The Charge

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According to Coloumb's Law, the electrostatic field $\vec{E}$ due to a charge q at the origin is given by:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

Get ready:

First we verify that $\frac{\vec{r}}{|\vec{r}|^3} = \frac{1}{r^2}\hat{r}$, where r is the radius in spherical coordinates. The general vector equation for a sphere in spherical coordinates is:

$$\vec{r} = r\dot{\hat{r}}$$

Substituting that in to the equation $\frac{\vec{r}}{|\vec{r}|^3}$ we get:

$$\frac{r}{r^3}\hat{r} \to \frac{1}{r^2}\hat{r}$$

Set:

Next we find the flux of the vector field $\vec{E}$ through the surface S, where S is the outward oriented sphere of radius $a > 0$ centered at the origin, including units. The flux of a vector field through some surface can be expressed as $\int_S \vec{E} \cdot d\vec{A}$. The electrostatic field $\vec{E}$ has units $\frac{N}{C}$ and the surface differential of the sphere has units m^2, taking the integral of the dot product of these two values will then give us units of:

$$\frac{Nm^2}{C}$$

Now we find the flux through the surface. Begin by finding the surface differential of the sphere:

$$r = a \to dr = 0 \qquad \vec{r} = a\hat{r} \to d\vec{r} = ad\theta\hat{\theta} + asin(\theta)d\phi\hat{\phi}$$

Hold $d\phi$ constant:

$$d\vec{r_1} = ad\theta\hat{\theta}$$

Hold $d\theta$ constant:

$$d\vec{r_2} = asin(\theta)d\phi\hat{\phi}$$

Take the cross product of these two vectors to find $d\vec{A}$:

$$d\vec{A} = d\vec{r_1} \times d\vec{r_2} = a^2 sin(\theta)d\theta d\phi\hat{r}$$

Now use $\int_S \vec{E} \cdot d\vec{A}$ to find the flux:

$$\int_S \vec{E} \cdot d\vec{A} \to \int_0^{2\pi}\int_0^{\pi}\left(\frac{q}{4\pi\epsilon_0} \cdot \frac{1}{a^2}\hat{r}\right) \bullet \left(a^2 sin(\theta)d\theta d\phi\hat{r}\right)$$

$$\frac{q}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi} sin(\theta)d\theta d\phi = \frac{q}{4\pi\epsilon_0}\int_0^{2\pi}(-cos(\theta)|_0^{\pi}) = \frac{q}{4\pi\epsilon_0}\int_0^{2\pi} 2d\phi = \frac{q}{2\pi\epsilon_0}(\phi|_0^{2\pi}) = \frac{q}{\epsilon_0}$$

The flux through a sphere of radius $a > 0$ is:

$$\frac{q}{\epsilon_0}\frac{Nm^2}{C}$$

Go:

This flux integral shows that for any constant point charge, the flux through a sphere of any radius, $r > 0$, with the charge at the center will be the value of the charge inversely proportional to the absolute dielectric permittivity of classical vacuum.

$$\Phi = \oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \frac{Nm^2}{C}$$