LTI SYSTEM PROPERTIES & IMPULSE RESPONSE

System properties

- Stability (BIBO)
- Memory (depend on current input only)
- Causality (does not depend on future inputs)
- Linearity
- Time invariance

LTI

Memoryless, LTI

Memoryless, Stable, LTI

For LTI systems:

$h(t)$

$h[n]$

Completely determine

Input-output behaviors

Thus, stability, memory, causality are related to h(t)/h[n].

a). If an LTI system is MEMORYLESS

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \cdots + \mathrm{h}[-2]\mathrm{x}[\mathrm{n}+2] + \mathrm{h}[-1]\mathrm{x}[\mathrm{n}+1] + \mathrm{h}[0]\mathrm{x}[\mathrm{n}]$$

$$+ \mathrm{h}[1]\mathrm{x}[\mathrm{n}-1] + \mathrm{h}[2]\mathrm{x}[\mathrm{n}-2] + \cdots$$

Memoryless: $y[n]$ depends on $x[n]$ ONLY. Thus, $h[k] = 0$ for $k \neq 0$.
The same is true for CT LTI systems.

<u>*Conclusion:*</u> LTI systems

$$\text{Memoryless} \xLeftrightarrow{\textit{iff}} \begin{matrix} h[k] = c\delta[k] \\ h(\tau) = c\delta(\tau) \end{matrix}$$

b) If an LTI system is CAUSAL:

$$y[n] = \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots$$

To make *y[n]* depend ONLY on *x[n]* and/or *x[k] k*<*n*, must have

$$\mathrm{DT}: h[k] = 0 \text{ for } k < 0$$

$$\mathrm{CT}: h(\tau) = 0 \text{ for } \tau < 0$$

c) If an LTI system is STABLE:
BIBO stable: if

$$|x[n]| \le M_x < \infty \quad \Rightarrow \quad |y[n]| \le M_y < \infty$$

$$\begin{aligned}
|y[n]| &= |x[n] * h[n]| \\
&= \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \\
&\le \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]| \\
&\le \left\{ \sum_{k=-\infty}^{\infty} |h[k]| \right\} M_x < \infty
\end{aligned}$$

If LTI BIBO stable $\leftrightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$: absolutely summable

- $\sum |\cdot| < \infty$ is a sufficient condition for BIBO
- $\sum |\cdot| < \infty$ is also a necessary condition for BIBO

Conclusion:

$$\boxed{\begin{array}{l} \sum_{k=-\infty}^{\infty} |h[k]| < \infty \xleftrightarrow{iff} \text{BIBO Stable} \\ \int_{-\infty}^{\infty} |\mathrm{h}(\tau)| d\tau < \infty \xleftrightarrow{iff} \text{BIBO Stable} \end{array}}$$

E First-order autoregressive (recursive) system

$$y[n] = \rho y[n-1] + x[n], \text{ with } h[n] = 0 \text{ for } n < 0$$

a) Impulse response?
b) Is system causal, memoryless, BIBO stable?

Solution:
a) Let $x[n] = \delta[n] \longrightarrow h[n] = \rho h[n-1] + \delta[n]$ Thus

$$h[k] = 0 \text{ for } k < 0,\ h[0] = 1,\ h[1] = \rho,\ h[2] = \rho^2,\ h[3] = \rho^3 \cdots\ h[n] = \rho^n\ (n > 0)$$

$$\boxed{h[n] = \rho^n u[n]}$$

b)
- Causal, because $h[n] = 0 \text{ for } k < 0$
- System is NOT memoryless, because $h[n] \neq c\delta[n]$

$$\sum_{k=\infty}^{\infty} |h[k]| = \sum_{k=\infty}^{\infty} |\rho^k| = \begin{cases} \infty \text{ if } |\rho| \geq 1 \\ < \infty \text{ if } |\rho| < 1 \end{cases}$$

- Therefore, system is stable if $|\rho| < 1$

STEP RESPONSE:

LTI

$$\begin{matrix} x[n] \\ x(t) \end{matrix} \rightarrow \boxed{H} \rightarrow \begin{matrix} y[n] \\ y(t) \end{matrix}$$

Step response: if $x[n] = u[n] \Rightarrow y[n] = s[n]$

$$\begin{aligned} s[n] &= h[n] * u[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \\ &= \sum_{k=-\infty}^{n} h[k]u[n-k] + \sum_{k=n+1}^{\infty} h[k]u[n-k] \end{aligned}$$

$$\boxed{s[n] = \sum_{k=-\infty}^{n} h[k]}$$

For a CT system: $\boxed{s(t) = \int_{-\infty}^{t} h(\tau)d\tau}$

Note: a) $h(t) = \frac{d}{dt} s(t)$

$$\delta(t) = \frac{d}{dt} u(t) \rightarrow \boxed{\mathrm{H}} \rightarrow h(t) = \frac{d}{dt} s(t)$$

b) $s[n] = h[n] + \underbrace{\sum_{k=-\infty}^{n-1} h[k]}_{=s[n-1]}$

$$= h[n] + s[n-1]$$

$\boxed{h[n] = s[n] - s[n-1]}$ COMPARE $\boxed{h(t) = \frac{d}{dt} s(t)}$

E

$$u[n] \rightarrow \boxed{h[n] = \rho^n u[n]} \rightarrow y[n] = ??$$

$$|\rho| < 1$$

- We know $h[n] = \rho^n u[n]$

- $$s[n] = \sum_{k=-\infty}^{n} h[k]$$
$$= \sum_{k=-\infty}^{n} \rho^k u[k] = \underbrace{\left(\sum_{k=0}^{n} \rho^k \right)}_{\frac{1-\rho^{n+1}}{1-\rho}} u[n] = \frac{1-\rho^{n+1}}{1-\rho} u[n]$$