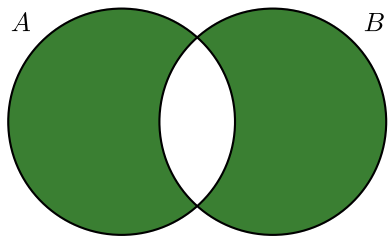
HW 5

Christopher Hunt

16. Describe a set in terms of A and B (using set notation) which has the following Venn diagram:



This Venn Diagram can be written as "The union of the sets $A - B$ and $B - A$". In set builder notation this will be:

$$A - B \cup B - A$$

To demonstrate this consider these two sets:

$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 4, 5, 6\}$$

Now subtract B from A and A from B:

$$A - B = \{1, 2\}$$
$$B - A = \{5, 6\}$$

Now take the union of these two values:

$$A - B \cup B - A = \{1, 2, 5, 6\}$$

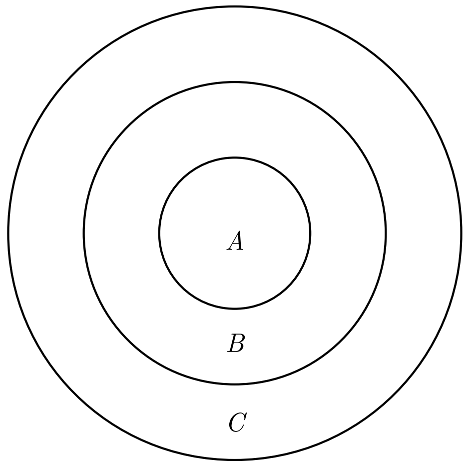
This produces the Venn diagram above.

25. Let A, B, and C be sets.

a. Suppose that $A \subseteq B$ and $B \subseteq C$. Does this mean that $A \subseteq C$?

Claim. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. *Assume $A \subseteq B$ and $B \subseteq C$ is true. By the definition of a subset we can state that every element x in A is an element in B and every element y in B is an element of C. This can be visualually demosntrated using this Venn diagram:*



Since, every element of B must be in C and every element of A must be in B it follows that every element of A must also be in C. Therefore, $A \subseteq C$ is true and the from that the original claim is true.

□

b. Suppose that $A \in B$ and $B \in C$. Does this mean that $A \in C$? Give an example to prove that this does NOT always happen.

Claim. *If $A \in B$ and $B \in C$, then $A \in C$.*

Proof. *For this claim to be false we must find a case where $A \in B$ and $B \in C$ is true but $A \in C$ is false. Consider the following sets, A, B, and C:*

$$A = \{1, 2\}$$
$$B = \{A, 3\}$$
$$C = \{B, 4\}$$

These sets fullfil the antecedent of the claim above. They can also be expressed like:

$$A = \{1, 2\}$$
$$B = \{\{1, 2\}, 3\}$$
$$C = \{\{\{1, 2\}, 3\}, 4\}$$

Viewing the expressions this way we can see that A is an element of an element of C but not directly an element itself, which violates the claim that A is an element of C if A is an element of B and B is an element of C. Therefore the claim is not always true.

□

SQ-4. For all sets A and B,

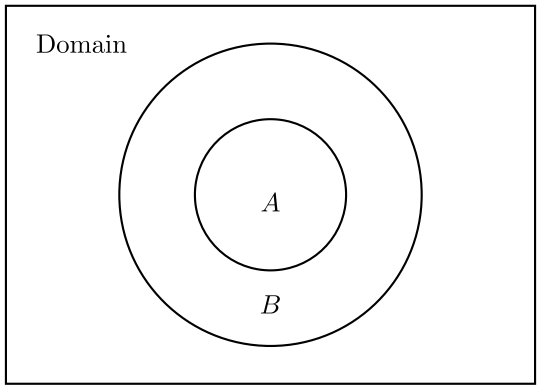
a. Prove that A is a subset of B if and only if B^c is a subset of A^c.

Claim. $A \subseteq B \leftrightarrow B^c \subseteq A^c$

Proof. *This bidirectional claim can be broken down into two implications:*

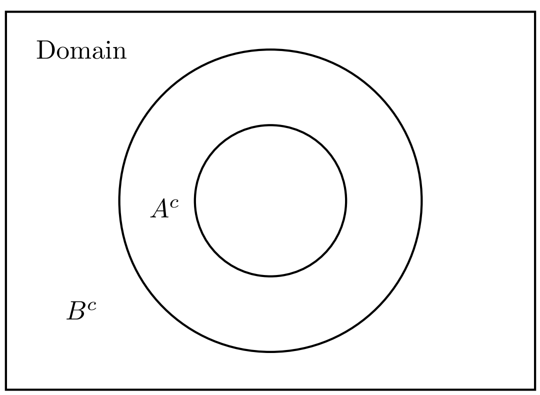
1. $A \subseteq B \rightarrow B^c \subseteq A^c$
2. $B^c \subseteq A^c \rightarrow A \subseteq B$

Let's assume the antecedant of claim one to be true. That is A is a subset of B.



Now consider B^c and A^c which are the areas in the domain which are outside the corresponding sets of A and B. Since every element of A is in B we know that A has less than or equal to the same number of elements as B. Taking their compliment we know that B^c is less than or equal to A^c and since all the elements in A are also in B, we know that any element which is not in B must also not be in A which meets the definition of a subset. Therefore the implication is true.

Now Let's assume the antecedent of the second claim is true. That is every element not in B is also not in A.



Due to this fact every element not in A^c must also not be in B^c, that is every element in A is also in B. Therefore the second claim holds. Since both claims are true then the original claim is also true.

□

b. Prove that if A is a subset of B and B is a subset of A, then $A = B$.

Claim. $A \subseteq B \land B \subseteq A \rightarrow A = B$

Proof. *Assume that $A \subseteq B$ and $B \subseteq A$ are both true. That is, every element of A is an element of B and every element of B is an element of A. Consider some arbitrary element x in the set A, this element will be in* *set B. Now, consider some other arbitrary element y in set B, this element will be in set A as well. Since there are no elements that can be found in A that are not in B and no elements that can be found in B that are not in A, therefore A and B contain exactly the same elements. Therefore the claim is true.*

$\square$

SQ-5. Prove or disprove: if A is a subset of B, the $\mathscr{P}(A)$ is a subset of $\mathscr{P}(B)$.

Claim. $A \subseteq B \rightarrow \mathscr{P}(A) \subseteq \mathscr{P}(B)$

Proof. *Assume $A \subseteq B$, that is, every element of A is an element of B. Now consider an arbitrary element $x \in \mathscr{P}(A)$. Since x is in the powerset of A we know that $x \subseteq A$ and due to the transitive property of subsets (proof 25a) x must be a subset of B as well. Because $x \subseteq B$ it follows that $x \in \mathscr{P}(B)$. Since x can be any arbitrary element of $\mathscr{P}(A)$, then it follows that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. Therefore, the claim is true.*

$\square$