PROPERTIES OF FOURIER REPRESENTATIONS

Time property	Periodic (t,n)	Nonperiodic (t,n)	
C.T. (t)	• Fourier Series (FS) $x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_0 t}, \omega_0=\frac{2\pi}{T}$ $X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt,$ (T: period)	• Four Transform (FT) $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$ $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$	Non-periodic (k,ω)
D.T. [n]	• Discrete-Time Fourier Series (DTFS) $x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n}, \Omega_0=\frac{2\pi}{N}$ $X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\Omega_0 n}$ (N: period)	• Discrete-Time Fourier Transform (DTFT) $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\Omega})e^{j\Omega n}d\Omega$ $X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}$	Periodic (k,Ω)
	Discrete [k]	Continuous (ω, Ω)	Freq. property

- <u>Linearity and symmetry</u>

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_0} Z[k] = aX[k] + bY[k]$$

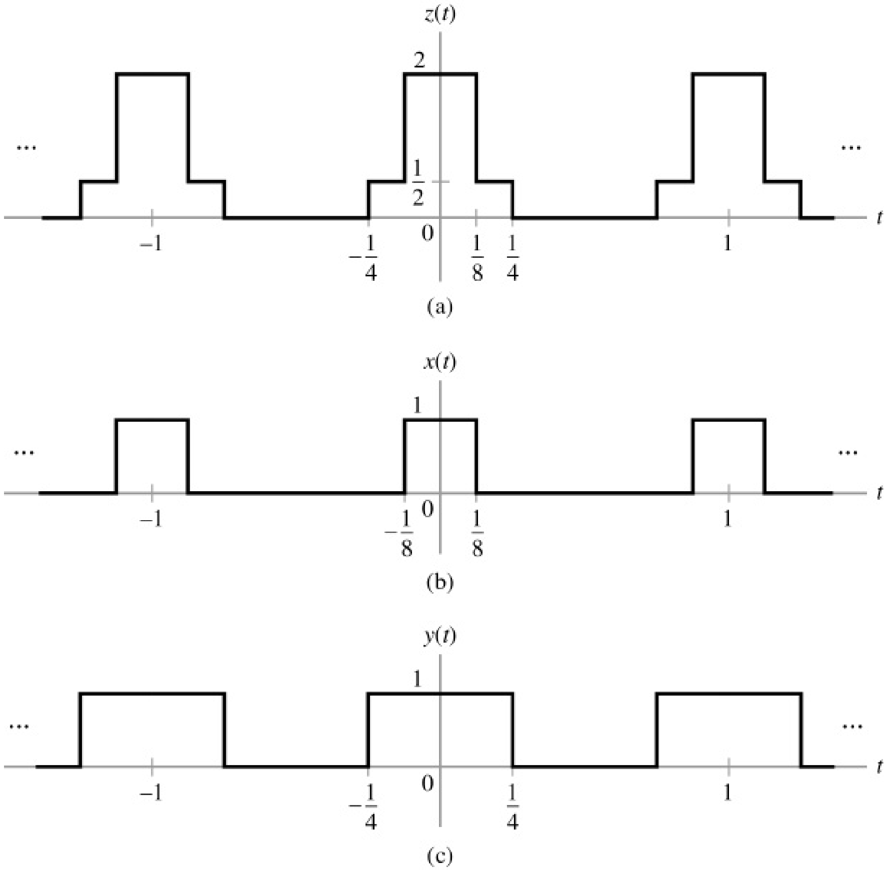
$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS;\Omega_0} Z[k] = aX[k] + bY[k]$$

$\boxed{\text{E}}$

$$z(t) = \frac{3}{2}x(t) + \frac{1}{2}y(t)$$

Find the frequency-domain representation of z(t).



(a)

(b)

(c)

Which type of freq.-domain representation?

- FT, FS, DTFT, DTFS ?

Periodic signals, continuous time. Thus, FS.

$$Z[k] = \frac{3}{2} X[k] + \frac{1}{2} Y[k]$$

$$X[k] = \frac{1}{T_x} \int_0^{T_x} x(t) e^{-jk\omega_{0x}t} dt$$

$$= \frac{1}{T_x} \int_{-T_x/2}^{T_x/2} x(t) e^{-jk\omega_{0x}t} dt$$

$$= \int_{-1/8}^{1/8} e^{-jk2\pi t} dt = \frac{1}{-jk2\pi} e^{-jk2\pi t} \Big|_{-\frac{1}{8}}^{\frac{1}{8}}$$

$$= \frac{1}{-jk2\pi} \left(-j2\sin\left(\tfrac{\pi}{4}k\right) \right)$$

$$= \frac{1}{k\pi} \sin\left(\tfrac{\pi}{4}k\right)$$

$$Y[k] = \frac{1}{k\pi} \sin\left(\tfrac{\pi}{2}k\right)$$

$$\boxed{Z[k] \xleftrightarrow{FS;2\pi} \frac{3}{2k\pi} \sin\left(\tfrac{\pi}{4}k\right) + \frac{1}{2k\pi} \sin\left(\tfrac{\pi}{2}k\right)}$$

Symmetry:

We will develop using continuous, non-periodic signals. Results for other 2 cases may be obtained in a similar way.

a) Assume $x(t)$ real $\Rightarrow x^*(t) = x(t)$

$$X^*\left(j\omega\right) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt$$

$$= X\left(-j\omega\right)$$

$$\boxed{\text{If } x(t) \text{ is real} \Rightarrow X(j\omega) \text{ is conjugate symmetric}}$$

√ Further assume $x(t)$ even (and real)

$x(-t) = x(t), x^*(t) = x(t) \Rightarrow x^*(t) = x(-t)$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x(-t) e^{-j\omega(-t)} dt$$

replace τ with $-t$ $\quad = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$ (why?)

$$= X(j\omega)$$

$$\int_a^b x(t)dt = -\int_b^a x(t)dt$$

$$\tau = -t \Rightarrow d\tau = -dt$$

$\boxed{\text{If } x(t) \text{ is real and even} \Rightarrow X(j\omega) \text{ is real.}}$

√ Further assume $x(t)$ odd (and real)

$$x(-t) = -x(t), x^*(t) = x(t) \Rightarrow x^*(t) = -x(-t)$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} \left(-x(-t)\right) e^{-j\omega(-t)} dt$$

$$= -X(j\omega)$$

$\boxed{\text{If } x(t) \text{ is real and odd} \Rightarrow X(j\omega) \text{ is purely imaginary.}}$

b) Assume $x(t)$ imaginary $\Rightarrow x^*(t) = -x(t)$

$$X^*(j\omega) = -\int_{-\infty}^{\infty} x(t) e^{-j(-\omega)t} dt$$
$$= -X(-j\omega)$$

If $x(t)$ is purely imaginary $\Rightarrow$

- Real part of $X(j\omega)$ has odd symmetry
- Imaginary part of $X(j\omega)$ has even symmetry

• <u>Convolution:</u> Applied to non-periodic signals.

Let $y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

If $x(t) \xleftrightarrow{FT} X(j\omega) \Rightarrow$

$$x(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-\tau)} d\omega$$

$$y(t)=\int_{-\infty}^{\infty}h(\tau)\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau$$

$$=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau\right]}_{H(j\omega)}X(j\omega)e^{j\omega t}d\omega$$

$$=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[H(j\omega)X(j\omega)\right]e^{j\omega t}d\omega$$

$$\boxed{y(t)=x(t)*h(t)\xleftrightarrow{FT}Y(j\omega)=X(j\omega)H(j\omega)}$$

<u>Conclusion:</u> Convolution in time domain $\rightarrow$ Multiplication in freq. domain.

$\boxed{\text{E}}$

Let $x(t)=\tfrac{1}{\pi t}\sin(\pi t)$ be input to a system with impulse response $h(t)=\tfrac{1}{\pi t}\sin(2\pi t)$. Find the system output $y(t)$

$$y(t) = x(t) * h(t)$$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$x(t) \xleftrightarrow{FT} X(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}$$

$$h(t) \xleftrightarrow{FT} H(j\omega) = \begin{cases} 1, & |\omega| < 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$$

$$Y(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases} \xleftrightarrow{FT} y(t) = \tfrac{1}{\pi t}\sin(\pi t)$$

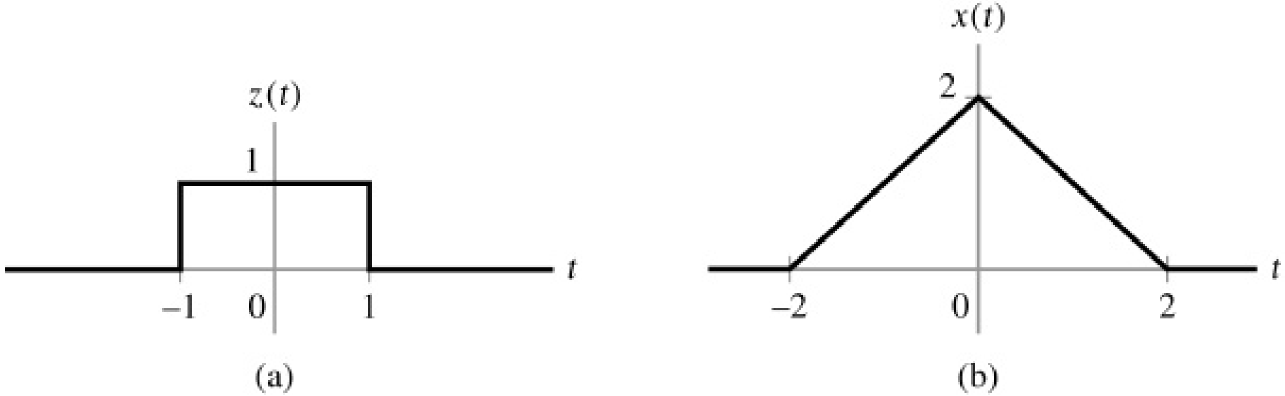
E

$$x(t) \xleftrightarrow{FT} X(j\omega) = \tfrac{4}{\omega^2}\sin^2(\omega). \qquad \text{Find } x(t).$$

$$z(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} \xleftrightarrow{FT} \tfrac{2}{\omega}\sin(\omega)$$

Let $\quad Z(j\omega) = \frac{2}{\omega}\sin(\omega)$. Then, $X(\mathrm{j}\omega) = Z(\mathrm{j}\omega)Z(\mathrm{j}\omega)$.

Thus, $\quad x(t) = z(t) * z(t)$.



The same convolution properties hold for discrete-time, non-periodic signals.

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

Convolution properties for periodic (DT or CT) and periodic with non-periodic signals will be discussed in Chapter 4.

- Differentiation and integration:
  - Applicable to continuous functions: time (t) or frequency (ω or Ω)
  - FT (t, ω) and DTFT (Ω)

Differentiation in time:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[j\omega X(j\omega)\right]e^{j\omega t}\,d\omega \qquad \text{it follows}$$

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)}$$

$\boxed{\text{E}}$ Find FT of $\dfrac{d}{dt}\left(e^{-at}u(t)\right), \quad a > 0$

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

$$\frac{d}{dt}\left(e^{-at}u(t)\right) \xleftrightarrow{FT} \frac{j\omega}{a+j\omega}$$

$\boxed{\text{E}}$ Find x(t) if $X(j\omega) = \begin{cases} j\omega, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$

Let $Z(j\omega) = \begin{cases} 1, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$ then,

$$X(j\omega) = j\omega Z(j\omega)$$

$$Z(j\omega) \xleftrightarrow{FT} z(t) = \frac{1}{\pi t}\sin(t)$$

Thus, $x(t) = \frac{d}{dt}z(t)$

$$= \frac{1}{\pi t}\cos(t) - \frac{1}{\pi t^2}\sin(t)$$

If x(t) is periodic, frequency-domain representation is Fourier Series (FS):

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$\boxed{\frac{d}{dt} x(t) \xleftrightarrow{FS;\omega_0} jk\omega_0 X[k]}$$

Differentiation in frequency:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} \left[(-jt)x(t)\right] e^{-j\omega t} dt, \qquad \text{thus}$$

$$\boxed{-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)}$$

Integration:

- In time: applicable to FT and FS
- In frequency: applicable to FT and DTFT

$$\text{Let } \ y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

$$\frac{d}{dt}y(t) = x(t) \Rightarrow$$

$$j\omega Y(j\omega) = X(j\omega) \Rightarrow Y(j\omega) = \frac{1}{j\omega}X(j\omega) \qquad \text{if } \omega \neq 0$$

For ω=0, this relationship is indeterminate. In general,

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)}$$

E Determine the Fourier transform of u(t).

$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau$$

$$\delta(t) \xleftrightarrow{FT} 1$$

$$U(j\omega) \xleftrightarrow{FT} \frac{1}{j\omega}+\pi\delta(j\omega)$$

E Problem 3.29, p279: Find x(t), given

$$X(j\omega)=\frac{1}{j\omega(j\omega+1)}+\pi\delta(j\omega)$$

$$X(j\omega)=\frac{1}{j\omega}-\frac{1}{j\omega+1}+\pi\delta(j\omega)=\frac{1}{j\omega}+\pi\delta(j\omega)-\frac{1}{1+j\omega}$$

$$\updownarrow$$

$$x(t)=u(t)-e^{-t}u(t)$$

$\boxed{\text{E}}$

$$x(t)=\frac{d}{dt}\left(2te^{-2t}u(t)\right). \qquad X(j\omega)=?$$

- $e^{-2t}u(t)\xleftrightarrow{FT}\dfrac{1}{2+j\omega}$
- $te^{-2t}u(t)\xleftrightarrow{FT}\dfrac{1}{(2+j\omega)^2}$
- $\dfrac{d}{dt}\left(2te^{-2t}u(t)\right)\xleftrightarrow{FT}\dfrac{2j\omega}{(2+j\omega)^2}$

$$\boxed{-jtx(t)\xleftrightarrow{FT}\frac{d}{d\omega}X(j\omega)}$$

$$\boxed{\frac{d}{dt}x(t)\xleftrightarrow{FT}j\omega X(j\omega)}$$

- Time and frequency shift

Time shift: Let $z(t) = x(t - t_0)$

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt$$

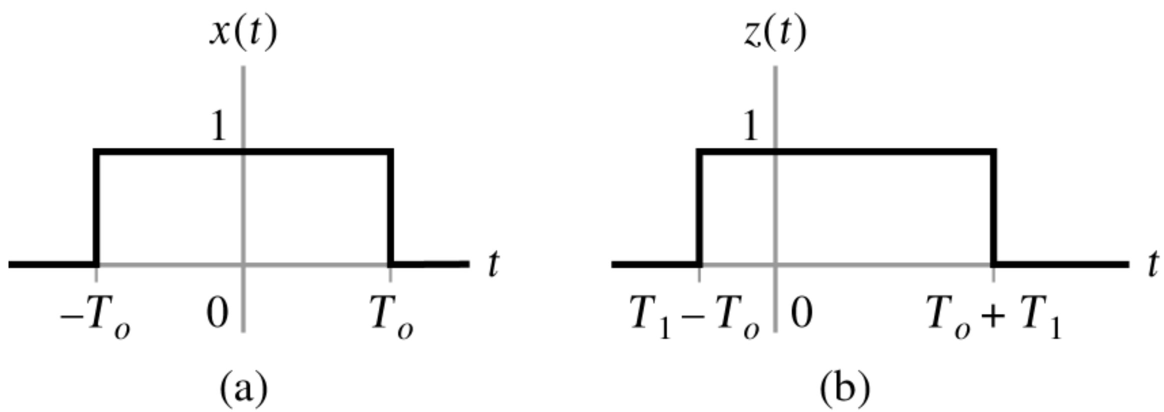
Replace $t - t_0$ with τ $= \left[\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \right] e^{-j\omega t_0}$

$$\boxed{Z(j\omega) = e^{-j\omega t_0} X(j\omega)}$$

Note: Time shift $\Rightarrow$ phase shift in frequency domain. Phase shift is a linear function of ω. Magnitude spectrum does not change.

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$
$$x(t-t_0) \xleftrightarrow{FS;\omega_0} e^{-jk\omega t_0} X[k]$$
$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0} X(e^{j\Omega})$$
$$x[n-n_0] \xleftrightarrow{DTFS;\Omega_0} e^{-jk\Omega n_0} X[k]$$

E Find Z(jω)



$$z(t) = x(t - T_1)$$

$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{2}{\omega}\sin(\omega T_0)$$

$$z(t) \xleftrightarrow{FT} Z(j\omega) = e^{-j\omega T_1}\frac{2}{\omega}\sin(\omega T_0)$$

E

$$X(j\omega) = \frac{e^{j4\omega}}{(2+j\omega)^2}. \qquad \text{Find } x(t)$$

- $Z(j\omega) = \dfrac{1}{2+j\omega} \xleftrightarrow{FT} z(t) = e^{-2t}u(t)$
- $-jtz(t) \xleftrightarrow{FT} \dfrac{d}{d\omega}Z(j\omega) = \dfrac{-j}{(2+j\omega)^2} \Rightarrow$

$$te^{-2t}u(t) \xleftrightarrow{FT} \frac{1}{(2+j\omega)^2}$$

- $(t+4)e^{-2(t+4)}u(t+4) \xleftrightarrow{FT} \dfrac{e^{j4\omega}}{(2+j\omega)^2}$

Frequency shift: $x(t) \xleftrightarrow{FT} X(j\omega)$

$$Z(j\omega) = X(j(\omega-\gamma)) \xleftrightarrow{FT} ??$$

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega-\gamma))e^{j\omega t}d\omega$$

$$= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\eta)e^{j\eta t}d\eta\right]e^{j\gamma t} \qquad (\text{let } \mu = \omega - \gamma)$$

$$\boxed{z(t) = e^{j\gamma t}x(t)}$$

Note:
– Frequency shift $\Rightarrow$ time signal multiplied by a complex sinusoid.
– Carrier modulation.

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$
$$e^{jk_0\omega_0 t}x(t) \xleftrightarrow{FS;\omega_0} X[k-k_0]$$
$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$
$$e^{jk_0\Omega_0 n}x[n] \xleftrightarrow{DTFS;\Omega_0} X[k-k_0]$$

E Find Z(jω).

$$z(t) = \begin{cases} e^{j10t}, & |t| < \pi \\ 0, & |t| > \pi \end{cases}$$

$$\text{Let } x(t) = \begin{cases} 1, & |t| < \pi \\ 0, & |t| > \pi \end{cases} \xleftrightarrow{FT} X(j\omega) = \frac{2}{\omega}\sin(\omega\pi)$$

$$z(t) = x(t)e^{j10t} \xleftrightarrow{FT} X(j(\omega-10)) = \frac{2}{\omega-10}\sin((\omega-10)\pi)$$

E

$$x(t) = \frac{d}{dt}\left\{\left(e^{-3t}u(t)\right) * \left(e^{-t}u(t-2)\right)\right\}. \quad \text{Find } X(j\omega).$$

$$\text{Let } w(t) = e^{-3t}u(t) \xleftrightarrow{FT} W(j\omega) = \frac{1}{3+j\omega}$$

$$v(t) = e^{-t}u(t-2)$$

$$= e^{-(t-2)}u(t-2)e^{-2} \xleftrightarrow{FT} V(j\omega) = e^{-2}e^{-j2\omega}\frac{1}{1+j\omega}$$

$$\text{Thus, } X(j\omega) = j\omega W(j\omega)V(j\omega) = e^{-2}\frac{j\omega e^{-j2\omega}}{(3+j\omega)(1+j\omega)}$$

- Multiplication

If $z(t), x(t)$ are non - periodic, what is the FT of $y(t) = x(t)z(t)$?

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jv)e^{jvt}dv$$

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\eta)e^{j\eta t}d\eta$$

$$y(t) = x(t)z(t) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} X(jv)Z(j\eta)e^{j(v+\eta)t}dvd\eta$$

Let $\omega = v + \mu$, fix v first $\Rightarrow d\eta = d\omega$. Then

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jv)Z(j(\omega - v))dv\right]}_{}e^{j\omega t}d\omega$$

Inverse FT of $\frac{1}{2\pi}X(j\omega)*Z(j\omega)$

$$\boxed{y(t) = x(t)z(t) \xleftrightarrow{FT} Y(j\omega) = \tfrac{1}{2\pi} X(j\omega) * Z(j\omega)}$$

$$\boxed{\text{Compare } x(t) * z(t) \xleftrightarrow{FT} X(j\omega)Z(j\omega)}$$

For discrete - time signals $x[n], z[n]$:

$$\boxed{y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \tfrac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})}$$

Periodic convolution: $X(e^{j\Omega}) \circledast Z(e^{j\Omega}) = \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Z\left(e^{j(\Omega-\theta)}\right) d\theta$

CT, periodic signals:

$$\boxed{\begin{aligned} &y(t) = x(t)z(t) \xleftrightarrow{FS;2\pi/T} Y[k] = X[k] * Z[k] \\ &X[k] * Z[k] = \sum_{m=-\infty}^{\infty} X[m]Z[k-m] \end{aligned}}$$

DT, periodic signals:

$$\boxed{\begin{aligned} & y[n] = x[n]z[n] \xleftrightarrow{DTFS;2\pi/N} Y[k] = X[k] \circledast Z[k] \\ & X[k] \circledast Z[k] = \sum_{m=0}^{N-1} X[m]Z[k-m] \end{aligned}}$$

- <u>Scaling</u> Let $z(t) = x(at)$

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

$$= \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau & a > 0 \\ \frac{1}{a}\int_{\infty}^{-\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau & a < 0 \end{cases} \quad (\text{let } \tau = at)$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau$$

$$= \frac{1}{|a|}X(j(\omega/a))$$

E

$$x(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

$$y(t) = \begin{cases} 1 & |t| < 2 \\ 0 & |t| > 2 \end{cases}$$

Find $Y(j\omega)$

$$y(t) = x(at)\big|_{a=\frac{1}{2}}$$

$$X(j\omega) = \frac{2}{\omega}\sin(\omega)$$

$$Y(j\omega) = 2X(j2\omega) = 2\frac{2}{2\omega}\sin(2\omega) = \frac{2}{\omega}\sin(2\omega)$$

E

Find $x(t)$ if $X(j\omega) = j\frac{d}{d\omega}\left\{\frac{e^{j2\omega}}{1 + j(\omega/3)}\right\}$

We know $s(t) = e^{-t}u(t) \xleftrightarrow{FT} S(j\omega) = \dfrac{1}{1+j\omega}$

– Time scaling: $z(t) = s(3t) \xleftrightarrow{FT} Z(j\omega) = \dfrac{1}{3}\dfrac{1}{1+j(\omega/3)}$

– Time shift: $v(t) = 3z(t+2) = 3s(3(t+2)) \xleftrightarrow{FT} \dfrac{e^{j2\omega}}{1+j(\omega/3)}$

– Differentiation: $tv(t) \xleftrightarrow{FT} j\dfrac{d}{d\omega}\left\{\dfrac{e^{j2\omega}}{1+j(\omega/3)}\right\}$

Thus,
$$
\begin{aligned}
x(t) &= tv(t) \\
&= 3tz(t+2) \\
&= 3ts(3(t+2)) \\
&= 3te^{-3(t+2)}u(3(t+2))
\end{aligned}
$$

$$x(at) \xleftrightarrow{FT} \tfrac{1}{|a|} X(j(\omega/a))$$
$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$
$$-jtx(t) \xleftrightarrow{FT} \tfrac{d}{d\omega} X(j\omega)$$

Note: $u(3(t+2)) = u(t+2)$. Thus, $x(t) = 3te^{-3(t+2)}u(t+2)$

- Parseval's relationship:

Energy of CT, non-periodic signal $x(t)$: $W_x = \int_{-\infty}^{\infty} | x(t) |^2 dt$

$$= \int_{-\infty}^{\infty} x^*(t) x(t) dt$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega$$

$$W_x = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega) d\omega$$

$$\boxed{W_x = \int_{-\infty}^{\infty} | x(t) |^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} | X(j\omega) |^2 \, d\omega}$$

Note: a) $| X(j\omega) |^2$: energy spectrum

b) Energy in time domain = energy in freq. domain

- DTFT: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$
- DTFS: $\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X[k]|^2$
- FS: $\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$

E

$x[n] = \frac{\sin(Wn)}{\pi n}$. Determine $E_x = \sum_{n=-\infty}^{\infty} \frac{\sin^2(Wn)}{(\pi n)^2}$

$$x[n] = \frac{\sin(Wn)}{\pi n} \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| \le \pi \end{cases}$$

$$E_x = \sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\Omega}\right)\right|^2 d\Omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} 1 d\Omega = W/\pi$$

• <u>Time-bandwidth product</u>

Compression in time domain $\Rightarrow$ expansion in frequency domain

<u>Bandwidth:</u> The extent of the signal's significant frequency content. It is in general a vague definition as "significant" is not mathematically defined. In practice, definitions of bandwidth include

– absolute bandwidth
– x% bandwidth
– first-null bandwidth.

If we define

$$T_d = \left[\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \right]^{1/2} \quad : \text{ RMS duration of an energy signal}$$

$$B_w = \left[\frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \right]^{1/2} \quad : \text{ RMS bandwidth, then}$$

$$\boxed{T_d B_w \geq 1/2}$$

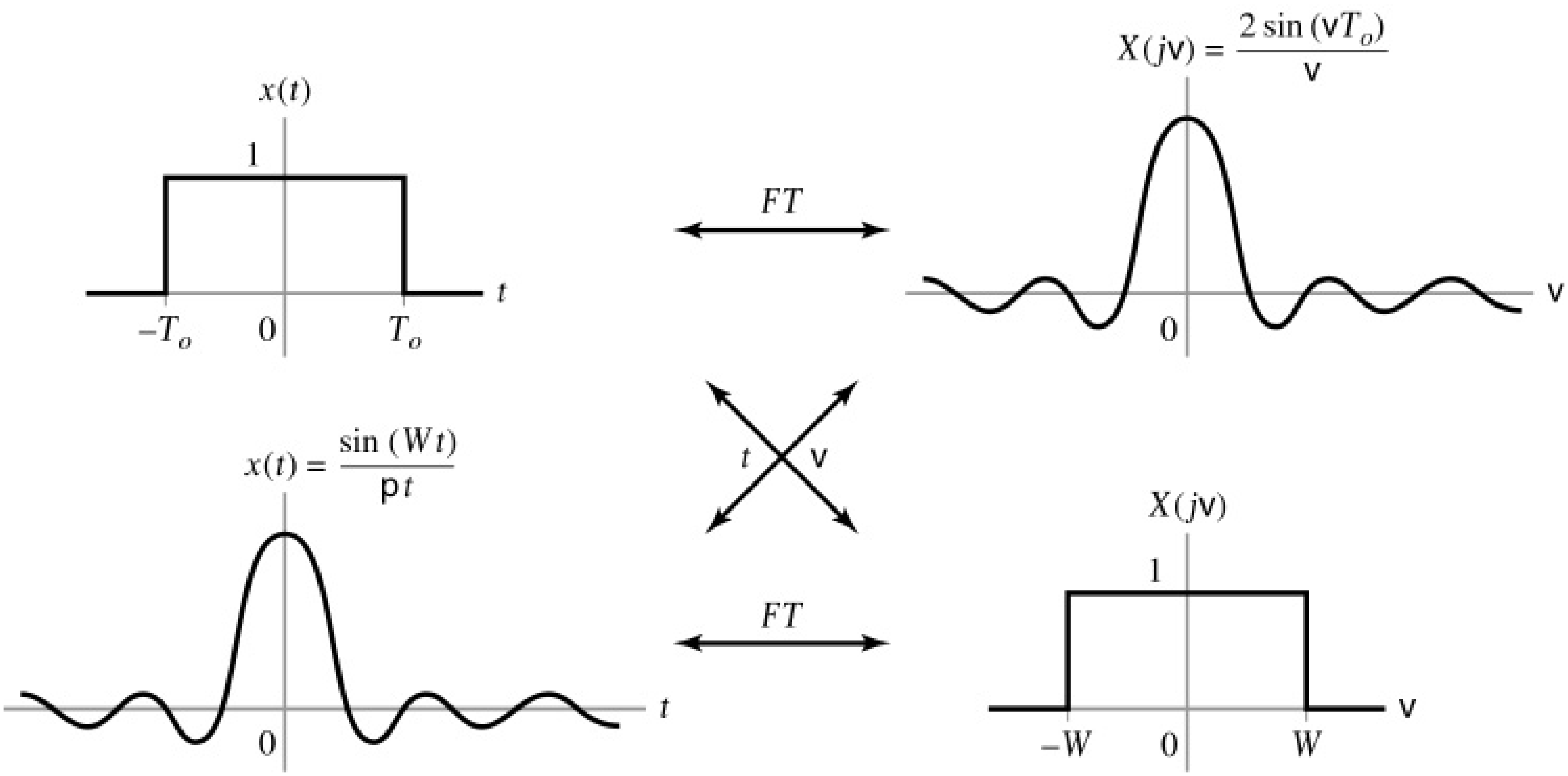
• <u>Duality</u>

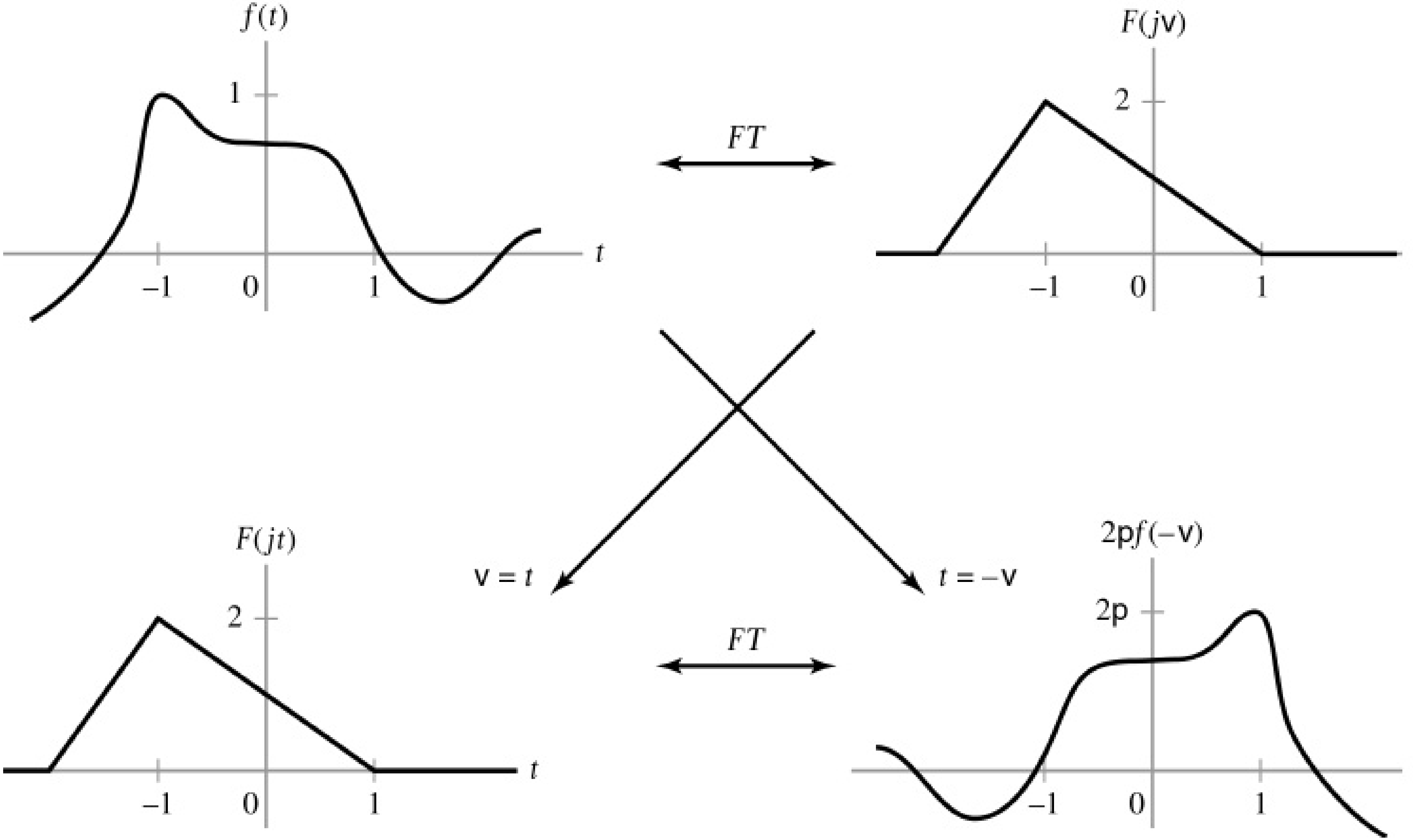
$$f(t) \xleftrightarrow{FT} F\left(j\omega\right)$$

$$F\left(jt\right) \xleftrightarrow{FT} 2\pi f(-\omega)$$

$$F(j\omega) = \int f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int F(j\omega) e^{j\omega t} d\omega$$




f(t)
1
–1
0
1
t
FT
F(jv)
2
–1
0
1
v
F(jt)
2
–1
0
1
t
v = t
t = –v
FT
2pf(–v)
2p
–1
0
1
v

E

Find $X(j\omega)$ if $x(t)=\dfrac{1}{1+jt}$

We know $f(t)=e^{-t}u(t)\xleftrightarrow{FT}\dfrac{1}{1+j\omega}=F(j\omega)$

Duality $F(jt)=\dfrac{1}{1+jt}\xleftrightarrow{FT}2\pi f(-\omega)\Rightarrow$

$$X(j\omega)=2\pi f(-\omega)=2\pi e^{\omega}u(-\omega)$$

Check $x(\mathrm{t})=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{-j\omega t}d\omega$

$$=\frac{1}{2\pi}\int_{-\infty}^{0}2\pi e^{\omega}e^{j\omega t}d\omega$$

$$=\int_{-\infty}^{0}e^{\omega(1+jt)}d\omega=\frac{1}{1+jt}$$

E

Find $x(t)$ if $X(j\omega)=u(\omega)$

$$X(j\omega) = u(\omega)$$

$$X(jt) = u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(\omega) = 2\pi x(-\omega) \Rightarrow$$

$$x(t) = \left[\frac{1}{j(-t)} + \pi\delta(-t)\right]\frac{1}{2\pi}$$

$$= \frac{-1}{2\pi jt} + \frac{1}{2}\delta(t)$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)}$$

- DTFS: $\begin{cases} x[n] \xleftrightarrow{DTFS;2\pi/N} X[k] \\ X[n] \xleftrightarrow{DTFS;2\pi/N} \frac{1}{N}x[-k] \end{cases}$
- DTFT and FS: $\begin{cases} x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) \\ X\left(e^{jt}\right) \xleftrightarrow{FS;1} x[-k] \end{cases}$
- *D*TFT and FS do not stay in their own class!