Surfaces

Christopher Hunt

(a)

Find the surface element of the following surfaces:

i

The hyperbolic paraboloid $z = x^2 - y^2$.

$$z = x^2 - y^2 \quad dz = 2xdx - 2ydy$$
$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} \rightarrow d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + (2xdx - 2ydy)\hat{k}$$

Find $d\vec{A}$:
Hold y constant: $d\vec{r_1} = dx\hat{\imath} + 2xdx\hat{k}$
Hold x constant: $d\vec{r_2} = dy\hat{\jmath} - 2ydy\hat{k}$

$$d\vec{A} = d\vec{r_1} \times d\vec{r_2} = \hat{\imath}(0 - 2xdxdy) - \hat{\jmath}(-2ydxdy - 0) + \hat{k}(dxdy - 0)$$
$$d\vec{A} = (-2x\hat{\imath} + 2y\hat{\jmath} + \hat{k})dxdy$$

ii

The ellipical paraboloid $z = x^2 + 4y^2$.

$$z = x^2 + 4y^2 \quad dz = 2xdx + 8ydy$$
$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} \rightarrow d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + (2xdx + 8ydy)\hat{k}$$

Find $d\vec{A}$:
Hold y constant: $d\vec{r_1} = dx\hat{\imath} + 2xdx\hat{k}$
Hold x constant: $d\vec{r_2} = dy\hat{\jmath} + 8ydy\hat{k}$

$$d\vec{A} = d\vec{r_1} \times d\vec{r_2} = \hat{\imath}(0 - 2xdxdy) - \hat{\jmath}(8ydxdy - 0) + \hat{k}(dxdy - 0)$$
$$d\vec{A} = (-2x\hat{\imath} - 8y\hat{\jmath} + \hat{k})dxdy$$

iii

The paraboloid $z = x^2 + y^2$ in cylindrical coordinates.

$$z = x^2 + y^2 \rightarrow z = r^2;\ Let\ r \geq 0 \rightarrow r = z^{\frac{1}{2}} \quad dr = \frac{1}{2}z^{-\frac{1}{2}}dz$$
$$d\vec{r} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{k} \rightarrow d\vec{r} = \frac{1}{2}z^{-\frac{1}{2}}dz\hat{r} + z^{\frac{1}{2}}d\phi\hat{\phi} + dz\hat{k}$$

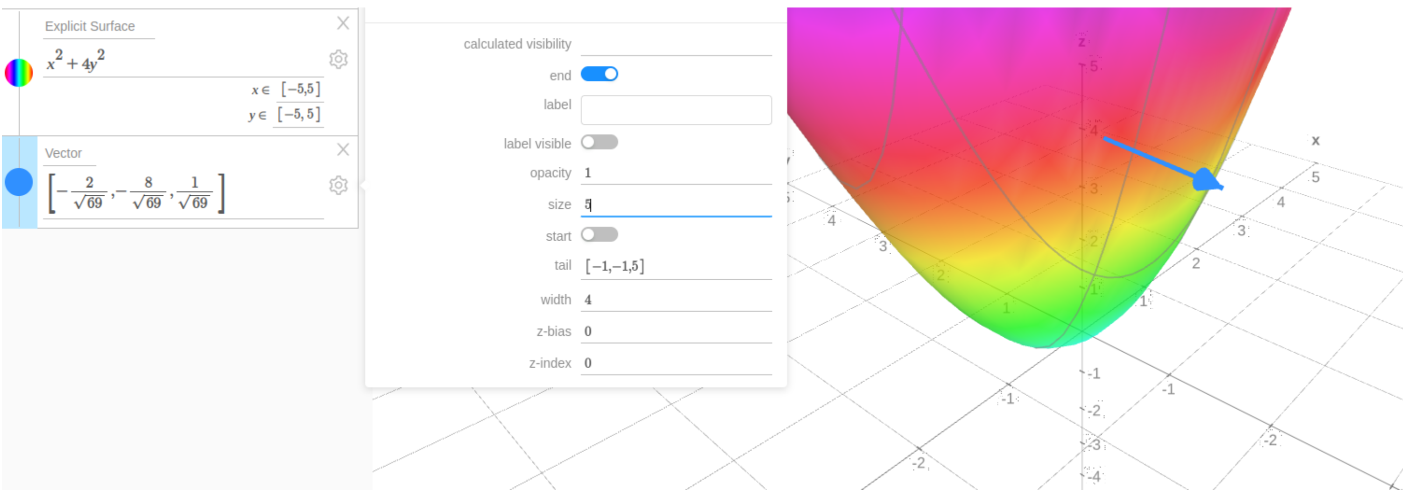
Find $d\vec{A}$:
Hold ϕ constant: $d\vec{r_1} = \frac{1}{2}z^{-\frac{1}{2}}dz\hat{r} + dz\hat{k}$
Hold z constant: $d\vec{r_2} = z^{\frac{1}{2}}d\phi\hat{\phi}$

$$d\vec{A} = d\vec{r_1} \times d\vec{r_2} = \hat{r}(0 - z^{\frac{1}{2}}d\phi dz) - \hat{\phi}(0-0) + \hat{k}(\frac{1}{2}d\phi dz - 0)$$

$$d\vec{A} = (-z^{\frac{1}{2}}\hat{r} + \frac{1}{2}\hat{k})d\phi dz$$

(b)

Plot of the surface $z = x^2 + 4y^2$ with $d\vec{A}$ at point $(-1,-1,5)$ scaled to a unit vector and sign adjusted to point away from the surface



(c)

Find the surface area of the hyperbolic paraboloid $z = x^2 - y^2$ on the domain $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ by setting up and evaluating the appropriate integral.
In Part A it was found that the surface differential for a hyperbolic paraboloid and the magnitude of it is:

$$d\vec{A} = (-2x\hat{i} + 2y\hat{j} + \hat{k})dxdy \quad |d\vec{A}| = \sqrt{4x^2 + 4y^2 + 1dxdy}$$

To find the surface area we can take the integral of the magnitude of the surface differential; $d\vec{A}$, and add it to the surface area of the base.
The surface area of the hyperbolic paraboloid is:

$$SA_{hp} = \int_S |d\vec{A}| \rightarrow \int_{-1}^{1}\int_{-1}^{1} \sqrt{4x^2 + 4y^2 + 1}\ dxdy$$

This integral will be evaluated using two different numerical methods.

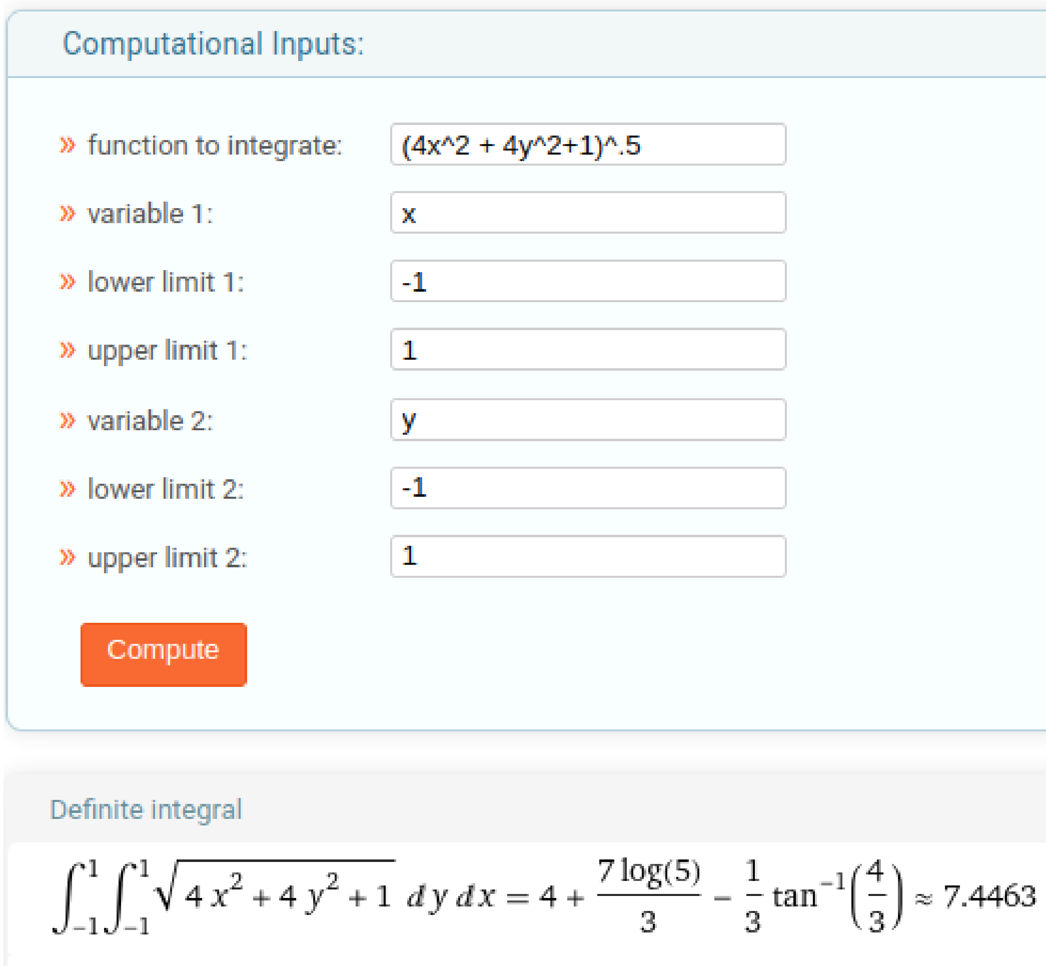
First is using this python code:

```python
import numpy as np

def func(x,y):
    return ((4*x**2+4*y**2+1)**.5)

x_1 = -1
x_2 = 1
y_1 = -1
y_2 = 1
steps = 10**4
dx = (x_2-x_1)/steps
dy = (y_2-y_1)/steps
x = np.linspace((x_1+dx/2),(x_2-dx/2),steps)
y = np.linspace((y_1+dy/2),(y_2-dy/2),steps)
sum=0

for step1 in x:
    for step2 in y:
        sum += (func(step1,step2)*dx*dx)

print(sum)
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL

```
chunt@ixuix:~/Code/Random/random$ python3 doubleintegral.py
7.44625670154616
```

Second using WolframAlpha:



The approximate value of the integral is 7.4463

The surface area of the base is:

$$SA_{base} = (1 - -1)(1 - -1) = 4$$

From these calculations, the total surface area of the hyperbolic paraboloid within the bounds described above is 11.4463.