FT Representation of DT Signals:

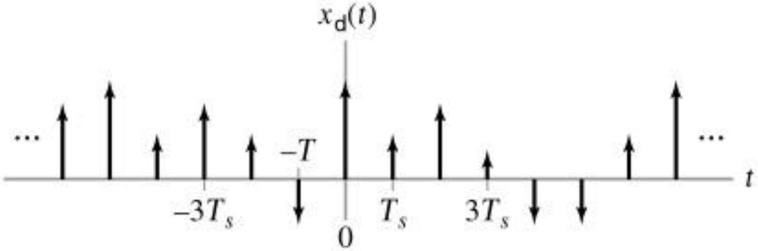
- Relating FT to DTFT

$$x_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

$$\boxed{x[n] = x(t)\,|_{t=nT_s}}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT_s) \leftarrow \text{CT representation of } x[n]$$



$$p(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

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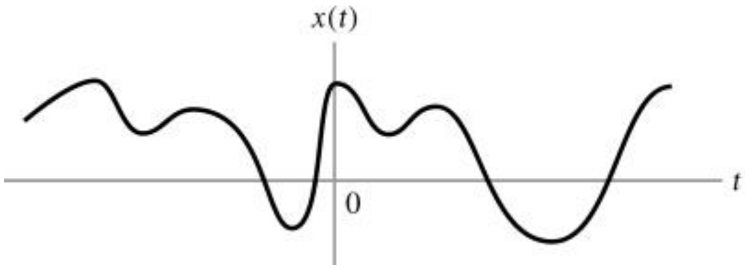

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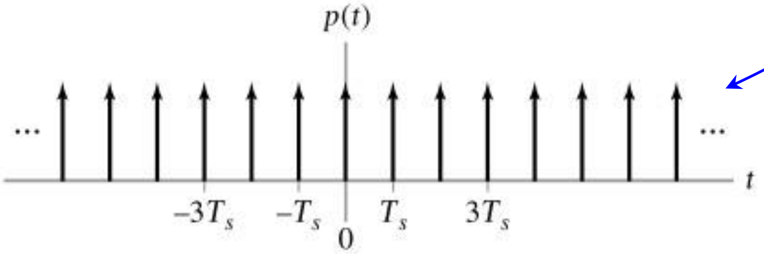


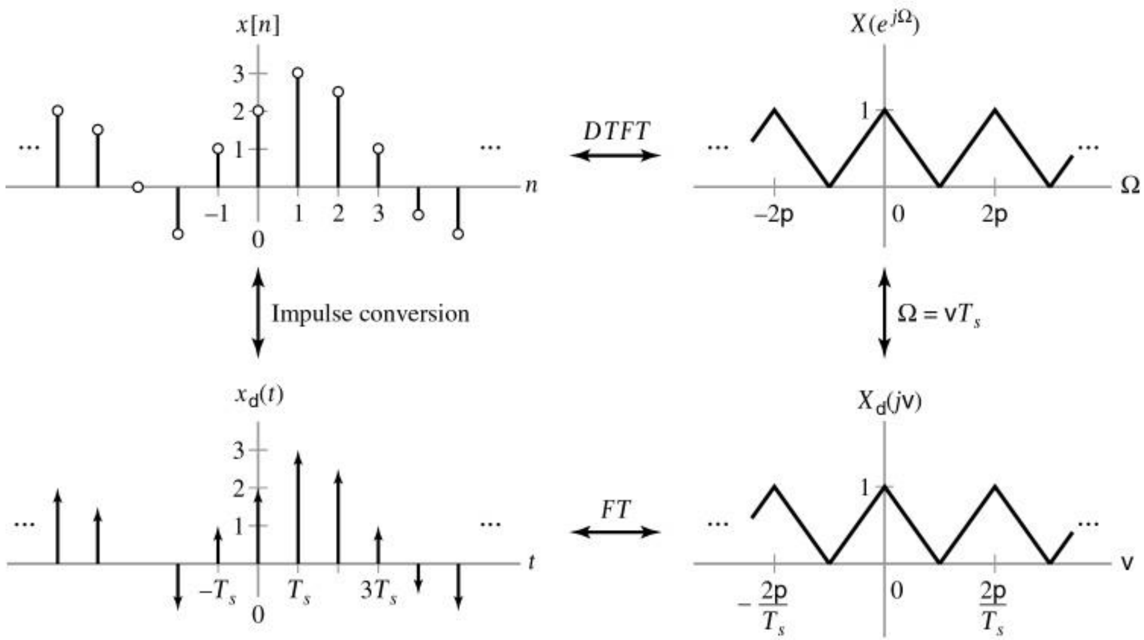
a) DTFT of x[n]: $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

b) FT of CT signal $x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$

$$X_\delta(j\omega) = \int_{-\infty}^{\infty} x_\delta(t) e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x[n]\delta(t-nT_s) \right] e^{-j\omega t} dt$$

$\Updownarrow$

$$\left.\begin{aligned} X_\delta(j\omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T_s n} \\ X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \end{aligned}\right\} \xrightarrow{\text{compare}} \boxed{\Omega = \omega T}$$



- Sampling. The figure shown 2 slides earlier:

- Continuous-time representation of discrete-time signal x[n]

$$
\begin{aligned}
x_\delta(t) &= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \\
&= x(t)\,p(t)
\end{aligned}
$$

(notice the difference between $x[n]$ & $x(nT_s)$)

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$ - impulse train

$$x_\delta(t)=x(t)\,p(t)$$

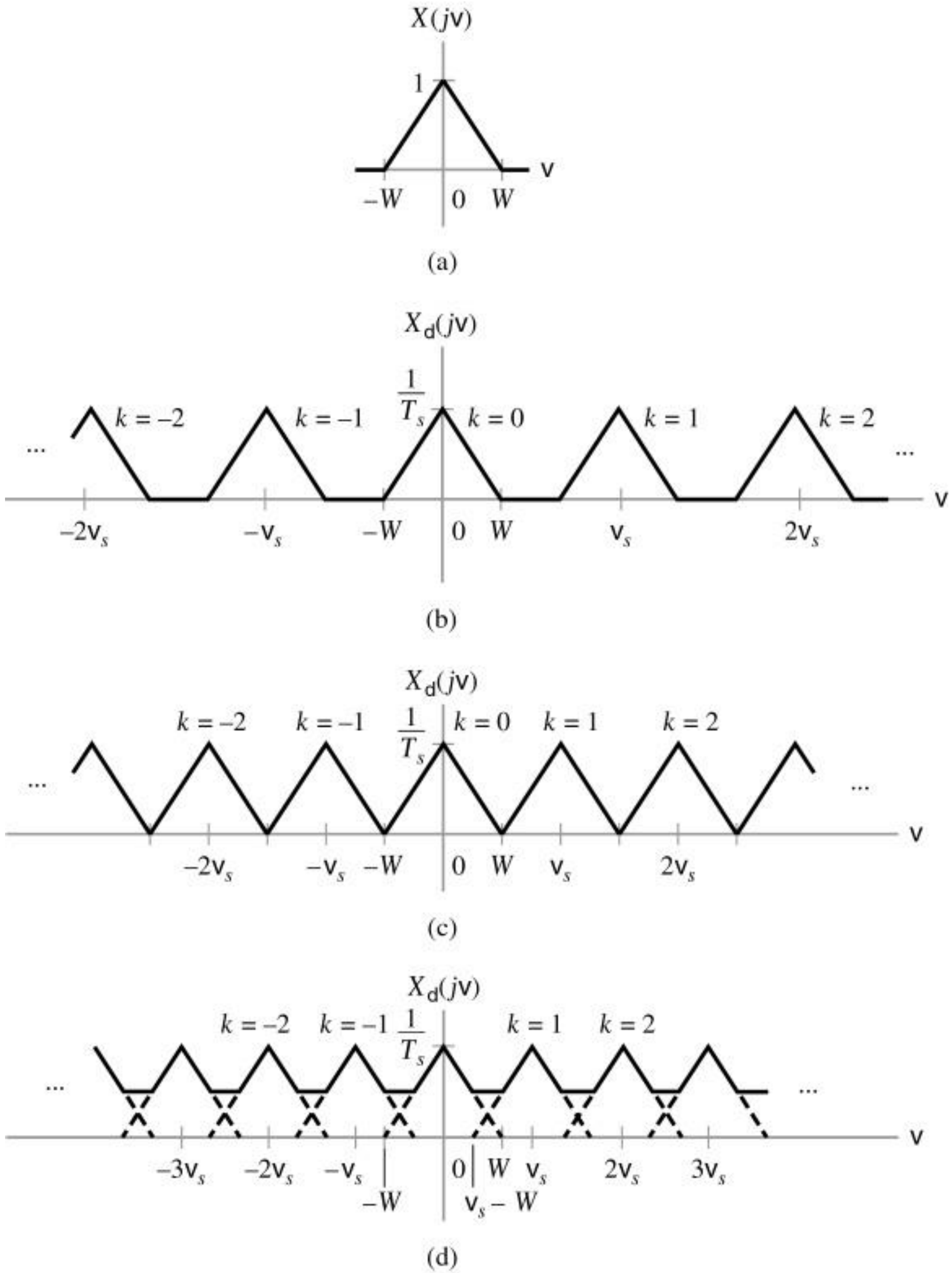
$$\updownarrow$$

$$X_\delta(j\omega)=\frac{1}{2\pi}X(j\omega)*P(j\omega)$$

$$=\frac{1}{2\pi}X(j\omega)*\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$

$$\boxed{X_\delta(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X(j(\omega-k\omega_s))}$$

The FT of a sampled signal for different sampling frequencies. Spectrum of continuous-time signal. Spectrum of sampled signal when ω_s = 3W. Spectrum of sampled signal when ω_s = 2W. (d) Spectrum of sampled signal when ω_s = 1.5W.



Observations:

1) FT of a sampled signal: x(jω) shifted by integer multiples of ω_s

2) Let $X(j\omega) = 0$ for $|\omega| > W$

$$\omega_s > 2W \rightarrow \text{no overlapping}$$

$$\omega_s = 2W \rightarrow \text{critical frequency (Nyquist rate)}$$

$$\omega_s < 2W \rightarrow \text{aliasing}$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{FT} P(j\omega) = ?$$

- $1 \xleftrightarrow{FT} 2\pi\delta(\omega)$ (p342)
- $e^{jk\omega_s t} \xleftrightarrow{FT} 2\pi\delta(\omega - k\omega_s)$
- FS representation of periodic signals

$$x(t) = \sum_{n=-\infty}^{\infty} X[k]e^{jk\omega_s t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_s)$$

- For $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$, which is periodic with period T_s,

$\omega_s = \frac{2\pi}{T_s}$, FT coefficients :

$$P[k] = \frac{1}{T_s} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}. \quad \text{Thus,}$$

$$P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{T_s} \delta(\omega - k\frac{2\pi}{T_s})$$

$$= \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

DTFT of sampled signal x[n] and FT of $x_\delta(t)$

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = X_\delta(j\omega)\Big|_{\omega=\frac{\Omega}{T_s}}$$

Note : $X(e^{j\Omega})$ is periodic with period 2π. $\qquad \omega_s = 2\pi f_s = 2\pi \frac{1}{T_s}$

$X_\delta(j\omega)$ is periodic with period $\omega_s = 2\pi f_s$

$\boxed{\text{E}}$

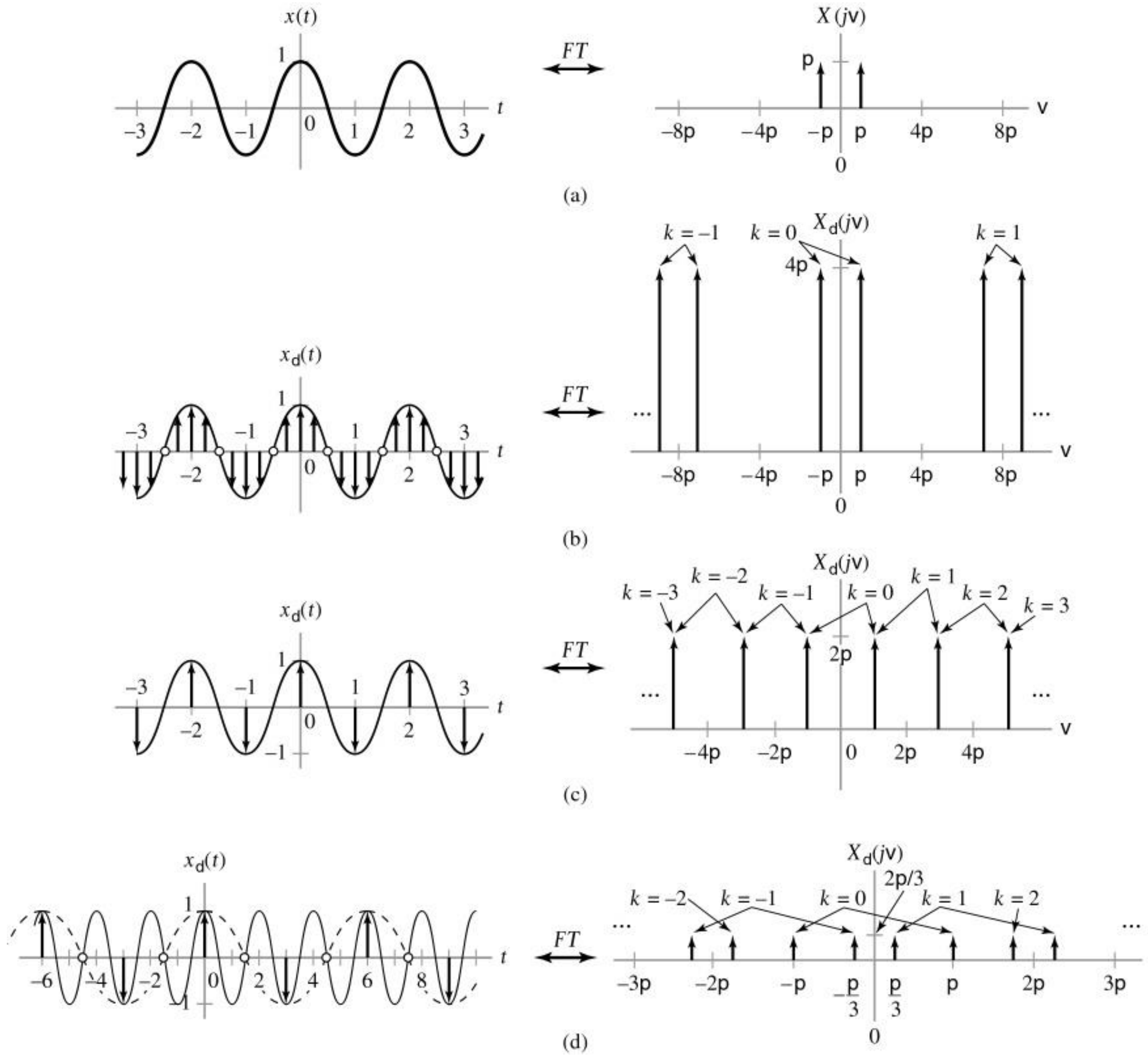
$x(t) = \cos(\pi t)$.

Find FT of $x_\delta(t)$ if (i) $T_s = 1/4$, (ii) $T_s = 1$, (iii) $T_s = 3/2$

$$x(t) \xleftrightarrow{\text{FT}} \pi\delta(\omega+\pi) + \pi\delta(\omega-\pi)$$

$$X_\delta(j\omega) = \frac{\pi}{T_s} \sum_{k=-\infty}^{\infty} \left[\delta\left(\omega+\pi-k\omega_s\right) + \delta\left(\omega-\pi-k\omega_s\right)\right]$$

The effect of sampling a sinusoid at different rates
(a) Original signal and FT.
(b) Original signal, impulse sampled representation and FT for $T_s = 1/4$.
(c) Original signal, impulse sampled representation and FT for $T_s = 1$.
(d) Original signal, impulse sampled representation and FT for $T_s = 3/2$. A cosine of frequency $\pi/3$ is shown as the dashed line.



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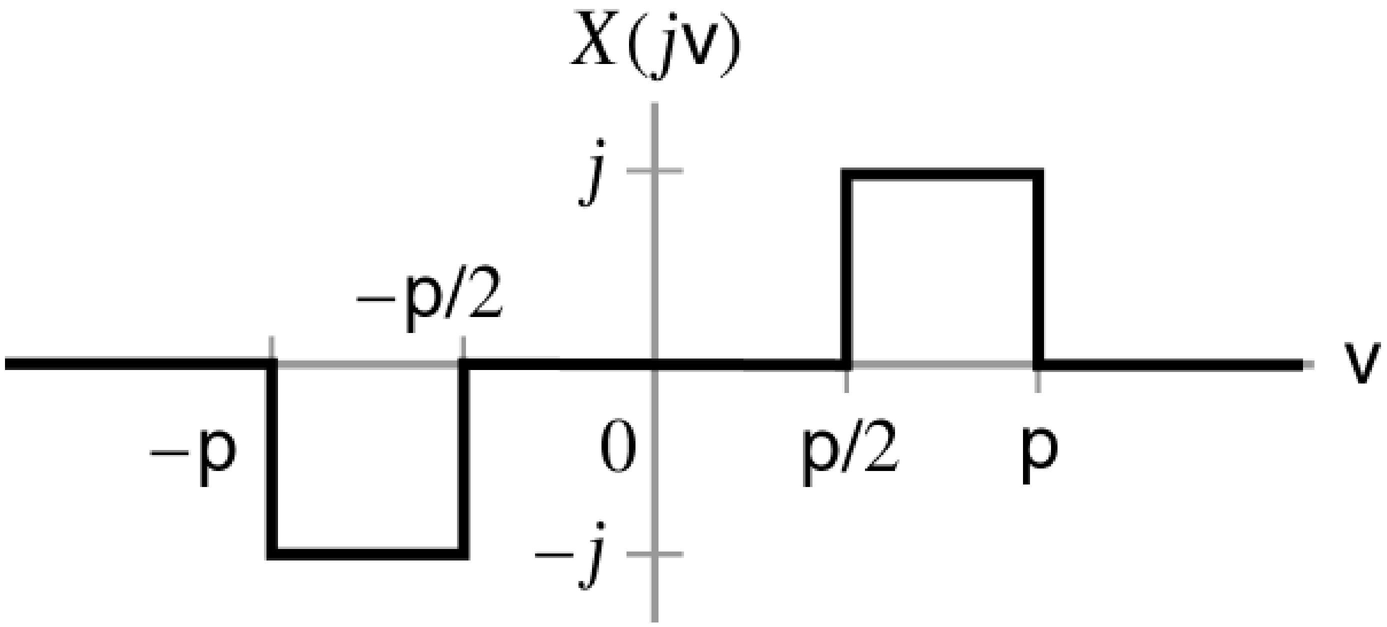
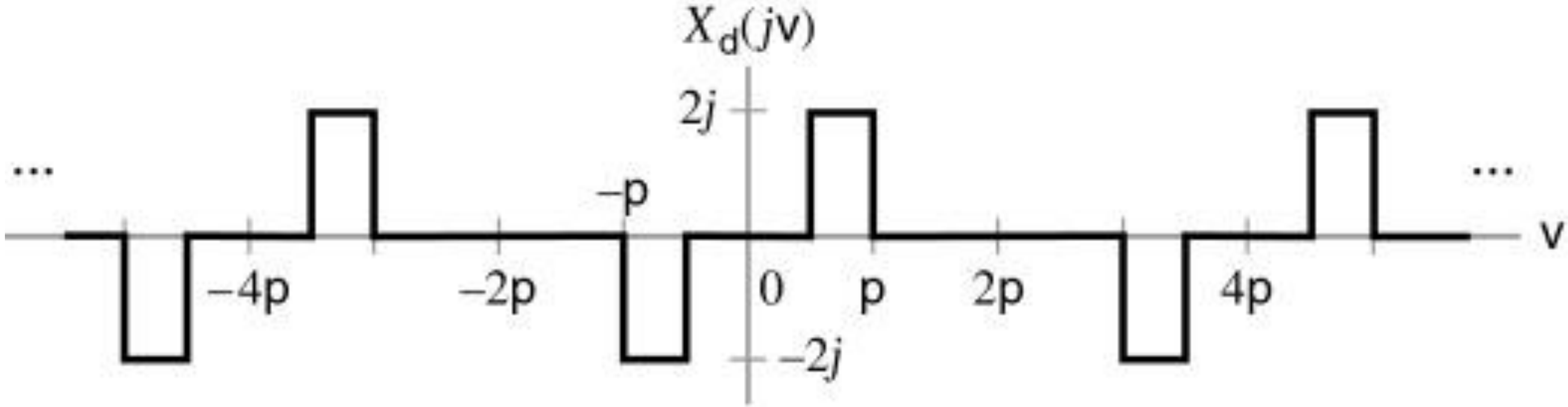
Draw the FT of a sampled version of the CT signal having the FT depicted By the following figure for (a) Ts=1/2 and (b) Ts=2.



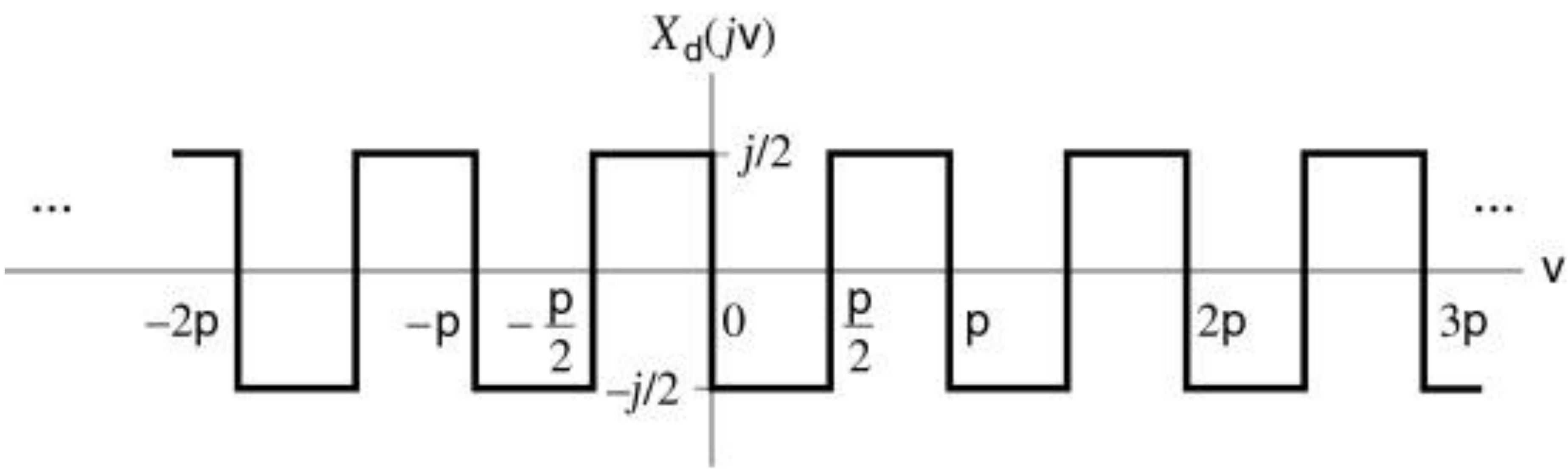
(a) Ts=1/2, $\omega_s=4\pi$.

(b) Ts=2, $\omega_s=\pi$.



$X_d(jv)$
2j
−2j
...
−p
−4p
−2p
0
p
2p
4p
v

(a)



$X_d(jv)$
j/2
−j/2
...
−2p
−p
$-\frac{p}{2}$
0
$\frac{p}{2}$
p
2p
3p
v

(b)

E Downsampling: Let $y[n] = x[qn]$

How is $Y\left(e^{j\Omega}\right)$ related to $X\left(e^{j\Omega}\right)$?

Let both x[n] and y[n] be obtained from sampling signal x(t).

$$x[n]: \qquad \Delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$y[n]: \qquad \Delta^{'}(t) = \sum_{n=-\infty}^{\infty} \delta(t - qnT_s)$$

$$= \sum_{n=-\infty}^{\infty} \delta(t - nT_s^{'}), \quad \text{where } T_s^{'} = qT_s$$

$$x_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$

$$y_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s^{'}), \quad T_s^{'} = qT_s \Rightarrow \omega_s^{'} = \tfrac{1}{q}\omega$$

$$\updownarrow$$

$$Y_\delta(j\omega) = \frac{1}{T_s^{'}}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s^{'})\right) = \frac{1}{qT_s}\sum_{k=-\infty}^{\infty} X\left(j(\omega - \tfrac{k}{q}\omega_s)\right)$$

Let $\frac{k}{q} = l + \frac{m}{q}$, $\quad l : (l : -\infty \sim \infty)$ integer portion of $\frac{k}{q}$,

m : remainder $(m : 0 \sim q-1)$

$$Y_\delta(j\omega) = \frac{1}{q}\sum_{m=0}^{q-1}\left\{\frac{1}{T_s}\sum_{l=-\infty}^{\infty} X\left(j(\omega - l\omega_s - \tfrac{m}{q}\omega_s)\right)\right\}.$$ Obviously,

$$\frac{1}{T_s}\sum_{l=-\infty}^{\infty} X\left(j(\omega - l\omega_s - \tfrac{m}{q}\omega_s)\right) = X_\delta\left(j(\omega - \tfrac{m}{q}\omega_s)\right)$$

Thus, $Y_\delta(j\omega) = \dfrac{1}{q}\sum_{m=0}^{q-1} X_\delta\left(j(\omega - \tfrac{m}{q}\omega_s)\right)$. Obviously,

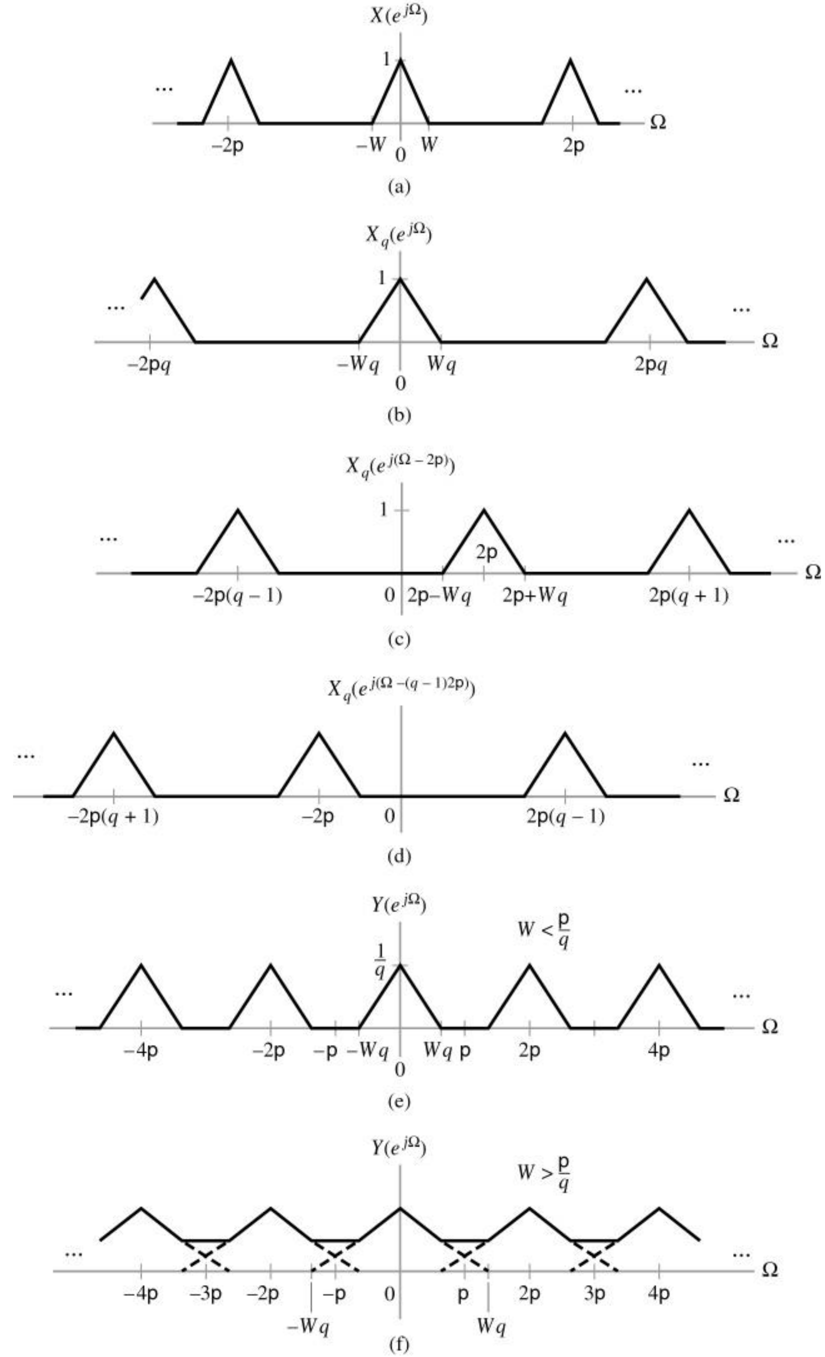
$$\frac{1}{T_s}\sum_{l=-\infty}^{\infty} X\left(j(\omega - l\omega_s - \tfrac{m}{q}\omega_s)\right) = X_\delta\left(j(\omega - \tfrac{m}{q}\omega_s)\right) \Rightarrow$$

$$\begin{cases} Y\left(e^{j\Omega}\right) = Y_\delta\left(j\omega\right)|_{\omega=(\Omega/(qT_s))} \\ X\left(e^{j\Omega}\right) = Y_\delta\left(j\omega\right)|_{\omega=(\Omega/T_s)} \end{cases} \Rightarrow$$

$$\boxed{Y\left(e^{j\Omega}\right) = \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)}$$

Note : sampling rate for y[n] is $1/(qT_s)$ and $\omega_s = 2\pi/T_s$

Effect of subsampling on the DTFT.
(a) Original signal spectrum.
(b) m = 0 term, $Xq(e^{j\Omega})$,
(c) m = 1 term
(d $m = q – 1$ term
(e) $Y(e^{j\Omega})$, assuming that $W < \pi/q$.
(f) $Y(e^{j\Omega})$, assuming that $W > \pi/q$.



Sampling theorem

Let $x(t) \xleftrightarrow{FT} X(j\omega)$ be a bandwidth limited signal such that $X(j\omega) = 0$ for $|\omega| > \omega_m$

$\omega_s = 2\omega_m$: Nyquist - rate sampling

$\omega_s > 2\omega_m$: No aliasing, original signal can be completely recovered from $x[n]$

$\omega_s < 2\omega_m$: aliasing

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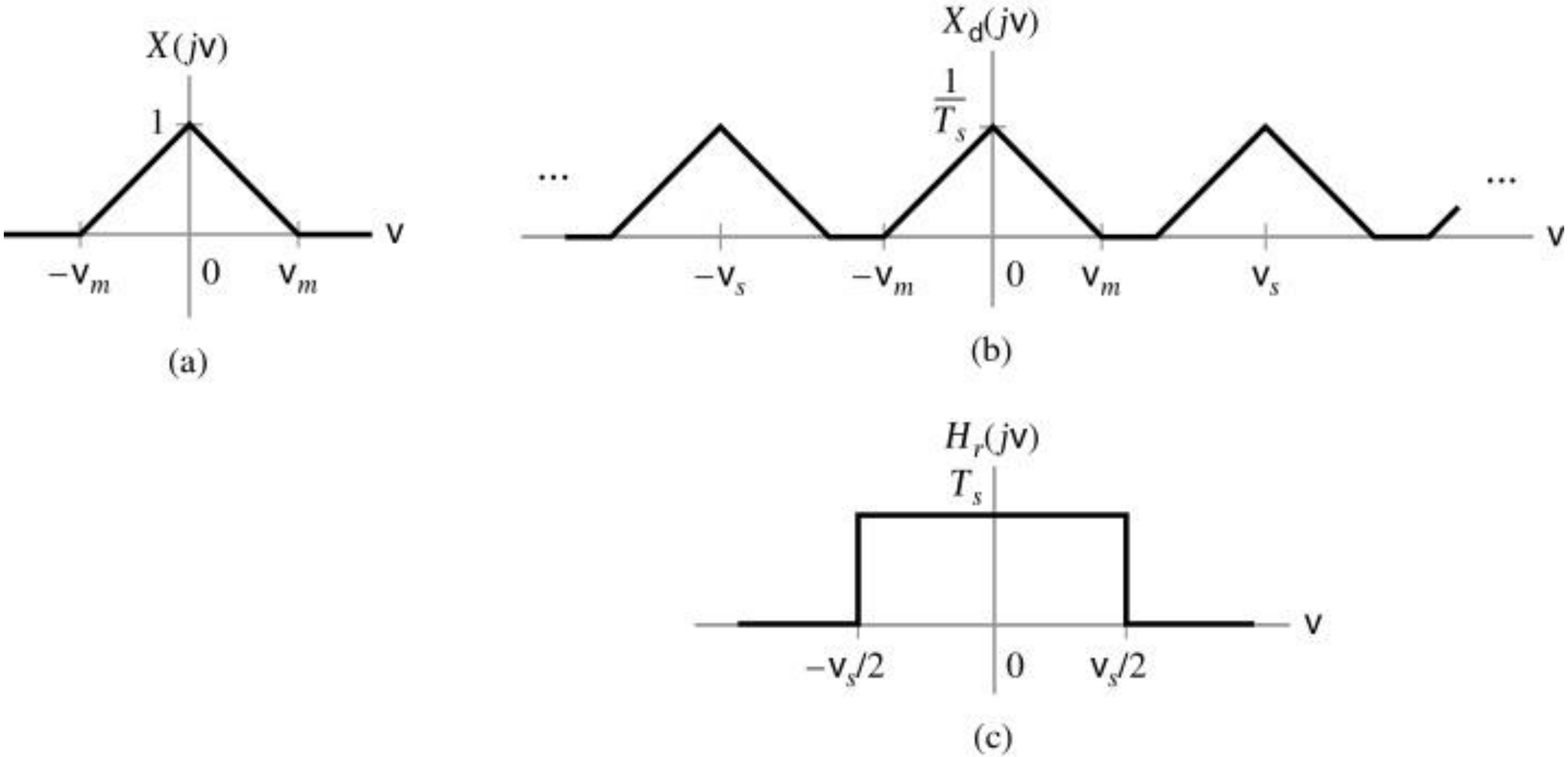
Suppose $x(t) = \dfrac{\sin(10\pi t)}{\pi t}$. Determine $\omega_s = 2\pi/(T_s)$ so that $x(t)$ can be uniquely represented by the DT sequence $x[n] = x(nT_s)$.

$$x(t) \xleftrightarrow{FT} X(j\omega) = \begin{cases} 1, & |\omega| \le 10\pi \\ 0, & |\omega| > 10\pi \end{cases}$$

Thus, $\omega_m = 10\pi$, minimum $\omega_s = 2\omega_m = 20\pi$ rads/s

Ideal reconstruction:



(a) Spectrum of original signal.
Spectrum of sampled signal.
(c) Frequency response of reconstruction filter.

$$H(j\omega) = \begin{cases} T_{s,} & |\omega| \le \omega_s / 2 \\ 0 & |\omega| > \omega_s / 2 \end{cases}$$

$$\updownarrow$$

$$h(t) = \frac{T_s \sin\left(\frac{\omega_s}{2} t\right)}{\pi t}$$

$$\boxed{\frac{\sin(Wt)}{Wt} = \frac{W}{\pi} \sin c\left(\frac{Wt}{\pi}\right)}$$

$$X(j\omega) = X_\delta(j\omega) H(j\omega)$$

$$\updownarrow$$

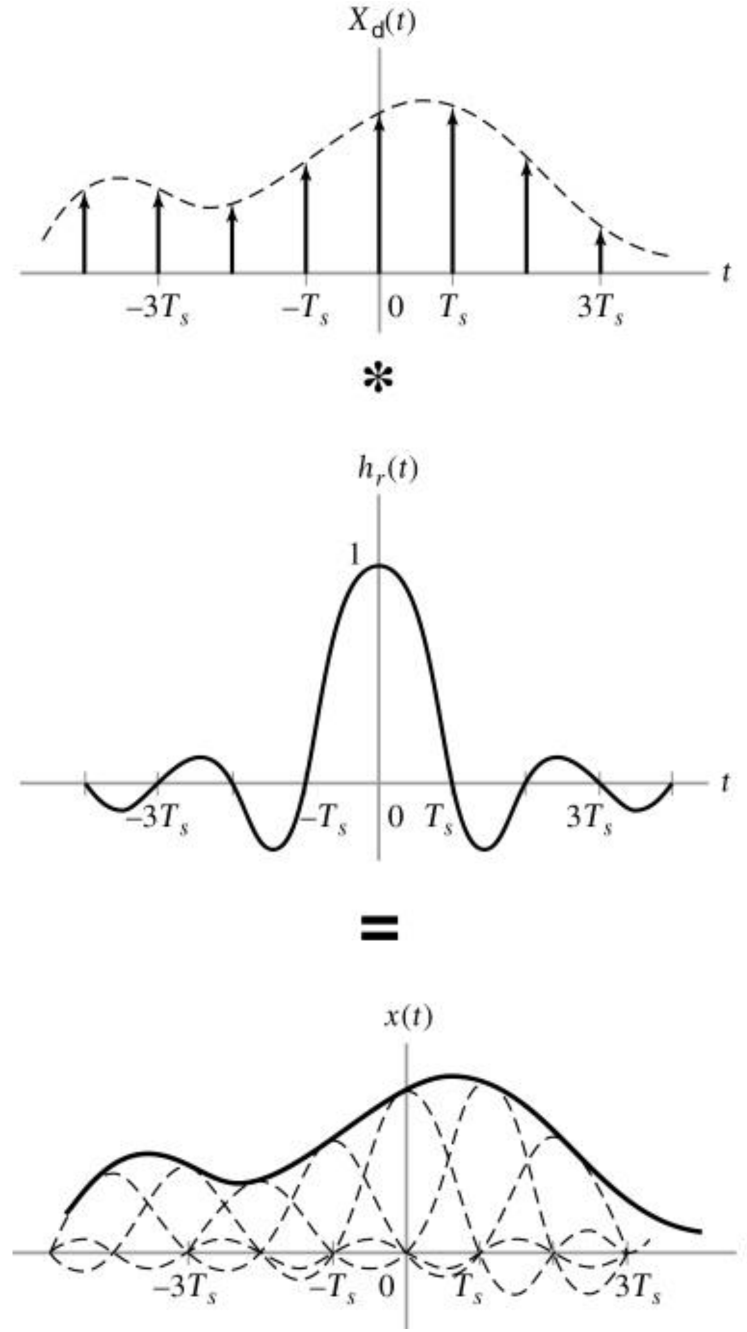
$$x(t) = x_\delta(t) * h(t)$$

$$= h(t) * \sum_{n=-\infty}^{\infty} x[n]\delta(t - bT_s)$$

$$= \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \sin c(\omega_s (t - nT_s)/(2\pi))$$

Ideal reconstruction in the time domain.



- **Ideal reconstruction is not realizable**
- **Practical systems could use a zero-order hold block**
- **This distorts signal spectrum, and compensation is needed**

Reconstruction via a zero-order hold.