Chris Hunt

DP4

MTH 231

22. Suppose that a particular real number x has the property that $x+\frac{1}{x}$ is an integer. Prove that $x^n+\frac{1}{x^n}$ is an integer for all natural numbers n.

__If,__ $\exists x \in \mathbb{R}\left(P(x): \text{“}x+\frac{1}{x} \text{ is } \in \mathbb{Z}\text{”}\right)$ is true

__Prove:__ $\forall n \in \mathbb{N}\left(Q(n,x): \text{“}x^n+\frac{1}{x^n} \text{ is an int for all } \mathbb{N}\ n\text{”}\right)$

Do I need to find some value x?

$X=1$ might satisfy the first sentence.

$$x+\frac{1}{x} = 1+\frac{1}{1}$$
$$= 2 \quad \checkmark \quad \boxed{2 \text{ is } \in \mathbb{Z}}$$

Therefore there exist some value in $\mathbb{R}$ that satisfies the first statement

__Base Case:__ $Q(0,1)$

$$x^n+\frac{1}{x^n} = 1^0+\frac{1}{1^0}$$
$$= 1+\frac{1}{1}$$
$$= 2$$

__Inductive Step:__

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28. Prove that the Fibonacci numbers satisfy the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$. One way is to prove the more general identity:

Let: $P(m,n): F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}$

When $m = n$ it is the same as above

a) Fix $m = 0$ $P(0,n)$ holds for all $n > 0$

Claim: $\forall n \in \mathbb{N} (P(0,n))$ is true

Base Case: Consider $P(0,0)$

$m = 0$ $n = 0$ Fib Sequence: 0, 1, 1, 2, 3, 5, 8

$F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}$

$0 \cdot 0 + 1 \cdot 1 = 1$ ✓ The identity holds

Inductive step: Assume $P(0,k)$ is true. For some $k \in \mathbb{Z}$

$F_0 F_k + F_1 F_{k+1} = F_{k+1}$ $F_0 = 0$ $F_1 = 1$

Prove $P(0,(k+1))$:

$F_0 F_{(k+1)} + F_1 F_{k+2} = F_{k+2}$

Def. of Fib Seq is $F_n + F_{n+1} = F_{n+2}$ when $F_0 = 0$ $F_1 = 1$

$F_{k+2} = F_k + F_{k+1}$ ✓ By the def of Fib Seq

$P(0,k)$ holds and $P(0,k+1)$ holds

By the principle of mathematical induction, $P(0,n)$ holds for all values $n \in \mathbb{N}$

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28. b) Prove for an arbitrary $n \in \mathbb{N}$ $P(n,m)$ holds for $\forall m \in \mathbb{N}$

Base case: We already proved $P(0,m)$, so let's chose $n=1$

Consider: $P(1,m)$:

$F_1 F_m + F_2 F_{m+1} = F_{m+2}$ $F_0 = 0$ $F_1 = 1$

$1 \cdot F_m + 1 \cdot F_{m+1} = F_{m+2}$ ✓ True by the def of Fib seq

Inductive Step: Still unsure about Strong Induction...

Assume $P(n,k)$ holds for all values up to k where $n, k \in \mathbb{N}$

I want to show that $P(n, k+1)$ is true.

... prove it!

Therefore, by strong induction $P(n,m)$ is true for all $m \geq 0$ for an arbitrary $n \in \mathbb{N}$

Chris Hunt $P(n)$: DP 4 MTH 231

SQ-3. Claim: $F_n < 2^n$ For all $n \geq 0$

Fib Seq: 0, 1, 1, 2, 3, 5 . . . $F_n + F_{n+1} = F_{n+2}$ $F_0 = 0$ $F_1 = 1$

<u>Base Case:</u> Concider $P(0)$

$0 < 2^0$

$0 < 1$ ✓ true!

<u>Inductive Step:</u> Assume $P(k)$ is true, for all N up to k

$F_k < 2^k$

I want to show that $P(k+1)$ is true.

$F_{k+1} < 2^{k+1}$ $F_{k+1} = F_k + F_{k-1}$ by def of Fib Seq

$F_{k+1} < 2^k \cdot 2^1$

$F_k + F_{k-1} < 2^k \cdot 2$

$F_k + F_{k-1} < 2^k + 2^k$ ✓

This is true since F_k is less than 2^k and F_{k-1} is less than 2^{k-1} which is less than 2^k.

Therefore by strong induction this statement is true.