FOURIER REPRESENTATIONS OF MIXED SIGNAL CLASSES

$$\text{CT}: \begin{array}{lll} \text{periodic}: & \text{FS} & X[k] \\ \text{non-periodic}: & \text{FT} & X(j\omega) \end{array}$$

$$\text{DT}: \begin{array}{lll} \text{periodic}: & \text{DTFS} & X[k] \\ \text{non-periodic}: & \text{DTFT} & X\left(e^{j\Omega}\right) \end{array}$$

- What about the Fourier representation of a mixture of
  a) periodic and non-periodic signals
  b) CT and DT signals?

Examples: $x(t) \rightarrow \boxed{H} \rightarrow y(t)$ such as $\cos(\omega_0 t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

(periodic)

$$x(t) \rightarrow \boxed{\text{sampler}} \rightarrow y[n]$$

- We will go through:

  a) FT of periodic signals, which we have used FS:

$$x(t) \xleftrightarrow{FS;\omega_0} X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T}$$

  We can take FT of x(t).

  b) Convolution and multiplication with mixture of periodic and non-periodic signals.

  c) *Fourier transform* of *discrete-time* signals.

<u>FT of periodic signals</u>

Previously, for CT periodic signals, we use FS representations. What happens if we take FT of periodic signals?

FS representation of periodic signal x(t):

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \qquad (*)$$

$$\text{FT of 1:} \qquad 1 \xleftarrow{FT} 2\pi\delta(\omega)$$

$$\text{Freq. shift:} \qquad 1 \cdot e^{jk\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega - k\omega_0)$$

Take FT of equation (*) $\rightarrow$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \xleftrightarrow{FT} 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k\omega_0)}$$

Note:

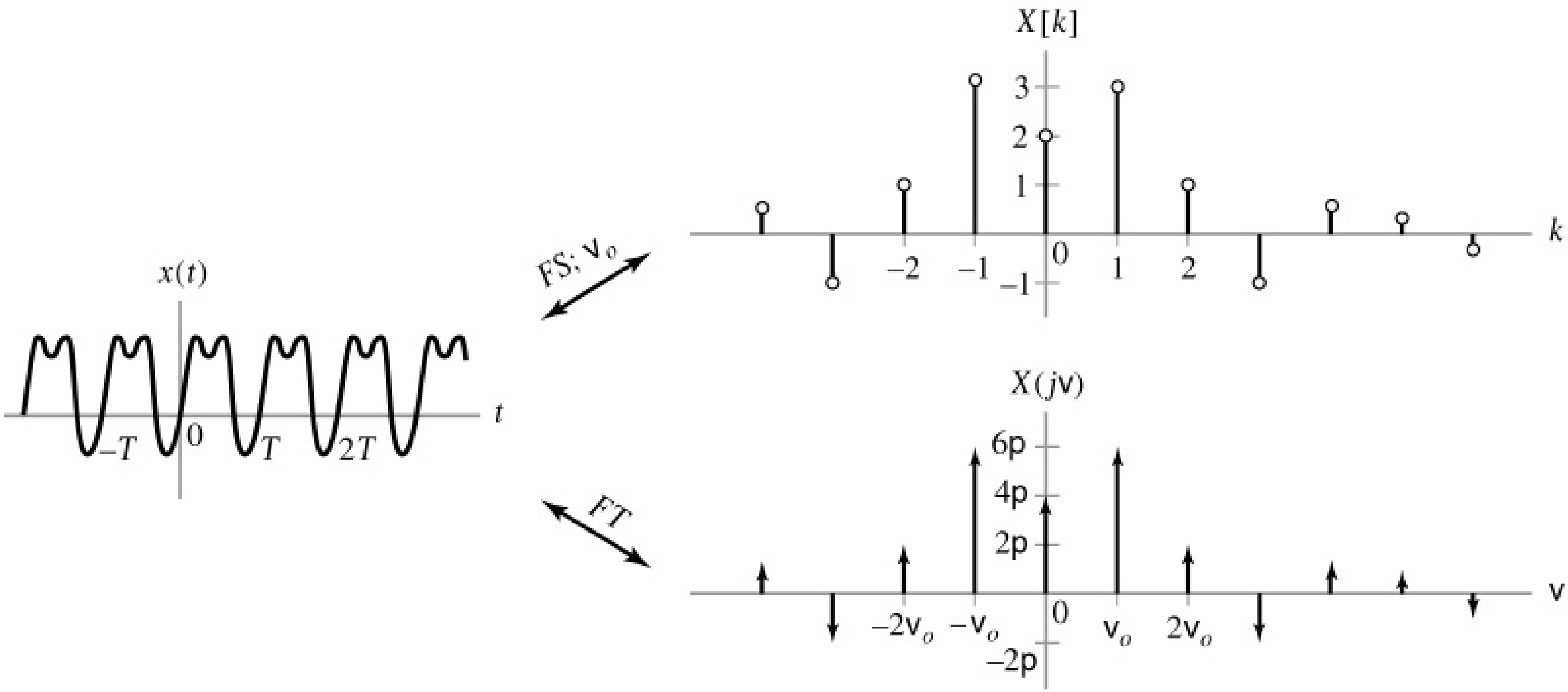
a) FT of a periodic signal is a series of impulses spaced by the fundamental frequency ω_0.

b) The k-th impulse has strength $2\pi X[k]$.

c) FT of $x(t)=\cos(\omega_0 t)$ can be obtained by replacing

$$e^{jk\omega_0 t} \text{ with } 2\pi\delta(\omega - k\omega_0), \text{ or vise versa}$$

FS and FT representation of a periodic continuous-time signal.

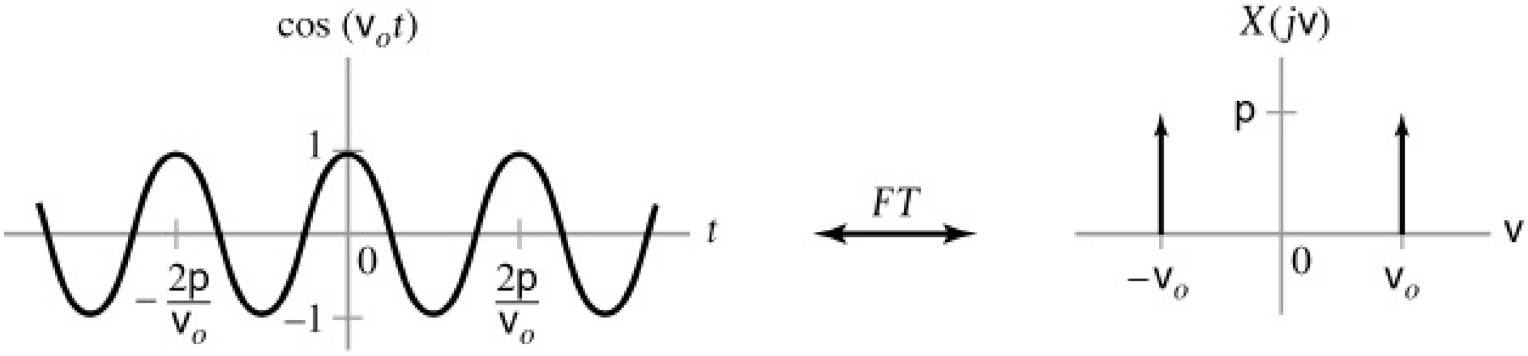


E

Find Fourier transform of $x(t) = \cos(\omega_0 t)$

$$\cos(\omega_0 t) \xleftrightarrow{FS} \begin{cases} 1/2, & k = \pm 1 \\ 0, & k \neq \pm 1 \end{cases}$$

Thus, $\cos(\omega_0 t) \xleftrightarrow{FT} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$



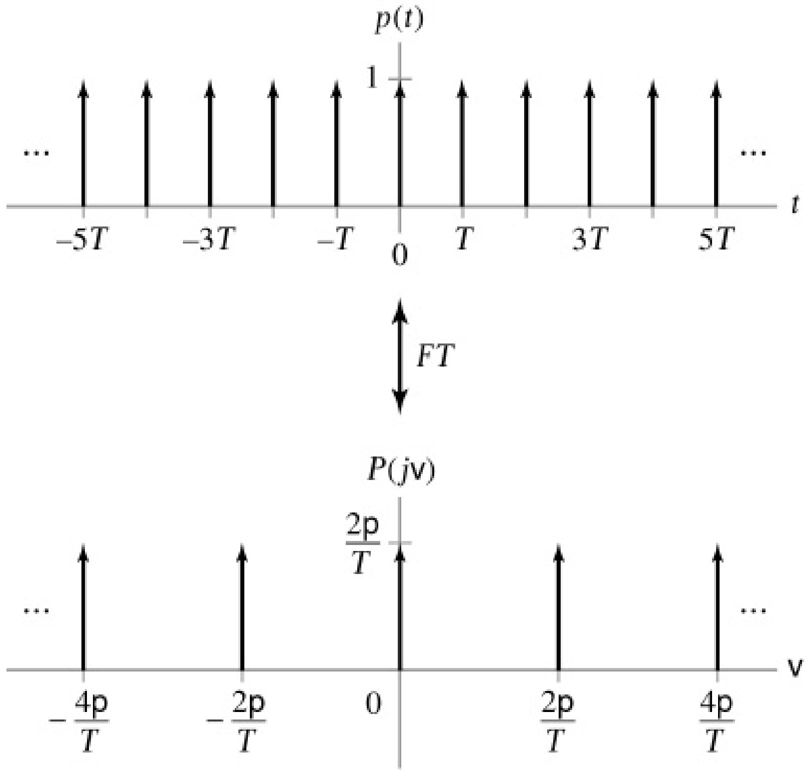
E

Determine the FT of the unit impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

p(t) is periodic with fundamental period T, fundamental frequency ω_0. FS coefficients:

$$P[k] = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}, \quad \forall k$$

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$



Relating DTFT to DTFS

N-periodic signal x[n] has DTFS expression

$$\begin{cases} x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \qquad\qquad (*) \\ \\ e^{jk\Omega_0 n} \xleftrightarrow{DTFT} 2\pi\delta(\Omega - k\Omega_0), \quad -\pi < \Omega < \pi, \quad -\pi < k\Omega_0 < \pi \end{cases}$$

Extending to any interval:

$$e^{jk\Omega_0 n} \xleftrightarrow{DTFT} 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - 2\pi m)$$

This, DTFT of x[n] given in (*) is expressed as:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = 2\pi \sum_{k=0}^{N-1} X[k] \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - 2\pi m)$$

Since X[k] is N periodic and $N\Omega_0=2\pi$, we have

$$\boxed{x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n}\xleftrightarrow{DTFT}X\left(e^{j\Omega}\right)=2\pi\sum_{k=-\infty}^{\infty}X[k]\delta(\Omega-k\Omega_0)}$$

Note:

a) DTFS $\rightarrow$ DTFT: $e^{jk\Omega_0 n}\Rightarrow\delta(\Omega-k\Omega_0)$ then scale by 2π

b) DTFT $\rightarrow$ DTFS: $\delta(\Omega-k\Omega_0)\Rightarrow e^{jk\Omega_0 n}$ then scale by $1/(2\pi)$

Also, replace sum intervals from 0~N-1 for DTFS to -∞ ~ ∞ for DTFT

$\boxed{\text{E}}$

$$x[n]=\sum_{k=-\infty}^{\infty}\delta[n-10k].\ \text{ Find the DTFT of x[n].}$$

Fundamental period? $N=10,\quad \Omega_0=\pi/5.\qquad \text{Use DTFS}:$

$$X[k] = \frac{1}{10}\sum_{n=0}^{9} x[n]e^{-jk\frac{\pi}{5}n} = \frac{1}{10}, \qquad \forall k$$

Use note a) last slide:

$$x[n] \xleftrightarrow{DTFT} 2\pi \frac{1}{10}\sum_{n=-\infty}^{\infty} \delta\left(\Omega - k\tfrac{\pi}{5}\right)$$

Question: if we take inverse DTFS of X[k], we get

$$x[n] = \sum_{k=0}^{9} X[k]e^{jk\frac{\pi}{5}n}$$

$$= \frac{1}{10}\sum_{k=0}^{9} e^{jk\frac{\pi}{5}n}$$

which does not seem to equal the original expression

$$x[n] = \sum_{n=-\infty}^{\infty} \delta[n - 10k].$$

Exercise: use Matlab to verify.

Convolution and multiplication with mixture of periodic and non-periodic signals

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad\qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

For periodic inputs:

$$y(t) = \underbrace{x(t)}_{\text{periodic}} * \underbrace{h(t)}_{\text{non-periodic}} \qquad\qquad y[n] = \underbrace{x[n]}_{\text{periodic}} * \underbrace{h[n]}_{\text{non-periodic}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\text{Use FT} \qquad\qquad\qquad\qquad \text{Use DTFT}$$

1) Convolution of periodic and non-periodic signals

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$$

$$x(t) \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta\left(\omega - k\omega_0\right),$$

$X[k]$: FS coefficients

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)H(j\omega)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} H(jk\omega_0)X[k]\delta(\omega - k\omega_0)$$

E LTI system has an impulse response $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$$h(t) = 2\cos(4\pi t)\sin(\pi t)/(\pi t).$$

For input signal $x(t) = 1 + \cos(\pi t) + \sin(4\pi t)$,

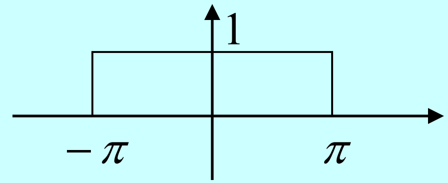
use the FT to demermine the system output $y(t)$.

$$h(t) = \frac{\sin(\pi t)}{\pi t}\left(e^{-j4\pi t} + e^{j4\pi t}\right)$$

$$\Updownarrow$$

$$H(j\omega) = rect\left(\frac{\omega + 4\pi}{2\pi}\right) + rect\left(\frac{\omega - 4\pi}{2\pi}\right)$$

$$h(t) = \frac{\sin(\pi t)}{\pi t} \xleftrightarrow{FT} rect\left(\frac{\omega}{2\pi}\right)$$



$$X(j\omega) = 2\pi\delta(\omega) + \pi\left(\delta(\omega-\pi) + \delta(\omega+\pi)\right) + \tfrac{\pi}{j}\left(\delta(\omega-4\pi) - \delta(\omega-4\pi)\right)$$

Because h(t) is an ideal bandpass filter with a bandwidth 2π centered at ±4π, the Fourier transform of the output signal is thus

$$Y(j\omega) = \tfrac{\pi}{j}\left(\delta(\omega-4\pi) - \delta(\omega-4\pi)\right)$$

which has a time-domain expression given as: $\boxed{y(t) = \sin(4\pi t)}$

For discrete-time signals:

$$\boxed{y[n] = \underbrace{x[n]}_{\text{periodic}} * h[n] \xleftrightarrow{DTFT} 2\pi \sum_{k=-\infty}^{\infty} H\left(e^{jk\Omega_0}\right) X[k]\,\delta\left(\Omega - k\Omega_0\right)}$$

2) Multiplication of periodic and non-periodic signals

$$y(t) = g(t)\underbrace{x(t)}_{\text{periodic}} \xleftrightarrow{FT}$$

$$Y(j\omega) = \tfrac{1}{2\pi} G(j\omega) * X(j\omega)$$
$$= G(j\omega) * \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

Note :
$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \xleftrightarrow{FT}$$
$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

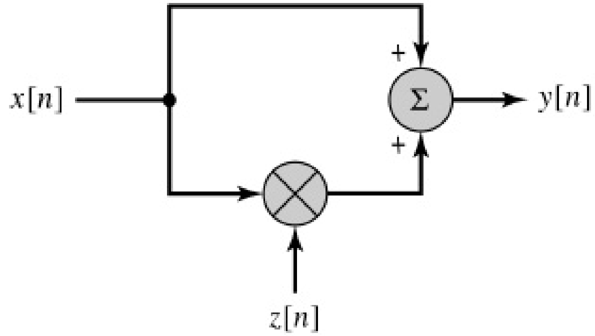
Carrying out the convolution yields:

$$\boxed{y(t) = g(t)x(t) \xleftrightarrow{FT} Y(j\omega) = \sum_{k=-\infty}^{\infty} X[k] G(\omega - k\omega_0)}$$

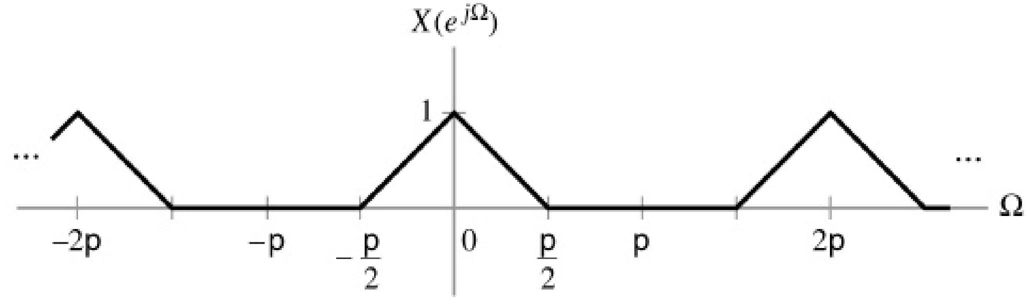
DT case:
$$\boxed{y[n] = \underbrace{x[n]}_{\text{periodic}} z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \sum_{k=0}^{N-1} X[k] Z\left(e^{j(\Omega - k\Omega_0)}\right)}$$

E

Consider the LTI system and input signal spectrum $X(e^{j\Omega})$ depicted by the figure below. Determine an expression for $Y(e^{j\Omega})$, the DTFT of the output y[n] assuming that $z[n]=2\cos(\pi n/2)$.



(a)

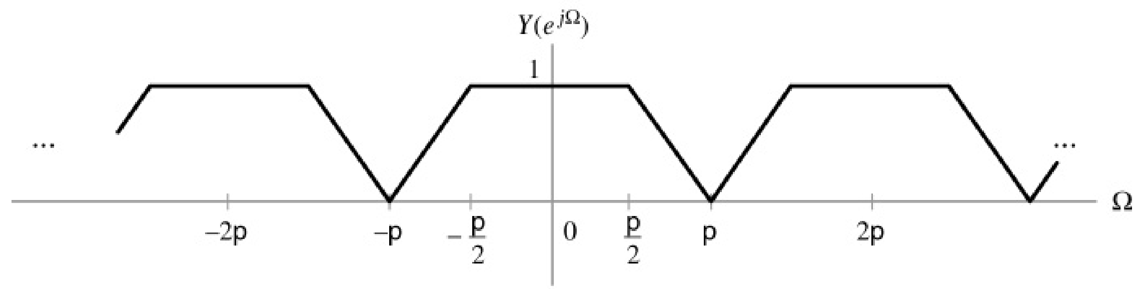


(b)

$$y[n] = x[n] + x[n]z[n]$$

$z[n]$: periodic with $\omega_0 = \frac{\pi}{2}$. FS coefficients of $z[n]$:

$$Z[k] = \begin{cases} 1, & k = \pm 1 \\ 0, & o.w. \end{cases}$$

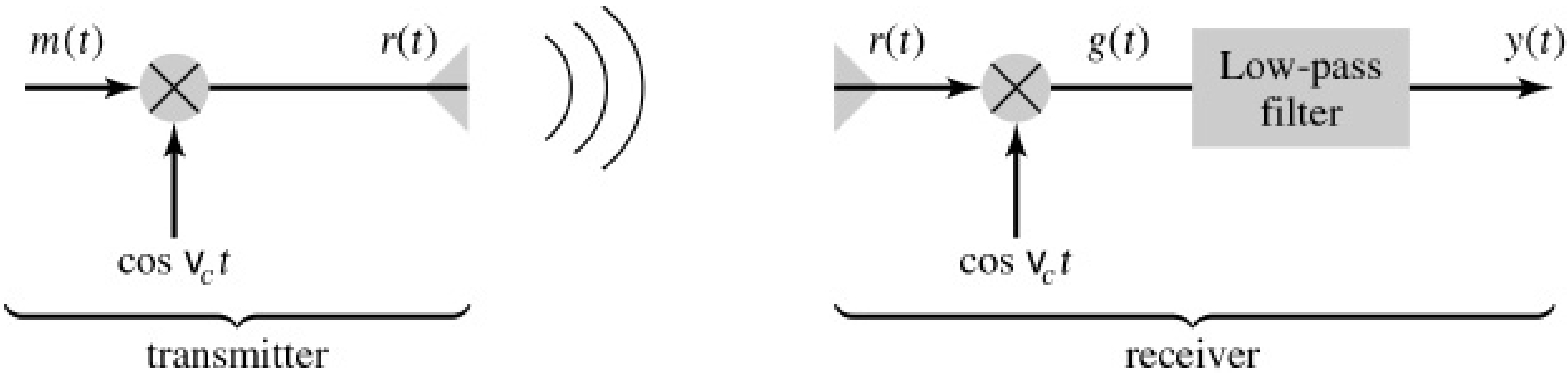


(b)

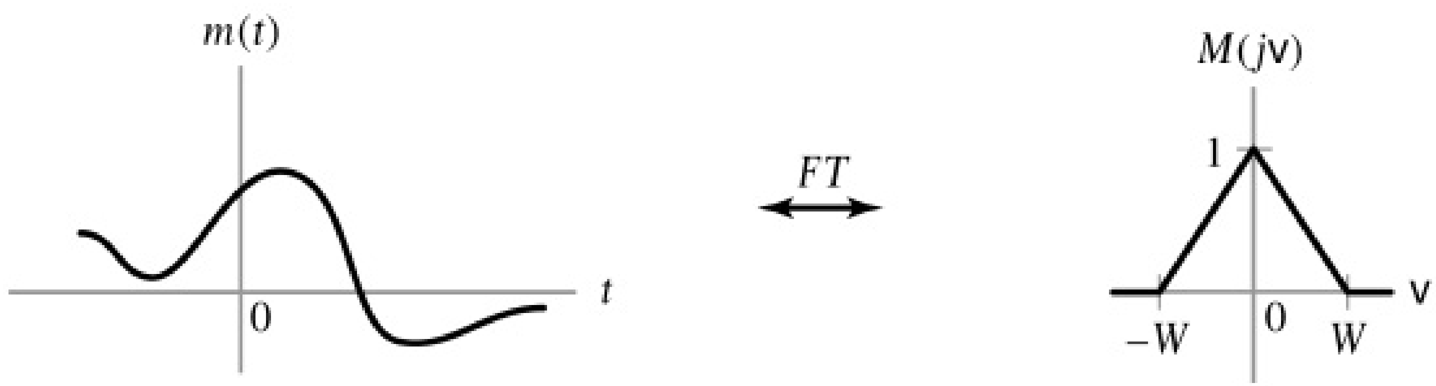
Thus,

$$Y\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right) + X\left(e^{j(\Omega - \frac{\pi}{2})}\right) + X\left(e^{j(\Omega + \frac{\pi}{2})}\right)$$

E AM Radio



(a)



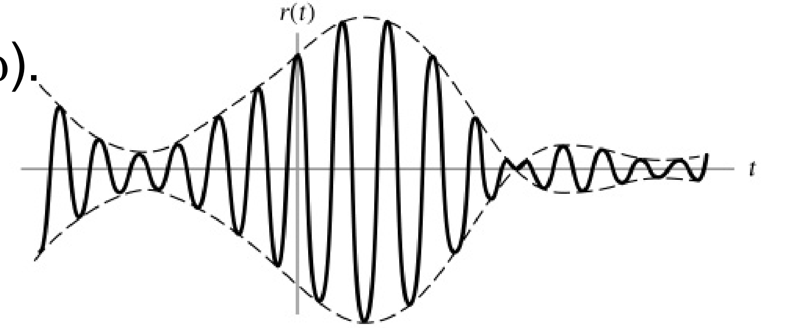
(b)

(a) Simplified AM radio transmitter & receiver.

(b) Spectrum of message signal.

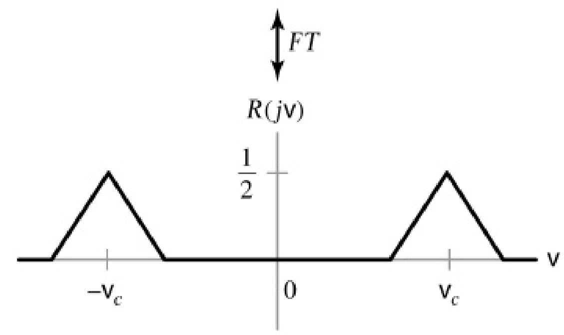
Analyze the system in the frequency domain.

Signals in the AM transmitter and receiver.
(a) Transmitted signal $r(t)$ and spectrum $R(j\omega)$.
(b) Spectrum of $q(t)$ in the receiver.
(c) Spectrum of receiver output y(t).



$$r(t) = m(t)\cos(\omega_c t) \xleftrightarrow{FT}$$

$$R(j\omega) = \tfrac{1}{2} M(j(\omega - \omega_c)) + \tfrac{1}{2} M(j(\omega + \omega_c))$$

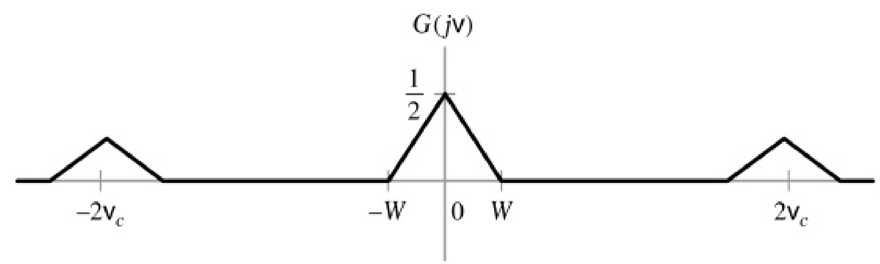


(a)

In the receiver, r(t) is multiplied by the identical cosine used in the transmitter to obtain:

$$g(t) = r(t)\cos(\omega_c t) \xleftrightarrow{FT}$$

$$G(j\omega) = \tfrac{1}{2} R(j(\omega - \omega_c)) + \tfrac{1}{2} R(j(\omega + \omega_c))$$

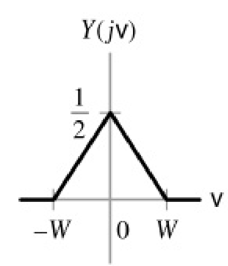


(b)

$$= \tfrac{1}{4} M(j(\omega - 2\omega_c)) + \tfrac{1}{2} M(j\omega) + \tfrac{1}{4} M(j(\omega + 2\omega_c))$$

After low-pass filtering:

$$Y(j\omega) = \tfrac{1}{2} M(j\omega)$$



(c)