Center of Cake

Christopher Hunt

Find the volume of a slice of cylindrical cake of height 2" and radius 5" between the planes $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$:

Recall that the volume of some surface can be found with this integral $V = \int_S dV$ and $dV = rdrdzd\phi$ in cylindrical.

$$0 \leq r \leq 5 \qquad 0 \leq z \leq 2 \qquad \pi/6 \leq \phi \leq \pi/3$$

$$V = \int_{\pi/6}^{\pi/3} \int_0^2 \int_0^5 rdrdzd\phi$$

When evaluated we get the volume:

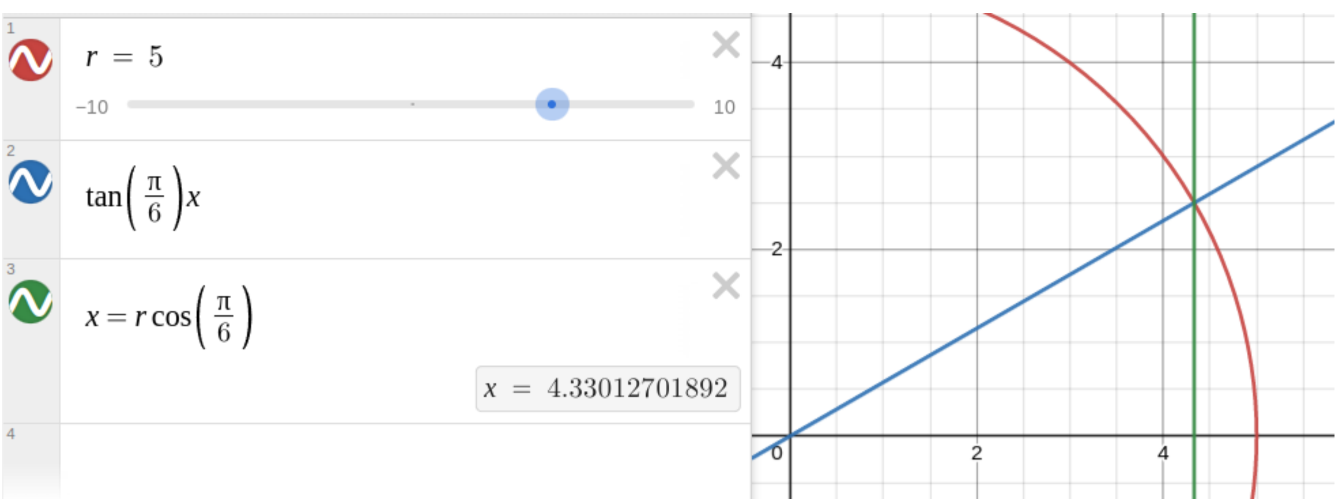
$$V = \frac{25\pi}{6} \, in^3$$

Assuming constant density, ρ, the mass of this slice is:

$$m = \frac{25\rho\pi}{6} \, g$$

To find the center of mass of a cylindrical wedge you must take into consideration that the value of r will approach zero as the angle phi approaches 2π. To solve this problem we use rectangular coordinates.

First, reorient the wedge angles ϕ by $-\frac{\pi}{6}$ and divide the wedge into two parts to be integrated. The projection on the xy-plane will look like this:



Now set up each integral to solve for $\bar{x}$, $\bar{y}$, $\bar{z}$:

$$\bar{x} = \frac{1}{m}\int_S x\rho dydxdz \rightarrow \frac{\rho}{m}\left(\int_0^2\int_0^{rcos(\frac{\pi}{6})}\int_0^{tan(\frac{\pi}{6})x} xdydxdz + \int_0^2\int_{rcos(\frac{\pi}{6})}^{r}\int_0^{\sqrt{25-x^2}} xdydxdz\right)$$

$$\bar{y} = \frac{1}{m}\int_S y\rho dydxdz \rightarrow \frac{\rho}{m}\left(\int_0^2\int_0^{rcos(\frac{\pi}{6})}\int_0^{tan(\frac{\pi}{6})x} ydydxdz + \int_0^2\int_{rcos(\frac{\pi}{6})}^{r}\int_0^{\sqrt{25-x^2}} ydydxdz\right)$$

$$\bar{z} = \frac{1}{m}\int_S z\rho dydxdz \rightarrow \frac{\rho}{m}\left(\int_0^2\int_0^{rcos(\frac{\pi}{6})}\int_0^{tan(\frac{\pi}{6})x} zdydxdz + \int_0^2\int_{rcos(\frac{\pi}{6})}^{r}\int_0^{\sqrt{25-x^2}} zdydxdz\right)$$

When evaluated we get:

$$\bar{x} = 3.1831\ in$$
$$\bar{y} = 0.8526\ in$$
$$\bar{z} = 1\ in$$

Converted to cylindrical and adjusted back to the original position:

$$\bar{r} = 3.2954\ in$$
$$\bar{\phi} = \frac{\pi}{4}\ rad$$
$$\bar{z} = 1\ in$$