ECE351: Signals and Systems I

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Fundamentals of Signals and Systems

- Two widely popular types of signals: speech and image/video
- Signal: a function of one or more variables (e.g., time, distance) that convey information on the nature of a physical phenomenon.
  - ⋆ Examples: heartbeat, blood pressure, temperature, vibration.
  - ⋆ One-dimensional signals: function depends on a single variable, e.g., $x(t)$
  - ⋆ Multi-dimensional signals: function depends on two or more variables, e.g., video - time and two spatial dimensions

Fundamentals of Signals and Systems (cont.)

- System: an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.
  - ⋆ Commonly encountered systems: communications systems.

Fundamentals of Signals and Systems (cont.)

- Real life examples:
  - Human speech communications.
  - Video communications and human vision.
  - Machine-machine communication.
  - Human-machine-human.
- Distortion: noise, interference, etc.
- Form: Analog and digital.

Fundamentals of Signals and Systems (cont.)

- Analog signal processing (ASP): use analog circuits such as resistors, capacitors, inductors, transistors, and diodes.
  - ⋆ Real time.
- Digital signal processing (DSP): adders, multiplers, memory.
  - ⋆ Flexible and repeatable.
- Notation:
  - ⋆ $x(t)$-Continuous time (CT) signals.
  - ⋆ $x[n]$-discrete time (DT) signals (n integers)

Classification of signals

1. CT and DT signals:

Classification of signals (cont.)

- For many cases, $x[n]$ is obtained by sampling $x(t)$ as:
  $x[n] = x(nT_s) = x(t)|_{t=nT_s},\ n = -\infty, \cdots, 0, \cdots, \infty$
- $x(t)$ must be recoverable from $x[n]$
- Are there any requirements for the sampling?

Classification of signals (cont.)

2. Periodic and non-periodic signals:
   CT signal: if $x(t) = x(t + T_p), \ \forall t$, then $x(t)$ is periodic.

   - Fundamental period: T_p
   - Fundamental frequency $f_p = 1/T_p$ (Hz or cycles/second)
   - Angular frequency: $\omega_p = 2\pi f_p = 2\pi/T_p$ (rad/seconds)

   DT signal: if $x[n] = x[n + N_p], \ \forall n$, then $x[n]$ is periodic.

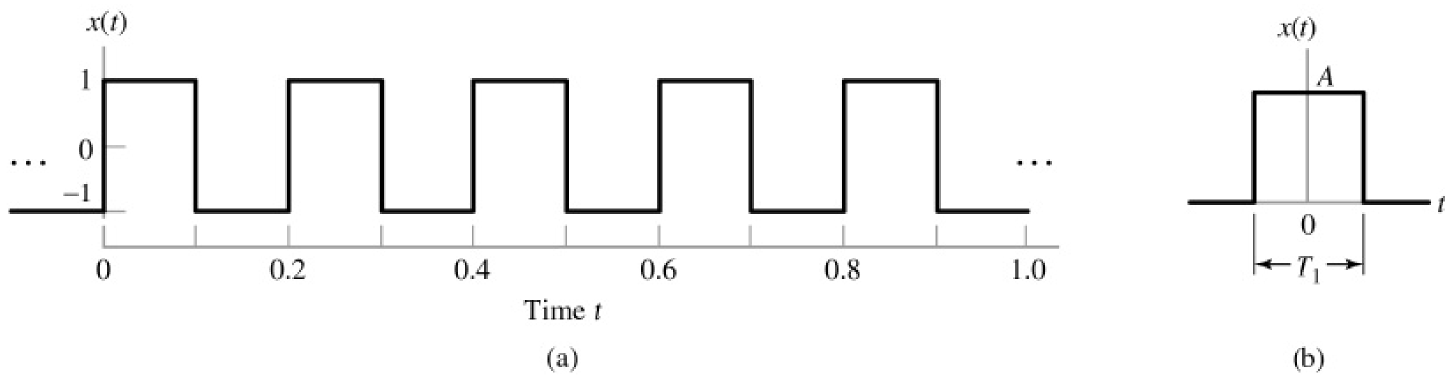
   - N_p: $N_p > 0$ integer. min(N_p): fundamental period
   - N_p: samples/period, if the unit of n is designated as samples.
   - $F_p = 1/N_p$ (cycles/sample)
   - $\Omega_p = 2\pi F_p$ (rads/sample). If the unit of n is designated as dimensionless, then Ω_p is simply in radians.

   Note: A sampled CT periodic signal may not be DT periodic.
   Condition for DT signals to be periodic:

   - $x[n] = x(nT_s) \Rightarrow N_p T_s = kT_p \Rightarrow$

$$N_p = \frac{kTp}{T_s}$$ must be integer, or $\frac{Tp}{T_s}$ is rational

Classification of signals (cont.)



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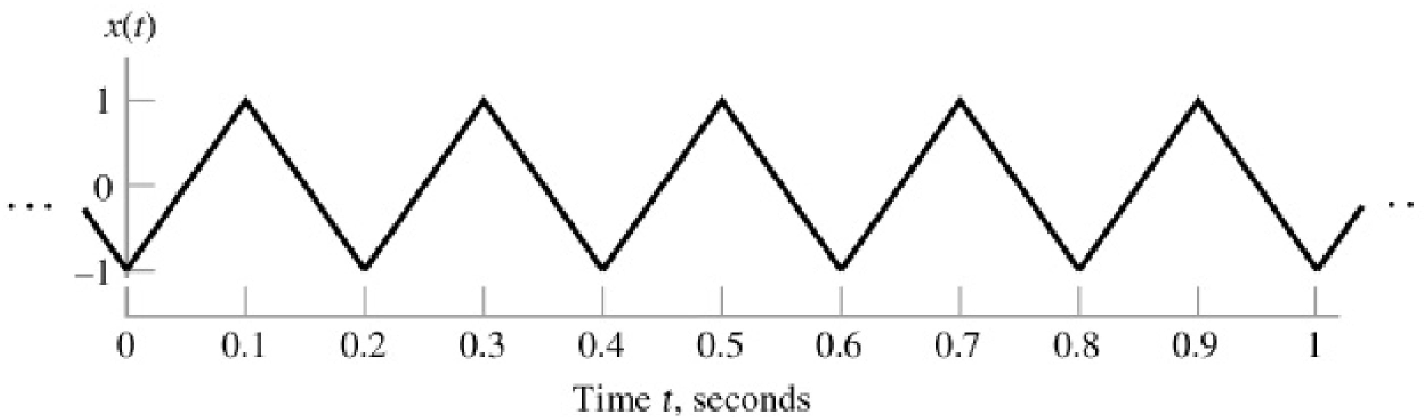

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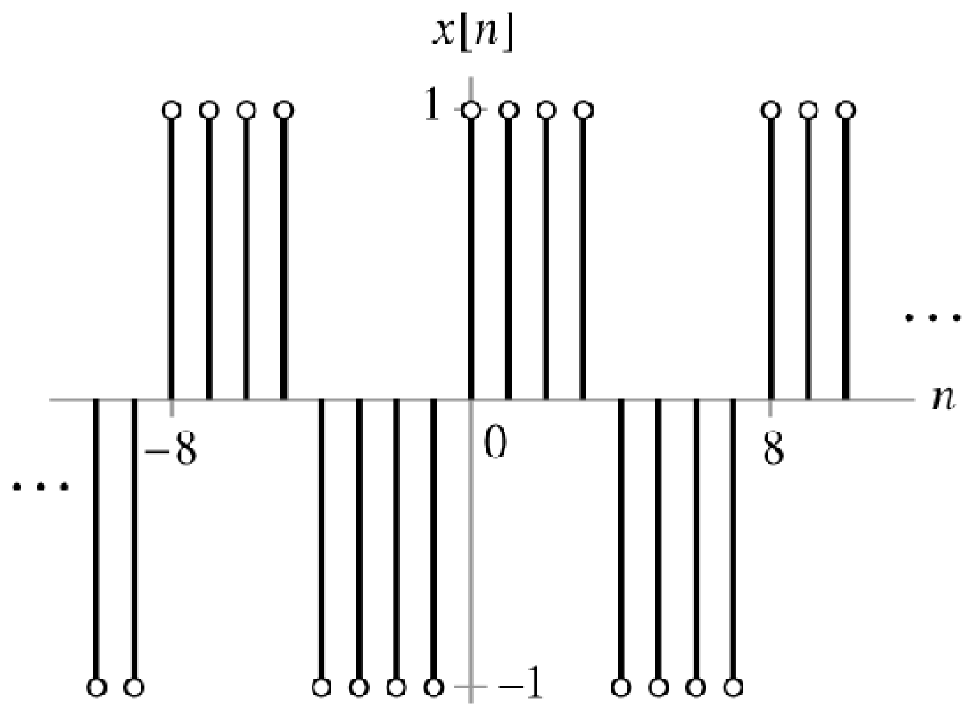
Classification of signals (cont.)



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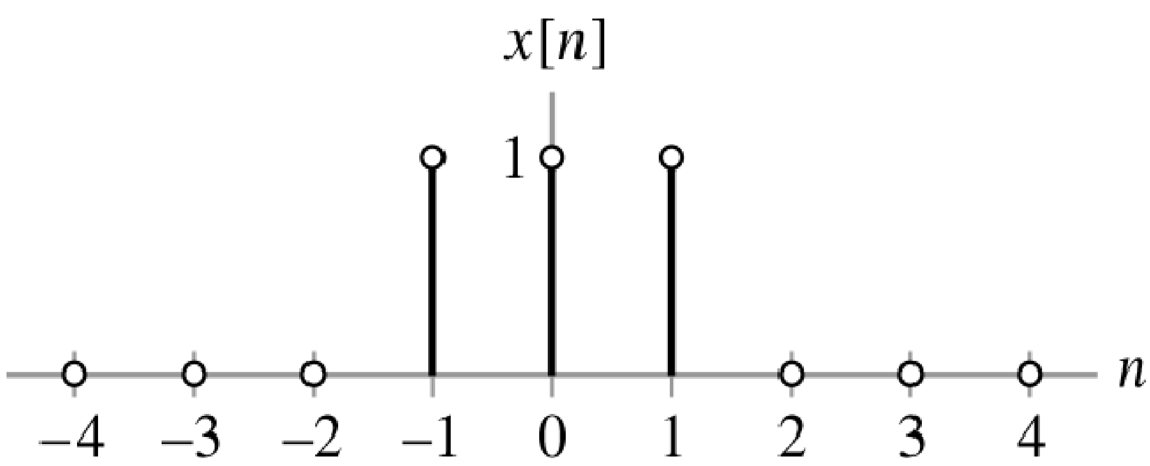

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Classification of signals (cont.)

E: $x[n] = A\cos(2\pi F_p n + \theta)$. Condition for $x[n]$ to be periodic?

$$\begin{aligned} x[n] &= A\cos(2\pi F_p n + \theta) \\ &= A\cos(2\pi F_p(n + N_p) + \theta) \Rightarrow \\ & 2\pi(F_p N_p) = 2\pi k \text{ for some integer } k. \end{aligned}$$

E: $x(t) = \sin^2(20\pi t)$ periodic? non-periodic?

$$\begin{aligned} x(t) &= \sin^2(20\pi t) \\ &= \frac{1}{2} - \frac{1}{2}\cos(40\pi t) \rightarrow \text{periodic} \\ f_p &= 20Hz,\ \omega_p = 2\pi f_p = 40\pi \text{ rads/sec},\ T_p = \frac{1}{20} \text{ seconds} \end{aligned}$$

Classification of signals (cont.)

- If $x(t)$ is sampled at $t = nT_s$ with $T_s = \frac{1}{4\pi} \rightarrow$

$$
\begin{aligned}
x[n] &= (1 - \cos(10n))/2 \\
&= \frac{1}{2} - \frac{1}{2}\cos\left(2\pi\left(\frac{10}{2\pi}\right)n\right)
\end{aligned}
$$

which is periodic? non-periodic?: non-periodic, because

$$\frac{T_p}{T_s} = \frac{1/20}{1/(4\pi)} = \frac{\pi}{5} \neq \text{integer}$$

- If $x(t)$ is sampled at $t = nT_s$ with $T_s = \frac{1}{40} \rightarrow$

$$x[n] = (1 - \cos(\pi n))/2$$

which is periodic? non-periodic?: periodic, because

$$\frac{T_p}{T_s} = 2 = \text{integer } (x[n] = [\cdots\ 0\ 1\ 0\ 1\ 0\ 1\ \cdots])$$

Classification of signals (cont.)

Sum of signals

CT signal:

If $x_1(t)$ and $x_2(t)$ are periodic with fundamental periods T_1 and T_2. How about $x(t) = x_1(t) + x_2(t)$?

$$\begin{cases} x_1(t) = x_1(t+T_1) \\ x_2(t) = x_2(t+T_2) \end{cases}$$

$$x_1(t) + x_2(t) = x_1(t+T_{sum}) + x_2(t+T_{sum}), \forall t$$

Must have: $T_{sum} = rT_1 = qT_2$ for integers r and q.

E:

$$x_1(t) = \cos(\pi t/2), \;\; x_2(t) = \cos(\pi t/3), \;\; x_1(t) + x_2(t)??$$

$$T_1 = 4, \;\; T_2 = 6, \;\; T_{sum} = 3T_1 = 2T_2 = 12.$$

Classification of signals (cont.)

DT signal:

If $\begin{cases} x_1[n] = x_1[n + N_1] \\ x_2[n] = x_2[n + N_2] \end{cases}$

$$x_1[n] + x_2[n] = x_1[n + N_{sum}] + x_2[n + N_{sum}], \forall n$$

Must have: $N_{sum} = rN_1 = qN_2$ for integers r and q.

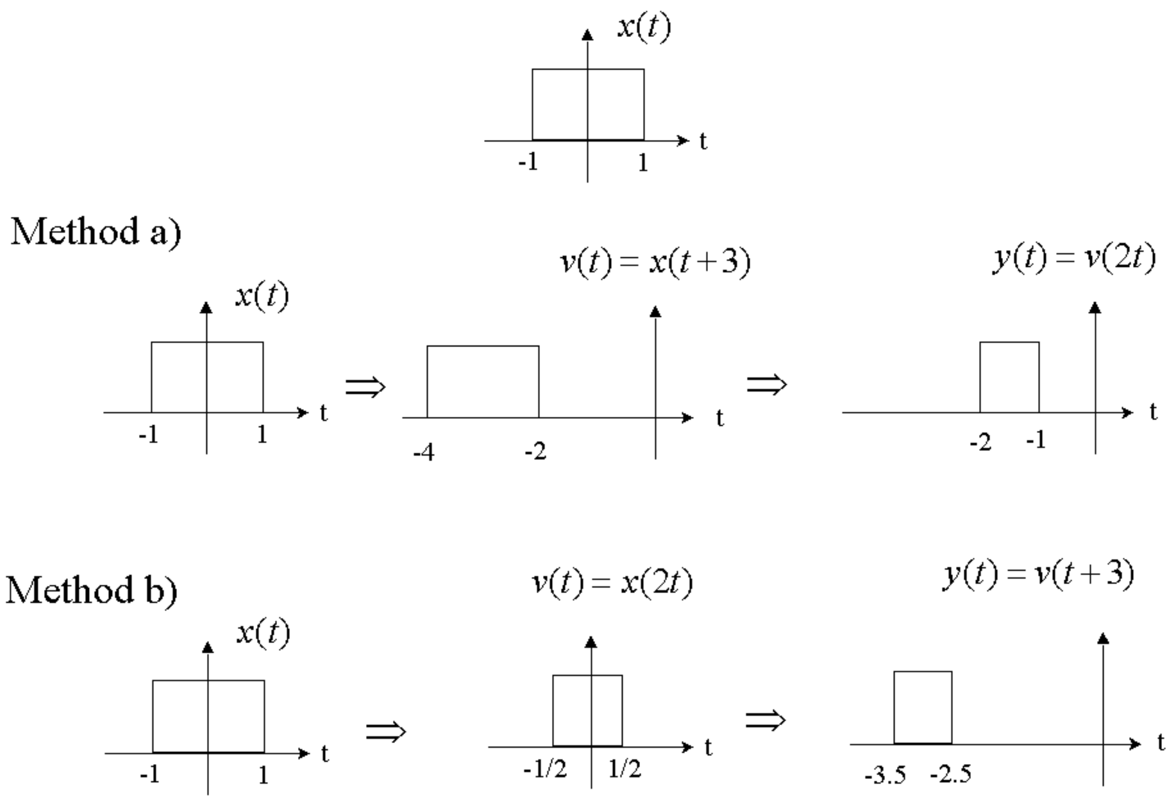
Basic operations on signals

Operation	CT	DT	Note
Amplitude scaling	$y(t) = cx(t)$	$y[n] = cx[n]$	$c > 1$: gain $c < 1$: atten
Addition	$y(t) = x_1(t) + x_2(t)$	$y[n] = x_1[n] + x_2[n]$	
Multiplication	$y(t) = x_1(t)x_2(t)$	$y[n] = x_1[n]x_2[n]$	
Differentiation	$y(t) = \frac{d}{dt}x(t)$	(NO DT case)	
Integration	$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$	(NO DT case)	
Time scaling	$y(t) = x(at)$ $\begin{cases} a > 1 : & \text{compression} \\ a < 1 : & \text{expansion} \end{cases}$	$y[n] = x[kn]$ $k > 0$ and integer only	
Reflection (time reversal)	$y(t) = x(-t)$	$y[n] = x[-n]$	
Time shifting	$y(t) = x(t - t_0)$ $\begin{cases} t_0 > 0 : & \text{right shift} \\ t_0 < 0 : & \text{left shift} \end{cases}$	$y[n] = x[n - n_0]$ $\begin{cases} n_0 > 0 : & \text{right shift} \\ n_0 < 0 : & \text{left shift} \end{cases}$	
Combination	$y(t) = x(at - t_0)$	$y[n] = x[kn - n_0]$	

Basic operations on signals (cont.)

Precedence rule for time shifting and time scaling:

E: See figure below. Find $y(t) = x(2t+3)$.



Elementary signals

1. Exponential

CT	DT
$x(t) = Be^{at}$, a, B real $\begin{cases} a < 0 : & \text{decaying} \\ a > 0 : & \text{growing} \\ a = 0 : & \text{DC} \end{cases}$	$x[n] = Br^n$ $\begin{cases} 0 < r < 1 : & \text{decaying} \\ r > 1 : & \text{growing} \\ r = 1 : & \text{DC} \end{cases}$

2. Sinusoidal

CT	DT
$x(t) = A\cos(\omega t + \phi)$	$x[n] = A\cos(\Omega n + \phi)$

Elementary signals (cont.)

Note:

- $x[n]$ May or may not be periodic
- If $\Omega N = 2\pi m$, m integer, or $\Omega = \frac{2\pi m}{N}$ (rads/sample), then $x[n]$ periodic: $x[n+N] = x[n]$
- Ω (unit?) rads/sample; N samples; ΩN radians (or simply radians if n is designated as dimensionless).

E: $x_1[n] = \sin\left(\frac{2\pi}{21}n\right)$, $x_2[n] = \sqrt{3}\cos\left(\frac{4\pi}{7}n\right)$. Fundamental period of $y[n] = x_1[n] + x_2[n]$?

$$\left\{\begin{array}{l} N_1 = \frac{2\pi}{\Omega_1}m_1 = 21m_1 \Rightarrow N_1 = 21(m_1 = 1) \\ N_2 = \frac{2\pi}{\Omega_2}m_2 = \frac{7}{2}m_2 \Rightarrow N_2 = 7(m_2 = 2) \end{array}\right. \Rightarrow N = 21.$$

Elementary signals (cont.)

3. Euler's identity

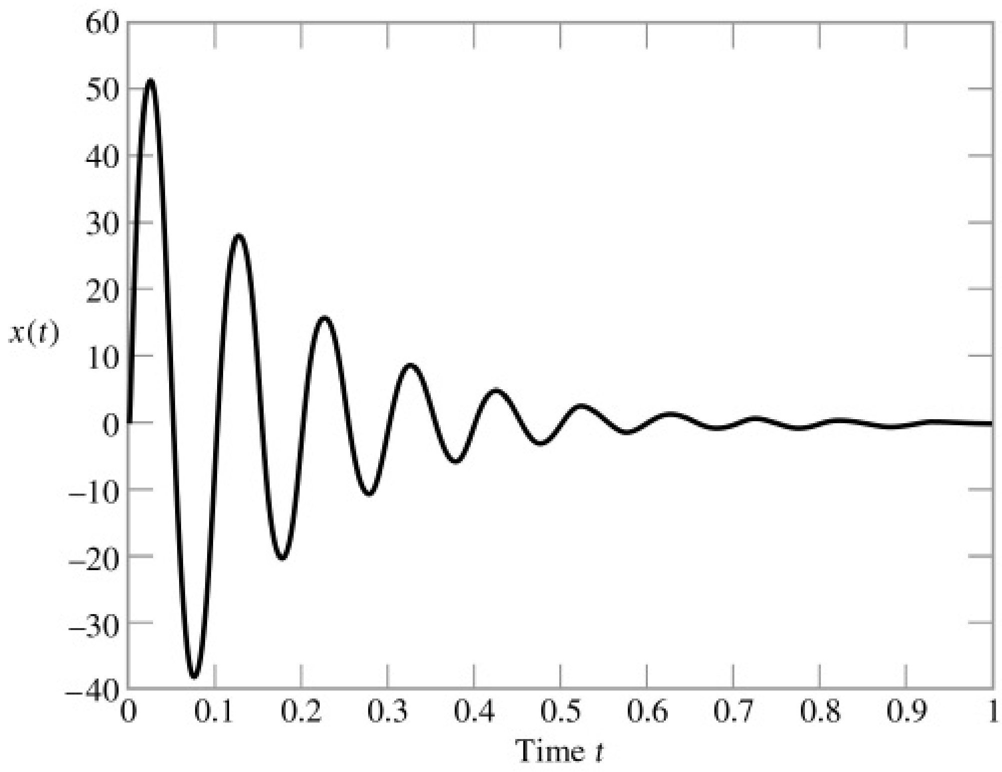
$$
\begin{aligned}
e^{j\theta} &= \cos(\theta) + j\sin\theta. \text{ Let } B = Ae^{j\phi} \\
Be^{j\omega t} &= Ae^{j\phi}e^{j\omega t} = Ae^{j(\omega t+\phi)} \\
&= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)
\end{aligned}
$$

$$
\begin{cases}
A\cos(\omega t+\phi) = \Re\{Be^{j\omega t}\} \\
A\sin(\omega t+\phi) = \Im\{Be^{j\omega t}\}
\end{cases}
$$

Elementary signals (cont.)

4. Exponentially damped sinusoidal

$$
\begin{aligned}
x(t) &= Ae^{-\alpha t}\sin(\omega t + \phi),\ \alpha > 0 \text{ for damped} \\
x[n] &= Br^{n}\sin(\Omega n + \phi),\ 0 < r < 1 \text{ for damped}
\end{aligned}
$$



Elementary signals (cont.)

5. Step function

CT	DT
$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$	$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Note: $u(0)$ is not defined. $u[0] = 1$.

Elementary signals (cont.)

E: Rectangular pulses in terms of $u(t)$ and $u[n]$.

$$x(t) = \begin{cases} A, & 0 \le |t| \le 0.5 \\ 0, & |t| > 0.5 \end{cases}$$

$x(t)$ can be expressed in terms of $u(t)$ as

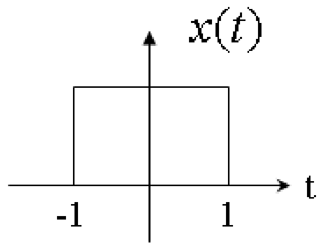
$$x(t) = Au(t + 1/2) - Au(t - 1/2)$$

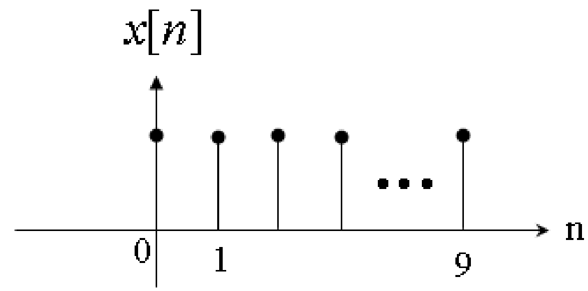
$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & o.w. \end{cases}$$

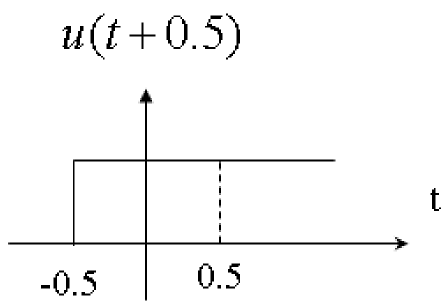
$x[n]$ can be expressed in terms of $u[n]$ as

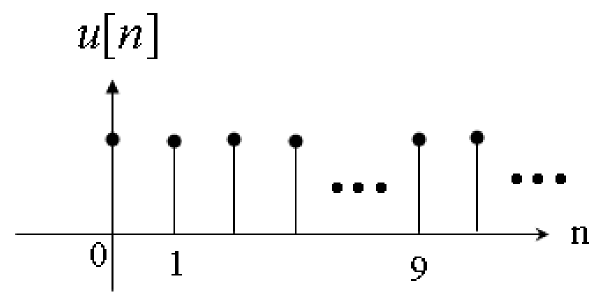
$$x[n] = u[n] - u[n - 10]$$

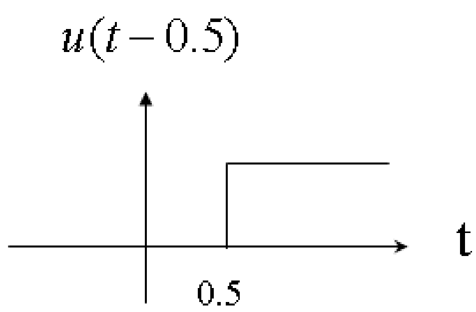
Elementary signals (cont.)

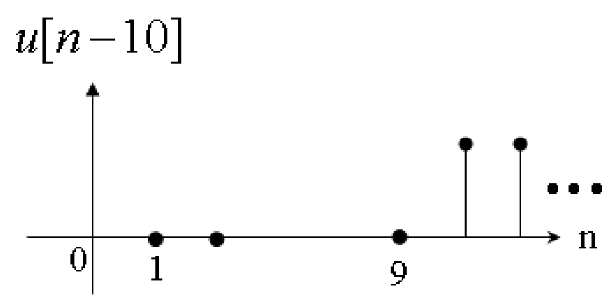












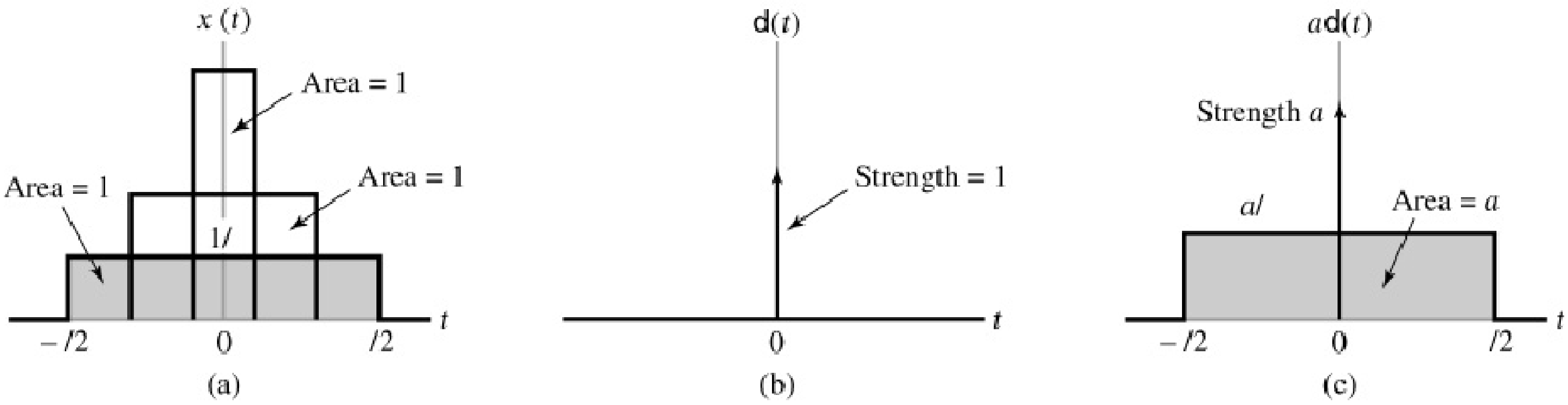
Elementary signals (cont.)

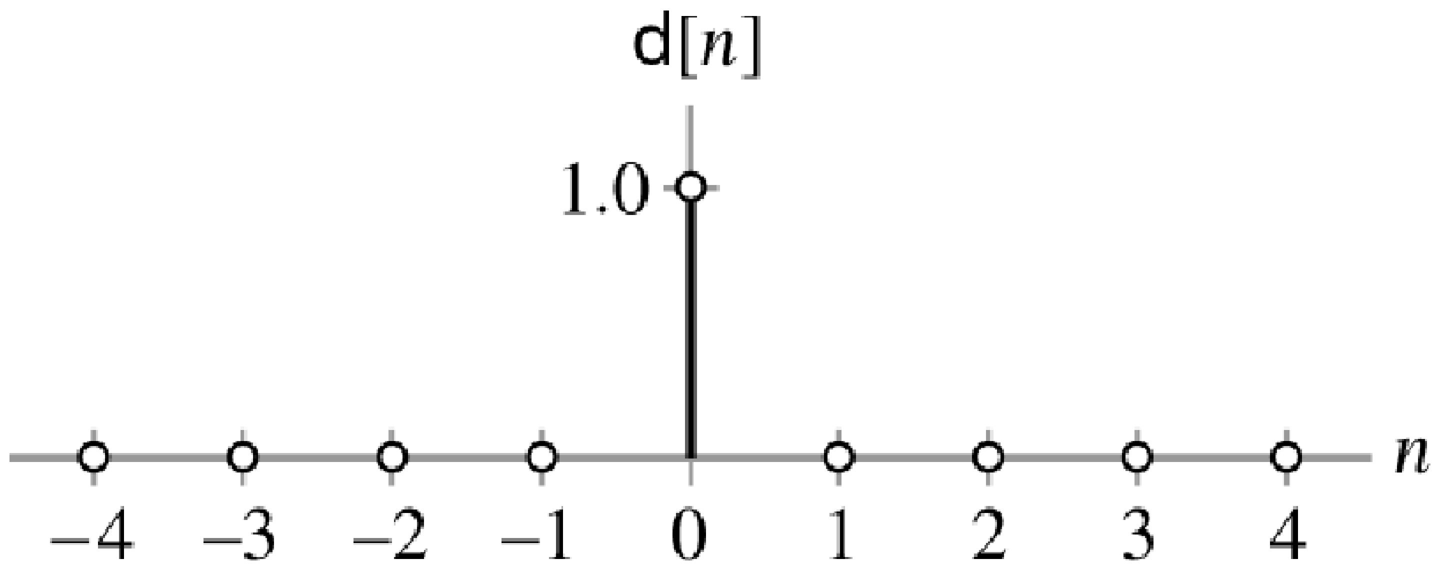
6. Impulse function

CT	DT
$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1 \end{cases}$	$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

- Let $x_\Delta(t)$ be a rectangle with area one and width Δ and height $\frac{1}{\Delta}$ ~~a~~. Then, $\delta(t) = \lim\limits_{\Delta \to 0} \cancel{a} x_\Delta(t)$
- $\delta(-t) = \delta(t)$
- $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$
- $\delta(t) = \frac{d}{dt}u(t) \Rightarrow u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$
- $\delta(at) = \frac{1}{a}\delta(t), \;\; a > 0$
- $\int_{-\infty}^{\infty} x(t)\frac{d}{dt}\delta(t - t_0)dt = \frac{d}{dt}x(t)|_{t=t_0}$ (p_{50} of text)
- $\int_{-\infty}^{\infty} x(t)\frac{d^n}{dt^n}\delta(t - t_0)dt = \frac{d^n}{dt^n}x(t)|_{t=t_0}$

Elementary signals (cont.)





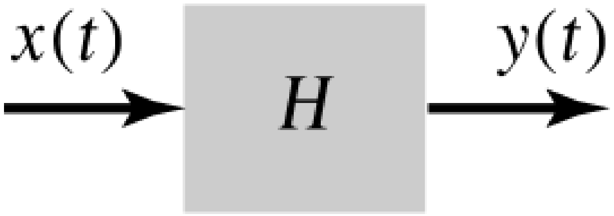
Elementary signals (cont.)

7. Unit ramp function

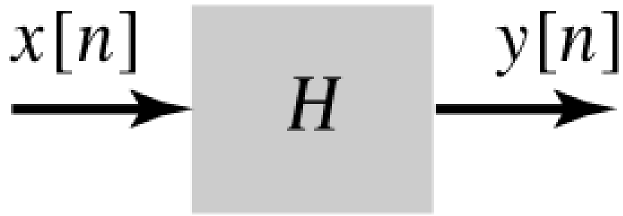
$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} = \int_{-\infty}^{t} u(\tau)d\tau = \int_{0}^{t} 1d\tau = tu(t)$$

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

System properties/characteristics



(a)



(b)

Notation: Let $\mathcal{H}$ represent the system, $x(t) \xrightarrow{\mathcal{H}} y(t)$ represent a system with input $x(t)$ and output $y(t)$.

1. Stability: BIBO (bounded input $\Rightarrow$ bounded output) stability

$$|x(t)| \leq M_x < \infty \implies |y(t)| \leq M_y < \infty$$
$$|x[n]| \leq M'_x < \infty \implies |y[n]| \leq M'_y < \infty$$

System properties/characteristics (cont.)

E: Is the system with $y(t) = e^{at}x(t)$ for $a > 0$ stable (BIBO)?

$$|y(t)| = |e^{at}||x(t)| \leq |e^{at}|M_x.$$

Since $|e^{at}|$ is not bounded, system is not BIBO stable.

E:

$$y[n] = \sum_{k=0}^{\infty} \rho^k x[n-k]$$

$$|y[n]| \leq \sum_{k=0}^{\infty} |\rho|^k |x[n-k]| \leq \sum_{k=0}^{\infty} |\rho|^k M_k < \infty \text{ iff } |\rho| < 1$$

Note that we have used the inequality $|A + B| \leq |A| + |B|$, and $|AB| \leq |A||B|$

Thus, the system is BIBO stable only if $|\rho| < 1$.

System properties/characteristics (cont.)

2. Memory/memoryless

- Memory system: present output value depends on future/past input.
- Memoryless system: present output value depends only on present input.

E: Memory systems:

$$\begin{aligned} y(t) &= 5x(t) + \int_{-\infty}^{t} x(\tau)d\tau \\ y[n] &= \sum_{m=n-5}^{n+5} x[m] \end{aligned}$$

Memoryless systems:

$$y[n] = x[n] + x^2[n]$$

System properties/characteristics (cont.)

3. Causal/noncausal

   - Causal: present output depends on present/past values of input.
   - Noncausal: present output depends on future values of input.

   Note: Memoryless $\Rightarrow$ causal, but causal not necessarily be memoryless.

4. Time invariance (TI): time delay or advance of input $\Rightarrow$ an identical time shift in the output.
   Let us define a system mapping $y(t) = \mathcal{H}(x(t))$. The system is time-invariant if

$$x(t - t_0) \xrightarrow{\mathcal{H}} y(t - t_0)$$

$$x[n - n_0] \xrightarrow{\mathcal{H}} y[n - n_0]$$

System properties/characteristics (cont.)

E: Is system $y[n] = r^n x[n]$ time invariant?

If $x[n-n_0] \longrightarrow r^n x[n-n_0]$.

$y[n-n_0] = r^{n-n_0} x[n-n_0] \neq r^n x[n-n_0]$. Thus, system is time variant.

- $y(t) = e^{at}x^2(t)$: TV, as
  $y(t-t_0) = e^{a(t-t_0)}x^2(t-t_0) \neq e^{at}x^2(t-t_0)$
- $y(t) = x^2(t)$: TI
- $y[n] = 5x[n] + 3x[n-3]$: TI
- $y[n] = u[n]x[n]$: TV, as
  $y[n-n_0] = u[n-n_0]x[n-n_0] \neq u[n]x[n-n_0]$
- $y(t) = \frac{d}{dt}x(t)$: TI, as $y(t-t_0) = \frac{d}{dt}x(t-t_0)$

System properties/characteristics (cont.)

5. Linearity

   Linear system: If $x_1(t) \xrightarrow{\mathcal{H}} y_1(t)$, $x_2(t) \xrightarrow{\mathcal{H}} y_2(t)$, then $ax_1(t) + bx_2(t) \xrightarrow{\mathcal{H}} ay_1(t) + by_2(t)$. Else, nonlinear.

   - Superposition property (addition)
   - Homogeneity (scaling)

   The following operations preserve linearity

   - $\frac{dx(t)}{dt} \xrightarrow{\mathcal{H}} \frac{dy(t)}{dt}$
   - $\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{\mathcal{H}} \int_{-\infty}^{t} y(\tau)d\tau$
   - $\displaystyle\sum_{m=-\infty}^{n} x[m] \xrightarrow{\mathcal{H}} \sum_{m=-\infty}^{n} y[m]$

System properties/characteristics (cont.)

E:

- $y[n] = nx[n-3]$: linear

$$\begin{aligned} n(ax_1[n-3] + bx_2[n-3]) &= a(nx_1[n-3]) + b(nx_2[n-3]) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

- $y(t) = 5x(t+t_0)$: linear

$$\begin{aligned} 5(ax_1(t+t_0) + bx_2(t+t_0)) &= a(5x_1(t+t_0)) + b(5x_2(t+t_0)) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

- $y(t) = |x(t)|$: nonlinear

$$|ax_1(t) + bx_2(t)| \neq a|x_1(t)| + b|x_2(t)| \text{ in general}$$

Basics of Matlab

1. Basic commands

   - 'help': e.g., 'help elfun;' 'help sinc;' 'help square'
   - 'lookfor': e.g., 'lookfor random;' 'lookfor round;' 'lookfor floor'
   - 'plot', 'stem', 'rand', 'randn', 'sin', 'cos', 'exp', 'sqrt', 'zeros', 'ones', 'find', 'xlabel', 'ylabel', 'title', 'legend'
   - '$\cdots$': continue next line

2. Before a variable is applied, it must have a value.

   - $t = p$; will not work if p does not have a value. $p = sin(2 * pi * 0.3)$; $t = p$; works.
   - Sampling on time axis

$$\begin{aligned} t &= 0 : 0.001 : 1;\ \%\text{sampling interval} T_s = 0.001s \\ x &= \sin(2 * pi * t);\ \%\ x \text{ is a row vector. } '\sin(2\ pi\ t)' \text{ will not work} \end{aligned}$$

Basics of Matlab (cont.)

3. Vector and matrix operations

- '$(\cdot)'$': complex conjugate transpose. E.g., $A = [1;\ 1+j0.7]$ is a 2×1 column vector. Then, $B = A'$ yields a 1×2 row vector as $B = [1\ \ 1-j0.7]$.
- '$(\cdot).'$': transpose.
- '$.*$': element-by-element multiplication. E.g., $A = [1\ \ 2;\ 3\ \ 4]$; $B = [4\ \ 5;\ 6\ \ 7]$; $C_1 = A. * B$; ($C_1 = [4\ \ 10;\ 18\ \ 28]$)
- '$*$': multiplication of matrices, vectors, and scalars. E.g, $C_2 = A * B$; ($C_2 = [16\ \ 19;\ 36\ \ 43]$).
- '$./$' and '$/$': element-by-element division and division, respectively. E.g., $A = [1\ \ 2\ \ 3]$; $B = [2\ \ 2\ \ 4]$; '$C = A/B$;' would NOT work. $C = A./B$; ($C = [0.2\ \ 1\ \ 0.75]$)
- '$(\cdot) \wedge n$': n-th power of a scalar. '$(\cdot). \wedge n$': n-th power of a vector or a matrix. E.g., $A = 1.4142$; $B = A \wedge 2$; ($B = 2$). $A = [1.4142\ \ 1.7321]$; '$B = A \wedge 2$;' would NOT work. $B = A. \wedge 2$; ($B = [2\ \ 3]$).