Final Portfolio

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Reflective Introduction

Throughout Math 231, Elements of Discrete Mathematics, I have had the opportunity to explore various concepts and problem-solving techniques. This final portfolio serves as a reflection of my growth and development as a learner and problem solver in the field of mathematics. In this reflective introduction, I will discuss my decision-making process in selecting the problems for my portfolio, the revisions and changes I made to these problems, and the strengths and weaknesses demonstrated in my portfolio. Finally, I will determine the final grade that best describes my learning and work in this course, supported by specific bullet points and my consistency in completing the course's requirements.

When deciding which problems to include in my portfolio, I aimed to showcase my understanding of different topics covered in the course. The prompt provided clear guidelines for the required contents, specifying the inclusion of problems related to set theory, propositional logic, proofs, combinatorics, and graph theory. I carefully reviewed my assignments, notes, and the textbook to identify problems that best exemplified my understanding of these topics. For each category, I selected problems that I found challenging yet rewarding to solve, ensuring they represented my growth as a learner throughout the course.

In preparing the problems for the portfolio, I made revisions and changes to improve their clarity, correctness, and presentation. I thoroughly reviewed each problem, ensuring that my solutions were well-organized and easy to follow. I revised the wording and structure of my proofs to enhance their logical flow and coherence. Additionally, I used visual aids, such as Venn diagrams, to provide clearer explanations in the set theory problem. By refining my solutions, I aimed to demonstrate my ability to communicate mathematical ideas effectively.

The portfolio reflects both strengths and weaknesses in my understanding of the course material. In terms of strengths, my problem on set theory illustrates my proficiency in using definitions and visual representations to explain the relationship between sets. The direct proof problem demonstrates my ability to construct logical arguments and provide clear justifications. Furthermore, my proof by mathematical induction showcases my understanding of the underlying principles and techniques involved in this type of proof. These strengths indicate my progress in developing a solid foundation in discrete mathematics.

However, weaknesses can also be identified in my portfolio. For instance, the problem on propositional logic, while correct, could have been presented more concisely. I realize that I need to practice condensing my logical statements to improve the clarity and readability of my solutions. Additionally, in the graph theory problem, I could have provided more explicit reasoning to support the claim that T spans all the vertices of G. Strengthening my ability to provide rigorous justifications will be an area of focus for future improvement.

Considering the grading criteria outlined in the course syllabus, I would assign myself the grade of A for this course. I have consistently engaged in the coursework, completed assignments on time, and actively participated in class discussions. My portfolio demonstrates a solid understanding of the major concepts covered in the course, including set theory, propositional logic, proofs, combinatorics, and graph theory. I have shown growth in my ability to construct clear and well-justified proofs, as well as my aptitude for problem solving in discrete mathematics. By consistently meeting the requirements of the course and consistently achieving satisfactory results, I believe an A grade accurately reflects my learning and work in this course.

In conclusion, this final portfolio represents my journey through Math 231, highlighting the progress I have made as a learner and problem solver in discrete mathematics. The careful selection, revision, and presentation of problems aimed to showcase my understanding of the course material. While strengths in communication and logical reasoning are evident, weaknesses in conciseness and explicit justifications have also been identified. With an overall assessment of my performance based on the grading criteria, I believe an A grade best describes my learning and work

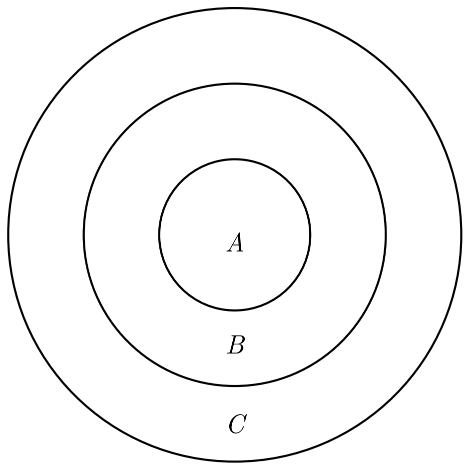
Set Theory

25. Let A, B, and C be sets.

a. Suppose that $A \subseteq B$ and $B \subseteq C$. Does this mean that $A \subseteq C$?

Claim. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. *Assume $A \subseteq B$ and $B \subseteq C$ is true. By the definition of a subset we can state that every element x in A is an element in B and every element y in B is an element of C. This can be visualually demosntrated using this Venn diagram:*



Since, every element of B must be in C and every element of A must be in B it follows that every element of A must also be in C. Therefore, $A \subseteq C$ is true and the from that the original claim is true.

$\square$

b. Suppose that $A \in B$ and $B \in C$. Does this mean that $A \in C$? Give an example to prove that this does NOT always happen.

Claim. *If $A \in B$ and $B \in C$, then $A \in C$.*

Proof. *For this claim to be false we must find a case where $A \in B$ and $B \in C$ is true but $A \in C$ is false. Consider the following sets, A, B, and C:*

$$A = \{1, 2\}$$
$$B = \{A, 3\}$$
$$C = \{B, 4\}$$

These sets fullfil the antecedent of the claim above. They can also be expressed like:

$$A = \{1, 2\}$$
$$B = \{\{1, 2\}, 3\}$$
$$C = \{\{\{1, 2\}, 3\}, 4\}$$

Viewing the expressions this way we can see that A is an element of an element of C but not directly an element itself, which violates the claim that A is an element of C if A is an element of B and B is an element of C. Therefore the claim is not always true.

$\square$

Propositional Logic

SQ-1. The notation ∃! means "there exists a unique." For example, "∃! x such that x is prime and x is even" means that there is one and only one even prime number. Suppose that P(x) is a predicate and D is the domain of discourse for x.

a. Rewrite the statement "$\exists! x \in D : P(x)$" without using the symbol ∃!.

Let:

$$Q(x, y) = \text{y is x}$$

The statement "$\exists! x \in D : P(x)$" can be rewritten as:

There exists some x in D such that P(x), and, for all y in D, if P(y), then y is x.

$$\exists x \in D : (P(x) \wedge (\forall y \in D : (P(y) \rightarrow Q(x, y))))$$

b. Write (and simplify) a negation of the statement "$\exists! x \in D : P(x)$." What does it mean in words?

The negation of the statement above is:

For all x in D P(x), or, there exists some y in D such that, P(y) and y is not x

Symbolically this can be done using de Morgan's Law:

$$\neg(\exists x \in D : (P(x) \wedge (\forall y \in D : (P(y) \rightarrow (y = x))))) \equiv \ \forall x \in D : (\neg P(x) \vee (\exists y \in D : (P(y) \wedge \neg Q(x, y))))$$

Direct Proof

7. Consider the statement: for all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6.

Let us begin by defining our terms.

Let: $a, b, n, m \in \mathbb{Z}$
$P(a, n) =$ "$a = 2n$" (by the definition of even numbers from class)
$Q(b, m) =$ "$3|b = m$" or "$b = 3m$"
$R(a, b, n, m) =$ "$6|ab = nm$" or "$ab = 6(nm)$"

a. Prove the statement. What sort of proof are you using?

This claim will be proven using a Direct Proof.

Claim. *For all integers a and b, if P and Q are true, then R is true. That is,* $\forall a, b \in \mathbb{Z}(P \wedge Q \to R)$.

Proof. *Suppose P and Q are true. Consider the product of a and b:*

$$\begin{aligned} ab &= (2n)(3m) \\ &= 6(nm) \end{aligned}$$

Since a, b, n, and m are all integers, and integers are closed under multiplication, we can conclude that the product of ab is divisible by 6*. Therefore, the claim is true.*

□

b. State the converse. Is it true? Prove or disprove.

Now consider the converse claim.

Claim. *For all integers a and b, if R is true, then P and Q are true. That is,* $\forall a, b \in \mathbb{Z}(R \to P \wedge Q)$.

Proof. *Suppose R is true. For the statement to be true a must be even and b a multiple of 3. To disprove this claim we need to show that there exists a product of a and b that is a multiple of 6 where either a is not even or b is not a multiple of 3. Consider this counterexample:*

$$\text{Let: } a = 6 \text{ and } b = 2$$

$$\begin{aligned} ab &= 6(2) \\ &= 12 \end{aligned}$$

Since the product of a and b is a multiple of 6, and b is not a multiple of 3 there exists a case where the statement R is true but the implication $P \wedge Q$ *is false. Therefore, the converse claim is false.*

□

Indirect Proof

SQ-4. For all sets A and B,

a. Prove that A is a subset of B if and only if B^c is a subset of A^c.

To begin this proof we must start with proving that the cardinality of a subset of a set is less than or equal to the cardinality of the set itself.

Claim. $A \subseteq B \to |A| \leq |B|$

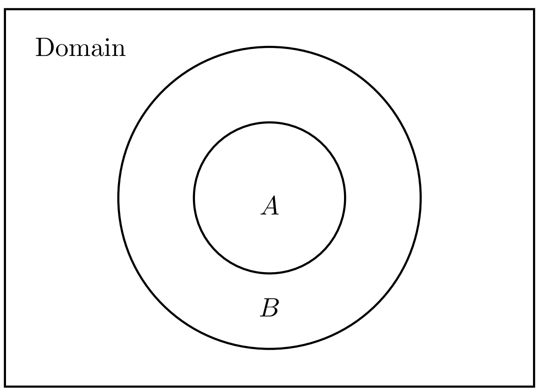
Lemma. *To prove this we will use proof by contradiction. Suppose the claim were not the case, then* $A \subseteq B \wedge |A| > |B|$. *This means that there must be at least one element in A that is not also in B, in order for this to be true A could not be a subset of B by the definition of subsets. Therefore the claim is true by contradiction.*

Claim. $A \subseteq B \leftrightarrow B^c \subseteq A^c$

Proof. *This bidirectional claim can be broken down into two implications:*

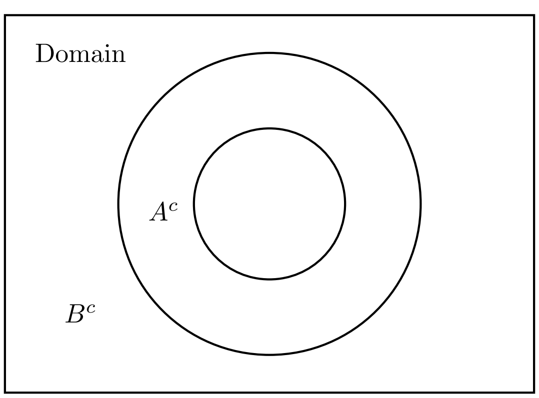
1. $A \subseteq B \to B^c \subseteq A^c$
2. $B^c \subseteq A^c \to A \subseteq B$

Let's assume the antecedant of claim one to be true. That is A *is a subset of* B.



Now consider B^c *and* A^c *which are the areas in the domain which are outside the corresponding sets of* A *and* B. *Since every element of* A *is in* B *we know that* A *has less than or equal to the same number of elements as* B *by Lemma 1. Taking their compliment we know that* B^c *is less than or equal to* A^c *and since all the elements in* A *are also in* B, *we know that any element which is not in* B *must also not be in* A *which meets the definition of a subset. Therefore the implication is true.*

Now Let's assume the antecedent of the second claim is true. That is every element not in B *is also not in* A.



Due to this fact every element not in A^c must also not be in B^c, that is every element in A is also in B. Therefore the second claim holds. Since both claims are true then the original claim is also true.

$\square$

b. Prove that if A is a subset of B and B is a subset of A, then $A = B$.

Claim. $A \subseteq B \wedge B \subseteq A \rightarrow A = B$

Proof. *Assume that $A \subseteq B$ and $B \subseteq A$ are both true. That is, every element of A is an element of B and every element of B is an element of A. Consider some arbitrary element x in the set A, this element will be in set B. Now, consider some other arbitrary element y in set B, this element will be in set A as well. Since there are no elements that can be found in A that are not in B and no elements that can be found in B that are not in A, therefore A and B contain exactly the same elements. Therefore the claim is true.*

$\square$

Proof By Mathematical Induction

22. Suppose that a particular real number has the property that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for all natural numbers.

Claim. *$x^n + \frac{1}{x^n}$ is an integer for all natural numbers.*

$$\forall n \in \mathbb{N}, x^n + \frac{1}{x^n} \text{ is an integer}$$

Proof. *Let's begin by considering the base case where $n = 1$.*

$$x^1 + \frac{1}{x^1}$$
$$x + \frac{1}{x}$$

Since $x + \frac{1}{x}$ is assumed to be an integer we can state that the claim where n=1, is true.

Now let us assume for all integers from 0 to k the claim holds true. Since we know that integers have closure under multiplication if we multiplied the integer $x^k + \frac{1}{x^k}$ by $x + \frac{1}{x}$ the product will be an integer. Consider the expansion of this product equals some integer j.

$$(x^k + \frac{1}{x^k})(x + \frac{1}{x}) = j$$
$$x^k x + \frac{x^k}{x} + \frac{x}{x^k} + \frac{1}{x^k x} = j$$
$$x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}} = j$$

Since we are assuming that the claim holds for all values up to k, it will then hold true for $k-1$. We can replace $x^{k-1} + \frac{1}{x^{k-1}}$ for some integer m

$$x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}} = j$$
$$x^{k+1} + \frac{1}{x^{k+1}} + m = j$$

Since we know that both j and m are integers and integers are closed under addition the sum of $x^{k+1} + \frac{1}{x^{k+1}}$ must also be an integer. Therefore by strong induction, the claim is true.

□

Combinatorics

SQ-8. Determine the number of five-card poker hands that contain three queens. How many of them contain, in addition to the three queens, another pair of cards?

To find the number of five-card poker hands that contain three queens we can consider two different choice cases. In the first case, out of the four queens available in the deck 3 are chosen, then from the remaining 48 cards we choose two.

$$\binom{4}{3} * \binom{48}{2}$$

In the second case, we choose the frist three queens, then from the remaining 48 we choose 1 and then we get the fourth queen free.

$$\binom{4}{3} * \binom{48}{1} * \binom{1}{1}$$

To find the total number of combinations, we add these two values together:

$$\binom{4}{3} * \binom{48}{2} + \binom{4}{3} * \binom{48}{1}\binom{1}{1} = 4704$$

There are 4704 possible poker hands that contain at least three queens.
Next, if we want to find the possible hands with three queens and two pairs we must first choose three of the four queens, then draw one of the remaining forty-eight, then from the remaining three suits of that previous card, draw one. This total must then be divided by half since order doesn't matter and we need to remove the double counted pairs.

$$\frac{\binom{4}{3} * \binom{48}{1} * \binom{3}{1}}{2!} = 288$$

There are 288 possible full house poker hands with three queens!

Graph Theory

SQ-11. Suppose G is a connected graph and T is a cycle-free subgraph of G. Suppose also that if any edge e of G that is not in T is added to T, the resulting graph contains a cycle. Prove that T is a spanning tree for G

Claim. *T is a spanning tree for G*

Proof. *First, we will show that T is a tree. Since T is a cycle-free subgraph of G, it is by definition acyclic. Now we need to prove that T is connected. Suppose T is not connected, which means there are two sets of vertices in T that are not adjacent to each other. Let's call these sets A and B. Since G is a connected graph, there exists at least one edge (u, v) in G that connects a vertex in set A to a vertex in set B. Adding this edge to T would create a cycle, contradicting the given condition that any edge not in T would create a cycle. Therefore, T must be connected, and hence a tree.*

Next, we will show that T spans all the vertices of G. Since T is a subgraph of G, it contains a subset of vertices from G. To prove that T spans all the vertices of G, we need to show that every vertex in G is also a vertex in T. Suppose there exists a vertex v in G that is not in T. Since G is a connected graph, there exists a path in G from any vertex u in T to vertex v. Along this path, there must be an edge e that is not in T. However, adding this edge e to T would create a cycle, which contradicts the given condition. Therefore, every vertex in G is also a vertex in T, and T spans all the vertices of G.

Based on the above arguments, we have shown that T is a connected, acyclic subgraph of G that spans all the vertices of G. Therefore, T is a spanning tree for G.

□