

22. Suppose that a particular real number x has the property that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for all natural numbers n .

If, $\exists x \in \mathbb{R} (P(x): "x + \frac{1}{x} \text{ is } \in \mathbb{Z}")$ is true

Prove: $\forall n \in \mathbb{N} (Q(n, x): "x^n + \frac{1}{x^n} \text{ is an int for all } \mathbb{N} n")$

Do I need to find some value x ?

$x=1$ might satisfy the first sentence.

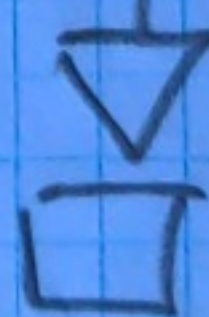
$$\begin{aligned} x + \frac{1}{x} &= 1 + \frac{1}{1} \\ &= 2 \end{aligned}$$

$$\checkmark \quad \boxed{2 \text{ is } \in \mathbb{Z}}$$

Therefore there exist some value in \mathbb{R} that satisfies the first statement

Base Case: $Q(0, 1)$

$$\begin{aligned} x^n + \frac{1}{x^n} &= 1^0 + \frac{1}{1^0} \\ &= 1 + \frac{1}{1} \\ &= 2 \end{aligned}$$



Inductive Step:

28. Prove that the Fibonacci numbers satisfy the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$. One way is to prove the more general identity:

Let: $P(m, n): F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}$

When $m = n$ it is the same as above

a) Fix $m = 0$ $P(0, n)$ holds for all $n \geq 0$

Claim: $\forall n \in \mathbb{N} (P(0, n))$ is true

Base Case: Consider $P(0, 0)$

$m = 0$ $n = 0$ Fib Sequence: 0, 1, 1, 2, 3, 5, 8

$$F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}$$

$$0 \cdot 0 + 1 \cdot 1 = 1$$

✓ The identity holds

Inductive step: Assume $P(0, k)$ is true. For some $k \in \mathbb{Z}$

$$F_0 F_k + F_1 F_{k+1} = F_{k+1} \quad F_0 = 0 \quad F_1 = 1$$

Prove $P(0, (k+1))$:

$$F_0 F_{(k+1)} + F_1 F_{k+2} = F_{k+2}$$

Def. of Fib Seq is $F_n + F_{n+1} = F_{n+2}$ when $F_0 = 0$ $F_1 = 1$

$$F_{k+2} = F_k + F_{k+1} \quad \checkmark \quad \text{By the def of Fib Seq}$$

$P(0, k)$ holds and $P(0, k+1)$ holds

By the principle of mathematical induction, $P(0, n)$ holds for all values $n \in \mathbb{N}$

28. b) Prove For an arbitrary $n \in \mathbb{N}$ $P(n, m)$ holds
for $\forall m \in \mathbb{N}$

Base case: We already proved $P(0, m)$, so let's
choose $n=1$

Consider: $P(1, m)$:

$$F_1 F_m + F_2 F_{m+1} = F_{m+2} \quad F_0 = 0 \quad F_1 = 1$$

$$1 \cdot F_m + 1 \cdot F_{m+1} = F_{m+2} \quad \checkmark \text{ True by the def of Fib seq}$$

Inductive Step: Still unsure about Strong Induction...

Assume $P(n, k)$ holds for all values up to k where
 $n, k \in \mathbb{N}$

I want to show that $P(n, k+1)$ is true.

... Prove it!

Therefore, by strong induction $P(n, m)$ is true for
all $m \geq 0$ for an arbitrary $n \in \mathbb{N}$

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$P(n)$:

DP 4

MTH 231

SQ-3. Claim: $F_n < 2^n$ For all $n \geq 0$

Fib Seq: 0, 1, 1, 2, 3, 5, ... $F_n + F_{n+1} = F_{n+2}$ $F_0 = 0$ $F_1 = 1$

Base Case: Consider $P(0)$

$$0 < 2^0$$

$$0 < 1 \quad \checkmark \quad \text{true!}$$

Inductive Step: Assume $P(k)$ is true, for all N up to k

$$F_k < 2^k$$

I want to show that $P(k+1)$ is true.

$$F_{k+1} < 2^{k+1}$$

$$F_{k+1} = F_k + F_{k-1} \quad \text{by def of Fib Seq}$$

$$F_{k+1} < 2^k \cdot 2^1$$

$$F_k + F_{k-1} < 2^k \cdot 2$$

$$F_k + F_{k-1} < 2^k + 2^k \quad \checkmark$$

This is true since

F_k is less than 2^k
and F_{k-1} is less than 2^{k-1} which is less than 2^k .

Therefore by strong induction this statement is true.