The Wire

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Consider the vector field created by the magnetic field around a wire along the z-axis carrying a constant current I in the z-direction.

$$\vec{B} = \frac{\mu_0 I}{2\pi}\left(\frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}\right) = \frac{\mu_0 I}{2\pi}\frac{\hat{\phi}}{r} \tag{1}$$

Ready:

(a)

Determine $\vec{B} \cdot d\vec{r}$ on any radial line of the form $y = mx$ where m is constant.

Find $d\vec{r}$:

$$\vec{r} = x\hat{i} + mx\hat{j} \to d\vec{r} = dx\hat{i} + mdx\hat{j}\ dx$$

Find $\vec{B} \cdot d\vec{r}$:

$$\vec{B} \cdot d\vec{r} = \frac{\mu_0 I}{2\pi}\left(\frac{-mx}{x^2 + y^2}\right) + \frac{\mu_0 I}{2\pi}\left(\frac{mx}{x^2 + y^2}\right) = 0$$

The dot product between the vector field $\vec{B}$ and the path derivative along any radial line will be:

$$0$$

(b)

Determine $\vec{B} \cdot d\vec{r}$ on any circle of the form $x^2 + y^2 = a^2$ where a is constant.

Find $d\vec{r}$:

$$\vec{r} = a\hat{r} \to d\vec{r} = a\hat{\phi}\ d\phi$$

Find $\vec{B} \cdot d\vec{r}$:

$$\vec{B} \cdot d\vec{r} = \frac{\mu_0 I}{2\pi a} * a d\phi = \frac{\mu_0 I}{2\pi}\ d\phi$$

The dot product between the vector field $\vec{B}$ and the path derivative along any circle will be:

$$\frac{\mu_0 I}{2\pi}\ d\phi$$

Set:

For each of the following curves, evaluate the line integral $\int_c \vec{B} \cdot d\vec{r}$.

C_1

The top half of the circle $r = 5$, traversed in a counterclockwise direction

$$C_1 = 5\hat{r} \qquad 0 \leq \phi \leq \pi$$

$$\int_0^\pi \frac{\mu_0 I}{2\pi}\,d\phi \rightarrow \frac{\mu_0 I \phi}{2\pi}|_0^\pi = \frac{\mu_0 I}{2}$$

C_2

The top half of the circle $r = 2$, traversed in a counterclockwise direction

$$C_2 = 2\hat{r} \qquad 0 \leq \phi \leq \pi$$

$$\int_0^\pi \frac{\mu_0 I}{2\pi}\,d\phi \rightarrow \frac{\mu_0 I \phi}{2\pi}|_0^\pi = \frac{\mu_0 I}{2}$$

C_3

The top half of the circle $r = 2$, traversed in a clockwise direction

$$C_3 = 2\hat{r} \qquad \pi \geq \phi \geq 0$$

$$\int_0^\pi \frac{\mu_0 I}{2\pi}\,d\phi \rightarrow \frac{\mu_0 I \phi}{2\pi}|_\pi^0 = -\frac{\mu_0 I}{2}$$

C_4

The bottom half of the circle $r = 2$, traversed in a clockwise direction

$$C_4 = 2\hat{r} \qquad 2\pi \geq \phi \geq \pi$$

$$\int_0^\pi \frac{\mu_0 I}{2\pi}\,d\phi \rightarrow \frac{\mu_0 I \phi}{2\pi}|_{2\pi}^\pi = -\frac{\mu_0 I}{2}$$

C_5

The radial line from $(2, 0)$ to $(5, 0)$

$$C_5 = x\hat{i} \qquad 2 \leq x \leq 5$$

From part one it was found that any radial path dotted with vector field will equal 0. The integral of 0 is 0, therefore the answer is 0.

C_6

The radial line from $(-5, 0)$ to $(-2, 0)$

$$C_6 = x\hat{i} \qquad -5 \leq x \leq -2$$

From part one it was found that any radial path dotted with vector field will equal 0. The integral of 0 is 0, therefore the answer is 0.

Go:

(a)

Find closed curves C_7 and C_8 such that this integral is nonzero over C_7 and zero over C_8

C_7

$$C_7 = 5\hat{r} \qquad 0 \leq \phi \leq 2\pi$$
$$\int_0^{\pi} \frac{\mu_0 I}{2\pi}\, d\phi \rightarrow \frac{\mu_0 I \phi}{2\pi}|_0^{2\pi} = \mu_0 I$$

C_8

$$C_8 = C_1 + C_6 + C_3 + C_5$$

Using the calculated line integral values from above we get:

$$\frac{\mu_0 I}{2} + 0 - \frac{\mu_0 I}{2} + 0 = 0$$

(b)

Ampere's Law says that, for any closed curve C, this integral is μ_0 times the current flowing through C. Use this fact to explain your results to part (a).

This vector field is interesting since there is a discontinuity at the point where the wire is (the origin for our example). When the path encircles this discontinuity we get Ampere's Law, $\mu_0 I$, but when a circulation is made *not* around this discontinuity we get 0, which is what we would expect from a conservative vector field.

(c)

Is $\vec{B}$ conservative?

No this field is not conservative because there is a discontinuity at the origin, therefore the vector field is not continuous on an open connected region.