The Puddle

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The Puddle

The depth of a puddle in millimeters is given by:

$$h(x, y) = \frac{1}{10}(1 + sin(\pi xy))\ mm$$

A path through the puddle is given by the parametric curve:

$$x(t) = 3t \qquad y(t) = 4t\ \rightarrow \vec{r}(t) = 3t\hat{\imath} + 4t\hat{\jmath}$$

With a current position of $x = 3\ mm \quad y = 4\ mm$

(a) At your current position, how fast is the depth of water through which you are walking changing per unit time?

Recall the master formula!

$$df = \nabla F \cdot d\vec{r}$$

Begin by finding the gradient vector ∇H and $d\vec{r}$

$$\frac{\partial h}{\partial x} = \frac{\pi y}{10} cos(\pi xy) \qquad \frac{\partial h}{\partial y} = \frac{\pi x}{10} cos(\pi xy) \qquad \nabla H = \frac{\partial h}{\partial x}\hat{\imath} + \frac{\partial h}{\partial y}\hat{\jmath}$$

$$\nabla H = \frac{\pi y}{10} cos(\pi xy)\hat{\imath} + \frac{\pi x}{10} cos(\pi xy)\hat{\jmath}\ \frac{mm}{s}$$

$$\frac{d\vec{r}}{dt} = 3\hat{\imath} + 4\hat{\jmath}\ \frac{mm}{s}$$

With this we can then use the master formula to find $\frac{dh}{dt}$.

$$\frac{dh}{dt} = \nabla H \cdot \frac{d\vec{r}}{dt} \rightarrow \frac{dh}{dt} = \frac{3\pi y}{10} cos(\pi xy)\hat{\imath} + \frac{4\pi x}{10} cos(\pi xy)\hat{\jmath}$$

Plug in our values of x and y to find the change in mm per unit time at that position.

$$dh = \frac{24\pi}{10} cos(12\pi) \rightarrow dh = 7.5398\ \frac{mm}{s}$$

(b) At your current position, how fast is the depth of water through which you are walking changing per unit distance?

For this question we are looking for the change in depth with respect to change per unit distance. This will be found by finding the magnitude of the vector $d\vec{r}$.

$$|\vec{r}| = \sqrt{3^2 + 4^2} = 5 = ds$$

Then to find $\frac{dh}{ds}$ we need just divide the master formula by it and then plug in our current position.

$$\frac{dh}{ds} = \frac{3\pi y}{5 * 10} cos(\pi xy)\hat{i} + \frac{4\pi x}{5 * 10} cos(\pi xy)\hat{j}$$
$$\frac{dh}{ds} = \frac{24\pi}{50} cos(12\pi) \rightarrow \frac{dh}{ds} = 1.5080 \, \frac{mm}{mm}$$

(c) Optional food for thought There is a walkway over the puddle at x = 10. At your current position, how fast is the depth of water through which you are walking changing per unit distance towards the walkway.

Using the master formula, we can find dh along any path in a gradient vector field. Since the bridge is located at x = 10 we know that we would simple need to move along a path such as $\vec{r}(t) = t \; \hat{i}$. We can then find $d\vec{r} = \hat{i}$ and $ds = |d\vec{r}| = 1$

Now to find this new $\frac{dh}{ds}$ use the master formula divided by ds and plug in our current position.

$$\frac{dh}{ds} = \frac{12\pi}{10} cos(12\pi) = 3.7699 \, \frac{mm}{mm}$$