HW 4

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17. Give a proof of the statement, "for all $n \in \mathbb{N}$, the number $n^2 + n$ is even."

Let P(n) be "$n^2 + n$ is even".

Claim. $\forall n \in \mathbb{N}$, *P(n)*

Proof. *Suppose $n = 1$, P(1) can be written as:*

$$\begin{aligned} 1^2 + 1 \\ 1 + 1 \\ 2 \end{aligned}$$

Since 2 is an even number, by the definition of even numbers, P(1) is true.

Now let us assume, for some natural number k, P(k) is true. Now consider P($k + 1$):

$$\begin{aligned} (k+1)^2 &+ k + 1 \\ k^2 + 2k + 1 &+ k + 1 \\ (k^2 + k) &+ 2k + 2 \\ (k^2 + k) &+ 2(k + 1) \end{aligned}$$

Since we know that $k^2 + k$ and $2(k + 1)$ are both even and the sum of two even numbers is even, P($k + 1$) will be even. Therefore by the principle of mathematical induction all natural numbers, n, the number $n^2 + n$ is even. $\square$

22. Suppose that a particular real number has the property that $x+\frac{1}{x}$ is an integer. Prove that $x^n+\frac{1}{x^n}$ is an integer for all natural numbers.

Claim. *$x^n + \frac{1}{x^n}$ is an integer for all natural numbers.*

$$\forall n \in \mathbb{N}, x^n + \frac{1}{x^n} \text{ is an integer}$$

Proof. *Let's begin by considering the base case where n = 1.*

$$x^1{+}\frac{1}{x^1}$$
$$x{+}\frac{1}{x}$$

Since $x + \frac{1}{x}$ is assumed to be an integer we can state that the claim where n=1, is true.

Now let us assume for all integers from 0 to k the claim holds true. Since we know that integers have closure under multiplication if we multiplied the integer $x^k + \frac{1}{x^k}$ by $x + \frac{1}{x}$ the product will be an integer. Consider the expansion of this product equals some integer j.

$$(x^k + \frac{1}{x^k})(x + \frac{1}{x}) = j$$
$$x^k x + \frac{x^k}{x} + \frac{x}{x^k} + \frac{1}{x^k x} = j$$
$$x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}} = j$$

Since we are assuming that the claim holds for all values up to k, it will then hold true for $k-1$. We can replace $x^{k-1} + \frac{1}{x^{k-1}}$ for some integer m

$$x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}} = j$$
$$x^{k+1} + \frac{1}{x^{k+1}} + m = j$$

Since we know that both j and m are integers and integers are closed under addition the sum of $x^{k+1} + \frac{1}{x^{k+1}}$ must also be an integer. Therefore by strong induction, the claim is true.

□

SQ-3. Claim: $\forall n \in \mathbb{N}$ such that $F_n < 2^n$. Prove this using induction.

To prove this first we must prove this lemma:

Lemma 1.

Claim. *All values, are natural numbers. If some x less than some n and some y is less than some m, the sum of x and y will be less than the sum of n and m.*

Proof. *Let us assume x is less than n and y is less than m. This means that there are some positive integer j and k such that $x + j = n$ and $y + k = m$. We can write this equality:*

$$x + y = n + m - (j + k)$$

Since both j and k are positive integers, the sum of n and m will have the value $j+k$ subtracted from it meaning it will be smaller. Therefore, the claim is true.

□

Let's begin our proof by defining the Fibonacci Sequence as follows:

$$\text{Fibonacci Sequence: } F_n + F_{n+1} = F_{n+2} \qquad F_0 = 0 \text{ ad } F_1 = 1 \qquad 0, 1, 1, 2, 3, 5...$$

Claim. *For all natural numbers n, $F_n < 2^n$*

Proof. *Consider the two base cases where n = 0 and n = 1:*

$$F_0 < 2^0 \text{ and } F_1 < 2^1$$
$$0 < 1 \text{ and } 1 < 2$$

Zero is less then one and one is less than two the claim holds true for n = 0 and n = 1.
Now assume that the claim holds true for all values from 0 to k. I want to show that the claim will hold true for n=k+1

$$F_{k+1} < 2^{k+1}$$
$$F_{k+1} < 2^k 2^1$$
$$F_{k+1} < 2^{k+1} 2$$
$$F_{k+1} < 2^k + 2^k$$

By the definition of the Fibonacci Sequence the F_{k+1} value in the sequence will be equal to the sum of the previous two values, $F_{k+1} = F_k + F_{k-1}$. Since we are assuming that F_k and F_{k-1} are both smaller than 2^k, then their sum will be smaller than the sum of $2_k + 2_k$ by lemma 1. This means that F_{k+1} is smaller than $2^k + 2^k$. Therefore, by strong induction the claim is true.

□