Solution for Quantum Computation and Quantum Information by Nielsen and Chuang

May 4, 2017

Contents

2 Introduction to quantum mechanics

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

2.2

$$A|0\rangle = A_{11}|0\rangle + A_{21}|1\rangle = |1\rangle \Rightarrow A_{11} = 0,\ A_{21} = 1 \tag{2}$$
$$A|1\rangle = A_{12}|0\rangle + A_{22}|1\rangle = |0\rangle \Rightarrow A_{12} = 1,\ A_{22} = 0 \tag{3}$$
$$\therefore A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|1\rangle,\ |0\rangle\} \tag{5}$$

2.3

$$A|v_i\rangle = \sum_j A_{ji}|w_j\rangle \tag{6}$$
$$B|w_j\rangle = \sum_k B_{kj}|x_k\rangle \tag{7}$$

Thus

$$BA|v_i\rangle = B\left(\sum_j A_{ji}|w_j\rangle\right) \tag{8}$$
$$= \sum_j A_{ji}B|w_j\rangle \tag{9}$$
$$= \sum_{j,k} A_{ji}B_{kj}|x_k\rangle \tag{10}$$
$$= \sum_k \left(\sum_j B_{kj}A_{ji}\right)|x_k\rangle \tag{11}$$
$$= \sum_k (BA)_{ki}|x_k\rangle \tag{12}$$

2.4

$$I\,|v_j\rangle = \sum_i I_{ij}\,|v_i\rangle = |v_j\rangle\,,\ \forall j. \tag{13}$$

$$\Rightarrow I_{ij} = \delta_{ij} \tag{14}$$

2.6

$$\left(\sum_i \lambda_i\,|w_i\rangle\,,\ |v\rangle\right) = \left(|v\rangle\,,\ \sum_i \lambda_i\,|w_i\rangle\right)^* \tag{15}$$

$$= \left[\sum_i \lambda_i\,(|v\rangle\,,\ |w_i\rangle)\right]^* \tag{16}$$

$$= \sum_i \lambda_i^*\,(|v\rangle\,,\ |w_i\rangle)^* \tag{17}$$

$$= \sum_i \lambda_i^*(|w_i\rangle\,,\ |v\rangle) \tag{18}$$

2.7

$$\langle w|v\rangle = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 1-1 = 0 \tag{19}$$

$$\frac{|w\rangle}{\||w\rangle\|} = \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \tag{20}$$

$$\frac{|v\rangle}{\||v\rangle\|} = \frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \tag{21}$$

2.9

$$\sigma_0 = I = |0\rangle\,\langle 0| + |1\rangle\,\langle 1| \tag{22}$$

$$\sigma_1 = X = |0\rangle\,\langle 1| + |1\rangle\,\langle 0| \tag{23}$$

$$\sigma_2 = Y = -i\,|0\rangle\,\langle 1| + i\,|1\rangle\,\langle 0| \tag{24}$$

$$\sigma_3 = Z = |0\rangle\,\langle 0| - |1\rangle\,\langle 1| \tag{25}$$

2.10

$$
\begin{aligned}
|v_j\rangle \langle v_k| &= I_V |v_j\rangle \langle v_k| I_V &(26)\\
&= \left(\sum_p |v_p\rangle \langle v_p|\right) |v_j\rangle \langle v_k| \left(\sum_q |v_q\rangle \langle v_q|\right) &(27)\\
&= \sum_{p,q} |v_p\rangle \langle v_p|v_j\rangle \langle v_k|v_q\rangle \langle v_q| &(28)\\
&= \sum_{p,q} \delta_{pj}\delta_{kq} |v_p\rangle \langle v_q| &(29)
\end{aligned}
$$

Thus

$$
(|v_j\rangle \langle v_k|)_{pq} = \delta_{pj}\delta_{kq} \tag{30}
$$

2.11

$$
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \det(X - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda \pm 1 \tag{31}
$$

If $\lambda = -1$,

$$
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{32}
$$

Thus

$$
|\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{33}
$$

If $\lambda = 1$

$$
|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{34}
$$

$$
X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|\lambda = -1\rangle, \ |\lambda = 1\rangle\} \tag{35}
$$

2.12

$$
\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I\right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1 \tag{36}
$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$
|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{37}
$$

Because $\ket{\lambda=1}\bra{\lambda=1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c\ket{\lambda=1}\bra{\lambda=1} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \tag{38}$$

2.13

Suppose $\ket{\psi}, \ \ket{\phi}$ are arbitrary vectors in V.

$$\begin{aligned} (\ket{\psi}, \ (\ket{w}\bra{v})\ket{\phi})^* &= \Big((\ket{w}\bra{v})^\dagger\ket{\psi}, \ \ket{\phi}\Big)^* & (39)\\ &= \Big(\ket{\phi}, \ (\ket{w}\bra{v})^\dagger\ket{\psi}\Big) & (40)\\ &= \bra{\phi}(\ket{w}\bra{v})^\dagger\ket{\psi}. & (41) \end{aligned}$$

On the other hand,

$$\begin{aligned} (\ket{\psi}, \ (\ket{w}\bra{v})\ket{\phi})^* &= (\braket{\psi|w}\braket{v|\phi})^* & (42)\\ &= \braket{\phi|v}\braket{w|\psi}. & (43) \end{aligned}$$

Thus

$$\begin{aligned} \bra{\phi}(\ket{w}\bra{v})^\dagger\ket{\psi} = \braket{\phi|v}\braket{w|\psi} \ \text{ for arbitrary vectors } \ket{\psi}, \ \ket{\phi} & \quad (44)\\ \therefore (\ket{w}\bra{v})^\dagger = \ket{v}\bra{w} & \quad (45) \end{aligned}$$

2.14

$$\begin{aligned} ((a_iA_i)^\dagger\ket{\phi}, \ \ket{\psi}) &= (\ket{\phi}, \ a_iA_i\ket{\psi}) & (46)\\ &= a_i(\ket{\phi}, \ A_i\ket{\psi}) & (47)\\ &= a_i(A_i^\dagger\ket{\phi}, \ \ket{\psi}) & (48)\\ &= (a_i^*A_i^\dagger\ket{\phi}, \ \ket{\psi}) & (49)\\ \therefore (a_iA_i)^\dagger = a_i^*A_i^\dagger & & (50) \end{aligned}$$

2.15

$$\begin{aligned} ((A^\dagger)^\dagger\ket{\psi}, \ \ket{\phi}) &= (\ket{\psi}, \ A^\dagger\ket{\phi}) & (51)\\ &= (A^\dagger\ket{\phi}, \ \ket{\psi})^* & (52)\\ &= (\ket{\phi}, \ A\ket{\psi})^* & (53)\\ &= (A\ket{\psi}, \ \ket{\phi}) & (54)\\ \therefore (A^\dagger)^\dagger = A & & (55) \end{aligned}$$

2.16

$$
\begin{aligned}
P &= \sum_i |i\rangle \langle i| \,. && (56)\\
P^2 &= \left(\sum_i |i\rangle \langle i|\right)\left(\sum_j |j\rangle \langle j|\right) && (57)\\
&= \sum_{i,j} |i\rangle \langle i|j\rangle \langle j| && (58)\\
&= \sum_i |i\rangle \langle j| \delta_{ij} && (59)\\
&= \sum_i |i\rangle \langle i| && (60)\\
&= P && (61)
\end{aligned}
$$

2.18

Suppose $|v\rangle$ is a eigenvector with corresponding eigenvalue λ.

$$
\begin{aligned}
U|v\rangle &= \lambda |v\rangle \,. && (62)\\
1 &= \langle v|v\rangle && (63)\\
&= \langle v| I |v\rangle && (64)\\
&= \langle v| U^\dagger U |v\rangle && (65)\\
&= \lambda\lambda^* \langle v|v\rangle && (66)\\
&= \|\lambda\|^2 && (67)\\
\therefore \lambda &= e^{i\theta} && (68)
\end{aligned}
$$

2.19

$$
X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad (69)
$$

2.20

$$
\begin{aligned}
U &\equiv \sum_i |w_i\rangle \langle v_i| && (70)\\
A'_{ij} &= \langle v_i|A|v_j\rangle && (71)\\
&= \langle v_i|UU^\dagger AUU^\dagger|v_j\rangle && (72)\\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \langle v_p|v_q\rangle \langle w_q|A|w_r\rangle \langle v_r|v_s\rangle \langle w_s|v_j\rangle && (73)\\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s|v_j\rangle && (74)\\
&= \sum_{p,r} \langle v_i|w_p\rangle \langle w_r|v_j\rangle A''_{pr} && (75)
\end{aligned}
$$

2.26