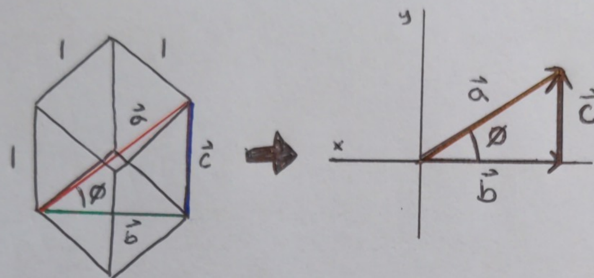
Chris Hunt HW1 MTH 255

1. Find the angle between two of the interior diagonals of a unit cube.



$\|\vec{a}\| = \sqrt{3}$

$\|\vec{b}\| = \sqrt{2}$

$\|\vec{c}\| = 1$

Use the law of cosines to find ϕ

$$\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\phi$$

$$1^2 = \sqrt{3}^2 + \sqrt{2}^2 - 2\sqrt{3}\sqrt{2}\cos\phi$$

$$1 = 5 - 2\sqrt{6}\cos\phi$$

$$-4 = -2\sqrt{6}\cos\phi$$

$$\cos\phi = \frac{2}{\sqrt{6}}$$

$$\phi = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$$

$$\phi = .6154797087$$

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2. Consider the polar position vector $\vec{r} = r(t)\hat{r}$
Find the velocity vector in polar.

Recall that $\hat{r} = \cos(\phi(t))\hat{\imath} + \sin(\phi(t))\hat{\jmath}$ and that

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\cos(\phi(t))\hat{\imath} + \sin(\phi(t))\hat{\jmath}\right)$$

$$\frac{d\hat{r}}{dt} = -\frac{d\phi}{dt}\sin(\phi(t))\hat{\imath} + \cos(\phi(t))\underbrace{\cancel{\frac{d\hat{\imath}}{dt}}}_{0} + \frac{d\phi}{dt}\cos(\phi(t))\hat{\jmath} + \sin(\phi(t))\underbrace{\cancel{\frac{d\hat{\jmath}}{dt}}}_{0}$$

$$\frac{d\hat{r}}{dt} = \frac{d\phi}{dt}\left(-\sin(\phi(t))\hat{\imath} + \cos(\phi(t))\hat{\jmath}\right)$$

Recall that $\hat{\phi} = -\sin(\phi(t))\hat{\imath} + \cos(\phi(t))\hat{\jmath}$

$$\frac{d\hat{r}}{dt} = \frac{d\phi}{dt}\hat{\phi}$$

Now Find $\frac{d\vec{r}}{dt}$:

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r(t)\hat{r}\right)$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r(t)\frac{d\hat{r}}{dt}$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r(t)\frac{d\phi}{dt}\hat{\phi}$$

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2. Bonus: Find the acceleration vector in polar.

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{r} + r(t)\frac{d\phi}{dt}\hat{\phi}\right)$$

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\phi}{dt}\hat{\phi}\right) + \left(\frac{dr}{dt}\frac{d\phi}{dt}\hat{\phi} + r(t)\frac{d^2\phi}{dt^2}\hat{\phi} - r(t)\left(\frac{d\phi}{dt}\right)^2\hat{r}\right)$$

$$\boxed{\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt} - r(t)\left(\frac{d\phi}{dt}\right)^2\right)\hat{r} + \left(2\frac{dr}{dt}\frac{d\phi}{dt} + r(t)\frac{d^2\phi}{dt^2}\right)\hat{\phi}}$$

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3.1 Find $d\vec{r}$ of the curve $y = x^2$

$\frac{dy}{dx} = 2x \rightarrow dy = 2x\,dx$

Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$

$\left(\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}\right) dt \rightarrow d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$

$d\vec{r} = dx\,\hat{\imath} + 2x\,dx\,\hat{\jmath} \rightarrow \boxed{d\vec{r} = (\hat{\imath} + 2x\hat{\jmath})\,dx}$

Find $d\vec{r}$ of the curve $x = y^2$

$\frac{dx}{dy} = 2y \rightarrow dx = 2y\,dy$

Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$

$\left(\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}\right) dt \rightarrow d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$

$d\vec{r} = 2y\,dy\,\hat{\imath} + dy\,\hat{\jmath} \rightarrow \boxed{d\vec{r} = (2y\hat{\imath} + \hat{\jmath})\,dy}$

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3. Continued: Find $d\vec{r}$ for the curve $4 = x^2 + y^2$

$4 = x^2 + y^2 \rightarrow y = (4 - x^2)^{1/2}$

$\frac{dy}{dx} = -x(4-x^2)^{1/2} \rightarrow dy = -x(4-x^2)^{-1/2}\, dx$

Let $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$

$\left(\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}\right) dt \rightarrow d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$

$d\vec{r} = dx\,\hat{\imath} + \left(-x(4-x^2)^{-1/2}\,dx\right)\hat{\jmath}$

$$\boxed{d\vec{r} = \left(\hat{\imath} - x(4-x^2)^{-1/2}\,\hat{\jmath}\right) dx}$$

Find $d\vec{r}$ for the curve $r(\phi) = 3 - \cos\phi$ in polar

$r(\phi) = 3 - \cos\phi \rightarrow \frac{dr}{d\phi} = \sin\phi \rightarrow dr = \sin\phi\, d\phi$

Let $\vec{r}(t) = r(t)\,\hat{r}$

$\left(\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r(t)\frac{d\phi}{dt}\hat{\phi}\right) dt \rightarrow d\vec{r} = dr\,\hat{r} + r(t)\,d\phi\,\hat{\phi}$

Let $r(t) = r(\phi)$

$d\vec{r} = \sin\phi\, d\phi\,\hat{r} + r(t)\,d\phi\,\hat{\phi} \rightarrow$ $\rightarrow d\vec{r} = (\sin\phi\,\hat{r} + (3-\cos\phi)\hat{\phi})d\phi$

$$\boxed{d\vec{r} = (\sin\phi\,\hat{r} + (3-\cos\phi)\hat{\phi})\, d\phi}$$