Stokes' Theorem

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Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ explicitly as a line integral, where $\vec{F} = r^3\hat{\phi}$, r and ϕ are cylindrical coordinates, and C is the circle of radius 3 in the xy-plane, oriented in the usual, counterclockwise direction.

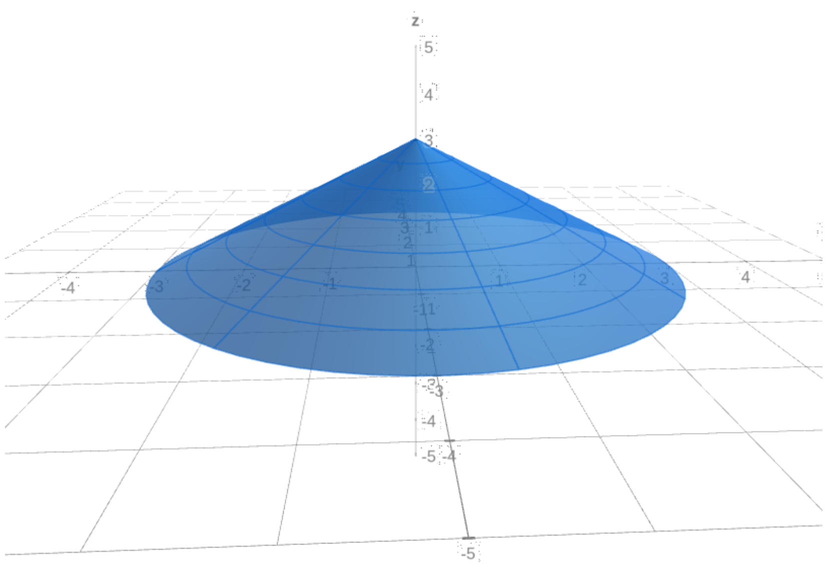
$$\vec{F} = r^3\hat{\phi} \qquad \vec{r} = 3\hat{r} \qquad d\vec{r} = 3d\phi\hat{\phi}$$

$$\oint_C \vec{F} \cdot d\vec{r} \to \int_0^{2\pi} (r^3\hat{\phi}) \cdot 3d\phi\ \hat{\phi} \to \int_0^{2\pi} 3r^3\ d\phi \to 162\pi$$

2 Stokes' Theorem

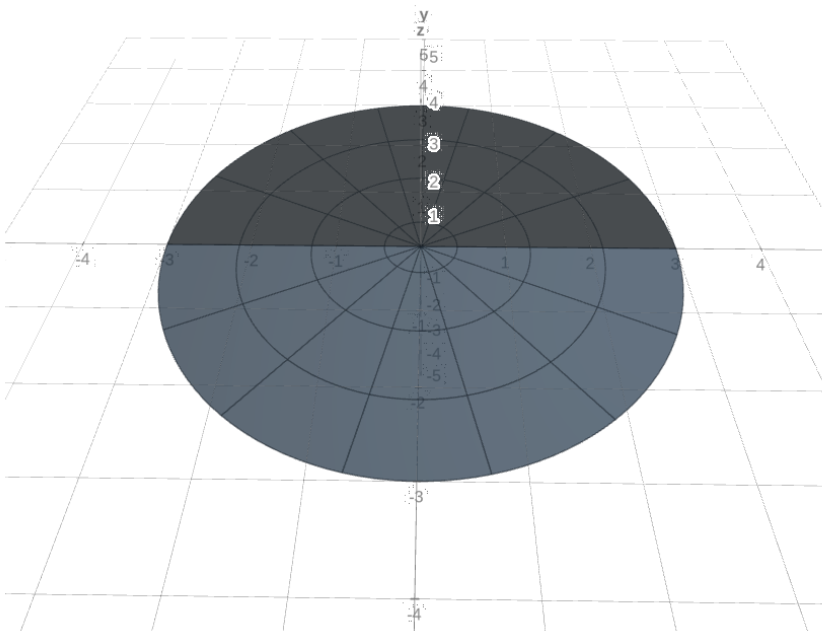
(a) List at least three different surfaces which you could use with Stokes's Theorem to evaluate the line integral in the previous problem.

Cone:



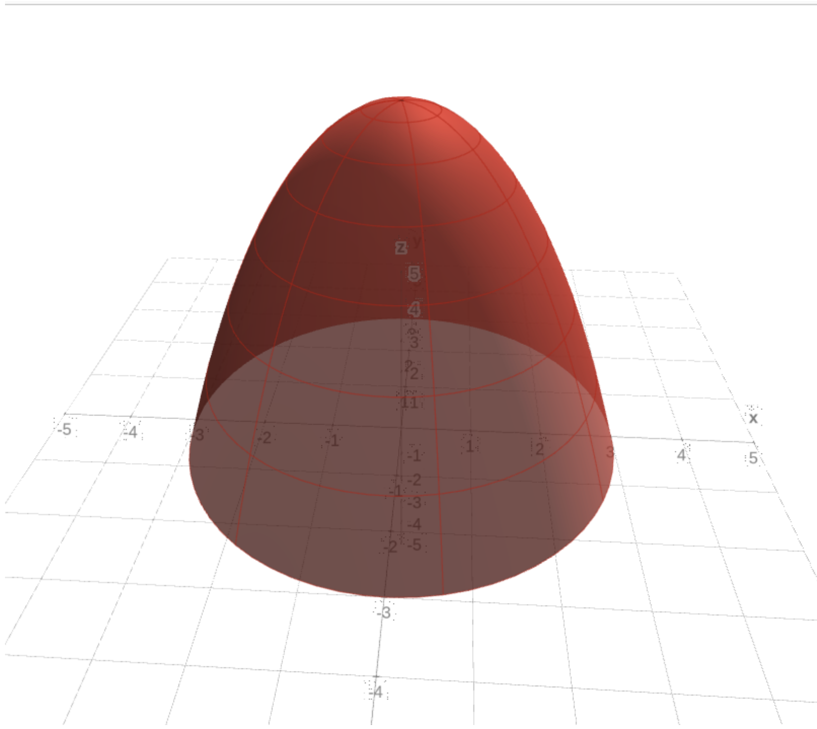
$$\vec{r} = r\ \hat{r} + (3 - r)\ \hat{k}$$

Disk:



$$\vec{r} = r\ \hat{r} + 0\ \hat{k}$$

Paraboloid:



$$\vec{r} = r\ \hat{r} + (9 - r^2)\ \hat{k}$$

(b) Evaluate the surface integral for any one of the surfaces on your list.

Cone:

$$\vec{r} = r\,\hat{r} + (3-r)\,\hat{k} \qquad d\vec{r} = dr\hat{r} + rd\phi\hat{\phi} - dr\hat{k} \quad \rightarrow \quad d\vec{r_1} = dr(\hat{r} - \hat{k}) \qquad d\vec{r_2} = rd\phi\hat{\phi}$$

$$d\vec{A} = d\vec{r_1} \times d\vec{r_2} = rdrd\phi\hat{r} + rdrd\phi\hat{k}$$

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = 4r^3 drd\phi$$

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} \rightarrow \int_0^{2\pi}\int_0^3 4r^3 drd\phi \rightarrow 162\pi$$

The surface integral is the same as the line integral of it's boundary, the circle at the origin with radius 3.