Math 120 - Fall 2014 - Midterm Exam - Version A

You have 110 minutes to complete this midterm exam. Books, notes and electronic devices are not permitted. Read and follow directions carefully. Show and check all work. If applicable, draw a box around your answers, use standard notation and write in complete sentences. If a problem is not clear, please ask for clarification.

I pledge that I have completed this midterm exam in compliance with the NYU CAS Honor Code. In particular, I have neither given nor received unauthorized assistance during this exam.

Name Solutions

Signature

Date

Section

1 Short Answer/Computations

1. Consider the statement: *If an integer is divisible by 2, then it is even.* (10 points)
   Note: The statements you write will not necessarily be true.

   a. Write the converse of the statement. (2 points)

   If an integer is even, then it is ~~divisible~~ divisible by 2.

   b. Write the contrapositive of the statement. (2 points)

   If an integer is not even, then it is not divisible by 2.

   c. Let A be the set of all integers divisible by 2. Write A in set-builder notation. (2 points)

   $A = \{x \in \mathbb{Z} : 2 \mid x\}$

   d. Write the statement in quantifier notation. (2 points)

   $\forall x \in \mathbb{Z}, (2 \mid x \Rightarrow x \text{ is even})$

   e. Write the negatation of the statement in quantifier notation. ($\neg$ should not be in your final answer.) (2 points)

   $\exists x \in \mathbb{Z}, (2 \mid x \wedge x \text{ is not even})$

2. Let $A = \{1, 2, 3, 4, 5\}$. (10 points)

a. How many lists of length 3 are there with repetition? (2 points)

no. of such lists $= 5^3 = 125$

b. How many lists of length 3 are there without repetition? (2 points)

no. of such lists $= (5)_3 = 5 \times 4 \times 3 = 60$

c. How many subsets does A have? (2 points)

Since A is finite,
no. of subsets of A $= 2^{|A|} = 2^5 = 32$

d. How many subsets of size 3 does A have? (2 points)

no. of subsets of A of size 3 $= \binom{5}{3} = \dfrac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$

e. What is the cardinality of the power set of A? (2 points)

$|2^A| = 2^{|A|} = 2^5 = 32$

3. Let $B \times A = \{(1,1),(1,2),(1,4),(3,1),(3,2),(3,4)\}$. (10 points)

a. Compute $A \cup B$. (2 points)

$A = \{1, 2, 4\}$, $B = \{1, 3\}$

$A \cup B = \{1, 2, 3, 4\}$

b. Compute $A \cap B$. (2 points)

$A \cap B = \{1\}$

c. Compute $B - A$. (2 points)

$B - A = \{3\}$

d. Compute $A \Delta B$. (2 points)

$A \Delta B = \{2, 3, 4\}$

e. Compute $A \times B$. (2 points)

$A \times B = \{(1,1), (2,1), (4,1), (1,3), (2,3), (4,3)\}$

4. Consider the following relations on the set $A = \{1, 2, 3\}$: (10 points)

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,2), (1,3), (2,1), (2,3)\}$
$R_4 = \{(1,1), (2,2), (3,3)\}$

a. Which of the above relations are reflexive? (2 points)

R_1, R_4

b. Which of the above relations are irreflexive? (2 points)

R_3

c. Which of the above relations are symmetric? (2 points)

R_1, R_2, R_4

d. Which of the above relations are antisymmetric? (2 points)

R_4

e. Which of the above relations are transitive? (2 points)

R_1, R_4

5. An office meeting consists of 3 managers and 7 staff. (10 points)

a. Everyone at the meeting shakes hands with everyone else. How many shakehands take place? (2 points)

$$\text{no. of handshakes} = \binom{3+7}{2} = \binom{10}{2} = \frac{10 \times 9}{1 \times 2} = 45$$

b. Everyone at the meeting gives a presentation. How many different presentation schedules are possible? (2 points)

$$\text{no. of possible presentation schedules} = 10!$$

c. A team consists of one manager and 3 staff. How many different teams are possible? (2 points)

$$\text{no. of possible teams} = \binom{3}{1}\binom{7}{3} = 3 \times \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 105$$

d. A photo is taken of the meeting participants standing in a row. How many different poses are possible in which all of the managers are side-by-side in the picture? (4 points)

$$\text{no. of possible poses} = \underbrace{(7+1)!}_{\text{arrange staff \& "group of managers" first}}\ \underbrace{3!}_{\text{then arrange managers within group}} = 8!\,3!$$

2 Long Answer/Proofs

6. Prove or disprove the following statements:

a. Let a, b, c be integers. If $a|c$ and $b|c$, then $(a+b)|c$. (5 points)

False. Counterexample: $a = b = c = 1$

$1|1$ (& $1|1$) but $(1+1) \nmid 1$.

b. Let a, b, c be integers. If $a|b$, then $(ac)|(bc)$. (5 points)

True.

Proof: Suppose $a|b$, then $b = ak$ for some $k \in \mathbb{Z}$.

Multiplying both sides by c, $bc = (ak)c = (ac)k$,

so $(ac)|(bc)$.

7. A relation R is defined on $\mathbb{Z}$ by xRy provided $x + 3y$ is even. (10 points)

a. Show that R is an equivalence relation. (6 points)

R is reflexive: $\forall x \in \mathbb{Z}$, $x + 3x = 4x = 2\underbrace{(2x)}_{\in \mathbb{Z}}$ is even, so $x R x$.

R is symmetric: $\forall x, y \in \mathbb{Z}$, if xRy, i.e. $x + 3y$ is even, so that it can be written as $x + 3y = 2k$ for some $k \in \mathbb{Z}$, then

$$y + 3x = (x+3y) + 2x - 2y = 2k + 2x - 2y = 2\underbrace{(k+x-y)}_{\in \mathbb{Z}}$$

So $y + 3x$ is even, i.e. $y R x$.

R is transitive: $\forall x, y, z \in \mathbb{Z}$, if $x R y$ & $y R z$, i.e. $x + 3y$ and $y + 3z$ are even, so that they can be written as $x + 3y = 2k$, $y + 3z = 2\ell$ for some $k, \ell \in \mathbb{Z}$, then $x + 3z = (x + 3y) + (y + 3z) - 4y = 2k + 2\ell - 4y = 2\underbrace{(k+\ell-2y)}_{\in \mathbb{Z}}$

So $x + 3z$ is even, i.e. $x R z$.

b. Determine the equivalence classes of R. Justify these are the only equivalence classes of R. (4 points)

- $[0] \supseteq \{2k : k \in \mathbb{Z}\} = \{\text{all even nos.}\}$

  because $(2k) + 3(0) = 2k$ is even, so $(2k) R\, 0 \quad \forall k \in \mathbb{Z}$.

- $[1] \supseteq \{2k+1 : k \in \mathbb{Z}\} = \{\text{all odd nos.}\}$

  because $(2k+1) + 3(1) = 2(k+2)$ is even, so $(2k+1) R\, 1 \quad \forall k \in \mathbb{Z}$.

- $0 \not R\, 1$ because $0 + 3(1) = 3$ is not even.

  $\therefore$ $[0] \neq [1]$ and hence $[0] \cap [1] = \phi$.

- Since $\{2k : k \in \mathbb{Z}\} \cup \{2k+1 : k \in \mathbb{Z}\} = \mathbb{Z}$, it follows that equalities hold above:

  $[0] = \{2k : k \in \mathbb{Z}\} = \{\text{all even nos.}\}$

  $[1] = \{2k+1 : k \in \mathbb{Z}\} = \{\text{all odd nos.}\}$

8. Let A, B and C be sets. Prove that if $A \cap B = A \cap C$ and $A \cup B = A \cup C$ then $B = C$. (10 points)

We first show that $B \subseteq C$.

Let $x \in B$.

Case 1: $x \in A$. Then $x \in A \cap B = A \cap C \Rightarrow x \in C$.

Case 2: $x \notin A$. Then $x \in A \cup B = A \cup C \Rightarrow x \in C$.

In either case, $x \in C$, so $B \subseteq C$.

By symmetry (exchanging the letters B & C in the argument above), $C \subseteq B$.

Since $B \subseteq C$ and $C \subseteq B$, it follows that $B = C$.

9. Let n be a positive integer. Prove that $1+4+7+\ldots+(3n-2)=\frac{3n^2-n}{2}$. (10 points)

*Proof by smallest counterexample:

Suppose for the sake of contradiction that the statement is not true for some $n \in \mathbb{Z}, n \geq 1$. By the Well-Ordering Principle, there exists a smallest counterexample, say when $n=k$, that is,

$$1+4+7+\ldots+(3k-2) \neq \frac{3k^2-k}{2}. \qquad ①$$

Note that $n=1$ is not a counterexample, since $1 = 3(1)-2 = \frac{3(1)^2-1}{2}$.

So $k>1$. Since the smallest counterexample is when $n=k$, $n=k-1 \geq 1$ is not a counterexample, so

$$1+4+7+\ldots+(3(k-1)-2) = \frac{3(k-1)^2-(k-1)}{2} \qquad ②$$

Adding $3k-2$ to both sides of ②,

$$1+4+7+\ldots+(3(k-1)-2)+3k-2 = \frac{3(k-1)^2-(k-1)}{2}+3k-2$$

$$= \frac{3k^2-6k+3-k+1}{2}+\frac{6k-4}{2}$$

$$= \frac{3k^2-1}{2}$$

This contradicts our choice of $n=k$ as a counterexample in ①.

*Proof by induction:

· Base case: When $n=1$,

$$1 = 3(1)-2 = \frac{3(1)^2-1}{2},$$

So the statement is true for $n=1$.

· Induction hypothesis: Assume that the statement is true for $n=k$, i.e

$$1+4+7+\ldots+(3k-2) = \frac{3k^2-k}{2}. \qquad ③$$

· Induction step: Adding $3(k+1)-2$ to both sides of ③,

$$1+4+7+\ldots+(3k-2)+3(k+1)-2 = \frac{3k^2-k}{2}+3(k+1)-2$$

$$= \frac{3k^2-k+6k+2}{2}$$

$$= \frac{3k^2+6k+3-k-1}{2}$$

$$= \frac{3(k+1)^2-(k+1)}{2}$$

∴ the statement is also true for $n=k+1$.
By induction, the statement is true for all n.

10. (10 points)

a. Show that $a \to b$ and $(a \wedge \neg b) \to \text{FALSE}$ are logically equivalent. (4 points)

$$(a \wedge \neg b) \rightarrow \text{FALSE} \equiv \neg(a \wedge \neg b) \vee \text{FALSE}$$
$$\equiv \neg(a \wedge \neg b)$$
$$\equiv \neg a \vee \neg(\neg b)$$
$$\equiv \neg a \vee b$$
$$\equiv a \rightarrow b$$

b. Prove that the sum of any four consecutive integers in not divisible by 4. (6 points)

Any four consecutive integers can be written in the form $k, k+1, k+2, k+3$, $k \in \mathbb{Z}$.

Their sum is

$$k + (k+1) + (k+2) + (k+3) = 4k+6$$

The only number x such that $4k+6 = 4x$ is $x = k + \frac{3}{2}$, but $k + \frac{3}{2} \notin \mathbb{Z}$ if $k \in \mathbb{Z}$. Hence there does not exist an integer ℓ such that $4k+6 = 4\ell$, so the sum of any 4 consecutive integers is not divisible by 4.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
Total	/100