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"Round" Basis Vectors

MTH 255

1. $\vec{r} = 3\cos\phi\,\hat{\imath} + 3\sin\phi\,\hat{\jmath}$

a) To find a unit vector tangent to the curve take the derivative of the parametric equation.

$$\frac{d\vec{r}}{d\phi} = -3\sin\phi\,\hat{\imath} + 3\cos\phi\,\hat{\jmath}$$

Then normalize by scaling by the magnitude

$$\hat{u} = \frac{-3\sin\phi\,\hat{\imath}}{3} + \frac{3\cos\phi\,\hat{\jmath}}{3} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

b) To find a unit vector normal to the curve pointing in the direction of bending find a vector that is perpendicular to $\hat{u}$ using the dot product.

$$\vec{r} \cdot \hat{u} = -3\cos\phi\sin\phi + 3\cos\phi\sin\phi = 0$$

$\vec{r} \cdot \hat{u}$ is perpendicular!

Therefore $\hat{n}$ will be $-\frac{\vec{r}}{\|\vec{r}\|}$

$$\hat{n} = -\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}$$

2. Consider a circle of radius 3 at the origin

a) $\hat{u} = \hat{\phi}$ $\qquad \hat{n} = -\hat{r}$

3. a) $\hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$

$\hat{r} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$

3. b) Find $\frac{d\hat{r}}{dt}$ and $\frac{d\hat{\phi}}{dt}$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\cos(\phi(t))\hat{i} + \sin(\phi(t))\hat{j}\right)$$

$$\frac{d\hat{r}}{dt} = -\frac{d\phi}{dt}\sin(\phi(t))\hat{i} + \underbrace{\cancel{\cos(\phi(t))\frac{d\hat{i}}{dt}}}_{0} + \frac{d\phi}{dt}\cos(\phi(t))\hat{j} + \underbrace{\cancel{\sin(\phi(t))\frac{d\hat{j}}{dt}}}_{0}$$

$$\frac{d\hat{r}}{dt} = \frac{d\phi}{dt}\left(-\sin(\phi(t))\hat{i} + \cos(\phi(t))\hat{j}\right)$$

$$\boxed{\frac{d\hat{r}}{dt} = \frac{d\phi}{dt}\hat{\phi}}$$

$$\frac{d\hat{\phi}}{dt} = \frac{d}{dt}\left(-\sin(\phi(t))\hat{i} + \cos(\phi(t))\hat{j}\right)$$

$$\frac{d\hat{\phi}}{dt} = -\frac{d\phi}{dt}\cos(\phi(t))\hat{i} - \underbrace{\cancel{\sin(\phi(t))\frac{d\hat{i}}{dt}}}_{0} - \frac{d\phi}{dt}\sin(\phi(t))\hat{j} + \underbrace{\cancel{\cos(\phi(t))\frac{d\hat{j}}{dt}}}_{0}$$

$$\frac{d\hat{\phi}}{dt} = -\frac{d\phi}{dt}\left(\cos(\phi(t))\hat{i} + \sin(\phi(t))\hat{j}\right)$$

$$\boxed{\frac{d\hat{\phi}}{dt} = -\frac{d\phi}{dt}\hat{r}}$$

c) As we move along the curve the $\hat{r}$ and $\hat{\phi}$ basis vectors change with respect to the rate of change of the time dependent variable, ϕ in our case.

d) For $\hat{i}$ and $\hat{j}$, as we move along the curve they remain constant, therefor their time derivatives will be zero.