b_i |i\rangle\langle i|$ .

$$\begin{aligned} \exp(A) \exp(B) &= \left( \sum_i \exp(a_i) |i\rangle\langle i| \right) \left( \sum_i \exp(b_i) |i\rangle\langle i| \right) \\ &= \sum_{i,j} \exp(a_i + b_j) |i\rangle\langle j| \langle j| \\ &= \sum_{i,j} \exp(a_i + b_j) |i\rangle\langle j| \delta_{i,j} \\ &= \sum_i \exp(a_i + b_i) |i\rangle\langle i| \\ &= \exp(A + B) \end{aligned}$$

**2.55**

$$H = \sum_E E |E\rangle\langle E|$$

$$\begin{aligned} U(t_2 - t_1) U^\dagger(t_2 - t_1) &= \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right) \exp\left(\frac{iH(t_2 - t_1)}{\hbar}\right) \\ &= \sum_{E,E'} \left( \exp\left(-\frac{iE(t_2 - t_1)}{\hbar}\right) |E\rangle\langle E| \right) \left( \exp\left(-\frac{iE'(t_2 - t_1)}{\hbar}\right) |E'\rangle\langle E'| \right) \\ &= \sum_{E,E'} \left( \exp\left(-\frac{i(E - E')(t_2 - t_1)}{\hbar}\right) |E\rangle\langle E'| \delta_{E,E'} \right) \\ &= \sum_E \exp(0) |E\rangle\langle E| \\ &= \sum_E |E\rangle\langle E| \\ &= I \end{aligned}$$

Similarly,  $U^\dagger(t_2 - t_1) U(t_2 - t_1) = I$ .

**2.56**

$$U = \sum_i \lambda_i |\lambda_i\rangle\langle \lambda_i| \quad (|\lambda_i| = 1).$$

$$\log(U) = \sum_j \log(\lambda_j) |\lambda_j\rangle\langle \lambda_j| = \sum_j i\theta_j |\lambda_j\rangle\langle \lambda_j| \quad \text{where } \theta_j = \arg(\lambda_j)$$

$$K = -i \log(U) = \sum_j \theta_j |\lambda_j\rangle\langle \lambda_j|.$$



$$K^\dagger = (-i \log U)^\dagger = \left( \sum_j \theta_j |\lambda_j\rangle\langle\lambda_j| \right)^\dagger = \sum_j \theta_j^* |\lambda_j\rangle\langle\lambda_j| = \sum_j \theta_j |\lambda_j\rangle\langle\lambda_j| = K$$

**2.57**

$$\begin{aligned} |\phi\rangle &\equiv \frac{L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \\ \langle\phi|M_m^\dagger M_m|\phi\rangle &= \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle} \\ \frac{M_m |\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} &= \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \cdot \frac{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}} \end{aligned}$$

**2.58**

$$\begin{aligned} \langle M \rangle &= \langle\psi|M|\psi\rangle = \langle\psi|m|\psi\rangle = m \langle\psi|\psi\rangle = m \\ \langle M^2 \rangle &= \langle\psi|M^2|\psi\rangle = \langle\psi|m^2|\psi\rangle = m^2 \langle\psi|\psi\rangle = m^2 \\ \text{deviation} &= \langle M^2 \rangle - \langle M \rangle^2 = m^2 - m^2 = 0. \end{aligned}$$

**2.59**

$$\begin{aligned} \langle X \rangle &= \langle 0|X|0\rangle = \langle 0|1\rangle = 0 \\ \langle X^2 \rangle &= \langle 0|X^2|0\rangle = \langle 0|X|1\rangle = \langle 0|0\rangle = 1 \\ \text{standard deviation} &= \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = 1 \end{aligned}$$

**2.60**

$$\begin{aligned} \vec{v} \cdot \vec{\sigma} &= \sum_{i=1}^3 v_i \sigma_i \\ &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(\vec{v} \cdot \vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\ &= \lambda^2 - 1 \quad (\because |\vec{v}| = 1) \end{aligned}$$

Eigenvalues are  $\lambda = \pm 1$ .

(i) if  $\lambda = 1$

$$\begin{aligned}\vec{v} \cdot \vec{\sigma} - \lambda I &= \vec{v} \cdot \vec{\sigma} - I \\ &= \begin{bmatrix} v_3 - 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - 1 \end{bmatrix}\end{aligned}$$

Eigenvector is  $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix}$ .

$$\begin{aligned}|\lambda_1\rangle\langle\lambda_1| &= \frac{1+v_3}{2} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1-v_3}{v_1+iv_2} \end{bmatrix} \\ &= \frac{1+v_3}{2} \begin{bmatrix} 1 & \frac{v_1-iv_2}{1+v_3} \\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\ &= \frac{1}{2} \left( I + \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\ &= \frac{1}{2} (I + \vec{v} \cdot \vec{\sigma})\end{aligned}$$

(ii) If  $\lambda = -1$ .

$$\begin{aligned}\vec{v} \cdot \vec{\sigma} - \lambda I &= \vec{v} \cdot \vec{\sigma} + I \\ &= \begin{bmatrix} v_3 + 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 + 1 \end{bmatrix}\end{aligned}$$

Eigenvector is  $|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$ .

$$\begin{aligned}|\lambda_{-1}\rangle\langle\lambda_{-1}| &= \frac{1-v_3}{2} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1+v_3}{v_1+iv_2} \end{bmatrix} \\ &= \frac{1-v_3}{2} \begin{bmatrix} 1 & -\frac{v_1-iv_2}{1-v_3} \\ -\frac{v_1+iv_2}{1-v_3} & \frac{1+v_3}{1-v_3} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1-v_3 & -(v_1-iv_2) \\ -(v_1+iv_2) & 1+v_3 \end{bmatrix} \\ &= \frac{1}{2} \left( I - \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\ &= \frac{1}{2} (I - \vec{v} \cdot \vec{\sigma}).\end{aligned}$$

## 2.61

$$\begin{aligned}\langle\lambda_1|0\rangle\langle 0|\lambda_1\rangle &= \langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle \\ &= \langle 0|\frac{1}{2}(I + \vec{v} \cdot \vec{\sigma})|0\rangle \\ &= \frac{1}{2}(1+v_3)\end{aligned}$$

Post-measurement state is

$$\begin{aligned}
\frac{|\lambda_1\rangle\langle\lambda_1|0\rangle}{\sqrt{\langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle}} &= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \cdot \frac{1}{2} \begin{bmatrix} 1+v_3 \\ v_1+iv_2 \end{bmatrix} \\
&= \sqrt{\frac{1}{2}(1+v_3)} \begin{bmatrix} 1 \\ \frac{v_1+iv_2}{1+v_3} \end{bmatrix} \\
&= \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \\
&= |\lambda_1\rangle.
\end{aligned}$$

## 2.62

Suppose  $M_m$  is an measurement operator. From the assumption,  $E_m = M_m^\dagger M_m = M_m$ . Then

$$\langle\psi|E_m|\psi\rangle = \langle\psi|M_m|\psi\rangle \geq 0.$$

for all  $|\psi\rangle$ .

Since  $M_m$  is positive operator,  $M_m$  is Hermitian. Therefore,

$$E_m = M_m^\dagger M_m = M_m M_m = M_m^2 = M_m.$$

Thus the measurement is a projective measurement.

## 2.63

$$\begin{aligned}
M_m^\dagger M_m &= \sqrt{E_m} U_m^\dagger U_m \sqrt{E_m} \\
&= \sqrt{E_m} I \sqrt{E_m} \\
&= E_m.
\end{aligned}$$

Since  $E_m$  is POVM, for arbitrary unitary  $U$ ,  $M_m^\dagger M_m$  is POVM.

## 2.64

Define  $E_i = |\psi_i\rangle\langle\psi_i|$  for  $1 \leq i \leq m$  and  $E_{m+1} = I - \sum_{i=1}^m E_i$ . Then  $\sum_{i=1}^{m+1} E_i = I$ . And  $\langle\psi_i|E_i|\psi_i\rangle = \langle\psi_i|\psi_i\rangle\langle\psi_i|\psi_i\rangle = 1$ .

## 2.65

$$|+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

## 2.66

$$X_1 Z_2 \left( \frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

$$\langle X_1 Z_2 \rangle = \left( \frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) X_1 Z_2 \left( \frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{\langle 00| + \langle 11|}{\sqrt{2}} \cdot \frac{|10\rangle - |01\rangle}{\sqrt{2}} = 0$$

**2.67**

Unsolved

$$W \subset V \rightarrow V = W \oplus W^\perp.$$

$$U : W \rightarrow V, U' : V \rightarrow V.$$

$$U' |w\rangle = U |w\rangle$$

$$U' \in \mathcal{L}(V)$$

$$U \in \mathcal{L}(W)$$

$$U' = U \oplus I \text{ ???}$$

**2.68**

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Suppose  $|a\rangle = a_0 |0\rangle + a_1 |1\rangle$  and  $|b\rangle = b_0 |0\rangle + b_1 |1\rangle$ .

$$|a\rangle |b\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle.$$

If  $|\psi\rangle = |a\rangle |b\rangle$ , then  $a_0 b_0 = 1$ ,  $a_0 b_1 = 0$ ,  $a_1 b_0 = 0$ ,  $a_1 b_1 = 1$  since  $\{|ij\rangle\}$  is an orthonormal basis.

If  $a_0 b_1 = 0$ , then  $a_0 = 0$  or  $b_1 = 0$ .

When  $a_0 = 0$ , this is contradiction to  $a_0 b_0 = 1$ . When  $b_1 = 0$ , this is contradiction to  $a_1 b_1 = 1$ .

Thus  $|\psi\rangle \neq |a\rangle |b\rangle$ .

**2.69**

Define Bell states as follows.

$$|\psi_1\rangle \equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|\psi_2\rangle \equiv \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$|\psi_3\rangle \equiv \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$|\psi_4\rangle \equiv \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

First, we prove  $\{|\psi_i\rangle\}$  is a linearly independent basis.

$$a_1 |\psi_1\rangle + a_2 |\psi_2\rangle + a_3 |\psi_3\rangle + a_4 |\psi_4\rangle = 0$$

$$\therefore \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 + a_2 \\ a_3 + a_4 \\ a_3 - a_4 \\ a_1 - a_2 \end{bmatrix} = 0$$

$$\therefore \begin{cases} a_1 + a_2 = 0 \\ a_3 + a_4 = 0 \\ a_3 - a_4 = 0 \\ a_1 - a_2 = 0 \end{cases}$$

$$\therefore a_1 = a_2 = a_3 = a_4 = 0$$

Thus  $\{|\psi_i\rangle\}$  is a linearly independent basis.

Moreover  $\| |\psi_i\rangle \| = 1$  and  $\langle \psi_i | \psi_j \rangle = \delta_{ij}$  for  $i, j = 1, 2, 3, 4$ . Therefore  $\{|\psi_i\rangle\}$  forms an orthonormal basis.

## 2.70

For any Bell states we get  $\langle \psi_i | E \otimes I | \psi_i \rangle = \frac{1}{2}(\langle 0 | E | 0 \rangle + \langle 1 | E | 1 \rangle)$ .

Suppose Eve measures the qubit Alice sent by measurement operators  $M_m$ . The probability that Eve gets result  $m$  is  $p_i(m) = \langle \psi_i | M_m^\dagger M_m \otimes I | \psi_i \rangle$ . Since  $M_m^\dagger M_m$  is positive,  $p_i(m)$  are same values for all  $|\psi_i\rangle$ . Thus Eve can't distinguish Bell states.

## 2.71

From spectral decomposition,

$$\begin{aligned} \rho &= \sum_i p_i |\psi_i\rangle \langle \psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1. \\ \rho^2 &= \sum_{i,j} p_i p_j |i\rangle \langle i| j\rangle \langle j| \\ &= \sum_{i,j} p_i p_j |i\rangle \langle j| \delta_{ij} \\ &= \sum_i p_i^2 |i\rangle \langle i| \end{aligned}$$

$$\text{Tr}(\rho^2) = \text{Tr} \left( \sum_i p_i^2 |i\rangle \langle i| \right) = \sum_i p_i^2 \text{Tr}(|i\rangle \langle i|) = \sum_i p_i^2 \langle i | i \rangle = \sum_i p_i^2 \leq \sum_i p_i = 1 \quad (\because p_i^2 \leq p_i)$$

Suppose  $\text{Tr}(\rho^2) = 1$ . Then  $\sum_i p_i^2 = 1$ . If  $0 \leq p_i < 1$ , then  $p_i^2 < p_i$ . Thus only one  $p_i = 1$  and otherwise are 0. Therefore  $\rho = |\psi_i\rangle \langle \psi_i|$  is pure state.

Conversely if  $\rho$  is pure, then  $\rho = |\psi\rangle \langle \psi|$ .

$$\text{Tr}(\rho^2) = \text{Tr}(|\psi\rangle \langle \psi| |\psi\rangle \langle \psi|) = \text{Tr}(|\psi\rangle \langle \psi|) = \langle \psi | \psi \rangle = 1.$$

**2.72**

(1) Since density matrix is Hermitian, matrix representation is  $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$ ,  $a, d \in \mathbb{R}$  and  $b \in \mathbb{C}$  w.r.t. standard basis. Because  $\rho$  is density matrix,  $\text{Tr}(\rho) = a + d = 1$ .

Define  $a = (1 + r_3)/2$ ,  $d = (1 - r_3)/2$  and  $b = (r_1 - ir_2)/2$ , ( $r_i \in \mathbb{R}$ ).

In this case,

$$\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{bmatrix} = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}).$$

Thus for arbitrary density matrix  $\rho$  can be written as  $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$ .

Next, we derive the condition that  $\rho$  is positive.

If  $\rho$  is positive, all eigenvalues of  $\rho$  should be non-negative.

$$\begin{aligned} \det(\rho - \lambda I) &= (a - \lambda)(b - \lambda) - |b|^2 = \lambda^2 - (a + d)\lambda + ad - |b|^2 = 0 \\ \lambda &= \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - |b|^2)}}{2} \\ &= \frac{1 \pm \sqrt{1 - 4\left(\frac{1-r_3^2}{4} - \frac{r_1^2+r_2^2}{4}\right)}}{2} \\ &= \frac{1 \pm \sqrt{1 - (1 - r_1^2 - r_2^2 - r_3^2)}}{2} \\ &= \frac{1 \pm \sqrt{|\vec{r}|^2}}{2} \\ &= \frac{1 \pm |\vec{r}|}{2} \end{aligned}$$

Since  $\rho$  is positive,  $\frac{1-|\vec{r}|}{2} \geq 0 \rightarrow |\vec{r}| \leq 1$ .

Therefore an arbitrary density matrix for a mixed state qubit is written as  $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$ .

(2)

$\rho = I/2 \rightarrow \vec{r} = 0$ . Thus  $\rho = I/2$  corresponds to the origin of Bloch sphere.

(3)

$$\begin{aligned} \rho^2 &= \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \\ &= \frac{1}{4} \left[ I + 2\vec{r} \cdot \vec{\sigma} + \sum_{j,k} r_j r_k \left( \delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l \right) \right] \\ &= \frac{1}{4} (I + 2\vec{r} \cdot \vec{\sigma} + |\vec{r}|^2 I) \\ \text{Tr}(\rho^2) &= \frac{1}{4} (2 + 2|\vec{r}|^2) \end{aligned}$$

If  $\rho$  is pure, then  $\text{Tr}(\rho^2) = 1$ .

$$\begin{aligned} 1 &= \text{Tr}(\rho^2) = \frac{1}{4} (2 + 2|\vec{r}|^2) \\ \therefore |\vec{r}| &= 1. \end{aligned}$$

Conversely, if  $|\vec{r}| = 1$ , then  $\text{Tr}(\rho^2) = \frac{1}{4}(2 + 2|\vec{r}|^2) = 1$ . Therefore  $\rho$  is pure.

### 2.73

#### Theorem 2.6

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j| = \sum_j q_j |\varphi_j\rangle\langle\varphi_j| \Leftrightarrow |\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$$

where  $u$  is unitary.

Transformation in theorem 2.6,  $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$ , corresponds to

$$\left[ |\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_k\rangle \right] = \left[ |\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle \right] U^T$$

where  $k = \text{rank}(\rho)$ .

From spectral theorem, density matrix  $\rho$  is decomposed as  $\rho = \sum_{k=1}^d \lambda_k |k\rangle\langle k|$  where  $d = \dim \mathcal{H}$ . Without loss of generality, we can assume  $p_k > 0$  for  $k = 1 \cdots l$  where  $l = \text{rank}(\rho)$  and  $p_k = 0$  for  $k = l + 1, \cdots, d$ . Thus  $\rho = \sum_{k=1}^l p_k |k\rangle\langle k| = \sum_{k=1}^l |\tilde{k}\rangle\langle\tilde{k}|$ , where  $|\tilde{k}\rangle = \sqrt{\lambda_k} |k\rangle$ .

Suppose  $|\psi_i\rangle$  is a state in support  $\rho$ . Then

$$|\psi_i\rangle = \sum_{k=1}^l c_{ik} |k\rangle, \quad \sum_k |c_{ik}|^2 = 1.$$

Define  $p_i = \frac{1}{\sum_k \frac{|c_{ik}|^2}{\lambda_k}}$  and  $u_{ik} = \frac{\sqrt{p_i} c_{ik}}{\sqrt{\lambda_k}}$ .

Now

$$\sum_k |u_{ik}|^2 = \sum_k \frac{p_i |c_{ik}|^2}{\lambda_k} = p_i \sum_k \frac{|c_{ik}|^2}{\lambda_k} = 1.$$

Next prepare an unitary operator <sup>1</sup> such that  $i$ th row of  $U$  is  $[u_{i1} \cdots u_{ik} \cdots u_{il}]$ . Then we can define another ensemble such that

$$\left[ |\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_i\rangle \cdots |\tilde{\psi}_l\rangle \right] = \left[ |\tilde{k}_1\rangle \cdots |\tilde{k}_l\rangle \right] U^T$$

where  $|\tilde{\psi}_i\rangle = \sqrt{p_i} |\psi_i\rangle$ . From theorem 2.6,

$$\rho = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|.$$

Therefore we can obtain a minimal ensemble for  $\rho$  that contains  $|\psi_i\rangle$ .

<sup>1</sup>By Gram-Schmidt procedure construct an orthonormal basis  $\{\mathbf{u}_j\}$  (row vector) with  $\mathbf{u}_i = [u_{i1} \cdots u_{ik} \cdots u_{il}]$ .

Then define unitary  $U = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_l \end{bmatrix}$ .

Moreover since  $\rho^{-1} = \sum_k \frac{1}{\lambda_k} |k\rangle\langle k|$ ,

$$\langle \psi_i | \rho^{-1} | \psi_i \rangle = \sum_k \frac{1}{\lambda_k} \langle \psi_i | k \rangle \langle k | \psi_i \rangle = \sum_k \frac{|c_{ik}|^2}{\lambda_k} = \frac{1}{p_i}.$$

Hence,  $\frac{1}{\langle \psi_i | \rho^{-1} | \psi_i \rangle} = p_i$ .

## 2.74

$$\begin{aligned} \rho_{AB} &= |a\rangle\langle a|_A \otimes |b\rangle\langle b|_B \\ \rho_A &= \text{Tr}_B \rho_{AB} = |a\rangle\langle a| \text{Tr}(|b\rangle\langle b|) = |a\rangle\langle a| \\ \text{Tr}(\rho_A^2) &= 1 \end{aligned}$$

Thus  $\rho_A$  is pure.

## 2.75

Define  $|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$  and  $|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ .

$$\begin{aligned} |\Phi_{\pm}\rangle\langle\Phi_{\pm}|_{AB} &= \frac{1}{2}(|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|) \\ \text{Tr}_B(|\Phi_{\pm}\rangle\langle\Phi_{\pm}|_{AB}) &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2} \\ |\Psi_{\pm}\rangle\langle\Psi_{\pm}| &= \frac{1}{2}(|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|) \\ \text{Tr}_B(|\Psi_{\pm}\rangle\langle\Psi_{\pm}|) &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2} \end{aligned}$$

## 2.76

Unsolved. I think the polar decomposition can only apply to square matrix  $A$ , not arbitrary linear operators. Suppose  $A$  is  $m \times n$  matrix. Then size of  $A^\dagger A$  is  $n \times n$ . Thus the size of  $U$  should be  $m \times n$ . Maybe  $U$  is isometry, but I think it is not unitary.

## 2.77

$$\begin{aligned} |\psi\rangle &= |0\rangle |\Phi_+\rangle \\ &= |0\rangle \left[ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right] \\ &= (\alpha |\phi_0\rangle + \beta |\phi_1\rangle) \left[ \frac{1}{\sqrt{2}}(|\phi_0\phi_0\rangle + |\phi_1\phi_1\rangle) \right] \end{aligned}$$

where  $|\phi_i\rangle$  are arbitrary orthonormal states and  $\alpha, \beta \in \mathbb{C}$ . We cannot vanish cross term. Therefore  $|\psi\rangle$  cannot be written as  $|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B |i\rangle_C$ .

## 2.78



*Proof.* Former part.

If  $|\psi\rangle$  is product, then there exist a state  $|\phi_A\rangle$  for system  $A$ , and a state  $|\phi_B\rangle$  for system  $B$  such that  $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$ .

Obviously, this Schmidt number is 1.

Conversely, if Schmidt number is 1, the state is written as  $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$ . Hence this is a product state.  $\square$

*Proof.* Later part.

( $\Rightarrow$ ) Proved by exercise 2.74.

( $\Leftarrow$ ) Let a pure state be  $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$ . Then  $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|) = \sum_i \lambda_i^2 |i\rangle\langle i|$ . If  $\rho_A$  is a pure state, then  $\lambda_j = 1$  and otherwise 0 for some  $j$ . It follows that  $|\psi_j\rangle = |j_A\rangle |j_B\rangle$ . Thus  $|\psi\rangle$  is a product state.  $\square$

## 2.79

Procedure of Schmidt decomposition.

Goal:  $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$

- Diagonalize reduced density matrix  $\rho_A = \sum_i \lambda_i |i_A\rangle\langle i_A|$ .
- Derive  $|i_B\rangle$ ,  $|i_B\rangle = \frac{(I \otimes \langle i_A|) |\psi\rangle}{\sqrt{\lambda_i}}$
- Construct  $|\psi\rangle$ .

(i)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ This is already decomposed.}$$

(ii)

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) = |\psi\rangle |\psi\rangle \text{ where } |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

(iii)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$

$$\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$$

$$\rho_A = \text{Tr}_B(\rho_{AB}) = \frac{1}{3} (2|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\det(\rho_A - \lambda I) = \left( \frac{2}{3} - \lambda \right) \left( \frac{1}{3} - \lambda \right) - \frac{1}{9} = 0$$

$$\lambda^2 - \lambda + \frac{1}{9} = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}/3}{2} = \frac{3 \pm \sqrt{5}}{6}$$

$$\text{Eigenvector with eigenvalue } \lambda_0 \equiv \frac{3 + \sqrt{5}}{6} \text{ is } |\lambda_0\rangle \equiv \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

Eigenvector with eigenvalue  $\lambda_1 \equiv \frac{3 - \sqrt{5}}{6}$  is  $|\lambda_1\rangle \equiv \frac{1}{\sqrt{\frac{5 - \sqrt{5}}{2}}} \begin{bmatrix} \frac{1 - \sqrt{5}}{2} \\ 1 \end{bmatrix}$ .

$$\rho_A = \lambda_0 |\lambda_0\rangle\langle\lambda_0| + \lambda_1 |\lambda_1\rangle\langle\lambda_1|.$$

$$|a_0\rangle \equiv \frac{(I \otimes \langle\lambda_0|) |\psi\rangle}{\sqrt{\lambda_0}}$$

$$|a_1\rangle \equiv \frac{(I \otimes \langle\lambda_1|) |\psi\rangle}{\sqrt{\lambda_1}}$$

Then

$$|\psi\rangle = \sum_{i=0}^1 \sqrt{\lambda_i} |a_i\rangle |\lambda_i\rangle.$$

(It's too tiresome to calculate  $|a_i\rangle$ )

### 2.80

Let  $|\psi\rangle = \sum_i \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B$  and  $|\varphi\rangle = \sum_i \lambda_i |\varphi_i\rangle_A |\varphi_i\rangle_B$ .  
Define  $U = \sum_i |\psi_j\rangle\langle\varphi_j|_A$  and  $V = \sum_j |\psi_j\rangle\langle\varphi_j|_B$ .

Then

$$\begin{aligned} (U \otimes V) |\varphi\rangle &= \sum_i \lambda_i U |\varphi_i\rangle_A V |\varphi_i\rangle_B \\ &= \sum_i \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B \\ &= |\psi\rangle. \end{aligned}$$

### 2.81

Suppose  $\rho_A = \text{Tr}_R |AR_2\rangle\langle AR_2| = \sum_i \lambda_i |i\rangle\langle i|$ . Define  $|AR_1\rangle = (I_A \otimes U_R) |AR_2\rangle$ .

$$\begin{aligned} \text{Tr}_R(|AR_1\rangle\langle AR_1|) &= \text{Tr}_R \left( (I_A \otimes U_R) |AR_2\rangle\langle AR_2| (I_A \otimes U_R^\dagger) \right) \\ &= \text{Tr}_R \left( |AR_2\rangle\langle AR_2| (I_A \otimes U_R^\dagger)(I_A \otimes U_R) \right) \\ &= \text{Tr}_R(|AR_2\rangle\langle AR_2|) \\ &= \rho_A. \end{aligned}$$

Thus  $|AR_1\rangle$  is also a purification of  $\rho_A$ .

### 2.82

(1)

Let  $|\psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$ .

$$\text{Tr}_R(|\psi\rangle\langle\psi|) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Thus  $|\psi\rangle$  is a purification of  $\rho$ .

(2) Probability

$$\text{Tr}[(I \otimes |i\rangle\langle i|) |\psi\rangle\langle\psi|] = \langle\psi|(I \otimes |i\rangle\langle i|)|\psi\rangle = p_i \langle\psi_i|\psi_i\rangle = p_i.$$

Post-measurement state

$$\frac{(I \otimes |i\rangle\langle i|) |\psi\rangle}{\sqrt{p_i}} = \frac{\sqrt{p_i} |\psi_i\rangle}{\sqrt{p_i}} = |\psi_i\rangle.$$

(3)

Suppose  $|AR\rangle$  is a purification of  $\rho$  such that  $|AR\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle$ . By exercise 2.81, the others purification is written as  $(I \otimes U) |AR\rangle$ .

$$\begin{aligned} (I \otimes U) |AR\rangle &= (I \otimes U) \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle U |r_i\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle \end{aligned}$$

where  $U = \sum_i |i\rangle\langle r_i|$ .

By (2), if we measure the system  $R$  w.r.t  $|i\rangle$ , post-measurement state for system  $A$  is  $|\psi_i\rangle$  with probability  $p_i$ , which prove the assertion.

**Problem 2.1**

From Exercise 2.35,  $\vec{n} \cdot \vec{\sigma}$  is decomposed as

$$\vec{n} \cdot \vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|$$

where  $|\lambda_{\pm 1}\rangle$  are eigenvector of  $\vec{n} \cdot \vec{\sigma}$  with eigenvalues  $\pm 1$ .

Thus

$$\begin{aligned} f(\theta \vec{n} \cdot \vec{\sigma}) &= f(\theta) |\lambda_1\rangle\langle\lambda_1| + f(-\theta) |\lambda_{-1}\rangle\langle\lambda_{-1}| \\ &= \left( \frac{f(\theta) + f(-\theta)}{2} + \frac{f(\theta) - f(-\theta)}{2} \right) |\lambda_1\rangle\langle\lambda_1| + \left( \frac{f(\theta) + f(-\theta)}{2} - \frac{f(\theta) - f(-\theta)}{2} \right) |\lambda_{-1}\rangle\langle\lambda_{-1}| \\ &= \frac{f(\theta) + f(-\theta)}{2} (|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + \frac{f(\theta) - f(-\theta)}{2} (|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\ &= \frac{f(\theta) + f(-\theta)}{2} I + \frac{f(\theta) - f(-\theta)}{2} \vec{n} \cdot \vec{\sigma} \end{aligned}$$

**Problem 2.2**

Unsolved

**Problem 2.3**

Unsolved



## Chapter 8

# Quantum noise and quantum operations

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