Solution for Quantum Computation and Quantum Information by Nielsen and Chuang

May 4, 2017

Contents

2 Introduction to quantum mechanics

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

2.2

$$A\left|0\right\rangle = A_{11}\left|0\right\rangle + A_{21}\left|1\right\rangle = \left|1\right\rangle \Rightarrow A_{11} = 0,\ A_{21} = 1 \tag{2}$$

$$A\left|1\right\rangle = A_{12}\left|0\right\rangle + A_{22}\left|1\right\rangle = \left|0\right\rangle \Rightarrow A_{12} = 1,\ A_{22} = 0 \tag{3}$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{\left|1\right\rangle,\ \left|0\right\rangle\} \tag{5}$$

2.3

$$A\left|v_i\right\rangle = \sum_j A_{ji}\left|w_j\right\rangle \tag{6}$$

$$B\left|w_j\right\rangle = \sum_k B_{kj}\left|x_k\right\rangle \tag{7}$$

Thus

$$\begin{aligned} BA\left|v_i\right\rangle &= B\left(\sum_j A_{ji}\left|w_j\right\rangle\right) && (8) \\ &= \sum_j A_{ji}B\left|w_j\right\rangle && (9) \\ &= \sum_{j,k} A_{ji}B_{kj}\left|x_k\right\rangle && (10) \\ &= \sum_k \left(\sum_j B_{kj}A_{ji}\right)\left|x_k\right\rangle && (11) \\ &= \sum_k (BA)_{ki}\left|x_k\right\rangle && (12) \end{aligned}$$

2.4

$$I\left|v_j\right\rangle = \sum_i I_{ij}\left|v_i\right\rangle = \left|v_j\right\rangle, \ \forall j. \tag{13}$$

$$\Rightarrow I_{ij} = \delta_{ij} \tag{14}$$

2.6

$$\left(\sum_i \lambda_i\left|w_i\right\rangle, \ \left|v\right\rangle\right) = \left(\left|v\right\rangle, \ \sum_i \lambda_i\left|w_i\right\rangle\right)^* \tag{15}$$

$$= \left[\sum_i \lambda_i\left(\left|v\right\rangle, \ \left|w_i\right\rangle\right)\right]^* \tag{16}$$

$$= \sum_i \lambda_i^*\left(\left|v\right\rangle, \ \left|w_i\right\rangle\right)^* \tag{17}$$

$$= \sum_i \lambda_i^*(\left|w_i\right\rangle, \ \left|v\right\rangle) \tag{18}$$

2.7

$$\langle w|v\rangle = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 1 - 1 = 0 \tag{19}$$

$$\frac{\left|w\right\rangle}{\|\left|w\right\rangle\|} = \frac{\left|w\right\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \tag{20}$$

$$\frac{\left|v\right\rangle}{\|\left|v\right\rangle\|} = \frac{\left|v\right\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \tag{21}$$

2.9

$$\sigma_0 = I = \left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right| \tag{22}$$

$$\sigma_1 = X = \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| \tag{23}$$

$$\sigma_2 = Y = -i\left|0\right\rangle\left\langle 1\right| + i\left|1\right\rangle\left\langle 0\right| \tag{24}$$

$$\sigma_3 = Z = \left|0\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right| \tag{25}$$

2.10

$$
\begin{aligned}
\left|v_j\right\rangle \left\langle v_k\right| &= I_V \left|v_j\right\rangle \left\langle v_k\right| I_V && (26)\\
&= \left(\sum_p \left|v_p\right\rangle \left\langle v_p\right|\right) \left|v_j\right\rangle \left\langle v_k\right| \left(\sum_q \left|v_q\right\rangle \left\langle v_q\right|\right) && (27)\\
&= \sum_{p,q} \left|v_p\right\rangle \left\langle v_p|v_j\right\rangle \left\langle v_k|v_q\right\rangle \left\langle v_q\right| && (28)\\
&= \sum_{p,q} \delta_{pj}\delta_{kq} \left|v_p\right\rangle \left\langle v_q\right| && (29)
\end{aligned}
$$

Thus

$$
\left(\left|v_j\right\rangle \left\langle v_k\right|\right)_{pq} = \delta_{pj}\delta_{kq} \tag{30}
$$

2.11

$$
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \det(X - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda \pm 1 \tag{31}
$$

If $\lambda = -1$,

$$
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{32}
$$

Thus

$$
\left|\lambda = -1\right\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{33}
$$

If $\lambda = 1$

$$
\left|\lambda = 1\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{34}
$$

$$
X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \left\{\left|\lambda = -1\right\rangle,\ \left|\lambda = 1\right\rangle\right\} \tag{35}
$$

2.12

$$
\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I\right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1 \tag{36}
$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$
\left|\lambda = 1\right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{37}
$$

Because $|\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c\, |\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \tag{38}$$