Homework 3

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7. Consider the statement: for all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6.

Let us begin by defining our terms.

$$\begin{aligned}
&\text{Let: } a, b, n, m \in \mathbb{Z} \\
&P = a = 2n \quad \text{(by the definition of even numbers from class)} \\
&Q = 3|b = m \text{ or } b = 3m \\
&R = 6|ab = nm \text{ or } ab = 6(nm)
\end{aligned}$$

a. Prove the statement. What sort of proof are you using?

This claim will be proven using a Direct Proof.

Claim. *For all integers a and b, if P and Q are true, then R is true. That is, $\forall a, b \in \mathbb{Z}(P \wedge Q \rightarrow R)$.*

Proof. *Suppose P and Q are true. Consider the product of a and b:*

$$\begin{aligned}
ab &= (2n)(3m) \\
&= 6(nm)
\end{aligned}$$

Since a, b, n, and m are all integers, and integers are closed under multiplication, we can conclude that the product of ab is divisible by 6. Therefore, the claim is true.

$\square$

b. State the converse. Is it true? Prove or disprove.

Now consider the converse claim.

Claim. *For all integers a and b, if R is true, then P and Q are true. That is, $\forall a, b \in \mathbb{Z}(R \rightarrow P \wedge Q)$.*

Proof. *Suppose R is true. For the statement to be true a must be even and b a multiple of 3. To disprove this claim we need to show that there exists a product of a and b that is a multiple of 6 where either a is not even or b is not a multiple of 3. Consider this counterexample:*

$$\text{Let: } a = 6 \text{ and } b = 2$$

$$\begin{aligned}
ab &= 6(2) \\
&= 12
\end{aligned}$$

Since the product of a and b is a multiple of 6, and b is not a multiple of 3 there exists a case where the statement R is true but the implication $P \wedge Q$ is false. Therefore, the converse claim is false.

$\square$

9. Prove the statement: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even.

To begin this proof we must first prove this lemma.

Lemma 1. *If n is an odd integer, then n^2 is an odd integer. Let $n = 2k + 1$ where k is some integer.*

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= (2k+1)(2k+1) \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since k is an integer and integers are closed under multiplication n^2 is odd. Therefore, the statement is true.

Claim. *For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even. That is, $\forall a, b, c \in \mathbb{Z}$ such that $(a^2 + b^2 = c^2 \to a \text{ or } b \text{ is even})$*

Proof. *Let us begin by finding the contradiction to the claim.*

$$\textit{Contradiction: } \exists a, b, c, \in \mathbb{Z} \text{ such that } (a^2 + b^2 = c^2 \text{ and } a \text{ and } b \text{ are odd.})$$

To disprove this claim we need to find a case where this contradiction is true. Suppose a and b are odd, that is $a = 2k + 1$ and $b = 2j + 1$ where k and j are some integer. Now consider $c^2 = a^2 + b^2$

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2j+1)^2 \\ &= 4k^2 + 4k + 1 + 4j^2 + 4j + 1 \\ &= 2(2k^2 + 2k + 2j^2 + 2j + 1) \end{aligned}$$

Since k and j are integers, and integers are closed under multiplication, this would mean that when a and b are some odd integer c^2 will always be an even. From lemma 1 we know that it's contra-positive is also true, that is if n^2 is an even integer, then n is an even integer. From the definition of an even integer we can rewrite c as $c = 2p$ where p is some integer. Now continue our work from above:

$$\begin{aligned} c^2 &= (2p)^2 \\ (2p)^2 &= 2(2k^2 + 2k + 2j^2 + 2j + 1) \\ 4p^2 &= 2(2k^2 + 2k + 2j^2 + 2j + 1) \\ 2p^2 &= 2k^2 + 2k + 2j^2 + 2j + 1 \\ 2(p^2) &\neq 2(k^2 + k + j^2 + j) + 1 \end{aligned}$$

Since we proved in class that integers must be either odd or even the above equality is a contradiction for all cases of odd integers a and b. Therefore, the original statement must be true.

$\square$

SQ-2. Use the definitions of even and odd integers to prove the following claim: "If k is any odd integer and m is any even integer, then $k^2 + m^2$ is odd."

Let us begin by defining our terms.

$$\begin{aligned}
&\text{Let: } k, m, a, b \in \mathbb{Z} \\
&P = k = 2a + 1 \\
&Q = m = 2b \\
&R = \text{ "}k^2 + m^2 \text{ is odd"}
\end{aligned}$$

Claim. *If k is any odd integer and m is any even integer, then $k^2 + m^2$ is odd, that is $\forall k, m \in \mathbb{Z}$ such that $(P \wedge Q \to R)$.*

Proof. *Suppose P and Q are both true. Consider $k^2 + m^2$:*

$$\begin{aligned}
k^2 + m^2 &= (2a + 1)^2 + (2b)^2 \\
&= 4a^2 + 4a + 1 + 4b^2 \\
&= 2(2a^2 + 2a + 2b^2) + 1
\end{aligned}$$

Since a and b are integers and integers are closed under multiplication we can conclude that $k^2 + m^2$ will always be an odd number. Therefore the claim is true.

□