Math 255 Write Up 2

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The Ant

Part 1

An ant crawls along the radius from the center to the edge of a circular disk of radius 1 meter, moving at a constant rate of .01 m/sec. Meanwhile, the disk is turning counterclockwise about its center at 1 revolution/sec.

From this problem statement we acquire two equations and a range of time:

$$r(t) = .01t\ m \qquad \phi(t) = 2\pi t\ radian \qquad 0 \leq t \leq 100\ second$$

Where r(t) is describing the motion of the ant and $\phi(t)$ is describing the motion of the turntable as it spins counter clockwise.

We have previously found that the general equation for a parametric curve in polar is $\vec{r}(t) = r(t)\hat{r}$ and that $\hat{r} = cos(\phi(t))\hat{\imath} + sin(\phi(t))\hat{\jmath}$ we can write our parametric curve in polar or rectangular as:

$$\vec{r}(t) = .01t\hat{r} = .01t\cos\phi(t)\hat{\imath} + .01t\sin\phi(t)\hat{\jmath}$$

To find the velocity of the ant we can use the derivative of the polar form of $\vec{r}(t)$ which we have already found to be $\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r(t)\frac{d\phi}{dt}\hat{\phi}$

Take the derivate of r(t) and $\phi(t)$: $\frac{dr}{dt} = .01\frac{m}{s} \qquad \frac{d\phi}{dt} = 2\pi rad$

$$\frac{d\vec{r}}{dt} = .01\hat{r} + .02\pi t\hat{\phi}\ \frac{m}{s}$$

This equation is the velocity of the ant as it moves, but what is it's speed? To find the speed we need to take the velocity and find it's magnitude using the Pythagorean Theorem.

$$|\frac{d\vec{r}}{dt}| = \sqrt{.0001 + .0004\pi^2 t^2}\ \frac{m}{s}$$

To find the ant's acceleration we take the general form for acceleration we defined in HW1 and plug in all of our expressions defined above:

$$\frac{d^2\vec{r}}{dt^2} = (\frac{d^2r}{dt^2} - r(t)(\frac{d\phi}{dt}^2))\hat{r} + (2\frac{dr}{dt}\frac{d\phi}{dt} + r(t)\frac{d^2\phi}{dt^2})\hat{\phi}$$

$$\frac{d^2r}{dt^2} = 0 \qquad \frac{d^2\phi}{dt^2} = 0$$

$$\frac{d^2\vec{r}}{dt^2} = (0 - .04\pi^2 t)\hat{r} + (.04\pi + 0)\hat{\phi}$$

$$\frac{d^2\vec{r}}{dt^2} = -.04\pi^2 t\hat{r} + .04\pi\hat{\phi}\ \frac{m}{s^2}$$

The magnitude of this acceleration is found the same way as above, by using the Pythagorean Theorem.

$$|\frac{d^2\vec{r}}{dt^2}| = \sqrt{.0016\pi^4 t^2 + .0004\pi^2}\ \frac{m}{s^2}$$

Part 2

Now consider the same problem but the turntable is also move upwards at .02 meters per second.

Our known values are:

$$r(t) = .01tm \qquad \phi(t) = 2\pi trad \qquad z(t) = .02tm \qquad 0 \leq t \leq 100$$

The differential, $d\vec{r}(t)$, in rectangular and cyclindrical coordinates is known to be:

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} = dr\hat{r} + r(t)d\phi\hat{\phi} + dz\hat{z}$$

All we are missing is the dz $\rightarrow dz = .02\ m$

Plug everything into $d\vec{r}$ in the equation for cylindrical coordinates.

$$d\vec{r} = .01\hat{r} + .02\pi t\hat{\phi} + .02\hat{z}$$

This would then give us a speed of:

$$|\frac{d\vec{r}}{dt}| = \sqrt{.0005 + .0004\pi^2 t^2}\ \frac{m}{s}$$

Acceleration would be the second derivative of $r(t)$. Since the second derivative of the z(t) is 0 $\rightarrow \frac{d^2z}{dt^2} = 0$ the ant's acceleration would be the same as Part 1:

$$\frac{d^2\vec{r}}{dt^2} = -.04\pi^2 t\hat{r} + .04\pi\hat{\phi}\ \frac{m}{s^2}$$

$$|\frac{d^2\vec{r}}{dt^2}| = \sqrt{.0016\pi^4 t^2 + .0004\pi^2}\ \frac{m}{s^2}$$

The Puddle

The depth of a puddle in millimeters is given by:

$$h(x,y) = \frac{1}{10}(1 + sin(\pi xy))\ mm$$

A path through the puddle is given by the parameterization:

$$x(t) = 3t \qquad y(t) = 4t\ \rightarrow \vec{r}(t) = 3t\hat{\imath} + 4t\hat{\jmath}$$

With a current position of $x = 1\ mm \quad y = 4\ mm$

Recall the master formula!

$$df = \nabla F \cdot d\vec{r}$$

(a) At your current position, how fast is the depth of water through which you are walking changing per unit time?

Begin by finding the gradient vector ∇H and $d\vec{r}$

$$\frac{\partial h}{\partial x} = \frac{\pi y}{10} cos(\pi xy) \qquad \frac{\partial h}{\partial y} = \frac{\pi x}{10} cos(\pi xy) \qquad \nabla H = \frac{\partial h}{\partial x}\hat{\imath} + \frac{\partial h}{\partial y}\hat{\jmath}$$
$$\nabla H = \frac{\pi y}{10} cos(\pi xy)\hat{\imath} + \frac{\pi x}{10} cos(\pi xy)\hat{\jmath}\ \frac{mm}{s}$$
$$\frac{d\vec{r}}{dt} = 3\hat{\imath} + 4\hat{\jmath}\ \frac{mm}{s}$$

With this we can then use the master formula to find $\frac{dh}{dt}$.

$$\frac{dh}{dt} = \nabla H \cdot \frac{d\vec{r}}{dt} \rightarrow \frac{dh}{dt} = \frac{3\pi y}{10} cos(\pi xy)\hat{\imath} + \frac{4\pi x}{10} cos(\pi xy)\hat{\jmath}$$

Plug in our values of x and y to find the change in mm per unit time at that position.

$$dh = \frac{24\pi}{10} cos(12\pi) \rightarrow dh = 7.5398\ \frac{mm}{s}$$

(b) At your current position, how fast is the depth of water through which you are walking changing per unit distance?

For this question we are looking for the change in depth with respect to change per unit distance. This will be found by finding the magnitude of the vector $d\vec{r}$.

$$|\vec{r}| = \sqrt{3^2 + 4^2} = 5 = ds$$

Then to find $\frac{dh}{ds}$ we need just divide the master formula by it and then plug in our current position.

$$\frac{dh}{ds} = \frac{3\pi y}{5 * 10} cos(\pi xy)\hat{\imath} + \frac{4\pi x}{5 * 10} cos(\pi xy)\hat{\jmath}$$
$$\frac{dh}{ds} = \frac{24\pi}{50} cos(12\pi) \rightarrow \frac{dh}{ds} = 1.5080\ \frac{mm}{mm}$$

(c) Optional food for thought There is a walkway over the puddle at x = 10. At your current position, how fast is the depth of water through which you are walking changing per unit distance towards the walkway.

Using the master formula, we can find dh along any path in a gradient vector field. Since the bridge is located at x = 10 we know that we would simple need to move along a path such as $\vec{r}(t) = t\ \hat{i}$. We can then find $d\vec{r} = \hat{i}$ and $ds = |d\vec{r}| = 1$

Now to find this new $\frac{dh}{ds}$ use the master formula divided by ds and plug in our current position.

$$\frac{dh}{ds} = \frac{12\pi}{10} cos(12\pi) = 3.7699\ \frac{mm}{mm}$$