Homework 1

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1 Suppose $P(x)$ is some predicate for which the statement $\forall x P(x)$ is true. Is it also the case that $\exists x P(x)$ is true? In other words, is the statement $\forall x P(x) \rightarrow \exists x P(x)$ always true? Is the converse always true? Assume the domain of discourse is non-empty.

Part 1

First we are to examine the statement $\forall x P(x) \rightarrow \exists x P(x)$. Since we are told that the antecedent is true we can conclude that there is some x that would make P(x) true. In fact, every value of x can act as evidence for the "True" truth value of the statement $\exists x P(x)$.

Part 2

Next we are to examine the converse statement, $\exists x P(x) \rightarrow \forall x P(x)$. Suppose that the antecedent is true, the fact that there exists one or more values of x that satisfy $P(x)$ does not provide enough evidence that all values of x will satisfy $P(x)$. For example, let the domain of discourse be "t-shirts at a thrift store" where x is a t-shirt at the thrift store, and P(x) is the predicate "x is blue." Now, we may find one (or many) blue t-shirts at the store but, from the information provided, we cannot conclude that every t-shirt is blue simply because one (or many) have been found.

2 Consider the statement, "For all natural numbers n, if n is prime, then n is solitary." You do not need to know what solitary means for this problem, just that it is a property that some numbers have and others do not.

Let:

$$P(n) = \text{"}n \text{ is prime"} \qquad Q(n) = \text{"}n \text{ is solitary"}$$

The statement can be rewritten as:

$$\forall n \in \mathbb{N}(P(n) \rightarrow Q(n))$$

b. Write the negation of the original statement. What would you need to show to prove that the statement is false?

The negation of an implication is equivalent to the statement P and not Q.

$$\neg(P \rightarrow Q) \equiv P \wedge \neg Q$$

The negation of the original statement can thus be written as: "There exists a natural number n such that n is prime and n is not solitary".

$$\exists n \in \mathbb{N} : (P(n) \land \neg Q(n))$$

To prove that the original statement is false, you would need to find at least one natural number n that satisfies the negation – an n that is both prime and not solitary. If such an n exists, it would contradict the original statement, proving it to be false.