Solution for "Quantum Computation and Quantum Information: 10th Anniversary Edition" by Nielsen and Chuang

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Chapter 2

Introduction to quantum mechanics

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2.2

$$\begin{aligned}
A\left|0\right\rangle &= A_{11}\left|0\right\rangle + A_{21}\left|1\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 0,\ A_{21} = 1 \\
A\left|1\right\rangle &= A_{12}\left|0\right\rangle + A_{22}\left|1\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 1,\ A_{22} = 0 \\
\therefore A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

input: $\{\left|0\right\rangle, \left|1\right\rangle\}$, output: $\{\left|1\right\rangle, \left|0\right\rangle\}$

$$\begin{aligned}
A\left|0\right\rangle &= A_{11}\left|1\right\rangle + A_{21}\left|0\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 1,\ A_{21} = 0 \\
A\left|1\right\rangle &= A_{12}\left|1\right\rangle + A_{22}\left|0\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 0,\ A_{22} = 1 \\
A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

2.3

From eq (2.12)

$$A\left|v_i\right\rangle = \sum_j A_{ji}\left|w_j\right\rangle$$

$$B\left|w_j\right\rangle = \sum_k B_{kj}\left|x_k\right\rangle$$

Thus

$$
\begin{aligned}
BA\left|v_i\right\rangle &= B\left(\sum_j A_{ji}\left|w_j\right\rangle\right) \\
&= \sum_j A_{ji} B\left|w_j\right\rangle \\
&= \sum_{j,k} A_{ji} B_{kj}\left|x_k\right\rangle \\
&= \sum_k \left(\sum_j B_{kj} A_{ji}\right)\left|x_k\right\rangle \\
&= \sum_k (BA)_{ki}\left|x_k\right\rangle \\
\therefore (BA)_{ki} &= \sum_j B_{kj} A_{ji}
\end{aligned}
$$

2.4

$$
\begin{aligned}
I\left|v_j\right\rangle = \sum_i I_{ij}\left|v_i\right\rangle = \left|v_j\right\rangle,\ \forall j. \\
\Rightarrow I_{ij} = \delta_{ij}
\end{aligned}
$$

2.5

Defined inner product on $\mathcal{C}^n$ is

$$
((y_1, \cdots, y_n), (z_1, \cdots, z_n)) = \sum_i y_i^* z_i.
$$

Verify (1) of eq (2.13).

$$
\begin{aligned}
\left((y_1, \cdots, y_n), \sum_i \lambda_i (z_{i1}, \cdots, z_{in})\right) &= \sum_i y_i^* \left(\sum_j \lambda_j z_{ji}\right) \\
&= \sum_{i,j} y_i^* \lambda_j z_{ji} \\
&= \sum_j \lambda_j \left(\sum_i y_i^* z_{ji}\right) \\
&= \sum_j \lambda_j \left((y_1, \cdots, y_n), (z_{j1}, \cdots, z_{jn})\right) \\
&= \sum_i \lambda_i \left((y_1, \cdots, y_n), (z_{i1}, \cdots, z_{in})\right).
\end{aligned}
$$

Verify (2) of eq (2.13),

$$
\begin{aligned}
((y_1, \cdots, y_n), (z_1, \cdots, z_n))^* &= \left(\sum_i y_i^* z_i \right)^* && (2.1)\\
&= \left(\sum_i y_i z_i^* \right) && (2.2)\\
&= \left(\sum_i z_i^* y_i \right) && (2.3)\\
&= ((z_1, \cdots, z_n), (y_1, \cdots, y_n)) && (2.4)
\end{aligned}
$$

.

Verify (3) of eq (2.13),

$$
\begin{aligned}
((y_1, \cdots, y_n), (y_1, \cdots, y_n)) &= \sum_i y_i^* y_i \\
&= \sum_i |y_i|^2
\end{aligned}
$$

Since $|y_i|^2 \geq 0$ for all i. Thus $\sum_i |y_i|^2 = ((y_1, \cdots, y_n), (y_1, \cdots, y_n)) \geq 0$.

From now on, I will show the following statement,

$$((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0 \text{ iff } (y_1, \cdots, y_n) = 0.$$

($\Leftarrow$) This is obvious.
($\Rightarrow$) Suppose $((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0$. Then $\sum_i |y_i|^2 = 0$.

Since $|y_i|^2 \geq 0$ for all i, if $\sum_i |y_i|^2 = 0$, then $|y_i|^2 = 0$ for all i. Therefore $|y_i|^2 = 0 \Leftrightarrow y_i = 0$ for all i. Thus,

$$(y_1, \cdots, y_n) = 0.$$

2.6

$$
\begin{aligned}
\left(\sum_i \lambda_i \left| w_i \right\rangle, \ \left| v \right\rangle \right) &= \left(\left| v \right\rangle, \ \sum_i \lambda_i \left| w_i \right\rangle \right)^* \\
&= \left[\sum_i \lambda_i \left(\left| v \right\rangle, \ \left| w_i \right\rangle \right) \right]^* \ (\because \text{ linearlity in the 2nd arg.}) \\
&= \sum_i \lambda_i^* \left(\left| v \right\rangle, \ \left| w_i \right\rangle \right)^* \\
&= \sum_i \lambda_i^* (\left| w_i \right\rangle, \ \left| v \right\rangle)
\end{aligned}
$$

2.7

$$\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0 \\
\frac{|w\rangle}{\||w\rangle\|} &= \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\frac{|v\rangle}{\||v\rangle\|} &= \frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

2.8

If $k = 1$,

$$\begin{aligned}
|v_2\rangle &= \frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle\|} \\
\langle v_1|v_2\rangle &= \langle v_1| \left(\frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle\|} \right) \\
&= \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle \langle v_1|v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle\|} \\
&= 0.
\end{aligned}$$

Suppose $\{v_1, \cdots v_n\}$ $(n \leq d-1)$ is a orthonormal basis. Then

$$\begin{aligned}
\langle v_j|v_{n+1}\rangle &= \langle v_j| \left(\frac{|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \right) \quad (j \leq n) \\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \langle v_j|v_i\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \delta_{ij}}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \\
&= \frac{\langle v_j|w_{n+1}\rangle - \langle v_j|w_{n+1}\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle\|} \\
&= 0.
\end{aligned}$$

Thus Gram-Schmidt procedure produces an orthonormal basis.

2.9

$$\begin{aligned}
\sigma_0 &= I = |0\rangle\langle 0| + |1\rangle\langle 1| \\
\sigma_1 &= X = |0\rangle\langle 1| + |1\rangle\langle 0| \\
\sigma_2 &= Y = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| \\
\sigma_3 &= Z = |0\rangle\langle 0| - |1\rangle\langle 1|
\end{aligned}$$

2.10

$$\begin{aligned}
|v_j\rangle\langle v_k| &= I_V |v_j\rangle\langle v_k| I_V \\
&= \left(\sum_p |v_p\rangle\langle v_p|\right) |v_j\rangle\langle v_k| \left(\sum_q |v_q\rangle\langle v_q|\right) \\
&= \sum_{p,q} |v_p\rangle\langle v_p|v_j\rangle\langle v_k|v_q\rangle\langle v_q| \\
&= \sum_{p,q} \delta_{pj}\delta_{kq} |v_p\rangle\langle v_q|
\end{aligned}$$

Thus

$$(|v_j\rangle\langle v_k|)_{pq} = \delta_{pj}\delta_{kq}$$

2.11

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \det(X - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda \pm 1$$

If $\lambda = -1$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus

$$|\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If $\lambda = 1$

$$|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|\lambda = -1\rangle, \ |\lambda = 1\rangle\}$$

2.12

$$\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I\right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because $|\lambda = 1\rangle \langle\lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c\,|\lambda = 1\rangle \langle\lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

2.13

Suppose $|\psi\rangle\,,\ |\phi\rangle$ are arbitrary vectors in V.

$$\begin{aligned} (|\psi\rangle\,,\ (|w\rangle\langle v|)\,|\phi\rangle)^* &= \Big((|w\rangle\langle v|)^\dagger\,|\psi\rangle\,,\ |\phi\rangle\Big)^* \\ &= \Big(|\phi\rangle\,,\ (|w\rangle\langle v|)^\dagger\,|\psi\rangle\Big) \\ &= \langle\phi|\,(|w\rangle\langle v|)^\dagger\,|\psi\rangle\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} (|\psi\rangle\,,\ (|w\rangle\langle v|)\,|\phi\rangle)^* &= (\langle\psi|w\rangle\langle v|\phi\rangle)^* \\ &= \langle\phi|v\rangle\langle w|\psi\rangle\,. \end{aligned}$$

Thus

$$\begin{aligned} \langle\phi|\,(|w\rangle\langle v|)^\dagger\,|\psi\rangle = \langle\phi|v\rangle\langle w|\psi\rangle \ \text{ for arbitrary vectors } |\psi\rangle\,,\ |\phi\rangle \\ \therefore (|w\rangle\langle v|)^\dagger = |v\rangle\langle w| \end{aligned}$$

2.14

$$\begin{aligned} ((a_iA_i)^\dagger\,|\phi\rangle\,,\ |\psi\rangle) &= (|\phi\rangle\,,\ a_iA_i\,|\psi\rangle) \\ &= a_i(|\phi\rangle\,,\ A_i\,|\psi\rangle) \\ &= a_i(A_i^\dagger\,|\phi\rangle\,,\ |\psi\rangle) \\ &= (a_i^*A_i^\dagger\,|\phi\rangle\,,\ |\psi\rangle) \\ \therefore (a_iA_i)^\dagger = a_i^*A_i^\dagger & \end{aligned}$$

2.15

$$\begin{aligned} ((A^\dagger)^\dagger\,|\psi\rangle\,,\ |\phi\rangle) &= (|\psi\rangle\,,\ A^\dagger\,|\phi\rangle) \\ &= (A^\dagger\,|\phi\rangle\,,\ |\psi\rangle)^* \\ &= (|\phi\rangle\,,\ A\,|\psi\rangle)^* \\ &= (A\,|\psi\rangle\,,\ |\phi\rangle) \\ \therefore (A^\dagger)^\dagger = A & \end{aligned}$$

2.16

$$
\begin{aligned}
P &= \sum_i |i\rangle \langle i| \, . \\
P^2 &= \left(\sum_i |i\rangle \langle i|\right) \left(\sum_j |j\rangle \langle j|\right) \\
&= \sum_{i,j} |i\rangle \langle i|j\rangle \langle j| \\
&= \sum_i |i\rangle \langle j| \, \delta_{ij} \\
&= \sum_i |i\rangle \langle i| \\
&= P
\end{aligned}
$$

2.18

Suppose $|v\rangle$ is a eigenvector with corresponding eigenvalue λ.

$$
\begin{aligned}
U|v\rangle &= \lambda |v\rangle \, . \\
1 &= \langle v|v\rangle \\
&= \langle v| I |v\rangle \\
&= \langle v| U^\dagger U |v\rangle \\
&= \lambda\lambda^* \langle v|v\rangle \\
&= \|\lambda\|^2 \\
\therefore \lambda &= e^{i\theta}
\end{aligned}
$$

2.19

$$
X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
$$

2.20

$$
\begin{aligned}
U &\equiv \sum_i |w_i\rangle \langle v_i| \\
A_{ij}^{'} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|UU^\dagger AUU^\dagger|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \langle v_p|v_q\rangle \langle w_q|A|w_r\rangle \langle v_r|v_s\rangle \langle w_s|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \, \delta_{pq} A_{qr}^{''} \delta_{rs} \langle w_s|v_j\rangle \\
&= \sum_{p,r} \langle v_i|w_p\rangle \langle w_r|v_j\rangle \, A_{pr}^{''}
\end{aligned}
$$

2.21
Suppose M be Hermitian. Then $M = M^\dagger$.

$$\begin{aligned} M &= IMI \\ &= (P+Q)M(P+Q) \\ &= PMP + QMP + PMQ + QMQ \end{aligned}$$

Now $PMP = \lambda P$, $QMP = 0$, $PMQ = PM^\dagger Q = (QMP)^* = 0$. Thus $M = PMP + QMQ$. Next prove QMQ is normal.

$$\begin{aligned} QMQ(QMQ)^\dagger &= QMQQM^\dagger Q \\ &= QM^\dagger QQMQ \quad (M = M^\dagger) \\ &= (QM^\dagger Q)QMQ \end{aligned}$$

Therefore QMQ is normal. By induction, QMQ is diagonal ... (following is same as Box 2.2)

2.22
Suppose A is a Hermitian operator and $|v_i\rangle$ are eigenvectors of A with eigenvalues λ_i. Then

$$\langle v_i|A|v_j\rangle = \lambda_j \langle v_i|v_j\rangle .$$

On the other hand,

$$\langle v_i|A|v_j\rangle = \langle v_i|A^\dagger|v_j\rangle = \langle v_j|A|v_i\rangle^* = \lambda_i^* \langle v_j|v_i\rangle^* = \lambda_i^* \langle v_i|v_j\rangle = \lambda_i \langle v_i|v_j\rangle$$

Thus

$$(\lambda_i - \lambda_j) \langle v_i|v_j\rangle = 0.$$

If $\lambda_i \neq \lambda_j$, then $\langle v_i|v_j\rangle = 0$.

2.23
Suppose P is projector and $|\lambda\rangle$ are eigenvectors of P with eigenvalues λ. Then $P^2 = P$.

$$P|\lambda\rangle = \lambda|\lambda\rangle \text{ and } P|\lambda\rangle = P^2|\lambda\rangle = \lambda P|\lambda\rangle = \lambda^2|\lambda\rangle .$$

Therefore

$$\begin{aligned} \lambda &= \lambda^2 \\ \lambda(\lambda-1) &= 0 \\ \lambda &= 0 \text{ or } 1. \end{aligned}$$

2.24
Def of positive $\langle v|A|v\rangle \geq 0$ for all $|v\rangle$.

Suppose A is a positive operator. A can be decomposed as follows.

$$\begin{aligned} A &= \frac{A+A^\dagger}{2} + i\frac{A-A^\dagger}{2i} \\ &= B + iC \quad \text{where } B = \frac{A+A^\dagger}{2}, \quad C = \frac{A-A^\dagger}{2i}. \end{aligned}$$

Now operators B and C are Hermitian.

$$\begin{aligned}\langle v|A|v\rangle &= \langle v|B+iC|v\rangle \\ &= \langle v|B|v\rangle + i\langle v|C|v\rangle \\ &= \alpha + i\beta \quad \text{where } \alpha = \langle v|B|v\rangle\,,\ \beta = \langle v|C|v\rangle\,.\end{aligned}$$

Since B and C are Hermitian, $\alpha,\ \beta \in \mathbb{R}$. From def of positive operator, β should be vanished. Hence $\beta = \langle v|C|v\rangle$ for all $|v\rangle$, i.e. $C = 0$.

Therefore $A = B$. Since B is Hermitian, positive operator A is also Hermitian.

2.25

$$\langle\psi|A^\dagger A|\psi\rangle = \|A\,|\psi\rangle\|^2 \geq 0 \text{ for all } \ |\psi\rangle\,.$$

Thus $A^\dagger A$ is positive.

2.26

$$\begin{aligned}|\psi\rangle^{\otimes 2} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}|\psi\rangle^{\otimes 3} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \\ &= \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \\ &= \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\end{aligned}$$

2.27

$$\begin{aligned}X \otimes Z &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \\ &= \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

In general, tensor product is not commutable.

2.28

$$
\begin{aligned}
(A \otimes B)^* &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* \\
&= \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
&= A^* \otimes B^*.
\end{aligned}
$$

$$
\begin{aligned}
(A \otimes B)^T &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{1m}^TB^T \\ \vdots & \ddots & \vdots \\ A_{n1}^TB^T & \cdots & A_{nm}^TB^T \end{bmatrix} \\
&= A^T \otimes B^T.
\end{aligned}
$$

$$
\begin{aligned}
(A \otimes B)^\dagger &= ((A \otimes B)^*)^T \\
&= (A^* \otimes B^*)^T \\
&= (A^*)^T \otimes (B^*)^T \\
&= A^\dagger \otimes B^\dagger.
\end{aligned}
$$

2.29

Suppose U_1 and U_2 are unitary operators. Then

$$\begin{aligned}(U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger &= U_1 U_1^\dagger \otimes U_2 U_2^\dagger \\ &= I \otimes I.\end{aligned}$$

Similarly,

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = I \otimes I.$$

2.30

Suppose A and B are Hermitian operators. Then

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B. \tag{2.5}$$

Thus $A \otimes B$ is Hermitian.

2.31

Suppose A and B are positive operators. Then

$$\langle\psi| \otimes \langle\phi| \, (A \otimes B) \, |\psi\rangle \otimes |\phi\rangle = \langle\psi|A|\psi\rangle \, \langle\phi|B|\phi\rangle \, .$$

Since A and B are positive operators, $\langle\psi|A|\psi\rangle \geq 0$ and $\langle\phi|B|\phi\rangle \geq 0$ for all $|\psi\rangle \, , |\phi\rangle$. Then $\langle\psi|A|\psi\rangle \, \langle\phi|B|\phi\rangle \geq 0$. Thus $A \otimes B$ is positive if A and B are positive.

2.32

Suppose P_1 and P_2 are projectors. Then

$$\begin{aligned}(P_1 \otimes P_2)^2 &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2.\end{aligned}$$

Thus $P_1 \otimes P_2$. is also projector.

2.33

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.6}$$

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

2.34

Suppose $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

$$\begin{aligned}\det(A - \lambda I) &= (4 - \lambda)^2 - 3^2 \\ &= \lambda^2 - 8\lambda + 7 \\ &= (\lambda - 1)(\lambda - 7)\end{aligned}$$

Eigenvalues of A are $\lambda = 1,\ 7$. Corresponding eigenvectors are $|\lambda = 1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $|\lambda = 7\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus

$$A = |\lambda = 1\rangle\langle\lambda = 1| + 7\,|\lambda = 7\rangle\langle\lambda = 7|\,.$$

$$\begin{aligned}
\sqrt{A} &= |\lambda = 1\rangle\langle\lambda = 1| + \sqrt{7}\,|\lambda = 7\rangle\langle\lambda = 7| \\
&= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\log(A) &= \log(1)\,|\lambda = 1\rangle\langle\lambda = 1| + \log(7)\,|\lambda = 7\rangle\langle\lambda = 7| \\
&= \frac{\log(7)}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

2.35

$$\begin{aligned}
\vec{v}\cdot\vec{\sigma} &= \sum_{i=1}^{3} v_i\sigma_i \\
&= v_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\det(\vec{v}\cdot\vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)
\end{aligned}$$

Eigenvalues are $\lambda = \pm 1$. Let $|\lambda_{\pm 1}\rangle$ be eigenvectors with eigenvalues ± 1.

Since $\vec{v}\cdot\vec{\sigma}$ is Hermitian, $\vec{v}\cdot\vec{\sigma}$ is diagonalizable. Then

$$\vec{v}\cdot\vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|$$

Thus

$$\begin{aligned}
\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right) &= e^{i\theta}\,|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= (\cos\theta + i\sin\theta)\,|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \cos\theta(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + i\sin\theta(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\
&= \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}.
\end{aligned}$$

$\because$ Since $\vec{v} \cdot \vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthogonal. Thus

$$|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}| = I.$$

2.36

$$\begin{aligned}
\operatorname{Tr}(\sigma_1) &= \operatorname{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0 \\
\operatorname{Tr}(\sigma_2) &= \operatorname{Tr}\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = 0 \\
\operatorname{Tr}(\sigma_3) &= \operatorname{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1 - 1 = 0
\end{aligned}$$

2.37

$$\begin{aligned}
\operatorname{Tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_{i,j} \langle i|A|j\rangle \langle j|B|i\rangle \\
&= \sum_{i,j} \langle j|B|i\rangle \langle i|A|j\rangle \\
&= \sum_j \langle j|BA|j\rangle \\
&= \operatorname{Tr}(BA)
\end{aligned}$$

2.38

$$\begin{aligned}
\operatorname{Tr}(A+B) &= \sum_i \langle i|A+B|i\rangle \\
&= \sum_i (\langle i|A|i\rangle + \langle i|B|i\rangle) \\
&= \sum_i \langle i|A|i\rangle + \sum_i \langle i|B|i\rangle \\
&= \operatorname{Tr}(A) + \operatorname{Tr}(B).
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tr}(zA) &= \sum_i \langle i|zA|i\rangle \\
&= \sum_i z \langle i|A|i\rangle \\
&= z \sum_i \langle i|A|i\rangle \\
&= z \operatorname{Tr}(A).
\end{aligned}$$

2.39

(1)

$(A, B) \equiv \mathrm{Tr}(A^\dagger B)$.

(i)

$$
\begin{aligned}
\left(A, \sum_i \lambda_i B_i\right) &= \mathrm{Tr}\left[A^\dagger \left(\sum_i \lambda_i B_i\right)\right] \\
&= \mathrm{Tr}(A^\dagger \lambda_1 B_1) + \cdots + \mathrm{Tr}(A^\dagger \lambda_n B_n) \quad (\because \text{Execise 2.38}) \\
&= \lambda_1 \mathrm{Tr}(A^\dagger B_1) + \cdots + \lambda_n \mathrm{Tr}(A^\dagger B_n) \\
&= \sum_i \lambda_i \mathrm{Tr}(A^\dagger B_i)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
(A, B)^* &= \left(\mathrm{Tr}(A^\dagger B)\right)^* \\
&= \left(\sum_{i,j} \langle i|A^\dagger|j\rangle \langle j|B|i\rangle\right)^* \\
&= \sum_{i,j} \langle i|A^\dagger|j\rangle^* \langle j|B|i\rangle^* \\
&= \sum_{i,j} \langle j|B|i\rangle^* \langle i|A^\dagger|j\rangle^* \\
&= \sum_{i,j} \langle i|B^\dagger|j\rangle \langle j|A|i\rangle \\
&= \sum_i \langle i|B^\dagger A|i\rangle \\
&= \mathrm{Tr}(B^\dagger A) \\
&= (B, A).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
(A, A) &= \mathrm{Tr}(A^\dagger A) \\
&= \sum_i \langle i|A^\dagger A|i\rangle
\end{aligned}
$$

Since $A^\dagger A$ is positive, $\langle i|A^\dagger A|i\rangle \geq 0$ for all $|i\rangle$.

Let a_i be i-th column of A. If $\langle i|A^\dagger A|i\rangle = 0$, then

$$
\langle i|A^\dagger A|i\rangle = a_i^\dagger a_i = \|a_i\|^2 = 0 \text{ iff } a_i = \mathbf{0}.
$$

Therefore $(A, A) = 0$ iff $A = \mathbf{0}$.

(2)

(3)

2.40

$$
\begin{aligned}
[X, Y] &= XY - YX \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \\
&= 2iZ
\end{aligned}
$$

$$
\begin{aligned}
[Y, Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\
&= 2iX
\end{aligned}
$$

$$
\begin{aligned}
[Z, X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 2i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 2iY
\end{aligned}
$$

2.41

$$
\begin{aligned}
\{\sigma_1, \sigma_2\} &= \sigma_1 \sigma_2 + \sigma_2 \sigma_1 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\{\sigma_2, \sigma_3\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\{\sigma_3, \sigma_1\} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sigma_0^2 &= I^2 = I \\
\sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I \\
\sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^2 = I \\
\sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I
\end{aligned}
$$

2.42

$$
\frac{[A,B] + \{A,B\}}{2} = \frac{AB - BA + AB + BA}{2} = AB
$$

2.43
From eq (2.75) and eq (2.76), $\{\sigma_j, \sigma_k\} = 2\delta_{jk} I$. From eq (2.77),

$$
\begin{aligned}
\sigma_j \sigma_k &= \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} \\
&= \frac{2i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l + 2\delta_{jk} I}{2} \\
&= \delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l
\end{aligned}
$$

2.44
By assumption, $[A,B] = 0$ and $\{A,B\} = 0$, then $AB = 0$. Since A is invertible, multiply by A^{-1} from left, then

$$
\begin{aligned}
A^{-1}AB &= 0 \\
IB &= 0 \\
B &= 0.
\end{aligned}
$$

2.45

$$
\begin{aligned}
[A,B]^\dagger &= (AB - BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= \left[B^\dagger, A^\dagger\right]
\end{aligned}
$$

2.46

$$
\begin{aligned}
[A,B] &= AB - BA \\
&= -(BA - AB) \\
&= -[B,A]
\end{aligned}
$$

2.47

$$
\begin{aligned}
(i\,[A,B])^\dagger &= -i\,[A,B]^\dagger \\
&= -i\left[B^\dagger, A^\dagger\right] \\
&= -i\,[B,A] \\
&= i\,[A,B]
\end{aligned}
$$

2.48

(Positive)
Since P is positive, it is diagonalizable. Then $P = \sum_i \lambda_i |i\rangle\langle i|$, $(\lambda_i \geq 0)$.

$$
J = \sqrt{P^\dagger P} = \sqrt{PP} = \sqrt{P^2} = \sum_i \sqrt{\lambda_i^2}\,|i\rangle\langle i| = \sum_i \lambda_i\,|i\rangle\langle i| = P.
$$

Therefore polar decomposition of P is $P = UP$ for all P. Thus $U = I$, then $P = P$.

(Unitary)
Suppose unitary U is decomposed by $U = WJ$ where W is unitary and J is positive, $J = \sqrt{U^\dagger U}$.

$$
J = \sqrt{U^\dagger U} = \sqrt{I} = I
$$

Since unitary operators are invertible, $W = UJ^{-1} = UI^{-1} = UI = U$. Thus polar decomposition of U is $U = U$.

(Hermitian)
Suppose $H = UJ$.

$$
J = \sqrt{H^\dagger H} = \sqrt{HH} = \sqrt{H^2}.
$$

Thus $H = U\sqrt{H^2}$.

In general, $H \neq \sqrt{H^2}$.
From spectral decomposition, $H = \sum_i \lambda_i |i\rangle\langle i|$, $\lambda_i \in \mathbb{R}$.

$$
\sqrt{H^2} = \sqrt{\sum_i \lambda_i^2\,|i\rangle\langle i|} = \sum_i \sqrt{\lambda_i^2}\,|i\rangle\langle i| = \sum_i |\lambda_i|\,|i\rangle\langle i| \neq H
$$

2.49
Normal matrix is diagonalizable, $A = \sum_i \lambda_i |i\rangle\langle i|$.

$$
\begin{aligned}
J &= \sqrt{A^\dagger A} = \sum_i |\lambda_i|\,|i\rangle\langle i|\,. \\
U &= \sum_i |e_i\rangle\langle i| \\
A &= UJ = \sum_i |\lambda_i|\,|e_i\rangle\langle i|\,.
\end{aligned}
$$

2.50

Define $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $A^\dagger A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation of $A^\dagger A$ is $\det(A^\dagger A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0$. Eigenvalues of $A^\dagger A$ are $\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$ and associated eigenvectors are $|\lambda_\pm\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}$.

$$A^\dagger A = \lambda_+ |\lambda_+\rangle\langle\lambda_+| + \lambda_- |\lambda_-\rangle\langle\lambda_-| .$$

$$\begin{aligned} J = \sqrt{A^\dagger A} &= \sqrt{\lambda_+} |\lambda_+\rangle\langle\lambda_+| + \sqrt{\lambda_-} |\lambda_-\rangle\langle\lambda_-| \\ &= \sqrt{\frac{3+\sqrt{5}}{2}} \cdot \frac{5-\sqrt{5}}{40} \begin{bmatrix} 4 & 2\sqrt{5}-2 \\ 2\sqrt{5}-2 & 6-2\sqrt{5} \end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{2}} \cdot \frac{5+\sqrt{5}}{40} \begin{bmatrix} 4 & -2\sqrt{5}-2 \\ -2\sqrt{5}-2 & 6+2\sqrt{5} \end{bmatrix} \end{aligned}$$

$$J^{-1} = \frac{1}{\sqrt{\lambda_+}} |\lambda_+\rangle\langle\lambda_+| + \frac{1}{\sqrt{\lambda_-}} |\lambda_-\rangle\langle\lambda_-| .$$

$$U = AJ^{-1}$$

I'm tired.

2.51

$$H^\dagger H = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

2.52

$$H^\dagger = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$$

Thus

$$H^2 = I.$$

2.53

$$\begin{aligned} \det(H - \lambda I) &= \left(\frac{1}{\sqrt{2}} - \lambda \right) \left(-\frac{1}{\sqrt{2}} - \lambda \right) - \frac{1}{2} \\ &= \lambda^2 - \frac{1}{2} - \frac{1}{2} \\ &= \lambda^2 - 1 \end{aligned}$$

Eigenvalues are $\lambda_\pm = \pm 1$ and associated eigenvectors are $|\lambda_\pm\rangle = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 \pm \sqrt{2} \end{bmatrix}$.

2.54
Since $[A, B] = 0$, A and B are simultaneously diagonalize, $A = \sum_i a_i \, |i\rangle\langle i|$, $B = \sum_i b_i \, |i\rangle\langle i|$.

$$
\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_i \exp(a_i) \, |i\rangle\langle i| \right) \left(\sum_i \exp(b_i) \, |i\rangle\langle i| \right) \\
&= \sum_{i,j} \exp(a_i + b_j) \, |i\rangle \, \langle i|j\rangle \, \langle j| \\
&= \sum_{i,j} \exp(a_i + b_j) \, |i\rangle\langle j| \, \delta_{i,j} \\
&= \sum_i \exp(a_i + b_i) \, |i\rangle\langle i| \\
&= \exp(A + B)
\end{aligned}
$$

2.55

$$
H = \sum_E E \, |E\rangle\langle E|
$$

$$
\begin{aligned}
U(t_2 - t_1)U^\dagger(t_2 - t_1) &= \exp\left(-\frac{iH(t_2 - t_1)}{\hbar} \right) \exp\left(\frac{iH(t_2 - t_1)}{\hbar} \right) \\
&= \sum_{E,E'} \left(\exp\left(-\frac{iE(t_2 - t_1)}{\hbar} \right) |E\rangle\langle E| \right) \left(\exp\left(-\frac{iE'(t_2 - t_1)}{\hbar} \right) |E'\rangle\langle E'| \right) \\
&= \sum_{E,E'} \left(\exp\left(-\frac{i(E - E')(t_2 - t_1)}{\hbar} \right) |E\rangle\langle E'| \, \delta_{E,E'} \right) \\
&= \sum_E \exp(0) \, |E\rangle\langle E| \\
&= \sum_E |E\rangle\langle E| \\
&= I
\end{aligned}
$$

Similarly, $U^\dagger(t_2 - t_1)U(t_2 - t_1) = I$.

2.56

$U = \sum_i \lambda_i \, |\lambda_i\rangle\langle \lambda_i| \quad (|\lambda_i| = 1)$.

$$
\log(U) = \sum_j \log(\lambda_j) \, |\lambda_j\rangle\langle \lambda_j| = \sum_j i\theta_j \, |\lambda_j\rangle\langle \lambda_j| \text{ where } \theta_j = \arg(\lambda_j)
$$

$$
K = -i \log(U) = \sum_j \theta_j \, |\lambda_j\rangle\langle \lambda_j| \, .
$$

$$
K^\dagger = (-i \log U)^\dagger = \left(\sum_j \theta_j \, |\lambda_j\rangle\langle \lambda_j| \right)^\dagger = \sum_j \theta_j^* \, |\lambda_j\rangle\langle \lambda_j| = \sum_j \theta_j \, |\lambda_j\rangle\langle \lambda_j| = K
$$

2.57

$$|\phi\rangle \equiv \frac{L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

$$\langle\phi|M_m^\dagger M_m|\phi\rangle = \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle}$$

$$\frac{M_m |\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \cdot \frac{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}}$$

2.58

$$\begin{aligned} \langle M\rangle &= \langle\psi|M|\psi\rangle = \langle\psi|m|\psi\rangle = m\,\langle\psi|\psi\rangle = m \\ \langle M^2\rangle &= \langle\psi|M^2|\psi\rangle = \langle\psi|m^2|\psi\rangle = m^2\,\langle\psi|\psi\rangle = m^2 \\ \text{deviation} &= \langle M^2\rangle - \langle M\rangle^2 = m^2 - m^2 = 0. \end{aligned}$$

2.59

$$\begin{aligned} \langle X\rangle &= \langle 0|X|0\rangle = \langle 0|1\rangle = 0 \\ \langle X^2\rangle &= \langle 0|X^2|0\rangle = \langle 0|X|1\rangle = \langle 0|0\rangle = 1 \\ \text{standard deviation} &= \sqrt{\langle X^2\rangle - \langle X\rangle^2} = 1 \end{aligned}$$

2.60

$$\begin{aligned} \vec{v}\cdot\vec{\sigma} &= \sum_{i=1}^{3} v_i\sigma_i \\ &= v_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(\vec{v}\cdot\vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\ &= \lambda^2 - 1 \quad (\because |\vec{v}| = 1) \end{aligned}$$

Eigenvalues are $\lambda = \pm 1$.

(i) if $\lambda = 1$

$$\begin{aligned} \vec{v}\cdot\vec{\sigma} - \lambda I &= \vec{v}\cdot\vec{\sigma} - I \\ &= \begin{bmatrix} v_3 - 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - 1 \end{bmatrix} \end{aligned}$$

Eigenvector is $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix}$.

$$
\begin{aligned}
|\lambda_1\rangle\langle\lambda_1| &= \frac{1+v_3}{2} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1-v_3}{v_1+iv_2} \end{bmatrix} \\
&= \frac{1+v_3}{2} \begin{bmatrix} 1 & \frac{v_1-iv_2}{1+v_3} \\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\
&= \frac{1}{2} \left(I + \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
&= \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})
\end{aligned}
$$

(ii) If $\lambda = -1$.

$$
\begin{aligned}
\vec{v}\cdot\vec{\sigma} - \lambda I = \vec{v}\cdot\vec{\sigma} + I \\
&= \begin{bmatrix} v_3+1 & v_1-iv_2 \\ v_1+iv_2 & -v_3+1 \end{bmatrix}
\end{aligned}
$$

Eigenvalue is $|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$.

$$
\begin{aligned}
|\lambda_{-1}\rangle\langle\lambda_{-1}| &= \frac{1-v_3}{2} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1+v_3}{v_1+iv_2} \end{bmatrix} \\
&= \frac{1-v_3}{2} \begin{bmatrix} 1 & -\frac{v_1-iv_2}{1-v_3} \\ -\frac{v_1+iv_2}{1-v_3} & \frac{1+v_3}{1-v_3} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1-v_3 & -(v_1-iv_2) \\ -(v_1+iv_2) & 1+v_3 \end{bmatrix} \\
&= \frac{1}{2} \left(I - \begin{bmatrix} v_3 & v_1-iv_2 \\ (v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
&= \frac{1}{2}(I - \vec{v}\cdot\vec{\sigma}).
\end{aligned}
$$

2.61

$$
\begin{aligned}
\langle\lambda_1|0\rangle\,\langle 0|\lambda_1\rangle &= \langle 0|\lambda_1\rangle\,\langle\lambda_1|0\rangle \\
&= \langle 0|\frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})|0\rangle \\
&= \frac{1}{2}(1+v_3)
\end{aligned}
$$

Post-measurement state is

$$
\begin{aligned}
\frac{|\lambda_1\rangle \langle\lambda_1|0\rangle}{\sqrt{\langle 0|\lambda_1\rangle \langle\lambda_1|0\rangle}} &= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \cdot \frac{1}{2} \begin{bmatrix} 1+v_3 \\ v_1+iv_2 \end{bmatrix} \\
&= \sqrt{\frac{1}{2}(1+v_3)} \begin{bmatrix} 1 \\ \frac{v_1+iv_2}{1+v_3} \end{bmatrix} \\
&= \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \\
&= |\lambda_1\rangle .
\end{aligned}
$$

2.62

Suppose M_m is an measurement operator. From the assumption, $E_m = M_m^\dagger M_m = M_m$. Then

$$\langle\psi|E_m|\psi\rangle = \langle\psi|M_m|\psi\rangle \geq 0.$$

for all $|\psi\rangle$.

Since M_m is positive operator, M_m is Hermitian. Therefore,

$$E_m = M_m^\dagger M_m = M_m M_m = M_m^2 = M_m.$$

Thus the measurement is a projective measurement.

2.63

$$
\begin{aligned}
M_m^\dagger M_m &= \sqrt{E_m} U_m^\dagger U_m \sqrt{E_m} \\
&= \sqrt{E_m} I \sqrt{E_m} \\
&= E_m.
\end{aligned}
$$

Since E_m is POVM, for arbitrary unitary U, $M_m^\dagger M_m$ is POVM.

2.64

Define $E_i = |\psi_i\rangle\langle\psi_i|$ for $1 \leq i \leq m$ and $E_{m+1} = I - \sum_{i=1}^{m} E_i$. Then $\sum_{i=1}^{m+1} E_i = I$. And $\langle\psi_i|E_i|\psi_i\rangle = \langle\psi_i|\psi_i\rangle \langle\psi_i|\psi_i\rangle = 1$.

2.65

$$|+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

2.66

$$X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

$$\langle X_1 Z_2 \rangle = \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{\langle 00| + \langle 11|}{\sqrt{2}} \cdot \frac{|10\rangle - |01\rangle}{\sqrt{2}} = 0$$

2.67

Unsolved
$W \subset V \to V = W \oplus W^\perp$.
$U : W \to V$, $U' : V \to V$.
$U' |w\rangle = U |w\rangle$
$U' \in \mathcal{L}(V)$
$U \in \mathcal{L}(W)$
$U' = U \oplus I$???

2.68

$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$.
Suppose $|a\rangle = a_0 |0\rangle + a_1 |1\rangle$ and $|b\rangle = b_0 |0\rangle + b_1 |1\rangle$.

$$|a\rangle |b\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle .$$

If $|\psi\rangle = |a\rangle |b\rangle$, then $a_0 b_0 = 1,\ a_0 b_1 = 0,\ a_1 b_0 = 0,\ a_1 b_1 = 1$ since $\{|ij\rangle\}$ is an orthonormal basis.

If $a_0 b_1 = 0$, then $a_0 = 0$ or $b_1 = 0$.

When $a_0 = 0$, this is contradiction to $a_0 b_0 = 1$. When $b_1 = 0$, this is contradiction to $a_1 b_1 = 1$.

Thus $|\psi\rangle \neq |a\rangle |b\rangle$.

2.69
Define Bell states as follows.

$$|\psi_1\rangle \equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|\psi_2\rangle \equiv \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$|\psi_3\rangle \equiv \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$|\psi_4\rangle \equiv \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

First, we prove $\{|\psi_i\rangle\}$ is a linearly independent basis.

$$a_1 |\psi_1\rangle + a_2 |\psi_2\rangle + a_3 |\psi_3\rangle + a_4 |\psi_4\rangle = 0$$
$$\therefore \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 + a_2 \\ a_3 + a_4 \\ a_3 - a_4 \\ a_1 - a_2 \end{bmatrix} = 0$$

$$\therefore \begin{cases} a_1 + a_2 = 0 \\ a_3 + a_4 = 0 \\ a_3 - a_4 = 0 \\ a_1 - a_2 = 0 \end{cases}$$

$$\therefore a_1 = a_2 = a_3 = a_4 = 0$$

Thus $\{|\psi_i\rangle\}$ is a linearly independent basis.

Moreover $\||\psi_i\rangle\| = 1$ and $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ for $i, j = 1, 2, 3, 4$. Therefore $\{|\psi_i\rangle\}$ forms an orthonormal basis.

2.70

For any Bell states we get $\langle\psi_i|E \otimes I|\psi_i\rangle = \frac{1}{2}(\langle 0|E|0\rangle + \langle 1|E|1\rangle)$.

Suppose Eve measures the qubit Alice sent by measurement operators M_m. The probability that Eve gets result m is $p_i(m) = \langle\psi_i|M_m^\dagger M_m \otimes I|\psi_i\rangle$. Since $M_m^\dagger M_m$ is positive, $p_i(m)$ are same values for all $|\psi_i\rangle$. Thus Eve can't distinguish Bell states.

2.71

From spectral decomposition,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1.$$
$$\rho^2 = \sum_{i,j} p_i p_j |i\rangle \langle i|j\rangle \langle j|$$
$$= \sum_{i,j} p_i p_j |i\rangle\langle j| \delta_{ij}$$
$$= \sum_i p_i^2 |i\rangle\langle i|$$

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}\left(\sum_i p_i^2 |i\rangle\langle i|\right) = \sum_i p_i^2 \mathrm{Tr}(|i\rangle\langle i|) = \sum_i p_i^2 \langle i|i\rangle = \sum_i p_i^2 \leq \sum_i p_i = 1 \quad (\because p_i^2 \leq p_i)$$

Suppose $\mathrm{Tr}(\rho^2) = 1$. Then $\sum_i p_i^2 = 1$. If $0 \leq p_i < 1$, then $p_i^2 < p_i$. Thus only one $p_i = 1$ and otherwise are 0. Therefore $\rho = |\psi_i\rangle\langle\psi_i|$ is pure state.

Conversely if ρ is pure, then $\rho = |\psi\rangle\langle\psi|$.

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}(|\psi\rangle \langle\psi|\psi\rangle \langle\psi|) = \mathrm{Tr}(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle = 1.$$

2.72

(1)

Since density matrix is Hermitian, matrix representation is $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$, $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$ w.r.t. standard basis. Because ρ is density matrix, $\mathrm{Tr}(\rho) = a + d = 1$.

Define $a = (1 + r_3)/2$, $d = (1 - r_3)/2$ and $b = (r_1 - ir_2)/2$, ($r_i \in \mathbb{R}$).

In this case,

$$\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{bmatrix} = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}).$$

Thus for arbitrary density matrix ρ can be written as $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

Next, we derive the condition that ρ is positive.

If ρ is positive, all eigenvalues of ρ should be non-negative.

$$\begin{aligned}
\det(\rho - \lambda I) = (a - \lambda)(b - \lambda) - |b|^2 &= \lambda^2 - (a + d)\lambda + ad - |b^2| = 0 \\
\lambda &= \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - |b|^2)}}{2} \\
&= \frac{1 \pm \sqrt{1 - 4\left(\frac{1 - r_3^2}{4} - \frac{r_1^2 + r_2^2}{4}\right)}}{2} \\
&= \frac{1 \pm \sqrt{1 - (1 - r_1^2 - r_2^2 - r_3^2)}}{2} \\
&= \frac{1 \pm \sqrt{|\vec{r}|^2}}{2} \\
&= \frac{1 \pm |\vec{r}|}{2}
\end{aligned}$$

Since ρ is positive, $\frac{1 - |\vec{r}|}{2} \geq 0 \rightarrow |\vec{r}| \leq 1$.

Therefore an arbitrary density matrix for a mixed state qubit is written as $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

(2)

$\rho = I/2 \rightarrow \vec{r} = 0$. Thus $\rho = I/2$ corresponds to the origin of Bloch sphere.

(3)

$$\begin{aligned}
\rho^2 &= \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})\ \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \\
&= \frac{1}{4}\left[I + 2\vec{r} \cdot \vec{\sigma} + \sum_{j,k} r_j r_k \left(\delta_{jk} I + i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l\right)\right] \\
&= \frac{1}{4}\left(I + 2\vec{r} \cdot \vec{\sigma} + |\vec{r}|^2 I\right) \\
\mathrm{Tr}(\rho^2) &= \frac{1}{4}(2 + 2|\vec{r}|^2)
\end{aligned}$$

If ρ is pure, then $\mathrm{Tr}(\rho^2) = 1$.

$$\begin{aligned}
1 = \mathrm{Tr}(\rho^2) &= \frac{1}{4}(2 + 2|\vec{r}|^2) \\
\therefore |\vec{r}| &= 1.
\end{aligned}$$

Conversely, if $|\vec{r}| = 1$, then $\mathrm{Tr}(\rho^2) = \frac{1}{4}(2 + 2|\vec{r}|^2) = 1$. Therefore ρ is pure.

2.73

Theorem 2.6

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j| = \sum_j q_j |\varphi_j\rangle\langle\varphi_j| \quad \Leftrightarrow \quad |\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$$

where u is unitary.
Transformation in theorem 2.6, $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$, corresponds to

$$\left[|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_k\rangle\right] = \left[|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle\right] U^T$$

where $k = \mathrm{rank}(\rho)$.

From spectral theorem, density matrix ρ is decomposed as $\rho = \sum_{k=1}^{d} \lambda_k |k\rangle\langle k|$ where $d = \dim \mathcal{H}$. Without loss of generality, we can assume $p_k > 0$ for $k = 1 \cdots, l$ where $l = \mathrm{rank}(\rho)$ and $p_k = 0$ for $k = l+1, \cdots, d$. Thus $\rho = \sum_{k=1}^{l} p_k |k\rangle\langle k| = \sum_{k=1}^{l} |\tilde{k}\rangle\langle\tilde{k}|$, where $|\tilde{k}\rangle = \sqrt{\lambda_k}\,|k\rangle$.

Suppose $|\psi_i\rangle$ is a state in support ρ. Then

$$|\psi_i\rangle = \sum_{k=1}^{l} c_{ik} |k\rangle, \quad \sum_k |c_{ik}|^2 = 1.$$

Define $p_i = \dfrac{1}{\sum_k \frac{|c_{ik}|^2}{\lambda_k}}$ and $u_{ik} = \dfrac{\sqrt{p_i} c_{ik}}{\sqrt{\lambda_k}}$.

Now

$$\sum_k |u_{ik}|^2 = \sum_k \frac{p_i |c_{ik}|^2}{\lambda_k} = p_i \sum_k \frac{|c_{ik}|^2}{\lambda_k} = 1.$$

Next prepare an unitary operator such that (i,k) component of U is u_{ik}. Then we can define another ensemble such that

$$\left[|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_i\rangle \cdots |\tilde{\psi}_l\rangle\right] = \left[|\tilde{k}_1\rangle \cdots |\tilde{k}_l\rangle\right] U^T$$

where $|\tilde{\psi}_i\rangle = \sqrt{p_i}\,|\psi_i\rangle$. From theorem 2.6,

$$\rho = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|.$$

Therefore we can obtain a minimal ensemble for ρ that contains $|\psi_i\rangle$.

Moreover since $\rho^{-1} = \sum_k \frac{1}{\lambda_k} |k\rangle\langle k|$,

$$\langle\psi_i|\rho^{-1}|\psi_i\rangle = \sum_k \frac{1}{\lambda_k} \langle\psi_i|k\rangle\langle k|\psi_i\rangle = \sum_k \frac{|c_{ik}|^2}{\lambda_k} = \frac{1}{p_i}.$$

Hence, $\frac{1}{\langle\psi_i|\rho^{-1}|\psi_i\rangle} = p_i$.

2.74

2.75

2.76

2.77

2.78

2.79

2.80

2.81

2.82