Time-Domain Representation of LTI Systems

Focus:

- System $\mathcal{H}$ is a linear time-invariant (LTI) system.
- How to analyze a system. Given an input, find system output.
- Impulse response of an LTI system $\mathcal{H}$:

Convolution sum

$$\begin{aligned} x(t) &= \delta(t) \xrightarrow{\mathcal{H}} y(t) = h(t) \\ x[n] &= \delta[n] \xrightarrow{\mathcal{H}} y[n] = h[n] \end{aligned}$$

where $h(t)$ (CT) and $h[n]$ (DT) are the system impulse responses.

- ⋆ $h(t)$ or $h[n]$ completely characterizes an LTI system.
- ⋆ By knowing $h(t)$ or $h[n]$, system output can be obtained for an arbitrary input signal $x(t)$ or $x[n]$.
- ⋆ How is $y(t)/y[n]$ related to $x(t)/x[n]$ and $h(t)/h[n]$?

Convolution sum (cont.)

We will start with DT systems, and then analyze CT systems.

- Any signal $x[n]$ can be expressed as a sum of time-shifted impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

By employing the properties LTI systems (superposition, homogeneity, shift-invariance (time-invariance for CT)):

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$

$$x[k]\delta[n-k] \xrightarrow{\mathcal{H}} x[k]h[n-k] \quad (x[k] \text{ is a constant})$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \; (\text{superposition of all terms } k = -\infty \cdots \infty)$$

Convolution sum (cont.)

- Convolution sum: $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

- Properties of convolution

  ⋆ $x[n] * h[n] = h[n] * x[n]$
  ⋆ $\delta[n] * h[n] = h[n]$
  ⋆ $\delta[n-k] * h[n] = h[n-k]$

E: A system with input-output relationship as

$$y[n] = x[n] + (1/2)x[n-1]$$

a) System impulse response?
b) Find $y[n]$ for

$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & o.w. \end{cases}$$

Convolution sum (cont.)

a) Let $x[n] = \delta[n] \longrightarrow h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

b)

$$
\begin{aligned}
x[n] &= 2\delta[n] + 4\delta[n-1] - 2\delta[n-2] \\
y[n] &= h[n] * x[n] \\
&= \left(\delta[n] + \frac{1}{2}\delta[n-1]\right) * (2\delta[n] + 4\delta[n-1] - 2\delta[n-2]) \\
&= 2\delta[n]+4\delta[n-1]-2\delta[n-2]+\delta[n-1]+2\delta[n-2]-\delta[n-3] \\
&= 2\delta[n] + 5\delta[n-1] - \delta[n-3]
\end{aligned}
$$

Convolution sum evaluation procedure

Let $w_n[k] = x[k]h[n-k]$. Then $y[n]$ is expressed as

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} w_n[k]$$

1. Graph both $x[k]$ and $h[k]$
2. Time reversal $h[k] \longrightarrow h[-k]$
3. Time shift $h[-k]$ by n shifts $\longrightarrow h[n-k]$ (left shift)
4. For a specific n, form product $x[k]h[n-k]$
5. Sum all samples of $x[k]h[n-k] \longrightarrow$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

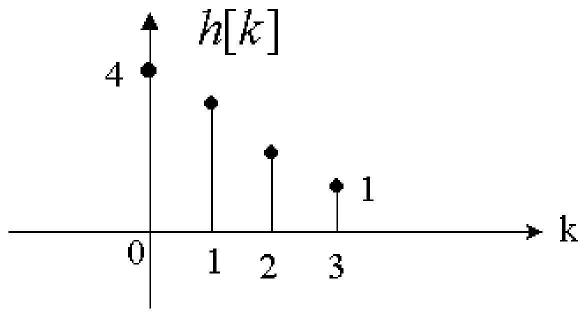
Convolution sum evaluation procedure (cont.)

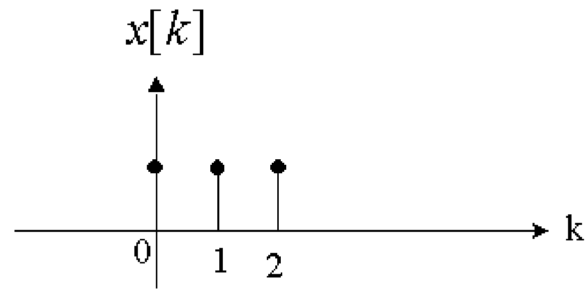
E: $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ is applied to an LTI system with impulse response
$h[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$. Find $y[n]$.

$$y[n] = x[n] * h[n]$$

Convolution sum evaluation procedure (cont.)

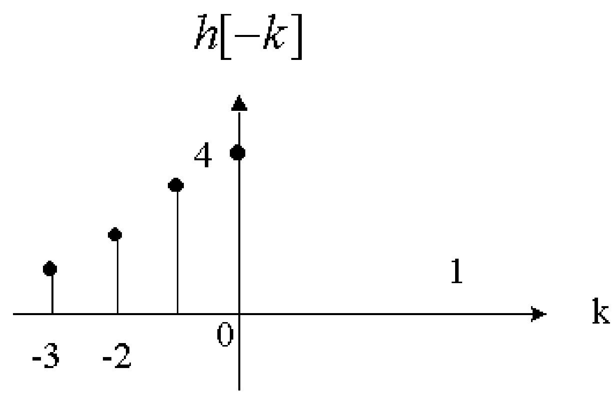
1)





- Graph x[k] & h[k]

2) $h[-k]$

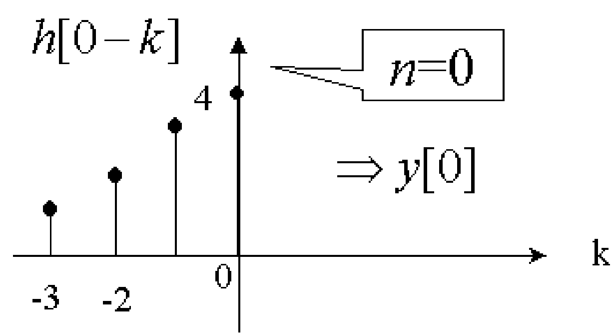


- Form h[-k] & time shift h[-k]

- For a specific n, form product

$$x[k]h[n-k] \Rightarrow \quad ...\ 0 \quad 0 \quad 0 \quad 4 \quad 0 \ ... \quad \Rightarrow y[0] = \sum \; = 4$$

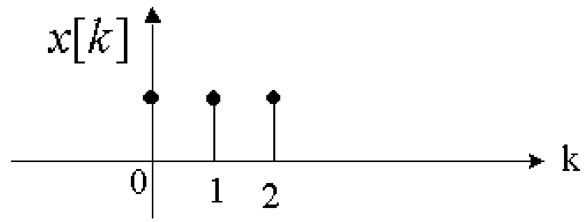
3) $h[n-k]$



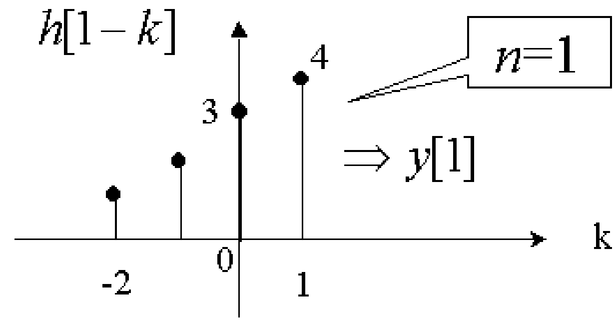
$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = 0 \quad \text{for } n < 0$$

- Sum all products of $x[k]h[n-k]$

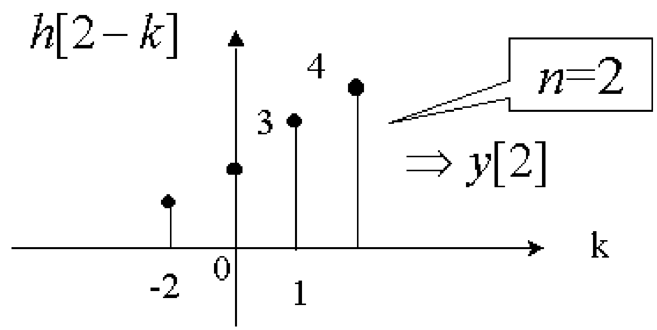
Convolution sum evaluation procedure (cont.)



$$x[k]h[n-k] \Rightarrow \quad \ldots 0 \quad 0 \quad 0 \quad 3 \quad 4 \quad 0 \quad 0 \quad \ldots \Rightarrow y[1] = \sum \;\; = 7$$



$$x[k]h[n-k] \Rightarrow \quad \ldots 0 \quad 0 \quad 0 \quad 2 \quad 3 \quad 4 \quad 0 \quad \ldots \Rightarrow y[2] = \sum \;\; = 9$$



$y[4] = 3$

$y[5] = 1$

$y[n] = 0 \text{ for } n \geq 6$

$$\ldots 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 0 \quad \ldots \Rightarrow y[3] = \sum \;\; = 6$$

Convolution integral

- For CT case.
- Recall DT case:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Note: Weighed SUM of time-shifted impulses. Similarly,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Note: Weighted superposition of time-shifted impulses.

$$x(t) \xrightarrow{\mathcal{H}} y(t)$$

Convolution integral (cont.)

$$\begin{aligned} y(t) &= \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \quad \text{linear operators} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\left\{\delta(t-\tau)\right\} d\tau \end{aligned}$$

$$\delta(t-\tau) \xrightarrow{\mathcal{H}} h(t-\tau)$$

Thus,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Note:

- $\star$ $x(t) * h(t) = h(t) * x(t)$
- $\star$ $\delta(t) * h(t) = h(t)$
- $\star$ $\delta(t-t_0) * h(t) = h(t-t_0)$

Convolution integral evaluation procedure

1. Graph $x(t)$ and $h(t)$

2. Time reverse $h(\tau) \Rightarrow h(-\tau)$

3. Time shift $h(-\tau)$ by $t \Rightarrow h(t-\tau)$

4. For a specific value of t, form product $x(\tau)h(t-\tau)$

5. Integrate $x(\tau)h(t-\tau) \Longrightarrow$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution integral evaluation procedure (cont.)

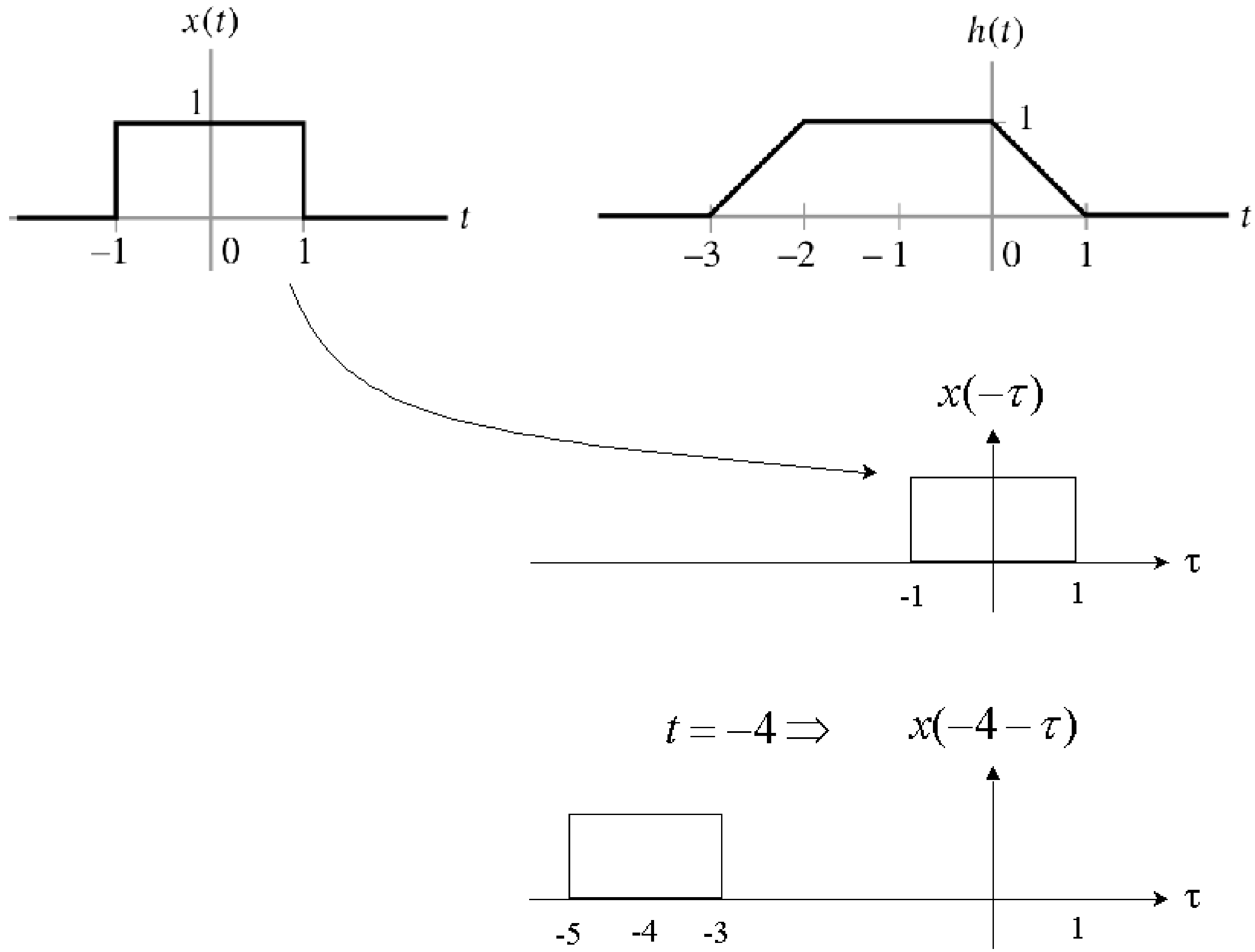
E: $x(t) \xrightarrow{\mathcal{H}} y(t) = ??$

$$y(t) = x(t) * h(t).$$

Graphical solution next slide. Solution:

$$y(t) = \begin{cases} 0, & t < -4, t > 2 \\ ? & -4 \leq t < -3 \\ ? & -3 \leq t < -2 \\ ? & -2 \leq t < -1 \\ ? & -1 \leq t < 0 \\ ? & 0 \leq t < 1 \\ ? & 1 \leq t < 2 \end{cases}$$

Convolution integral evaluation procedure (cont.)



Direct convolution integral evaluation

E: RADAR range measurement: RADAR-Radio Detection And Ranging:

Tx: $x(t) = \begin{cases} \sin(w_c t), & 0 \leq t \leq T_0 \\ 0, & o.w. \end{cases}$

Typically,

$$h(t) = \alpha\delta(t-\beta),\ \beta > 0$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution integral evaluation

E: $x(t) = u(t) \xrightarrow{\mathcal{H}} y(t)$ where LTI system $\mathcal{H}$ has an impulse response $h(t) = e^{-t}u(t)$. Determine $y(t)$.

$$\begin{aligned} \delta(t) &= \frac{d}{dt}u(t) \\ u(t) &= \int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1, & \text{if } t > 0 \text{ (integral includes } t = 0) \\ 0, & t < 0 \end{cases} \end{aligned}$$

Thus, for $x(t) = u(t)$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} h(\tau)d\tau = \int_{-\infty}^{t} e^{-\tau}u(\tau)d\tau = \int_{0}^{t} e^{-\tau}d\tau \\ &= \left(1 - e^{-t}\right) u(t) \end{aligned}$$

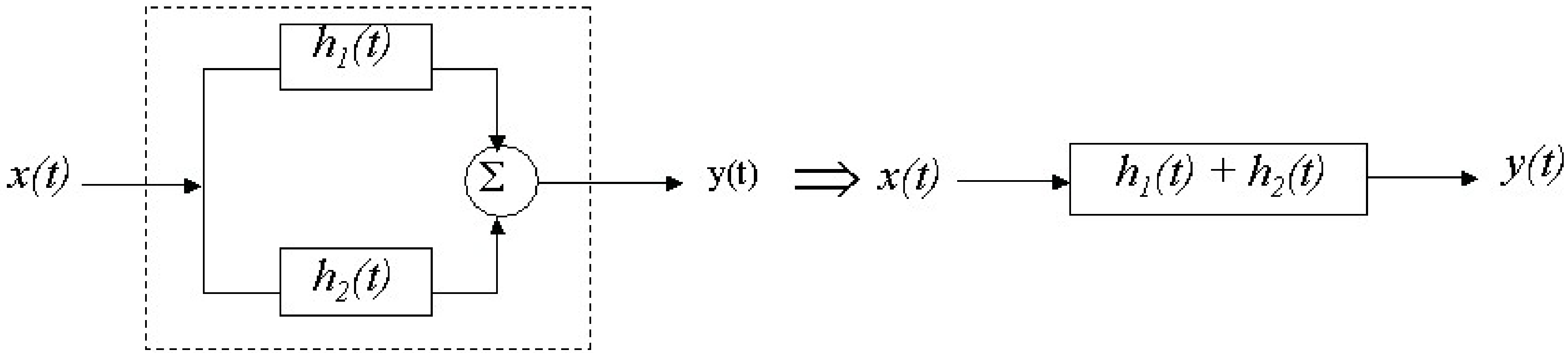
Interconnection of LTI systems

Given:

$$\left.\begin{array}{ccc} h_1(t) & & \xrightarrow{\mathcal{H}_1} \\ \vdots & & \vdots \\ h_N(t) & & \xrightarrow{\mathcal{H}_N} \end{array}\right\} \Longrightarrow \text{ form a bigger system } \xrightarrow{\mathcal{H}}$$

Question: How is $h(t)$ related to $h_1(t) \cdots h_N(t)$?

- Parallel Connection



Interconnection of LTI systems (cont.)

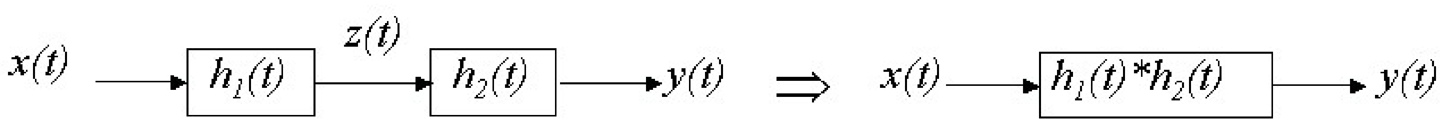
$$\begin{aligned} y(t) &= y_1(t) + y_1(t) = x(t) * h_1(t) + x(t) * h_2(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)[\underbrace{h_1(t-\tau) + h_2(t-\tau)}_{h(t-\tau)}]d\tau \\ &= x(t) * h(t) \end{aligned}$$

⋆ Distribution property of convolution process:

$$\begin{aligned} \text{CT: } x(t) * h_1(t) + x(t) * h_2(t) &= x(t) * \{h_1(t) + h_2(t)\} \\ \text{DT: } x[n] * h_1[n] + x[n] * h_2[n] &= x[n] * \{h_1[n] + h_2[n]\} \end{aligned}$$

Interconnection of LTI systems (cont.)

- Cascade Connection



$$
\begin{aligned}
z(\tau) &= x(\tau) * h_1(\tau) \\
&= \int_{-\infty}^{\infty} x(\nu) h_1(\tau - \nu) d\nu \\
y(t) &= z(t) * h_2(t) \\
&= \int_{-\infty}^{\infty} z(\tau) h_2(t - \tau) d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{x(\nu) h_1(\tau - \nu)} h_2(t - \tau) d\nu d\tau
\end{aligned}
$$

Interconnection of LTI systems (cont.)

Let $\eta = \tau - \nu,\ d\eta = d\tau$ (for fixed ν). Then,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\nu) \underbrace{\left[\int_{-\infty}^{\infty} h_1(\eta) h_2(t-\nu-\eta) d\eta\right]}_{=h_1(p)*h_2(p)|_{p=t-\nu}} d\nu \\
&= \int_{-\infty}^{\infty} x(\nu) \int_{-\infty}^{\infty} h_1(\eta) h_2(p-\eta) d\eta|_{p=t-\nu} \\
&= \int_{-\infty}^{\infty} x(\nu) h(t-\nu) d\nu
\end{aligned}$$

Thus,

$$y(t) = \int_{-\infty}^{\infty} x(\nu) h(t-\nu) d\nu = x(t) * h(t)$$

where $h(t) = h_1(t) * h_2(t)$.

Interconnection of LTI systems (cont.)

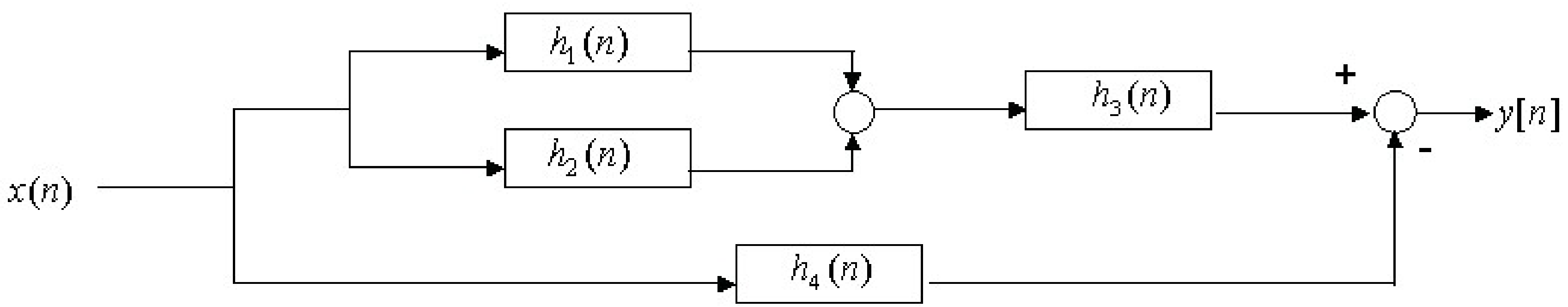
- Associative Property (Same for DT)

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

- Commutative Property (Same for DT

$$h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

E: . Find the impulse response $h[n]$ of the overall system.



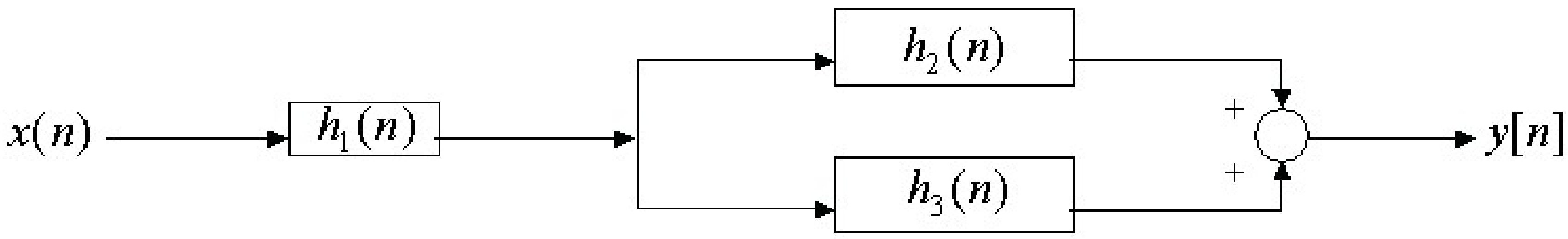
Interconnection of LTI systems (cont.)

$$\begin{cases} h_1[n] = u[n] \\ h_2[n] = u[n+2] - u[n] \\ h_3[n] = \delta[n-2] \\ h_4[n] = \alpha^n u[n] \end{cases}$$

$$\begin{aligned} h[n] &= [h_1[n] + h_2[n]] * h_3[n] - h_4[n] \\ &= u[n+2] * \delta[n-2] - \alpha^n u[n] \\ &= u[n] - \alpha^n u[n] \\ &= \{1 - \alpha^n\} u[n] \end{aligned}$$

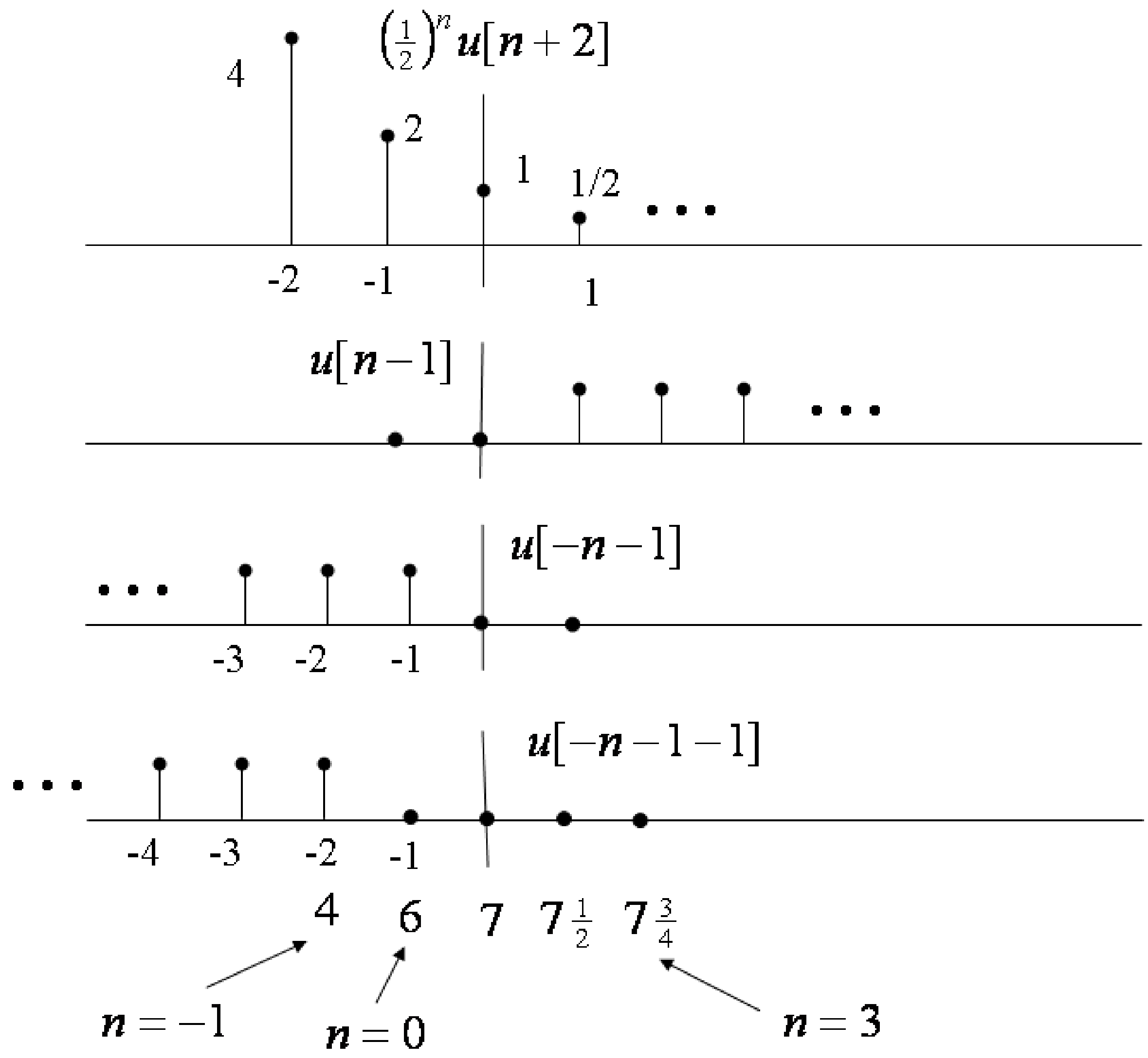
Interconnection of LTI systems (cont.)

E: An interconnection of LTI system is depicted in the figure below. $h_1[n] = (\frac{1}{2})^n u[n+2]$, $h_2[n] = \delta[n]$, and $h_3[n] = u[n-1]$. Find the impulse response $h[n]$ of the overall system.



$$\begin{aligned} h[n] &= h_1[n] * [h_2[n] + h_3[n]] \\ &= (1/2)^n u[n+2] * \{\delta[n] + u[n-1]\} \\ &= (1/2)^n u[n+2] + \underbrace{(1/2)^n u[n+2] * u[n-1]}_{=?} \\ &= (1/2)^n u[n+2] + (8 - (1/2)^{(n-1)}) u[n+1] \end{aligned}$$

Interconnection of LTI systems (cont.)



$$= \left(8 - \left(\tfrac{1}{2}\right)^{n-1}\right) u[n-1]$$