Homework 3

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7. Consider the statement: for all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6.

Let us begin by defining our terms.

$$\begin{aligned}
&\text{Let: } a, b, n, m \in \mathbb{Z} \\
&P(a, n) = \text{"}a = 2n\text{"} \quad \text{(by the definition of even numbers from class)} \\
&Q(b, m) = \text{"}3|b = m\text{" or "}b = 3m\text{"} \\
&R(a, b, n, m) = \text{"}6|ab = nm\text{" or "}ab = 6(nm)\text{"}
\end{aligned}$$

a. Prove the statement. What sort of proof are you using?

This claim will be proven using a Direct Proof.

Claim. *For all integers a and b, if P and Q are true, then R is true. That is,* $\forall a, b \in \mathbb{Z}(P \land Q \to R)$.

Proof. *Suppose P and Q are true. Consider the product of a and b:*

$$\begin{aligned}
ab &= (2n)(3m) \\
&= 6(nm)
\end{aligned}$$

Since a, b, n, and m are all integers, and integers are closed under multiplication, we can conclude that the product of ab is divisible by 6*. Therefore, the claim is true.*

□

b. State the converse. Is it true? Prove or disprove.

Now consider the converse claim.

Claim. *For all integers a and b, if R is true, then P and Q are true. That is,* $\forall a, b \in \mathbb{Z}(R \to P \land Q)$.

Proof. *Suppose R is true. For the statement to be true a must be even and b a multiple of 3. To disprove this claim we need to show that there exists a product of a and b that is a multiple of 6 where either a is not even or b is not a multiple of 3. Consider this counterexample:*

$$\textit{Let: } a = 6 \textit{ and } b = 2$$

$$\begin{aligned}
ab &= 6(2) \\
&= 12
\end{aligned}$$

Since the product of a and b is a multiple of 6, and b is not a multiple of 3 there exists a case where the statement R is true but the implication $P \land Q$ *is false. Therefore, the converse claim is false.*

□

9. Prove the statement: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even.

To begin this proof we must first prove this lemma.

Lemma 1. *If n is an odd integer, then n^2 is an odd integer. Let $n = 2k + 1$ where k is some integer.*

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= (2k+1)(2k+1) \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since k is an integer and integers are closed under multiplication n^2 is odd. Therefore, the statement is true.

Claim. *For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even. That is, $\forall a, b, c \in \mathbb{Z}(a^2 + b^2 = c^2 \rightarrow$ a or b is even)*

Proof. *Let us begin by finding the contradiction to the claim.*

$$\textit{Contradiction: } \exists a, b, c, \in \mathbb{Z} \textit{ such that } (a^2 + b^2 = c^2 \textit{ and } a \textit{ and } b \textit{ are odd.})$$

To disprove this claim we need to find a case where this contradiction is true. Suppose a and b are odd, that is $a = 2k + 1$ and $b = 2j + 1$ where k and j are some integer. Now consider $c^2 = a^2 + b^2$

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2j+1)^2 \\ &= 4k^2 + 4k + 1 + 4j^2 + 4j + 1 \\ &= 2(2k^2 + 2k + 2j^2 + 2j + 1) \end{aligned}$$

Since k and j are integers, and integers are closed under multiplication, this would mean that when a and b are some odd integer c^2 will always be an even. From lemma 1 we know that it's contra-positive is also true, that is if n^2 is an even integer, then n is an even intege. From the definition of an even integer we can rewrite c as $c = 2p$ where p is some integer. So, $c^2 = (2p)^2$. Now continue our work from above:

$$\begin{aligned} c^2 &= (2p)^2 \\ &= 4p^2 \end{aligned}$$

Since c^2 is equal to $4p^2$ we can substitute that in for c^2 in the above equation.

$$\begin{aligned} &4p^2 = 2(2k^2 + 2k + 2j^2 + 2j + 1) \ \textit{ Divide both sides by two.} \\ &2p^2 = 2k^2 + 2k + 2j^2 + 2j + 1 \ \textit{ Pull out a two from the right side of the equation.} \\ &2p^2 \neq 2(k^2 + k + j^2 + j) + 1 \ \textit{ The left side is even and the right side is odd} \end{aligned}$$

Since we proved in class that integers must be either odd or even the above equality is a contradiction for all cases of odd integers a and b. Therefore, the original statement must be true.

□

SQ-2. Use the definitions of even and odd integers to prove the following claim: "If k is any odd integer and m is any even integer, then $k^2 + m^2$ is odd."

Let us begin by defining our terms.

Let: $k, m \in \mathbb{Z}$
$P(k) =$ "k is any odd integer"
$Q(m) =$ "m is any even integer"
$R(k, m) =$ "$k^2 + m^2$ is odd"

Claim. *If k is any odd integer and m is any even integer, then $k^2 + m^2$ is odd, that is $\forall k, m \in \mathbb{Z}(P \wedge Q \rightarrow R)$.*

Proof. *Suppose P and Q are both true. By the definition of odd integers there must be some integer a such that $k = 2a + 1$ and by the definition of even integers there must be some integer b such that $m = 2b$. Consider $k^2 + m^2$:*

$$\begin{aligned} k^2 + m^2 &= (2a + 1)^2 + (2b)^2 \\ &= 4a^2 + 4a + 1 + 4b^2 \\ &= 2(2a^2 + 2a + 2b^2) + 1 \end{aligned}$$

Since a and b are integers and integers are closed under multiplication we can conclude that $k^2 + m^2$ will always be an odd number. Therefore the claim is true.

□