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4.1) Graphs are made up of a collection of vertices and connections between them (edges).

When two vertices are connected they are said to be adjacent

Graphs can thus be represented by two sets

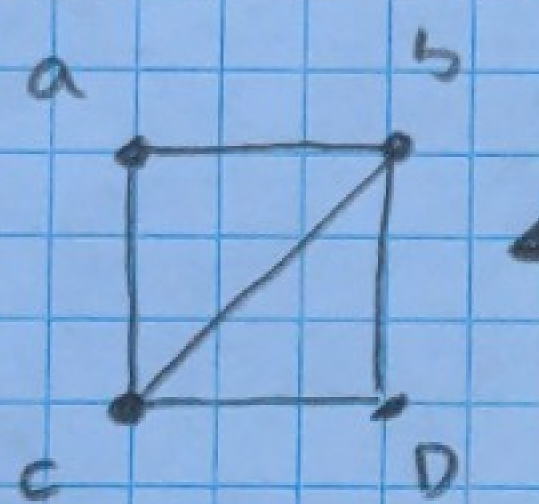
V - vertices

E - edges $\rightarrow$ set of 2 element subsets of V

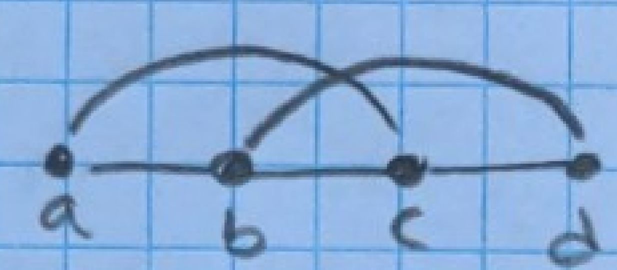
A graph is an ordered pair $G = (V, E)$

V is nonempty E is of 2-element subsets of V

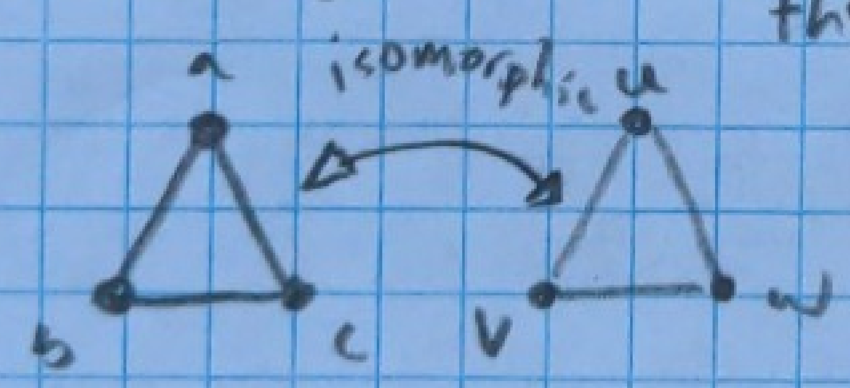
Ex. $G = (\underbrace{\{a, b, c, d\}}_{4V}, \underbrace{\{\{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}\}}_{5E})$



or



Isomorphic graphs $\rightarrow$ same except for the names of the vertices.



An isomorphism between two graphs G_1 and G_2 is a bijection $f: V_1 \rightarrow V_2$ between the vertices of the graphs such that $\{a,b\}$ is an edge in G_1 if and only if $\{f(a), f(b)\}$ is an edge in G_2

$$G_1 \cong G_2$$

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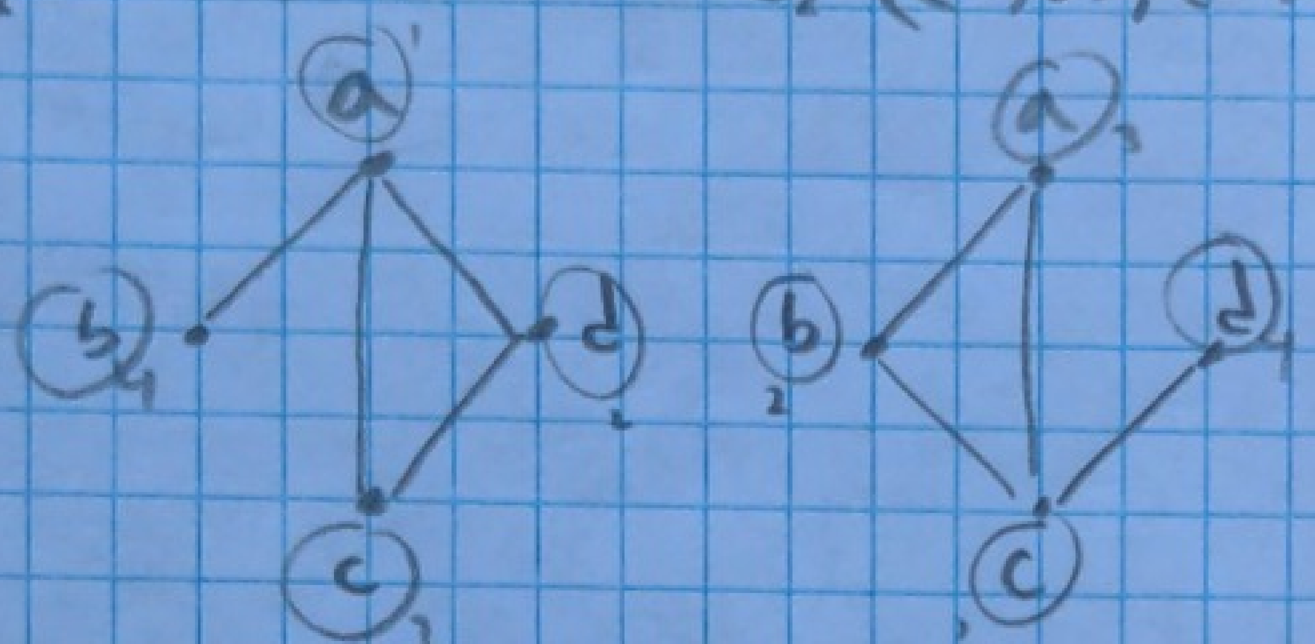
Finding isomorphisms:

$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$

$V_1 = \{a, b, c, d\}$ $E_1 = \{\{a,b\}, \{a,c\}, \{a,d\}, \{c,d\}\}$

$V_2 = \{a, b, c, d\}$ $E_2 = \{\{a,b\}, \{a,c\}, \{b,c\}, \{c,d\}\}$



$F: a \to c$

$F: d \to b$

$F: c \to a$

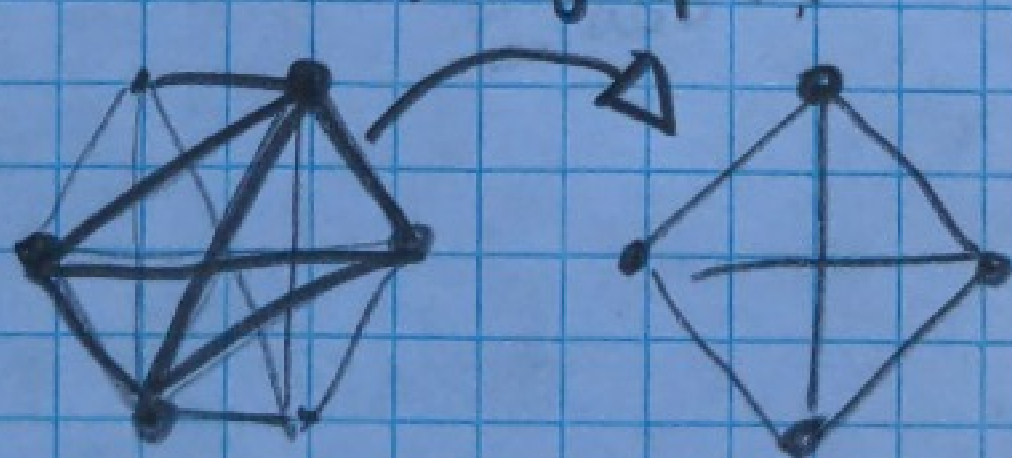
$F: b \to d$

$$\left.\begin{array}{l} F(\{a,b\}) \to \{c,d\} \checkmark \\ F(\{a,c\}) \to \{c,a\} \checkmark \\ F(\{a,d\}) \to \{c,b\} \checkmark \\ F(\{c,d\}) \to \{a,b\} \checkmark \end{array}\right\} G_1 \cong G_2$$

Special Graph: Peterson Graph or K_n = all graphs isomorphic to any copy of that particular graph

Subgraphs!

sub graph



$G' = (V', E')$ is a subgraph of $G = (V, E)$ provided that $V' \subseteq V$ and $E' \subseteq E$

$G' = (V', E')$ is an induced subgraph of $G = (V, E)$ provided $V' \subseteq V$ and every edge in E whose vertices are still in V' is also an edge in E'

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Every induced graph is also an ordinary subgraph but the converse is not always true

Multigraphs - a graph with double edges

Connected - get from any vertex to any other vertex by following some path of edges

Complete - If every pair of vertices is connected by an edge

K_n is the complete graph on n vertices

Degree - # of edges from a vertices

Handshake Lemma: In any graph, the sum of the degrees of vertices in the graph is always twice the # of edges

$$\sum_{v \in V} d(v) = 2|E|$$

sum $(4,4,3,3,3,2,1) = \frac{20}{2} = 10e$

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Bipartite: The vertices can be divided into two sets, A and B with no two vertices in A adjacent and no two vertices in B adjacent

IF each vertex in A is adjacent to all the vertices in B, then the graph is a complete bipartite graph.

K_n edges

K_5:

How many edges? $\frac{5 \cdot 4}{2} \to \frac{n(n-1)}{2}$

$4,4,4,4,4 = 20$ counted twice

Can't have an odd # of odd degree V

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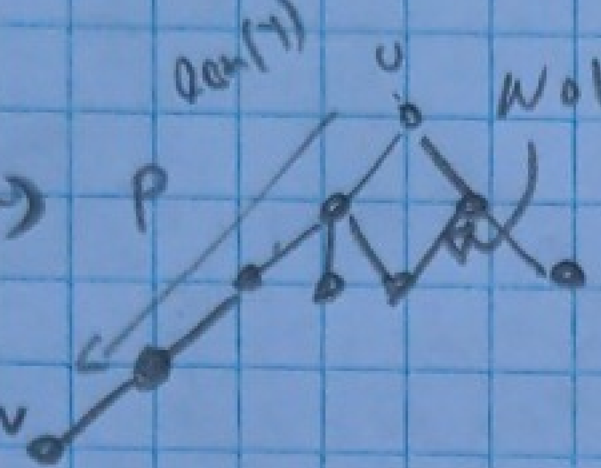
4.2) Tree: a connected graph containing no cycles

Forest: a graph containing no cycles

Rooted tree: A tree with one vertex as root!

Not a tree no cycles!

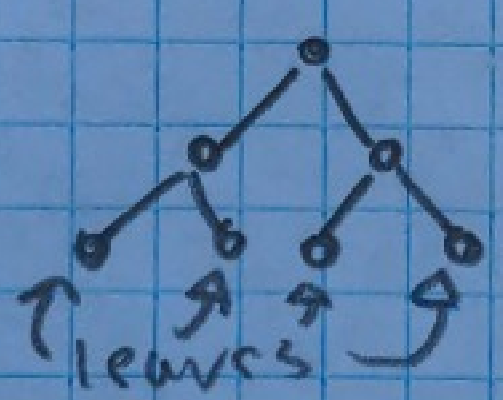


If you wanna prove something is a tree

A graph T is a Tree if and only if between every pair of distinct vertices of T there is a unique path.

A graph F is a Forest if and only if between any pair of vertices in F there is at most one path

All trees have leaves → vertices of degree one



Any tree with at least 2 vertices has at least 2 vertices of degree 1

Let T be a tree with v vertices and e edges

$$e = V - 1$$

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6. What is the largest number of edges possible in a graph with 10 vertices? What about for a bipartite graph? What about a tree?

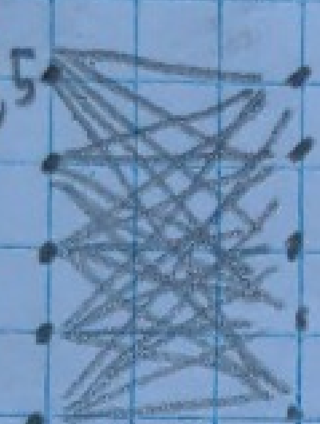
a)

$\frac{10v \cdot 9a}{2}$? 10 vertices, each with 9 adjacencys

$9+8+7+6+5+4+3+2+1$?

Each V is adjacent to all other V. Begin with V_1 and count each edge, move to V_2 don't count the edge to V_1, count the rest - continue for all V_n

b) $K_{5,5}$



$5 \cdot 5$?

$K_{9,1}$

$9 \cdot 1 = 9$

$K_{8,2} = 8 \cdot 2 = 16$ $\quad K_{6,4} = 6 \cdot 4 = 24$ $\quad K_{4,6} = 24$ $\quad$ 9

$K_{7,3} = 7 \cdot 3 = 21$ $\quad K_{5,5} = 5 \cdot 5 = 25$ $\quad$ 16

25 is the max amount of edges

c) The edges in a tree is one less than the number of vertices

$e = V - 1 \rightarrow 9 = 10 - 1$

9 edges in a tree of 10 vertices

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11) Let $k_1, k_2, \ldots, k_j$ be a list of $\mathbb{Z}^+$ that sum to n use two graphs containing n vertices to explain why

$$\sum_{i=1}^{j} \binom{k_i}{2} \leq \binom{n}{2}$$

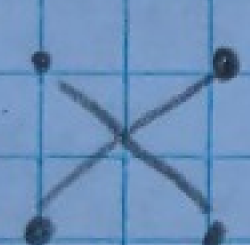
Triangle

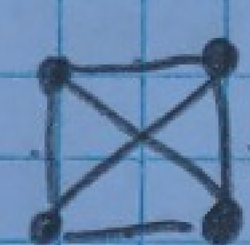
K_n has $\frac{n \cdot (n-1)}{2}$ edges

$1 + 4 + 5 + 7 = 17$ $n = 17$

A graph with n vertices contains $\frac{n(n-1)}{2}$ or $\binom{n}{2}$ edges

$1 + 1 = 2$ $\binom{1}{2} + \binom{1}{2} \leq \binom{2}{2}$ ✓





6 edges $= \binom{4}{2}$

$3 + 2 = 5$ $\binom{5}{2} = 10$

$\binom{3}{2} + \binom{2}{2} = 3 + 1 = 3$

Consider two graphs, on K_n and another with n vertices which is the sum of all completely connected vertices $K - K_j$. Since this graph is not connected it can not exceed the number of the complete graph K_n and since the number of edges in a complete graph can be found by $\binom{x}{2}$ where x are the # of vertices in a complete graph $\binom{n}{2}$ will always be greater than or equal to $\sum^{j} \binom{k_i}{2}$

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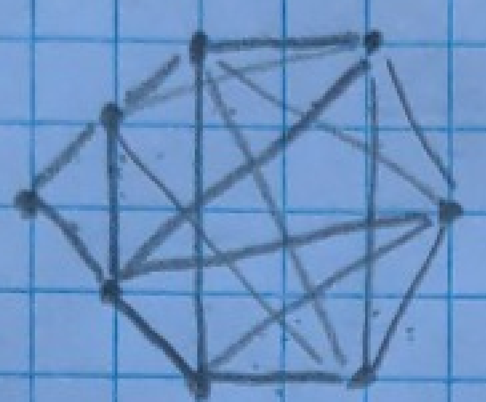
15) Prove that any graph with at least 2 vertices must have 2 vertices of the same degree (or more?)

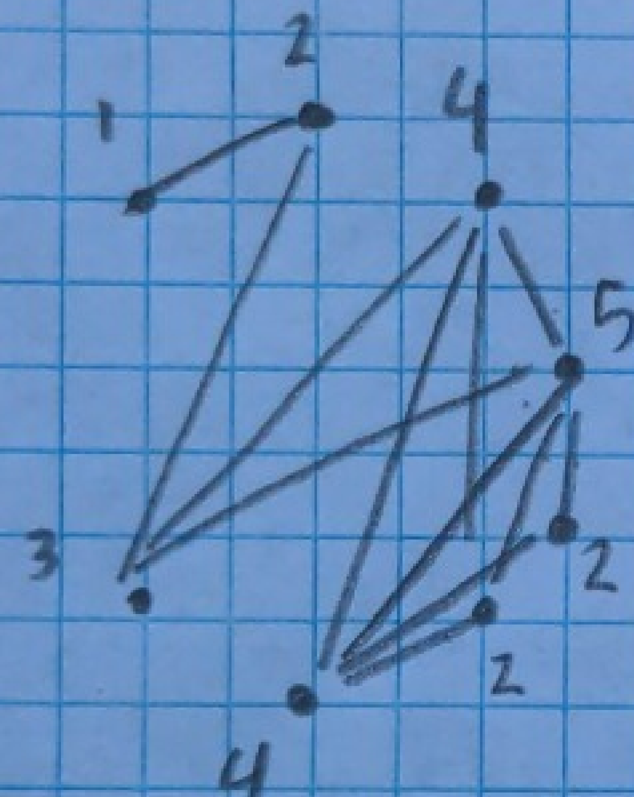
Claim: A graph of size 2 or more must have 2 vertices of the same degree

Proof:



($\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$, $\underline{5}$, $\underline{6}$, $\underline{7}$, $\underline{8}$)

is this possible?



Hmm, the max degree value for a graph size n is $(n-1)$

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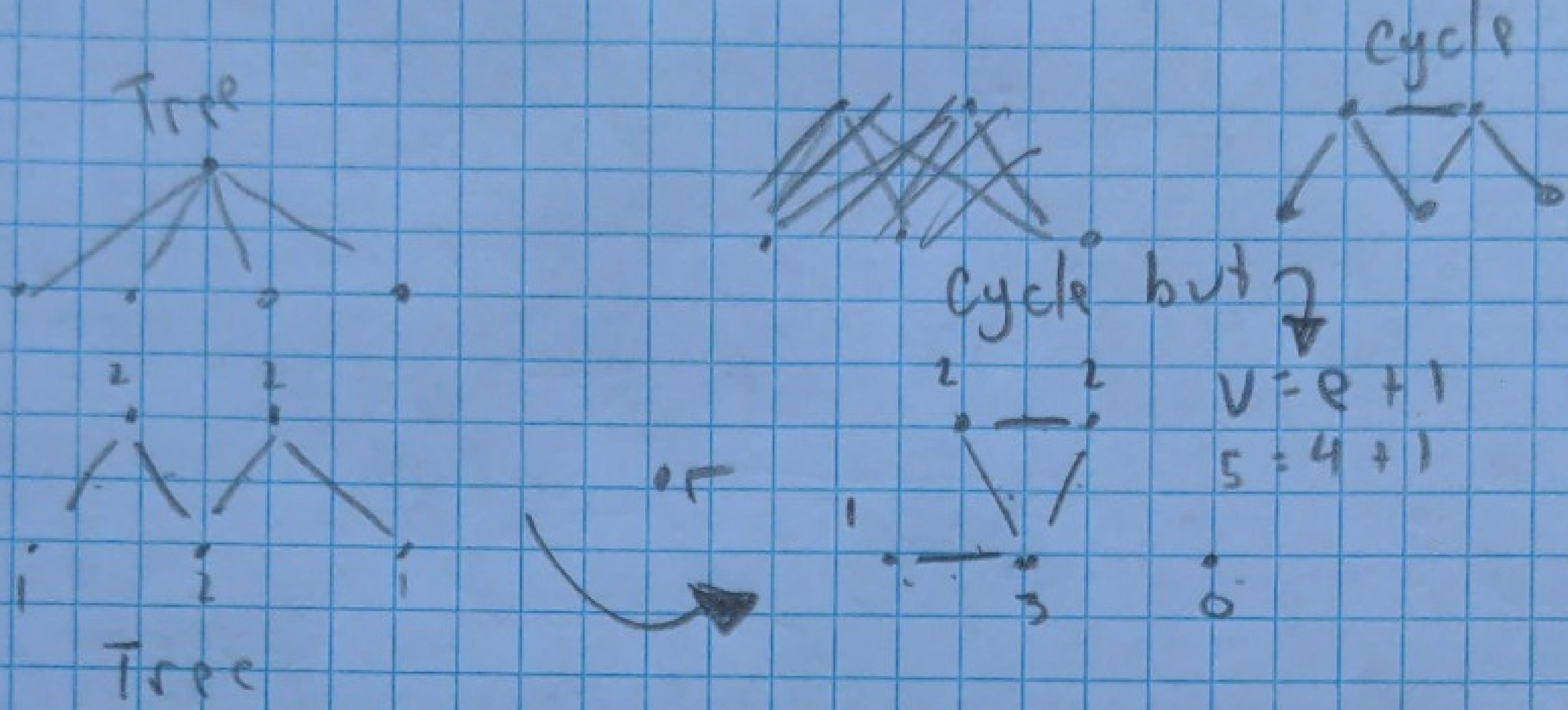
4. Suppose you have a graph with v vertices and e edges that satisfies $v = e + 1$. Must the graph be a tree?



5. Prove that any graph with v vertices and e edges that satisfy ~~[illegible]~~ $v > e + 1$ will NOT be connected

What are the min amount of edges to make a connected graph?

$v \geq e + 1$ $3 \geq 2 + 1$ ✓

$2 = 1 + 1$

$3 = 1 + 1$

Start with 1 vertices

• This is inherently connected and does not satisfy $v > e + 1$ because $1 > 1$ is not true

• • $2 > 1$ yes but not connected

Every time we add a vertex we must also add an edge to connect it, therefore all graphs $v > e + 1$ will not be connected.