Chris Hunt DP4 MTH231

2.5 Induction

Induction is a style of proof to convince a mathematical statement is always true.

Stamps: e = 8 cent stamp f = 5 cent Stamp

Let E be the set of 8 cent stamps
F be the set of 5 cent stamps

T be the set of all possible combinations

$t = e \cdot a + f \cdot b$

Find all possible combination of values

Sequences: Let $P(n)$: "you can make n cents of postage using just 8 cent stamps and 5 cent stamps

$P(1), P(2), P(3), P(4), P(5), P(6), \ldots$

F F F F T F F T ...

1. Demonstrate that $P(28)$ is true
2. Prove that if $P(k)$ is true, then $P(k+1)$ is true For any $k \geq 28$ ↑recursion

Step 1 is the base case
Step 2 is the inductive case

Induction is used when there is a way to go from one case to the next - how to always do "one more"

Think about the problem dynamical; how does increasing n change the problem?

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Induction Proof Structure

State the claim: $P(n)$

1. Base case: Prove $P(0)$
2. Inductive case: Prove that $P(k) \rightarrow P(k+1)$ for all $k \geq 0$

   $P(k)$ is called the inductive hypothesis

Therefore, claim is true for all $n \geq 0$

This is like falling dominoes

Easier to find recursive formulas rather than closed formulas

Ex. 2.5.1

Claim: For all $n \geq 1$ $1+2+3+\dots+n = \frac{n(n+1)}{2}$

1. Base case: Assume $n=1$

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \checkmark$$

2. Inductive case: Assume: $n = n+1$

$$1+2+3+\dots+n+(n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+1+1)}{2} = \frac{(n+1)(n+2)}{2}$$

$$\frac{n(n+1)}{2} + \frac{2(n+1)}{2} \rightarrow \frac{n(n+1)+2n+2}{2}$$

$$\frac{n^2+n+2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+2)(n+1)}{2} = \checkmark$$

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Strong Induction:

Let $P(n)$ be the statement..."

1. Base Case: Prove that $P(0)$ is true

2. Inductive case: Assume $P(k)$ is true for all $k < n$
prove that $P(n)$ is true

10. Prove that the sum of n squares can be found as follows

$$P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Let $P(n)$ be the statement above.

Consider this base case. Let $n = 1$ or 0?

$$P(1): \quad 1^2 = \frac{1(1+1)(2+1)}{6}$$

$$= \frac{6}{6}$$

$$= 1 \checkmark$$

Now assume $P(k)$ is true.

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now consider $P(k+1)$:

$$\underbrace{1^2 + 2^2 + 3^2 + \ldots + k^2}_{} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 = \frac{6k^2 + 12k + 6}{6}$$

$$(k+1)^2 = \frac{6(k+1)^2}{6}$$

$$(k+1)^2 = (k+1)^2 \checkmark$$

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17. Claim: For all $n \in \mathbb{N}$, the number n^2+n is even
$\quad P(n)$

$\forall n \in \mathbb{N}(P(n))$

Let j be some natural number.

$n^2+n=2j$ By the def. of even numbers

Consider $P=(1)$

$1^2+1=2j$

$1+1=2j$

$2=2j$ ✓ $2|2$

Now assume $P(k)$ is true

$k^2+k=2j$

Now consider $P(k+1)$ and some natural number m

$(k+1)^2+k+1=2m$

$k^2+2k+1+k+1=2m$

$\underbrace{k^2+k}+2k+2=2m$

$2j+2k+2=2m$

$2(j+k+1)=2m$ ✓

Since $(j+k+1)$ is an integer, it is even

Therefore it's true for all n.

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21. **Claim:** If n people all shake hands with each other, then the total number of handshakes is $\frac{n(n-1)}{2}$

Base:
Let's suppose $n = 1$. There's only one person so there should be no hand shakes.

$$0 = \frac{1(1-1)}{2}$$
$$= \frac{0}{2}$$
$$= 0 \checkmark$$

Induction: Assume $P(k)$ is true.

For $n = k$ there are $\frac{k(k-1)}{2}$ handshakes

Now consider $P(k+1)$. Adding one more person means they will shake k many hands so the total number of hand shakes is

$$\frac{k(k-1)}{2} + k = \frac{(k+1)(k)}{2}$$

$$\frac{k(k-1)+2k}{2} = \frac{(k+1)(k)}{2}$$

$$\frac{k^2+k}{2} = \frac{k^2+k}{2} \checkmark$$

Since it is true for $k+1$ and we know $P(1)$ is true it is true for all values of n.

$\frac{3(2)}{2}$

$3+3=6$

$\frac{4(3)}{2}$ 6

$\frac{5(4)}{2} = 10$

$\frac{6(5)}{2}$ 15

$\frac{7(6)}{2} = 21$