Class Portfolio

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22. Suppose that a particular real number has the property that $x+\frac{1}{x}$ is an integer. Prove that $x^n+\frac{1}{x^n}$ is an integer for all natural numbers.

Claim. *$x^n + \frac{1}{x^n}$ is an integer for all natural numbers.*

$$\forall n \in \mathbb{N}, x^n + \frac{1}{x^n} \text{ is an integer}$$

Proof. *Let's begin by considering the base case where n = 1.*

$$x^1{+}\frac{1}{x^1}$$
$$x{+}\frac{1}{x}$$

Since $x + \frac{1}{x}$ is assumed to be an integer we can state that the claim where n=1, is true.

Now let us assume for all integers from 0 to k the claim holds true. Since we know that integers have closure under multiplication if we multiplied the integer $x^k + \frac{1}{x^k}$ by $x + \frac{1}{x}$ the product will be an integer. Consider the expansion of this product equals some integer j.

$$(x^k + \frac{1}{x^k})(x + \frac{1}{x}) = j$$
$$x^k x + \frac{x^k}{x} + \frac{x}{x^k} + \frac{1}{x^k x} = j$$
$$x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}} = j$$

Since we are assuming that the claim holds for all values up to k, it will then hold true for $k-1$. We can replace $x^{k-1} + \frac{1}{x^{k-1}}$ for some integer m

$$x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}} = j$$
$$x^{k+1} + \frac{1}{x^{k+1}} + m = j$$

Since we know that both j and m are integers and integers are closed under addition the sum of $x^{k+1} + \frac{1}{x^{k+1}}$ must also be an integer. Therefore by strong induction, the claim is true.

□

SQ-4. For all sets A and B,

a. Prove that A is a subset of B if and only if B^c is a subset of A^c.

To begin this proof we must start with proving that the cardinality of a subset of a set is less than or equal to the cardinality of the set itself.

Claim. $A \subseteq B \rightarrow |A| \leq |B|$

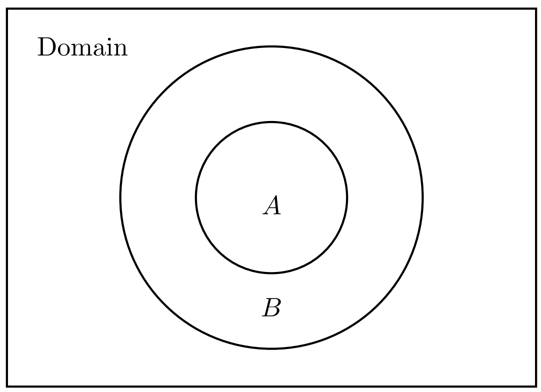
Lemma 1. *To prove this we will use proof by contradiction. Suppose the claim were not the case, then $A \subseteq B \wedge |A| > |B|$. This means that there must be at least one element in A that is not also in B, in order for this to be true A could not be a subset of B by the definition of subsets. Therefore the claim is true by contradiction.*

Claim. $A \subseteq B \leftrightarrow B^c \subseteq A^c$

Proof. *This bidirectional claim can be broken down into two implications:*

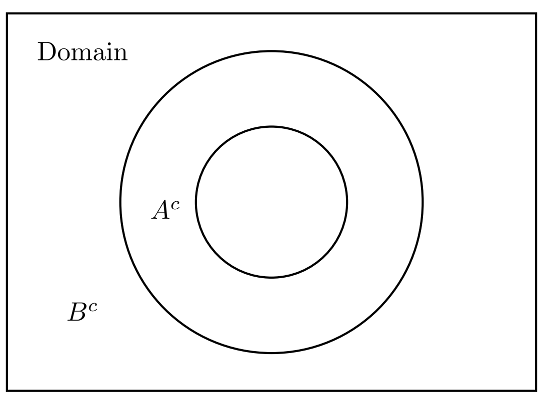
1. $A \subseteq B \rightarrow B^c \subseteq A^c$
2. $B^c \subseteq A^c \rightarrow A \subseteq B$

Let's assume the antecedant of claim one to be true. That is A is a subset of B.



Now consider B^c and A^c which are the areas in the domain which are outside the corresponding sets of A and B. Since every element of A is in B we know that A has less than or equal to the same number of elements as B by Lemma 1. Taking their compliment we know that B^c is less than or equal to A^c and since all the elements in A are also in B, we know that any element which is not in B must also not be in A which meets the definition of a subset. Therefore the implication is true.

Now Let's assume the antecedent of the second claim is true. That is every element not in B is also not in A.



Due to this fact every element not in A^c must also not be in B^c, that is every element in A is also in B. Therefore the second claim holds. Since both claims are true then the original claim is also true.

$\square$

b. Prove that if A is a subset of B and B is a subset of A, then $A = B$.

Claim. $A \subseteq B \wedge B \subseteq A \rightarrow A = B$

Proof. *Assume that $A \subseteq B$ and $B \subseteq A$ are both true. That is, every element of A is an element of B and every element of B is an element of A. Consider some arbitrary element x in the set A, this element will be in set B. Now, consider some other arbitrary element y in set B, this element will be in set A as well. Since there are no elements that can be found in A that are not in B and no elements that can be found in B that are not in A, therefore A and B contain exactly the same elements. Therefore the claim is true.*

□