Solution for "Quantum Computation and Quantum Information: 10th Anniversary Edition" by Nielsen and Chuang

goropikari

July 11, 2018

Contents

Chapter 2

Introduction to quantum mechanics

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2.2

$$\begin{aligned} A\left|0\right\rangle &= A_{11}\left|0\right\rangle + A_{21}\left|1\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 0,\ A_{21} = 1 \\ A\left|1\right\rangle &= A_{12}\left|0\right\rangle + A_{22}\left|1\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 1,\ A_{22} = 0 \\ \therefore A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

input: $\{\left|0\right\rangle, \left|1\right\rangle\}$, output: $\{\left|1\right\rangle, \left|0\right\rangle\}$

$$\begin{aligned} A\left|0\right\rangle &= A_{11}\left|1\right\rangle + A_{21}\left|0\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 1,\ A_{21} = 0 \\ A\left|1\right\rangle &= A_{12}\left|1\right\rangle + A_{22}\left|0\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 0,\ A_{22} = 1 \\ A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

2.3

From eq (2.12)

$$A\left|v_i\right\rangle = \sum_j A_{ji}\left|w_j\right\rangle$$

$$B\left|w_j\right\rangle = \sum_k B_{kj}\left|x_k\right\rangle$$

Thus

$$
\begin{aligned}
BA\ket{v_i} &= B\left(\sum_j A_{ji}\ket{w_j}\right) \\
&= \sum_j A_{ji}B\ket{w_j} \\
&= \sum_{j,k} A_{ji}B_{kj}\ket{x_k} \\
&= \sum_k \left(\sum_j B_{kj}A_{ji}\right)\ket{x_k} \\
&= \sum_k (BA)_{ki}\ket{x_k} \\
\therefore (BA)_{ki} &= \sum_j B_{kj}A_{ji}
\end{aligned}
$$

2.4

$$
\begin{aligned}
I\ket{v_j} = \sum_i I_{ij}\ket{v_i} = \ket{v_j},\ \forall j. \\
\Rightarrow I_{ij} = \delta_{ij}
\end{aligned}
$$

2.5

Defined inner product on $\mathcal{C}^n$ is

$$((y_1, \cdots, y_n), (z_1, \cdots, z_n)) = \sum_i y_i^* z_i.$$

Verify (1) of eq (2.13).

$$
\begin{aligned}
\left((y_1, \cdots, y_n), \sum_i \lambda_i (z_{i1}, \cdots, z_{in})\right) &= \sum_i y_i^* \left(\sum_j \lambda_j z_{ji}\right) \\
&= \sum_{i,j} y_i^* \lambda_j z_{ji} \\
&= \sum_j \lambda_j \left(\sum_i y_i^* z_{ji}\right) \\
&= \sum_j \lambda_j \left((y_1, \cdots, y_n), (z_{j1}, \cdots, z_{jn})\right) \\
&= \sum_i \lambda_i \left((y_1, \cdots, y_n), (z_{i1}, \cdots, z_{in})\right).
\end{aligned}
$$

Verify (2) of eq (2.13),

$$
\begin{align}
((y_1, \cdots, y_n), (z_1, \cdots, z_n))^* &= \left(\sum_i y_i^* z_i \right)^* \tag{2.1} \\
&= \left(\sum_i y_i z_i^* \right) \tag{2.2} \\
&= \left(\sum_i z_i^* y_i \right) \tag{2.3} \\
&= ((z_1, \cdots, z_n), (y_1, \cdots, y_n)) \tag{2.4}
\end{align}
$$

.

Verify (3) of eq (2.13),

$$
\begin{aligned}
((y_1, \cdots, y_n), (y_1, \cdots, y_n)) &= \sum_i y_i^* y_i \\
&= \sum_i |y_i|^2
\end{aligned}
$$

Since $|y_i|^2 \geq 0$ for all i. Thus $\sum_i |y_i|^2 = ((y_1, \cdots, y_n), (y_1, \cdots, y_n)) \geq 0$.

From now on, I will show the following statement,

$$((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0 \text{ iff } (y_1, \cdots, y_n) = 0.$$

($\Leftarrow$) This is obvious.

($\Rightarrow$) Suppose $((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0$. Then $\sum_i |y_i|^2 = 0$.

Since $|y_i|^2 \geq 0$ for all i, if $\sum_i |y_i|^2 = 0$, then $|y_i|^2 = 0$ for all i. Therefore $|y_i|^2 = 0 \Leftrightarrow y_i = 0$ for all i. Thus,

$$(y_1, \cdots, y_n) = 0.$$

2.6

$$
\begin{aligned}
\left(\sum_i \lambda_i \left| w_i \right\rangle, \ \left| v \right\rangle \right) &= \left(\left| v \right\rangle, \ \sum_i \lambda_i \left| w_i \right\rangle \right)^* \\
&= \left[\sum_i \lambda_i \left(\left| v \right\rangle, \ \left| w_i \right\rangle \right) \right]^* \ (\because \text{linearlity in the 2nd arg.}) \\
&= \sum_i \lambda_i^* \left(\left| v \right\rangle, \ \left| w_i \right\rangle \right)^* \\
&= \sum_i \lambda_i^* (\left| w_i \right\rangle, \ \left| v \right\rangle)
\end{aligned}
$$

2.7

$$
\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0 \\
\frac{|w\rangle}{\||w\rangle\|} &= \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\frac{|v\rangle}{\||v\rangle\|} &= \frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

2.8

If $k = 1$,

$$
\begin{aligned}
|v_2\rangle &= \frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle\|} \\
\langle v_1|v_2\rangle &= \langle v_1| \left(\frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle\|} \right) \\
&= \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle \langle v_1|v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle\|} \\
&= 0.
\end{aligned}
$$

Suppose $\{v_1, \cdots v_n\}$ $(n \leq d-1)$ is a orthonormal basis. Then

$$
\begin{aligned}
\langle v_j|v_{n+1}\rangle &= \langle v_j| \left(\frac{|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \right) \quad (j \leq n) \\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \langle v_j|v_i\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \delta_{ij}}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \\
&= \frac{\langle v_j|w_{n+1}\rangle - \langle v_j|w_{n+1}\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \\
&= 0.
\end{aligned}
$$

Thus Gram-Schmidt procedure produces an orthonormal basis.

2.9

$$
\begin{aligned}
\sigma_0 &= I = |0\rangle \langle 0| + |1\rangle \langle 1| \\
\sigma_1 &= X = |0\rangle \langle 1| + |1\rangle \langle 0| \\
\sigma_2 &= Y = -i\,|0\rangle \langle 1| + i\,|1\rangle \langle 0| \\
\sigma_3 &= Z = |0\rangle \langle 0| - |1\rangle \langle 1|
\end{aligned}
$$

2.10

$$
\begin{aligned}
|v_j\rangle \langle v_k| &= I_V |v_j\rangle \langle v_k| I_V \\
&= \left(\sum_p |v_p\rangle \langle v_p| \right) |v_j\rangle \langle v_k| \left(\sum_q |v_q\rangle \langle v_q| \right) \\
&= \sum_{p,q} |v_p\rangle \langle v_p|v_j\rangle \langle v_k|v_q\rangle \langle v_q| \\
&= \sum_{p,q} \delta_{pj}\delta_{kq} |v_p\rangle \langle v_q|
\end{aligned}
$$

Thus

$$
\left(|v_j\rangle \langle v_k|\right)_{pq} = \delta_{pj}\delta_{kq}
$$

2.11

$$
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \det(X - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) = 0 \Rightarrow \lambda = \pm 1
$$

If $\lambda = -1$,

$$
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Thus

$$
|\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}
$$

If $\lambda = 1$

$$
|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

$$
X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|\lambda = -1\rangle, \ |\lambda = 1\rangle\}
$$

2.12

$$
\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I \right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1
$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$
|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

Because $|\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c\, |\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}
$$

2.13

Suppose $|\psi\rangle\,,\ |\phi\rangle$ are arbitrary vectors in V.

$$\begin{aligned}(|\psi\rangle\,,\ (|w\rangle\langle v|)\,|\phi\rangle)^* &= \left((|w\rangle\langle v|)^\dagger\,|\psi\rangle\,,\ |\phi\rangle\right)^* \\ &= \left(|\phi\rangle\,,\ (|w\rangle\langle v|)^\dagger\,|\psi\rangle\right) \\ &= \langle\phi|\,(|w\rangle\langle v|)^\dagger\,|\psi\rangle\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}(|\psi\rangle\,,\ (|w\rangle\langle v|)\,|\phi\rangle)^* &= (\langle\psi|w\rangle\langle v|\phi\rangle)^* \\ &= \langle\phi|v\rangle\langle w|\psi\rangle\,.\end{aligned}$$

Thus

$$\begin{aligned}\langle\phi|\,(|w\rangle\langle v|)^\dagger\,|\psi\rangle = \langle\phi|v\rangle\langle w|\psi\rangle\ \text{ for arbitrary vectors }\ |\psi\rangle\,,\ |\phi\rangle \\ \therefore (|w\rangle\langle v|)^\dagger = |v\rangle\langle w|\end{aligned}$$

2.14

$$\begin{aligned}((a_iA_i)^\dagger\,|\phi\rangle\,,\ |\psi\rangle) &= (|\phi\rangle\,,\ a_iA_i\,|\psi\rangle) \\ &= a_i(|\phi\rangle\,,\ A_i\,|\psi\rangle) \\ &= a_i(A_i^\dagger\,|\phi\rangle\,,\ |\psi\rangle) \\ &= (a_i^*A_i^\dagger\,|\phi\rangle\,,\ |\psi\rangle) \\ \therefore (a_iA_i)^\dagger &= a_i^*A_i^\dagger\end{aligned}$$

2.15

$$\begin{aligned}((A^\dagger)^\dagger\,|\psi\rangle\,,\ |\phi\rangle) &= (|\psi\rangle\,,\ A^\dagger\,|\phi\rangle) \\ &= (A^\dagger\,|\phi\rangle\,,\ |\psi\rangle)^* \\ &= (|\phi\rangle\,,\ A\,|\psi\rangle)^* \\ &= (A\,|\psi\rangle\,,\ |\phi\rangle) \\ \therefore (A^\dagger)^\dagger &= A\end{aligned}$$

2.16

$$
\begin{aligned}
P &= \sum_i |i\rangle \langle i| . \\
P^2 &= \left(\sum_i |i\rangle \langle i|\right) \left(\sum_j |j\rangle \langle j|\right) \\
&= \sum_{i,j} |i\rangle \langle i|j\rangle \langle j| \\
&= \sum_{i,j} |i\rangle \langle j| \delta_{ij} \\
&= \sum_i |i\rangle \langle i| \\
&= P
\end{aligned}
$$

2.17

Proof. ($\Rightarrow$) Suppose A is Hermitian. Then $A = A^\dagger$. Let $|\lambda\rangle$ be eigenvectors of A with eigenvalues λ, that is,

$$A\,|\rangle = \lambda\,|\lambda\rangle .$$

Therefore

$$\langle\lambda|A|\lambda\rangle = \lambda\,\langle\lambda|\lambda\rangle = \lambda.$$

On the other hand,

$$\lambda^* = \langle\lambda|A|\lambda\rangle^* = \langle\lambda|A^\dagger|\lambda\rangle = \langle\lambda|A|\lambda\rangle = \lambda\,\langle\lambda|\lambda\rangle = \lambda.$$

Hence eigenvalues of Hermitian matrix are real.

($\Leftarrow$) Suppose eigenvalues of A are real. From spectral theorem, normal matrix A can be written by

$$A = \sum_i \lambda_i\,|\lambda_i\rangle\langle\lambda_i| \tag{2.5}$$

where λ_i are real eigenvalues with eigenvectors $|\lambda_i\rangle$. By taking adjoint, we get

$$
\begin{aligned}
A^\dagger &= \sum_i \lambda_i^*\,|\lambda_i\rangle\langle\lambda_i| \\
&= \sum_i \lambda_i\,|\lambda_i\rangle\langle\lambda_i| \quad (\because \lambda_i \text{ are real}) \\
&= A
\end{aligned}
$$

Thus A is Hermitian. $\square$

2.18

Suppose $|v\rangle$ is a eigenvector with corresponding eigenvalue λ.

$$\begin{aligned}
U|v\rangle &= \lambda|v\rangle.\\
1 &= \langle v|v\rangle\\
&= \langle v|I|v\rangle\\
&= \langle v|U^\dagger U|v\rangle\\
&= \lambda\lambda^* \langle v|v\rangle\\
&= \|\lambda\|^2\\
\therefore \lambda &= e^{i\theta}
\end{aligned}$$

2.19

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2.20

$$\begin{aligned}
U &\equiv \sum_i |w_i\rangle\langle v_i|\\
A'_{ij} &= \langle v_i|A|v_j\rangle\\
&= \langle v_i|UU^\dagger AUU^\dagger|v_j\rangle\\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle\langle v_p|v_q\rangle\langle w_q|A|w_r\rangle\langle v_r|v_s\rangle\langle w_s|v_j\rangle\\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle\delta_{pq}A''_{qr}\delta_{rs}\langle w_s|v_j\rangle\\
&= \sum_{p,r} \langle v_i|w_p\rangle\langle w_r|v_j\rangle A''_{pr}
\end{aligned}$$

2.21

Suppose M be Hermitian. Then $M = M^\dagger$.

$$\begin{aligned}
M &= IMI\\
&= (P+Q)M(P+Q)\\
&= PMP + QMP + PMQ + QMQ
\end{aligned}$$

Now $PMP = \lambda P$, $QMP = 0$, $PMQ = PM^\dagger Q = (QMP)^* = 0$. Thus $M = PMP + QMQ$. Next prove QMQ is normal.

$$\begin{aligned}
QMQ(QMQ)^\dagger &= QMQQM^\dagger Q\\
&= QM^\dagger QQMQ \quad (M = M^\dagger)\\
&= (QM^\dagger Q)QMQ
\end{aligned}$$

Therefore QMQ is normal. By induction, QMQ is diagonal ... (following is same as Box 2.2)

2.22

Suppose A is a Hermitian operator and $|v_i\rangle$ are eigenvectors of A with eigenvalues λ_i. Then

$$\langle v_i|A|v_j\rangle = \lambda_j \langle v_i|v_j\rangle .$$

On the other hand,

$$\langle v_i|A|v_j\rangle = \langle v_i|A^\dagger|v_j\rangle = \langle v_j|A|v_i\rangle^* = \lambda_i^* \langle v_j|v_i\rangle^* = \lambda_i^* \langle v_i|v_j\rangle = \lambda_i \langle v_i|v_j\rangle$$

Thus

$$(\lambda_i - \lambda_j) \langle v_i|v_j\rangle = 0.$$

If $\lambda_i \neq \lambda_j$, then $\langle v_i|v_j\rangle = 0$.

2.23

Suppose P is projector and $|\lambda\rangle$ are eigenvectors of P with eigenvalues λ. Then $P^2 = P$.

$$P|\lambda\rangle = \lambda|\lambda\rangle \text{ and } P|\lambda\rangle = P^2|\lambda\rangle = \lambda P|\lambda\rangle = \lambda^2|\lambda\rangle .$$

Therefore

$$\begin{aligned} \lambda &= \lambda^2 \\ \lambda(\lambda-1) &= 0 \\ \lambda &= 0 \text{ or } 1. \end{aligned}$$

2.24

Def of positive $\langle v|A|v\rangle \geq 0$ for all $|v\rangle$.

Suppose A is a positive operator. A can be decomposed as follows.

$$\begin{aligned} A &= \frac{A+A^\dagger}{2} + i\frac{A-A^\dagger}{2i} \\ &= B + iC \quad \text{where } B = \frac{A+A^\dagger}{2}, \quad C = \frac{A-A^\dagger}{2i}. \end{aligned}$$

Now operators B and C are Hermitian.

$$\begin{aligned} \langle v|A|v\rangle &= \langle v|B+iC|v\rangle \\ &= \langle v|B|v\rangle + i\langle v|C|v\rangle \\ &= \alpha + i\beta \quad \text{where } \alpha = \langle v|B|v\rangle, \; \beta = \langle v|C|v\rangle . \end{aligned}$$

Since B and C are Hermitian, $\alpha, \; \beta \in \mathbb{R}$. From def of positive operator, β should be vanished because $\langle v|A|v\rangle$ is real. Hence $\beta = \langle v|C|v\rangle = 0$ for all $|v\rangle$, i.e. $C = 0$.

Therefore $A = A^\dagger$.

Reference: MIT 8.05 Lecture note by Prof. Barton Zwiebach.
`https://ocw.mit.edu/courses/physics/8-05-quantum-physics-ii-fall-2013/lecture-notes/MIT8_05F13_Chap_03.pdf`

Proposition. 2.0.1. *Let T be a linear operator in a complex vector space V. If $(u, Tv) = 0$ for all $u, v \in V$, then $T = 0$.*

Proof. Suppose $u = Tv$. Then $(Tv, Tv) = 0$ for all v implies that $Tv = 0$ for all v. Therefore $T = 0$. □

Theorem. 2.0.1. *If $(v, Av) = 0$ for all $v \in V$, then $A = 0$.*

Proof. First, we show that $(u, Tv) = 0$ if $(v, Av) = 0$. Then apply proposition 2.0.1 Suppose $u, v \in V$. Then (u, Tv) is decomposed as

$$
\begin{aligned}
(u, Tv) = \frac{1}{4}\Bigg[&(u+v, T(u+v)) - (u-v, T(u-v)) + \frac{1}{i}(u+iv, T(u+iv)) \\
&- \frac{1}{i}(u-iv, T(u-iv))\Bigg].
\end{aligned}
$$

If $(v, Tv) = 0$ for all $v \in V$, the right hand side of above eqn vanishes. Thus $(u, Tv) = 0$ for all $u, v \in V$. Then $T = 0$. □

2.25

$$
\langle\psi|A^\dagger A|\psi\rangle = \|A\,|\psi\rangle\|^2 \geq 0 \text{ for all } |\psi\rangle\,.
$$

Thus $A^\dagger A$ is positive.

2.26

$$
\begin{aligned}
|\psi\rangle^{\otimes 2} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
|\psi\rangle^{\otimes 3} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \\
&= \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \\
&= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

2.27

$$
\begin{aligned}
X \otimes Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

In general, tensor product is not commutable.

2.28

$$
\begin{aligned}
(A \otimes B)^* &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* \\
&= \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
&= A^* \otimes B^*.
\end{aligned}
$$

$$
\begin{aligned}
(A \otimes B)^T &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{1m}B^T \\ \vdots & \ddots & \vdots \\ A_{n1}B^T & \cdots & A_{nm}B^T \end{bmatrix} \\
&= A^T \otimes B^T.
\end{aligned}
$$

$$
\begin{aligned}
(A \otimes B)^\dagger &= ((A \otimes B)^*)^T \\
&= (A^* \otimes B^*)^T \\
&= (A^*)^T \otimes (B^*)^T \\
&= A^\dagger \otimes B^\dagger.
\end{aligned}
$$

2.29

Suppose U_1 and U_2 are unitary operators. Then

$$
\begin{aligned}
(U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger &= U_1 U_1^\dagger \otimes U_2 U_2^\dagger \\
&= I \otimes I.
\end{aligned}
$$

Similarly,

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = I \otimes I.$$

2.30

Suppose A and B are Hermitian operators. Then

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B. \tag{2.6}$$

Thus $A \otimes B$ is Hermitian.

2.31

Suppose A and B are positive operators. Then

$$\langle\psi| \otimes \langle\phi| \, (A \otimes B) \, |\psi\rangle \otimes |\phi\rangle = \langle\psi|A|\psi\rangle \, \langle\phi|B|\phi\rangle \, .$$

Since A and B are positive operators, $\langle\psi|A|\psi\rangle \geq 0$ and $\langle\phi|B|\phi\rangle \geq 0$ for all $|\psi\rangle, |\phi\rangle$. Then $\langle\psi|A|\psi\rangle \, \langle\phi|B|\phi\rangle \geq 0$. Thus $A \otimes B$ is positive if A and B are positive.

2.32

Suppose P_1 and P_2 are projectors. Then

$$\begin{aligned}(P_1 \otimes P_2)^2 &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2.\end{aligned}$$

Thus $P_1 \otimes P_2$. is also projector.

2.33

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.7}$$

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

2.34

Suppose $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

$$\begin{aligned}\det(A - \lambda I) &= (4-\lambda)^2 - 3^2 \\ &= \lambda^2 - 8\lambda + 7 \\ &= (\lambda - 1)(\lambda - 7)\end{aligned}$$

Eigenvalues of A are $\lambda = 1,\ 7$. Corresponding eigenvectors are $|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $|\lambda = 7\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus

$$A = |\lambda = 1\rangle\langle\lambda = 1| + 7\, |\lambda = 7\rangle\langle\lambda = 7|\,.$$

$$\begin{aligned}\sqrt{A} &= |\lambda = 1\rangle\langle\lambda = 1| + \sqrt{7}\, |\lambda = 7\rangle\langle\lambda = 7| \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\log(A) &= \log(1)\, |\lambda = 1\rangle\langle\lambda = 1| + \log(7)\, |\lambda = 7\rangle\langle\lambda = 7| \\ &= \frac{\log(7)}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

2.35

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma} &= \sum_{i=1}^{3} v_i\sigma_i \\
&= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\det(\vec{v}\cdot\vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)
\end{aligned}$$

Eigenvalues are $\lambda = \pm 1$. Let $|\lambda_{\pm 1}\rangle$ be eigenvectors with eigenvalues ± 1.

Since $\vec{v}\cdot\vec{\sigma}$ is Hermitian, $\vec{v}\cdot\vec{\sigma}$ is diagonalizable. Then

$$\vec{v}\cdot\vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|$$

Thus

$$\begin{aligned}
\exp(i\theta\vec{v}\cdot\vec{\sigma}) &= e^{i\theta}\,|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= (\cos\theta + i\sin\theta)\,|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \cos\theta(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + i\sin\theta(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\
&= \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}.
\end{aligned}$$

$\because$ Since $\vec{v}\cdot\vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthogonal. Thus

$$|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}| = I.$$

2.36

$$\begin{aligned}
\mathrm{Tr}(\sigma_1) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0 \\
\mathrm{Tr}(\sigma_2) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = 0 \\
\mathrm{Tr}(\sigma_3) &= \mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1 - 1 = 0
\end{aligned}$$

2.37

$$\begin{aligned}
\operatorname{Tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_{i,j} \langle i|A|j\rangle \langle j|B|i\rangle \\
&= \sum_{i,j} \langle j|B|i\rangle \langle i|A|j\rangle \\
&= \sum_j \langle j|BA|j\rangle \\
&= \operatorname{Tr}(BA)
\end{aligned}$$

2.38

$$\begin{aligned}
\operatorname{Tr}(A+B) &= \sum_i \langle i|A+B|i\rangle \\
&= \sum_i (\langle i|A|i\rangle + \langle i|B|i\rangle) \\
&= \sum_i \langle i|A|i\rangle + \sum_i \langle i|B|i\rangle \\
&= \operatorname{Tr}(A) + \operatorname{Tr}(B).
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tr}(zA) &= \sum_i \langle i|zA|i\rangle \\
&= \sum_i z \langle i|A|i\rangle \\
&= z \sum_i \langle i|A|i\rangle \\
&= z \operatorname{Tr}(A).
\end{aligned}$$

2.39

(1) $(A, B) \equiv \operatorname{Tr}(A^\dagger B)$.

(i)

$$\begin{aligned}
\left(A, \sum_i \lambda_i B_i\right) &= \operatorname{Tr}\left[A^\dagger \left(\sum_i \lambda_i B_i\right)\right] \\
&= \operatorname{Tr}(A^\dagger \lambda_1 B_1) + \cdots + \operatorname{Tr}(A^\dagger \lambda_n B_n) \quad (\because \text{Execise 2.38}) \\
&= \lambda_1 \operatorname{Tr}(A^\dagger B_1) + \cdots + \lambda_n \operatorname{Tr}(A^\dagger B_n) \\
&= \sum_i \lambda_i \operatorname{Tr}(A^\dagger B_i)
\end{aligned}$$

(ii)

$$
\begin{aligned}
(A, B)^* &= \left(\mathrm{Tr}(A^\dagger B)\right)^* \\
&= \left(\sum_{i,j} \langle i|A^\dagger|j\rangle \langle j|B|i\rangle\right)^* \\
&= \sum_{i,j} \langle i|A^\dagger|j\rangle^* \langle j|B|i\rangle^* \\
&= \sum_{i,j} \langle j|B|i\rangle^* \langle i|A^\dagger|j\rangle^* \\
&= \sum_{i,j} \langle i|B^\dagger|j\rangle \langle j|A|i\rangle \\
&= \sum_{i} \langle i|B^\dagger A|i\rangle \\
&= \mathrm{Tr}(B^\dagger A) \\
&= (B, A).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
(A, A) &= \mathrm{Tr}(A^\dagger A) \\
&= \sum_i \langle i|A^\dagger A|i\rangle
\end{aligned}
$$

Since $A^\dagger A$ is positive, $\langle i|A^\dagger A|i\rangle \geq 0$ for all $|i\rangle$.

Let a_i be i-th column of A. If $\langle i|A^\dagger A|i\rangle = 0$, then

$$
\langle i|A^\dagger A|i\rangle = a_i^\dagger a_i = \|a_i\|^2 = 0 \text{ iff } a_i = \mathbf{0}.
$$

Therefore $(A, A) = 0$ iff $A = \mathbf{0}$.

(2)

(3)

2.40

$$
\begin{aligned}
[X, Y] &= XY - YX \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \\
&= 2iZ
\end{aligned}
$$

$$
\begin{aligned}
[Y, Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\
&= 2iX
\end{aligned}
$$

$$
\begin{aligned}
[Z, X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 2i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 2iY
\end{aligned}
$$

2.41

$$
\begin{aligned}
\{\sigma_1, \sigma_2\} &= \sigma_1\sigma_2 + \sigma_2\sigma_1 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\{\sigma_2, \sigma_3\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\{\sigma_3, \sigma_1\} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sigma_0^2 &= I^2 = I \\
\sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I \\
\sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^2 = I \\
\sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I
\end{aligned}
$$

2.42

$$\frac{[A,B]+\{A,B\}}{2} = \frac{AB-BA+AB+BA}{2} = AB$$

2.43

From eq (2.75) and eq (2.76), $\{\sigma_j, \sigma_k\} = 2\delta_{jk}I$. From eq (2.77),

$$\begin{aligned}\sigma_j\sigma_k &= \frac{[\sigma_j,\sigma_k]+\{\sigma_j,\sigma_k\}}{2} \\ &= \frac{2i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l + 2\delta_{jk}I}{2} \\ &= \delta_{jk}I + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l\end{aligned}$$

2.44

By assumption, $[A,B]=0$ and $\{A,B\}=0$, then $AB=0$. Since A is invertible, multiply by A^{-1} from left, then

$$\begin{aligned}A^{-1}AB &= 0 \\ IB &= 0 \\ B &= 0.\end{aligned}$$

2.45

$$\begin{aligned}[A,B]^\dagger &= (AB-BA)^\dagger \\ &= B^\dagger A^\dagger - A^\dagger B^\dagger \\ &= \left[B^\dagger, A^\dagger\right]\end{aligned}$$

2.46

$$\begin{aligned}[A,B] &= AB-BA \\ &= -(BA-AB) \\ &= -[B,A]\end{aligned}$$

2.47

$$\begin{aligned}(i[A,B])^\dagger &= -i[A,B]^\dagger \\ &= -i\left[B^\dagger, A^\dagger\right] \\ &= -i[B,A] \\ &= i[A,B]\end{aligned}$$

2.48

(Positive)

Since P is positive, it is diagonalizable. Then $P = \sum_i \lambda_i |i\rangle\langle i|$, $(\lambda_i \geq 0)$.

$$J = \sqrt{P^\dagger P} = \sqrt{PP} = \sqrt{P^2} = \sum_i \sqrt{\lambda_i^2}\, |i\rangle\langle i| = \sum_i \lambda_i\, |i\rangle\langle i| = P.$$

Therefore polar decomposition of P is $P = UP$ for all P. Thus $U = I$, then $P = P$.

(Unitary)

Suppose unitary U is decomposed by $U = WJ$ where W is unitary and J is positive, $J = \sqrt{U^\dagger U}$.

$$J = \sqrt{U^\dagger U} = \sqrt{I} = I$$

Since unitary operators are invertible, $W = UJ^{-1} = UI^{-1} = UI = U$. Thus polar decomposition of U is $U = U$.

(Hermitian)

Suppose $H = UJ$.

$$J = \sqrt{H^\dagger H} = \sqrt{HH} = \sqrt{H^2}.$$

Thus $H = U\sqrt{H^2}$.

In general, $H \neq \sqrt{H^2}$.

From spectral decomposition, $H = \sum_i \lambda_i |i\rangle\langle i|$, $\lambda_i \in \mathbb{R}$.

$$\sqrt{H^2} = \sqrt{\sum_i \lambda_i^2\, |i\rangle\langle i|} = \sum_i \sqrt{\lambda_i^2}\, |i\rangle\langle i| = \sum_i |\lambda_i|\, |i\rangle\langle i| \neq H$$

2.49

Normal matrix is diagonalizable, $A = \sum_i \lambda_i |i\rangle\langle i|$.

$$J = \sqrt{A^\dagger A} = \sum_i |\lambda_i|\, |i\rangle\langle i|\,.$$

$$U = \sum_i |e_i\rangle\langle i|$$

$$A = UJ = \sum_i |\lambda_i|\, |e_i\rangle\langle i|\,.$$

2.50

Define $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $A^\dagger A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation of $A^\dagger A$ is $\det(A^\dagger A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0$. Eigenvalues of $A^\dagger A$ are $\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$ and associated eigenvectors are $|\lambda_\pm\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}$.

$$A^\dagger A = \lambda_+ \left|\lambda_+\right\rangle\left\langle\lambda_+\right| + \lambda_- \left|\lambda_-\right\rangle\left\langle\lambda_-\right|.$$

$$\begin{aligned}
J = \sqrt{A^\dagger A} &= \sqrt{\lambda_+} \left|\lambda_+\right\rangle\left\langle\lambda_+\right| + \sqrt{\lambda_-} \left|\lambda_-\right\rangle\left\langle\lambda_-\right| \\
&= \sqrt{\frac{3+\sqrt{5}}{2}} \cdot \frac{5-\sqrt{5}}{40} \begin{bmatrix} 4 & 2\sqrt{5}-2 \\ 2\sqrt{5}-2 & 6-2\sqrt{5} \end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{2}} \cdot \frac{5+\sqrt{5}}{40} \begin{bmatrix} 4 & -2\sqrt{5}-2 \\ -2\sqrt{5}-2 & 6+2\sqrt{5} \end{bmatrix}
\end{aligned}$$

$$J^{-1} = \frac{1}{\sqrt{\lambda_+}} \left|\lambda_+\right\rangle\left\langle\lambda_+\right| + \frac{1}{\sqrt{\lambda_-}} \left|\lambda_-\right\rangle\left\langle\lambda_-\right|.$$

$$U = AJ^{-1}$$

I'm tired.

2.51

$$H^\dagger H = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^\dagger \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

2.52

$$H^\dagger = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$$

Thus

$$H^2 = I.$$

2.53

$$\begin{aligned}
\det\left(H - \lambda I\right) &= \left(\frac{1}{\sqrt{2}} - \lambda\right)\left(-\frac{1}{\sqrt{2}} - \lambda\right) - \frac{1}{2} \\
&= \lambda^2 - \frac{1}{2} - \frac{1}{2} \\
&= \lambda^2 - 1
\end{aligned}$$

Eigenvalues are $\lambda_\pm = \pm 1$ and associated eigenvectors are $\left|\lambda_\pm\right\rangle = \frac{1}{\sqrt{4\mp2\sqrt{2}}}\begin{bmatrix} 1 \\ -1\pm\sqrt{2} \end{bmatrix}$.

2.54

Since $[A, B] = 0$, A and B are simultaneously diagonalize, $A = \sum_i a_i |i\rangle\langle i|$, $B = \sum_i b_i |i\rangle\langle i|$.

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_i \exp(a_i)\, |i\rangle\langle i|\right)\left(\sum_i \exp(b_i)\, |i\rangle\langle i|\right) \\
&= \sum_{i,j} \exp(a_i + b_j)\, |i\rangle \langle i|j\rangle \langle j| \\
&= \sum_{i,j} \exp(a_i + b_j)\, |i\rangle\langle j|\, \delta_{i,j} \\
&= \sum_i \exp(a_i + b_i)\, |i\rangle\langle i| \\
&= \exp(A + B)
\end{aligned}$$

2.55

$$H = \sum_E E\, |E\rangle\langle E|$$

$$\begin{aligned}
U(t_2 - t_1)U^\dagger(t_2 - t_1) &= \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right)\exp\left(\frac{iH(t_2 - t_1)}{\hbar}\right) \\
&= \sum_{E,E'}\left(\exp\left(-\frac{iE(t_2 - t_1)}{\hbar}\right)|E\rangle\langle E|\right)\left(\exp\left(-\frac{iE'(t_2 - t_1)}{\hbar}\right)|E'\rangle\langle E'|\right) \\
&= \sum_{E,E'}\left(\exp\left(-\frac{i(E - E')(t_2 - t_1)}{\hbar}\right)|E\rangle\langle E'|\, \delta_{E,E'}\right) \\
&= \sum_E \exp(0)\, |E\rangle\langle E| \\
&= \sum_E |E\rangle\langle E| \\
&= I
\end{aligned}$$

Similarly, $U^\dagger(t_2 - t_1)U(t_2 - t_1) = I$.

2.56

$U = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \quad (|\lambda_i| = 1)$.

$$\begin{aligned}
\log(U) &= \sum_j \log(\lambda_j)\, |\lambda_j\rangle\langle\lambda_j| = \sum_j i\theta_j\, |\lambda_j\rangle\langle\lambda_j| \;\text{ where } \theta_j = \arg(\lambda_j) \\
K &= -i\log(U) = \sum_j \theta_j\, |\lambda_j\rangle\langle\lambda_j|\,.
\end{aligned}$$

$$K^\dagger = (-i\log U)^\dagger = \left(\sum_j \theta_j\, |\lambda_j\rangle\langle\lambda_j|\right)^\dagger = \sum_j \theta_j^*\, |\lambda_j\rangle\langle\lambda_j| = \sum_j \theta_j\, |\lambda_j\rangle\langle\lambda_j| = K$$

2.57

$$|\phi\rangle \equiv \frac{L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

$$\langle\phi|M_m^\dagger M_m|\phi\rangle = \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle}$$

$$\frac{M_m |\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \cdot \frac{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}}$$

2.58

$$\begin{aligned}
\langle M\rangle &= \langle\psi|M|\psi\rangle = \langle\psi|m|\psi\rangle = m\langle\psi|\psi\rangle = m \\
\langle M^2\rangle &= \langle\psi|M^2|\psi\rangle = \langle\psi|m^2|\psi\rangle = m^2\langle\psi|\psi\rangle = m^2 \\
\text{deviation} &= \langle M^2\rangle - \langle M\rangle^2 = m^2 - m^2 = 0.
\end{aligned}$$

2.59

$$\begin{aligned}
\langle X\rangle &= \langle 0|X|0\rangle = \langle 0|1\rangle = 0 \\
\langle X^2\rangle &= \langle 0|X^2|0\rangle = \langle 0|X|1\rangle = \langle 0|0\rangle = 1 \\
\text{standard deviation} &= \sqrt{\langle X^2\rangle - \langle X\rangle^2} = 1
\end{aligned}$$

2.60

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma} &= \sum_{i=1}^{3} v_i\sigma_i \\
&= v_1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + v_2\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + v_3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \\
&= \begin{bmatrix}v_3 & v_1 - iv_2\\ v_1 + iv_2 & -v_3\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\det(\vec{v}\cdot\vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)
\end{aligned}$$

Eigenvalues are $\lambda = \pm 1$.

(i) if $\lambda = 1$

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma} - \lambda I &= \vec{v}\cdot\vec{\sigma} - I \\
&= \begin{bmatrix}v_3 - 1 & v_1 - iv_2\\ v_1 + iv_2 & -v_3 - 1\end{bmatrix}
\end{aligned}$$

Normalized eigenvector is $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1 - iv_2} \end{bmatrix}$.

$$\begin{aligned}
|\lambda_1\rangle\langle\lambda_1| &= \frac{1+v_3}{2} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1-v_3}{v_1+iv_2} \end{bmatrix} \\
&= \frac{1+v_3}{2} \begin{bmatrix} 1 & \frac{v_1-iv_2}{1+v_3} \\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\
&= \frac{1}{2} \left(I + \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
&= \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})
\end{aligned}$$

(ii) If $\lambda = -1$.

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma} - \lambda I &= \vec{v}\cdot\vec{\sigma} + I \\
&= \begin{bmatrix} v_3+1 & v_1-iv_2 \\ v_1+iv_2 & -v_3+1 \end{bmatrix}
\end{aligned}$$

Normalized eigenvalue is $|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$.

$$\begin{aligned}
|\lambda_{-1}\rangle\langle\lambda_{-1}| &= \frac{1-v_3}{2} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1+v_3}{v_1+iv_2} \end{bmatrix} \\
&= \frac{1-v_3}{2} \begin{bmatrix} 1 & -\frac{v_1-iv_2}{1-v_3} \\ -\frac{v_1+iv_2}{1-v_3} & \frac{1+v_3}{1-v_3} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1-v_3 & -(v_1-iv_2) \\ -(v_1+iv_2) & 1+v_3 \end{bmatrix} \\
&= \frac{1}{2} \left(I - \begin{bmatrix} v_3 & v_1-iv_2 \\ (v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
&= \frac{1}{2}(I - \vec{v}\cdot\vec{\sigma}).
\end{aligned}$$

While I review my proof, I notice that my proof has a defect. The case $(v_1, v_2, v_3) = (0, 0, 1)$, second component of eigenstate, $\frac{1-v_3}{v_1-iv_2}$, diverges. So I implicitly assume $v_1 - iv_2 \neq 0$. Hence my proof is incomplete.

Since the exercise doesn't require explicit form of projector, we should prove the problem more abstractly. In order to prove, we use the following properties of $\vec{v}\cdot\vec{\sigma}$

- $\vec{v}\cdot\vec{\sigma}$ is Hermitian
- $(\vec{v}\cdot\vec{\sigma})^2 = I$ where $\vec{v}$ is a real unit vector.

We can easily check above conditions.

$$\begin{aligned}
(\vec{v}\cdot\vec{\sigma})^\dagger &= (v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3)^\dagger \\
&= v_1\sigma_1^\dagger + v_2\sigma_2^\dagger + v_3\sigma_3^\dagger \\
&= v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \quad (\because \text{Pauli matrices are Hermitian.}) \\
&= \vec{v}\cdot\vec{\sigma}
\end{aligned}$$

$$\begin{aligned}
(\vec{v}\cdot\vec{\sigma})^2 &= \sum_{j,k=1}^{3} (v_j\sigma_j)(v_k\sigma_k) \\
&= \sum_{j,k=1}^{3} v_j v_k \sigma_j \sigma_k \\
&= \sum_{j,k=1}^{3} v_j v_k \left(\delta_{jk} I + i\sum_{l=1}^{3} \epsilon_{jkl}\sigma_l\right) \quad (\because \text{eqn}(2.78)\ \text{page}78) \\
&= \sum_{j,k=1}^{3} v_j v_k \delta_{jk} I + i \sum_{j,k,l=1}^{3} \epsilon_{jkl} v_j v_k \sigma_l \\
&= \sum_{j=1}^{3} v_j^2 I \\
&= I \quad \left(\because \sum_j v_j^2 = 1\right)
\end{aligned}$$

Proof. Suppose $|\lambda\rangle$ is an eigenstate of $\vec{v}\cdot\vec{\sigma}$ with eigenvalue λ. Then

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma}\,|\lambda\rangle &= \lambda\,|\lambda\rangle \\
(\vec{v}\cdot\vec{\sigma})^2\,|\lambda\rangle &= \lambda^2\,|\lambda\rangle
\end{aligned}$$

On the other hand $(\vec{v}\cdot\vec{\sigma})^2 = I$,

$$\begin{aligned}
(\vec{v}\cdot\vec{\sigma})^2\,|\lambda\rangle &= I\,|\lambda\rangle = |\lambda\rangle \\
\therefore \lambda^2\,|\lambda\rangle &= |\lambda\rangle\,.
\end{aligned}$$

Thus $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$. Therefore $\vec{v}\cdot\vec{\sigma}$ has eigenvalues ± 1.

Let $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are eigenvectors with eigenvalues 1 and -1, respectively. I will prove that $P_\pm = |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|$.

In order to prove above equation, all we have to do is prove following condition. (see Theorem 2.0.1)

$$\langle\psi|(P_\pm - |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|)|\psi\rangle = 0 \text{ for all } |\psi\rangle \in \mathbb{C}^2. \tag{2.8}$$

Since $\vec{v}\cdot\vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthonormal vector ($\because$ Exercise 2.22). Let $|\psi\rangle \in \mathbb{C}^2$ be an arbitrary state. $|\psi\rangle$ can be written as

$$|\psi\rangle = \alpha\,|\lambda_1\rangle + \beta\,|\lambda_{\pm 1}\rangle \quad (|\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C}).$$

$$
\begin{aligned}
\langle\psi|(P_\pm - |\lambda_\pm\rangle\langle\lambda_\pm|)|\psi\rangle &= \langle\psi|P_\pm|\psi\rangle - \langle\psi|\lambda_\pm\rangle\langle\lambda_\pm|\psi\rangle. \\
\langle\psi|P_\pm|\psi\rangle &= \langle\psi|\frac{1}{2}(I \pm \vec{v}\cdot\vec{\sigma})|\psi\rangle \\
&= \frac{1}{2} \pm \frac{1}{2}\langle\psi|\vec{v}\cdot\vec{\sigma})|\psi\rangle \\
&= \frac{1}{2} \pm \frac{1}{2}(|\alpha|^2 - |\beta|^2) \\
&= \frac{1}{2} \pm \frac{1}{2}(2|\alpha|^2 - 1) \quad (\because |\alpha|^2 + |\beta|^2 = 1) \\
\langle\psi|\lambda_1\rangle\langle\lambda_1|\psi\rangle &= |\alpha|^2 \\
\langle\psi|\lambda_{-1}\rangle\langle\lambda_{-1}|\psi\rangle &= |\beta|^2 = 1 - |\alpha|^2
\end{aligned}
$$

Therefore $\langle\psi|(P_\pm - |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|)|\psi\rangle = 0$ for all $|\psi\rangle \in \mathbb{C}^2$. Thus $P_\pm = |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|$. □

2.61

$$
\begin{aligned}
\langle\lambda_1|0\rangle\langle 0|\lambda_1\rangle &= \langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle \\
&= \langle 0|\frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})|0\rangle \\
&= \frac{1}{2}(1 + v_3)
\end{aligned}
$$

Post-measurement state is

$$
\begin{aligned}
\frac{|\lambda_1\rangle\langle\lambda_1|0\rangle}{\sqrt{\langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle}} &= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \cdot \frac{1}{2}\begin{bmatrix} 1 + v_3 \\ v_1 + iv_2 \end{bmatrix} \\
&= \sqrt{\frac{1}{2}(1+v_3)}\begin{bmatrix} 1 \\ \frac{v_1+iv_2}{1+v_3} \end{bmatrix} \\
&= \sqrt{\frac{1+v_3}{2}}\begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \\
&= |\lambda_1\rangle.
\end{aligned}
$$

2.62

Suppose M_m is a measurement operator. From the assumption, $E_m = M_m^\dagger M_m = M_m$. Then

$$\langle\psi|E_m|\psi\rangle = \langle\psi|M_m|\psi\rangle \geq 0.$$

for all $|\psi\rangle$.

Since M_m is positive operator, M_m is Hermitian. Therefore,

$$E_m = M_m^\dagger M_m = M_m M_m = M_m^2 = M_m.$$

Thus the measurement is a projective measurement.

2.63

$$\begin{aligned} M_m^\dagger M_m &= \sqrt{E_m} U_m^\dagger U_m \sqrt{E_m} \\ &= \sqrt{E_m} I \sqrt{E_m} \\ &= E_m. \end{aligned}$$

Since E_m is POVM, for arbitrary unitary U, $M_m^\dagger M_m$ is POVM.

2.64

Define $E_i = |\psi_i\rangle\langle\psi_i|$ for $1 \le i \le m$ and $E_{m+1} = I - \sum_{i=1}^m E_i$. Then $\sum_{i=1}^{m+1} E_i = I$. And $\langle\psi_i|E_i|\psi_i\rangle = \langle\psi_i|\psi_i\rangle \langle\psi_i|\psi_i\rangle = 1$.

2.65

$$|+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

2.66

$$X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

$$\langle X_1 Z_2 \rangle = \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{\langle 00| + \langle 11|}{\sqrt{2}} \cdot \frac{|10\rangle - |01\rangle}{\sqrt{2}} = 0$$

2.67

Unsolved
$W \subset V \to V = W \oplus W^\perp$.
$U : W \to V$, $U' : V \to V$.
$U' |w\rangle = U |w\rangle$
$U' \in \mathcal{L}(V)$
$U \in \mathcal{L}(W)$
$U' = U \oplus I$???

2.68

$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$.
Suppose $|a\rangle = a_0 |0\rangle + a_1 |1\rangle$ and $|b\rangle = b_0 |0\rangle + b_1 |1\rangle$.

$$|a\rangle |b\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle .$$

If $|\psi\rangle = |a\rangle |b\rangle$, then $a_0 b_0 = 1,\ a_0 b_1 = 0,\ a_1 b_0 = 0,\ a_1 b_1 = 1$ since $\{|ij\rangle\}$ is an orthonormal basis.

If $a_0 b_1 = 0$, then $a_0 = 0$ or $b_1 = 0$.

When $a_0 = 0$, this is contradiction to $a_0 b_0 = 1$. When $b_1 = 0$, this is contradiction to $a_1 b_1 = 1$.

Thus $|\psi\rangle \neq |a\rangle\,|b\rangle$.

2.69

Define Bell states as follows.

$$|\psi_1\rangle \equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|\psi_2\rangle \equiv \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$|\psi_3\rangle \equiv \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$|\psi_4\rangle \equiv \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

First, we prove $\{|\psi_i\rangle\}$ is a linearly independent basis.

$$a_1 |\psi_1\rangle + a_2 |\psi_2\rangle + a_3 |\psi_3\rangle + a_4 |\psi_4\rangle = 0$$

$$\therefore \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 + a_2 \\ a_3 + a_4 \\ a_3 - a_4 \\ a_1 - a_2 \end{bmatrix} = 0$$

$$\therefore \begin{cases} a_1 + a_2 = 0 \\ a_3 + a_4 = 0 \\ a_3 - a_4 = 0 \\ a_1 - a_2 = 0 \end{cases}$$

$$\therefore a_1 = a_2 = a_3 = a_4 = 0$$

Thus $\{|\psi_i\rangle\}$ is a linearly independent basis.

Moreover $\| |\psi_i\rangle \| = 1$ and $\langle \psi_i | \psi_j \rangle = \delta_{ij}$ for $i, j = 1, 2, 3, 4$. Therefore $\{|\psi_i\rangle\}$ forms an orthonormal basis.

2.70

For any Bell states we get $\langle \psi_i | E \otimes I | \psi_i \rangle = \frac{1}{2}(\langle 0|E|0\rangle + \langle 1|E|1\rangle)$.

Suppose Eve measures the qubit Alice sent by measurement operators M_m. The probability that Eve gets result m is $p_i(m) = \langle \psi_i | M_m^\dagger M_m \otimes I | \psi_i \rangle$. Since $M_m^\dagger M_m$ is positive, $p_i(m)$ are same values for all $|\psi_i\rangle$. Thus Eve can't distinguish Bell states.

2.71

From spectral decomposition,

$$\begin{aligned}\rho &= \sum_i p_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right|, \quad p_i \geq 0, \quad \sum_i p_i = 1.\\ \rho^2 &= \sum_{i,j} p_i p_j \left|i\right\rangle \left\langle i|j\right\rangle \left\langle j\right| \\ &= \sum_{i,j} p_i p_j \left|i\right\rangle\!\left\langle j\right| \delta_{ij} \\ &= \sum_i p_i^2 \left|i\right\rangle\!\left\langle i\right|\end{aligned}$$

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}\left(\sum_i p_i^2 \left|i\right\rangle\!\left\langle i\right|\right) = \sum_i p_i^2 \,\mathrm{Tr}(\left|i\right\rangle\!\left\langle i\right|) = \sum_i p_i^2 \left\langle i|i\right\rangle = \sum_i p_i^2 \leq \sum_i p_i = 1 \quad (\because p_i^2 \leq p_i)$$

Suppose $\mathrm{Tr}(\rho^2) = 1$. Then $\sum_i p_i^2 = 1$. If $0 \leq p_i < 1$, then $p_i^2 < p_i$. Thus only one $p_i = 1$ and otherwise are 0. Therefore $\rho = \left|\psi_i\right\rangle\!\left\langle\psi_i\right|$ is pure state.

Conversely if ρ is pure, then $\rho = \left|\psi\right\rangle\!\left\langle\psi\right|$.

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\left|\psi\right\rangle \left\langle\psi|\psi\right\rangle \left\langle\psi\right|) = \mathrm{Tr}(\left|\psi\right\rangle\!\left\langle\psi\right|) = \left\langle\psi|\psi\right\rangle = 1.$$

2.72

(1) Since density matrix is Hermitian, matrix representation is $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$, $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$ w.r.t. standard basis. Because ρ is density matrix, $\mathrm{Tr}(\rho) = a + d = 1$.

Define $a = (1 + r_3)/2$, $d = (1 - r_3)/2$ and $b = (r_1 - ir_2)/2$, $(r_i \in \mathbb{R})$.

In this case,

$$\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{bmatrix} = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}).$$

Thus for arbitrary density matrix ρ can be written as $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

Next, we derive the condition that ρ is positive.

If ρ is positive, all eigenvalues of ρ should be non-negative.

$$\det(\rho - \lambda I) = (a - \lambda)(b - \lambda) - |b|^2 = \lambda^2 - (a + d)\lambda + ad - |b^2| = 0$$

$$\begin{aligned}\lambda &= \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - |b|^2)}}{2} \\ &= \frac{1 \pm \sqrt{1 - 4\left(\frac{1 - r_3^2}{4} - \frac{r_1^2 + r_2^2}{4}\right)}}{2} \\ &= \frac{1 \pm \sqrt{1 - (1 - r_1^2 - r_2^2 - r_3^2)}}{2} \\ &= \frac{1 \pm \sqrt{|\vec{r}|^2}}{2} \\ &= \frac{1 \pm |\vec{r}|}{2}\end{aligned}$$

Since ρ is positive, $\frac{1-|\vec{r}|}{2} \geq 0 \to |\vec{r}| \leq 1$.
Therefore an arbitrary density matrix for a mixed state qubit is written as $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$.

(2)
$\rho = I/2 \to \vec{r} = 0$. Thus $\rho = I/2$ corresponds to the origin of Bloch sphere.

(3)

$$
\begin{aligned}
\rho^2 &= \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})\ \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \\
&= \frac{1}{4}\left[I + 2\vec{r}\cdot\vec{\sigma} + \sum_{j,k} r_j r_k \left(\delta_{jk} I + i\sum_{l=1}^{3} \epsilon_{jkl}\sigma_l\right)\right] \\
&= \frac{1}{4}\left(I + 2\vec{r}\cdot\vec{\sigma} + |\vec{r}|^2 I\right) \\
\mathrm{Tr}(\rho^2) &= \frac{1}{4}(2 + 2|\vec{r}|^2)
\end{aligned}
$$

If ρ is pure, then $\mathrm{Tr}(\rho^2) = 1$.

$$
\begin{aligned}
1 = \mathrm{Tr}(\rho^2) &= \frac{1}{4}(2 + 2|\vec{r}|^2) \\
&\therefore |\vec{r}| = 1.
\end{aligned}
$$

Conversely, if $|\vec{r}| = 1$, then $\mathrm{Tr}(\rho^2) = \frac{1}{4}(2 + 2|\vec{r}|^2) = 1$. Therefore ρ is pure.

2.73

Theorem 2.6

$$
\rho = \sum_i p_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right| = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j| = \sum_j q_j \left|\varphi_j\right\rangle\!\left\langle\varphi_j\right| \quad \Leftrightarrow \quad |\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle
$$

where u is unitary.
Transformation in theorem 2.6, $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$, corresponds to

$$
\left[|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_k\rangle\right] = \left[\,|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle\,\right] U^T
$$

where $k = \mathrm{rank}(\rho)$.

From spectral theorem, density matrix ρ is decomposed as $\rho = \sum_{k=1}^{d} \lambda_k \left|k\right\rangle\!\left\langle k\right|$ where $d = \dim\mathcal{H}$. Without loss of generality, we can assume $p_k > 0$ for $k = 1\cdots, l$ where $l = \mathrm{rank}(\rho)$ and $p_k = 0$ for $k = l+1, \cdots, d$. Thus $\rho = \sum_{k=1}^{l} p_k \left|k\right\rangle\!\left\langle k\right| = \sum_{k=1}^{l} |\tilde{k}\rangle\langle\tilde{k}|$, where $|\tilde{k}\rangle = \sqrt{\lambda_k}\left|k\right\rangle$.

Suppose $\left|\psi_i\right\rangle$ is a state in support ρ. Then

$$
\left|\psi_i\right\rangle = \sum_{k=1}^{l} c_{ik}\left|k\right\rangle, \quad \sum_k |c_{ik}|^2 = 1.
$$

Define $p_i = \dfrac{1}{\sum_k \frac{|c_{ik}|^2}{\lambda_k}}$ and $u_{ik} = \dfrac{\sqrt{p_i}c_{ik}}{\sqrt{\lambda_k}}$.

Now

$$\sum_k |u_{ik}|^2 = \sum_k \frac{p_i|c_{ik}|^2}{\lambda_k} = p_i \sum_k \frac{|c_{ik}|^2}{\lambda_k} = 1.$$

Next prepare an unitary operator [1] such that ith row of U is $[u_{i1} \cdots u_{ik} \cdots u_{il}]$. Then we can define another ensemble such that

$$\Big[\, |\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_i\rangle \cdots |\tilde{\psi}_l\rangle \,\Big] = \Big[\, |\tilde{k}_1\rangle \cdots |\tilde{k}_l\rangle \,\Big] U^T$$

where $|\tilde{\psi}_i\rangle = \sqrt{p_i}\,|\psi_i\rangle$. From theorem 2.6,

$$\rho = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|\,.$$

Therefore we can obtain a minimal ensemble for ρ that contains $|\psi_i\rangle$.
Moreover since $\rho^{-1} = \sum_k \frac{1}{\lambda_k}\,|k\rangle\langle k|$,

$$\langle\psi_i|\rho^{-1}|\psi_i\rangle = \sum_k \frac{1}{\lambda_k}\,\langle\psi_i|k\rangle\langle k|\psi_i\rangle = \sum_k \frac{|c_{ik}|^2}{\lambda_k} = \frac{1}{p_i}.$$

Hence, $\frac{1}{\langle\psi_i|\rho^{-1}|\psi_i\rangle} = p_i$.

2.74

$$\begin{aligned}
\rho_{AB} &= |a\rangle\langle a|_A \otimes |b\rangle\langle b|_B \\
\rho_A &= \mathrm{Tr}_B\, \rho_{AB} = |a\rangle\langle a|\,\mathrm{Tr}(|b\rangle\langle b|) = |a\rangle\langle a| \\
\mathrm{Tr}(\rho_A^2) &= 1
\end{aligned}$$

Thus ρ_A is pure.

2.75
Define $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

$$\begin{aligned}
|\Phi_\pm\rangle\langle\Phi_\pm|_{AB} &= \frac{1}{2}(|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|) \\
\mathrm{Tr}_B(|\Phi_\pm\rangle\langle\Phi_\pm|_{AB}) &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2} \\
|\Psi_\pm\rangle\langle\Psi_\pm| &= \frac{1}{2}(|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|) \\
\mathrm{Tr}_B(|\Psi_\pm\rangle\langle\Psi_\pm|) &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2}
\end{aligned}$$

2.76

[1] By Gram-Schmidt procedure construct an orthonormal basis $\{\boldsymbol{u}_j\}$ (row vector) with $\boldsymbol{u}_i = [u_{i1} \cdots u_{ik} \cdots u_{il}]$.

Then define unitary $U = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_i \\ \vdots \\ \boldsymbol{u}_l \end{bmatrix}$.

Unsolved. ~~I think the polar decomposition can only apply to square matrix A, not arbitrary linear operators. Suppose A is $m \times n$ matrix. Then size of $A^\dagger A$ is $n \times n$. Thus the size of U should be $m \times n$. Maybe U is isometry, but I think it is not unitary.~~

I misunderstand linear operator.

> Quoted from "Advanced Liner Algebra" by Steven Roman, ISBN 0387247661.
>
> A linear transformation $\tau : V \to V$ is called a **linear operator** on V.[2]

Thus coordinate matrices of linear operator are square matrices. And Nielsen and Chaung say at Theorem 2.3, "Let A be a linear operator on a vector space V." Therefore A is a linear transformation such that $A : V \to V$.

2.77

$$
\begin{aligned}
|\psi\rangle &= |0\rangle |\Phi_+\rangle \\
&= |0\rangle \left[\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \right] \\
&= (\alpha |\phi_0\rangle + \beta |\phi_1\rangle) \left[\frac{1}{\sqrt{2}} (|\phi_0 \phi_0\rangle + |\phi_1 \phi_1\rangle) \right]
\end{aligned}
$$

where $|\phi_i\rangle$ are arbitrary orthonormal states and $\alpha, \beta \in \mathbb{C}$. We cannot vanish cross term. Therefore $|\psi\rangle$ cannot be written as $|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B |i\rangle_C$.

2.78

Proof. Former part.

If $|\psi\rangle$ is product, then there exist a state $|\phi_A\rangle$ for system A, and a state $|\phi_B\rangle$ for system B such that $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$.

Obviously, this Schmidt number is 1.

Conversely, if Schmidt number is 1, the state is written as $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$. Hence this is a product state. $\square$

Proof. Later part.

($\Rightarrow$) Proved by exercise 2.74.

($\Leftarrow$) Let a pure state be $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$. Then $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \sum_i \lambda_i^2 |i\rangle\langle i|$. If ρ_A is a pure state, then $\lambda_j = 1$ and otherwise 0 for some j. It follows that $|\psi_j\rangle = |j_A\rangle |j_B\rangle$. Thus $|\psi\rangle$ is a product state. $\square$

2.79

[2] According to Roman, some authors use the term linear operator for any linear transformation from V to W.

Procedure of Schmidt decomposition.
Goal: $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$

- Diagonalize reduced density matrix $\rho_A = \sum_i \lambda_i |i_A\rangle\langle i_A|$.
- Derive $|i_B\rangle$, $|i_B\rangle = \dfrac{(I \otimes \langle i_A|) |\psi\rangle}{\sqrt{\lambda_i}}$
- Construct $|\psi\rangle$.

(i)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ This is already decomposed.}$$

(ii)

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = |\psi\rangle |\psi\rangle \text{ where } |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

(iii)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$
$$\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$$

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{3}\left(2 |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$
$$\det(\rho_A - \lambda I) = \left(\frac{2}{3} - \lambda\right)\left(\frac{1}{3} - \lambda\right) - \frac{1}{9} = 0$$
$$\lambda^2 - \lambda + \frac{1}{9} = 0$$
$$\lambda = \frac{1 \pm \sqrt{5}/3}{2} = \frac{3 \pm \sqrt{5}}{6}$$

Eigenvector with eigenvalue $\lambda_0 \equiv \dfrac{3+\sqrt{5}}{6}$ is $|\lambda_0\rangle \equiv \dfrac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$.

Eigenvector with eigenvalue $\lambda_1 \equiv \dfrac{3-\sqrt{5}}{6}$ is $|\lambda_1\rangle \equiv \dfrac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.

$$\rho_A = \lambda_0 |\lambda_0\rangle\langle\lambda_0| + \lambda_1 |\lambda_1\rangle\langle\lambda_1| .$$

$$|a_0\rangle \equiv \frac{(I \otimes \langle\lambda_0|) |\psi\rangle}{\sqrt{\lambda_0}}$$
$$|a_1\rangle \equiv \frac{(I \otimes \langle\lambda_1|) |\psi\rangle}{\sqrt{\lambda_1}}$$

Then

$$|\psi\rangle = \sum_{i=0}^{1} \sqrt{\lambda_i} |a_i\rangle |\lambda_i\rangle .$$

(It's too tiresome to calculate $|a_i\rangle$)

2.80

Let $|\psi\rangle = \sum_i \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B$ and $|\varphi\rangle = \sum_i \lambda_i |\varphi_i\rangle_A |\varphi_i\rangle_B$.
Define $U = \sum_i |\psi_j\rangle\langle\varphi_j|_A$ and $V = \sum_j |\psi_j\rangle\langle\varphi_j|$.
Then

$$\begin{aligned}(U \otimes V) |\varphi\rangle &= \sum_i \lambda_i U |\varphi_i\rangle_A V |\varphi_i\rangle_B \\ &= \sum_i \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B \\ &= |\psi\rangle .\end{aligned}$$

2.81

Suppose $\rho_A = \mathrm{Tr}_R |AR_2\rangle\langle AR_2| = \sum_i \lambda_i |i\rangle\langle i|$. Define $|AR_1\rangle = (I_A \otimes U_R) |AR_2\rangle$.

$$\begin{aligned}\mathrm{Tr}_R(|AR_1\rangle\langle AR_1|) &= \mathrm{Tr}_R\left((I_A \otimes U_R) |AR_2\rangle\langle AR_2| (I_A \otimes U_R^\dagger)\right) \\ &= \mathrm{Tr}_R\left(|AR_2\rangle\langle AR_2| (I_A \otimes U_R^\dagger)(I_A \otimes U_R)\right) \\ &= \mathrm{Tr}_R(|AR_2\rangle\langle AR_2|) \\ &= \rho_A.\end{aligned}$$

Thus $|AR_1\rangle$ is also a purification of ρ_A.

2.82

(1)
Let $|\psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$.

$$\mathrm{Tr}_R(|\psi\rangle\langle\psi|) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Thus $|\psi\rangle$ is a purification of ρ.

(2) Probability

$$\mathrm{Tr}\left[(I \otimes |i\rangle\langle i|) |\psi\rangle\langle\psi|\right] = \langle\psi|(I \otimes |i\rangle\langle i|)|\psi\rangle = p_i \langle\psi_i|\psi_i\rangle = p_i.$$

Post-measurement state

$$\frac{(I \otimes |i\rangle\langle i| |\psi\rangle)}{\sqrt{p_i}} = \frac{\sqrt{p_i} |\psi_i\rangle}{\sqrt{p_i}} = |\psi_i\rangle .$$

(3)

Suppose $|AR\rangle$ is a purification of ρ such that $|AR\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle$. By exercise 2.81, the others purification is written as $(I \otimes U) |AR\rangle$.

$$
\begin{aligned}
(I \otimes U) |AR\rangle &= (I \otimes U) \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle \\
&= \sum_i \sqrt{p_i} |\psi_i\rangle U |r_i\rangle \\
&= \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle
\end{aligned}
$$

where $U = \sum_i |i\rangle\langle r_i|$.

By (2), if we measure the system R w.r.t $|i\rangle$, post-measurement state for system A is $|\psi_i\rangle$ with probability p_i, which prove the assertion.

Problem 2.1

From Exercise 2.35, $\vec{n} \cdot \vec{\sigma}$ is decomposed as

$$\vec{n} \cdot \vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|$$

where $|\lambda_{\pm 1}\rangle$ are eigenvector of $\vec{n} \cdot \vec{\sigma}$ with eigenvalues ± 1.

Thus

$$
\begin{aligned}
f(\theta \vec{n} \cdot \vec{\sigma}) &= f(\theta) |\lambda_1\rangle\langle\lambda_1| + f(-\theta) |\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \left(\frac{f(\theta) + f(-\theta)}{2} + \frac{f(\theta) - f(-\theta)}{2} \right) |\lambda_1\rangle\langle\lambda_1| + \left(\frac{f(\theta) + f(-\theta)}{2} - \frac{f(\theta) - f(-\theta)}{2} \right) |\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \frac{f(\theta) + f(-\theta)}{2} (|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + \frac{f(\theta) - f(-\theta)}{2} (|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\
&= \frac{f(\theta) + f(-\theta)}{2} I + \frac{f(\theta) - f(-\theta)}{2} \vec{n} \cdot \vec{\sigma}
\end{aligned}
$$

Problem 2.2
Unsolved

Problem 2.3
Unsolved

Chapter 8

Quantum noise and quantum operations

8.1
Density operator of initial state is written by $|\psi\rangle\langle\psi|$ and final state is written by $U|\psi\rangle\langle\psi|U^\dagger$. Thus time development of $\rho = |\psi\rangle\langle\psi|$ can be written by $\mathcal{E}(\rho) = U\rho U^\dagger$.

8.2
From eqn (2.147) (on page 100),

$$\rho_m = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}(M_m^\dagger M_m \rho)} = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}(M_m \rho M_m^\dagger)} = \frac{\mathcal{E}_m(\rho)}{\mathrm{Tr}\,\mathcal{E}_m(\rho)}.$$

And from eqn (2.143) (on page 99), $p(m) = \mathrm{Tr}(M_m^\dagger M_m \rho) = \mathrm{Tr}(M_m \rho M_m^\dagger) = \mathrm{Tr}\,\mathcal{E}_m(\rho)$.

8.3

8.4

8.5

8.6

8.7

8.8

8.9

8.10

8.11

8.12

8.13

8.14

8.15

8.16

8.17

8.18

8.19

8.20

8.21

8.22

8.23

8.24

8.25

8.26

8.27

8.28

8.29

8.30

8.31

8.32

8.33

8.34

8.35