Chris Hunt HW10 MTH255

3. a) The fundamental theorem of line integrals states that for any conservative vector field the work done by the vector field along a path will equal the difference between the vector field's potential function evaluated at the end and begining.

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{begin}) \text{ if } \vec{F} = \vec{\nabla} f$$

For example: $\vec{F} = y\hat{i} + x\hat{j}$ and $f(x,y) = xy$

Let our path be $y = x^2$ from $1 \le x \le 3$

$\vec{r} = x\hat{i} + y\hat{j} \rightarrow \vec{F} = x\hat{i} + x^2\hat{j} \rightarrow d\vec{r} = dx\hat{i} + 2x\,dx\hat{j}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^3 3x^2\,dx = \boxed{26}$$

$$f(\text{end}) - f(\text{begin}) = f(3,9) - f(1,1) = 3 \cdot 9 - 1 \cdot 1 = \boxed{26}$$

b) The Divergence Theorem states that the flux through a closed surface area is equal to the divergence of that vector field times the volume enclosed by the surface

$$\iint_{\partial V} \vec{F} \cdot d\vec{A} = \iiint_V \vec{\nabla} \cdot \vec{F}\,dV$$

Consider a unit sphere, $\vec{r} = \hat{r}$, and a spherical vector field $\vec{F} = 3r\hat{r} + 0\hat{\theta} + 0\hat{\phi}$

Surface Differential: $d\vec{A} = r^2 \sin(\theta)\,d\theta\,d\phi\,\hat{r}$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

$0 \le r \le 1$

$$\iint_{\partial V} \vec{F} \cdot d\vec{A} = \int_0^{2\pi} \int_0^{\pi} 3(1)^3 \sin(\theta)\,d\theta\,d\phi = \boxed{12\pi}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} = 9$$

$$\iiint_V \vec{\nabla} \cdot \vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 9r^2 \sin\theta\,d\theta\,d\phi\,dr = \boxed{12\pi}$$