MAP-UA.120: Discrete Mathematics
Midterm Exam II

Spring 2013

Name: ______________________________ C

This exam is scheduled for 110 minutes. No calculators, notes, or other outside materials are permitted. **Show all work to receive full credit, except where specified.** The exam is worth 80 points.

Please read all directions thoroughly.

Please read and sign the following statement.

> *I, the undersigned, hereby affirm that I will not share the contents of this exam with any person or animal, living or deceased. Furthermore, I have not received outside assistance in the midst of nor prior to taking this exam.*

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(1) (20 points) For each term below, give a precise mathematical definition or a precise mathematical statement of the result, as appropriate.

(a) permutation
SOLUTION:
A permutation $\pi : A \to A$ is a bijection from a set to itself.

(b) antisymmetric relation
SOLUTION:
An antisymmetric relation $R \subseteq A \times A$ is a relation such that

$$\forall x, y \in A, xRy \wedge yRx \to x = y.$$

(c) Well-Ordering Principle
SOLUTION:
Every nonempty subset of the natural numbers has a least element. That is,

$$\forall S \in 2^{\mathbb{N}} \setminus \{\emptyset\}, \exists x \in S, \forall y \in S, x \leq y.$$

(d) one-to-one function
A function $f : A \to B$ is a function such that

$$\forall x, y \in A, f(x) = f(y) \to x = y.$$

(e) the big theorem about sequences generated by polynomials
There exists a natural number k such that $\Delta^{k+1} a_n = 0$ for all n if and only if a_n is a polynomial sequence. Furthermore,

$$a_n = \sum_{\ell=0}^{k} \binom{n}{\ell} \Delta^{\ell} a_0$$

(2) (25 points)

(a) Write the permutation $(2, 4, 5, 3)(1, 6) \circ (1, 5, 2, 6)(3, 4)$ as a composition of disjoint cycles.
SOLUTION:

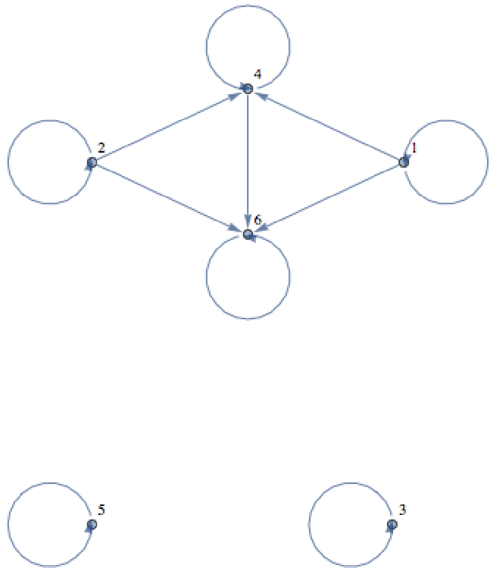
$$(1, 3, 5, 4, 2)$$

(b) Use the Euclidean Algorithm to compute the greatest common divisor of 42 and 140.
SOLUTION:

$$140 = 42 \cdot 3 + \mathbf{14}$$
$$42 = 14 \cdot 3 + 0$$

(c) How many digits of the decimal expansion of π would one have to check to be guaranteed to find a repeated 7-digit sequence (overlapping like $\ldots \underline{12372\overline{12}}\overline{37212} \ldots$ counts as a repeated sequence)? Support your answer.
SOLUTION:
There 10^7 possibilities. We there fore need $10^7 + 1$ sequences of seven digits. This takes 10000007 digits.

(d) Which properties among the five discussed in class does the following relation on the set $\{1, 2, 3, 4, 5, 6\}$ have? *No justification is required.*



SOLUTION:
reflexive, antisymmetric, and transitive.

(e) Recall that $\lfloor x \rfloor$ stands for the "the greatest integer less than or equal to x". Let R be the following equivalence relation on the integers.

$$R = \{(x, y) : x, y \in \mathbb{Z} \land \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{y} \rfloor\}$$

Write out the elements of the equivalence class [27].

SOLUTION:
$\lfloor \sqrt{27} \rfloor = \lfloor 5.196\ldots \rfloor = 5$ so the equivalence class of 27 is all number whose square root is between 5 (included) and 6 (excluded).

$$[27] = \{25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35\}$$

(3) (5 points each) Prove or disprove each statement.

(a) Recall that S_{11} is the set of all permutations on the set $\{1, 2, 3, \dots, 11\}$. Define the function $f : S_{11} \to S_{11}$ by the formula

$$f(\pi) = (1, 2, 3) \circ \pi.$$

Then, f is an injective function on S_{11}.

SOLUTION:

We wish to prove: $\forall \pi, \sigma \in S_{11}, f(\pi) = f(\sigma) \to \pi = \sigma$.

Suppose $f(\pi) = f(\sigma)$. Then,

$$(1, 2, 3) \circ \pi = (1, 2, 3) \circ \sigma$$

Now compose with the permutation $(1, 3, 2) = (1, 2, 3)^{-1}$ on both sides.

$$\begin{aligned}(1, 3, 2) \circ (1, 2, 3) \circ \pi &= (1, 3, 2) \circ (1, 2, 3) \circ \sigma \\ \iota \circ \pi &= \iota \circ \sigma \\ \pi &= \sigma\end{aligned}$$

Note: ι here is the identity function. $\square$

(b) Recall that S_{11} is the set of all permutations on the set $\{1, 2, 3, \dots, 11\}$. Define the function $f : S_{11} \to S_{11}$ by the formula

$$f(\pi) = \pi \circ \pi.$$

Then, f is an injective function on S_{11}.

SOLUTION:

We show this is not one-to-one by exhibiting two permutations that get mapped to the same permutation under f.

$$f(\iota) = \iota \circ \iota = \iota$$

$$f((1, 2)) = (1, 2) \circ (1, 2) = \iota$$

Note: ι here is the identity function. $\square$

(4) (10 points) Let $a, b \in \mathbb{Z}$, and let $b > 0$. Show that the smallest element of the set

$$\{a - qb : q \in \mathbb{Z} \wedge a - qr \geq 0\}$$

exists and is necessarily less than b.

(bonus) (5 points) Let $\sim$ be the relation on sets where $A \sim B$ provided there exists a bijection $f : A \to B$. Prove or disprove the statement, "If A, B and C are sets and $A \sim B$, then $A \cup C \sim B \cup C$."

(For grading purposes. Please do not write on this page.)

Problem Number	Problem Points	Points Earned
1	25	
2	25	
3	20	
4	10	
bonus	5	
Total	80	