Chris Hunt   DP6   MTH 231

$\{2, 3, 5, 6\}$

2. Suppose you own x Fezzes and y bow ties. x and y are greater than 1

a. How many combinations of Fez and bow ties can you make? You can only wear one fez and one bow tie at a time. Explain.

This can be solved using the multiplicative principle for sets. The amount of different combinations is the product of fezzes and bow ties:

$$x \cdot y = xy$$

Another way is to line the Fezzes up in a row and the bow ties in a column each cell of x and y is a combo. The area is the amount of combos.

b. Explain why the answer is also $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2}$

$$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = \frac{(x+y)!}{2!(x+y-2)!} - \frac{x!}{2!(x-2)!} - \frac{y!}{2!(y-2)!}$$

$\binom{x+y}{2}$ is the # of 2 element subsets of a set with the union of set X and Y ~~where~~ X is the set of fezzes and Y is the set of bow ties

$\binom{X}{2}$ is the # of 2 element subsets of a set X

$\binom{Y}{2}$ is the # of 2 element subsets of a set Y

The number of subsets of size 2 of both X and Y contain all possible combinations of the elements of X and Y. We want this but without the pairs from the same set. So we then subtract the XX and YY pairs

c) Let F be the set of fezzes with cardinality x and let B be the set of bow ties with cardinality y. Arranging each element of F in a row and B as a column, each intersection of the grid they make is a fez-tie pair. Count each cell, the sum is the product of $x \cdot y$. We can also count subsets as follows: Find all subsets of $F \cup B$ of weight 2, then subtract

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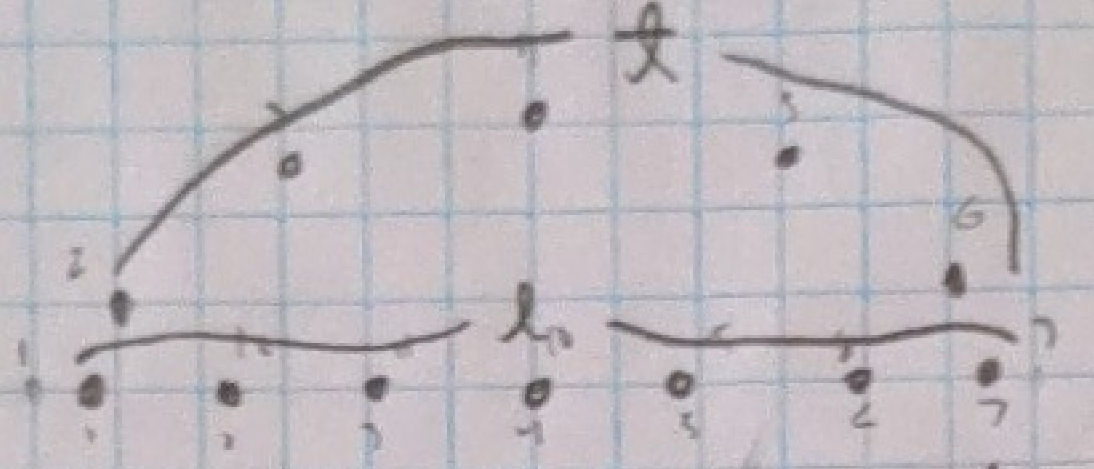
2.C) cont.

all subsets of weight 2 of the set F and all the subsets of weight 2 of the set B.

3. How many triangles can you draw using the dots below as vertices?



Vertices
$\ell = 7$
$t = 5$

a) Two vertices in ℓ and one in t:

$\binom{7}{2} \cdot \binom{5}{1}$

Two vertices in t and one in ℓ:

$\binom{5}{2} \cdot \binom{7}{1}$

Three vertices in t:

$\binom{5}{3}$

$\binom{7}{2} \cdot \binom{5}{1} + \binom{7}{1}\binom{5}{2} + \binom{5}{3} = 185$ triangles

b) Of the set of 12 vertices choose 3, then subtract all the cases where 3 vertices were also all in ℓ

$\binom{12}{3} - \binom{7}{3} = 185$ triangles

c) $\binom{x+y}{3} - \binom{x}{3} = \binom{x}{2}\binom{y}{1} + \binom{y}{2}\binom{x}{1} + \binom{y}{3}$

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5. Let BF be the set of best Friends.

$BF = 15$ 6 must be bridesmaids
1 must be maid of honor

a. $\binom{15}{6} \cdot 6 = 30030$

b. $\binom{15}{1} \cdot \binom{14}{5} = 30030$

c) $6\binom{15}{6} = 15\binom{14}{5}$

Choosing 6 out of 15 then choosing 1 of six something something... they are the same...

6. Consider: $k\binom{n}{k} = n\binom{n-1}{k-1}$

a)

$n=3 \quad k=2$

$2\binom{3}{2} = 6 \qquad 3\binom{2}{1} = 6$ ✓

$n=5 \quad k=3$

$3\binom{5}{3} = 30 \qquad 5\binom{4}{2} = 30$ ✓

b)

$$k\left(\frac{n!}{k!(n-k)!}\right) = n\left(\frac{(n-1)!}{(k-1)!(n-1-(k-1))!}\right)$$

$$\frac{k}{k}\left(\frac{n!}{(k-1)!(n-k)!}\right) = \frac{n\cdot(n-1)!}{(k-1)!(n-k)!}$$

$$\frac{n!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

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6. Cont.

c) Give a combinatorial proof of the identity.

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

The amount of combinations of k weighted subsets of n with one of those values emphasized is the same as picking one of the total elements and then finding all the subsets of size k-1 of the remaining, n-1, elements.