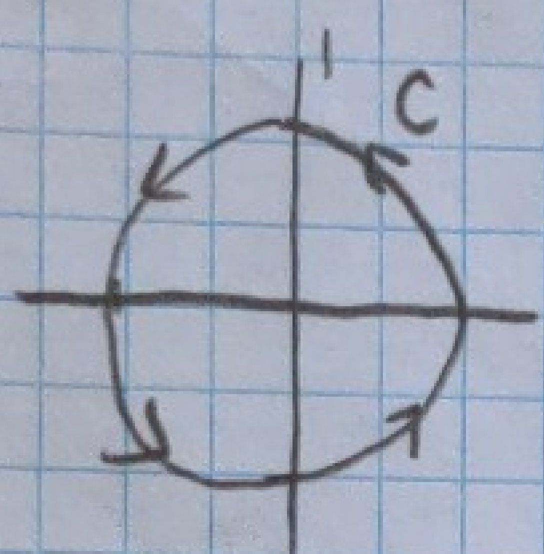
Chris Hunt HW10 MTH 255

2. Green's Theorem states that the circulation of a vector field around a simple closed curve in 2D is equal to the flux of the curl of the vector field through the region enclosed by the curve.

a) $$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F})\cdot d\vec{A} \quad \text{in 2D only}$$

For example, consider the unit circle, oriented counterclockwise, centered at the origin and a 2D vector field $\vec{F}(x,y)$:



$$\vec{F}(x,y) = (x+2y)\hat{\imath} + x^2\hat{\jmath} \qquad x = r\cos(\phi) \quad y = r\sin(\phi)$$
$$r = 1 \qquad 0 \le \phi \le 2\pi$$
$$\vec{F}(\phi) = (\cos(\phi) + 2\sin\phi)\hat{\imath} + \cos^2(\phi)\hat{\jmath}$$
$$\vec{r}(\phi) = \cos(\phi)\hat{\imath} + \sin(\phi)\hat{\jmath}$$
$$d\vec{r}(\phi) = (-\sin(\phi)\hat{\imath} + \cos(\phi)\hat{\jmath})\, d\phi$$

b) $$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} -\sin(\phi)\cos(\phi) - 2\sin^2(\phi) + \cos^3(\phi)\, d\phi$$

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = -2\pi}$$

$$\iint_R (\vec{\nabla}\times\vec{F})\cdot d\vec{A} \qquad r \to 0 \le r \le 1 \quad \phi \to 0 \le \phi \le 2\pi$$
$$d\vec{A} = r\,dr\,d\phi\,\hat{k} \to dr\,d\phi\,\hat{k}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y & x^2 & 0 \end{vmatrix} = (2x-2)\hat{k}$$
$$\downarrow$$
$$(2\cos\phi - 2)\hat{k}$$

$$\iint_R (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = \int_0^{2\pi}\int_0^1 2r\cos(\phi) - 2r\, dr\, d\phi$$

$$\boxed{\iint_R (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = -2\pi}$$