FOURIER REPRESENTATION OF SIGNALS & LTI SYSTEMS

CT: f cycle/second (Hz)

$\omega = 2\pi f$ rads/s

DT: F cycles/sample

$\Omega = 2\pi F$ rads/sample

- Basic signals as weighted superposition of impulses

$$x[n] = \underbrace{\sum_{k=-\infty}^{\infty}}_{\text{superposition}} \underbrace{x[k]}_{\text{weight}}\,\underbrace{\delta[n-k]}_{\text{delay}} \xrightarrow[h[n]]{LTI} y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

(LTI property)

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \xrightarrow[h(t)]{LTI} y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

• Time-domain waveform represents how fast signal changes.
Signals in terms different frequency components or weighted superpositions of complex sinusoids.

$$\begin{cases} \text{CT: } X(f) \text{ or } X(\omega) \\ \text{DT: } X[k] \end{cases}$$

Why signals represented as weighted superpositions of complex sinusoids?

DT: $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

If $x[n] = e^{j\Omega n}$ ⟵ single frequency

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\underbrace{e^{j\Omega n}e^{-j\Omega k}}_{x[n-k]}$$

$$= e^{j\Omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

$$= e^{j\Omega n}H(e^{j\Omega})$$

$$\boxed{H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}}$$ ~ related to h[k]

Note:

a). $H(e^{j\Omega})$ is NOT a function of *n*, only a function of Ω.

$H(e^{j\Omega})$ Is called the frequency response.

b). System modifies the amplitude of input by $\left|H(e^{j\Omega})\right|$.

$\left|H(e^{j\Omega})\right|$: magnitude response.

c). System introduces a phase lag $\angle H(e^{j\Omega})$.

(the book uses $\arg\{H(e^{j\Omega})\}$)

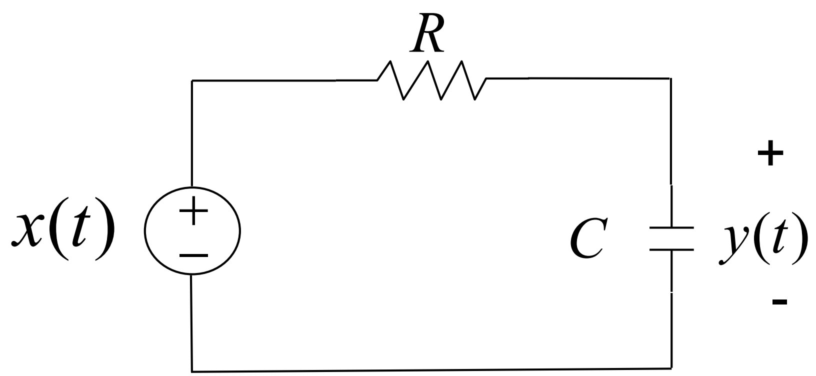
$$H(e^{j\Omega}) = \left|H(e^{j\Omega})\right| e^{j\angle H(e^{j\Omega})}$$

CT: $$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}$$

with $x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega) e^{j\omega t}$

$$= \left|H(j\omega)\right| e^{j(\omega t + \angle H(e^{j\Omega}))}$$

E



Impulse response: $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$

Find frequency response.

Solution:

$$H(j\omega)=\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau=\int_{-\infty}^{\infty}\frac{1}{RC}e^{-\frac{\tau}{RC}}u(\tau)e^{-j\omega\tau}d\tau$$

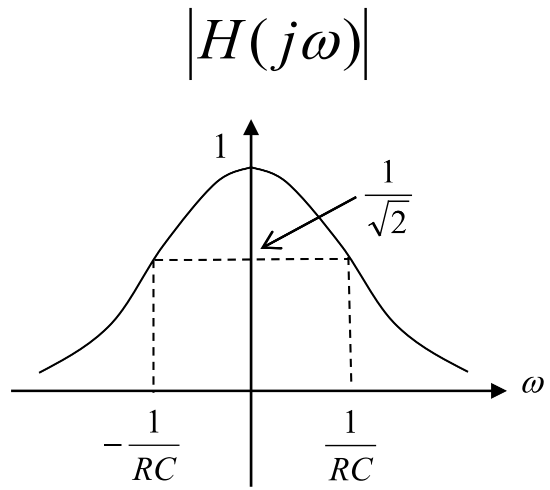
$$=\frac{1}{RC}\int_{0}^{\infty}e^{-\left(j\omega+1/(RC)\right)\tau}d\tau$$

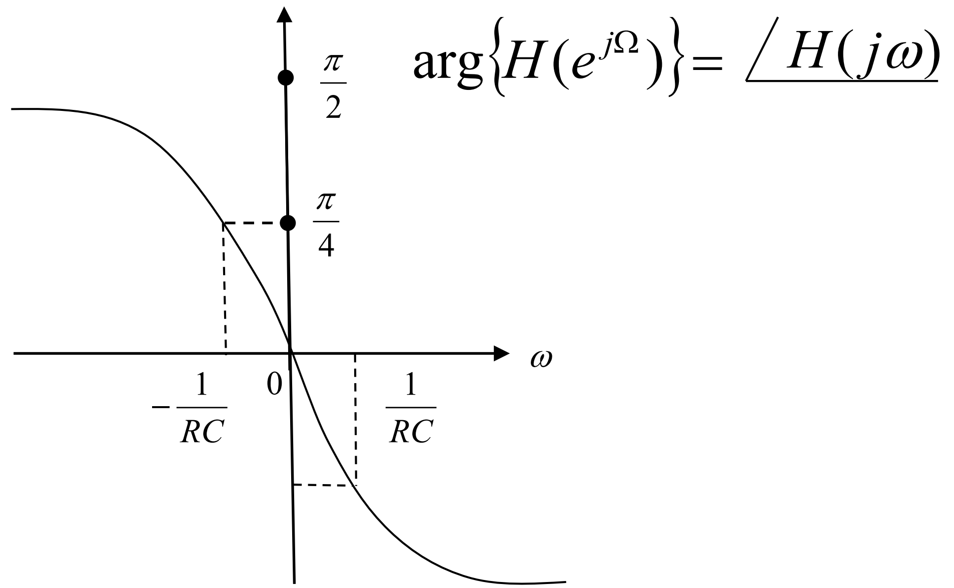
$$=\frac{1}{RC}\cdot\frac{-1}{j\omega+1/(RC)}e^{-\left(j\omega+1/(RC)\right)\tau}\Big|_{0}^{\infty}$$

$$=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$

- Magnitude response: $$\left|H(j\omega)\right|=\frac{1/RC}{\sqrt{\omega^2+(1/RC)^2}}$$

- Phase response: $$\angle H(j\omega)=-\arctan(\omega RC)$$





$e^{j\omega t}$: eigenfunction of the LTI system (eigen value $\lambda = H(j\omega)$)

$$(H\{e^{j\omega t}\} = \lambda e^{j\omega t})$$

Now, if the input to an LTI system is expressed as a weighted sum of <u>*M* complex sinusoids</u>:

$$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}, \quad \text{then}$$

$$y(t) = \sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t}$$

Fourier representations of four classes of signals

Time property	Periodic	Nonperiodic
Continuous time (t)	• Fourier Series (FS) $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T}$ $X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt,$ (T: period)	• Fourier Transform (FT) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
Discrete time [n]	• Discrete-Time Fourier Series (DTFS) $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}, \Omega_0 = \frac{2\pi}{N}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n}$ (N: period)	• Discrete-Time Fourier Transform (DTFT) $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

DTFS: *x[n]* periodic with period *N*, fundamental freq. $\Omega_0 = 2\pi/N$
DTFS coefficients of *x[n]: X[k].* Then

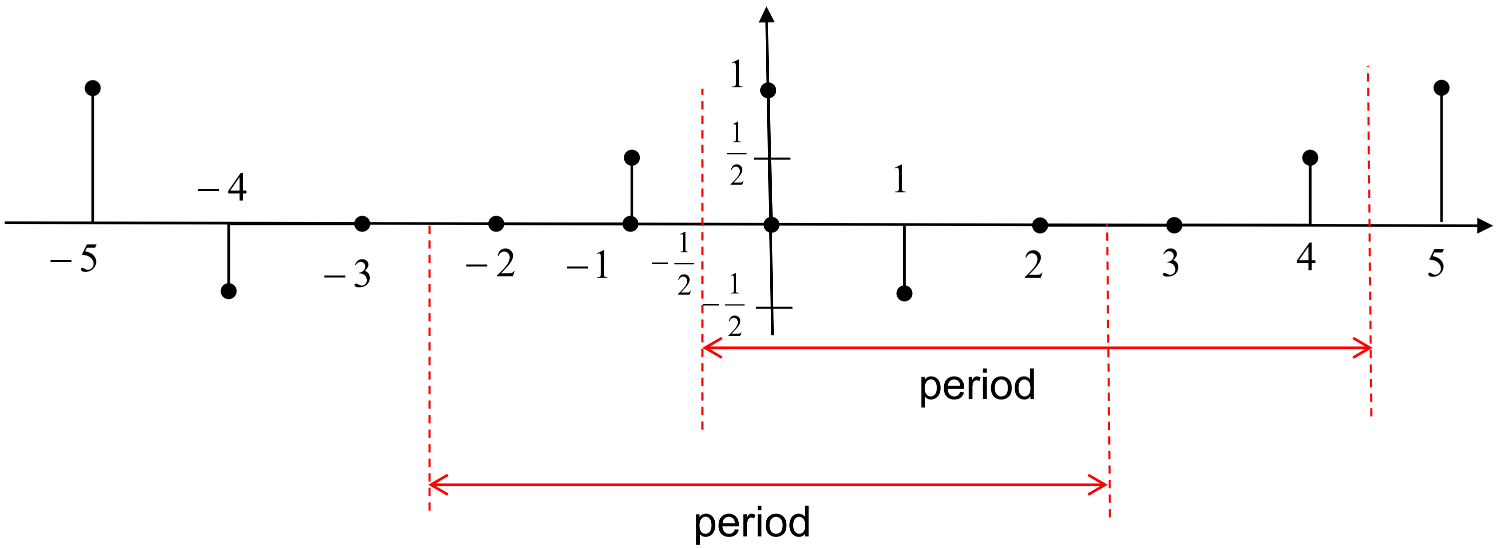
$$\begin{cases} x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \\ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} \end{cases}$$

Freq-domain representation of x[n]

x[n] and X[k] are a DTFS pair: $\boxed{x[n] \xleftrightarrow{DTFS;\ \Omega_0} X[k]}$

Note: a). Either x[n] or X[k] provides a complete description of the signal.
b). The limits on sums of x[n] or X[k] may be chosen differently from 0 to N-1.

E   Find the freq-domain representation of x[n] given below



Solution: $N = 5$ (Period)

$\Omega_0 = 2\pi/N = 2\pi/5$ (Fundamental frequency)

$$X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\Omega_0 n}$$

$$=\frac{1}{5}\sum_{n=-2}^{2}x[n]e^{-jk\frac{2\pi}{5}n}$$

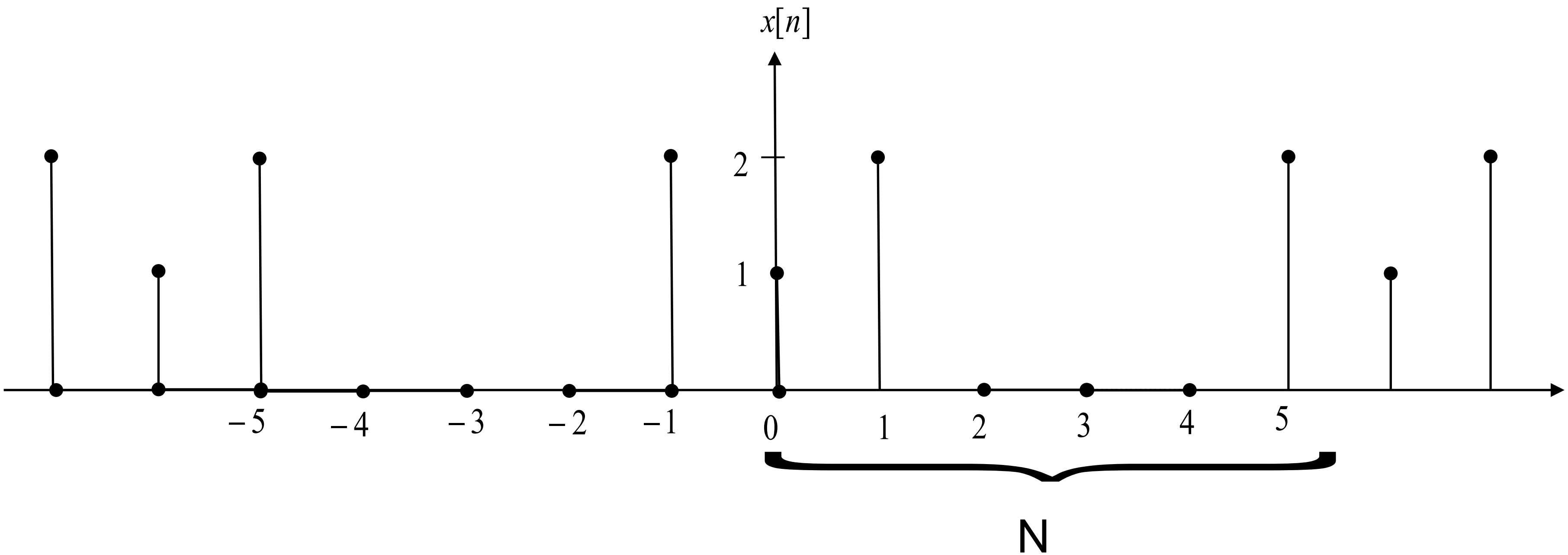
Yields the same result as $\sum_{n=0}^{4}$. Use Matlab to verify.

Thus $$X[k]=\frac{1}{5}\left\{x[-2]e^{jk\frac{4\pi}{5}}+x[-1]e^{jk\frac{2\pi}{5}}+x[0]e^{j0}+x[1]e^{-jk\frac{2\pi}{5}}+x[2]e^{-jk\frac{4\pi}{5}}\right\}$$

$$=\frac{1}{5}\left\{1+\frac{1}{2}e^{jk\frac{2\pi}{5}}-\frac{1}{2}e^{-jk\frac{2\pi}{5}}\right\}$$

$$=\frac{1}{5}\left\{1+j\sin(k\frac{2\pi}{5})\right\}$$

E



Find DTFS coefficients X[k] of periodic signal x[n]

Solution:

$$X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-j\Omega_0 nk}=\frac{1}{N}\sum_{n=-N/2}^{N/2-1}x[n]e^{-j\Omega_0 nk}$$ If N even

$$\begin{cases} N=6 \\ \Omega_0=\dfrac{2\pi}{6}=\dfrac{\pi}{3} \end{cases}$$ thus,

$$X[k]=\frac{1}{6}\sum_{n=-1}^{1}x[n]e^{-j\frac{\pi}{3}\cdot n\cdot k}=\frac{1}{6}+\frac{1}{3}e^{j\frac{\pi}{3}k}+\frac{1}{3}e^{-j\frac{\pi}{3}k}=\frac{1}{6}+\frac{2}{3}\cos\left(\frac{\pi}{3}\cdot k\right)$$

Note: Sometimes when x[n] is composed of real or complex sinusoids, it might be easier to calculate X[k] by inspection.

E

$$x[n] = \cos(\pi n/3 + \phi), \quad \text{Find } X[k].$$

$$x[n] = \cos\left(\frac{\pi}{3}n + \phi\right) = \cos\left(\frac{\pi}{3}(n+N) + \phi\right) \quad \Rightarrow \begin{cases} \mathrm{N} = 6 \\ \Omega_0 = \dfrac{\pi}{3} \end{cases}$$

$$= \frac{1}{2}\left[e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}\right]$$

$$= \frac{1}{2}e^{j\phi} \cdot e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{-j\phi} \cdot e^{-j\frac{\pi}{3}n}$$

General: $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$ (compare these two x[n]'s)

$$\Rightarrow \quad X[k] = \begin{cases} \dfrac{1}{2}e^{j\phi} & k = 1 \\ \dfrac{1}{2}e^{-j\phi} & k = -1 \\ 0 & \text{elsewhere} \end{cases}$$

E

$$x[n] = 1 + \sin(n\pi/12 + 3\pi/8)$$

$$\begin{cases} N = 24 \\ \Omega_0 = \pi/12 \end{cases}$$

$$x[n] = 1 + \frac{1}{2j}\left\{ e^{j\left(\frac{\pi}{12}n + \frac{3\pi}{8}\right)} - e^{-j\left(\frac{\pi}{12}n + \frac{3\pi}{8}\right)} \right\}$$

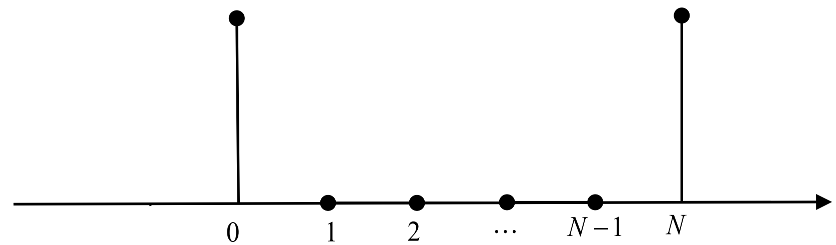
$$= \underset{\substack{\uparrow \\ k=0}}{1} + \underbrace{\frac{1}{2j} e^{j\cdot\frac{3\pi}{8}}}_{\substack{\Uparrow \\ k=1}} \cdot e^{j\cdot\frac{\pi}{12}n} - \underbrace{\frac{1}{2j} e^{-j\cdot\frac{3\pi}{8}}}_{\substack{\Uparrow \\ k=-1}} \cdot e^{-j\cdot\frac{\pi}{12}n} \Rightarrow$$

$$x[n] = \xleftrightarrow{DTFS;\ 2\pi/24} X[k] = \begin{cases} 1, & k = 0 \\ e^{j3\pi/8}/(2j), & k = 1 \\ -e^{j3\pi/8}/(2j), & k = -1 \\ 0, & e.w. \end{cases}$$

E DTFS of an Impulse train:

$$x[n]=\sum_{l=-\infty}^{\infty}\delta[n-lN]$$

$$N=?\quad (N),\quad \Omega_0=\frac{2\pi}{N}$$

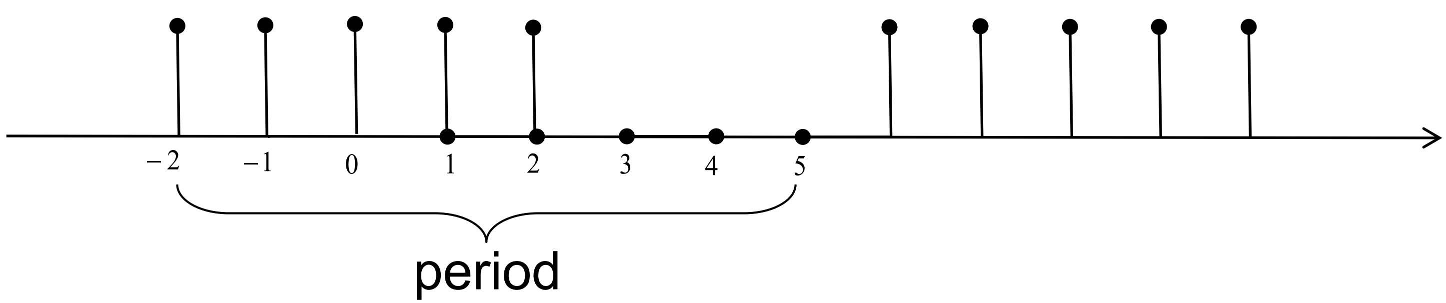


$$X[k]=\frac{1}{N}\sum_{n=0}^{N-1}\delta[n]e^{-jkn2\pi/N}=\frac{1}{N}$$

E DTFS of a square signal:

$$x[n]=\begin{cases}1, & -M\le n\le M\\ 0, & M<n<N-M\end{cases}$$

$$\text{Example}\begin{cases}M=2\\ N=8\end{cases}$$



Period N, so $\Omega_0 = 2\pi/N$

$n < N - M$

not $n \leq N\text{-}M$

$$X[k] = \frac{1}{N}\sum_{n=-M}^{N-M-1} x[n]e^{-jk\Omega_0 n}$$

$$= \frac{1}{N}\sum_{n=-M}^{M} e^{-jk\Omega_0 n}$$

Let $m = n + M$ $\qquad$ $$= \frac{1}{N}\sum_{m=0}^{2M} e^{-jk\Omega_0 m} \cdot e^{jk\Omega_0 M}$$

$$= \begin{cases} (2M+1)/N & k = 0, \pm N, \pm 2N, \cdots \\ \dfrac{e^{jk\Omega_0 m}}{N}\left(\dfrac{1-e^{-jk\Omega_0(2M+1)}}{1-e^{jk\Omega_0}}\right), & k \neq 0, \pm N, \pm 2N, \cdots \end{cases}$$

For $k \neq 1, \pm N, \pm 2N, \cdots$

$$X[k] = \frac{1}{N}\left(\frac{e^{jk\Omega_0(2M+1)/2} - e^{-jk\Omega_0(2M+1)/2}}{e^{jk\Omega_0/2} - e^{-jk\Omega_0/2}}\right)$$

$$= \frac{1}{N}\frac{\sin(k\Omega_0(2M+1)/2)}{\sin(k\Omega_0/2)}$$

$$\boxed{\Omega_0 = \frac{2\pi}{N}}$$

$$\mathrm{X[k]} = \frac{1}{N}\frac{\sin(k\pi(2M+1)/N)}{\sin(k\pi/N)},$$ if values of $X[k]$ for $k = 0, \pm N, \pm 2N \cdots$ are obtained from the limit as $k \to 0, \pm N, \pm 2N \cdots$

E

One period of DTFS coefficients

$$X[k] = \left(\frac{1}{2}\right)^k, \; 0 \le k \le 9$$

Determine $x[n]$ assuming $N = 10$

<u>Solution</u>:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\Omega_0 kn}$$

$$\begin{cases} N = 10 \\ \Omega_0 = \pi/5 \end{cases}$$

$$= \sum_{k=0}^{9} \left(\frac{1}{2}\right)^k e^{j\frac{\pi}{5}\cdot n\cdot k}$$

$$= \sum_{k=0}^{9} \left(\frac{1}{2} e^{j\frac{\pi}{5}\cdot n}\right)^k$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2} e^{j\frac{\pi}{5}\cdot n}}$$

$$\boxed{e^{j\frac{\pi}{5}\cdot n\cdot 10} = ? \quad (1)}$$

F.S. (CT, periodic). x(t): fundamental period T
fundamental frequency

$$\omega_0 = 2\pi/T$$

$$\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} & (*) \\ X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt \end{cases}$$

x(t) and X[k] are an FS pair: $\boxed{x(t) \xleftrightarrow{FS;\omega_0} X[k]}$

FS coefficients X[k] are a freq-domain representation of x(t).

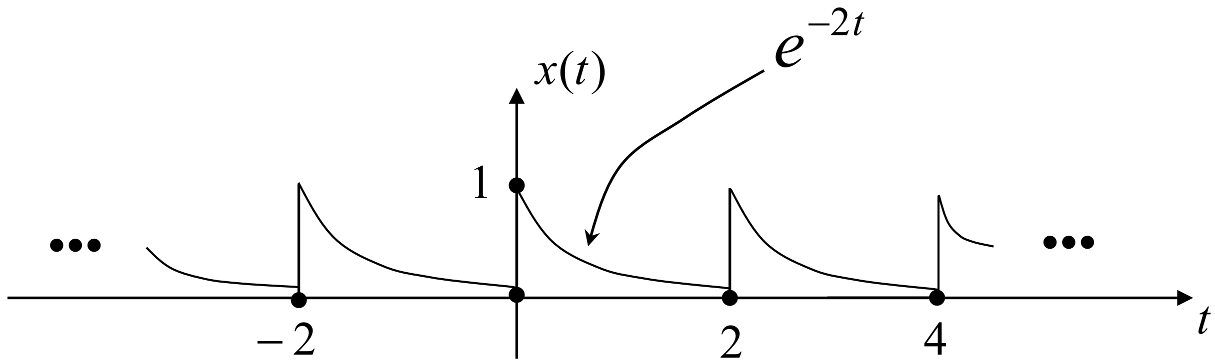
Note: (*) is NOT guaranteed to converge for all possible signals.

However, if $\frac{1}{T}\int_0^T |x(t)|^2 dt < \infty \Rightarrow \frac{1}{T}\int_0^T |x(t) - \hat{x}(t)|^2 dt = 0$ (**)

Original signal

(*) representation

$\boxed{\text{E}}$ x(t) given as



$$X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \qquad \text{where } \begin{cases} T=2 \\ \omega_0=\pi=\dfrac{2\pi}{T} \end{cases}$$

$$=\frac{1}{2}\int_0^2 e^{-2t}e^{-j\pi kt}dt$$

$$=\frac{1}{2}\int_0^2 e^{-(2+j\pi k)t}dt$$

$$=\frac{-1}{2(2+j\pi k)}e^{-(2+j\pi k)t}\Big|_0^2$$

$$=\frac{1}{4+j2\pi k}\left(1-e^{-4}\underbrace{e^{-j2\pi k}}_{=1}\right)$$

$$=\frac{1-e^{-4}}{4+j2\pi k}$$

Note: As with DTFS, $|X[k]|$: magnitude spectrum of x(t)

$\angle X[k]$: phase spectrum of x(t)

E

Determine X[k] of $x(t)=\sum_{l=-\infty}^{\infty}\delta(t-4l)$

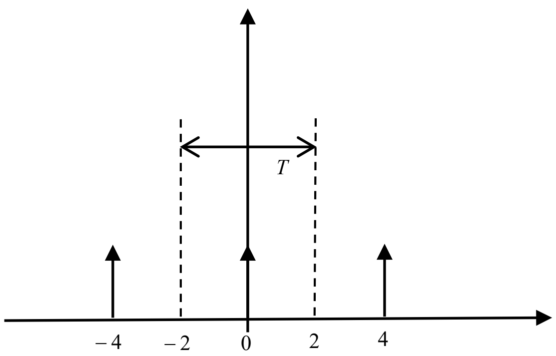
Solution: $\begin{cases} T=4 \\ \omega_0 = 2\pi/T = \pi/2 \end{cases}$

It is easier to evaluate from -2 to 2.

$$X[k]=\frac{1}{T}\int_{-2}^{2}x(t)e^{-jk\omega_0 t}dt$$

$$=\frac{1}{4}\int_{-2}^{2}\delta(t)e^{-jk\frac{\pi}{2}t}dt$$

$$=\frac{1}{4}$$



So we can write $x(t)=\sum_k X[k]e^{jk\omega_0 t}=\frac{1}{4}\sum_k e^{jk\frac{\pi}{2}t}$

FS coefficients by inspection.

$\boxed{\text{E}}$ $$x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$$ Find X[k]

Solution:

$$x(t) = \frac{3}{2}\left\{e^{j\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)}\right\}$$

$$\begin{cases} \text{T} = 4 \\ \omega_0 = \pi/2 \end{cases}$$

$$= \underbrace{\frac{3}{2}e^{j\frac{\pi}{4}}}_{k=1} \cdot e^{j\frac{\pi}{2}t} + \underbrace{\frac{3}{2}e^{-j\frac{\pi}{4}}}_{k=-1} \cdot e^{-j\frac{\pi}{2}t}$$

$$X[k] = \begin{cases} \frac{3}{2}e^{j\frac{\pi}{4}}, & \text{k} = 1 \\ \frac{3}{2}e^{-j\frac{\pi}{4}}, & \text{k} = -1 \\ 0, & \text{e.w.} \end{cases}$$

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} X[k]e^{jk(\pi/2)t} \end{aligned}}$$

$\boxed{\text{E}}$

$$x(t) = 2\sin(2\pi t - 3) + \sin(6\pi t). \quad \text{Find } X[k]$$

Solution:

$$x(t) = \frac{1}{j}\left[e^{j(2\pi t - 3)} - e^{-j(2\pi t - 3)}\right] + \frac{1}{j2}\left[e^{j6\pi t} - e^{-j6\pi t}\right]$$

$$= \underbrace{-je^{-j3}}_{k=1} e^{j2\pi t} + \underbrace{je^{j3}}_{k=-1} e^{-j2\pi t} - \underbrace{\frac{j}{2}}_{k=3} e^{+j6\pi t} + \underbrace{\frac{j}{2}}_{k=-3} e^{-j6\pi t}$$

$$x(t) = \xleftrightarrow{FS;2\pi} X[k] = \begin{cases} j/2, & k = -3 \\ -j/2, & k = 3 \\ -je^{-j3}, & k = 1 \\ je^{j3}, & k = -1 \\ 0, & e.w. \end{cases}$$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}}$$

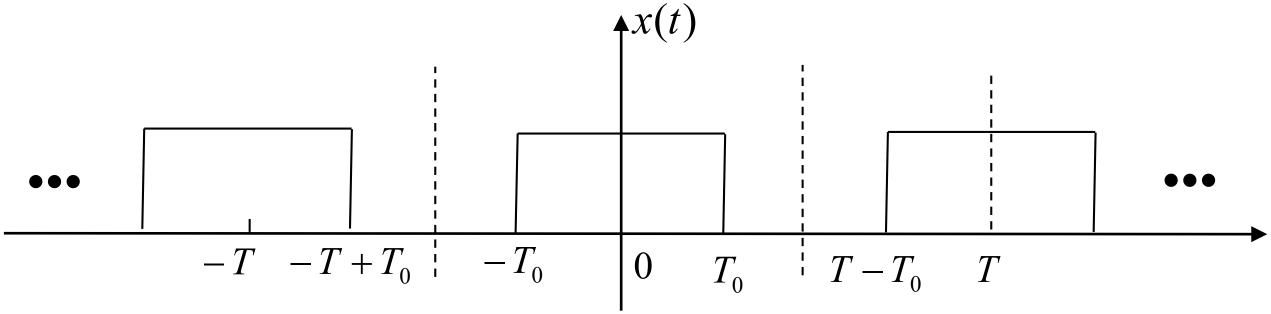
$\boxed{\text{E}}$ Inverse FS.

$$X[k] = -j\delta[k-2] + j\delta[k+2] + 2\delta[k-3] + 2\delta[k+3],\ \omega_0 = \pi.\ \text{Find } x(t)$$

Solution:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \\ &= -je^{j2\pi t} + je^{-j2\pi t} + 2e^{j3\pi t} + 2e^{-j3\pi t} \\ &= 2\sin(2\pi t) + 4\cos(3\pi t) \end{aligned}$$

E FS of a square wave.



Period is T, so $\omega_0 = 2\pi/T$

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\omega_0 t}dt$$

$$= \frac{2}{Tk\omega_0}\left(\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{2j}\right)$$

$$= \frac{2\sin(k\omega_0 T_0)}{k\omega_0 T}, \qquad \text{for } k \neq 0$$

$$\boxed{\lim_{k\to 0}\frac{2\sin(\mathrm{k}\omega_0 T_0)}{\mathrm{k}\omega_0 T} = \frac{2T_0}{T}}$$

For $\mathrm{k}=0$, $X[0]=\dfrac{1}{T}\int_{-T_0}^{T_0} dt = \dfrac{2T_0}{T}$

Thus, we can write $X[k]=\dfrac{2\sin(k\omega_0 T_0)}{k\omega_0 T}$, where X[0] is obtained as a limit $k \to 0$

$$= \frac{2\sin(k2\pi \frac{T_0}{T})}{k2\pi}$$

Observations: a). $T_0/T \downarrow \Rightarrow$ energy of X[k] distributed over a broader frequency interval.

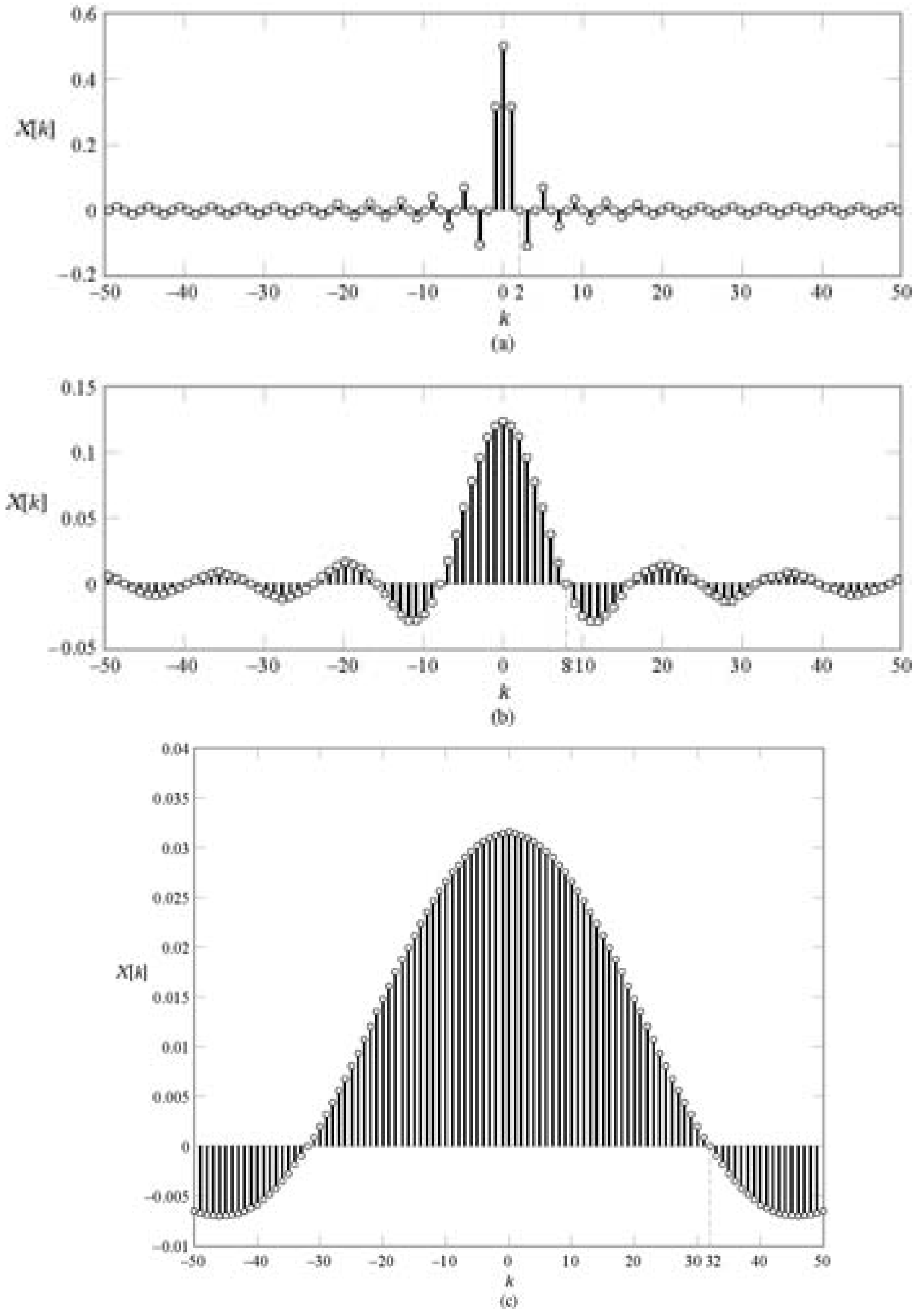
b). $T_0/T \downarrow \Rightarrow$ energy of x(t) is concentrated over a narrower time interval.

Define $\boxed{\mathrm{sinc}(u)=\dfrac{\sin(\pi u)}{\pi u}}$ Then $X[k]=\dfrac{2T_0}{T}\mathrm{sinc}\left(k\dfrac{2T_0}{T}\right)$

$T_0 / T = 1/4$

$T_0 / T = 1/6$

$T_0 / T = 1/64$



DTFT D.T., nonperiodic

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

DTFT of signal x[n], also Freq-domain representation of x[n].

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$$

$$\boxed{X(e^{j(\Omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \cdot e^{-j2\pi n} = X(e^{j\Omega})}$$

Note: a). If x[n] is of infinite duration, $X(e^{j\Omega})$ may or may not converge.

b). If x[n] is of finite duration,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}, \text{ converges everywhere.}$$

c). $X(e^{j(\Omega+2\pi)}) = X(e^{j\Omega})$

$\boxed{\text{E}}$

$$x[n] = \alpha^n u[n], \quad X(e^{j\Omega}) = ?$$

<u>Solution</u>:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n$$

$$= \frac{1}{1-\alpha e^{-j\Omega}}, \text{ if } |\alpha| < 1$$

$\boxed{\text{E}}$ Rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases} \qquad X(e^{j\Omega}) = ?$$

Solution:

$$X(e^{j\Omega}) = \sum_{n=-M}^{M} 1 \cdot e^{-j\Omega n} \qquad \text{let } m = n + M, \quad m = 0 \to 2M$$

$$= \sum_{m=0}^{2M} e^{-j\Omega(m-M)}$$

$$= e^{j\Omega M} \sum_{m=0}^{2M} e^{-j\Omega m}$$

$$= \begin{cases} e^{j\Omega M} \dfrac{1-e^{-j\Omega(2M+1)}}{1-e^{-j\Omega}}, & \Omega \neq 0, \pm 2\pi, \pm 4\pi, \cdots \\ 2M+1, & \Omega = 0, \pm 2\pi, \pm 4\pi, \cdots \end{cases}$$

$$= \begin{cases} \dfrac{e^{j\Omega(2M+1)/2} - e^{-j\Omega(2M+1)/2}}{e^{j\Omega/2} - e^{-j\Omega/2}}, & \text{otherwise} \\ 2M+1, & \Omega = 0, \ \pm 2\pi, \ \pm 4\pi, \cdots \end{cases}$$

$$= \frac{\sin[\Omega(2M+1)/2]}{\sin(\Omega/2)} \qquad \forall \Omega$$

where $X(e^{j\Omega})$ for $\Omega = 0, \pm 2\pi, \cdots$ is obtained as a limit.

E

Inverse DTFT of a rectangular spectrum

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}$$

$X(e^{j\Omega})$ is defined over $(-\pi, \pi)$ periodic in Ω

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi \cdot jn} e^{j\Omega n}\Big|_{-W}^{W}$$

$$= \frac{1}{\pi n}\sin(Wn) \quad \text{for n} \neq 0$$

$$\lim_{n\to 0}\frac{1}{\pi n}\sin(Wn) = \frac{W}{\pi},$$ Thus, we can write

$$x[n] = \frac{1}{\pi n}\sin(Wn) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right)$$

E

DTFT of unit impulse $x[n] = \delta[n]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\Omega n} = 1$$

- What about inverse DTFT of a unit impulse spectrum?

$$X(e^{j\Omega}) = \delta(\Omega), \quad -\pi < \Omega \le \pi \qquad \text{(defined only one period)}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi}$$ over one period

$$\text{For all } \Omega,\ X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$ This is a common definition too!

$$\boxed{X(e^{j\Omega}) \text{ is periodic with period} = 2\pi}$$

E

$$x[n] = \begin{cases} 2^n, & 0 \le n \le 9 \\ 0, & o.w. \end{cases} \qquad X(e^{j\Omega}) = ?$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{9} (2e^{-j\Omega})^n = \frac{1 - 2^{10} e^{-j\Omega 10}}{1 - 2e^{-j\Omega}}$$

E

$$X(e^{j\Omega}) = 2\cos(2\Omega), \quad x[n] = ?$$

Use inspection! $X(e^{j\Omega}) = e^{j2\Omega} + e^{-j2\Omega}$

By definition: $$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \Rightarrow$$

$$x[n] = \begin{cases} 1, & n = \pm 2 \\ 0, & \text{otherwise} \end{cases}$$

E

Multipath channel: $y[n] = x[n] + ax[n-1]$

Impulse response: $h[n] = \delta[n] + a\delta[n-1]$

$$H\left(e^{j\Omega}\right) = 1 + ae^{-j\Omega}$$

Freq.-domain representation of the system impulse response.

To equalize the system, cascade it with $H^{inv}\left(e^{j\Omega}\right) = \dfrac{1}{1 + ae^{-j\Omega}}$

FT C.T., nonperiodic signals

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ X(j\omega) = \displaystyle\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \end{cases}$$

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

E

$$x(t) = e^{-at}u(t). \quad \text{Find} \quad X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

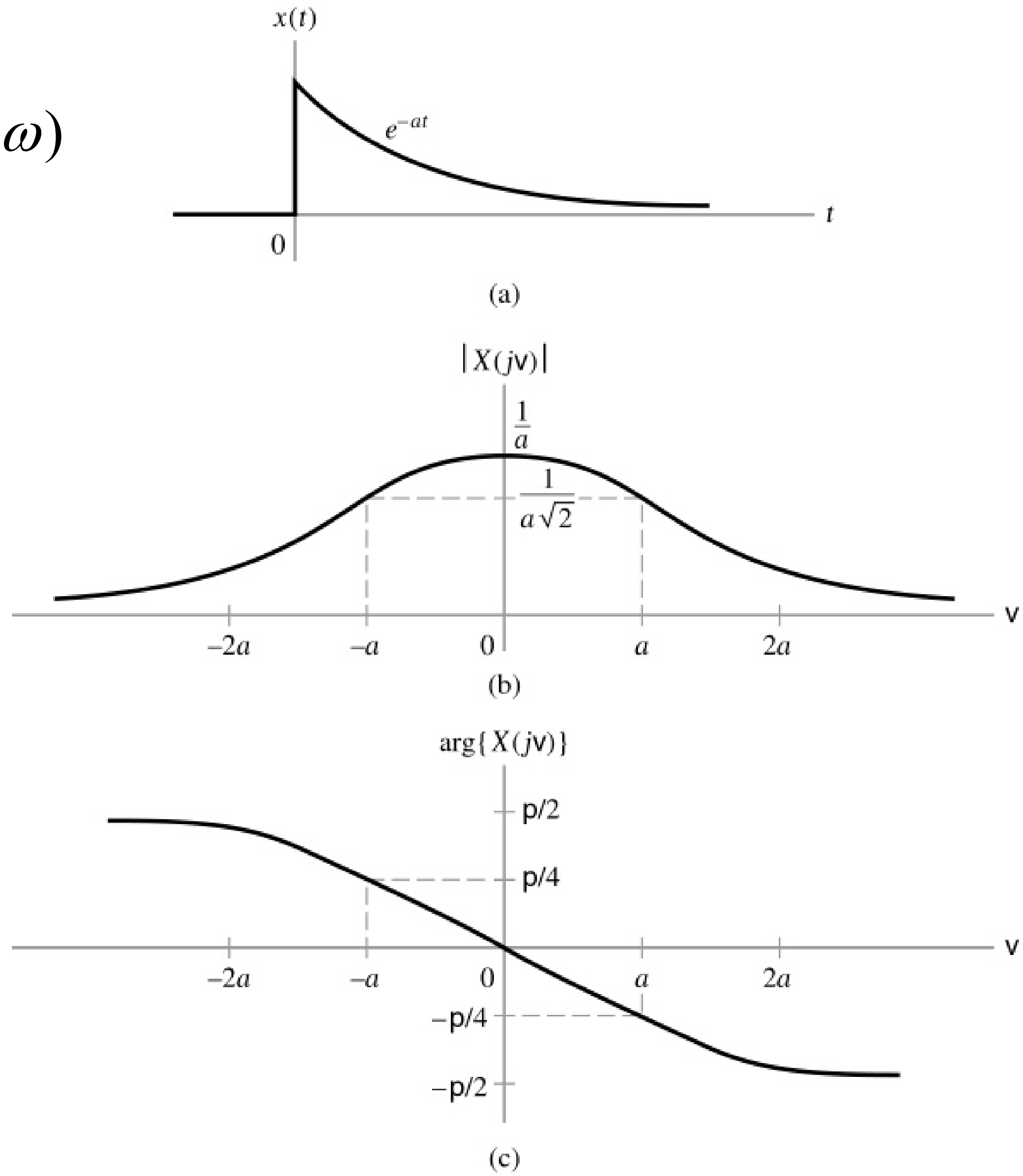
For a>0, we have

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}u(t)dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega}$$



E

Rectangular pulse: $$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & |t| > T_0 \end{cases}$$

x(t) is absolutely integrable for $T_0 < \infty$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T_0}^{T_0} e^{-j\omega t} dt = -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-T_0}^{T_0}$$

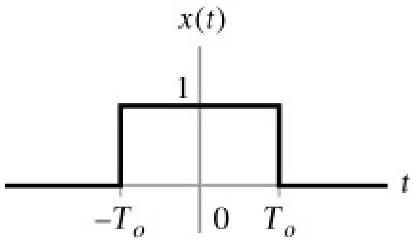
$$= -\frac{1}{j\omega}\left[e^{-j\omega T_0} - e^{j\omega T_0} \right]$$

$$= \frac{2}{\omega} \sin(\omega T_0) \qquad \text{for} \quad \omega \neq 0$$

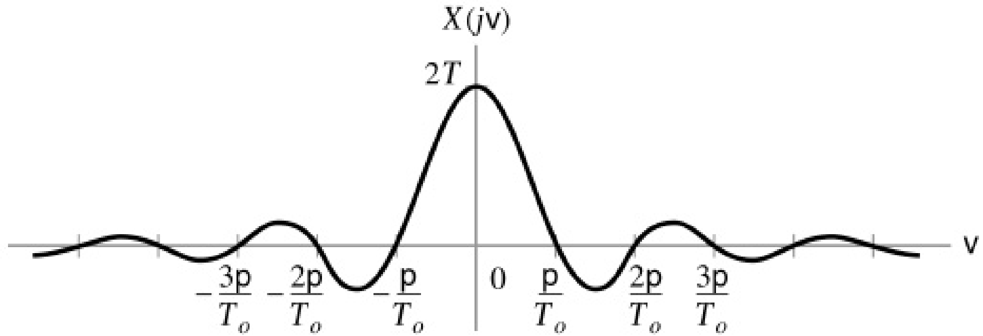
$$= \frac{1}{\pi f} \sin(2\pi f T_0)$$

$$= 2T_0 \frac{\sin(\pi 2 f T_0)}{\pi 2 f T_0}$$

$$= 2T_0 \operatorname{sinc}(2 f T_0)$$



(a)



(b)

Note: a) 1st zero-crossing point of $X(j\omega):\quad f=\frac{1}{2T_0}$ or $\omega=\frac{\pi}{T_0}$

b) $T_0\downarrow\Rightarrow$ 1st zero-crossing point $\uparrow$

$T_0\uparrow\Rightarrow$ 1st zero-crossing point $\downarrow$

E

Inverse FT of a rectangular spectrum:

$$X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega$$

$$= \frac{1}{\pi t}\sin(Wt)$$

$$= \frac{W}{\pi}\frac{\sin(\pi\frac{Wt}{\pi})}{\pi\frac{Wt}{\pi}}$$

$$= \frac{W}{\pi}\mathrm{sinc}(\frac{Wt}{\pi})$$

$$\mathrm{x(t)} = rect\left(\frac{1}{2T_0}\right)$$

Compare $\updownarrow$

$$X(j\omega) = 2T_0 \sin c(2fT_0) = 2T_0\mathrm{sinc}\left(\frac{\omega T_0}{\pi}\right)$$

E

Unit impulse: $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$
$$= 1$$

$$\boxed{\delta(t) \xleftrightarrow{FT} 1}$$

E

Inverse FT of an impulse spectrum: $X(j\omega) = 2\pi\delta(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega = 1$$

$$\boxed{1 \xleftrightarrow{FT} 2\pi\delta(\omega)}$$