ECE351: Signals and Systems I - Fall 2018 - Dr. Thinh Nguyen
Final Examination

Name:
Student ID:

Instruction: Please write your work clearly. Credits will not be given to the correct answers without proper derivations. You are allowed a 2-sided 8.5x11" sheet of notes. No calculator is allowed. Your answers should not contain the symbols for integration or sum. You have 110 minutes to do the exam.

1. A system is characterized by the following input-output relationship:

$$y(t) = \int_{t-2}^{t+1} (t-\tau)^2 x(\tau) d\tau$$

   (a) Show that the system is an LTI system (10 pts)
   (b) Determine the impulse response $h(t)$ of the system. (4 pts)
   (c) Is $h(t)$ BIBO stable? Causal? Justify your answers. (6 pts)

2. Let $x[n] = u[n-1]$ and $h[n] = 2^n(u[n] - u[n-6])$.

   (a) Sketch $h[k]$, $x[k]$, $x[n-k]$ and carefully label the values on the axes. (6 pts)
   (b) Determine $y[n] = x[n] * h[n]$ by performing **graphical convolution**. No need to sketch $y[n]$. (14 pts)

3. Using either the definition or inspection method,

   (a) Compute $X[k]$ for $x[n] = \cos^2(\frac{2\pi n}{3}) + e^{\frac{j\pi n}{2}}$ (15 pts)
   (b) Compute $X(j\omega)$ for $x(t) = \sum_{i=-\infty}^{\infty} 2^{-|t|}\delta(t-i)$ (15 pts)

4. Let

$$x(t) = \left(\frac{d(\cos(\pi t)e^{-|t|})}{dt}\right) * e^{-\frac{t}{2}}u(t-2) \overset{FT}{\longleftrightarrow} X(j\omega), \tag{1}$$

use the properties of FT and the well-known FT pairs to find $X(j\omega)$ (15 pts).

5. You are interested in studying a unique periodic extraterresial (ET) signal. This ET signal has all its energy in the two following frequency ranges: 100MHz - 110MHz and 900MHz - 930MHz. However, the signal you observe, is the sum of the ET signal and other signals such as TV and radio transmissions. To obtain a clean ET signal, you want to design an LTI system to filter out the unwanted signals.

   (a) Sketch an ideal frequency response $H(j\omega)$ of an LTI system that allows signals whose energies are in the frequency ranges 100MHz-110MHz and 900MHz-930MHz to pass through unchanged while other signals are zeroed out. (5 pts)
   (b) Your friend show you a neat way to implement a subsystem with the impulse response:

$$s_W(t) = sinc(\frac{Wt}{\pi}) \overset{FT}{\longleftrightarrow} S_W(j\omega) = \begin{cases} 1, & \omega < |W| \\ 0, & \text{otherwise,} \end{cases}$$

   for any value of W. Can you write the frequency response $H(j\omega)$ in part (a) as a linear combination of shifted versions of $S_W(j\omega)$ with different values of W? (5 pts)
   (c) Determine the impulse response $h(t)$ of the system in part (a) in terms of $s_W(t)$. You can perform any mathematical operations on $s_W(t)$ (such as multiplying or adding $s_W(t)$ with any function) to obtain $h(t)$. (5 pts)

C Tables of Fourier Representations and Properties

C.1 *Basic Discrete-Time Fourier Series Pairs*

Time Domain	*Frequency Domain*
$x[n] = \sum_{k=0}^{N-1} X[k] e^{jkn\Omega_o}$ *Period* $= N$	$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jkn\Omega_o}$ $\Omega_o = \frac{2\pi}{N}$
$x[n] = \begin{cases} 1, & \lvert n\rvert \le M \\ 0, & M < \lvert n\rvert \le N/2 \end{cases}$ $x[n] = x[n+N]$	$X[k] = \dfrac{\sin\left(k\frac{\Omega_o}{2}(2M+1)\right)}{N\sin\left(k\frac{\Omega_o}{2}\right)}$
$x[n] = e^{jp\Omega_o n}$	$X[k] = \begin{cases} 1, & k = p, p \pm N, p \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \cos(p\Omega_o n)$	$X[k] = \begin{cases} \frac{1}{2}, & k = \pm p, \pm p \pm N, \pm p \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \sin(p\Omega_o n)$	$X[k] = \begin{cases} \frac{1}{2j}, & k = p, p \pm N, p \pm 2N, \ldots \\ \frac{-1}{2j}, & k = -p, -p \pm N, -p \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = 1$	$X[k] = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \sum_{p=-\infty}^{\infty} \delta[n - pN]$	$X[k] = \frac{1}{N}$

C.2 *Basic Fourier Series Pairs*

Time Domain	*Frequency Domain*
$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ *Period* $= T$	$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$ $\omega_o = \frac{2\pi}{T}$
$x(t) = \begin{cases} 1, & \lvert t\rvert \le T_o \\ 0, & T_o < \lvert t\rvert \le T/2 \end{cases}$	$X[k] = \frac{\sin(k\omega_o T_o)}{k\pi}$
$x(t) = e^{jp\omega_o t}$	$X[k] = \delta[k-p]$
$x(t) = \cos(p\omega_o t)$	$X[k] = \frac{1}{2}\delta[k-p] + \frac{1}{2}\delta[k+p]$
$x(t) = \sin(p\omega_o t)$	$X[k] = \frac{1}{2j}\delta[k-p] - \frac{1}{2j}\delta[k+p]$
$x(t) = \sum_{p=-\infty}^{\infty}\delta(t-pT)$	$X[k] = \frac{1}{T}$

C.3 *Basic Discrete-Time Fourier Transform Pairs*

Time Domain	*Frequency Domain*
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$	$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
$x[n] = \begin{cases} 1, & \lvert n\rvert \le M \\ 0, & \text{otherwise} \end{cases}$	$X(e^{j\Omega}) = \dfrac{\sin\left[\Omega\left(\frac{2M+1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$
$x[n] = \alpha^n u[n], \quad \lvert\alpha\rvert < 1$	$X(e^{j\Omega}) = \frac{1}{1-\alpha e^{-j\Omega}}$
$x[n] = \delta[n]$	$X(e^{j\Omega}) = 1$
$x[n] = u[n]$	$X(e^{j\Omega}) = \frac{1}{1-e^{-j\Omega}} + \pi\sum_{p=-\infty}^{\infty}\delta(\Omega - 2\pi p)$
$x[n] = \frac{1}{\pi n}\sin(Wn), \quad 0 < W \le \pi$	$X(e^{j\Omega}) = \begin{cases} 1, & \lvert\Omega\rvert \le W \\ 0, & W < \lvert\Omega\rvert \le \pi \end{cases}$ $\quad X(e^{j\Omega})$ is 2π periodic
$x[n] = (n+1)\alpha^n u[n]$	$X(e^{j\Omega}) = \frac{1}{(1-\alpha e^{-j\Omega})^2}$

C.4 Basic Fourier Transform Pairs

Time Domain	Frequency Domain
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$
$x(t) = \begin{cases} 1, & \lvert t\rvert \le T_o \\ 0, & \text{otherwise} \end{cases}$	$X(j\omega) = \frac{2\sin(\omega T_o)}{\omega}$
$x(t) = \frac{1}{\pi t}\sin(Wt)$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert \le W \\ 0, & \text{otherwise} \end{cases}$
$x(t) = \delta(t)$	$X(j\omega) = 1$
$x(t) = 1$	$X(j\omega) = 2\pi\delta(\omega)$
$x(t) = u(t)$	$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$
$x(t) = e^{-at}u(t), \quad \mathrm{Re}\{a\} > 0$	$X(j\omega) = \frac{1}{a + j\omega}$
$x(t) = te^{-at}u(t), \quad \mathrm{Re}\{a\} > 0$	$X(j\omega) = \frac{1}{(a + j\omega)^2}$
$x(t) = e^{-a\lvert t\rvert}, \quad a > 0$	$X(j\omega) = \frac{2a}{a^2 + \omega^2}$
$x(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$	$X(j\omega) = e^{-\omega^2/2}$

C.5 Fourier Transform Pairs for Periodic Signals

Periodic Time-Domain Signal	Fourier Transform
$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$	$X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_o)$
$x(t) = \cos(\omega_o t)$	$X(j\omega) = \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$
$x(t) = \sin(\omega_o t)$	$X(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_o) - \frac{\pi}{j}\delta(\omega + \omega_o)$
$x(t) = e^{j\omega_o t}$	$X(j\omega) = 2\pi\delta(\omega - \omega_o)$
$x(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_s)$	$X(j\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T_s}\right)$
$x(t) = \begin{cases} 1, & \lvert t\rvert \le T_o \\ 0, & T_o < \lvert t\rvert < T/2 \end{cases}$ $x(t + T) = x(t)$	$X(j\omega) = \sum_{k=-\infty}^{\infty}\frac{2\sin(k\omega_o T_o)}{k}\delta(\omega - k\omega_o)$

C.6 *Discrete-Time Fourier Transform Pairs for Periodic Signals*

Periodic Time-Domain Signal	*Discrete-Time Fourier Transform*
$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}$	$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o)$
$x[n] = \cos(\Omega_1 n)$	$X(e^{j\Omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi) + \delta(\Omega + \Omega_1 - k2\pi)$
$x[n] = \sin(\Omega_1 n)$	$X(e^{j\Omega}) = \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi) - \delta(\Omega + \Omega_1 - k2\pi)$
$x[n] = e^{j\Omega_1 n}$	$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi)$
$x[n] = \sum_{k=-\infty}^{\infty} \delta(n - kN)$	$X(e^{j\Omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{k2\pi}{N}\right)$

C.7 Properties of Fourier Representations

Property	Fourier Transform $x(t) \xleftrightarrow{FT} X(j\omega)$ $y(t) \xleftrightarrow{FT} Y(j\omega)$	Fourier Series $x(t) \xleftrightarrow{FS;\omega_o} X[k]$ $y(t) \xleftrightarrow{FS;\omega_o} Y[k]$ Period $= T$
Linearity	$ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$	$ax(t) + by(t) \xleftrightarrow{FS;\omega_o} aX[k] + bY[k]$
Time shift	$x(t - t_o) \xleftrightarrow{FT} e^{-j\omega t_o}X(j\omega)$	$x(t - t_o) \xleftrightarrow{FS;\omega_o} e^{-jk\omega_o t_o}X[k]$
Frequency shift	$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega - \gamma))$	$e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\omega_o} X[k - k_o]$
Scaling	$x(at) \xleftrightarrow{FT} \frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$	$x(at) \xleftrightarrow{FS;a\omega_o} X[k]$
Differentiation in time	$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$	$\frac{d}{dt}x(t) \xleftrightarrow{FS;\omega_o} jk\omega_o X[k]$
Differentiation in frequency	$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$	—
Integration/ Summation	$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$	—
Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau \xleftrightarrow{FT} X(j\omega)Y(j\omega)$	$\int_{0}^{T} x(\tau)y(t-\tau)\,d\tau \xleftrightarrow{FS;\omega_o} TX[k]Y[k]$
Multiplication	$x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\nu)Y(j(\omega-\nu))\,d\nu$	$x(t)y(t) \xleftrightarrow{FS;\omega_o} \sum_{l=-\infty}^{\infty} X[l]Y[k-l]$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$	$\frac{1}{T}\int_{0}^{T} \lvert x(t)\rvert^2\,dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$
Duality	$X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$	$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ $X(e^{jt}) \xleftrightarrow{FS;1} x[-k]$
Symmetry	$x(t)$ real $\xleftrightarrow{FT} X^*(j\omega) = X(-j\omega)$ $x(t)$ imaginary $\xleftrightarrow{FT} X^*(j\omega) = -X(-j\omega)$ $x(t)$ real and even $\xleftrightarrow{FT} \text{Im}\{X(j\omega)\} = 0$ $x(t)$ real and odd $\xleftrightarrow{FT} \text{Re}\{X(j\omega)\} = 0$	$x(t)$ real $\xleftrightarrow{FS;\omega_o} X^*[k] = X[-k]$ $x(t)$ imaginary $\xleftrightarrow{FS;\omega_o} X^*[k] = -X[-k]$ $x(t)$ real and even $\xleftrightarrow{FS;\omega_o} \text{Im}\{X[k]\} = 0$ $x(t)$ real and odd $\xleftrightarrow{FS;\omega_o} \text{Re}\{X[k]\} = 0$

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C.7 *(continued)*

Property	Discrete-Time FT $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ $y[n] \xleftrightarrow{DTFT} Y(e^{j\Omega})$	Discrete-Time FS $x[n] \xleftrightarrow{DTFS;\,\Omega_o} X[k]$ $y[n] \xleftrightarrow{DTFS;\,\Omega_o} Y[k]$ Period $= N$
Linearity	$ax[n] + by[n] \xleftrightarrow{DTFT} aX(e^{j\Omega}) + bY(e^{j\Omega})$	$ax[n] + by[n] \xleftrightarrow{DTFS;\,\Omega_o} aX[k] + bY[k]$
Time shift	$x[n - n_o] \xleftrightarrow{DTFT} e^{-j\Omega n_o}X(e^{j\Omega})$	$x[n - n_o] \xleftrightarrow{DTFS;\,\Omega_o} e^{-jk\Omega_o n_o}X[k]$
Frequency shift	$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$	$e^{jk_o\Omega_o n}x[n] \xleftrightarrow{DTFS;\,\Omega_o} X[k - k_o]$
Scaling	$x_z[n] = 0, \quad n \neq 0, \pm p, \pm 2p, \pm 3p, \ldots$ $x_z[pn] \xleftrightarrow{DTFT} X_z(e^{j\Omega/p})$	$x_z[n] = 0, \quad n \neq 0, \pm p, \pm 2p, \pm 3p, \ldots$ $x_z[pn] \xleftrightarrow{DTFS;\,p\Omega_o} pX_z[k]$
Differentiation in time	—	—
Differentiation in frequency	$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X(e^{j\Omega})$	—
Integration/ Summation	$\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{DTFT} \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$	—
Convolution	$\sum_{l=-\infty}^{\infty} x[l]y[n - l] \xleftrightarrow{DTFT} X(e^{j\Omega})Y(e^{j\Omega})$	$\sum_{l=0}^{N-1} x[l]y[n - l] \xleftrightarrow{DTFS;\,\Omega_o} NX[k]Y[k]$
Multiplication	$x[n]y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Gamma})Y(e^{j(\Omega-\Gamma)})\,d\Gamma$	$x[n]y[n] \xleftrightarrow{DTFS;\,\Omega_o} \sum_{l=0}^{N-1} X[l]Y[k - l]$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\Omega})\rvert^2\,d\Omega$	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert X[k]\rvert^2$
Duality	$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ $X(e^{jt}) \xleftrightarrow{FS;1} x[-k]$	$X[n] \xleftrightarrow{DTFS;\,\Omega_o} \frac{1}{N}x[-k]$
Symmetry	$x[n]$ real $\xleftrightarrow{DTFT} X^*(e^{j\Omega}) = X(e^{-j\Omega})$ $x[n]$ imaginary $\xleftrightarrow{DTFT} X^*(e^{j\Omega}) = -X(e^{-j\Omega})$ $x[n]$ real and even $\xleftrightarrow{DTFT} \mathrm{Im}\{X(e^{j\Omega})\} = 0$ $x[n]$ real and odd $\xleftrightarrow{DTFT} \mathrm{Re}\{X(e^{j\Omega})\} = 0$	$x[n]$ real $\xleftrightarrow{DTFS;\,\Omega_o} X^*[k] = X[-k]$ $x[n]$ imaginary $\xleftrightarrow{DTFS;\,\Omega_o} X^*[k] = -X[-k]$ $x[n]$ real and even $\xleftrightarrow{DTFS;\,\Omega_o} \mathrm{Im}\{X[k]\} = 0$ $x[n]$ real and odd $\xleftrightarrow{DTFS;\,\Omega_o} \mathrm{Re}\{X[k]\} = 0$

C.8 *Relating the Four Fourier Representations*

Let

$$g(t) \xleftrightarrow{FS;\,\omega_o = 2\pi/T} G[k]$$

$$v[n] \xleftrightarrow{DTFT} V(e^{j\Omega})$$

$$w[n] \xleftrightarrow{DTFS;\,\Omega_o = 2\pi/N} W[k]$$

■ C.8.1 FT Representation for a Continuous-Time Periodic Signal

$$g(t) \xleftrightarrow{FT} G(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} G[k]\delta(\omega - k\omega_o)$$

■ C.8.2 DTFT Representation for a Discrete-Time Periodic Signal

$$w[n] \xleftrightarrow{DTFT} W(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} W[k]\delta(\Omega - k\Omega_o)$$

■ C.8.3 FT Representation for a Discrete-Time Nonperiodic Signal

$$v_\delta(t) = \sum_{n=-\infty}^{\infty} v[n]\delta(t - nT_s) \xleftrightarrow{FT} V_\delta(j\omega) = V(e^{j\Omega})\Big|_{\Omega = \omega T_s}$$

■ C.8.4 FT Representation for a Discrete-Time Nonperiodic Signal

$$w_\delta(t) = \sum_{n=-\infty}^{\infty} w[n]\delta(t - nT_s) \xleftrightarrow{FT} W_\delta(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} W[k]\delta\left(\omega - \frac{k\Omega_o}{T_s}\right)$$

C.9 *Sampling and Aliasing Relationships*

Let

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$v[n] \xleftrightarrow{DTFT} V(e^{j\Omega})$$

■ C.9.1 Impulse Sampling for Continuous-Time Signals

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k\frac{2\pi}{T_s}\right)\right)$$

Sampling interval T_s, $X_\delta(j\omega)$ is $2\pi/T_s$ periodic.

■ C.9.2 Sampling a Discrete-Time Signal

$$y[n] = v[qn] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} V(e^{j(\Omega - m2\pi)/q})$$

$Y(e^{j\Omega})$ is 2π periodic.

■ C.9.3 Sampling the DTFT in Frequency

$$w[n] = \sum_{m=-\infty}^{\infty} v[n + mN] \xleftrightarrow{DTFS;\,\Omega_o = 2\pi/N} W[k] = \frac{1}{N}V(e^{jk\Omega_o})$$

$w[n]$ is N periodic.

■ C.9.4 Sampling the FT in Frequency

$$g(t) = \sum_{m=-\infty}^{\infty} x(t + mT) \xleftrightarrow{FS;\,\omega_o = 2\pi/T} G[k] = \frac{1}{T}X(jk\omega_o)$$

$g(t)$ is T periodic.