DERIVING DIFFERENTIAL APPROXIMATION RESULTS FOR K CSPS FROM COMBINATORIAL DESIGNS *

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Abstract. Traditionally, inapproximability results for $\mathsf{Max\,k\,CSP{-}q}$ have been established using balanced t-wise independent distributions, which are related to orthogonal arrays of strength t. We contribute to the field by exploring such combinatorial designs in the context of the differential approximation measure. First, we establish a connection between the average differential ratio and orthogonal arrays. We deduce a differential approximation ratio of $1/q^k$ on $(k+1)$-partite instances of $\mathsf{k\,CSP{-}q}$, $\Omega(1/n^{k/2})$ on boolean instances, $\Omega(1/n)$ when $k=2$, and $\Omega(1/\nu^{k-\lceil \log_{p^\kappa} k\rceil})$ when $k \geq 3$ and $q \geq 3$, where p^κ is the smallest prime power greater than q. Secondly, by considering pairs of arrays related to balanced k-wise independence, we establish a reduction from $\mathsf{k\,CSP{-}q}$ to $\mathsf{k\,CSP{-}k}$ (with $q > k$), with an expansion factor of $1/(q-k/2)^k$ on the differential approximation guarantee. This, combined with the result of Yuri Nesterov, implies a lower differential approximability bound of $0.429/(q-1)^2$ for $\mathsf{2\,CSP{-}q}$. Finally, we prove that every Hamming ball with radius k provides a $\Omega(1/n^k)$-approximation of the instance diameter. Our work highlights the utility of combinatorial designs in establishing approximation results.

Key words. approximation algorithms, differential approximation, constraint satisfaction problems, k CSPs, combinatorial designs, balanced k-wise independence

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1. Introduction.

A wide variety of combinatorial optimization problems can be considered as *Constraint Satisfaction Problems*, CSPs over a finite domain. In a CSP over an alphabet Σ, we have a set $\{x_1, \ldots, x_n\}$ of Σ-valued variables and a set $\{C_1, \ldots, C_m\}$ of constraints. Each constraint C_i consists of the application of a (non constant) predicate $P_i : \Sigma^{k_i} \to \{0, 1\}$ to a tuple of variables. The goal is to assign values to the variables so as to optimize the number of satisfied constraints.

In general, the alphabet is either Boolean or of size q, denoted by $\Sigma_q := \{0, 1, 2, \ldots, q-1\}$. It typically represents a set of symbols, but sometimes we will use algebraic operations on it, such as addition or multiplication. In these cases, we associate Σ_q with the ring $\mathbb{Z}/q\mathbb{Z}$. When q is prime, this ring is also a field. Arithmetic operations on elements of Σ_q are always performed modulo q and arithmetic operations over Σ_q^ν are interpreted componentwise modulo q.

For example, the *Maximum Satisfiability Problem* ($\mathsf{Max\,Sat}$) is the boolean CSP where the objective is to satisfy as many disjunctive clauses $(\ell_{i_1} \vee \ldots \vee \ell_{i_{k_i}})$ as possible. Here, a literal ℓ_j represents either the boolean variable x_j or its negation $\bar{x}_j$. In $\mathsf{Lin{-}q}$, where the alphabet is Σ_q, constraints are linear equations of the form $(\alpha_{i,1}x_{i_1} + \ldots + \alpha_{i,k_i}x_{i_{k_i}} \equiv \alpha_{i,0} \bmod q)$, where $\alpha_{i,1}, \ldots, \alpha_{i,k_i}, \alpha_{i,0}$ are constant integers of $\mathbb{Z}_q$.

In the most general case, for each $i \in [m] := \{1, 2, \ldots, m\}$, the constraint C_i is associated with a positive weight w_i, and the functions P_i can take real values (*e.g.* see [43, 6] for the latter generalization). The goal is then to optimize an objective function of the form

$$v(I, x) \quad = \sum_{i=1}^m w_i P_i(x_{J_i}) = \sum_{i=1}^m w_i P_i(x_{i_1}, \ldots, x_{i_{k_i}})$$

over Σ^n, where for all $i \in [m]$, $k_i \leq n$, $P_i : \Sigma^{k_i} \to \mathbb{R}$, $J_i = \{i_1, \ldots, i_{k_i}\} \subseteq [n]$ and

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$w_i > 0$. For example, if $q = 3$ and $n = m = 4$, I might consist of minimizing the objective function $v(I, (x_1, x_2, x_3, x_4))$ below over Σ_3^4:

$$(x_1 + x_2 \equiv 1 \bmod 3) + 2.3 \times (x_1 = x_3) + 1.4 \times (x_2 = 1 \wedge x_4 = 2) + 7 \times x_4 \tag{1.1}$$

In this example, we have $w_1 = 1$, $w_2 = 2.3$, $w_3 = 1.4$, $w_4 = 7$, $k_1 = k_2 = k_3 = 2$, $k_4 = 1$, $J_1 = \{1, 2\}$, $J_2 = \{1, 3\}$, $J_3 = \{2, 4\}$ and $J_4 = \{4\}$.

We use $\mathsf{CSP}{-}\mathsf{q}$ to denote the optimization problem with the alphabet $\Sigma_q = \{0, \ldots, q-1\}$ of size q. It is worth noting that real-valued functions of boolean variables are often referred to as *pseudo-boolean* in the literature.

For a family $\mathcal{F}$ of functions, $\mathsf{CSP}(\mathcal{F})$ denotes the CSP in which all constraints P_i are elements of $\mathcal{F}$ (for all $i \in [m]$). For example, $\mathsf{Lin}{-}2$ corresponds to

$$\mathsf{CSP}(\{\mathsf{XNOR}^{\mathsf{k}}, \mathsf{XOR}^{\mathsf{k}} \mid \mathsf{k} \in \mathbb{N}\backslash\{0\}\})$$

where, for a positive integer k, $XNOR^k$ and XOR^k refer to the k-ary boolean predicates that are true for entries having an even and an odd number of nonzero coordinates, respectively.

In this paper, we focus on k-CSPs, which are CSPs where each constraint involves at most k variables, that is, where $k_i \leq k$ for all $i \in [m]$. For a specific CSP Π (*e.g.*, $\mathsf{Lin}{-}\mathsf{q}$), its restriction to instances where each constraint depends on respectively *at most* and *exactly* k variables is commonly denoted by $\mathsf{k}\,\Pi$ (*e.g.*, $\mathsf{k}\,\mathsf{Lin}{-}\mathsf{q}$) and $\mathsf{Ek}\,\Pi$ (*e.g.*, $\mathsf{Ek}\,\mathsf{Lin}{-}\mathsf{q}$); its restrictions to instances where the goal is to maximize or to minimize, are denoted by $\mathsf{Max}\,\Pi$ and $\mathsf{Min}\,\Pi$, respectively.

Even when $q = k = 2$, $\mathsf{Max}\,2\,\mathsf{Sat}$ and $\mathsf{Min}\,2\,\mathsf{Sat}$ are $\mathbf{NP}{-}\mathbf{hard}$ [24, 35]. Hence, a major issue in the optimization of CSPs is to provide a characterization of their computational complexity by addressing their *approximation degree*.

1.1. Approximation measures. Given an optimization CSP Π, we denote by $\mathcal{I}_\Pi$ its instance set. For an instance $I \in \mathcal{I}_\Pi$, we denote by $\mathrm{opt}(I)$ and $\mathrm{wor}(I)$ respectively the optimum and the worst solution values on I. The purpose of an approximation measure is to compare in some way the value of approximate solutions to the optimum solution value. Most approximation results are related to the *standard approximation measure*, which compares the value of a given solution to the optimum solution value. Namely, the standard ratio performed by a solution x of an instance I is defined by[1]:

$$\min\{v(I, x)/\mathrm{opt}(I), \mathrm{opt}(I)/v(I, x)\}$$

A solution x is *ρ-standard approximate* on I for some $\rho \in (0, 1]$ if this ratio is greater than or equal to ρ. Given $\rho : \mathcal{I}_\Pi \to (0, 1]$, a polynomial time algorithm $\mathcal{A}$ is a *ρ-standard approximation algorithm* for Π if, when applied to any instance I of Π, it returns a solution that is $\rho(I)$-standard approximate. Π is *approximable within a standard factor of ρ* whenever such an algorithm exists.

It is convenient to think of the average solution value on I as the expected value $\mathbb{E}_X[v(I, X)]$ of a random solution where $X = (X_1, \ldots, X_n)$ is a vector of pairwise independent random variables, each uniformly distributed over Σ_q. In [31], Håstad and Venkatesh introduced an approximation measure, that we here call *gain approximation measure*. This measure is based on the *optimum advantage over a random*

[1] The standard ratio is also commonly defined as the inverse $\max\{v(I, x)/\mathrm{opt}(I), \mathrm{opt}(I)/v(I, x)\}$ of $\min\{v(I, x)/\mathrm{opt}(I), \mathrm{opt}(I)/v(I, x)\}$ in the literature.

assignment, which is the quantity $|\mathrm{opt}(I) - \mathbb{E}_X[v(I,X)]|$. The *gain ratio* performed by a solution x on an instance I is defined by:

$$\frac{v(I,x) - \mathbb{E}_X[v(I,X)]}{\mathrm{opt}(I) - \mathbb{E}_X[v(I,X)]}$$

This measure was motivated by the fact that, for numerous boolean CSPs, for all constant $\varepsilon > 0$, finding solutions with value at least $\mathbb{E}_X[v(I,X)] + \varepsilon \times \sum_{i=1}^m w_i$ on almost satisfiable instances is **NP−hard**. For example, $\mathsf{E3\,Lin{-}2}$ is such a CSP [30].

The *differential approximation measure* is based on the distance to the worst solution value. Specifically, the differential ratio achieved by solution x on instance I is defined by:

$$\frac{v(I,x) - \mathrm{wor}(I)}{\mathrm{opt}(I) - \mathrm{wor}(I)}$$

The quantity $|\mathrm{opt}(I) - \mathrm{wor}(I)|$, which can be seen as the optimum advantage over a worst solution value, is commonly referred to as the *diameter* of I. The differential ratio gained prominence in approximation theory due to its stability under affine transformations of the objective function [1, 4, 21].

ρ-differential and *ρ-gain approximate solutions* of a given instance, *approximation algorithms* for a given problem, and *approximable problems* are defined in the same way as for the standard approximation measure. On an instance I where the goal is to maximize and $v(I,.)$ is nonnegative, any solution x satisfies:

$$\frac{v(I,x)}{\mathrm{opt}(I)} \geq \frac{v(I,x) - \mathrm{wor}(I)}{\mathrm{opt}(I) - \mathrm{wor}(I)} \geq \frac{v(I,x) - \mathbb{E}_X[v(I,X)]}{\mathrm{opt}(I) - \mathbb{E}_X[v(I,X)]}$$

In particular, for all positive integers q, k, if $\mathsf{k\,CSP{-}q}$ is approximable within gain factor ρ, then it is approximable within differential factor ρ and, if it is, $\mathsf{Max\,k\,CSP{-}q}$ is approximable within standard factor ρ *on instances with nonnegative solution values.*

We here address three questions regarding the differential approximability of $\mathsf{k\,CSP{-}q}$, about which, unlike the standard approximation, only a few facts are known.

1.2. Differential approximability of CSPs. The *Conjunctive Constraint Satisfaction Problem*, CCSP for short, is the boolean CSP whose constraints are conjunctive clauses. For all constants $\varepsilon > 0$, the restriction of $\mathsf{Max\,CCSP}$ to unweighted instances is **NP−hard** to approximate within standard ratio $1/m^{1-\varepsilon}$, where m is the number of constraints of the CSP instance. This is due to the standard inapproximability bound of [27, 46] for the *Maximum Independent set problem*, which extends by reduction to $\mathsf{Max\,CCSP}$ [9]. $\mathsf{Max\,Sat}$, for its part, is inapproximable within any constant differential factor assuming $\mathbf{P} \neq \mathbf{NP}$, as Escoffier and Paschos argue in [22]. The same authors observe that the conditional expectation technique [32], though, provides $(1/m)$-differential approximate solutions on unweighted instances of Sat [22] (see section 2 for more details). For $\mathsf{Lin{-}2}$, Håstad and Venkatesh show that combining this technique with exhaustive search allows to approximate the optimum gain over a random assignment within a factor of $\Omega(1/m)$. $\mathsf{Lin{-}2}$ consequently is $\Omega(1/m)$-differentially approximable.

The differential approximability bound of $\Omega(1/m)$ for $\mathsf{Lin{-}2}$ extends to $\mathsf{k\,CSP{-}q}$ for all constant integers k, q, using a binary encoding of the variables and the discrete Fourier transform [16]. When $q = k = 2$, $\mathsf{2\,CSP{-}2}$ is approximable within differential factor $2 - \pi/2 > 0.429$ combining the semidefinite programming based algorithm of Goemans and Williamson [25] with derandomization techniques such as the one proposed in [26]. This is a straightforward consequence of a result by Nesterov, which establishes this approximation guarantee for *Unconstrained Binary Quadratic Programming* [41]. The approximability bound of $2 - \pi/2$ extends by reduction to $3\,\mathsf{CSP}{-}2$, although up to a factor $1/2$ on the approximation guarantee [17]. The question whether $\mathsf{k}\,\mathsf{CSP}{-}\mathsf{q}$ is approximable within any constant differential factor, though, remains open as soon as $q \geq 3$ or $k \geq 4$.

Standard inapproximability bounds are known for $\mathsf{k}\,\mathsf{CSP}{-}\mathsf{q}$, that even hold when restricting to *k-partite instances*. The *primary* hypergraph of a CSP instance I is the hypergraph G_I where for each variable x_j of I, there is a vertex j in G_I and for each constraint $C_i = P_i(x_{i_1}, \ldots, x_{i_{k_i}})$ of I, there is in G_I a hyperedge $e_i = (i_1, \ldots, i_{k_i})$. Assimilating I to its primary hypergraph, a *strong coloring* of I is a partition $V_1 \sqcup \ldots \sqcup V_\nu$ of $[n]$ such that the support $J_i = (i_1, \ldots, i_{k_i})$ of any constraint intersects each color set V_c in at most one index. We say that I is ν-partite when such a partition of size ν exists. The smallest integer ν for which I is ν-partite is called the *strong chromatic number* of I (*e.g.* see [7]). Let $q \geq 2$, $k \geq 3$ be two integers. In [13], Chan establishes that the restriction of $\mathsf{Max}\,\mathsf{k}\,\mathsf{CSP}{-}\mathsf{q}$ to k-partite instances with nonnegative solution values is $\mathbf{NP}{-}\mathbf{hard}$ to standardly approximate within any constant factor greater than $(q-1)k/q^{k-1}$ if q is a prime power, $O(k/q^{k-1})$ if $k \geq q$ and $O((q-1)k/q^{k-1})$ otherwise. We infer that the same inapproximability bounds apply to k-partite instances of $\mathsf{k}\,\mathsf{CSP}{-}\mathsf{q}$ and the differential approximation measure. Notice that these results reinforce those provided by Håstad in [30]. Finally, we observe that the 6-gadget from [30] that reduces $\mathsf{E3}\,\mathsf{Lin}{-}2$ to $\mathsf{E2}\,\mathsf{Lin}{-}2$ implies a differential inapproximability bound of $7/8+\varepsilon$ for $\mathsf{Bipartite}\,\mathsf{Lin}{-}2$ for all constant $\varepsilon > 0$ assuming $\mathbf{P} \neq \mathbf{NP}$[2].

1.3. Approximability of CSPs and balanced t-wise independence. Numerous inapproximability bounds for k CSPs, including the ones of [13], involve *balanced t-wise independent distributions* or *balanced t-wise independent subsets* (notably see [8, 7]). Given three integers $q \geq 1$, $t \geq 1$ and $\nu \geq t$, a *probability distribution* μ on Σ_q^ν is *balanced t-wise independent* whenever the probability that any t coordinates $(Y_{c_1}, \ldots, Y_{c_t})$ of a vector Y from the probability space (Σ_q^ν, μ) take any t values $(v_1, \ldots, v_t)$ equals $1/q^t$. By extension (*e.g.* see [13]), a *subset* $\mathcal{U}$ of Σ_q^ν is said to be *balanced t-wise independent* if for each sequence $J = (c_1, \ldots, c_t)$ of t indices from $[\nu]$ and each $v \in \Sigma_q^t$, $\mathcal{U}$ contains exactly $|\mathcal{U}|/q^t$ vectors u such that $(u_{c_1}, \ldots, u_{c_t}) = (v_1, \ldots, v_t)$. For instance, let $ZeroSum^{\nu,q}$ refer to the predicate on Σ_q^ν that is true for entries $(y_1, \ldots, y_\nu)$ with $y_1 + \ldots + y_\nu \equiv 0 \bmod q$. Then fixing any $\nu - 1$ variables of the equation to any values $v_1, \ldots, v_{\nu-1}$, there is a single assignment $(-v_1 - \ldots - v_{\nu-1}) \bmod q$ to the remaining variable so as to satisfy the equation. The $q^{\nu-1}$ accepting entries of $ZeroSum^{\nu,q}$ therefore constitute a balanced $(\nu-1)$-wise independent subset of Σ_q^ν. Furthermore, the distribution μ on Σ_q^ν that associates with any $u \in \Sigma_q^\nu$ the probability $1/q^{\nu-1}$ if $ZeroSum^{\nu,q}(u) = 1$ and 0 otherwise clearly is balanced $(\nu-1)$-wise independent.

Given a function P on Σ_q^k, we denote by r_P the average value of P over Σ_q^k, *i.e.*:

$$r_P \quad = \textstyle\sum_{y \in \Sigma_q^k} P(y)/q^k$$

Furthermore, given $v \in \Sigma_q^k$, the *shift by v of P* is the function, denoted by P_v, that

[2]See section SM4 of the supplement for more details.

associates with each $y \in \Sigma_q^k$ the value of P taken at $y + v$. Formally:

$$P_v(y_1, \ldots, y_k) \quad = P\left((y_1 + v_1) \bmod q, \ldots, (y_k + v_k) \bmod q\right), \quad y_1, \ldots, y_k \in \Sigma_q$$

For any element $a \in \Sigma_q$, $\mathbf{a} = (a, \ldots, a)$ denotes the vector where all coordinates are equal to a (the dimension depends on the context), and $P_{\mathbf{a}}$ denotes the translation of function P by the vector $\mathbf{a}$. For instance, we recognize in the first term of (1.1) the function $ZeroSum_v^{2,3}$ where $v = (2, 0)$. Moreover, let $AllZeros^{k,q}$ refer to the predicate on Σ_q^k that is only true on input $\mathbf{0}$; then in the same expression, P_3 is $AllZeros_v^{2,3}$ where $v = (2, 1)$. Furthermore, for $(x_2, x_4) \in \Sigma_3^2$, we have:

$$\begin{aligned} P_{3\mathbf{1}}(x_2, x_4) \quad &= AllZeros^{2,3}_{v+\mathbf{1}}(x_2, x_4) \quad = AllZeros^{2,3}_{0,2}(x_2, x_4) \quad = (x_2 = 0 \wedge x_4 = 1) \\ P_{3\mathbf{2}}(x_2, x_4) \quad &= AllZeros^{2,3}_{v+\mathbf{2}}(x_2, x_4) \quad = AllZeros^{2,3}_{1,0}(x_2, x_4) \quad = (x_2 = 2 \wedge x_4 = 0) \end{aligned}$$

The inapproximability bounds of [13] specifically follow from the Theorem below[3].

THEOREM 1.1 ([13]). *Let $k \geq 3$ and $q \geq 2$ be two constant integers, and let P be a predicate on $\mathbb{Z}_q^k$ such that $P^{-1}(1)$ is a balanced pairwise independent subgroup of $\mathbb{Z}_q^k$. Then* $\mathsf{Max\,CSP}(\{\mathsf{P_v} \,|\, \mathsf{v} \in \mathbb{Z}_\mathsf{q}^\mathsf{k}\})$ *in k-partite instances is* **NP−hard** *to approximate within any constant standard factor greater than r_P.*

For example, given two integers $q \geq 2$ and $k \geq 3$, the accepting entries of $ZeroSum^{k,q}$ form a subgroup of $\mathbb{Z}_q^k$. Since this subgroup is balanced $(k-1)$-wise independent, it is in also balanced pairwise independent. Theorem 1.1 induces a differential inapproximability bound of $1/q$ for k-partite instances of $\mathsf{k\,CSP{-}q}$ assuming $\mathbf{P} \neq \mathbf{NP}$. When $(k, q) = (3, 2)$, this yields an approximability upper bound of $1/2$ for $\mathsf{3\,CSP{-}2}$. For others values of (k, q), Chan deduces the bounds of $O(k/q^{k-1})$ or $O(k/q^{k-2})$ we mentioned earlier from more elaborate predicates and a standard approximation preserving reduction that allows to reduce to a smaller alphabet size which is a prime power (more details on the reduction can be found in subsection 3.2).

With a slight misuse of language, for two positive integers q and t, we will subsequently refer to the real-valued functions P of variables with domain Σ_q as *balanced t-wise independent* if they satisfy the following property: their mean value remains constant when any t of their variables are fixed to any t values. Formally, if P depends on k variables, it must satisfy:

$$\sum_{y \in \Sigma_q^k : y_J = v} P(y)/q^{k-t} = r_P, \qquad J \subseteq [k], |J| = t,\ v \in \Sigma_q^t \tag{1.2}$$

From now onwards, we denote the set of such functions by $\mathcal{I}_q^t$. Balanced t-wise independent functions provide a natural extension of balanced t-wise independent distributions over Σ_q^k, which satisfy (1.2). Conversely, let P be a function on Σ_q^k with minimum value P_*; then P satisfies (1.2) *if an only if* the function

$$y \mapsto \tilde{P}(y) := \frac{P(y) - P_*}{\sum_{u \in \Sigma_q^k}(P(u) - P_*)}, \qquad y \in \Sigma_q^k$$

defines a balanced t-wise independent distribution on Σ_q^k.[4] For example, if P is $ZeroSum^{k,q}$, then $\tilde{P}$ is $ZeroSum^{k,q}/q^{k-1}$. Balanced t-wise independent functions

[3] We expose in Theorem 1.1 a simplified version of the result of [13], which actually applies to CSPs over finite abelian groups.

[4] By construction $\tilde{P}$ takes values in $[0, 1]$, has a mean value of $1/q^k$, and satisfies (1.2) *iff* P does.

TABLE 1

Differential approximation bounds that are already known for $\mathsf{k\,CSP{-}q}$ *and* $\mathsf{CSP}(\mathcal{O}_\mathsf{q})$ *where* $k \geq 2$ *and* $q \geq 2$. *Inapproximability bounds hold for all constant* $\varepsilon > 0$ *assuming* $\mathbf{P} \neq \mathbf{NP}$. *The bounds marked by* * *are commented in section SM4 of the supplement.*

Restriction	Approximation bound
$\mathsf{CSP}(\mathcal{O}_\mathsf{q})$	$1/q$ (trivial) $\neg\, 1/q + \varepsilon$, even for $\mathsf{3{-}partite\,E3\;CSP}(\mathcal{O}_\mathsf{q} \cap \mathcal{I}^2_\mathsf{q})$ [13]
$\mathsf{2\,CSP{-}2}$, $\mathsf{3\,CSP}(\mathcal{E}_2)$	$2 - \pi/2$ (> 0.429) [41, 17] $\neg\, 7/8 + \varepsilon^*$, even for $\mathsf{Bipartite\,Lin{-}2}$, due to the gadget of [30] from $\mathsf{E3\,Lin{-}2}$
$\mathsf{3\,CSP{-}2}$	$1 - \pi/4$ (> 0.214) using [41], by reduction to $\mathsf{2\,CSP{-}2}$ [17] $\neg\, 1/2 + \varepsilon$, even for $\mathsf{3{-}partite\,E3\,Lin{-}2}$ [13]
$\mathsf{k\,CSP{-}q}$	$\Omega(1/m)$ using [31], by reduction to $\mathsf{Lin{-}2}$ [16]
$\mathsf{k\,CSP{-}q}$ where $k \geq 3$	$\neg\, O(k/q^{k-1}) + \varepsilon$ if $k \geq q$, $\neg\,(q-1)k/q^{k-1} + \varepsilon$ if q is a prime power, $\neg\, O((q-1)k/q^{k-1}) + \varepsilon$ otherwise, even for $\mathsf{k{-}partite\,CSP}(\mathcal{I}^2_\mathsf{q})$ [13]

also provide a natural extension of balanced t-wise subsets of Σ_q^k. Here consider that a predicate whose accepting entries are defined as the elements of a given balanced t-wise independent subset of Σ_q^k clearly satisfies (1.2). For instance, function $ZeroSum^{k,q}$ is balanced $(k-1)$-wise independent for all integers $q \geq 2$ and $k \geq 2$.

1.4. Outline. We identify new connections between balanced t-wise independence and optimization CSPs over q-ary alphabets, which allow to establish new positive and conditional differential approximation results for $\mathsf{k\,CSP{-}q}$.

To establish the inapproximation bounds of [13], balanced t-wise independence restricts the functions that can express the constraints of the CSP. In the present article, balanced t-wise independence essentially concerns distributions on the solution set of the CSP instance. We more specifically manipulate arrays or pairs of arrays which are interpreted as multisets of solutions by identifying each possible row of the arrays with a solution of the CSP instance. Balanced t-wise independence precisely constraints the frequency of rows in the arrays. Our results notably involve a famous family of combinatorial designs termed *Orthogonal arrays* (OAs for short), which can be seen as rational balanced t-wise independent measures on Σ_q^ν (see subsection 2.4 for a complete definition of OAs).

Note that when functions with arbitrary sign values are permitted to express the constraints of CSP instances, $\mathsf{Max\,CSP{-}q}$ and $\mathsf{Min\,CSP{-}q}$ can be considered as equivalent optimization problems. This equivalence arises from the fact that, for any instance of $\mathsf{CSP{-}q}$, replacing each constraint function with its opposite is equivalent to reversing the optimization objective. The paper is organized as follows:

- In section 2, we seek valid lower bounds for the differential ratio taken at the average solution value of $\mathsf{k\,CSP{-}q}$ instances. We exhibit a connection between this ratio and OAs or related designs that involves the strong chromatic number of the instance (Theorem 2.4). Arrays from the literature then allow to deduce that the average differential ratio is $\Omega(1)$ when restricting to instances with a bounded strong chromatic number, $\Omega(1/n^{k/2})$ when $q = 2$ and $\Omega(1/n^{k-\lceil \log_{p^\kappa} k \rceil})$ where p^κ is the smallest prime power $\geq q$ otherwise.
- In section 3, we introduce a family of combinatorial designs, which allows to find solutions of CSPs over an alphabet of size q by solving CSPs over an alphabet of a smaller size p, provided that every constraint depends on at most p variables (Theorem 3.2). These designs consist of pairs of arrays that can be viewed as some

TABLE 2

New differential approximability bounds for $\mathsf{k\,CSP{-}q}$*,* $\mathsf{CSP}(\mathcal{O}_\mathsf{q})$*,* $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$ *and* $\mathsf{k\,CSP}(\mathcal{I}^\mathsf{t}_\mathsf{q})$ *where* $k \geq 2$*,* $q \geq 2$ *and* $t \in [k-1]$*: we denote by* p^κ *the smallest prime power* $\geq q$*, by* ν *the strong chromatic number of the instance, by* $\tilde{B}^1$ *if* $q = 2$ *the neighbourhood function that associates with any solution* x *the set of the solutions that coincide with* x *on all, all but one, none, or one coordinates.*

Lower bounds for the average differential ratio (section 2)

Restriction	Conditions on ν, q, k, t	Lower bound
$\mathsf{CSP}(\mathcal{O}_\mathsf{q})$		$1/q$
$\mathsf{k\,CSP{-}q}$	$\nu \leq k+1$	$1/q^k$
$\mathsf{k\,CSP{-}q}$	$p^\kappa > k$ and $\nu \leq p^\kappa + 1$	$1/p^{\kappa k}$ ($\geq 1/(2q-2)^k$) (using [12])
$\mathsf{3\,CSP{-}q}$	$q \geq 3$ and $\nu \leq 2^{\lceil \log_2 q \rceil} + 2$	$1/2^{\lceil \log_2 q \rceil k}$ ($\geq 1/(2q-2)^3$) (using [12])
$\mathsf{k\,CSP}(\mathcal{I}^\mathsf{t}_\mathsf{q})$	$\nu \leq k+t+1$	$1/q^{\min\{\nu-t,k\}}$
$\mathsf{k\,CSP}(\mathcal{I}^\mathsf{t}_\mathsf{q})$	q prime power, $q > k$ and $\nu \leq q+1+t$	$1/q^k$ (using [12])
$\mathsf{3\,CSP}(\mathcal{I}^\mathsf{t}_\mathsf{q})$	q power of 2, $q > 3$ and $\nu \leq q+2+t$	$1/q^3$ (using [12])
$\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$	q or k is odd and $\nu \leq k+1$	$1/q^{k-1}$
$\mathsf{k\,CSP{-}2}$		$\Omega(1/\nu^{\lfloor k/2 \rfloor})$ (using [15, 29, 11, 20])
$\mathsf{k\,CSP{-}q}$	$q \geq 3$	$\Omega(1/\nu^{k-\lceil \log_{p^\kappa} k \rceil})$ (using [15, 10])

Differential approximability bounds obtained using [41] by reduction to $\mathsf{2\,CSP{-}2}$ *(section 3)*

Restriction	Conditions on q	Approximation bound
$\mathsf{2\,CSP{-}q}$	$q \geq 3$	$(2-\pi/2)/(q-1)^2$
$\mathsf{2\,CSP}(\mathcal{E}_\mathsf{q})$	$q \in \{3,4,5,7,8\}$	$(2-\pi/2)/q$

Approximability bounds related to Hamming balls of fixed radius (section 4)

Restriction	Approximation guarantee
$\mathsf{CSP}(\mathcal{O}_\mathsf{q})$	for all solutions x, the highest differential ratio achieved over $\{x, x+\mathbf{1}, \ldots, x+\mathbf{q-1}\}$ is $\geq 1/q$
$\mathsf{2\,CSP{-}2}$, $\mathsf{3\,CSP}(\mathcal{E}_2)$	local optima w.r.t. $\tilde{B}^1$ perform a differential ratio of $\Omega(1/\nu)$ and for all solutions x, the highest differential ratio achieved on $\tilde{B}^1(x)$ is $\Omega(1/n)$ times this bound
$\mathsf{Ek\,CSP}(\mathcal{I}^\mathsf{k-1}_\mathsf{q})$	the same lower bounds hold for the differential ratio reached at local optima w.r.t. Hamming balls of radius 1 as for the average differential ratio, and the highest differential ratio reached over any such ball is $\Omega(1/n)$ times this ratio
$\mathsf{k\,CSP{-}q}$	over any Hamming ball of radius k, the ratio of the maximum distance between two solution values to the instance diameter is $\Omega(1/n^k)$

constrained decomposition of a balanced k-wise independent function on Σ_q^q. By exhibiting such pairs (Theorem 3.8), we show that whenever $\mathsf{k\,CSP{-}k}$ is approximable within differential factor ρ, given any $q > k$, $\mathsf{k\,CSP{-}q}$ is approximable within differential factor $\rho/(q-k/2)^k$. It thus follows from [41] that for all constant integers $q \geq 2$, $\mathsf{2\,CSP{-}q}$ is differentially approximable within some constant factor.

• In section 4, we use similar designs as those of section 3 to evaluate the differential ratio reached by solutions with extremal value over Hamming balls with fixed radius k. We obtain a combinatorial identity that allows to express any solution value as a linear combination of solution values over any such ball (Theorem 4.8). Consequently, for $\mathsf{k\,CSP{-}q}$, every Hamming ball of radius k provides a pair of solutions whose difference in value is a fraction $\Omega(1/n^k)$ of the instance diameter $|\mathrm{opt}(I) - \mathrm{wor}(I)|$.

As is customary, we discuss the obtained results and the prospects they offer in a concluding section. We group together some technical arguments and side issues in a dedicated supplementary document.

We summarize in Tables 1 and 2 the resulting knowledge of the differential approximability of $\mathsf{k\,CSP{-}q}$, it restriction $\mathsf{k\,CSP}(\mathcal{I}^{\mathsf{t}}_{\mathsf{q}})$, and the restrictions $\mathsf{CSP}(\mathcal{O}_{\mathsf{q}})$ and $\mathsf{k\,CSP}(\mathcal{E}_{\mathsf{q}})$ of $\mathsf{CSP{-}q}$ and $\mathsf{k\,CSP{-}q}$ to function families $\mathcal{O}_q$ and $\mathcal{E}_q$ which we define immediately afterwards.

1.5. Notations that will be used in the rest of the paper. ***Function families $\mathcal{E}_q$ and $\mathcal{O}_q$.*** Given a positive integer k, function XOR^k is remarkable in that, given any two k-dimensional boolean vectors y and $\bar{y}$, we have either $XOR^k(y) = XOR^k(\bar{y})$, or $XOR^k(y) + XOR^k(\bar{y}) = 1$, depending on k mod 2. Consider that the number of nonzero coordinates in $\bar{y}$ has the same parity as the number of nonzero coordinates in y *if and only if* k is even. $\mathcal{E}_q$ and $\mathcal{O}_q$ provide some generalization to q-ary alphabets of such boolean predicates, namely: functions of $\mathcal{E}_q$ are stable under the shift by the same quantity of all their entries, while functions of $\mathcal{O}_q$ satisfy that their mean value over any q inputs $y, y+\mathbf{1}, \ldots, y+\mathbf{q-1}$ is equal to their mean value. Formally, given an integer $k \geq 1$, a function $P : \Sigma_q^k \to \mathbb{R}$ belongs to $\mathcal{E}_q$ and $\mathcal{O}_q$ whenever it satisfies respectively (1.3) and (1.4) below:

$$(1.3) \qquad P_{\mathbf{a}}(y) := P(y_1 + a, \ldots, y_k + a) = P(y_1, \ldots, y_k), \qquad y \in \Sigma_q^k,\ a \in \Sigma_q$$

$$(1.4) \qquad \textstyle\sum_{a=0}^{q-1} P(y_1 + a, \ldots, y_k + a)/q = r_P, \qquad y \in \Sigma_q^k$$

For example, let $AllEqual^{k,q}$ refer to the predicate on Σ_q^k that is true for entries $(y_1, \ldots, y_k)$ with $y_1 = \ldots = y_k$. Then $AllEqual^{k,q}$ clearly belongs to $\mathcal{E}_q$. Now consider equation $(y_1 + \ldots + y_k \equiv 0 \bmod q)$ over Σ_q^k. Given any $y = (y_1, \ldots, y_k) \in \Sigma_q^k$ and any $a \in \Sigma_q$, we have:

$$(y_1 + a) + \ldots + (y_k + a) \quad = (y_1 + \ldots + y_k) + ka$$

We deduce that $ZeroSum^{k,q} \in \mathcal{E}_q$ *iff* k is a multiple of q, and that $ZeroSum^{k,q} \in \mathcal{O}_q$ *iff* k and q are mutually prime[5].

Let $P \in \mathcal{E}_q$. By (1.3), the average value of P when one of its variables is set to a certain $a \in \Sigma_q$ does not depend on a specific choice of a; this value consequently is equal to r_P. Functions of $\mathcal{E}_q$ therefore belong to $\mathcal{I}_q^1$.

For some insight on families $\mathcal{E}_q$ and $\mathcal{O}_q$ and the corresponding CSPs, we invite the reader to refer to section SM1 of the supplement.

Arrays. Let ν and q be two positive integers, where $q \geq 2$. An array with ν columns on symbol set Σ_q is a multisubset of Σ_q^ν. Given an array M, a row index r and a column index c, M_r and M^c refer to respectively the row with index r and the column with index c of M. Given a sequence $J = (c_1, \ldots, c_t)$ of column indices, M^J refers to the subarray $(M^{c_1}, \ldots, M^{c_t})$. M is *simple* if no word $u \in \Sigma_q^\nu$ occurs more than once as a row in M.

For an $R \times \nu$ array M on Σ_q, we denote by μ^M the frequency of words of Σ_q^ν in M, *i.e.*:

$$(1.5) \qquad \mu^M(u) := |\{r \in [R] \,|\, M_r = u\}| / R, \qquad u \in \Sigma_q^\nu$$

μ^M defines a probability distribution on Σ_q^ν. We will also consider the function $\mu_E^M \in \mathcal{E}_q$ which associates with each $u \in \Sigma_q^\nu$ a fraction $1/q$ of the overall frequency in M of

[5]For $\mathcal{O}_q$, the number of integers $a \in \Sigma_q$ such that $ZeroSum^{k,q}(y + \mathbf{a})$ is true must be the same for all $y \in \Sigma_q^k$. Equivalently, the number of integers $a \in \Sigma_q$ such that $ak \equiv b \bmod q$ should be the same for all integers $b \in \Sigma_q$. If d refers to the greatest common divisor of k and q, then this number is d if d is a divisor of b and 0 otherwise. Since d is always a divisor of 0, we conclude that d must be 1.

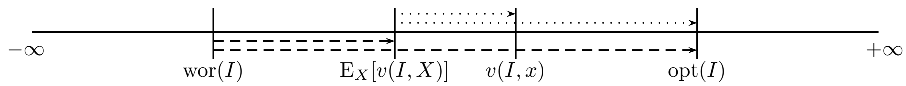

FIG. 1. *Quantities involved in the gain ratio achieved by a given solution (in dotted lines) and the average differential ratio (in dashed lines).*

words of the form $u + \mathbf{a}$. Formally:

$$\mu_E^M(u) := \textstyle\sum_{a=0}^{q-1} \mu^M(u + \mathbf{a})/q, \qquad u \in \Sigma_q^\nu \tag{1.6}$$

μ_E^M defines an alternate probability distribution on Σ_q^ν.

2. Differential approximation quality of the average solution value. Solutions with value at least $\mathrm{E}_X[v(I,X)]$ are computationally easy to find, using the *conditional expectation technique* [32]. The method, when applied to an instance I of $\mathsf{Max\,CSP-q}$, consists in associating a (new) random variable X_j to each variable x_j of I, and then iteratively fixing variables $x_j, j = 1, \dots, n$ to a symbol $a \in \Sigma_q$ that maximizes the conditional expectation:

$$\mathbb{E}_X\left[v(I,X) \,|\, (X_1, X_2, \dots, X_{j-1}, X_j) = (x_1, x_2, \dots, x_{j-1}, a)\right].$$

By proceeding in this way, provided that the variables X_j, $j \in [n]$ are independently distributed, we obtain a solution x with value:

$$\begin{aligned} v(I,x) &= \mathbb{E}_X\left[v(I,X) \,|\, X = x\right] \\ &\geq \mathbb{E}_X\left[(X_1, \dots, X_{n-1}) = (x_1, \dots, x_{n-1})\right] \quad \geq \dots \geq \quad \mathbb{E}_X[v(I,X)] \end{aligned}$$

In particular, the method returns a solution with value at least the average solution value when the variables X_j, $j \in [n]$ are uniformly distributed. Therefore, two questions can naturally be asked: is it possible to compute within polynomial time solutions that beat $\mathbb{E}_X[v(I,X)]$, and what is the gain of $\mathbb{E}_X[v(I,X)]$ over the worst solution value?

Approximating the optimum advantage over a random assignment is one way to address the former question. This involves determining the highest ρ for which a ρ-gain approximation algorithm exists. For example, $\mathsf{3\,Lin-2}$ is **NP−hard** to approximate to within any constant gain factor [30], but is approximable within an expected gain factor of $\Omega(\sqrt{1/m})$ [31]. The latter question is related to the *advantage of a random solution over a worst assignment*, which precisely is the average differential ratio. The issue here consists in determining the tightest possible lower bound for the average differential ratio. For example, on an instance I of $\mathsf{E3\,Lin-2}$, any equation is satisfied exactly once over any pair $\{x, \bar{x}\}$ of solutions. Hence, for this specific CSP, the average advantage over wor(I) is exactly one half of the instance diameter [22]. Figure 1 pictures the quantities involved in these two measures. The two questions are complementary, and the second has potential to shed light on the first. In particular, one might think that the further $\mathbb{E}_X[v(I,X)]$ is from wor(I), the harder it is to get away from it.

We here address the second question. We specifically seek lower bounds for the average differential ratio on instances of $\mathsf{k\,CSP-q}$ and its restrictions $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$ and $\mathsf{k\,CSP}(\mathcal{I}_\mathsf{q}^\mathsf{t})$. Note that such lower bounds also provide an estimate of the differential approximation guarantee offered by the conditional expectation technique.

2.1. Previous related works and preliminary remarks. We discuss three restrictions under which some lower bound for the average differential ratio is either already known, or obvious. In [22], Escoffier and Paschos analyze the differential ratio achieved by solutions returned by the conditional expectation technique on unweighted instances of Sat. They observed that on such an instance I on which the goal is to maximize, provided that $\mathrm{opt}(I) \neq \mathrm{wor}(I)$, we have:

$$\lceil \mathbb{E}_X[v(I,X)] \rceil \quad \geq \mathrm{wor}(I) + 1 \quad \geq \mathrm{wor}(I) + (\mathrm{opt}(I) - \mathrm{wor}(I))/m$$

The solution returned by the conditional expectation technique therefore is $(1/m)$-differential approximate on I. Besides, the argument extends for all integers $q \geq 2$ to the restriction of $\mathsf{CSP}{-}\mathsf{q}$ with integer solution values and a polynomially bounded diameter. However, the argument does not apply to instances that manipulate arbitrary weights.

Given a positive integer n, a function $P : \{0,1\}^n \to \mathbb{R}$ is *submodular* if and only if it satisfies:

$$P(y) + P(z) \quad \geq P(y_1 \vee z_1, \ldots, y_n \vee z_n) + P(y_1 \wedge z_1, \ldots, y_n \wedge z_n), \quad y, z \in \{0,1\}^n$$

Feige et al. demonstrated in [23], that for all submodular functions P, if x^* refers to a maximizer of P, then :

$$\mathbb{E}_X[P(X)] \geq P(x^*)/4 + P(\bar{x}^*)/4 + P(\mathbf{0})/4 + P(\mathbf{1})/4 \tag{2.1}$$

Since a conical combination of submodular pseudo-boolean functions is submodular, inequality (2.1) notably holds when P is the objective function $v(I,.)$ of an instance of $\mathsf{Max\,CSP}{-}2$ in which functions P_i all are submodular. Considering that none of the solution values $v(I,\bar{x}^*)$, $v(I,\mathbf{0})$, $v(I,\mathbf{1})$ can be less than $\mathrm{wor}(I)$, we deduce that the average differential ratio is at least $1/4$ on such instances. This ratio is even bounded below by $1/2$ on submodular instances of $\mathsf{Max\,CSP}(\mathcal{E}_2)$, on which $v(I,\bar{x}^*) = \mathrm{opt}(I)$. The *Maximum Directed Cut*, $\mathsf{Max\,Di\,Cut}$ is the restriction of $\mathsf{Max\,2\,CCSP}{-}2$ to clauses of the form $(x_{i_1} \wedge \bar{x}_{i_2})$, while the boolean *Not-All-Equal Satisfiability Problem* ($\mathsf{NAE\,Sat}$) is the restriction of $\mathsf{CSP}{-}2$ to constraints of the form $\neg(\ell_{i_1} = \ldots = \ell_{i_{k_i}})$. Then submodular CSPs notably include $\mathsf{Max\,Di\,Cut}$, the restriction — known as the *satisfiability problem with no mixed clause* — of $\mathsf{Max\,Sat}$ to constraints of the form $(x_{i_1} \vee \ldots \vee x_{i_{k_i}})$ or $(\bar{x}_{i_1} \vee \ldots \vee \bar{x}_{i_{k_i}})$, and the restriction — known as the *monotone not-all-equal satisfiability problem* — of $\mathsf{Max\,NAE\,Sat}$ to constraints of the form $\neg(x_{i_1} = \ldots = x_{i_{k_i}})$.

We now consider an optimization problem where the objective is to optimize a function $P \in \mathcal{O}_q$ over Σ_q^n. According to the definition of $\mathcal{O}_q$, we have:

$$\mathbb{E}_X[P(X)] = r_P = (P(x) + P(x+\mathbf{1}) + \ldots + P(x + \mathbf{q} - \mathbf{1}))/q, \qquad x \in \Sigma_q^n \tag{2.2}$$

Let I be an instance of $\mathsf{Max\,CSP}(\mathcal{O}_\mathsf{q})$. Since any linear combination of functions of $\mathcal{O}_q$ still belongs to $\mathcal{O}_q$, relation (2.2) particularly holds when $P = v(I,.)$ and x is an optimal solution of I. This implies that the differential ratio achieved at the average solution value on I is at least $1/q$.

The assumptions of uniformity of weightings, submodularity or membership in $\mathcal{O}_q$ are quite restrictive. Let us shift our focus to instance I of $\mathsf{Max\,Ek\,CSP}{-}\mathsf{q}$. On I, the average solution value expresses as:

$$\mathbb{E}_X[v(I,X)] \quad = \textstyle\sum_{i=1}^m w_i \times \sum_{v \in \Sigma_q^k} P_i(v)/q^k \quad = \sum_{i=1}^m w_i r_{P_i}$$

For example, if I is an instance of $\mathsf{Lin}-2$, then the average solution value on I is equal to $\sum_{i=1}^m w_i \times 1/2$. Denoting x^* a solution with optimal value on I, we observe:

$$\sum_{i=1}^m w_i r_{P_i} = \sum_{i=1}^m w_i \left(P_i(x^*_{J_i}) + \sum_{v\in\Sigma_q^k : v\neq x^*_{J_i}} P_i(v)\right)/q^k \quad \Leftrightarrow$$

$$\mathbb{E}_X[v(I,X)] = v(I,x^*)/q^k + \sum_{i=1}^m w_i \sum_{v\in\Sigma_q^k : v\neq x^*_{J_i}} P_i(v)/q^k \tag{2.3}$$

Hence, on I, provided that $w_i P_i \geq 0$, $i\in[m]$, the average solution value is a fraction at least $1/q^k$ of the optimum value. This value is an even greater fraction of $\mathrm{opt}(I)$ if we restrict the functions that occur in the constraints to $\mathcal{E}_q \cup_{t=1}^{k-1} \mathcal{I}_q^t$. Assume first that functions P_i, $i\in[m]$ are all balanced t-wise independent, where t is some positive integer less than or equal to k. Given $i\in[m]$, if $J_i = (i_1,\ldots,i_k)$, then we denote by $L_i = (i_1,\ldots,i_t)$ and by $R_i = (i_{t+1},\ldots,i_k)$ respectively the t first and the $k-t$ last elements of J_i. Substituting into (1.2), we deduce that the average solution value on I satisfies:

$$\begin{aligned}\sum_{i=1}^m w_i r_{P_i} &= \sum_{i=1}^m w_i \times \sum_{v\in\Sigma_q^{k-t}} P_i(x^*_{L_i}, v)/q^{k-t} \qquad \text{by (1.2)}\\ &= \sum_{i=1}^m w_i \left(P_i(x^*_{L_i}, x^*_{R_i}) + \sum_{v\in\Sigma_q^{k-t} : v\neq x^*_{R_i}} P_i(x^*_{L_i}, v)\right)/q^{k-t} \quad \Leftrightarrow\end{aligned}$$

$$\mathbb{E}_X[v(I,X)] = v(I,x^*)/q^{k-t} + \sum_{i=1}^m w_i \sum_{v\in\Sigma_q^{k-t} : v\neq x^*_{R_i}} P_i(x^*_{L_i}, v)/q^{k-t} \tag{2.4}$$

Since $\mathcal{E}_q \subseteq \mathcal{I}_q^1$, equality (2.4) in particular holds with $t=1$ on instances of $\mathsf{Ek\,CSP}(\mathcal{E}_\mathsf{q})$.

From the equalities (2.3) and (2.4), we deduce that for any three integers $q\geq 2$, $k\geq 2$ and $t\in[k-1]$, on any instance of $\mathsf{Max\,Ek\,CSP}-\mathsf{q}$, $\mathsf{Max\,Ek\,CSP}(\mathcal{I}_\mathsf{q}^\mathsf{t})$ and $\mathsf{Max\,Ek\,CSP}(\mathcal{E}_\mathsf{q})$ in which the constraints are all nonnegative, the average standard ratio is at least $1/q^k$, $1/q^{k-t}$ and $1/q^{k-1}$, respectively.

Nevertheless, a similar deduction cannot be made for the average differential ratio. Specifically, in the most general case, we can not assert that the quantities $\mathbb{E}_X[v(I,X)] - v(I,x^*)/q^k$ and $\mathbb{E}_X[v(I,X)] - v(I,x^*)/q^{k-t}$ which appear in the right-hand side of these equalities are the average of the values of q^k-1 and $q^{k-t}-1$ solutions respectively.

The $1/q$ ratio for $\mathsf{CSP}(\mathcal{O}_\mathsf{q})$ results from the fact that q solutions are sufficient to evaluate the average value of all the solutions, and that for any solution x, there exists such a set of q solutions containing x. Taking inspiration from this singular case, we evaluate the average differential ratio on instances of $\mathsf{k\,CSP}-\mathsf{q}$ and its restrictions $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$ and $\mathsf{k\,CSP}(\mathcal{I}_\mathsf{q}^\mathsf{t})$. We adopt a kind of neighbourhood approach: we associate with each solution x of I a multiset $\mathcal{X}(I,x)$ of solutions having the same mean solution value as the set of solutions, with relatively small size R, in which x appears a certain number $R^*>0$ of times. Taking $\mathcal{X}(I,.)$ at an optimal solution x^*, we deduce that the average differential ratio on I is at least R^*/R.

2.2. Partition-based solution multisets. Given an instance I of $\mathsf{CSP}-\mathsf{q}$ on n variables, we consider the following framework in order to construct our solution multisets $\mathcal{X}(I,x)$, $x\in\Sigma_q^n$.

• **Solution multiset association.** With a partition $\mathcal{V} = \{V_1,\ldots,V_\nu\}$ of $[n]$, a solution $x\in\Sigma_q^n$ and a vector $u\in\{0,\ldots,q-1\}^\nu$, we associate the solution $y(\mathcal{V},x,u)$ defined by:

$$(y(\mathcal{V},x,u)_{V_1},\ldots,y(\mathcal{V},x,u)_{V_\nu}) = (x_{V_1}+\mathbf{u_1},\ldots,x_{V_\nu}+\mathbf{u}_\nu)$$

That is, the solution $y(\mathcal{V},x,u)$ is obtained from x by shifting each of its coordinates in V_c by u_c, for each $c\in[\nu]$. Notably, when u is the all-zeros vector, $y(\mathcal{V},x,\mathbf{0})$ coincides with x. We then consider arrays with ν columns on Σ_q. To such an $R\times\nu$ array M, we associate the solution multiset:

$$\mathcal{X}(I,x) \quad = (y(\mathcal{V},x,M_r)\,|\,r\in[R])$$

• **Conditions.** To ensure that solution values $(v(I,y(\mathcal{V},x,M_r))\,|\,r\in[R])$ cover the optimum solution value provided that x is optimal, the all-zeros vector must occur at least once as a row in M. Note that for any $(u,a)\in\{0,\dots,q-1\}^{\nu+1}$, $y(\mathcal{V},x,u+\mathbf{a})$ equals $y(\mathcal{V},x,u)+\mathbf{a}$. Hence, when considering the restriction $\mathsf{CSP}(\mathcal{E}_\mathsf{q})$ of $\mathsf{CSP}{-}\mathsf{q}$, we only require that M contains at least one row of the form $\mathbf{a}$.

Since our ultimate goal is to connect the average solution value to the optimum solution value, the mean of the solution values over $(y(\mathcal{V},x,M_r)\,|\,r\in[R])$ should match the mean of the solution values over Σ_q^n. Specifically, we need M to satisfy:

$$\textstyle\sum_{r=1}^R v(I,y(\mathcal{V},x,M_r))/R=\mathbb{E}_X[v(I,X)], \qquad x\in\Sigma_q^n \tag{2.5}$$

When such a case occurs, the average differential ratio on I is at least $\mu^M(\mathbf{0})$. Indeed, let x^* be an optimal solution of I. We assume *w.l.o.g.* that the goal on I is to maximize. Then we have:

$$\begin{aligned}\mathbb{E}_X[v(I,X)] \quad &= \textstyle\sum_{r=1}^R v(I,y(\mathcal{V},x^*,M_r))/R \qquad \text{by (2.5)}\\ &\geq \mu^M(\mathbf{0})\times v(I,x^*)+\left(1-\mu^M(\mathbf{0})\right)\times \mathrm{wor}(I)\end{aligned}$$

In case where $v(I,.)\in\mathcal{E}_q$, we similarly obtain a lower bound of $\sum_{a=0}^{q-1}\mu^M(\mathbf{a})$ for the average differential ratio.

Assuming that $\mathcal{V}$ is either given or computable within polynomial time, picking a solution with maximum value over $\{y(\mathcal{V},\mathbf{0},M_r)\,|\,r\in[R]\}$ yields the same differential approximation guarantee as the average solution value. For example, on an instance I of $\mathsf{CSP}(\mathcal{O}_\mathsf{q})$, relation (2.2) suggests to consider the partition $\mathcal{V}=\{[n]\}$ of $[n]$ and the array M on Σ_q defined by:

$$M \quad = \begin{pmatrix} 0\\ 1\\ \vdots\\ q-1\end{pmatrix}$$

This array satisfies $\mu^M(\mathbf{0})=1/q$.

Notice that given any $q\geq 2$, the predicate on Σ_q^3 that accepts solutions to equation $(y_1+y_2-y_3\equiv 0 \bmod q)$ belongs to $\mathcal{O}_q\cap\mathcal{I}_q^2$. As the q^2 accepting entries of this predicate constitute a subgroup of $\mathbb{Z}_q^3$, it follows from [13] that $3{-}\mathsf{partite\,E3\,CSP}(\mathcal{O}_\mathsf{q}\cap\mathcal{I}_\mathsf{q}^2)$ is **NP−hard** to differentially approximate within any constant factor greater than $1/q$. Thus, under the assumption $\mathbf{P}\neq\mathbf{NP}$, with regard to the approximability of $\mathsf{CSP}(\mathcal{O}_\mathsf{q})$ within some constant differential factor, no polynomial time algorithm can outperforme the trivial strategy of selecting a solution with maximum value over $\{\mathbf{0},\dots,\mathbf{q-1}\}$.

2.3. Orthogonal arrays and Difference schemes.

Definition 2.1 (*e.g.* see [28]). *Let q, t, $\nu\geq t$ be three positive integers, and R be a multiple of q^t. Then an $R\times\nu$ array M with entries in Σ_q is a q-*levels Orthogonal Array of strength t with ν factors and R runs*, an $OA(R,\nu,q,t)$ for short, if given*

TABLE 3
An $OA(3^2, 3, 3, 2)$ (on the left) and a $D_2(3^1, 2, 3)$ (on the right).

M^1	M^2	M^3
0	0	0
0	1	2
0	2	1
1	0	2
1	1	1
1	2	0
2	0	1
2	1	0
2	2	2

M^1	M^2
0	0
0	1
0	2

any sequence $J = (c_1, \ldots, c_t)$ of pairwise distinct column indices, the rows of subarray M^J coincide equally often with every $u \in \Sigma_q^t$. Formally, M shall satisfy:

$$(2.6)\qquad |\{r \in [R] \,|\, M_r^J = v\}| = R/q^t, \qquad J \subseteq [\nu], |J| = t,\ v \in \Sigma_q^t$$

Equivalently, M is an $OA(R, \nu, q, t)$ *iff* μ^M is balanced t-wise independent (what implies $\mu^M \in \mathcal{I}_q^t$). Hence, rational-valued balanced t-wise independent distributions over Σ_q^ν and orthogonal arrays of strength t with ν columns on the symbol set Σ_q are essentially equivalent, viewing the frequency of the words of Σ_q^ν in the array as a distributions over Σ_q^ν. Specifically, balanced t-wise independent subsets $\mathcal{Y}$ of Σ_q^ν correspond precisely to the simple orthogonal arrays of strength t with ν columns and coefficients in Σ_q. For instance, the accepting entries of $ZeroSum^{t+1,q}$ form the rows of a simple $OA(q^t, t+1, q, t)$ on the symbol set Σ_q. (The corresponding array for $q = 3$ and $t = 2$ is illustrated on the left-hand side of Table 3.)

DEFINITION 2.2 (*e.g.* see [28]). *Let $t \geq 1$, $\nu \geq t$, $R \geq 1$, $q \geq 2$ where R is a multiple of q^{t-1} be four integers. Then an $R \times \nu$ array M with entries in Σ_q is a* Difference schemes of strength t *based on $(\mathbb{Z}_q, +)$, a $D_t(R, \nu, q)$ for short, if given any sequence $J = (c_1, \ldots, c_t)$ of pairwise distinct column indices, the rows of subarray M^J lie equally often on each subset $\{u, u + \mathbf{1}, \ldots, u + \mathbf{q-1}\}, u \in \Sigma_q^t$ of words. Formally, M shall satisfy:*

$$(2.7)\quad \textstyle\sum_{a=0}^{q-1} |\{r \in [R] \,|\, M_r^J = v + \mathbf{a}\}| = R/q^{t-1}, \quad J \subseteq [\nu], |J| = t,\ v \in \{0\} \times \Sigma_q^{t-1}$$

Equivalently, M is a $D_t(R, \nu, q)$ *iff* μ_E^M is balanced t-wise independent (what implies $\mu_E^M \in \mathcal{E}_q^t \cap \mathcal{I}_q^t$). Table 3 pictures the trivial $D_t(q^{t-1}, t, q)$ when $q = 3$ and $t = 2$. Difference schemes can be seen as a slight relaxation of Orthogonal Arrays. For some insight on such arrays and their connections to orthogonal arrays, we invite the reader to refer to [28].

2.4. Connecting the average differential ratio to orthogonal arrays. Consider an instance I of $\mathsf{k\,CSP{-}q}$. A sufficient condition for the average solution value over $(y(\mathcal{V}, x, M_r) \,|\, r \in [R])$ to coincide with the average solution value over Σ_q^n is that for each constraint $P_i(x_{J_i})$ of I, its average value over $(y(\mathcal{V}, x, M_r)_{J_i} \,|\, r \in [R])$ coincides with r_{P_i}. Hence, from now on, we will be looking for pairs $(\mathcal{V}, M)$ that satisfy:

$$(2.8)\qquad \textstyle\sum_{r=1}^R P_i(y(\mathcal{V}, x, M_r)_{J_i})/R = r_{P_i}, \qquad i \in [m],\ x \in \Sigma_q^n$$

Let $P_i(x_{J_i}) = P_i(x_{i_1}, \ldots, x_{i_{k_i}})$ be a constraint of I. A sufficient condition for a pair $(\mathcal{V}, M)$ to satisfy (2.8) at i is that over the solution multiset $(y(\mathcal{V}, x, M_r) \,|\, r \in [R])$,

P_i is evaluated the same number of times on each of its possible entries. Firstly, we observe that such a condition requires that any two distinct indices $j, h \in J_i$ belong to two distinct sets of $\mathcal{V}$ (as otherwise, over $[R]$, the difference $y(\mathcal{V}, x, M_r)_j - y(\mathcal{V}, x, M_r)_h$ between the corresponding coordinates is the constant $x_j - x_h$). Therefore, we assume that $\mathcal{V} = \{V_1, \ldots, V_\nu\}$ is a strong coloring of I, and M is an array on ν columns. We denote by $H = (c_1, \ldots, c_{k_i})$ the sequence of color indexes satisfying that $(i_1, \ldots, i_{k_i}) \in V_{c_1} \times \ldots \times V_{c_{k_i}}$. Thus given $r \in [R]$, on solution $y(\mathcal{V}, x, M_r)$, P_i is evaluated at $y(\mathcal{V}, x, M_r)_{J_i} = x_{J_i} + M_r^H$. Then, secondly, we observe that the vectors of $(x_{J_i} + M_r^H \,|\, r \in [R])$ coincide equally often with each $v \in \Sigma_q^{k_i}$ *iff* the words of $(M_r^H \,|\, r \in [R])$ coincide equally often with each $v \in \Sigma_q^{k_i}$. Since H can be any at most k-cardinality subset of $[\nu]$, we deduce that $(\mathcal{V}, M)$ satisfies (2.8) provided that M is an orthogonal array of strength k.

Now assume that $P_i \in \mathcal{E}_q$, which means that P_i evaluates the same on any two entries $(y_1, \ldots, y_{k_i})$ and $(y_1 + a, \ldots, y_{k_i} + a)$. In this case, a sufficient condition for $(\mathcal{V}, M)$ to satisfy (2.8) at i is that the vectors of $(x_{J_i} + M_r^H \,|\, r \in [R])$ belong equally often to each subset $\{v, v + \mathbf{1}, \ldots, v + \mathbf{q-1}\}$, $v \in \Sigma_q^{k_i}$. Now, this occurs *iff* the vectors of $(M_r^H \,|\, r \in [R])$ belong equally often to each subset $\{v, v+\mathbf{1}, \ldots, v+\mathbf{q-1}\}$, $v \in \Sigma_q^{k_i}$. We deduce that $(\mathcal{V}, M)$ satisfies (2.8) provided that M is a difference scheme of strength k.

Finally assume that P_i is balanced t-wise independent where t is some positive integer strictly less than k. This means that we can fix (to x_j) the value of up to t variables with index $j \in J_i$, and still obtain the average value of P_i when averaging the value taken by P_i over all possible assignments for the remaining variables. Hence, rather than $\mathcal{V}$, we consider the partition $\mathcal{U} = \{V_1, \ldots, V_{\nu-t}, U_{\nu-t+1}\}$ of $[n]$ where $U_{\nu-t+1} = V_{\nu-t+1} \cup \ldots \cup V_\nu$. Moreover, M is an array on $\nu - t + 1$ columns, the last of which contains only zeros. Under these assumptions, given $r \in [R]$, $y(\mathcal{V}, x, M_r)_{J_i}$ can be described as the vector $(v_{i_1}, \ldots, v_{k_i})$ of $\Sigma_q^{k_i}$ defined for $s \in [k_i]$ by $v_{i_s} = x_{i_s}$ if $i_s \in U_{\nu-t+1}$, and $v_{i_s} = x_{i_s} + M_r^{c_s}$ otherwise. By construction, at most t indexes $j \in J_i$ can belong to $U_{\nu-t+1}$ (as these indexes originate from at most t colors sets $V_{\nu-t+1}, \ldots, V_\nu$). Let $L = H \cap [\nu - t]$ and $s = |L|$. Then we deduce from the preceding observations that a sufficient condition for $(\mathcal{U}, M)$ to satisfy (2.8) at i is that the vectors of $(M_r^L \,|\, r \in [R])$ coincide equally often with each $v \in \Sigma_q^s$. Since L can be any subset of $[\nu]$ with cardinality at most $\min\{k, \nu - t\}$, we conclude that $(\mathcal{U}, M)$ satisfies (2.8) provided that the $\nu - t$ first columns of M constitute an orthogonal array of strength $\min\{k, \nu - t\}$.

For example, on a k-partite instance I of $\mathsf{k\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k-1}})$, we consider the partition $\mathcal{U} = \{V_1, [n] \backslash V_1\}$ of $[n]$ where V_1 is a color set of a strong coloring of I, and the array M on Σ_q defined by:

$$M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ q-1 & 0 \end{pmatrix}$$

We deduce from the preceding analysis that for all integers $q \geq 2$ and $k \geq 2$, $\mathsf{k-partite\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k-1}})$ is trivially approximable within differential factor $\mu^M(\mathbf{0}) = 1/q$. Moreover, as $P = ZeroSum^{k,q}$ fulfills the requirements of Theorem 1.1, this constant factor is optimal assuming $\mathbf{P} \neq \mathbf{NP}$.

Notice that by shifting every row of an $OA(R, \nu, q, t)$ M on Σ_q by $-u^*$ where u^* is a maximizer of μ^M, one obtains an $OA(R, \nu, q, t)$ N on which $\mathbf{0}$ is a maximizer of μ^N.

TABLE 4
Arrays that achieve $\rho(\nu,3,2)$ and $\rho_E(\nu,3,2)$ when $\nu \in \{3,4\}$. As regards relation (2.15), observe that $\rho_E(4,3,2) > \rho(3,3,2)$ and $3\times\rho(4,3,2) > \rho_E(4,3,2)$, whereas $3\times\rho(3,3,2) = \rho_E(3,3,2)$.

$\rho_E(3,3,2) = 1/3$

M^1	M^2	M^3
0	0	0
0	1	2
0	2	1

$\rho(3,3,2) = 1/9$

M^1	M^2	M^3
0	0	0
0	1	2
0	2	1
1	1	1
1	2	0
1	0	2
2	2	2
2	0	1
2	1	0

$\rho_E(4,3,2) = 1/5$

M^1	M^2	M^3	M^4
0	0	0	0
0	0	0	0
0	0	0	0
0	0	1	2
0	0	2	1
0	1	0	2
0	1	1	2
0	1	2	0
0	1	2	1
0	1	2	2
0	2	0	1
0	2	1	0
0	2	1	1
0	2	1	2
0	2	2	1

$\rho(4,3,2) = 1/9$

M^1	M^2	M^3	M^4
0	0	0	0
0	1	2	2
0	2	1	1
1	1	1	0
1	2	0	2
1	0	2	1
2	2	2	0
2	0	1	2
2	1	0	1

Therefore, we can always assume given an $OA(R,\nu,q,t)$ M that μ^M is maximized at $\mathbf{0}$. We can similarly assume *w.l.o.g.* given an $R\times\nu$ difference scheme M on Σ_q that μ_E^M is maximized at $\mathbf{0}$. Hence, as regards such arrays, we are interested in *maximal* frequencies rather that in the frequency of a precise vector v or a precise vector family $\{v, v+\mathbf{1},\ldots,v+\mathbf{q}-\mathbf{1}\}$. We introduce the following numbers:

DEFINITION 2.3. *For three positive integers q, ν and $t\in[\nu]$, we denote by $\rho(\nu,q,t)$ the greatest number ρ for which there exists an orthogonal array M with ν columns of strength t on symbol set Σ_q such that:*

$$\max_{v\in\Sigma_q^\nu}\left\{\mu^M(v) := |\{r\in[R] : M_r = v\}|/R\right\} = \rho \tag{2.9}$$

Similarly, we denote by $\rho_E(\nu,q,t))$ the greatest number ρ for which there exists a difference scheme M with ν columns of strength t on Σ_q such that:

$$\max_{v\in\{0\}\times\Sigma_q^{\nu-1}}\left\{\mu_E^M(v) := \mu^M(v)+\mu^M(v+\mathbf{1})+\ldots+\mu^M(v+\mathbf{q}-\mathbf{1})\right\} = \rho \tag{2.10}$$

Tables 4 and 5 depict a few arrays that achieve number either $\rho(\nu,q,t)$ or $\rho_E(\nu,q,t)$. The preceding discussion establishes the following connection between these numbers and the average differential ratio on $\mathsf{k\,CSP{-}q}$ instances:

THEOREM 2.4. *For all integers $q\geq 2$, $k\geq 2$, $t\in[k-1]$ and $\nu\geq k$, on every ν-partite instance of $\mathsf{k\,CSP{-}q}$, $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$ and $\mathsf{k\,CSP}(\mathcal{I}_\mathsf{q}^\mathsf{t})$, the average differential ratio is at least $\rho(\nu,q,k)$, $\rho_E(\nu,q,k)$ and $\rho(\nu-t,q,\min\{k,\nu-t\})$, respectively.*

2.5. Lower bounds for numbers $\rho(\nu,q,k)$ and $\rho_E(\nu,q,k)$. We deduce lowers bounds for numbers $\rho(\nu,q,t)$ and $\rho_E(\nu,q,t)$ from orthogonal arrays and difference schemes, most often simple, of the literature. The minimum number of rows in an orthogonal array of strength t with ν columns on a set of q symbols is referred to as $F(\nu,q,t)$ in the literature [28]. We similarly denote by $E(\nu,q,t)$ the minimum number of rows in a difference scheme of strength t with ν columns on Σ_q. It is worth noting that for all triples (ν,q,t) of positive integers, we have the obvious inequalities:

$$\rho(\nu,q,t) \geq 1/F(\nu,q,t), \qquad \rho_E(\nu,q,t) \geq 1/E(\nu,q,t) \tag{2.11}$$

Table 5

Arrays that achieve $\rho(\nu-1,2,2)$ and $\rho_E(\nu,2,2)=\rho_E(\nu,2,3)$ when $\nu\in\{4,5\}$. For $\nu=4$ (on the left) and $\nu=5$ (on the right), the two arrays M and (M,u) illustrate equality $\rho_E(\nu,2,2t)=\rho(\nu-1,2,2t)$ of relation (2.16) with $t=1$. In both cases, if we applied transformation (2.14) to array (M,u), we would obtain an orthogonal array that achieves $\rho(\nu,2,3)=\rho_E(\nu,2,2)/2$.

$\rho(3,2,2)=1/4$

M^1	M^2	M^3
0	0	0
0	1	1
1	0	1
1	1	0

$\rho_E(4,2,2)=\rho_E(4,2,3)=1/4$

M^1	M^2	M^3	u
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	0

$\rho(4,2,2)=1/6$

M^1	M^2	M^3	M^4
0	0	0	0
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0

$\rho_E(5,2,2)=\rho_E(5,2,3)=1/6$

M^1	M^2	M^3	M^4	u
0	0	0	0	0
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0

We first present some well-known facts about such arrays that provide useful relationships between numbers $F(\nu,q,t)$ and $E(\nu,q,t)$ on the one hand, $\rho(\nu,q,t)$ and $\rho_E(\nu,q,t)$ on the other hand.

Property 2.5 (*e.g.* see [28]). With an $R\times\nu$ array M on Σ_q, associate the three arrays $A(M),B(M),C(M)$ on Σ_q defined by:

$$A(M) := (M_r^{[\nu-1]} \,|\, r\in[R] : M_r^{\nu}=0) \tag{2.12}$$
$$B(M) := ((M_r,0) \,|\, r\in[R]) \tag{2.13}$$
$$C(M) := (M_r+\mathbf{a} \,|\, r\in[R], a\in\{0,\dots,q-1\}) \tag{2.14}$$

The following facts hold given four arrays M, $A(M)$, $B(M)$ and $C(M)$:

1. if M is an $OA(R,\nu,q,t)$, then $A(M)$ is an $OA(R/q,\nu-1,q,t-1)$;
2. if M is an $OA(R,\nu,q,t)$, then $B(M)$ is a $D_t(R,\nu+1,q)$;
3. M is a $D_t(R,\nu,q)$ *iff* $C(M)$ is an $OA(q\times R,\nu,q,t)$.

For example, on both sides of Table 5, array (M,u) is the map by B of array M. In Table 4, the second array is the map by C of the first array.

Property 2.5 implies for all integers $q\geq 2$, $t\geq 1$ and $\nu\geq t$ the two relations below:

$$\left\{\begin{array}{llll} E(\nu,q,t) & \leq F(\nu-1,q,t) & \leq 1/q\times F(\nu,q,t+1) & \leq E(\nu,q,t+1) \\ \rho_E(\nu,q,t) & \geq \rho(\nu-1,q,t) & \geq q\times\rho(\nu,q,t+1) & \geq \rho_E(\nu,q,t+1) \end{array}\right. \tag{2.15}$$

Over a binary alphabet, numbers $F(\nu,2,t)$ and $E(\nu,2,t)$ on the one hand, $\rho(\nu,2.t)$ and $\rho_E(\nu,2,t)$ on the other hand, are more closely related.

Property 2.6 (*e.g.* see [28]). If M is a difference scheme of even strength $2t$ on a binary alphabet, then M actually has strength $2t+1$.

Accordingly, given any two integers $t\geq 1$ and $\nu\geq 2t$, we have:

$$E(\nu,2,2t+1) \quad = E(\nu,2,2t) \quad \text{and} \quad \rho_E(\nu,2,2t+1) \quad = \rho_E(\nu,2,2t)$$

Property 2.6 and inequalities (2.15) consequently imply for all integers $t\geq 1$ and $\nu\geq 2t+1$ the two relations below:

$$\left\{\begin{array}{llll} E(\nu,2,2t) & = F(\nu-1,2,2t) & = F(\nu,2,2t+1)/2 & = E(\nu,2,2t+1) \\ \rho_E(\nu,2,2t) & = \rho(\nu-1,2,2t) & = 2\rho(\nu,2,2t+1) & = \rho_E(\nu,2,2t+1) \end{array}\right. \tag{2.16}$$

(A proof of relations (2.15) and Property 2.6 can be found in section SM3 of the supplement.) Arrays of Tables 4 and 5 provide some illustration of relations (2.15) and (2.16).

We now review upper bounds for the quantities $F(q,\nu,t)$, $E(q,\nu,t)$, $\rho(q,\nu,t)$ and $\rho_E(q,\nu,t)$ from the literature, starting with small values of ν. For the initial values of ν, we observe the equalities $F(t+1,q,t) = F(t,q,t) = q^t$ and $E(t,q,t) = q^{t-1}$. Bush exhibits in [12] other triples (ν,q,t) for which $F(\nu,q,t)$ is still equal to q^t:

THEOREM 2.7 ([12]). *Let $q \geq 2$, $t \geq 2$ and $\nu \geq t$ be three integers. Then $F(\nu,q,t) = q^t$ if $\nu \leq t+1$, or q is a prime power greater than t and $\nu \leq q+1$, or $t = 3$, q is a power of two greater than 3 and $\nu \leq q+2$.*

For greater integers ν, Colbourn et al. explicitly study in [15] the existence of orthogonal arrays that maximize their maximum frequency in the restrictive case of $t = 2$. They notably prove the following theorem:

THEOREM 2.8 ([15]). *Let $q \geq 2$ and $\nu \geq q$ be two integers such that ν is 1 or 0 modulo q. Then $1/\rho(\nu,q,2)$ is equal to:*

$$\begin{cases} \nu(q-1)+1 & \text{if } \nu \equiv 1 \bmod q \\ \nu(q-1)+q & \text{if } \nu \equiv 0 \bmod q \end{cases}$$

According to Theorem 2.8, $\rho(\nu,2,2)$ equals $1/(\nu+1)$ if ν is odd and $1/(\nu+2)$ otherwise. In addition, Property 2.6 allows us to derive the exact values of $\rho(\nu,2,3)$ and $\rho_E(\nu,2,3)$ from Theorem 2.8:

COROLLARY 2.9. *For all integers $\nu \geq 3$, $1/\rho(\nu,2,3) = 2\nu$ if ν is even and $2(\nu+1)$ otherwise. Equivalently, $1/\rho_E(\nu,2,3) = \nu$ if ν is even and $\nu+1$ otherwise.*

For values of t greater than 2, upper bounds for $F(\nu,q,t)$ and $E(\nu,q,t)$ are derived from infinite families of *linear codes.*

DEFINITION 2.10. *Given two positive integers L, r and a prime power q, a q-ary* linear code *$\mathcal{C}$ of* length *L and* dimension *r is a r-dimensional subspace of $\mathbb{F}_q^L$. The* distance *of $\mathcal{C}$ is the minimal Hamming distance between two vectors of $\mathcal{C}$. The vectors $v \in \mathbb{F}_q^L$ such that $\sum_{j=1}^L v_j c_j = 0$, $c \in C$ are the codewords of a linear code termed the* dual code *of $\mathcal{C}$.*

For binary alphabets (i.e. with $q = 2$), binary *BCH codes* — where BCH stands for Bose, Ray-Chaudhuri and Hocquenghem — provide the following upper bound of $F(\nu,2,2t)$:

THEOREM 2.11 ([29, 11, 20]). *For all positive integers $t \geq 1$ and $\nu \geq \max\{2t+1,7\}$ such that $\nu+1$ is a power of 2,*

$$F(\nu,2,2t) \leq (\nu+1)^t \tag{2.17}$$

Accordingly, for all positive integers $t \geq 1$ and $\nu \geq \max\{2t+2,8\}$ such that ν is a power of 2, $E(\nu,2,2t+1) \leq \nu^t$ and $F(\nu,2,2t+1) \leq 2\nu^t$.

Proof. Given two positive integers $\kappa \geq 3$ and t such that $2^\kappa - 1 \geq 2t+1$, the *primitive binary BCH code* of length $2^\kappa - 1$ and *design distance* $2t+1$ is a binary linear code with dimension at least $2^\kappa - 1 - t\kappa$, of distance at least $2t+1$ (*e.g.* see [36]). Delsarte's Theorem [20] states that, if $\mathcal{C}$ is a linear code of length L, dimension r and distance d over $\mathbb{F}_q$, then the codewords of its dual form a simple $OA(q^{L-r}, L, q, d-1)$. Considering $q = 2$, $L = 2^\kappa - 1$, $r \geq 2^\kappa - 1 - t\kappa$ and $d \geq 2t+1$, there thus exists an

$OA(R, 2^\kappa - 1, 2, 2t)$ where $R \le 2^{t\kappa}$. Applying transformation (2.13) of Property 2.5 to this orthogonal array, one obtains a $D_{2t}(R, 2^\kappa, 2)$ which, according to Property 2.6, is a $D_{2t+1}(R, 2^\kappa, 2)$. Utilizing the transformation (2.14) of Property 2.5 on this difference scheme results in an $OA(2R, 2^\kappa, 2, 2t+1)$. ∎

For greater prime powers q, Bierbrauer demonstrates in [10] that trace-codes of Reed-Solomon codes yield a simple $OA(q \times q^{\nu(t-1-\lambda)}, q^\nu, q, t)$ for all natural numbers ν, t, λ such that $q^\nu \ge t > q^\lambda$. We observe that these arrays additionally provide an upper bound for $E(\nu, q, t)$ in case where q is prime. Specifically, Bierbrauer defines in [10] the orthogonal array B as follows:

$$B^c_{(a,z)} = \phi\left(\sum_{j=1}^{k-1} a_j c^j\right) + z, \quad a = (a_1, \ldots, a_{k-1}) \in (\mathbb{F}_q^\kappa)^{k-1},\ z \in \mathbb{F}_q,\ c \in \mathbb{F}_q^\kappa$$

where ϕ is some surjective map from $\mathbb{F}_q^\kappa$ to $\mathbb{F}_q$. Let $c_1, \ldots, c_{q^\kappa}$ refer to the elements of $\mathbb{F}_q^\kappa$. Given $a = (a_1, \ldots, a_{k-1}) \in (\mathbb{F}_q^\kappa)^{k-1}$, we define $u(a)$ as:

$$u(a) = \left(\phi\left(\sum_{j=1}^{k-1} a_j c^1\right), \ldots, \phi\left(\sum_{j=1}^{k-1} a_j c^{q^\kappa}\right)\right)$$

Then B precisely is the union, over all $a \in (\mathbb{F}_q^\kappa)^{k-1}$, of the set

$$\{u(a), u(a) + \mathbf{1}, \ldots, u(a) + \mathbf{q-1}\}$$

of rows. Thus assume that q is prime, in which case $\mathbb{F}_q \simeq \mathbb{Z}_q$. We deduce from Item 3 of Property 2.5 that the restriction of B to rows with index $(a, 0), a \in (\mathbb{F}_q^\nu)^{k-1}$ is a $D_k(q^{\nu(k-1)}, q^\nu, q)$.

Theorem 2.12 ([10]). *For all integers $q \ge 2$, $t \ge 2$, $\nu \ge t$ where q is a prime power and ν is a power of q, we have:*

$$F(\nu, q, t) \le q \times \nu^{t - \lceil \log_q t \rceil} \tag{2.18}$$

$$\text{if } q \text{ is a prime, then } E(\nu, q, t) \le \nu^{t - \lceil \log_q t \rceil} \tag{2.19}$$

2.6. Approximation guarantees for $\mathsf{k\,CSP{-}q}$. We derive lower bounds for the average differential ratio from Theorem 2.4 and the arrays of subsection 2.5. While these arrays typically demand q to be a prime power, we can still establish lower bounds for the average differential ratio when q is not a prime power by reducing it to the prime power case.

Theorem 2.13. *Let $q, k \ge 2$ and $d > q$ be three integers. Assume that on all instances I of $\mathsf{k\,CSP{-}d}$, the average differential ratio is bounded below by ρ, where ρ possibly depends on the primary hypergraph of I; then this also holds for $\mathsf{k\,CSP{-}q}$.*

Proof. Consider an instance I of $\mathsf{k\,CSP{-}q}$. We denote by $\mathcal{M}$ the set of surjective maps from Σ_d to Σ_q. Given a sequence $\pi = (\pi_1, \ldots, \pi_n)$ of n maps from $\mathcal{M}$, we interpret I as the instance $f_\pi(I)$ of $\mathsf{CSP{-}d}$ where:

1. for each $j \in [n]$, there is in $f_\pi(I)$ a variable z_j with domain Σ_d;
2. for each $i \in [m]$, there is in $f_\pi(I)$ a constraint $C_i = P_i(\pi_{i_1}(z_{i_1}), \ldots, \pi_{i_{k_i}}(z_{i_{k_i}}))$ with the same weight w_i as C_i in I.

By construction, two instances $f_\pi(I)$ and I share the same primary hypergraph. To retrieve solutions of I from solutions of $f_\pi(I)$, we define $g_\pi(I, .)$ by $g_\pi(I, z) = (\pi_1(z_1), \ldots, \pi_n(z_n))$, $z \in \Sigma_d^n$. This function is surjective, and satisfies for all $z \in \Sigma_d^n$

that $v(I, g_\pi(I, z)) = v(f_\pi(I), z)$. The extremal solution values on I and $f_\pi(I)$ therefore satisfy:

$$\text{opt}(f_\pi(I)) = \text{opt}(I), \quad \text{wor}(f_\pi(I)) = \text{wor}(I) \tag{2.20}$$

By contrast, $\mathbb{E}_Z[v(f_\pi(I), Z)]$ may differ from $\mathbb{E}_X[v(I, X)]$, due to the fact that $g_\pi(I, .)$ possibly associates with two distinct vectors $x, x' \in \Sigma_q^n$ a distinct number of vectors from Σ_d^n. Hence, rather than n specific maps $\pi_1, \ldots, \pi_n$, we consider a collection $\Pi = (\Pi_1, \ldots, \Pi_n)$ of random maps that are independently and uniformly distributed over $\mathcal{M}$. Let $b \in \Sigma_q$ and $b' \in \Sigma_q \backslash \{b\}$. We consider on $\mathcal{M}$ the function σ that associates with any $\tau \in \mathcal{M}$ the map $\sigma(\tau) \in \mathcal{M}$ defined by:

$$\sigma(\tau)(c) = \begin{cases} b' & \text{if } c \in \tau^{-1}(b) \\ b & \text{if } c \in \tau^{-1}(b') \\ \tau(c) & \text{otherwise} \end{cases}$$

σ clearly is a bijection on $\mathcal{M}$. Given any $a \in \Sigma_d$, we have:

$$\begin{aligned} |\{\tau \in \mathcal{M} \mid \tau(a) = b\}| &= |\{\tau \in \mathcal{M} \mid \sigma(\tau)(a) = b\}| \quad \text{since } \sigma \text{ is a bijective} \\ &= |\{\tau \in \mathcal{M} \mid \tau(a) = b'\}| \end{aligned}$$

We deduce that cardinalities $|\{\tau \in \mathcal{M} \mid \tau(a) = b\}|$, $(a, b) \in \Sigma_d \times \Sigma_q$ are all equal to $1/q$. Probabilities $\mathrm{P}_\Pi[g_\Pi(I, z) = x]$, $z \in \Sigma_d^n$, $x \in \Sigma_q^n$ therefore are all equal to:

$$\textstyle\prod_{j=1}^n \mathrm{P}_{\Pi_j}[\Pi_j(z_j) = x_j] = \prod_{j=1}^n \left(\frac{|\{\tau \in \mathcal{M} \mid \tau(z_j) = x_j\}|}{|\mathcal{M}|} \right) = 1/q^n$$

Accordingly, given any $z \in \Sigma_d^n$, we have:

$$\mathbb{E}_\Pi[v(I, g_\Pi(I, z))] = \textstyle\sum_{x \in \Sigma_q^n} v(I, x) \times \mathrm{P}_\Pi[g_\Pi(I, z) = x] = \mathbb{E}_X[v(I, X)]$$

Finally, the expected average solution value on $f_\Pi(I)$ satisfies:

$$\mathbb{E}_\Pi\left[\mathbb{E}_Z[v(f_\Pi(I), Z)]\right] = \mathbb{E}_Z\left[\mathbb{E}_\Pi[v(I, g_\Pi(I, Z))]\right] = \mathbb{E}_X[v(I, X)] \tag{2.21}$$

Referring to (2.21), there exists a vector $\pi_* \in \mathcal{M}^n$ such that $\mathbb{E}_Z[v(f_{\pi_*}(I), Z)] \leq \mathbb{E}_X[v(I, X)]$, while given such a π_*, we have:

$$\begin{aligned} \frac{\mathbb{E}_X[v(I, X)] - \text{wor}(I)}{\text{opt}(I) - \text{wor}(I)} &\geq \frac{\mathbb{E}_Z[v(f_{\pi_*}(I), Z)] - \text{wor}(I)}{\text{opt}(I) - \text{wor}(I)} \\ &= \frac{\mathbb{E}_Z[v(f_{\pi_*}(I), Z)] - \text{wor}(f_{\pi_*}(I))}{\text{opt}(f_{\pi_*}(I)) - \text{wor}(f_{\pi_*}(I))} \quad \text{by (2.20)} \end{aligned}$$

This completes the proof. ∎

Notice that, in the most general case, transformation f_π does not map an initial instance of $\mathsf{CSP}(\mathcal{E}_\mathsf{q})$ or $\mathsf{CSP}(\mathcal{I}_\mathsf{q}^\mathsf{t})$ to an instance of $\mathsf{k}\,\mathsf{CSP}(\mathcal{E}_\mathsf{d})$ or $\mathsf{k}\,\mathsf{CSP}(\mathcal{I}_\mathsf{d}^\mathsf{t})$. For example, assume that $d = 3$, $q = 2$, $\pi_1 = \pi_2$ maps each $a \in \Sigma_3$ to $a \bmod 2$, and consider the function $XNOR^2$. While the function $XNOR^2$ belongs to $\mathcal{E}_2$ (and thus, to $\mathcal{I}_2^1$), the function P on Σ_3^2 that assigns to each $(a, b) \in \Sigma_3^2$ the value $XNOR^2(a \bmod 2, b \bmod 2)$ does not belong to $\mathcal{I}_3^1$ (and thus, to $\mathcal{E}_3$).[6]

[6] We have $P(0,0) + P(0,1) + P(0,2) = 2XNOR^2(0,0) + XNOR^2(0,1) = 2$ and $P(1,0) + P(1,1) + P(1,2) = 2XNOR^2(1,0) + XNOR^2(1,1) = 1$. P therefore cannot satisfy (1.2) at rank $t = 1$.

Consider an instance I of $\mathsf{k\,CSP\!-\!q}$. If I is ν-partite, we know from Theorem 2.4 that $\rho(\nu, q, \min\{\nu, k\})$ is a proper lower bound for the average differential ratio on I. We argue that for all integers $s \geq \nu$, $\rho(s, q, \min\{\nu, s\})$ is a proper lower bound for this ratio. On the one hand, the ν first columns of an $OA(R, s, q, t)$ in which row $\mathbf{0}$ occurs R^* times form an $OA(R, \nu, q, t)$ in which row $\mathbf{0}$ occurs at least R^* times. On the other hand, an $OA(R, \nu, q, t)$ obviously is an $OA(R, \nu, q, t')$ for all positive integers $t' \leq t$. From Theorem 2.13, we eventually deduce that for all integers $s \geq \nu$ and $q' \geq q$, $\rho(s, q', \min\{s, k\})$ is a proper lower bound for the average differential on I. In the light of these observations, using Theorems 2.4 and 2.13, we derive from Theorems 2.7, 2.8, 2.11, and 2.12 and Corollary 2.9 the following lower bounds for the average differential ratio for $\mathsf{k\,CSP\!-\!q}$:

Corollary 2.14 (Consequence of Theorems 2.4, 2.7, and 2.13). *Let $q \geq 2$, $k \geq 2$, $\nu \geq k$ with $\nu \in O(\max\{q, k\})$ be three integers, and I be a ν-partite instance of $\mathsf{k\,CSP\!-\!q}$. We denote by p^κ the smallest prime power greater than or equal to q. On I, the average differential ratio is bounded below by:*

1. *$1/q^k$ if $\nu \leq k+1$, or q is a prime power $> k$ and $\nu \leq q+1$, or $k = 3$, q is a power of 2 > 3 and $\nu \leq q+2$;*
2. *$1/p^{\kappa k} \geq 1/(2q-2)^k$ if $p^\kappa > k$ and $\nu \leq p^\kappa + 1$;*
3. *$1/2^{3\lceil \log_2 q \rceil} \geq 1/(2q-2)^3$) if $k = 3$, $q \geq 3$ and $\nu \leq 2^{\lceil \log_2 q \rceil} + 2$.*

Corollary 2.15 (Consequence of Theorems 2.4 and 2.8). *Let $q \geq 2$ and $\nu \geq 2$ be two integers. Then on all ν-partite instances of $\mathsf{2\,CSP\!-\!q}$, the average differential ratio is bounded below by:*

$$\frac{1}{q\lceil (\nu-1)/q \rceil (q-1) + q} \geq \frac{1}{(q-1)\nu + (q-1)(q-2) + 1} \sim \frac{1}{(q-1)\nu}$$

In particular, when $q = 2$, this ratio is at least $1/(\nu+1)$ if ν is odd and $1/(\nu+2)$ otherwise.

Corollary 2.16 (Consequence of Theorem 2.4 and Corollary 2.9). *On all instances of $\mathsf{3\,CSP\!-\!2}$ with a strong chromatic number $\nu \geq 3$, the average differential ratio is bounded below by $1/\left(4\lceil \nu/2 \rceil\right) \sim 1/(2\nu)$.*

Corollary 2.17 (Consequence of Theorems 2.4 and 2.11). *Let $k \geq 4$ and $\nu \geq k$ be two integers. Then on all ν-partite instances of $\mathsf{k\,CSP\!-\!2}$, the average differential ratio is at least*

$$\begin{array}{llll} 1/2^{\lceil \log_2(\nu+1) \rceil k/2} & \geq 1/(2\nu)^{k/2} & \sim 1/(2\nu)^{\lfloor k/2 \rfloor} & \text{if } k \text{ is even,} \\ 1/2^{1+\lceil \log_2 \nu \rceil (k-1)/2} & \geq 1/2 \times 1/(2\nu-2)^{(k-1)/2} & \sim 1/(2^{\lceil k/2 \rceil} \nu^{\lfloor k/2 \rfloor}) & \text{if } k \text{ is odd.} \end{array}$$

Corollary 2.18 (Consequence of Theorems 2.4, 2.12, and 2.13). *Let $q \geq 3$, $k \geq 2$, $\nu \geq k$ be three integers, and I be a ν-partite instance of $\mathsf{k\,CSP\!-\!q}$. We denote by p^κ the smallest prime power such that $p^\kappa \geq q$ (thus $p^\kappa = q$ provided that q is a prime power). Then the average differential ratio on I is at least:*

$$\frac{1}{p^{\kappa\left(1 + \lceil \log_{p^\kappa} \nu \rceil (k - \lceil \log_{p^\kappa} k \rceil)\right)}} \geq \frac{1}{p^\kappa} \times \frac{1}{(p^\kappa \nu - p^\kappa)^{k - \lceil \log_{p^\kappa} k \rceil}} \sim \frac{1}{p^\kappa} \times \frac{1}{(p^\kappa \nu)^{k - \lceil \log_{p^\kappa} k \rceil}}$$

If each constraint of I involves a function of $\mathcal{I}_q^t$ where t is some positive integer, according to Theorem 2.4, we can consider for the average differential ratio on I the tighter lower bound of $\rho(\nu - t, q, \min\{\nu - t, k\})$. We can more generally consider the bound $\rho(s - t, q, \min\{s - t, k\})$ for all integers $s \geq \nu$. The result is a slight improvement in the estimates of the average differential ratio for the restriction $\mathsf{k\,CSP}(\mathcal{I}_\mathsf{q}^\mathsf{t})$ of $\mathsf{k\,CSP\!-\!q}$:

COROLLARY 2.19 (Consequence of Theorems 2.4, 2.7, 2.8, 2.11, and 2.12 and Corollary 2.9). *Let $q \geq 2$, $k \geq 2$, $t \in [k-1]$, $\nu \geq k$ be four integers, and I be a ν-partite instance of* $\mathsf{k\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{t}})$.

- *When $\nu = O(\max\{q,k\})$, the average differential ratio on I is bounded below by $1/q^{\nu-t}$ if $\nu - t < k$, and by $1/q^k$ if $\nu - t \leq k+1$, or q is a prime power $> k$ and $\nu - t \leq q+1$, or $k = 3$, q is a power of 2 > 3 and $\nu - t \leq q+2$.*
- *When $k = 2$ (thus $t = 1$), this ratio is at least*

$$1/\left(q\lceil(\nu-2)/q\rceil(q-1)+q\right) \quad \geq 1/\left((q-1)\nu+(q-1)(q-2)+1\right)$$

- *When $q = 2$ and $k \geq 3$, this ratio is at least*

$$\begin{array}{lll} 1/\left(4\lceil(\nu-t)/2\rceil\right) & \geq 1/(2\nu-2t+2) & \textit{if } k=3,\\ 1/2^{\lceil\log_2(\nu-t+1)\rceil k/2} & \geq 1/(2\nu-2t)^{k/2} & \textit{if } k\geq 4 \textit{ and } k \textit{ is even},\\ 1/2^{1+\lceil\log_2(\nu-t)\rceil(k-1)/2} & \geq 1/2\times 1/(2\nu-2t-2)^{(k-1)/2} & \textit{if } k\geq 4 \textit{ and } k \textit{ is odd.}\end{array}$$

- *When $q \geq 3$ and $k \geq 3$, it is at least*

$$1/q^{1+\lceil\log_q(\nu-t)\rceil(k-\lceil\log_q k\rceil)} \quad \geq 1/q\times 1/(q\nu-qt-q)^{(k-\lceil\log_q k\rceil)}$$

Similarly to numbers $\rho(\nu,q,t)$, inequality $\rho_E(\nu,q,\min\{t,\nu\}) \geq \rho_E(s,q,\min\{t,s\})$ holds for all integers $t,q,\nu > 0$ and $s \geq \nu$. Hence, it follows from Theorem 2.4 that, on a ν-partite instance I of $\mathsf{CSP}(\mathcal{E}_{\mathsf{q}})$, for all integers $s \geq \nu$, $\rho_E(s,q,\min\{k,s\})$ is a proper lower bound for the average differential ratio on I. Considering this fact, for $\mathsf{CSP}(\mathcal{E}_{\mathsf{q}})$ in case where $q = 2$ and k is odd, or q is a prime number ≥ 3, we obtain bounds that increase by a multiplicative factor of q the bounds already obtained for $\mathsf{CSP{-}q}$ and $\mathsf{CSP}(\mathcal{I}_{\mathsf{q}}^{1})$:

COROLLARY 2.20 (Consequence of (2.16) and Corollary 2.9 and Theorems 2.4, 2.7, 2.11, and 2.12). *Let q be a prime number, $k \geq 2$, $\nu \geq k$ be two integers, and I be a ν-partite instance of* $\mathsf{k\,CSP}(\mathcal{E}_{\mathsf{q}})$. *Then:*

- *If $q = 2$ and k is odd, then the average differential ratio on I is at least*

$$\begin{array}{llll} 1/2^{k-1} & & & \textit{if } \nu\leq k+1,\\ 1/\left(2\lceil\nu/2\rceil\right) & \geq 1/(\nu+1) & \sim 1/\nu & \textit{if } k=3,\\ 1/2^{\lceil\log_2\nu\rceil(k-1)/2} & \geq 1/\left(2(\nu-1)\right)^{(k-1)/2} & \sim 1/(2\nu)^{\lfloor k/2\rfloor} & \textit{if } k\geq 5.\end{array}$$

- *If $q > 3$, then this ratio is bounded below by*

$$1/q^{\lceil\log_q\nu\rceil(k-\lceil\log_q k\rceil)} \quad \geq 1/\left(q(\nu-1)\right)^{(k-\lceil\log_q k\rceil)} \quad \sim 1/(q\nu)^{(k-\lceil\log_q k\rceil)}.$$

2.7. Concluding remarks. We ask the following questions: is the average solution value a good approximation of the optimum value? Have we provided good estimates of the differential approximation ratio? How good are our estimates of numbers $\rho(\nu,q,t)$ and $\rho_E(\nu,q,t)$? In doing so, we identify potential areas of improvement and directions for future research.

For all integers $k \geq 3$ and $q \geq 2$, the inapproximability bound from [13] for k-partite instances of $\mathsf{CSP}(\mathcal{I}_{\mathsf{q}}^{2})$ is a factor $O(k)$ of the lower bound we obtain for the average differential ratio on these instances. For the rather restrictive case of k-partite instances of $\mathsf{CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k-1}})$, the average solution value is even basically optimal in terms of differential approximation within a constant factor.

For instances with an unbounded strong chromatic number, we observe that our analysis is either tight or asymptotically tight for $\mathsf{k\,CSP}(\mathcal{I}_{\mathsf{q}}^{1})$, when either $k = 2$, or $k = 3$ and $q = 2$. Given three positive integers q, k, $n \geq k$, we denote by $I_n^{q,k}$ the instance of $\mathsf{CSP}(\{\mathsf{AllEqual}^{\mathsf{k,q}}\})$ which considers all the k-ary constraints that can be formed over a set of nq variables. Instance $I_n^{q,k}$ is trivially satisfiable and its strong chromatic number is qn. Furthermore, a worst solution on $I_n^{q,k}$ assigns exactly n variables to each $a \in \Sigma_q$. Thus considering that $\mathrm{opt}(I_n^{q,k}) = \binom{qn}{k}$, $\mathrm{wor}(I_n^{q,k}) = q \times \binom{n}{k}$

and $\mathbb{E}_X[v(I_n^{q,k}, X)] = \binom{qn}{k}/q^{k-1}$, the average differential ratio on $I_n^{q,k}$ is equal to:

$$\frac{\binom{qn}{k}/q^{k-1} - q \times \binom{n}{k}}{\binom{qn}{k} - q \times \binom{n}{k}} = \frac{1}{q^{k-1}} \times \frac{\prod_{i=0}^{k-1}(qn-i) - q^k \prod_{i=0}^{k-1}(n-i)}{\prod_{i=0}^{k-1}(qn-i) - q \prod_{i=0}^{k-1}(n-i)}$$

We observe that the above fraction is equal to $1/(qn)$ when either $k = 2$, or $k = 3$ and $q = 2$. Hence, when $k = 2$ or $(k, q) = (3, 2)$, the average differential ratio on $I_n^{q,k}$ is asymptotically a factor respectively $(q-1)$ or 2 of the lower bound Corollary 2.19 provides for this ratio. This ratio and the bound of Corollary 2.19 even coincide in case where $k = q = 2$.

As noted by Stinson in [45], Mukerjee, Qian and Wu provide in [39] an upper bound for $\rho(\nu, q, t)$ for all integers $q, t \geq 2$, $\nu \geq k$. In their work, an $OA(R, \nu, q, t)$ is termed *nested* if it contains an $OA(R', \nu, q', t)$ as a subarray for some positive integers $R' < R$ and $q' \leq q$. The authors provide a lower bound for R/R' which generalizes the Rao bound for R in an $OA(R, \nu, q, t)$ [44]. Viewing R^* identical rows of an $OA(R, \nu, q, t)$ as an $OA(R^*, \nu, 1, t)$, the bound of [39] when $q' = 1$ defines an upper bound for $1/\rho(\nu, q, t)$ [45]. It precisely follows from [39] that $1/\rho(\nu, q, t)$ is at most:

$$\begin{array}{rll} \sum_{j=0}^{t/2}(q-1)^j\binom{\nu}{j} & \sim ((q-1)\nu)^{\lfloor t/2 \rfloor}/\lfloor t/2 \rfloor! & \text{if } t \text{ is even,} \\ \sum_{j=0}^{\lfloor t/2 \rfloor}(q-1)^j\binom{\nu}{j} + (q-1)^{\lceil t/2 \rceil}\binom{\nu-1}{\lfloor t/2 \rfloor} & \sim q \times ((q-1)\nu)^{\lfloor t/2 \rfloor}/\lfloor t/2 \rfloor! & \text{if } t \text{ is odd.} \end{array}$$

When $q = 2$ and $k \geq 4$, our estimate of $\rho(\nu, 2, k)$ is asymptotically a multiplicative factor $1/\left(\lfloor k/2 \rfloor! \times 2^{\lceil k/2 \rceil}\right)$ of this bound. For $2\,\mathsf{CSP}{-}\mathsf{q}$ and $3\,\mathsf{CSP}{-}2$, a ratio of respectively $1/((q-1)\nu)$ and $1/(2\nu)$ is asymptotically the best lower bound we can derive from our approach for the average differential ratio.

Nevertheless, our estimate of quantities $\rho(\nu, q, t)$ and $\rho_E(\nu, q, t)$ could potentially be improved for most triples (ν, q, t). First, with the exception of the case $t = 2$, the lower bounds that we considered for $\rho(\nu, q, t)$ come from simple arrays. By definition of $\rho(\nu, q, t)$ and $F(\nu, q, t)$, inequality $\rho(\nu, q, t) \geq 1/F(\nu, q, t)$ holds for all triples (ν, q, t). One thus may wonder how much better $\rho(\nu, q, t)$ can be compared to $1/F(\nu, q, t)$ depending on ν, q, t. Table 6 provides the exact value of $F(\nu, q, t)$ and $\rho(\nu, q, t)$ (and the corresponding numbers for difference schemes) for some triples (ν, q, t). For a fair comparison, in this table, we indicate the minimal number of rows in an array that realizes $\rho(\nu, q, t)$, as well as the maximal number of rows of zeros in an array that realizes $F(\nu, q, t)$.

Second, we found few results on difference schemes in the literature. The analysis carried out suggests the search for difference schemes maximizing the overall frequency of the words $\mathbf{a}$, $a \in \Sigma_q$. Considering in (2.15) inequality $\rho_E(\nu, q, t) \leq q \times \rho(\nu, q, t)$, we are more specifically interested in the search for such arrays when $\nu = \Theta(t)$. Given three integers R, q, t, $f(R, q, t)$ refers to the greatest integer ν for which an $OA(R, \nu, q, t)$ exists. Bush notably investigated numbers $f(q^t, q, t)$ in [12]. As with the numbers $f(q^t, q, t)$, one should seek, for two integers q, t, the greatest $\nu \geq t$ such that $E(\nu, q, t) = q^{t-1}$. For instance, consider the two equations below:

$$y_1 + \ldots + y_\nu - y_{\nu+1} - \ldots - y_{2\nu} \equiv 0 \bmod q \tag{2.22}$$

$$y_1 + \ldots + y_{\nu-1} + 2y_\nu - y_{\nu+1} - \ldots - y_{2\nu+1} \equiv 0 \bmod q \tag{2.23}$$

Let P refer to the predicate whose accepting entries are the solutions to equation (2.22). If we fix in this equation the value of any $2\nu - 1$ variables, there is one and only one assignment for the remaining variable to satisfy the equation. Thus

$P \in \mathcal{I}_q^{2\nu-1}$, and the accepting entries of P form the rows of an $OA(q^{2\nu-1}, 2\nu, q, 2\nu-1)$. Furthermore, a vector $y \in \Sigma_q^{2\nu}$ is solution to (2.22) *iff* vectors of the form $y + \mathbf{a}$ are all solutions to (2.22). We deduce from Item 3 of Property 2.5 that solutions y to equation (2.22) that additionally satisfy, *e.g.*, $y_1 = 0$ constitute a $D_{2\nu-1}(q^{2\nu-2}, 2\nu, q)$.

The predicate whose accepting entries are the solutions to equation (2.23) similarly belongs to $\mathcal{E}_q$, and to $\mathcal{I}_q^{2\nu}$ provided that q is odd. Hence, assuming that q is odd, the solutions y to equation (2.23) that additionally satisfy $y_1 = 0$ form the rows of a $D_{2\nu}(q^{2\nu-1}, 2\nu+1, q)$. Therefore, we have:

$$E(t+1, q, t) = q^{t-1}, \qquad q, t \in \mathbb{N}\backslash\{0\},\ t \text{ or } q \text{ is odd} \tag{2.24}$$

We deduce that provided that k or q is odd, the average differential ratio on $(k+1)$-partite instances of $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$ is at least $1/q^{k-1}$.

3. Reducing the alphabet size. In the general case, CSPs become harder as the alphabet size increases. Specifically, the alphabet size has a logarithmic impact on the constraints arity. Let $p \geq 2$, $q \geq p$ and $k \geq 1$ be three integers, and define $\kappa = \lceil \log_p q \rceil$. Then a function P of k variables $x_1, \ldots, x_k \in \Sigma_q$ can be interpreted as a function of k strings $y_1, \ldots, y_k \in \Sigma_p^\kappa$ where for each $j \in [k]$, $y_j = (y_{j,1}, \ldots, y_{j,\kappa})$ is the base p encoding of x_j. We can therefore encode an instance of $\mathsf{k\,CSP{-}q}$ by an instance of $(\lceil \log_\mathsf{p} \mathsf{q} \rceil \mathsf{k})\mathsf{CSP{-}p}$. Now, in the most general case, CSPs become harder as the constraints arity increases. Here consider that a function of h variables can be interpreted as a function of $k > h$ variables, whose value depends only on its h first inputs.

We ask whether it is possible to reduce the alphabet size without increasing the arity of the constraints, at the possible cost of a reduced approximation guarantee. In other words, we are looking for a reduction from $\mathsf{k\,CSP{-}q}$ to $\mathsf{k\,CSP{-}p}$ that preserves the differential approximation ratio given three positive integers k, q and $p < q$.

3.1. Differential approximation preserving reductions. Consider two optimization CSPs Π and Π'. A *reduction* from Π' to Π can be seen as a polynomial time algorithm $\mathcal{A}'$ for Π' which uses a hypothetical algorithm $\mathcal{A}$ for Π as a subroutine. Such an algorithm is a *differential approximation preserving reduction* (D-reduction for short) if there exists $\gamma > 0$ such that $\mathcal{A}'$ is a $(\gamma \times \rho)$-approximation algorithm for Π' provided that $\mathcal{A}$ is a ρ-approximation algorithm for Π. When this occurs, we write $\Pi' \leq_D^\gamma \Pi$. For example, we have $\mathsf{3\,CSP{-}2} \leq_D^{1/2} \mathsf{E2\,Lin{-}2}$ [17] and $\mathsf{k\,CSP{-}q} \leq_D^1 \mathsf{k}\lceil \log_2 \mathsf{q} \rceil\,\mathsf{Lin{-}2}$ [16].

Most often, $\mathcal{A}'$ relies on a pair (f, g) of polynomial time algorithms, where f associates with each instance I of Π' an instance $f(I)$ of Π, and g associates with each instance I of Π' and each solution x of $f(I)$ a solution of I. $\mathcal{A}'$ then consists in computing $f(I)$, a solution x of $f(I)$ by running $\mathcal{A}$ on $f(I)$, and finally, $g(I, x)$. Such a pair (f, g) defines a D-reduction with expansion γ if, for all instances I of Π' and all solutions x of $f(I)$, we have:

$$\frac{v(I, g(I,x)) - \mathrm{wor}(I)}{\mathrm{opt}(I) - \mathrm{wor}(I)} \geq \gamma \times \frac{v(f(I), x) - \mathrm{wor}(f(I))}{\mathrm{opt}(f(I)) - \mathrm{wor}(f(I))}$$

In this section, we present a reduction in which the transformation f associates not a single, but multiple instances $f_1(I), \ldots, f_R(I)$ of Π to an input instance I of Π'. In this case, the solution returned by $\mathcal{A}'$ is the best that can be derived from the solutions $x_1, \ldots, x_R$ that algorithm $\mathcal{A}$ returns on instances $f_1(I), \ldots, f_R(I)$.

Table 6

The ratio of the maximum multiplicity R^* of a row to the total number R of rows in orthogonal arrays and difference schemes that realize $\rho(\nu,q,t)$, $F(\nu,q,t)$, $\rho_E(\nu,q,t)$ or $E(\nu,q,t)$. The corresponding arrays were calculated by computer solving linear programs (see section SM2 *of the supplement for more details). We use grey color to identify cases where a same array realizes both bounds either $E(\nu,q,t)$ and $\rho_E(\nu,q,t)$, or $F(\nu,q,t)$ and $\rho(\nu,q,t)$.*

R^*/R in $D_t(R,\nu,q)$ that minimize R among those that achieve $\rho_E(\nu,q,t)$

		ν											
q	t	2	3	4	5	6	7	8	9	10	11	12	13
2	2	1/2	1/4	1/4	2/12	2/12	1/8	1/8	2/20	2/20	1/12	1/12	2/28
	4	–	–	1/8	1/16	1/16	3/80	4/144	6/240	6/336	6/336		
	6	–	–	–	–	1/32	1/64	1/64	4/448	6/960	25/5184		
3	2	1/3	1/3	3/15	1/6	1/6	3/24	1/9	1/9	3/33			
	3	–	1/9	1/9	2/27	2/27	8/162	8/162					
	4	–	–	1/27	1/27	7/297	5/243						
	5	–	–	–	1/81	1/81	27/3240						
4	2	1/4	2/8	2/8	4/24	14/104	2/16	2/16					
	3	–	1/16	1/16	2/32	2/32							
	4	–	–	1/64	2/128								

R^*/R in $D_t(R,\nu,q)$ that maximize R^* among those that achieve $E(\nu,q,t)$

		ν											
q	t	2	3	4	5	6	7	8	9	10	11	12	13
2	2	1/2	1/4	1/4	1/8	1/8	1/8	1/8	1/12	1/12	1/12	1/12	1/16
	4	–	–	1/8	1/16	1/16	1/32	1/64	1/64				
	6	–	–	–	–	1/32	1/64	1/64	1/128	1/256			
3	2	1/3	1/3	1/6	1/6	1/6	1/9	1/9	1/9	1/12			
	3	–	1/9	1/9	1/18	2/27	1/27	1/27					
	4	–	–	1/27	1/27	1/81	1/81						
	5	–	–	–	1/81	1/81	1/243						
4	2	1/4	2/8	2/8	2/16	2/16	2/16	2/16					
	3	–	1/16	1/16	2/32	2/32							
	4	–	–	1/64	2/128								

R^*/R in $OA(R,\nu,q,t)$ that minimize R among those that achieve $\rho(\nu,q,t)$

		ν						
q	t	2	3	4	5	6	7	8
3	2	1/9	1/9	1/9	2/27	3/45	3/45	7/135
	3	–	1/27	1/27	2/54	2/81		
	4	–	–	1/81	1/81	4/324		
4	2	1/16	1/16	1/16	1/16	7/160		

R^*/R in $OA(R,\nu,q,t)$ that maximize R^* among those that achieve $F(\nu,q,t)$

		ν					
q	t	2	3	4	5	6	7
3	2	1/9	1/9	1/9	1/18	1/18	1/18
	3	–	1/27	1/27	2/54	2/81	
	4	–	–	1/81	1/81	2/243	
4	2	1/16	1/16	1/16	1/16	1/32	

3.2. Previous related works and preliminary remarks. Given an instance I of $\mathsf{k\,CSP-q}$, our objective is to derive an approximate solution of I from approximate solutions of instances of $\mathsf{k\,CSP-p}$. Thereafter, given two natural numbers q and $p \le q$, we denote by $\mathcal{P}_p(\Sigma_q)$ the set of the p-cardinality subsets of Σ_q.

For any $S = (S_1, \ldots, S_n) \in \mathcal{P}_p(\Sigma_q)^n$, the restriction of I to solution set S, which we denote by $I(S)$, can be interpreted as an instance of $\mathsf{k\,CSP-p}$. Namely, pick any family $(\pi_j : \Sigma_p \to S_j \,|\, j \in [n])$ of bijections. Then associate with $I(S)$ instance $f_S(I)$ of $\mathsf{k\,CSP{-}p}$ defined as follows:

1. for each variable x_j of I, there is in $f_S(I)$ a variable z_j with domain Σ_p;
2. for each constraint $C_i = P_i(x_{i_1}, \ldots, x_{i_{k_i}})$ of I, there is in $f_S(I)$ a constraint $P_i(\pi_{i_1}(z_{i_1}), \ldots, \pi_{i_{k_i}}(z_{i_{k_i}}))$ with the same associated weight w_i as C_i.

By construction, given any $z \in \Sigma_p^n$, solution

$$g_S(I, z) := (\pi_1(z_1), \ldots, \pi_n(z_n)) \tag{3.1}$$

of $I(S)$ performs on $I(S)$ — and thus, on I — the same objective value as solution z on $f(I)$. This implies that the worst solution on $f(I)$ is no worse than the worst solution on I.

The consideration of sub-instances $I(S)$ seems quite natural having in view a reduction from $\mathsf{k\,CSP{-}q}$ to $\mathsf{k\,CSP{-}p}$. In fact, if we impose the constraints $w_i \times P_i$ to be nonnegative, there is a standard approximation preserving reduction from $\mathsf{Max\,Ek\,CSP{-}q}$ to $\mathsf{Max\,Ek\,CSP{-}p}$ [14]. This reduction precisely consists in choosing a solution subset S by randomly selecting n subsets $S_1, \ldots, S_n$ independently and uniformly over $\mathcal{P}_p(\Sigma_q)$. The argument essentially relies on the fact that, by doing so, the value $\mathrm{opt}(I(S))$ is, in expectation, at least a constant fraction of $\mathrm{opt}(I)$; namely:

$$\mathbb{E}_S[\mathrm{opt}(I(S))] \geq p^k/q^k \times \mathrm{opt}(I) \tag{3.2}$$

Assume that for each $S \in \mathcal{P}_p(\Sigma_q)^n$, we can compute within polynomial time a solution $x(S)$ which is ρ-standard approximate on $I(S)$; such solutions are in expectation $(p^k/q^k \times \rho)$-standard approximate on I, considering:

$$\begin{aligned} \mathbb{E}_S[v(I, x(S))] &\geq \mathbb{E}_S[\rho \times \mathrm{opt}(I(S))] \quad \text{by assumption on } x(S),\ S \in \mathcal{P}_p(\Sigma_q)^n \\ &= \rho \times \mathbb{E}_S[\mathrm{opt}(I(S))] \quad \geq \rho \times p^k/q^k \times \mathrm{opt}(I) \quad \text{by (3.2)} \end{aligned}$$

The reduction can be derandomized using an alternate distribution over $\mathcal{P}_p(\Sigma_q)^n$, although up to a factor $(1 - \varepsilon)$ on the approximation guarantee [14].

To establish (3.2), the authors of [14] associate with a supposed optimal solution x^* a family $x^*(S)$, $S \in \mathcal{P}_p(\Sigma_q)^n$ of solutions where, for a solution subset S, $x^*(S)$ can be any solution that matches with x^* on its coordinates indexed by j such that $x_j^* \in S_j$. They observe that a constraint $P_i(x_{J_i})$ evaluates the same on $x^*(S)$ as on x^* provided that $x_j^* \in S_j$ holds for each $j \in J_i$, what occurs with probability:

$$\prod_{j=1}^{k} \binom{q-1}{p-1} / \binom{q}{p} \quad = (p/q)^k$$

They deduce that the expected value of $v(I, x^*(S))$ over $\mathcal{P}_p(\Sigma_q)^n$ satisfies:

$$\begin{aligned} \mathbb{E}_S[v(I, x^*(S))] &= \textstyle\sum_{i=1}^{m} w_i \times \mathbb{E}_S\left[P_i(x^*(S)_{J_i})\right] \\ &\geq (p^k/q^k) \times \mathrm{opt}(I) + (1 - p^k/q^k) \textstyle\sum_{i=1}^{m} w_i \times \mathbb{E}_S\left[P_i(x^*(S)_{J_i}) \,|\, x_{J_i}^* \notin S_{J_i}\right] \end{aligned} \tag{3.3}$$

Since $\mathrm{opt}(I(S)) \geq v(I, x^*(S))$, $S \in \mathcal{P}_p(\Sigma_q)^n$, inequality (3.2) follows from (3.3) under the condition that $w_i \times P_i \geq 0$, $i \in [m]$. We may not, however, derive a lower bound for the expected differential ratio reached at $\mathrm{opt}(I(S))$ from inequality (3.3), because there is no straightforward way to compare the quantity $\mathbb{E}_S[v(I, x^*(S))] - (p^k/q^k) \times \mathrm{opt}(I)$ to a solution value.

Hence, our objective is to find a lower bound for the differential ratio reached at $\mathrm{opt}(I(S))$ for some solution subset S for which we can compute a ρ-differential approximate solution on $I(S)$, provided that a ρ-differential approximation algorithm for $\mathsf{k\,CSP-p}$ exists. We restrict our analysis to solution sets of the form T^n where T is a p-cardinality subset of Σ_q. Identifying T^n with T, from now on, we denote by $I(T)$ the restriction of I to solution set T^n. Notice that choosing a best solution among hypothetical approximate solutions $x(T)$ of $I(T)$, $T \in \mathcal{P}_p(\Sigma_q)$ requires comparing only $\binom{q}{p}$ solution values. Although $\binom{q}{p}$ may be large, it remains a constant number.

Similarly to how we estimated the average differential ratio, we associate with each solution x of a given instance I two multisets $\mathcal{X}(I,x)$ and $\mathcal{Y}(I,x)$ of solutions of the same size R. Here, $\mathcal{X}(I,x)$ is a subset of $\{T^n \,|\, T \in \mathcal{P}_p(\Sigma_q)\}$, $\mathcal{Y}(I,x)$ contains some number $R^* > 0$ of occurrences of x, and the sum of the solution values over $\mathcal{X}(I,x)$ is the same as for $\mathcal{Y}(I,x)$, i.e.:

$$\textstyle\sum_{y\in\mathcal{X}(I,x)} v(I,y) \quad = \sum_{y\in\mathcal{Y}(I,x)} v(I,y)$$

Taking $\mathcal{X}(I,.)$ and $\mathcal{Y}(I,.)$ at an optimal solution x^*, we deduce that a best solution on $\mathcal{X}(I,x^*)$, and therefore a best solution on $\{T^n \,|\, T \in \mathcal{P}_p(\Sigma_q)\}$, is R^*/R-differential approximate on I.

3.3. Partition-based solution multisets. We define our solution multisets $\mathcal{X}(I,.)$ and $\mathcal{Y}(I,.)$ for an instance I of $\mathsf{k\,CSP-q}$ on n variables as follows.

• **Solution multisets association.** Any solution x induces a partition of $[n]$ into q subsets based on the possible values taken by its coordinates. Given a solution $x \in \Sigma_q^n$, solution $y(x,u)$ is defined by assigning the value u_c to coordinates equal to c in x. Formally, for every $j \in [n]$, $y(x,u)_j = u_{x_j}$. In particular, $y(x,(0,1,\ldots,q-1))$ coincides with x. Furthermore, the components of a solution $y(x,u)$ take as many distinct values as the components of u.

We consider two arrays Ψ and Φ with q columns on Σ_q. These arrays have the same number of rows, denoted by R. We associate with (Ψ,Φ) and x the solution multisets $\mathcal{X}(I,x)$ and $\mathcal{Y}(I,x)$ defined by:

$$\mathcal{X}(I,x) \quad = (y(x,\Psi_r) \,|\, r \in [R]), \qquad \mathcal{Y}(I,x) \quad = (y(x,\Phi_r) \,|\, r \in [R])$$

• **Conditions.** We denote by R^* the number of occurrences of row $(0,1,\ldots,q-1)$ in Φ. Arrays Ψ and Φ must satisfy certain conditions. First, the coefficients of every row of Ψ must take at most p distinct values. This ensures that $\mathcal{X}(I,x)$ exclusively considers solutions of sub-instances $I(T)$. Then, R^* must be positive, for $\mathcal{Y}(I,x)$ to cover $\mathrm{opt}(I)$ provided that x is optimal. Finally, since our ultimate goal is to connect $\mathrm{opt}(I)$ to solution values on sub-instances $I(T)$, the sum of solution values must be the same over both sets $\mathcal{X}(I,x)$ and $\mathcal{Y}(I,x)$. Formally:

$$\textstyle\sum_{r=1}^{R} v(I,y(x,\Psi_r)) = \sum_{r=1}^{R} v(I,y(x,\Phi_r)), \qquad x \in \Sigma_q^n \tag{3.4}$$

Assume that arrays Ψ and Φ fulfill the above requirements, and x^* refers to an optimal solution on I. We assume *w.l.o.g.* that the goal on I is to maximize. Furthermore, we denote by $\mathcal{T}$ a subset of $\mathcal{P}_p(\Sigma_q)$ such that the coefficients of each row of Ψ are contained in some $T \in \mathcal{T}$. Then, we have:

$$\begin{aligned}
\max\nolimits_{T\in\mathcal{T}} \{\mathrm{opt}\,(I(T))\} &\geq \textstyle\sum_{r=1}^{R} v\,(I,y(x^*,\Psi_r))\,/R \\
&= \textstyle\sum_{r=1}^{R} v\,(I,y(x^*,\Phi_r))\,/R \qquad \text{by (3.4)} \\
&\geq (R^* \times \mathrm{opt}(I) + (R-R^*)\mathrm{wor}(I))\,/R \qquad (3.5)
\end{aligned}$$

Assume that ρ-differential approximate solutions $x(T)$ can be computed for each $T \in \mathcal{T}$. We observe:

$$\begin{aligned} \max_{T\in\mathcal{T}} \{v\,(I, x(I(T)))\} &\geq \max_{T\in\mathcal{T}} \{\rho\,\mathrm{opt}(I(T)) + (1-\rho)\mathrm{wor}(I(T))\} \\ &\geq \rho\,\max_{T\in\mathcal{T}} \{\mathrm{opt}(I(T))\} + (1-\rho)\mathrm{wor}(I) \end{aligned} \tag{3.6}$$

We deduce from (3.5) and (3.6) that solutions $x(T)$ achieving the best objective value over $\mathcal{T}$ have value at least $\rho\,R^*/R \times \mathrm{opt}(I) + (1 - \rho\,R^*/R)\mathrm{wor}(I)$. They thus are $(\rho\,R^*/R)$-differential approximate on I.

3.4. Connection to combinatorial designs. We need to identify the conditions under which the arrays Ψ and Φ satisfy (3.4). Clearly, a sufficient condition for the mean value of solutions to be the same on $(y(x, \Psi_r))\,|\,r \in [R])$ and $(y(x, \Phi_r))\,|\,r \in [R])$ is that each constraint of I takes, on average, the same value over these two solution multisets. In other words, Ψ and Φ satisfy (3.4) as soon as they verify:

$$\textstyle\sum_{r=1}^{R} P_i(y(x, \Psi_r)_{J_i}) = \sum_{r=1}^{R} P_i(y(x, \Phi_r)_{J_i}), \qquad i \in [m],\ x \in \Sigma_q^n \tag{3.7}$$

Consider a solution $x \in \Sigma_q^n$ and a constraint $C_i = P_i(x_{J_i})$ of I. The coordinates x_j, $j \in J_i$ of x take at most $\min\{q, k\}$ distinct values. Let $H = (c_1, \ldots, c_t)$ be the set of these values. A sufficient condition for (Ψ, Φ) to satisfy (3.7) at (i, x) is that, over solution multisets $(y(x, \Psi_r)\,|\,r \in [R])$ and $(y(x, \Phi_r)\,|\,r \in [R])$, function P_i is evaluated on the same multisets of entries. By definition of solutions $y(x, u)$, this happens *iff* $(\Psi_r^H\,|\,r \in [R])$ and $(\Phi_r^H\,|\,r \in [R])$ define the same multisets of words from Σ_q^t. Observe, though, that this cannot occur unless $|H| \leq p$, considering that $\Phi_r^H = (c_1, \ldots, c_t)$ must hold for at least one row of Φ, while $|\Psi_r^H| \leq p$ must hold for all rows of Ψ. Since H can be any at most $\min\{q, k\}$-cardinality subset of Σ_q, we conclude that (Ψ, Φ) satisfies (3.7) provided that $k \leq p \leq q$, and arrays Ψ and Φ verify for all k-cardinality subsets H of Σ_q that subarrays Ψ^H and Φ^H coincide, up to the ordering of their rows.

From now on, we assume $k \leq p \leq q$. Consider the function $\mu^\Psi - \mu^\Phi : \Sigma_q^q \to [-1, 1]$. Its mean value clearly is zero. Moreover, let H be a k-cardinality subset of Σ_q. Then subarrays Ψ^H and Φ^H define the same collection of words from Σ_q^q *iff* for all $v \in \Sigma_q^k$, the overall frequency of words $u \in \Sigma_q^q$ such that $u_H = v$ is the same in Ψ as in Φ. In other words, two subarrays Ψ^H and Φ^H define the same collection of rows given any $H \subseteq \Sigma_q$ with $|H| \leq k$ *iff* $\mu^\Psi - \mu^\Phi$ is balanced k-wise independent.

We introduce the following family of combinatorial designs.

DEFINITION 3.1. *Let $k > 0$, $p \geq k$ and $q \geq p$ be three integers. Then given two positive integers R and $R^* \leq R$, we define $\Gamma(R, R^*, q, p, k)$ as the (possibly empty) set of pairs (Ψ, Φ) of $R \times q$ arrays on Σ_q that verify:*

1. $|\{r \in [R]\,|\,\Phi_r = (0, 1, \ldots, q-1)\}| = R^*$*;*
2. *the components of each row of Ψ take at most p distinct values;*
3. *$\mu^\Psi - \mu^\Phi$ is balanced k-wise independent.*

Furthermore, we define $\gamma(q, p, k)$ to be the largest number γ for which there exist two positive integers R and $R^* \leq R$ such that $R^*/R = \gamma$ and $\Gamma(R, R^*, q, p, k) \neq \emptyset$.

Table 7 illustrates a few pairs of arrays that achieve $\gamma(q, p, k)$ (calculated using a computer). The preceding discussion establishes the following connection between these combinatorial designs and the reducibility of $\mathsf{k\,CSP{-}q}$ to $\mathsf{k\,CSP{-}p}$.

THEOREM 3.2. *For all constant integers $k \geq 2$, $p \geq k$ and $q \geq p$, $\mathsf{k\,CSP{-}q}$ D-reduces to $\mathsf{k\,CSP{-}p}$ with an expansion of $\gamma(q, p, k)$ on the approximation guarantee.*

TABLE 7
Pairs of arrays that achieve $\gamma(4,3,2)$, $\gamma(5,3,2)$ and $\gamma(5,4,3)$. We use the $*$ mark to emphasis rows of the form $(0,\ldots,q-1)$.

$\gamma(4,3,2) = 1/3$

Ψ^0	Ψ^1	Ψ^2	Ψ^3	Φ^0	Φ^1	Φ^2	Φ^3	
0	0	2	3	0	0	2	2	
0	1	0	3	0	1	0	2	
0	1	2	2	0	1	2	3	$*$
0	1	2	2	0	1	2	3	$*$
3	0	0	2	3	0	0	3	
3	1	2	3	3	1	2	2	

$\gamma(5,3,2) = 1/6$

Ψ^0	Ψ^1	Ψ^2	Ψ^3	Ψ^4	Φ^0	Φ^1	Φ^2	Φ^3	Φ^4	
0	1	3	3	3	0	1	2	3	4	$*$
0	2	2	2	4	0	2	3	2	3	
1	1	2	1	4	1	1	3	1	3	
1	2	3	1	3	1	2	2	1	4	
3	3	2	3	4	3	3	2	2	4	
3	3	3	2	3	3	3	3	3	3	

$\gamma(5,4,3) = 1/5$

Ψ^0	Ψ^1	Ψ^2	Ψ^3	Ψ^4	Φ^0	Φ^1	Φ^2	Φ^3	Φ^4	
0	0	1	0	3	0	0	1	3	4	
0	0	2	3	4	0	0	2	0	4	
0	0	2	3	4	0	0	2	3	3	
0	1	1	3	4	0	1	1	0	4	
0	1	1	3	4	0	1	1	3	3	
0	1	2	0	4	0	1	2	0	3	
0	1	2	0	4	0	1	2	3	4	$*$
0	1	2	3	3	0	1	2	3	4	$*$
0	1	2	3	3	0	1	2	3	4	$*$
4	0	1	0	4	4	0	1	0	3	
4	0	1	3	3	4	0	1	0	3	
4	0	2	0	3	4	0	2	3	4	
4	1	1	0	3	4	1	1	3	4	
4	1	2	3	4	4	1	2	0	4	
4	1	2	3	4	4	1	2	3	3	

The reduction requires solving $\binom{q}{p}$ instances of $\mathsf{k\,CSP{-}p}$*. It more precisely requires to solve $|\mathcal{T}|$ instances of* $\mathsf{k\,CSP{-}p}$ *where $\mathcal{T}$ is a minimal-size subset of $\mathcal{P}_p(\Sigma_q)$ for which we know there exists a pair $(\Psi,\Phi) \in \Gamma(R,R^*,q,p,k)$ where $R^*/R = \gamma(q,p,k)$ such that the components of every row of Ψ are contained in some $T \in \mathcal{T}$.*

We emphasize that this connection relies on the following relationship between the optimal values on sub-instances $I(T)$ and the optimal value on I:

PROPOSITION 3.3. *For all constant integers $k \geq 2$, $p \geq k$, $q \geq p$, on all instances I of* $\mathsf{k\,CSP{-}q}$*, the best solution among those whose components take at most p distinct values is $\gamma(q,p,k)$-approximate. Formally, let $\mathrm{opt}_p(I\,|\,\mathcal{P}_p(\Sigma_q))$ refer to the quantity:*

$$\begin{cases} \max_{T\subseteq\Sigma_q:|T|=p}\{\max_{x\in T^n} v(I,x)\} & \textit{if the goal on } I \textit{ is to maximize,} \\ \min_{T\subseteq\Sigma_q:|T|=p}\{\min_{x\in T^n} v(I,x)\} & \textit{otherwise.} \end{cases}$$

Then we have:

$$\frac{\mathrm{opt}_p(I\,|\,\mathcal{P}_p(\Sigma_q)) - \mathrm{wor}(I)}{\mathrm{opt}(I) - \mathrm{wor}(I)} \geq \gamma(q,p,k) \tag{3.8}$$

3.5. Some lower bound for numbers $\gamma(q,p,k)$. Our task is to find pairs of arrays that meet the conditions of Definition 3.1 with the largest possible ratio R^*/R.

When $p = q \geq k$, $\Gamma(R^*,R^*,q,q,k) \neq \emptyset$ for all integers $R^* > 0$, considering for Ψ as well as for Φ the array that consists of R^* occurrences of the row $(0,\ldots,q-1)$:

Property 3.4. For all positive integers k, $q \geq k$ and R^*, $\Gamma(R^*,R^*,q,q,k) \neq \emptyset$.

When $q \geq p > k$, for any two integers $R^* > 0$ and $R \geq R^*$, $\Gamma(R,R^*,q,p,k) \neq \emptyset$ provided that $\Gamma(R,R^*,q-p+k,k,k) \neq \emptyset$. Assume the latter family contains a pair of arrays. Then extending each row of these arrays by

$$(q-p+k, q-p+k+1, \ldots, q-1)$$

trivially yields a design of the former family. The following property thus holds:

Property 3.5. For all positive integers k, $p > k$, $q \geq p$, R^* and $R \geq R^*$, if $\Gamma(R, R^*, q-p+k, k, k) \neq \emptyset$, then $\Gamma(R, R^*, q, p, k) \neq \emptyset$.

When $q > p = k$, for a fixed value of k, we show how to derive designs on symbol set Σ_q from designs on symbol set Σ_{q-1}.

LEMMA 3.6. *For all positive integers k, $q > k$, R^*, $R \geq R^*$ and $T = \sum_{r=0}^{k-1} \binom{q-1}{r}\binom{q-2-r}{k-1-r}$, if $\Gamma(R, R^*, q-1, k, k) \neq \emptyset$, then $\Gamma(R + R^* \times T, R^*, q, k, k) \neq \emptyset$.*

Algorithm 3.1 Mapping a pair $(\Psi, \Phi) \in \Gamma(R, R^*, q-1, k, k)$ to a design of $\Gamma(R + T \times R^*, q, k, k)$ where $T = \sum_{r=0}^{k-1} \binom{q-1}{r}\binom{q-2-r}{k-1-r}$

```
Duplicate the first column of each array Ψ and Φ in a qth column
Substitute with each row (0, 1, ..., q−2, 0) of Φ the row (0, 1, ..., q−2, q−1)
for h = k−1 down to 0 do
  for all J ⊆ Σ_{q−1} with |J| = h do
    if h ≡ k−1 mod 2 then
      Insert binom(q−h−2, k−h−1) × R* copies of (α(J), q−1) in Ψ, of (α(J), 0) in Φ
    else
      Insert binom(q−h−2, k−h−1) × R* copies of (α(J), 0) in Ψ, of (α(J), q−1) in Φ
    end if
  end for
end for
```

Proof. With each $J \subseteq \Sigma_{q-1}$, we associate the word $\alpha(J) = (\alpha(J)_0, \ldots, \alpha(J)_{q-2})$ of Σ_q^{q-1} defined by:

$$\alpha(J)_j := \begin{cases} j & \text{if } j \in J \\ q-1 & \text{otherwise} \end{cases}, \qquad j \in \Sigma_{q-1} \tag{3.9}$$

We transform a given pair $(\Psi, \Phi) \in \Gamma(R, R^*, q-1, k, k)$ of arrays into a design of $\Gamma(R + R^* \times T, R^*, q, k, k)$ using Algorithm 3.1. Table 8 illustrates the construction when either $k = 2$ and $q \in \{3, 4, 5, 6\}$, or $k = 3$ and $q \in \{4, 5\}$, starting with the basic design of $\Gamma(1, 1, k, k, k)$. We must prove that, at the end of the construction, (Ψ, Φ) is an element of $\Gamma(R + R^* \times T, R^*, q, k, k)$.

For all natural numbers $h \leq k-1$, the number of h-cardinality subsets of Σ_{q-1} is equal to $\binom{q-1}{h}$. The construction therefore indeed inserts $R^* \times T$ new rows in each of the two arrays. Furthermore, array Φ clearly contains R^* occurrences of the row $(0, 1, \ldots, q-1)$. Moreover, each row of Ψ contains at most k distinct values. For rows Ψ_r with indices $r \leq R$, this property holds due to the initial definition of Ψ, and the way the qth column of Ψ is initialized. For greater indices r, consider that vectors $(\alpha(q, J), q-1)$ and $(\alpha(q, J), 0)$ take respectively $|J| + 1$ and at most $|J| + 2$ distinct values, where $|J| \leq k-1$. Additionally, rows of the form $(\alpha(q, J), 0)$ with $|J| = k-1$ are inserted into Φ, but not into Ψ.

It remains for us to show that the difference $\mu_\Psi - \mu_\Phi$ of the frequencies of rows occurring in Ψ and Φ is balanced k-wise independent. Formally, we must show that arrays Ψ and Φ satisfy:

$$\begin{aligned} |\{r \in [R + R^*T] \mid \Psi_r^J = v\}| - |\{r \in [R + R^*T] \mid \Phi_r^J = v\}| &= 0, \\ J = (j_1, \ldots, j_k) \in \Sigma_q^k,\ j_1 < \ldots < j_k,\ v \in \Sigma_q^k \end{aligned} \tag{3.10}$$

By assumption $(\Psi, \Phi) \in \Gamma(R, R^*, q-1, k, k)$, this is true before proceeding to Line 2 of the algorithm. Right after this line, (Ψ, Φ) violates (3.10) at pairs (J, v) such that $j_k = q-1$ and $v \in \{(j_1, \dots, j_{k-1}, 0), (j_1, \dots, j_{k-1}, j_k)\}$. The details of the proof being technical, we invite the reader to refer to subsection SM5.1 of the supplement. Nevertheless, we set out the principle. First, iteration $h = k-1$ of the external **for** loop fixes the violations of (3.10) induced by Line 2. However, doing so, it generates new violations of (3.10) at pairs (J, v) such that $j_k = q-1$, $v_k \in \{0, q-1\}$, $v_s \in \{j_s, q-1\}, s \in [k-1]$, and $v_s = j_s$ holds for at most $k-2$ integers $s \in [k-1]$. Iterations $h = k-2$ down to 0 of the external **for** loop then iteratively fix the violations of (3.10) at pairs (J, v) such that $j_k = q-1$, $v_k \in \{0, q-1\}$, $v_s \in \{j_s, q-1\}, s \in [k-1]$, and $v_s = j_s$ holds for exactly h integers $s \in [k-1]$. ∎

We derive from Lemma 3.6 a lower bound for numbers $\gamma(q, p, k)$. To proceed, we introduce a quantity central to the upcoming construction:

Property 3.7. For two natural numbers a and $b < a$, we define:

$$T(a, b) := \sum_{r=0}^{b} \binom{a}{r} \binom{a-1-r}{b-r} \tag{3.11}$$

These numbers satisfy the following relations:

$$T(b+1, b) = 2^{b+1} - 1, \qquad b \in \mathbb{N} \tag{3.12}$$

$$T(a, b) = 2^b \binom{a-1}{b} + T(a-1, b-1), \qquad a, b \in \mathbb{N},\ a > b > 0 \tag{3.13}$$

$$T(a, b) = 2^b \binom{a}{b} - T(a, b-1), \qquad a, b \in \mathbb{N},\ a > b > 0 \tag{3.14}$$

$$T(a, b) = 2^{b+1} \binom{a-1}{b} - T(a-1, b), \qquad a, b \in \mathbb{N},\ a > b+1 \tag{3.15}$$

$$T(a, b) = 2T(a-1, b-1) + T(a-1, b), \qquad a, b \in \mathbb{N},\ a > b+1,\ b > 0 \tag{3.16}$$

Proof (sketch). Recursions (3.13) and (3.14) are obtained applying Pascal's rule to coefficients of the form respectively $\binom{a}{r}$ and $\binom{a-1-r}{b-r}$. Relation (3.12) is obvious, since $T(b+1, b)$ is equal to $\sum_{r=0}^{b} \binom{b+1}{r}$. We derive identity (3.15) by subtracting (3.13) from (3.14), both evaluated at $(a, b+1)$. Identity (3.16) is $2 \times (3.13) - (3.15)$. ∎

Algorithm 3.2 Construction for $\Gamma\left((T(q,k)+1)/2, q, k, k\right)$ given two positive integers k and $q > k$

$R \leftarrow 1$
$\Psi, \Phi \leftarrow \{(0, 1, \dots, k-1)\}$
for $i = k+1$ **to** q **do**
  $R \leftarrow R + T(i-1, k-1)$
  Map (Ψ, Φ) into a design of $\Gamma(R, 1, i, k, k)$ using Algorithm 3.1 (with the value i for parameter q)
end for

By iterating the construction of Lemma 3.6 from the basic pair $\Psi = \Phi = \{(0, 1, \dots, k-1)\}$ of arrays and, if $p > k$, then applying the transformation of Property 3.5, we obtain the following lower bounds on numbers $\gamma(q, p, k)$:

THEOREM 3.8. *Let $k > 0$, $p \geq k$ and $q \geq p$ be three integers. If $p = q$, then $\gamma(q, q, k) = 1$. Otherwise, we have:*

$$\gamma(q, p, k) \geq 2 / \left(\sum_{r=0}^{k} \binom{q-p+k}{r} \binom{q-p+k-1-r}{k-r} + 1 \right) \tag{3.17}$$

TABLE 8
Construction for $\Gamma\left((T(q,k)+1)/2, 1, q, k, k\right)$*: illustration when* $(k,q) \in \{(2,6),(3,5)\}$*.*

$2/(T(6,2)+1) = 1/25$

Ψ^0	Ψ^1	Ψ^2	Ψ^3	Ψ^4	Ψ^5
0	1	0	0	0	0
0	2	2	0	0	0
2	1	2	2	2	2
2	2	0	2	2	2
0	3	3	3	0	0
3	1	3	3	3	3
3	3	2	3	3	3
3	3	3	0	3	3
3	3	3	0	3	3
0	4	4	4	4	0
4	1	4	4	4	4
4	4	2	4	4	4
4	4	4	3	4	4
4	4	4	4	0	4
4	4	4	4	0	4
4	4	4	4	0	4
0	5	5	5	5	5
5	1	5	5	5	5
5	5	2	5	5	5
5	5	5	3	5	5
5	5	5	5	4	5
5	5	5	5	5	0
5	5	5	5	5	0
5	5	5	5	5	0
5	5	5	5	5	0

Φ^0	Φ^1	Φ^2	Φ^3	Φ^4	Φ^5
0	1	2	3	4	5
0	2	0	0	0	0
2	1	0	2	2	2
2	2	2	2	2	2
0	3	3	0	0	0
3	1	3	0	3	3
3	3	2	0	3	3
3	3	3	3	3	3
3	3	3	3	3	3
0	4	4	4	0	0
4	1	4	4	0	4
4	4	2	4	0	4
4	4	4	3	0	4
4	4	4	4	4	4
4	4	4	4	4	4
4	4	4	4	4	4
0	5	5	5	5	0
5	1	5	5	5	0
5	5	2	5	5	0
5	5	5	3	5	0
5	5	5	5	4	0
5	5	5	5	5	5
5	5	5	5	5	5
5	5	5	5	5	5
5	5	5	5	5	5

$2/(T(5,3)+1) = 1/25$

Ψ^0	Ψ^1	Ψ^2	Ψ^3	Ψ^4
0	1	2	0	0
0	1	3	3	0
0	3	2	3	0
3	1	2	3	3
0	3	3	0	0
3	1	3	0	3
3	3	2	0	3
3	3	3	3	3
0	1	4	4	4
0	4	2	4	4
0	4	4	3	4
4	1	2	4	4
4	1	4	3	4
4	4	2	3	4
0	4	4	4	0
0	4	4	4	0
4	1	4	4	0
4	1	4	4	0
4	4	2	4	0
4	4	2	4	0
4	4	4	3	0
4	4	4	3	0
4	4	4	4	4
4	4	4	4	4
4	4	4	4	4

Φ^0	Φ^1	Φ^2	Φ^3	Φ^4
0	1	2	3	4
0	1	3	0	0
0	3	2	0	0
3	1	2	0	3
0	3	3	3	0
3	1	3	3	3
3	3	2	3	3
3	3	3	0	3
0	1	4	4	0
0	4	2	4	0
0	4	4	3	0
4	1	2	4	0
4	1	4	3	0
4	4	2	3	0
0	4	4	4	4
0	4	4	4	4
4	1	4	4	4
4	1	4	4	4
4	4	2	4	4
4	4	2	4	4
4	4	4	3	4
4	4	4	3	4
4	4	4	4	0
4	4	4	4	0
4	4	4	4	0

Proof. The case where $p = q$ is trivial (see Property 3.4). The case where $q > p = k$ then follows from Lemma 3.6 and identity (3.16). Consider Algorithm 3.2 (see Table 8 for an illustration when $(k,q) \in \{(2,6),(3,5)\}$). The first call to Algorithm 3.1 derives from the basic design of $\Gamma(1,1,k,k)$ a design of $\Gamma(1+T(k,k-1),1,k,k)$ where, according to relation (3.12):

$$T(k,k-1)+1 \quad = 2^k \quad = (T(k+1,k)+1)/2$$

Then, at each further iteration $i \in \{k+2,\dots,q\}$, the call to Algorithm 3.1 derives from a design of $\Gamma((T(i-1,k)+1)/2,1,k,k)$ a design of $\Gamma((T(i-1,k)+1)/2+T(i-1,k-1),1,k,k)$ where, according to identity (3.16):

$$(T(i-1,k)+1)/2 + T(i-1,k-1) \quad = 1/2 + T(i,k)/2 \quad = (T(i,k)+1)/2$$

The case where $q > p > k$ eventually follows from Property 3.5. ∎

3.6. Approximation results. Theorem 3.8 together with Theorem 3.2 implies that for any two integers $k \geq 2$ and $q > k$, $\mathsf{k\,CSP{-}q}$ D-reduces to $\mathsf{k\,CSP{-}k}$ with an expansion of at most $2/(T(q,k)+1)$ on the approximation guarantee. Furthermore, when combined with Property 3.5, these theorems imply that for any three integers $k \geq 2$, $p \geq k$ and $q > p$, $\mathsf{k\,CSP{-}q}$ D-reduces to $\mathsf{k\,CSP{-}p}$ with an expansion on the approximation guarantee of:

$$\gamma(q,p,k) \quad \geq 2/(T(q-p+k,k)+1)$$

We seek a simple estimate of $(T(a,b)+1)/2$. Applying first recursion (3.14) to

$T(a,b)$ and then recursion (3.13) to $T(a,b-1)$, one gets:

$$\begin{aligned} T(a,b)+1 &= 2^b\binom{a}{b} - T(a,b-1)+1 && \text{by (3.14)}\\ &= 2^b\binom{a}{b} - 2^{b-1}\binom{a-1}{b-1} - T(a-1,b-2)+1 && \text{by (3.13)} \end{aligned}$$

First, we deduce from recursion (3.13) and definition (3.11) that we have:

$$T(a-1,b-2)-1 \quad \geq T(a-b+1,0)-1 \quad = \binom{a-b+1}{0}\binom{a-b-0}{0}-1 \quad = 0$$

Thus $T(a,b)+1 \leq 2^b\binom{a}{b} - 2^{b-1}\binom{a-1}{b-1}$. Second we observe that $2^b\binom{a}{b} - 2^{b-1}\binom{a-1}{b-1}$ equivalently writes as:

$$\begin{aligned} 2^b\binom{a}{b} - 2^{b-1}\binom{a-1}{b-1} &= 2^b\binom{a-1}{b-1} \times (a/b - 1/2)\\ &= (2^b/b!) \times (a-1) \times \ldots \times (a-b+1) \times (a-b/2) \end{aligned}$$

Finally, we deduce from the inequality of arithmetic and geometric means that we have:

$$\prod_{i=1}^{b-1}(a-i) \quad \leq \left(\sum_{i=1}^{b-1}(a-i)/(b-1)\right)^{b-1} \quad = (a-b/2)^{b-1}$$

In conclusion, we obtain for $(T(a,b)+1)/2$ the following proper upper bound:

$$(T(a,b)+1)/2 \leq 2^{b-1}/b! \times (a-b/2)^b, \qquad a,b\in\mathbb{N},\ a>b \tag{3.18}$$

We thus can derive from Theorems 3.2 and 3.8 the following corollary:

COROLLARY 3.9. *Given any constant integers $k\geq 2$, $p\geq k$ and $q>p$, if* $\mathsf{k\,CSP{-}p}$ *is approximable within some differential factor ρ, then* $\mathsf{k\,CSP{-}q}$ *is approximable within differential factor $\gamma\times\rho$, where:*

$$\gamma \quad = 2/\left(\sum_{r=0}^{k}\binom{q-p+k}{r}\binom{q-p+k-1-r}{k-r}+1\right) \quad \geq 2(k!)/(2q-2p+k)^k$$

In particular, for all $q\geq 3$, $\mathsf{2\,CSP{-}q}$ D-reduces to $\mathsf{2\,CSP{-}2}$ with an expansion of $1/(q-1)^2$ on the approximation guarantee. Therefore, the result of [41] implies for all $q\geq 2$ that $\mathsf{2\,CSP{-}q}$ is differentially approximable within some constant factor using semidefinite programming along with derandomization techniques.

COROLLARY 3.10. *For all constant integers $q\geq 2$,* $\mathsf{2\,CSP{-}q}$ *is approximable within differential factor $(2-\pi/2)/(q-1)^2 \geq 0.429/(q-1)^2$.*

3.7. Concluding remarks. In [18], we show that the construction of Algorithm 3.2 is optimal. Therefore, the obtained estimate of the expansion of the reduction from $\mathsf{k\,CSP{-}q}$ to $\mathsf{k\,CSP{-}k}$ — or of the differential ratio reached at the optimum value over $\{T^n\,|\,T\subseteq\Sigma_q : |T|=k\}$ — is the best estimate we can obtain using our approach.

For the case where $p>k$, our estimate of the expansion of the reduction from $\mathsf{k\,CSP{-}q}$ to $\mathsf{k\,CSP{-}p}$ relies on the relation $\gamma(q,p,k)\geq\gamma(q-p+k,k,k)$. However, given three such integers $k>0$, $p>k$ and $q>p$, $\gamma(q+1,p+1,k)$ is the most likely $>\gamma(q,p,k)$. For instance, we have (see Table 10):

$$\begin{aligned} \gamma(6,4,2)=1/4 &\quad >\gamma(5,3,2)=1/6 &\quad >\gamma(4,2,2)=1/9\\ \gamma(6,5,3)=1/4 &\quad >\gamma(5,4,3)=1/5 &\quad >\gamma(4,3,3)=1/8 \end{aligned}$$

Thus, a closer study of the families $\Gamma(R,R^*,q,p,k)$ of designs when $p>k$ could provide a finer estimate of the expansion of the reduction, by providing a finer estimate

TABLE 9

Pairs of arrays that achieve $\gamma_E(4,3,2)$ and $\gamma_E(5,4,3)$. We use the $*$ mark to emphasis rows of the form $(0,\ldots,q-1)$.

$\gamma_E(4,3,2) = 1/2$

Ψ^0	Ψ^1	Ψ^2	Ψ^3	Φ^0	Φ^1	Φ^2	Φ^3	
0	0	1	2	0	0	0	0	
0	0	1	2	0	0	3	1	
0	1	0	3	0	1	2	3	$*$
0	1	2	0	0	1	2	3	$*$
0	1	2	0	0	1	2	3	$*$
0	1	2	1	0	1	2	3	$*$
0	1	3	3	0	1	2	3	$*$
0	1	3	3	0	1	2	3	$*$
0	2	2	3	0	2	0	2	
0	2	2	3	0	2	1	1	
0	3	0	1	0	3	1	0	
0	3	2	3	0	3	3	2	

$\gamma_E(5,4,3) = 1/3$

Ψ^0	Ψ^1	Ψ^2	Ψ^3	Ψ^4	Φ^0	Φ^1	Φ^2	Φ^3	Φ^4	
0	0	1	2	3	0	0	1	3	4	
0	0	2	3	4	0	0	2	2	4	
0	0	2	3	4	0	0	2	3	3	
0	1	1	3	4	0	1	1	2	3	
0	1	2	2	4	0	1	2	3	4	$*$
0	1	2	3	0	0	1	2	3	4	$*$
0	1	2	3	3	0	1	2	3	4	$*$
0	1	2	4	4	0	1	2	3	4	$*$
0	1	3	3	4	0	1	3	4	0	
0	2	2	3	4	0	2	2	3	0	
0	2	2	3	4	0	2	2	4	4	
0	2	3	4	0	0	2	3	3	4	

of the differential ratio reached at a best solution among those whose coordinates take at most p distinct values.

Furthermore, similarly to the designs of the preceding section, when restricting to $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$, the constraints on arrays Φ, Ψ can be slightly relaxed. On the one hand, any function P_i that occurs in an instance of $\mathsf{CSP}(\mathcal{E}_\mathsf{q})$ takes the same value when evaluated at any two entries $v \in \Sigma_q^{k_i}$ and $v + \mathbf{a}$. On the other hand, given any $u \in \Sigma_q^q$ and any $a \in \Sigma_q$, the solution $y(x, u + \mathbf{a})$ precisely consists of $y(x,u) + \mathbf{a}$. Hence, when reducing from $\mathsf{CSP}(\mathcal{E}_\mathsf{q})$, one rather should consider the following relaxation of families $\Gamma(R, R^*, q, p, k)$ of designs:

DEFINITION 3.11. *For five positive integers k, $p \geq k$, $q \geq p$, R^* and $R \leq R^*$, we define $\Gamma_E(R, R^*, q, p, k)$ as the set of pairs (Ψ, Φ) of $R \times q$ arrays on Σ_q that satisfy:*

1. $\sum_{a=0}^{q-1} |\{r \in [R] \,|\, \Phi_r = (a, 1+a, \ldots, q-1+a)\}| = R^*$*;*
2. *the components of each row of Ψ take at most p distinct values;*
3. $\mu_E^\Psi - \mu_E^\Phi$ *is balanced k-wise independent.*

Moreover, we define $\gamma_E(q, p, k)$ as the greatest number γ for which there exist two positive integers R and $R^* \leq R^*$ such that $R^*/R = \gamma$ and $\Gamma_E(R, R^*, q, p, k) \neq \emptyset$.

Table 9 depicts a few pairs of arrays that achieve $\gamma_E(q,p,k)$ (and that have been calculated by computer). Similarly to $\gamma(q,p,k)$, $\gamma_E(q,p,k)$ provides a lower bound for the expansion of the reduction from $\mathsf{k\,CSP{-}q}$ to $\mathsf{k\,CSP{-}p}$ when applied to instances of $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$, especially because this number is, for such instances of $\mathsf{k\,CSP{-}q}$, a lower bound for the differential ratio reached at a best solution among those whose coordinates take at most p distinct values.

THEOREM 3.12. *For all constant integers $k \geq 2$, $p \geq k$ and $q \geq p$, $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$ D-reduces to $\mathsf{k\,CSP{-}p}$ with an expansion of $\gamma_E(q,p,k)$ on the approximation guarantee.*

Proof. Assume that a pair $(\Psi, \Phi) \in \Gamma_E(R, R^*, q, p, k)$ exists for some positive integers R and $R^* \leq R$. Consider then an instance I of $\mathsf{CSP}(\mathcal{E}_\mathsf{q})$. We adopt the same notations as in subsection 3.4. Since I is an instance of $\mathsf{CSP}(\mathcal{E}_\mathsf{q})$, P_i is stable under a shift by $(a, \ldots, a)$ of its entry, $a \in \Sigma_q$. Therefore, a sufficient condition for (Ψ, Φ) to satisfy (3.7) at (i, x) is that, for each $v \in \Sigma_q^{k_i}$, solutions from the multisets $(y(x, \Psi_r) \,|\, r \in [R])$ and $(y(x, \Phi_r) \,|\, r \in [R])$ coincide as frequently with a vector from $\{v, v + \mathbf{1}, \ldots, v + \mathbf{q-1}\}$ on their coordinates in J_i. By definition of solutions $y(x,u)$,

Table 10

*Value of numbers $\gamma(q,p,k)$ and $\gamma_E(q,p,k)$ for some triples (q,p,k). These values have been calculated by computer solving linear programs in continuous and in integer variables (see section SM2 for more details). The * mark indicates the triples for which we have not yet been able to calculate an optimal integer solution, in which case the result possibly is inaccurate, due to numerical approximation.*

			q					
k	p	γ or γ_E	3	4	5	6	7	8
2	2	γ	1/4	1/9	1/16	1/25	1/36	1/49
		γ_E	1/3	1/4	1/5	9/59	1/7	1/8
	3	γ	–	1/3	1/6	1/10	1/15	1/21
		γ_E	–	1/2	2/5	4/13	2/7	93/404
	4	γ	–	–	4/9	1/4	4/25	1/9
		γ_E	–	–	3/5	7/15	3/7	3/8
	5	γ	–	–	–	1/2	3/10	1/5
		γ_E	–	–	–	2/3	11/21	13/28
	6	γ	–	–	–	–	9/16	9/25
		γ_E	–	–	–	–	5/7	4/7
	7	γ	–	–	–	–	–	3/5
		γ_E	–	–	–	–	–	3/4
3	3	γ	–	1/8	1/25	1/56	1/105	1/176
		γ_E	–	1/4	1/11	38425/701342	3676/107221*	
	4	γ	–	–	1/5	2/27	1/28	1/50
		γ_E	–	–	1/3	1/6	5/52*	
	5	γ	–	–	–	1/4	5/49	5/96
		γ_E	–	–	–	4/9	2/9*	
	6	γ	–	–	–	–	2/7	1/8
		γ_E	–	–	–	–	1/2	
	7	γ	–	–	–	–	–	7/20
		γ_E	–	–	–	–	–	9/16
4	4	γ	–	–	1/16	1/65	1/176	1/385
		γ_E	–	–	1/11	~ 0.03159029059*	~ 0.013964734*	
	5	γ	–	–	–	1/10	1/36	1/91
		γ_E	–	–	–	1/6	~ 0.058898*	
	6	γ	–	–	–	–	5/33	1/21
		γ_E	–	–	–	–	~ 0.2088948787*	
	7	γ	–	–	–	–	–	9/49
5	5	γ	–	–	–	1/32	1/161	1/512
		γ_E	–	–	–	1/16	~ 0.01281777623*	
	6	γ	–	–	–	–	2/35	1/78
		γ_E	–	–	–	–	1/10	
	7	γ	–	–	–	–	–	5/64
6	6	γ	–	–	–	–	1/64	1/385
		γ_E	–	–	–	–	1/42*	
	7	γ	–	–	–	–	–	1/35

this occurs if and only if for each $v \in \Sigma_q^t$, the multisets $(\Psi_r^H \mid r \in [R])$ and $(\Phi_r^H \mid r \in [R])$ of words coincide as many often with a word from $\{v, v+\mathbf{1}, \ldots, v+\mathbf{q-1}\}$. Equivalently, given any $v \in \Sigma_q^t$, the overall frequency of words $u \in \Sigma_q^q$ such that $u_H \in \{v, v+\mathbf{1}, \ldots, v+\mathbf{q-1}\}$ should be the same in Ψ as in Φ. This condition is precisely verified by the assumption that $\mu_E^\Psi - \mu_E^\Phi$ is balanced k-wise independent, considering that $k \geq t$. Thus, arrays Ψ, Φ do satisfy relation (3.7).

We assume *w.l.o.g.* that the goal on I is to maximize. Similarly to the general case, we denote by $\mathcal{T}$ a subset of $\mathcal{P}_p(\Sigma_q)$ that satisfies for all $r \in [R]$ that $\{\Psi_r^0, \ldots, \Psi_r^{q-1}\}$ is included in some $T \in \mathcal{T}$. Notice that $|\mathcal{T}|$ obviously is at most $\binom{q}{p}$. Assume there exists a ρ-differential approximation algorithm $\mathcal{A}$ for $\mathsf{k\,CSP-p}$. Given

$T \in \mathcal{T}$, we denote by $x(T)$ the solution of $I(T)$ function $g_{T^n}(I,.)$ associates with the solution algorithm $\mathcal{A}$ returns on $f_{T^n}(I)$. On the one hand, by assumption on $\mathcal{A}$, these solutions satisfy relation (3.6). On the other hand, considering that at least R^* of the solutions $y(x, \Phi_r)$, $r \in [R]$ are optimal provided that x is optimal, we deduce from (3.7) that sub-instances $I(T)$, $T \in \mathcal{T}$ satisfy relation (3.5). We conclude that a solution with optimal value over $\{x(T) \,|\, T \in \mathcal{T}\}$ achieves on I a differential ratio at least $\rho \times R^*/R$. ∎

Table 10 provides the value of $\gamma_E(q,p,k)$ and $\gamma(q,p,k)$ for a few triples (q,p,k). Although for all integers $k \geq 2$, $p \geq k$ and $q > p$, $\mathsf{k\,CSP}(\mathcal{E}_\mathsf{q})$ D-reduces to $\mathsf{k\,CSP{-}p}$ with an expansion of $\gamma(q,p,k) \geq 2/(T(q{-}p{+}k,k){+}1)$ on the approximation guarantee, it is most likely the case, for three such integers, that $\gamma_E(q,p,k) > \gamma(q,p,k)$ (this is in fact true for all the cases we have computed). For instance (see Table 10), we have:

$$\begin{array}{llll} \gamma_E(5,2,2)/\gamma(5,2,2) & = 16/5 & \gamma_E(5,3,2)/\gamma(5,3,2) & = 12/5 \\ \gamma_E(5,3,3)/\gamma(5,3,3) & = 25/11 & \gamma_E(5,4,3)/\gamma(5,4,3) & = 5/3 \end{array}$$

Notably, $\gamma_E(q,2,2)$ equals $1/q$ for all $q \in \{3,4,5,7,8\}$. Thus, for $q \in \{3,4,5,7,8\}$, it follows from [41] that $\mathsf{2\,CSP}(\mathcal{E}_\mathsf{q})$ is approximable within differential factor $0.429/q$ (rather than $0.429/(q{-}1)^2$). Families $\Gamma_E(R,R^*,q,p,k)$ of designs thus deserve further study, in particular for the case where $q = k = 2$. These families seem to be more challenging than families $\Gamma(R,R^*,q,p,k)$ of designs. This is notably due to the fact that two vectors $u,v \in \mathbb{Z}_q^k$ may satisfy $u_j - u_1 = v_j - v_1 \bmod q$, $j \in \{2,\dots,k\}$, but $u_j - u_1 \neq v_j - v_1 \bmod (q+1)$ for some $j \in \{2,\dots,k\}$. Consequently, in the most general case, a pair of arrays from $\Gamma_E(R,R^*,q,p,k)$ cannot be interpreted as a pair of subarrays of a design from $\Gamma_E(S,S^*,q+1,p,k)$ where $S \geq R$ and $S^* \in [R^*]$, and vice-versa. (By contrast, two vectors $u,v \in \mathbb{Z}_q^k$ satisfying $u = v$ can always be viewed as two vectors of $\mathbb{Z}_{q+1}^k$ satisfying $u = v$. Designs from $\Gamma(R,R^*,q,p,k)$ therefore can be interpreted as partial designs of $\Gamma(S,S^*,q+1,p,k)$ for some integers $S \geq R$ and $S^* > 0$.)

4. At the neighbourhood of any solution.

We explored in section 2 the question whether the average solution value yields any differential approximation guarantee. In this section, we address a similar question: we examine the differential ratio reached at solutions with optimum value over Hamming balls of a given radius $d \geq k$, or over the union of the shifts of such balls by vectors of the form $\mathbf{a}$.

4.1. Definitions and previous related works.

Given three natural numbers $q > 1$, d and $\nu \geq d$, the *Hamming distance* between two vectors x, y of Σ_q^ν, which we denote by $d_H(x,y)$, is the number of coordinates on which x and y differ, i.e.:

$$d_H(x,y) = |\{j \in [\nu] \,:\, x_j \neq y_j\}|, \qquad x,y \in \Sigma_q^\nu \tag{4.1}$$

For $x \in \Sigma_q^\nu$, the *Hamming ball of radius d centered at x*, which we denote by $B^d(x)$, is the set of the ν-dimensional q-ary vectors y that are at Hamming distance at most d from x, i.e.:

$$B^d(x) = \{y \in \Sigma_q^\nu \,:\, d_H(x,y) \leq d\}, \qquad x \in \Sigma_q^\nu \tag{4.2}$$

In particular, $x \in B^d(x)$. For any $a \in \Sigma_q$, we denote by $B^d_\mathbf{a}$ the function that associates with any $x \in \Sigma_q^n$ the set of the shifts by $\mathbf{a}$ of vectors from $B^d(x)$ or, equivalently, the Hamming ball of radius d centered at $x + \mathbf{a}$; namely:

$$B^d_\mathbf{a}(x) = B^d(x + \mathbf{a}), \qquad x \in \Sigma_q^\nu,\ a \in \Sigma_q \tag{4.3}$$

Finally, we denote by $\tilde{B}^d$ the function that associates with any $x \in \Sigma_q^n$ the union of the sets $B^d_{\mathbf{a}}$, $a \in \Sigma_q$, i.e.:

$$\tilde{B}^d(x) = B^d(x) \cup B^d_{\mathbf{1}}(x) \cup \ldots \cup B^d_{\mathbf{q-1}}(x), \qquad x \in \Sigma_q^\nu \tag{4.4}$$

A common heuristic approach for CSPs consists in fixing some radius d, and then computing a local optimum with respect to B^d, that is, a solution x with value $v(I,x) \geq v(I,y)$, $y \in B^d(x)$ if the goal is to maximize, with value $v(I,x) \leq v(I,y)$, $y \in B^d(x)$ if the goal is to minimize. Many articles address the question whether such solutions yield any approximation guarantee [33, 38, 5, 2]. Notably, Khanna et al. showed that for $\mathsf{Max\,2\,CCSP}$, there is no constant integer d for which local search with respect to B^d yields any constant standard approximation guarantee [33]. In the differential approximation paradigm, this fact extends by reduction to $\mathsf{Max\,Ek\,Sat}$ for all $k \geq 2$, [38]. On the other hand, we observed in a previous section that for $\mathsf{CSP}(\mathcal{O}_\mathsf{q})$, local optima with respect to $\tilde{B}^0$ are $1/q$-differential approximate.

4.2. From the average solution value to solution values over Hamming balls of radius one. The results of section 2 allow to identify a few cases where local optima with respect to B^1 or $\tilde{B}^1$, as well as solutions with optimum value at the neighbourhood B^1 or $\tilde{B}$ of any solution, yield some differential approximation guarantee.

Property 4.1. Let I be an instance of $\mathsf{Ek\,CSP}(\mathcal{I}_\mathsf{q}^{\mathsf{k}-1})$ where $q \geq 2$ and $k \geq 2$. We denote by $\mathrm{avd}(I)$ the average differential ratio on I. Then local optima with respect to B^1 and solutions with optimum value over Hamming balls of radius 1 achieve a differential ratio at least, respectively, $\mathrm{avd}(I)$ and $1/n \times kq/(q-1) \times \mathrm{avd}(I)$.

Proof. Let $x \in \Sigma_q^n$. We aim at evaluating the sum of the solution values over $B^1(x)\backslash\{x\}$. Consider a constraint $P_i(x_{J_i}) = P_i(x_{i_1}, \ldots, x_{i_k})$ of I. Over $B^1(x)\backslash\{x\}$, P_i is evaluated:

- for all $s \in [k]$, once on each $(x_{i_1}, \ldots, x_{i_{s-1}}, x_{i_s} + a, x_{i_{s+1}}, \ldots, x_{i_k})$, $a \in [q-1]$;
- $(q-1) \times (n-k)$ times on x_{J_i}.

Equivalently, P_i is taken

- for all $s \in [k]$, once at each $(x_{i_1}, \ldots, x_{i_{s-1}}, a, x_{i_{s+1}}, \ldots, x_{i_k})$, $a \in \Sigma_q$, plus
- $(q-1) \times (n-k) - k = (q-1)n - qk$ times at x_{J_i}.

As P_i is balanced $(k-1)$-wise independent, we deduce:

$$\textstyle\sum_{y \in B^1(x): y \neq x} P_i(y_{J_i}) \quad = k \times q\, r_{P_i} + ((q-1)n - qk) \times P_i(x_{J_i})$$

Hence, considering that $|B^1(x)\backslash\{x\}| = (q-1)n$, we find for the sum of solution values over $B^1(x)\backslash\{x\}$ the following expression:

$$\textstyle\sum_{y \in B^1(x): y \neq x} v(I,y) = qk \times \mathbb{E}_X[v(I,X)] + \left(|B^1(x)\backslash\{x\}| - qk\right) \times v(I,x) \tag{4.5}$$

From (4.5), it follows that the average differential ratio over $B^1(x)\backslash\{x\}$ is bounded below by $qk/|B^1(x)\backslash\{x\}| \times \mathrm{avd}(I)$. Furthermore, if x is a local optimum with respect to B^1, then assuming *w.l.o.g.* that the goal on I is to maximize, we have:

$$|B^1(x)\backslash\{x\}| \times v(I,x) \geq \textstyle\sum_{y \in B^1(x): y \neq x} v(I,y) \tag{4.6}$$

Combining (4.6) with (4.5), we deduce that $v(I,x) \geq \mathbb{E}_X[v(I,X)]$. Equivalently, the differential ratio taken at x is at least the average differential ratio on I. ∎

From Property 4.1, it follows that the differential guarantees discussed in section 2 for $\mathsf{Ek\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k}-1})$, which hold at the average solution value, also apply to local optima with respect to B^1. Furthermore, these guarantees extend to solutions with optimum value over Hamming balls of radius one, although up to an expansion of $O(1/n)$ on the approximation guarantee. Note that $\mathsf{Ek\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k}-1})$ notably covers the restriction of $\mathsf{Lin}-\mathsf{q}$ to equations of the form $x_{i_1}+\ldots+x_{i_k}\equiv\alpha_0 \bmod q$. In particular, for $q=2$, according to Corollary 2.19, local optima with respect to B^1 and solutions with optimum value over Hamming balls with radius 1 yield a differential approximation guarantee of respectively $\Omega(1/n^k)$ and $\Omega(1/n^{k+1})$ for $\mathsf{E(2k)Lin}-2$.

For $\mathsf{E2\,Lin}-2$, we precisely obtain the ratios $1/(2\lceil n/2\rceil)$ and $2/(\lceil n/2\rceil\times n)$. We observe that these ratios are asymptotically tight. Instances $I_n^{2,2}$ discussed in subsection 2.7 serve as evidence for the highest differential ratio attained over $B^1(x)$ for any $x\in\{0,1\}^n$. Let n be a positive integer. Given any $d\in\{0,\ldots,n\}$, on $I_n^{2,2}$, any boolean vector with $n\pm d$ nonzero coordinates satisfies all but $(n-d)(n+d)=n^2-d^2$ of the constraints. In particular, we have $\operatorname{opt}(I)=\binom{2n}{2}$ and $\operatorname{wor}(I)=\binom{2n}{2}-n^2$. Moreover, provided that x_* is a balanced vector, given any $d\in[n]$, the maximum solution value over $\tilde{B}^d(x_*)$ equals $\binom{2n}{2}-n^2+d^2$. For such a vector x_*, the highest differential ratio reached over $\tilde{B}^d(x_*)$ therefore is equal to d^2/n^2. When $d=1$, this ratio coincides with the lower bound deduced from Property 4.1 and Corollary 2.19.

Observe that local search with respect to B^1 will return on $I_n^{2,2}$ one of the two optimal solutions $\mathbf{0}$ and $\mathbf{1}$. Consider then the instance $\tilde{I}_n$ of $\mathsf{CSP}(\{\mathsf{XNOR}^2\})$ obtained from $I_n^{2,2}$ by removing the constraints $(x_{2j-1}=x_{2j})$, $j\in[n]$. This instance still is trivially satisfiable (for example, by the all-zeros vector), and thus $\operatorname{opt}(\tilde{I}_n)=\binom{2n}{2}-n=n(2n-2)$. Furthermore, solutions that violate a maximum number of constraints are balanced vectors x_* satisfying for all (but one if n is odd) $j\in[n]$ that $x_{*2j-1}=x_{*2j}$. Therefore, we have:

$$\operatorname{wor}(\tilde{I}_n) \quad = 2\binom{n}{2}-n+(n \bmod 2) \quad = n(n-2)+(n \bmod 2)$$

Eventually note that the solution $\tilde{x}$ whose nonzero coordinates are the coordinates with odd index clearly is a local optimum with respect to $\tilde{B}^1$, with value $2\binom{n}{2}=n(n-1)$. The differential ratio reached at $\tilde{x}$ therefore is equal to:

$$\frac{n(n-1)-n(n-2)-(n \bmod 2)}{n(2n-2)-n(n-2)-(n \bmod 2)} \quad = \frac{n-(n \bmod 2)}{n^2-n \bmod 2)} \quad = \frac{1}{n+(n \bmod 2)}$$

Since $\tilde{I}_n$ manipulates $2n$ variables, this ratio is asymptotically a factor 2 of the ratio deduced from Property 4.1 and Corollary 2.19.

Similar conclusions can be drawn for $\mathsf{3\,CSP}(\mathcal{E}_2)$ as those we derived from section 2 for $\mathsf{Ek\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k}-1})$:

Property 4.2. Let I be an instance of $\mathsf{3\,CSP}(\mathcal{E}_2)$, and $\operatorname{avd}(I)$ be the average differential ratio on I. Then, local optima with respect to B^1 and solutions with optimum value over Hamming balls of radius 1 achieve a differential ratio of at least, respectively, $\operatorname{avd}(I)$ and $4/n\times\operatorname{avd}(I)$.

Proof. Over $B^1(x)\backslash\{x\}$, a constraint of the form $P_i(x_{i_1},x_{i_2})$ is evaluated once on $(\bar{x}_{i_1},x_{i_2})$ and $(x_{i_1},\bar{x}_{i_2})$, and $n-2$ times on (x_{i_1},x_{i_2}). Since P_i evaluates the same on any two entries y and $\bar{y}$, equivalently, P_i is taken twice at $(0,0)$ and $(0,1)$, plus $n-4$ times at (x_{i_1},x_{i_2}). If we now consider a constraint of the form $P_i(x_{i_1},x_{i_2},x_{i_3})$, it is evaluated once on $(\bar{x}_{i_1},x_{i_2},x_{i_3})$, $(x_{i_1},\bar{x}_{i_2},x_{i_3})$ and $(x_{i_1},x_{i_2},\bar{x}_{i_3})$, and $n-3$ times on $(x_{i_1}, x_{i_2}, x_{i_3})$. Equivalently, P_i is taken once at each $v \in \{0\} \times \{0,1\}^2$, plus $n-4$ times at $(x_{i_1}, x_{i_2}, x_{i_3})$. We deduce that the sum of the solution values over $B^1(x)\backslash\{x\}$ satisfies:

$$\textstyle\sum_{y \in B^1(x):y\neq x} v(I,y) \quad = 4 \times \mathbb{E}_X[v(I,X)] + \big(|B^1(x)\backslash\{x\}| - 4\big) \times v(I,x)$$

The end of the argument is the same as for Property 4.1. ∎

We finally observe that such guarantees somehow extend by reduction to $2\,\mathsf{CSP}-2$.

Property 4.3. On ν-partite instances of $2\,\mathsf{CSP}-2$, local optima with respect to $\tilde{B}^1$ and solutions with optimum value over the neighbourhood $B^1(x) \cup B^1_{\mathbf{1}}(x)$ of any solution x yield a differential approximation guarantee of respectively $1/(\nu+1)$ and $4/\left((\nu+1)(n+1)\right)$ if ν is odd, of $1/(\nu+2)$ and $4/\left((\nu+2)(n+1)\right)$ otherwise.

Proof. Consider an instance I of $2\,\mathsf{CSP}-2$. From I, we derive an instance J of $3\,\mathsf{CSP}(\mathcal{E}_2)$ as follows: first, we introduce a new binary variable z_0; then, we substitute for each constraint $P_i(x_{i_1})$ or $P_i(x_{i_1}, x_{i_2})$ of I the new constraint respectively $P_i(x_{i_1} + z_0)$ or $P_i(x_{i_1} + z_0, x_{i_2} + z_0)$. The strong chromatic number of J is $\nu+1$. Furthermore, the objective functions on I and J satisfy:

$$v(I,(x_1+z_0,\ldots,x_n+z_0)) \quad = v(J,(x,z_0)), \quad x \in \{0,1\}^n,\ z_0 \in \{0,1\}$$

Two solutions $x + \mathbf{z_0}$ of I and (x, z_0) of J thus achieve the same differential ratio on their respective instances. Let $(x, z_0) \in \{0,1\}^n \times \{0,1\}$, and $\tilde{x} \in B^1(x)$. Assume w.l.o.g. that the goal on I (and thus, on J) is to maximize. We observe:

- $v(I,\tilde{x}) \geq v(I,y)$, $y \in B^1(x)$ *iff* $v(J,(\tilde{x},0)) \geq v(J,(y,0))$, $y \in B^1(x)$;
- $v(I,\tilde{x}) \geq v(I,\bar{x})$ *iff* $v\left(J,(\tilde{x},0)\right) \geq v\left(J,(x,1)\right)$;
- $B^1\left((x,0)\right) = \{(y,0) \,|\, y \in B^1(x)\} \cup \{(x,1)\}$.

We deduce that solution $(\tilde{x}, 0)$ of J is optimal over $B^1(x,0)$ provided that solution $\tilde{x}$ is optimal over $B^1(x) \cup B^1(\bar{x})$. In particular, if x is a local optimum with respect to $\tilde{B}^1$ on I, then $(\tilde{x},0)$ is a local optimum with respect to B^1 on J. Now we know from Property 4.2 that on J, the differential ratio reached at local optima with respect to B^1, or at solutions with optimum value over Hamming balls with radius 1, is at least a factor respectively 1 or $4/(n+1)$ of the average differential ratio. According to Corollary 2.20, this ratio is bounded below by $1/(\nu+1)$ if $\nu+1$ is even, by $1/(\nu+2)$ otherwise. The result is straightforward. ∎

Table 11 summarizes the approximation guarantees induced by Properties 4.1 to 4.3.

4.3. Hamming balls with radius at least k: preliminary remarks. We explore the following inquiry: for $\mathsf{k\,CSP-q}$, do solutions with extremal values over Hamming balls of fixed radius $d \geq k$ offer any differential approximation guarantees?

Consider an instance I of $\mathsf{k\,CSP-q}$, and two solutions x^*, x of I where x^* is optimal. We denote by ν the Hamming distance between x^* and x. Assuming $\nu \geq k$, consider an integer $d \in \{k,\ldots,\nu\}$. We are interested in the vectors of $\{x^*_1, x_1\} \times \ldots \times \{x^*_n, x_n\}$ that are at Hamming distance d from x, and denote by $N^d(x^*,x)$ this solution set. The average solution value over $N^d(x^*,x)$ expresses as:

$$\begin{aligned}
&\textstyle\sum_{y\in N^d(x^*,x)} v(I,y)/|N^d(x^*,x)| \\
&\quad = \textstyle\sum_{i=1}^m w_i \times \sum_{y\in N^d(x^*,x)} P_i(y_{J_i})/\binom{\nu}{d} \\
&\quad = \textstyle\sum_{i=1}^m w_i \left(\sum_{y\in N^d(x^*,x):y_{J_i}=x^*_{J_i}} P_i(x^*_{J_i}) + \sum_{y\in N^d(x^*,x):y_{J_i}\neq x^*_{J_i}} P_i(y_{J_i})\right)/\binom{\nu}{d}
\end{aligned}$$

Let $i \in [m]$, and ν_i refer to the number of indices $j \in J_i$ such that $x_j \neq x^*_j$. Over $N^d(x^*,x)$, P_i is evaluated $\binom{\nu-\nu_i}{d-\nu_i}$ times on $x^*_{J_i}$. We deduce that, provided that the

TABLE 11

For a neighbourhood function $g \in \{B^1, \tilde{B}^1\}$, lower bounds on the differential ratio reached at local optima (loc. opt.) w.r.t. g, and at solutions with optimal value (opt. sol.) over $g(x)$ given any solution x, on instances of $\mathsf{Ek\,CSP}(\mathcal{I}_\mathsf{q}^{\mathsf{k}-1})$, $\mathsf{3\,CSP}(\mathcal{E}_2)$ and $\mathsf{2\,CSP}-2$. We denote by p^κ the smallest prime power $\geq q$, by ν the chromatic number of the instance.

restriction	g	conditions on ν, q, k	loc. opt. w.r.t. g	opt. sol. over $g(x)$, $x \in \mathbb{Z}_q^n$
$\mathsf{Ek\,CSP}(\mathcal{I}_\mathsf{q}^{\mathsf{k}-1})$, $q, k \geq 2$	B^1	$\nu \leq 2k - 1$	$1/q^{\nu-k+1}$	$k/(q^{\nu-k}(q-1)n)$
		$\nu \leq 2k$, $q = p^\kappa \wedge \nu \leq q + k$	$1/q^k$	$k/(q^{k-1}(q-1)n)$
		$\nu \leq p^\kappa + 1 \wedge p^\kappa > k$	$1/(2(q-1))^k$	$kq/(2^k(q-1)^{k+1}n)$
		$q \geq 3$	$1/O\left(\nu^{k-\lceil \log_{p^\kappa} k \rceil}\right)$	$1/O\left(\nu^{k-\lceil \log_{p^\kappa} k \rceil} \times n\right)$
$\mathsf{E3\,CSP}(\mathcal{I}_\mathsf{q}^2)$, $q \geq 3$	B^1	$q = 2^\kappa \wedge \nu \leq q + 4$	$1/q^3$	$3/(q^2(q-1)n)$
		$\nu \leq 2^{\lceil \log_2 q \rceil} + 2$	$1/\left(2(q-1)\right)^3$	$3q/\left(2^3(q-1)^4 n\right)$
$\mathsf{Ek\,CSP}(\mathcal{I}_2^{\mathsf{k}-1})$	B^1	$k \geq 3$	$1/O\left(\nu^{\lfloor k/2 \rfloor}\right)$	$1/O\left(\nu^{\lfloor k/2 \rfloor} \times n\right)$
$\mathsf{E2\,CSP}(\mathcal{I}_\mathsf{q}^1)$	B^1	$q \geq 2$	$\dfrac{1}{q\lceil \frac{\nu-2}{q} \rceil (q-1) + q}$	$\dfrac{2}{\left(\lceil \frac{\nu-2}{q} \rceil (q-1) + 1\right)(q-1)n}$
$\mathsf{3\,CSP}(\mathcal{E}_2)$	B^1	$\nu \geq 3$	$\dfrac{1}{2\lceil \nu/2 \rceil}$	$\dfrac{4}{2\lceil \nu/2 \rceil \times n}$
$\mathsf{2\,CSP}-2$	$\tilde{B}^1$	$\nu \geq 2$	$\dfrac{1}{2\lceil (\nu+1)/2 \rceil}$	$\dfrac{4}{2\lceil (\nu+1)/2 \rceil \times (n+1)}$

goal on I is to maximize and I is such that $w_i P_i \geq 0, i \in [m]$, the average solution value over $N^d(x^*, x)$ is bounded below by:

$$\frac{\min_{i=1}^m \binom{\nu-\nu_i}{d-\nu_i}}{\binom{\nu}{d}} \times v(I, x^*) \quad \geq \frac{\binom{\nu-k}{d-k}}{\binom{\nu}{d}} \times \mathrm{opt}(I) \quad = \frac{d(d-1)\dots(d-k+1)}{\nu(\nu-1)\dots(\nu-k+1)} \times \mathrm{opt}(I)$$

Since $N^d(x^*, x)$ is a subset of $B^d(x)$ and $\nu \leq n$, we conclude that the standard ratio reached at solutions that perform the best objective value over $B^d(x)$ is at least $k!\binom{d}{k}/n^k$.

In contrast, deriving a similar deduction for the differential ratio poses a challenge. Specifically, the comparison between the sum of solution values over $N^d(x^*, x)$ minus $\binom{\nu-k}{d-k} v(I, x^*)$ on the one hand, and the sum of $\binom{\nu}{d} - \binom{\nu-k}{d-k}$ solution values on the other hand, is not straightforward in the general case. Let $g \in \{B^d, \tilde{B}^d\}$. Our approach for evaluating the highest differential ratio reached at the neighbourhood $g(x)$ of every solution x is to associate with each $(x^*, x) \in \Sigma_q^n \times \Sigma_q^n$ a pair $(\mathcal{X}(I, x^*, x), \mathcal{Y}(I, x^*, x))$ of solutions multisets. These multisets must have the same size R, and satisfy that $\mathcal{X}(I, x^*, x)$ is a subset of $g(x)$, x^* occurs a certain number $R^* > 0$ of times in $\mathcal{Y}(I, x^*, x)$, and the sum of solution values is the same over both multisets. Provided that x^* is optimal, the optimum solution over $\mathcal{X}(I, x^*, x)$ and hence, over $g(x)$ is R^*/R-differential approximate on I.

4.4. Partition-based solution multisets. Our goal is to evaluate the highest differential ratio reached over Hamming balls with radius d given an integer $d \geq k$. As the case where $B^d(x)$ contains an optimum solution is trivial, we consider pairs (x^*, x) of solutions that are at Hamming distance at least $d+1$ the one to each other. Furthermore, we restrict $\mathcal{X}(I, x^*, x)$ and $\mathcal{Y}(I, x^*, x)$ to solutions from the set $\{x_1^*, x_1\} \times, \dots, \times \{x_n^*, x_n\}$. Following a similar approach to that described in section 3, we precisely define our solution multisets $\mathcal{X}$ and $\mathcal{Y}$ by considering the following framework.

• **Solution multisets association.** Let $\nu \in \{d+1, \ldots, n\}$ and let x^*, x be two vectors of Σ_q^n that disagree on ν coordinates. We denote by $\mathcal{J}(x^*, x) = \{j_1, \ldots, j_\nu\}$ the set of indices $j \in [n]$ such that $x_j \neq x_j^*$. We associate with (x^*, x) and each $u \in \{0,1\}^\nu$ a solution $y(x^*, x, u)$ which is defined by:

$$y(x^*, x, u)_j = \begin{cases} x_j^* & \text{if } j = j_c \in \mathcal{J}(x^*, x) \text{ and } u_c = 1 \\ x_j & \text{otherwise} \end{cases} \tag{4.7}$$

Thus, solution $y(x^*, x, u)$ is obtained from x by switching for each index $c \in [\nu]$ such that $u_c = 1$ coordinate x_{j_c} to $x_{j_c}^*$. Specifically, $y(x^*, x, \mathbf{1})$ coincides with x^*, whereas $d_H\left(y(x^*, x, u), x\right)$ is the number of nonzero coordinates of u.

We consider two arrays Ψ and Φ with ν columns and binary coefficients. These arrays have the same number of rows, which we denoted by R. With such a pair (Ψ, Φ) of arrays, we associate solution multisets:

$$\mathcal{X}(I, x^*, x) = \left(y(x^*, x, \Psi_r) \,|\, r \in [R]\right), \ \mathcal{Y}(I, x^*, x) = \left(y(x^*, x, \Phi_r) \,|\, r \in [R]\right)$$

• **Conditions.** In order to model solutions of $B^d(x)$, every row of array Ψ shall have at most d nonzero coordinates. In order to modelize solution x^*, rows of array Φ shall coincide at least once with the all-ones vector. Finally, since our goal is to relate solution values over $B^d(x)$ to $\mathrm{opt}(I)$ provided that x^* is optimal, arrays Ψ, Φ shall satisfy:

$$\textstyle\sum_{r=1}^R v(I, y(x^*, x, \Psi_r)) = \sum_{r=1}^R v(I, y(x^*, x, \Phi_r)) \tag{4.8}$$

Without loss of generality, assume that the goal on I is to maximize. We denote by R^* the number of times the all-ones vector occurs as a row in Φ. Then solutions with optimum value over $B^d(x)$ satisfy:

$$\begin{aligned} \max_{y \in B^d(x)} v(I, y) &\geq \textstyle\sum_{r=1}^R v(I, y(x^*, x, \Psi_r))/R \qquad \text{as } y(x^*, x, \Psi_r) \in B^d(x),\ r \in [R] \\ &= \textstyle\sum_{r=1}^R v(I, y(x^*, x, \Phi_r))/R \qquad \text{by (4.8)} \\ &\geq R^* \times v(I, x^*)/R + (R - R^*)\mathrm{wor}(I)/R \qquad (4.9) \end{aligned}$$

Such solutions therefore are R^*/R-differential approximate provided that x^* is optimal.

4.5. Connection to combinatorial designs. Similarly to the case we considered in the preceding section, one way to ensure that arrays Ψ and Φ satisfy (4.8) is, again, to require that $\mu^\Psi - \mu^\Phi$ is balanced k-wise independent.

Preliminary observe that relation (4.8) is implied by relation (4.10) below:

$$\textstyle\sum_{r=1}^R P_i(y(x^*, x, \Psi_r)_{J_i}) = \sum_{r=1}^R P_i(y(x^*, x, \Phi_r)_{J_i}), \qquad i \in [m] \tag{4.10}$$

Let $i \in [m]$. A sufficient condition for (Ψ, Φ) to satisfy (4.10) at i is that, over solution multisets $(y(x^*, x, \Psi_r) \,|\, r \in [R])$ and $(y(x^*, x, \Phi_r) \,|\, r \in [R])$, function P_i is evaluated on the same multisets of entries. Recall that solutions $y(x^*, x, u)$, where $u \in \Sigma_2^\nu$, coincide with x on coordinates outside $\mathcal{J}(x^*, x)$, and with either x_j or x_j^* depending on u_j for coordinates in $\mathcal{J}(x^*, x)$. Let $H = J_i \cap \mathcal{J}(x^*, x)$, and $t = |H|$. Then we deduce that the multisets $\left(y(x^*, x, \Psi_r) \,|\, r \in [R]\right)_{J_i}$ and $\left(y(x^*, x, \Phi_r) \,|\, r \in [R]\right)_{J_i}$ of vectors coincide *iff* $(\Psi^H_r \,|\, r \in [R])$ and $(\Phi^H_r \,|\, r \in [R])$ define the same multisets of words of Σ_2^t. Since $t \leq \min\{|J_i|, \nu\} \leq k$, this condition indeed is verified provided that $\mu^\Psi - \mu^\Phi$ is balanced k-wise independent.

Our objective is to find pairs (Ψ, Φ) of arrays that satisfy all the conditions described above, ideally maximizing $\mu^\Phi(\mathbf{1})$.

DEFINITION 4.4. *Let $k \geq 1$, $d \geq k$ and $\nu \geq k$ be three integers. Then given any two positive integers R and $R^* \leq R$, we define $\Delta(R, R^*, \nu, d, k)$ as the (possibly empty) set of pairs (Ψ, Φ) of $R \times \nu$ arrays on $\{0, 1\}$ that satisfy:*

1. $|\{r \in [R] \,|\, \Phi_r = \mathbf{1}\}| = R^*$*;*
2. *each row of Ψ has at most d nonzero coordinates;*
3. $\mu^\Psi - \mu^\Phi$ *is balanced k-wise independent.*

Moreover, we define $\delta(\nu, d, k)$ as the greatest number δ for which there exist two positive integers R, R^* such that $R^*/R = \delta$ and $\Delta(R, R^*, \nu, d, k) \neq \emptyset$. We also define $\bar{\delta}(\nu, d, k)$ as the greatest number δ for which there exist two positive integers R, R^* and a pair $(\Psi, \Phi) \in \Delta(R, R^*, \nu, d, k)$ such that $R^*/R = \delta$, and the rows of Φ all have either ν, or at most d, nonzero coordinates.

The previous discussion establishes the following link between these combinatorial designs and the approximation guarantees offered for $\mathsf{k\,CSP\!-\!q}$ by Hamming ball with radius k.

THEOREM 4.5. *Let $q, k \geq 2$, $d \geq k$ be three integers, and I be an instance of $\mathsf{k\,CSP\!-\!q}$. We denote by n the number of variables in I, and by B an Hamming ball with radius d of Σ_q^n. Then solutions with optimum value over B and $\cup_{a=0}^{q-1}\{y + \mathbf{a} \,|\, y \in B\}$ reach a differential ratio at least, respectively:*

$$\delta(n, d, k) \text{ and } \delta\left(\lfloor n(q-1)/q \rfloor, d, k\right)$$

Furthermore, the ratio of the maximum difference between two solution values over B and $\cup_{a=0}^{q-1}\{y + \mathbf{a} \,|\, y \in B\}$ to the diameter $|\mathrm{opt}(I) - \mathrm{wor}(I)|$ of I is at least, respectively:

$$\frac{\bar{\delta}(n, d, k)}{2 - \bar{\delta}(n, d, k)} \text{ and } \frac{\bar{\delta}\left(\lfloor n(q-1)/q \rfloor, d, k\right)}{2 - \bar{\delta}\left(\lfloor n(q-1)/q \rfloor, d, k\right)}$$

Proof. We assume *w.l.o.g.* that the goal on I is to maximize. Consider two vectors $x, x^* \in \Sigma_q^n$ where x^* is optimal, and let ν refer to the number of coordinates on which x and x^* differ. If $\nu \leq d$, then $B^d(x)$ contains an optimal solution, and thus the highest differential ratio achieved on $B^d(x)$ is $1 \geq \max\{\delta(n, d, k), \delta(\lfloor n(q-1)/q \rfloor, d, k)\}$. We thus assume $\nu > d$, and there exists a pair $(\Psi, \Phi) \in \Delta(R, R^*, \nu, d, k)$ where $R \geq R^* \geq 1$. By Item 3 of Definition 4.4, (Ψ, Φ) satisfies (4.8), what we deduce together with Items 1 and 2 of Definition 4.4 that inequality (4.9) holds. Quantity $\delta(\nu, d, k)$ consequently is a proper lower bound for the highest differential ratio reached over $B^d(x)$. More generality, let ν_a given $a \in \Sigma_q$ refer to the Hamming distance between x^* and $x + \mathbf{a}$. Then for all $a \in \Sigma_q$, $\delta(\nu_a, d, k)$ is a proper lower bound for the highest differential ratio reached over $B^d(x + \mathbf{a})$. The quantity $\max_{a=0}^{q-1} \delta(\nu_a, d, k)$ consequently is a proper lower bound for the highest differential ratio reached over $\tilde{B}^d(x)$. Now, we observe that numbers $\delta(n, d, k)$ are non increasing in n, considering that the $\nu \leq n$ first columns of a pair $(\Psi, \Phi) \in \Delta(R, R^*, n, d, k)$ of arrays constitute an element of $\Delta(R, R^*, \nu, d, k)$. It thus follows from inequalities $\nu \leq n$ and

$$\min_{a=0}^{q-1} \nu_a \quad \leq \textstyle\sum_{a=0}^{q-1} \nu_a / q \quad = (q-1)n/q$$

that $\delta(\nu, d, k) \geq \delta(n, d, k)$, and $\max_{a=0}^{q-1} \delta(\nu_a, d, k) \geq \delta(\min_{a=0}^{q-1} \nu_a, d, k) \geq \delta(\lfloor n(q-1)/q \rfloor, d, k)$.

Regarding the instance diameter, we observe that inequality (4.9) can be strengthened provided that every row of Φ has either exactly n, or at most d nonzero coordinates. Indeed, when this occurs, solutions $y(x^*, x, \Phi_r)$ that do not coincide with x^* all belong to $B^d(x)$. We thus in (4.9) can replace $\mathrm{wor}(I)$ by the expression $\min_{y \in B^d(x)} v(I, y)$. By doing so, we obtain the following inequality:

$$\max_{y \in B^d(x)} v(I, y) \geq R^*/R \times \mathrm{opt}(I) + (1 - R^*/R) \min_{y \in B^d(x)} v(I, y) \tag{4.11}$$

Now consider a solution x_* with worst value on I. We denote by μ the number of coordinates on which x disagrees with x_*. We assume $\mu > d$ as otherwise, we already know that quantity

$$\max_{y \in B^d(x)} v(I, y) - \min_{y \in B^d(x)} v(I, y) \quad = \max_{y \in B^d(x)} v(I, y) - \mathrm{wor}(I)$$

is a fraction at least $\delta(\nu, d, k) \geq \delta(n, d, k)$ of the instance diameter. We symmetrically assume $\nu > d$, and that there exist two pairs $(\Psi, \Phi) \in \Delta(R, R^*, \nu, d, k)$ and $(\zeta, \xi) \in \Delta(S, S^*, \mu, d, k)$ of arrays where $R \geq R^* \geq 1$, $S \geq S^* \geq 1$, and the rows of arrays Φ and ξ, at the exception of the rows af all-ones, have at most d nonzero coordinates. Symmetrically to (4.11), we have:

$$\min_{y \in B^d(x)} v(I, y) \leq S^*/S \times \mathrm{wor}(I) + (1 - S^*/S) \max_{y \in B^d(x)} v(I, y) \tag{4.12}$$

Regarding the approximation guarantee over $B^d(x)$, we observe that $R/R^* \times$ (4.11) $- S/S^* \times$ (4.12) yields inequality:

$$(R/R^* + S/S^* - 1) \times \left(\max_{y \in B^d(x)} v(I, y) - \min_{y \in B^d(x)} v(I, y) \right) \geq \mathrm{opt}(I) - \mathrm{wor}(I) \tag{4.13}$$

Hence, provided that the pairs (Ψ, Φ) and (ζ, ξ) of arrays realize respectively $\bar{\bar{\delta}}(\nu, d, k)$ and $\bar{\bar{\delta}}(\mu, d, k)$, quantity $\max_{y \in B^d(x)} v(I, y) - \min_{y \in B^d(x)} v(I, y)$ is a fraction at least

$$\frac{1}{R/R^* + S/S^* - 1} \quad = \frac{1}{1/\bar{\bar{\delta}}(\nu, d, k) + 1/\bar{\bar{\delta}}(\mu, d, k) - 1} \tag{4.14}$$

of the diameter of I. This allows to conclude for $B^d(x)$, considering:

$$\min\{\bar{\bar{\delta}}(\nu, d, k), \bar{\bar{\delta}}(\mu, d, k)\} \quad \geq \bar{\bar{\delta}}\left(\max\{\nu, \mu\}, d, k\right) \quad \geq \bar{\bar{\delta}}(n, d, k)$$

Now let x^+ and x^- refer to respectively a maximizer and a minimizer of $v(I, .)$ over $\cup_{a=0}^{q-1} B^d(x + \mathbf{a})$. By definition, such solutions satisfy:

$$\begin{aligned} v(I, x^+) \quad &\geq \max_{y \in B^d(x+\mathbf{a})} v(I, y) \\ &\geq \min_{y \in B^d(x+\mathbf{a})} v(I, y) \quad \geq v(I, x^-), \quad a \in \Sigma_q \end{aligned} \tag{4.15}$$

First, we deduce from inequality (4.11) taken at $x + \mathbf{a}$, $a \in \Sigma_q$ and inequalities (4.15) that we have:

$$v(I, x^+) \geq \bar{\bar{\delta}}(\nu_a, d, k)\mathrm{opt}(I) + \left(1 - \bar{\bar{\delta}}(\nu_a, d, k)\right) v(I, x^-), \qquad a \in \Sigma_q \tag{4.16}$$

Let $\mu_a = d_H(x_*, x + \mathbf{a})$, $a \in \Sigma_q$. Then inequality (4.12) taken at $x + \mathbf{a}$, $a \in \Sigma_q$ and inequalities (4.15) symmetrically imply:

$$v(I, x^-) \leq \bar{\delta}(\mu_a, d, k)\mathrm{wor}(I) + \big(1 - \bar{\delta}(\mu_a, d, k)\big)\, v(I, x^+), \qquad a \in \Sigma_q \tag{4.17}$$

Given $(b, c) \in \Sigma_q^2$, $1/\bar{\delta}(\nu_b, d, k) \times$ (4.16) taken at b, minus $1/\bar{\delta}(\mu_c, d, k) \times$ (4.17) taken at c, yields inequality:

$$\frac{v(I, x^+) - v(I, x^-)}{\mathrm{opt}(I) - \mathrm{wor}(I)} \geq \frac{1}{1/\bar{\delta}(\nu_b, d, k) + 1/\bar{\delta}(\mu_c, d, k) - 1}, \qquad b, c \in \Sigma_q \tag{4.18}$$

In order to conclude, we observe that integers $\min_{b=0}^{q-1} \nu_b$ and $\min_{c=0}^{q-1} \mu_c$ both are $\leq \lfloor n(q-1)/q \rfloor$. ∎

4.6. Estimation of numbers $\delta(\nu, d, k)$. It remains for us to prove the existence of pairs of arrays that satisfy the conditions of Definition 4.4. In fact, this can be done by relating given five integers $k \geq 2$, $d \geq k$, $\nu \geq d$, $R^* \geq 1$, $R \geq R^*$ the two families $\Delta(R, R^*, \nu, d, k)$ and $\Gamma(R, R^*, \nu, d, k)$ of designs. Specifically, a pair $(\Psi, \Phi) \in \Gamma(R, R^*, \nu, d, k)$ of arrays can be interpreted as an element of $\Delta(R, R^*, \nu, d, k)$ by interpreting the coefficients M_r^j that occur in the column with index $j \in \Sigma_\nu$ of Ψ or Φ as the binary relation $(M_r^j = j)$.

PROPOSITION 4.6. *Numbers $\delta(\nu, d, k)$ satisfy:*

$$\delta(\nu, d, k) \geq \gamma(\nu, d, k), \qquad \nu, d, k \in \mathbb{N},\ \nu \geq d \geq k \geq 1 \tag{4.19}$$

Furthermore, if a pair $(\Psi, \Phi) \in \Gamma(R, R^*, \nu, d, k)$ of arrays where $R \geq R^* \geq 1$ and $\nu \geq d \geq k \geq 1$ satisfies

$$|\{j \in \Sigma_\nu \mid \Phi_r^j = j\}| \in \{0, \ldots, d\} \cup \{\nu\}, \qquad r \in [R] \tag{4.20}$$

then $\bar{\delta}(\nu, d, k) \geq R^*/R$.

Proof. For a positive integer ν, we define a surjective map σ_ν from the set of arrays with ν columns on symbol set Σ_ν to the set of arrays with ν columns on symbol set Σ_2. We index the columns of the former and the latter arrays by Σ_ν and $[\nu]$ respectively. Let R be a positive integer and M be an $R \times \nu$ array on Σ_ν. Then σ_ν maps M to the $R \times \nu$ array on $\{0, 1\}$ defined by:

$$\sigma_\nu(M)_r^j \;= \begin{cases} 1 & \text{if } M_r^{j-1} = j - 1 \\ 0 & \text{otherwise} \end{cases}, \quad r \in [R], j \in [\nu] \tag{4.21}$$

Let $(\Psi, \Phi) \in \Gamma(R, R^*, \nu, d, k)$ where $R \geq R^* > 1$. We want to show that $(\sigma_\nu(\Psi), \sigma_\nu(\Phi)) \in \Delta(R, R^*, \nu, d, k)$. First, given any row index r, $\sigma_\nu(\Phi)_r$ coincides with the all-ones vector *iff* Φ_r coincides with $(0, 1, \ldots, \nu - 1)$. Second, the number of nonzero coefficients in $\sigma_\nu(\Psi)_r$ precisely is the number of indices $j \in \Sigma_\nu$ for which $\Psi_r^j = j$. In particular, since the coordinates of Ψ_r take at most d distinct values, $\Psi_r^j = j$ cannot occur for more than d indices $j \in \Sigma_\nu$. The number of nonzero coordinates of $\sigma_\nu(\Psi)_r$ is therefore less than or equal to d. Furthermore, provided that Φ satisfies (4.20), every row of $\sigma_\nu(\Psi)$ has either ν, or at most d nonzero coordinates.

Now consider a sequence $J = (j_1, \ldots, j_k)$ of pairwise distinct integers from $[\nu]$ and a vector $u \in \{0, 1\}^k$. We denote by $\mathcal{V}(u)$ the set of vectors $v \in \Sigma_\nu^k$ such that:

$$\begin{cases} v_s & = j_s - 1 \quad \text{if } u_s = 1 \\ v_s & \neq j_s - 1 \quad \text{if } u_s = 0 \end{cases}, \quad s \in [k]$$

Table 12
Construction for $\Delta\left((T(\nu,k)+1)/2, 1, \nu, k, k\right)$*: illustration when* $(k,\nu) \in \{(2,6),(3,5)\}$.

$2/(T(6,2)+1) = 1/25$

Ψ^1	Ψ^2	Ψ^3	Ψ^4	Ψ^5	Ψ^6
1	1	0	0	0	0
1	0	1	0	0	0
0	1	1	0	0	0
0	0	0	0	0	0
1	0	0	1	0	0
0	1	0	1	0	0
0	0	1	1	0	0
0	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	1	0
0	1	0	0	1	0
0	0	1	0	1	0
0	0	0	1	1	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	0	1
0	1	0	0	0	1
0	0	1	0	0	1
0	0	0	1	0	1
0	0	0	0	1	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Φ^1	Φ^2	Φ^3	Φ^4	Φ^5	Φ^6
1	1	1	1	1	1
1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	1	0	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	1	0
0	0	0	0	1	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	1
0	0	0	0	0	1
0	0	0	0	0	1

$2/(T(5,3)+1) = 1/25$

Ψ^1	Ψ^2	Ψ^3	Ψ^4	Ψ^5
1	1	1	0	0
1	1	0	1	0
1	0	1	1	0
0	1	1	1	0
1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
1	1	0	0	1
1	0	1	0	1
1	0	0	1	1
0	1	1	0	1
0	1	0	1	1
0	0	1	1	1
1	0	0	0	0
1	0	0	0	0
0	1	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	1	0
0	0	0	0	1
0	0	0	0	1
0	0	0	0	1

Φ^1	Φ^2	Φ^3	Φ^4	Φ^5
1	1	1	1	1
1	1	0	0	0
1	0	1	0	0
0	1	1	0	0
1	0	0	1	0
0	1	0	1	0
0	0	1	1	0
0	0	0	0	0
1	1	0	0	0
1	0	1	0	0
1	0	0	1	0
0	1	1	0	0
0	1	0	1	0
0	0	1	1	0
1	0	0	0	1
1	0	0	0	1
0	1	0	0	1
0	1	0	0	1
0	0	1	0	1
0	0	1	0	1
0	0	0	1	1
0	0	0	1	1
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Let $H = (j_1 - 1, \ldots, j_k - 1)$. Given $M \in \{\Psi, \Phi\}$, by definition of σ_ν, the frequency of u in subarray $\sigma_\nu(M)^J$ corresponds to the overall frequency of vectors from $\mathcal{V}(u)$ in subarray M^H. Since $\mu^\Psi - \mu^\Phi$ is balanced k-wise independent, each $v \in \mathcal{V}(u)$ occurs the same number of times as a row in both subarrays Ψ^H and Φ^H. The overall frequency of vectors from $\mathcal{V}(u)$ consequently is the same in both subarrays Ψ^H and Φ^H. Equivalently, u occurs the same number of times as a row in both subarrays $\sigma_\nu(\Psi)^J$ and $\sigma_\nu(\Phi)^J$. We conclude that the pair $(\sigma_\nu(\Psi), \sigma_\nu(\Phi))$ of arrays indeed is an element of $\Delta(R, R^*, \nu, d, k)$ provided that the initial pair (Ψ, Φ) of arrays is an element of $\Gamma(R, R^*, \nu, d, k)$. ∎

Table 12 depicts the pair of arrays we obtain by applying transformation σ_ν of Proposition 4.6 to the pair (Ψ, Φ) of arrays produced by Algorithm 3.2 on values $(k, \nu) \in \{(2,6),(3,5)\}$. Let us take a closer look at arrays $\sigma_\nu(\Psi)$ and $\sigma_\nu(\Phi)$ given two positive integers k and $\nu > k$. First, we observe that the rows of $\sigma_\nu(\Phi)$, except for the rows of all-ones, are all of weight at most k: Algorithm 3.2 initializes Φ to the single row $(0, 1, \ldots, k-1)$, which is then completed into $(0, 1, \ldots, q-1)$. Later, at each iteration $i \in \{k+1, \ldots, \nu\}$, Algorithm 3.1 inserts in Φ rows $u = (u_0, \ldots, u_{i-1}) \in \Sigma_i^i$ that satisfy $u_j = j$ for at most k coordinates $j \in \Sigma_i$, which are then completed by $\nu - i$ coefficients with value $u_0 \neq i, \ldots, \nu - 1$. At most k coordinates of each such rows of Φ therefore coincide with their index.

The following lower bounds thus hold on numbers $\delta(\nu, d, k)$ and $\bar{\delta}(\nu, d, k)$:

Proposition 4.7. *Let* $k > 0$*,* $d \geq k$ *and* $\nu \geq d$ *be three integers. If* $d = \nu$*, then*

$\delta(\nu,\nu,k) = \bar{\delta}(\nu,\nu,k) = 1$. *Otherwise, we have:*

$$\delta(\nu,d,k) \quad \geq \bar{\delta}(\nu,d,k) \quad \geq 2/\left(\sum_{r=0}^{k}\binom{\nu-d+k}{r}\binom{\nu-d+k-1-r}{k-r}+1\right) \tag{4.22}$$

Proof. Inequality $\delta(\nu,d,k) \geq \bar{\delta}(\nu,d,k)$ holds by definition of numbers $\delta(\nu,d,k)$ and $\bar{\delta}(\nu,d,k)$. As regards $\bar{\delta}(\nu,d,k)$, when $d=\nu$, $\bar{\delta}(\nu,\nu,k)$ is trivially equal to 1. When $\nu > d = k$, we deduce from the preceding discussion and Theorem 3.8 that $\bar{\delta}(\nu-d+k,k,k) \geq 2/(T(\nu-d+k,k)+1)$. When $\nu > d > k$, we observe that exending each row of each array of a design of $\Delta(R,R^*,\nu-d+k,k,k)$ by the $(d-k)$-length word of all-ones yields a pair of arrays of $\Delta(R,R^*,\nu,d,k)$. Thus, we conclude that $\bar{\delta}(\nu,d,k) \geq \bar{\delta}(\nu-d+k,k,k)$, which completes the proof. ∎

Furthermore, given any $h \in \{0,\ldots,k\}$, ν-dimensional binary vectors having exactly h nonzero coordinates all occur the same number of times, in the same array. Precisely, $(\sigma_\nu(\Psi),\sigma_\nu(\Phi))$ can be described as follows (a detailed proof is presented in subsection SM5.2 of the supplement):

- the word of all-ones occurs exactly once as a row in $\sigma_\nu(\Phi)$;
- every $u \in \{0,1\}^\nu$ with a number $a \in \{0,\ldots,k\}$ of nonzero coordinates such that $a \equiv k \bmod 2$ occurs exactly $\binom{\nu-1-a}{k-a}$ times as a row in $\sigma_\nu(\Psi)$;
- every $u \in \{0,1\}^\nu$ with a number $a \in \{0,\ldots,k\}$ of nonzero coordinates such that $a \not\equiv k \bmod 2$ occurs exactly $\binom{\nu-1-a}{k-a}$ times as a row in $\sigma_\nu(\Phi)$.

We deduce that the following identity holds.

THEOREM 4.8. *Let $q \geq 2$, $k \geq 2$ be two integers, I be an instance of* $\mathsf{k\,CSP{-}q}$*, x and x^* be two solutions of I that are at a Hamming distance $\nu > k$ from each other. Moreover, let $N^h(x^*,x)$ where $h \in \{0,\ldots,\nu\}$ refer to the restriction of $B^h(x)$ to vectors $y \in \Sigma_q^n$ that coincide with x^* or x on each of their coordinates, and are at Hamming distance d from x. Formally:*

$$N^h(x^*,x) := \{y \in \{x_1^*,x_1\} \times \ldots \times \{x_n^*,x_n\} \,:\, d_H(x,y) = h\}$$

Then solution sets $N^h(x^*,x)$, $h \in \{0,\ldots,k\}$ satisfy:

$$v(I,x^*) = \sum_{h=0}^{k}(-1)^{k-h}\binom{\nu-1-h}{k-h}\sum_{y \in N^h(x^*,x)} v(I,y) \tag{4.23}$$

Notice that some generalization of this identity for the case where $d > k$ was communicated in [19]. However, because the corresponding pairs of arrays fail to achieve either $\bar{\delta}(\nu,d,k)$ or $\delta(\nu,d,k)$ unless $d=k$, we opt not to present the more general identity in this paper.

4.7. Approximation guarantees. Consider an instance I of $\mathsf{k\,CSP{-}q}$ along with a Hamming ball B of radius k on Σ_q^n. According to Theorem 4.5 and Proposition 4.7 and relation (3.18), the highest differential ratio reached over B and $\cup_{a=0}^{q-1}\{y + \mathbf{a} \,|\, y \in B\}$ is respectively at least

$$\begin{aligned} \delta(n,k,k) &\geq \frac{2}{T(n,k)+1} &&\geq \frac{2(k!)}{(2n-k)^k} \\ \text{and} \quad \delta(\lfloor n(q-1)/q \rfloor,k,k) &\geq \frac{2}{T(\lfloor n(q-1)/q) \rfloor,k)+1} &&\geq \frac{2(k!)}{(2(q-1)n/q-k)^k} \end{aligned}$$

Theorem 4.5 and Proposition 4.7 additionally imply an approximation guarantee for the instance diameter, which is a stronger result. Specifically, the ratio of the maximum difference between two solution values over B and $\cup_{a=0}^{q-1}\{y+\mathbf{a} \,|\, y \in B\}$ to the instance diameter is respectively at least

$$\begin{array}{rlll} & \dfrac{1}{2/\delta(n,k,k)-1} & \geq \dfrac{1}{T(n,k)} & \geq \dfrac{k!}{(2n-k)^k} \\ \text{and} & \dfrac{1}{2/\delta\left(\lfloor n(q-1)/q\rfloor,k,k\right)-1} & \geq \dfrac{1}{T\left(\lfloor n(q-1)/q)\rfloor,k\right)} & \geq \dfrac{k!}{(2n(q-1)/q-k)^k} \end{array}$$

We summarize the obtained approximability bounds in the Corollary below.

Corollary 4.9. *Let $q \geq 2$, $k \geq 2$ be two integers, I be an instance of* $\mathsf{k\,CSP-q}$ *on $n \geq k$ variables, and $x \in \Sigma_q^n$. The ratio of the maximum difference between any two solution values over $B^k(x)$ and $\tilde{B}^k(x)$ to the diameter of I is at least $k!/(2n-k)^k$ and $k!/(2(q-1)n/q-k)^k$, respectively. Furthermore, the highest differential ratio reached on $B^k(x)$ and $\tilde{B}^k(x)$ is at least twice these bounds.*

4.8. Concluding remarks. We ask the following questions: how good are our estimates of numbers $\delta(\nu,d,k)$ and $\bar{\delta}(\nu,d,k)$? How tight is our analysis of the maximum differential ratio reached at any Hamming ball with radius k for $\mathsf{k\,CSP-q}$?

Recently, we established that the lower bound provided for numbers $\bar{\delta}(\nu,d,k)$ and $\delta(\nu,d,k)$ matches their exact value when $d=k$ [18]. As a result, the estimate we obtain for the differential approximation guarantees on any Hamming ball of radius k stands as the most accurate outcome that can be derived from the proposed analysis.

Furthermore, the best approximation guarantee we can expect from the optimal solution value on Hamming balls of fixed radius $d \geq k$, as well as from the greatest difference between two solution values over such balls, is $\Omega(1/n^k)$. Similar to instances $I_n^{q,k}$ of $\mathsf{CSP}(\{\mathsf{AllEqual}^{\mathsf{k,q}}\})$, let $J_n^{q,k}$ denote the instance of $\mathsf{CSP}(\{\mathsf{AllZero}^{\mathsf{k,q}}\})$ that considers all the k-ary constraints that can be formulated on a set of n variables, given three positive integers q, k, $n \geq k$. For any $d \in \{0,\dots,n\}$, any vector with exactly d coordinates equal to zero satisfies $\binom{d}{k}$ of the constraints. In particular, we have $\mathrm{opt}(J_n^{q,k}) = \binom{n}{k}$ (the all-zeros vector satisfies all the constraints) and $\mathrm{wor}(J_n^{q,k}) = 0$ (for example, the all-ones vector satisfies no constraint). Moreover, given any $d \in \{k,\dots,n\}$, the maximum solution value over $B^d(\mathbf{1})$ equals $\binom{d}{k}$. The best differential ratio reached over $B^d(\mathbf{1})$ therefore is:

$$\frac{\binom{d}{k}-0}{\binom{n}{k}-0} = \frac{d(d-1)\dots(d-k+1)}{n(n-1)\dots(n-k+1)} \sim \frac{k!\binom{d}{k}}{n^k}$$

Since $B^d(\mathbf{1})$ contains $\mathbf{1}$, which is a worst solution, this ratio coincides with the ratio of $\max_{y\in B^d(\mathbf{1})} v(J_n^{q,k},y) - \min_{y\in B^d(\mathbf{1})} v(J_n^{q,k},y)$ to the diameter of $J_n^{q,k}$. When $d=k$, this ratio is asymptotically a factor 2^{k-1} and 2^k of the lower bounds given by Corollary 4.9 for, respectively, the differential approximation of $\mathrm{opt}(I)$ and the approximation of the instance diameter.

Regarding the best differential ratio reached over the shifts by some integer $a \in \Sigma_q$ of Hamming balls of a fixed radius k, we observe that the guarantee we obtain is asymptotically tight when $k=2$. Here, we consider instances $(I_n^{q,k} \,|\, n \in \mathbb{N}\backslash\{0\})$ of $\mathsf{Max\,CSP}\{\mathsf{AllEqual}^{\mathsf{k,q}}\}$ we introduced in subsection 2.7. As $\mathit{AllEqual}^{k,q}$ is stable under a shift by a same quantity of all its variables, on $I_n^{q,k}$, a solution y performs the best solution value over a Hamming ball $B^k(x)$ *if and only if* y performs the best solution value over the union of the Hamming balls $B^k(x+\mathbf{a})$, $a \in \Sigma_q$. Consider three positive integers n, q and k. Given any partition of n into q natural numbers $n_0,\dots,n_{q-1}$, any vector with n_a coordinates equal to a, $a \in \Sigma_q$ satisfies $\sum_{a=0}^{q-1}\binom{n_a}{k}$ constraints. Recall that $\mathrm{opt}(I_n^{q,k}) = \binom{qn}{k}$ and $\mathrm{wor}(I_n^{q,k}) = q \times \binom{n}{k}$. Moreover, let x_* be a vector with n coordinates equal to a for each $a \in \Sigma_q$; then one may easily check that the maximum solution value over $\tilde{B}^k(x_*)$ equals $\binom{n+k}{k} + \binom{n-k}{k} + (q-2)\binom{n}{k}$. The highest differential ratio reached on $\tilde{B}^k(x_*)$ therefore is equal to:

$$\frac{\binom{n+k}{k} + \binom{n-k}{k} - 2\binom{n}{k}}{\binom{qn}{k} - q \times \binom{n}{k}} = \frac{\prod_{i=0}^{k-1}(n+k-i) + \prod_{i=0}^{k-1}(n-k-i) - 2\prod_{i=0}^{k-1}(n-i)}{\prod_{i=0}^{k-1}(qn-i) - q\prod_{i=0}^{k-1}(n-i)}$$

We denote by $num(q,n)$ and $den(q,n)$ respectively the numerator polynomial and the denominator polynomial of the above right-hand side rational fraction. Polynomials $num(q,n)$ and $den(q,n)$ are of degree respectively $k-2$ and k. The coefficient of n^k in $den(q,n)$ is $q^k - q$, while the coefficient of n^{k-2} in $num(q,n)$ is equal to $2k^2\binom{k}{2}$[7]. Accordingly, the highest differential ratio reached on $\tilde{B}^k(x_*)$ asymptotically is:

$$\frac{num(q,n)}{den(q,n)} \sim \frac{2k^2\binom{k}{2}}{q^k - q} \times \frac{1}{n^2} = \frac{2k^2\binom{k}{2}q}{q^{k-1} - 1} \times \frac{1}{(qn)^2}$$

Secondly, according to Corollary 4.9, on $I_n^{q,k}$, for any $x \in \Sigma_q^{qn}$, solutions achieving the maximum value over $\tilde{B}^k(x)$ attain a differential ratio of at least:

$$\frac{2(k!)}{(2(q-1)(qn)/q - k)^k} \sim \frac{k! q^k}{2^{k-1}(q-1)^k} \times \frac{1}{(qn)^k}$$

Hence, when $k = 2$, the maximal differential ratio over $\tilde{B}^2(x_*)$ asymptotically is a factor $8(q-1)/q$ of the lower bound provided by Corollary 4.9 for this ratio. If k and n are constant integers while q can be arbitrarily large, this ratio and the lower bound provided by Corollary 4.9 both are in $\Theta(1/q^k)$.

5. Conclusion.

Combinatorial designs and CSPs. While our investigations span different contexts, the underlying approach in the presented results remains the same. First, we identify partitions $\mathcal{V} = (V_1, \ldots, V_\nu)$ of $[n]$ and multisubsets M of Σ_q^ν having the "right" properties as regards the goal pursued. Then, we consider solutions of the form $(y_{V_1}^r, \ldots, y_{V_\nu}^r) = (x_{V_1} + \mathbf{M}_\mathbf{r}^\mathbf{1}, \ldots, x_{V_\nu} + \mathbf{M}_\mathbf{r}^\nu)$. When not imposing further restrictions on q-ary CSPs beyond the arity of their constraints, the consideration of balanced k-wise independent functions for their analysis arises naturally. Note that it suffices to establish the existence, without necessarily exhibiting, the pairs $(\mathcal{V}, M)$ that underlie the multisets of solutions supporting the argument. The neighborhood analysis inherently establishes that ρ-approximate solutions can be found throughout the solution set for a given ρ. However, due to the methodology employed, the lower bounds derived for the average differential ratio not only signify that $\mathrm{E}_X[v(I,X)]$ attains a certain differential ratio ρ, but also suggest that ρ-differential approximate solutions are distributed across the solution set. Namely, if $\mathcal{V}$ and M denote the partition of $[n]$ and the array utilized to establish the approximation guarantee, then for any x, among the solutions $y(\mathcal{V}, x, M_r)$, $r \in [R]$, there exists at least one solution that achieves at least the average differential ratio. Thus, the proposed analysis of the average differential ratio offers further insights into the distribution of solution values.

[7]As $\sum_{0\le i<j\le k-1}((k-i)(k-j) + (k+i)(k+j) - 2ij) = \sum_{0\le i<j\le k-1} 2k^2 = 2k^2\binom{k}{2}$.

Table 13

Differential (dapx) and gain (gapx) approximability bounds for k CSP−q *that are achievable by either deterministic (det.) or randomized (exp.) algorithms, and their comparison to the lower bounds known for the average differential ratio (avd): p^κ refers to the smallest prime power $\geq q$; ν refers to the strong chromatic number of the instance primary hypergraph. Inapproximability bounds hold for all constant $\varepsilon > 0$ assuming $\mathbf{P} \neq \mathbf{NP}$. The bounds marked by * are commented in section SM4 of the supplement.*

Approximability bounds in k-partite instances of $\mathsf{Ek\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{t}})$

k	q	t	gapx det.	dapx det.	avd
$=2$	$=2$	$=1$	0.561 [3]	0.78 [3]*	$=1/2$
≥ 3	≥ 2	$=k-1$	$\neg\, \varepsilon$ [13]	$\neg\, 1/q+\varepsilon$ [13]	$\geq 1/q$
≥ 3	$\geq 2,\ \leq k$	$=2$	$\neg\, \varepsilon$ [13]	$\neg\, O(k/q^{k-1})+\varepsilon$ [13]	$\geq 1/q^{k-2}$
≥ 3	$\geq k$	$=2$	$\neg\, \varepsilon$ [13]	$\neg\, O(k/q^{k-2})+\varepsilon$ [13]	$\geq 1/q^{k-2}$

Gain approximability bounds for $\mathsf{E3\,CSP}(\mathcal{I}_{\mathsf{q}}^{2})$ *(row 1) and* k CSP−q *(the other rows)*

k	q	t	gapx det.	gapx exp.	avd
$=3$	$=2$	$=2$	$\Omega(1/m)$ [31]	$\Omega(\sqrt{\ln n/n})$ [34]*	$=1/2$
$=2$	$=2$		$\Omega(1/\ln n)$ [40]		$\Omega(1/\nu)$
≥ 4	$=2$		$\Omega(1/m)$ [31]	$\Omega(1/\sqrt{m})$ [31]	$\Omega(1/\nu^{\lfloor k/2\rfloor})$
≥ 2	$=2^\kappa,\ \geq 4$		$\Omega(1/m)$ [31, 16]	$\Omega(1/\sqrt{m})$ [31, 16]	$\Omega(1/\nu^{k-\lceil \log_q k\rceil})$

Other differential approximability bounds for k CSP−q

$k,\ q$	dapx det.	dapx exp.	avd
$k=2$ or $(k,q)=(3,2)$	$\Omega(1)$ [41, 17]		$\Omega(1/\nu)$
$k\geq 3$ and $q\geq 3$	$\Omega(1/m)$ [31, 16]	$\Omega(1/\sqrt{m})$ [31, 16]	$\Omega(1/\nu^{k-\lceil \log_{p^\kappa} k\rceil})$

In our opinion, this work highlights the power of combinatorial designs when it comes to studying the differential approximability of k CSPs. Nevertheless, further investigation is needed to assess the tightness of the ratios obtained. We think of the reduction from k CSP−q to k CSP−p: on instances of k CSP−q, how close to $\gamma(q,p,k)$ is the differential gap between the optimum solution value when restricted to solutions whose coordinates take at most p distinct values and the optimum solution value? Additionally, we think of examining the differential ratio achieved at the average solution value, as well as at solutions with the optimum value over the union of shifts by a, $a \in \{0,\dots,q-1\}$ of Hamming balls with radius k when $k \geq 3$. In the worst-case scenarios, are the quantities $\Omega(1/n^b)$, where b varies across integers in $\{\lfloor k/2\rfloor,\dots,k-1\}$ depending on k and q, for the former case (assuming $k\geq 4$ or $q \geq 3$), and $\Omega(1/n^k)$ for the latter case, indicative of the correct order of magnitude?

We also highlight that the question of whether a reduction from k CSP−q to k CSP−p exists, preserving the differential approximation ratio within a constant multiplicative factor, remains entirely unresolved when $k > p$.

Approximability bounds. We review the estimates we obtain for the average differential ratio in light of the gain and differential approximability bounds of the literature. We summarize in Table 13 the approximability bounds we are aware of. On the one hand, we compare the differential approximation guarantee offered by $\mathbb{E}_X[v(I,X)]$ to the ones offered by dedicated algorithms. On the other hand, we find it interesting to compare the approximation of the optimum advantage over $\mathrm{wor}(I)$ offered by $\mathbb{E}_X[v(I,X)]$, to the appproximabilty of the optimum advantage over $\mathbb{E}_X[v(I,X)]$.

For such symptomatic CSPs as the restriction of $\mathsf{Ek\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k-1}})$ to k-partite instances given any $k \geq 3$, $\mathbb{E}_X[v(I,X)]$ trivially brings the differential approximation guarantee of $1/q$ which, according to [30, 13], is the best constant factor one may expect assuming $\mathbf{P} \neq \mathbf{NP}$. For this specific CSP, the optimum advantage over a random assignment $\mathbf{NP-hard}$ to approximate to within any positive constant. (Notice that the same facts hold for $\mathsf{CSP}(\mathcal{O}_{\mathsf{q}})$.) By contrast, for $2\,\mathsf{CSP-q}$, $\mathbb{E}_X[v(I,X)]$ is of rather low quality. On the one hand, we proved that $\Omega(1/n)$ is a tight lower bound for the average differential ratio. On the other hand, $2\,\mathsf{CSP-q}$ is approximable within a constant differential factor [41, 17] and, in the boolean case, $2\,\mathsf{CSP-2}$ is approximable within a gain factor of $\Omega(1/\ln n)$ [40, 42, 37]. It is worth noting that, in both cases, the approximation guarantee is achievable by combining semidefinite programming with derandomization techniques. For $\mathsf{k\,CSP-2}$ when $k > 2$, $|\mathbb{E}_X[v(I,X)] - \mathrm{wor}(I)|$ approximates the instance diameter within a factor of $\Omega(1/n^{\lfloor k/2 \rfloor})$ which, in dense instances, is comparable the expected gain approximability bound of $\Omega(1/\sqrt{m})$ given in [31] for this problem.

We now compare the approximation guarantees brought by Hamming balls with fixed radius to the lower bounds we obtained for the average differential ratio. For $2\,\mathsf{CSP-2}$ and $\mathsf{Ek\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k-1}})$, we know that local optima with respect to respectively $\tilde{B}^1$ and B^1 bring the same differential approximation guarantee as the average solution value, and that solutions with optimal value at the neighbourhood respectively $\tilde{B}^1(x)$ and $B^1(x)$ of any x achieve a differential ratio at least $\Omega(1/n)$ times the average differential ratio. By contrast, for $\mathsf{k\,CSP-q}$, the differential approximation guarantee of $\Omega(1/n^k)$ brought at solutions with optimal value at the neighbourhood $\tilde{B}^k(x)$ of any x is a factor $\Theta(1/n^{\lceil \log_{p^\kappa} k \rceil})$ if $q \geq 3$, $\Theta(1/n^{\lceil k/2 \rceil})$ if $q = 2$, of the lower bound we provide for the average differential ratio. Moreover, this guarantee is tight when considering neighbourhood function B^k rather than $\tilde{B}^k$. For $2\,\mathsf{CSP-2}$, Hamming balls of radius 2 yield an approximation guarantee of $\Omega(1/n^2)$ for the instance diameter, while this latter is approximable within factor $4/\pi - 1 > 0.273$ [41]. Still, on dense instances of $\mathsf{k\,CSP-2^\kappa}$, the approximation by a factor $\Omega(1/n^k)$ of the instance diameter is comparable to the gain approximability bound of $\Omega(1/m)$ implied by [31].[8]

Distribution of solution values. Although it brings a rather poor approximation guarantee, the neighbourhood analysis for $\mathsf{k\,CSP-q}$ roughly tells us that out of all solutions, $1/n^k$ of the solutions provide a differential approximation guarantee of $\Omega(1/n^k)$. The analysis of the average solution value similarly indicates that picking a solution uniformly at random yields a solution with an expected differential ratio of $\Omega(1/\nu^b)$, and that a solution realizing such a ratio exists at *some* $O(1/\nu^b)$-cardinality neighbourhood of every solution, where $b = \lfloor k/2 \rfloor$ if $q = 2$ and $\lceil \log_{p^\kappa} k \rceil$ otherwise. For $\mathsf{Ek\,CSP}(\mathcal{I}_{\mathsf{q}}^{\mathsf{k-1}})$, $1/(kq)$ of the solutions achieve a differential ratio $\Omega(1/n)$ times the average differentia ratio. In the very special cases of $\mathsf{E(2k+1)Lin-2}$ and $\mathsf{CSP}(\mathcal{O}_{\mathsf{q}})$, respectively one half and $1/q$ of the solutions are $1/2$ and $1/q$-differential approximate.

Furthermore, exhibiting solutions that bring the approximation guarantees we established, either for the average solution value, or with respect to Hamming balls with radius 1 or k, can be easily done by evaluating $B^k(\mathbf{0})$, $\tilde{B}^k(\mathbf{0})$, $B^1(\mathbf{0})\backslash\{\mathbf{0}\}$ or

[8] Approximating the optimum advantage over a random assignment for $\mathsf{k\,CSP-2^\kappa}$ reduces to approximating the optimum gain over a random assignment for $\mathsf{Lin-2}$ with no loss on the approximation guarantee [16]. Now, if we can compute within polynomial time a solution x^+ such that $|v(I,x^+) - \mathbb{E}_X[v(I,X)]| \geq \rho \times |\mathrm{opt}(I) - \mathbb{E}_X[v(I,X)]|$, then we can symmetrically compute within polynomial time a solution x^- such that $|v(I,x^-) - \mathbb{E}_X[v(I,X)]| \geq \rho \times |\mathrm{wor}(I) - \mathbb{E}_X[v(I,X)]|$, in which case we have $|v(I,x^+) - v(I,x^-)| \geq \rho|\mathrm{opt}(I) - \mathrm{wor}(I)|$.

$\tilde{B}^1(\mathbf{0})\backslash\{\mathbf{0},\mathbf{1}\}$, by computing a local optimum with respect to B^1 or $\tilde{B}^1$, or by running the conditional expectation technique, depending on the considered case. Notice that the $\Omega(1/m)$-gain approximation algorithm of [31] for $\mathsf{Lin}-2$ basically picks a constraint with maximum weight, and then runs the conditional expectation technique on the resulting instance. Hence: performing *some* approximation guarantee with respect to the differential ratio, or even approximating the instance diameter or the optimum gain over a random assignment is *structurally* easy. This somehow contrasts with the fact than only a few algorithms are known that provide significantly better approximation bounds, which involve sophisticate (and time consuming) techniques.

Note that algorithms that approximate the diameter of the instance (or the optimum advantage over a random assignment) to within a positive factor can be used to decide whether the solutions of a given instance all have the same value. In particular, our result concerning the instance diameter tells us that deciding whether an instance of $\mathsf{k}\,\mathsf{CSP}-\mathsf{q}$ is constant reduces to deciding whether on this instance, the solutions of $B^k(\mathbf{0})$ all perform the same objective value.

The average differential ratio. We think that the average differential ratio has potential to provide new insights into CSPs. In our analysis, we took into account a very few characteristics of the input instance, namely: the strong chromatic number of the instance primary hypergraph, the possible restriction to the — general enough — function families $\mathcal{E}_q$ and $\mathcal{I}_q^t$, the possible restriction to the — rather restrictive — function family $\mathcal{O}_q$, and the maximum arity of the constraints. Hence, a next step should be the identification of hypergraphs and function properties that allow to build partition-based solution multisets of low cardinality that satisfy (2.8). More generally, it would be worthwhile to characterize such functions families $\mathcal{F}$ as the set of submodular functions for which $\mathsf{Max}\,\mathsf{CSP}(\mathcal{F})$ or $\mathsf{Min}\,\mathsf{CSP}(\mathcal{F})$ admits a constant lower bound for the average differential ratio.

Beyond these aspects, the properties of the average differential ratio viewed as a complexity measure, including its connections to other measures, should be investigated. Notably, because for $\mathsf{E3}\,\mathsf{Lin}-2$, this ratio is in $O(1)$, the authors of [22] could derive from the hardness result of [30] as regards $\mathsf{E3}\,\mathsf{Lin}-2$ a constant inapproximability bound of 0 for the diameter of $\mathsf{3}\,\mathsf{Sat}$ instances, assuming $\mathbf{P} \neq \mathbf{NP}$.

Combinatorial designs. Besides, the analysis raises new questions regarding orthogonal arrays and difference schemes, and introduces new families of combinatorial designs. The estimate of the average differential ratio involves an uncommon criterion on orthogonal arrays and difference schemes of given strength and number of columns. Namely, the analysis led in section 2 suggests the search for such arrays that maximize their highest frequency (rather than minimizing the number of their rows). Such researchs have been recently led in [15] for orthogonal arrays of strength 2. Furthermore, the reduction from q-ary CSPs to p-ary CSPs and the neighbourhood analysis suggest to further investigate the families of designs we introduced in sections 3 and 4. We are in particular interested in pairs of arrays that achieve numbers $\gamma_E(q,k,k)$: although such pairs are only a slight relaxation of pairs that achieve numbers $\gamma(q,k,k)$, they the most likely do not admit such a regular construction as for $\gamma(q,k,k)$.

Regarding the families $\Gamma(R,R^*,q,p,k)$ and $\Delta(R,R^*,\nu,d,k)$ of designs, we recently proved that numbers $\gamma(q,p,k)$, $\delta(q,p,k)$ and $\bar{\delta}(q,p,k)$ all coincide for all triples (q,p,k) [18]. Plus, we know the exact value of these numbers in case where $p=k$. However, for $\delta(q,p,k)$ in case where $q>p$, we only provided the naive lower bound of $\delta(q-p+k,k,k)$. Therefore, we should further study numbers $\delta(q,p,k)$ in this case.

In particular, this would allow us to obtain a better estimate of the differential ratio achieved by a best solution among those whose coordinates take at most p distinct values, and hence of the expansion of our reduction from $\mathsf{k\,CSP-q}$ to $\mathsf{k\,CSP-p}$ when $p > k$.

Other questions regarding the families $\Gamma(R, R^*, q, p, k)$ and $\Delta(R, R^*, \nu, d, k)$ of designs and their restrictions should be addressed, starting with their connections when considering an alternate optimization criterion, as well as additional constraints. We notably are curious about the construction of such *simple* designs (i.e., arrays with no repeated row) having a minimum number of rows.

REFERENCES

[1] A. Aiello, E. Burattini, A. Massaroti, and F. Ventriglia, *Towards a general principle of evaluation for approximate algorithms*, RAIRO – Theoretical Informatics and Applications, 13 (1979), pp. 227–239, https://doi.org/10.1051/ita/1979130302271.

[2] P. Alimonti, *Non-oblivious local search for max 2-ccsp with application to max dicut*, in Graph-Theoretic Concepts in Computer Science, R. H. Möhring, ed., Berlin, Heidelberg, 1997, Springer Berlin Heidelberg, pp. 2–14.

[3] N. Alon and A. Naor, *Approximating the cut-norm via grothendieck's inequality*, SIAM Journal on Computing, 35 (2006), pp. 787–803, https://doi.org/10.1137/S0097539704441629.

[4] G. Ausiello, A. D'Atri, and M. Protasi, *Structure preserving reductions among convex optimization problems*, Journal of Computational System Sciences, 21 (1980), pp. 136–153, https://doi.org/10.1016/0022-0000(80)90046-X.

[5] G. Ausiello and M. Protasi, *Local search, reducibility and approximability of np-optimization problems*, Information Processing Letters, 54 (1995), pp. 73–79, https://doi.org/https://doi.org/10.1016/0020-0190(95)00006-X.

[6] P. Austrin, *Towards sharp inapproximability for any 2-csp*, SIAM Journal on Computing, 39 (2010), pp. 2430–2463, https://doi.org/10.1137/070711670.

[7] P. Austrin and S. Khot, *A characterization of approximation resistance for even k-partite csps*, in Proceedings of the 4th Conference on Innovations in Theoretical Computer Science, ITCS '13, New York, NY, USA, 2013, Association for Computing Machinery, p. 187–196, https://doi.org/10.1145/2422436.2422459.

[8] P. Austrin and E. Mossel, *Approximation resistant predicates from pairwise independence*, Computational complexity, 18 (2009), pp. 249–271, https://doi.org/10.1007/s00037-009-0272-6.

[9] M. Bellare and P. Rogaway, *The complexity of approximating a nonlinear program*, Mathematical Programming, 69 (1995), pp. 429–441, https://doi.org/10.1007/BF01585569.

[10] J. Bierbrauer, *Construction of orthogonal arrays*, Journal of Statistical Planning and Inference, 56 (1996), pp. 39–47, https://doi.org/https://doi.org/10.1016/S0378-3758(96)00007-9. Orthogonal Arrays and Affine Designs, Part I.

[11] R. Bose and D. Ray-Chaudhuri, *On a class of error correcting binary group codes*, Information and Control, 3 (1960), pp. 68–79, https://doi.org/https://doi.org/10.1016/S0019-9958(60)90287-4.

[12] K. A. Bush, *Orthogonal Arrays of Index Unity*, The Annals of Mathematical Statistics, 23 (1952), pp. 426 – 434, https://doi.org/10.1214/aoms/1177729387.

[13] S. O. Chan, *Approximation resistance from pairwise-independent subgroups*, J. ACM, 63 (2016), https://doi.org/10.1145/2873054.

[14] M. Charikar, K. Makarychev, and Y. Makarychev, *Near-optimal algorithms for maximum constraint satisfaction problems*, ACM Trans. Algorithms, 5 (2009), https://doi.org/10.1145/1541885.1541893.

[15] C. J. Colbourn, D. R. Stinson, and S. Veitch, *Constructions of optimal orthogonal arrays with repeated rows*, Discrete Mathematics, 342 (2019), pp. 2455–2466, https://doi.org/https://doi.org/10.1016/j.disc.2019.05.021.

[16] J. Culus and S. Toulouse, *How far from a worst solution a random solution of a kcsp instance can be?*, in Combinatorial Algorithms - 29th International Workshop, IWOCA 2018, Singapore, July 16-19, 2018, Proceedings, C. S. Iliopoulos, H. W. Leong, and W. Sung, eds., vol. 10979 of Lecture Notes in Computer Science, Springer, 2018, pp. 374–386, https://doi.org/10.1007/978-3-319-94667-2_31.

[17] J.-F. Culus and S. Toulouse, *2 csps all are approximable within a constant differential factor*,

in Combinatorial Optimization, J. Lee, G. Rinaldi, and A. R. Mahjoub, eds., vol. 10856 of Lecture Notes in Computer Science, Cham, 2018, Springer International Publishing, pp. 389–401, https://doi.org/10.1007/978-3-319-96151-4_33.

[18] J.-F. Culus and S. Toulouse, *Optimizing alphabet reduction pairs of arrays*, CoRR, abs/TODO (2024), https://arxiv.org/abs/TODO, https://arxiv.org/abs/TODO.

[19] J.-F. Culus, S. Toulouse, and F. Roupin, *Differential approximation of SNP optimization problems*, in International Symposium on Combinatorial Optimization (ISCO) 2012 (short paper), 2012.

[20] P. Delsarte, *An algebraic approach to the association schemes of coding theory*, tech. report, Philips Research Reports, Supplement No. 10, 1973.

[21] M. Demange and V. T. Paschos, *On an approximation measure founded on the links between optimization and polynomial approximation theory*, Theoretical Computer Science, 158 (1996), pp. 117–141, https://doi.org/https://doi.org/10.1016/0304-3975(95)00060-7.

[22] B. Escoffier and V. T. Paschos, *Differential approximation of min sat, max sat and related problems*, European Journal of Operational Research, 181 (2007), pp. 620–633, https://doi.org/https://doi.org/10.1016/j.ejor.2005.04.057.

[23] U. Feige, V. S. Mirrokni, and J. Vondrák, *Maximizing non-monotone submodular functions*, SIAM Journal on Computing, 40 (2011), pp. 1133–1153, https://doi.org/10.1137/090779346.

[24] M. Garey, D. Johnson, and L. Stockmeyer, *Some simplified np-complete graph problems*, Theoretical Computer Science, 1 (1976), pp. 237–267, https://doi.org/https://doi.org/10.1016/0304-3975(76)90059-1.

[25] M. X. Goemans and D. P. Williamson, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. ACM, 42 (1995), p. 1115–1145, https://doi.org/10.1145/227683.227684.

[26] R. Hariharan and S. Mahajan, *Derandomizing approximation algorithms based on semidefinite programming*, SIAM Journal on Computing, 28 (1999), pp. 1641–1663, https://doi.org/10.1137/S0097539796309326.

[27] J. Håstad, *Clique is hard to approximate within $n^{1-\varepsilon}$*, Acta mathematica, 182 (1999), pp. 105–142, https://doi.org/10.1007/BF02392825.

[28] A. Hedayat, N. J. A. Sloane, and J. Stufken, *Orthogonal Arrays: Theory and Applications*, Springer, 1999, https://doi.org/https://doi.org/10.1007/978-1-4612-1478-6.

[29] A. Hocquenghem, *Codes Correcteurs d'Erreurs*, Chiffres (Paris), 2 (1959), pp. 147–156.

[30] J. Håstad, *Some optimal inapproximability results*, J. ACM, 48 (2001), p. 798–859, https://doi.org/10.1145/502090.502098.

[31] J. Håstad and S. Venkatesh, *On the advantage over a random assignment*, Random Structures & Algorithms, 25 (2004), pp. 117–149, https://doi.org/https://doi.org/10.1002/rsa.20031.

[32] D. S. Johnson, *Approximation algorithms for combinatorial problems*, Journal of Computer and System Sciences, 9 (1974), pp. 256–278, https://doi.org/https://doi.org/10.1016/S0022-0000(74)80044-9.

[33] S. Khanna, R. Motwani, M. Sudan, and U. Vazirani, *On syntactic versus computational views of approximability*, SIAM Journal on Computing, 28 (1998), pp. 164–191, https://doi.org/10.1137/S0097539795286612.

[34] S. Khot and A. Naor, *Linear equations modulo 2 and the l_1 diameter of convex bodies*, SIAM Journal on Computing, 38 (2008), pp. 1448–1463, https://doi.org/10.1137/070691140.

[35] R. Kohli, R. Krishnamurti, and P. Mirchandani, *The minimum satisfiability problem*, SIAM Journal on Discrete Mathematics, 7 (1994), pp. 275–283, https://doi.org/10.1137/S0895480191220836.

[36] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-holland Publishing Company, 1977.

[37] A. Megretski, *Relaxations of quadratic programs in operator theory and system analysis*, in Systems, Approximation, Singular Integral Operators, and Related Topics, A. A. Borichev and N. K. Nikolski, eds., Basel, 2001, Birkhäuser Basel, pp. 365–392.

[38] J. Monnot, V. T. Paschos, and S. Toulouse, *Approximation polynomiale des problèmes NP-difficiles - Optima locaux et rapport différentiel*, Hermes Science, Paris, 2003.

[39] R. Mukerjee, P. Z. Qian, and C. Jeff Wu, *On the existence of nested orthogonal arrays*, Discrete Mathematics, 308 (2008), pp. 4635–4642, https://doi.org/https://doi.org/10.1016/j.disc.2007.08.096.

[40] A. S. Nemirovski, C. Roos, and T. Terlaky, *On maximization of quadratic form over intersection of ellipsoids with common center*, Mathematical Programming, 86 (1999),

pp. 463–473, https://doi.org/10.1007/s101070050100.

[41] Y. Nesterov, *Semidefinite relaxation and nonconvex quadratic optimization*, Optimization Methods and Software, 9 (1998), pp. 141–160, https://doi.org/10.1080/10556789808805690.

[42] Y. Nesterov, H. Wolkowicz, and Y. Ye, *Semidefinite programming relaxations of nonconvex quadratic optimization*, in Handbook of Semidefinite Programming: Theory, Algorithms, and Applications, H. Wolkowicz, R. Saigal, and L. Vandenberghe, eds., Springer US, Boston, MA, 2000, pp. 361–419, https://doi.org/10.1007/978-1-4615-4381-7_13.

[43] P. Raghavendra, *Optimal algorithms and inapproximability results for every csp?*, in Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing, STOC '08, New York, NY, USA, 2008, Association for Computing Machinery, p. 245–254, https://doi.org/10.1145/1374376.1374414.

[44] C. R. Rao, *Factorial experiments derivable from combinatorial arrangements of arrays*, Supplement to the Journal of the Royal Statistical Society, 9 (1947), pp. 128–139.

[45] D. R. Stinson, *Bounds for orthogonal arrays with repeated rows*, Bulletin of the ICA, 85 (2019), pp. 60 – 73.

[46] D. Zuckerman, *Linear degree extractors and the inapproximability of max clique and chromatic number*, Theory of Computing, 3 (2007), pp. 103–128, https://doi.org/10.4086/toc.2007.v003a006.