# MATH-UA.120: Discrete Mathematics
# Midterm Exam I

## Spring 2014

Name: Solution ____________________ NetID: ____________

This exam is scheduled for 110 minutes. No calculators, notes, or other outside materials are permitted. **Show all work to receive full credit, except where specified.** The exam is worth 80 points.

**Please read all directions thoroughly.**

Please read and sign the following statement.

*I, the undersigned, hereby affirm that I will not share the contents of this exam with any person or animal, living or deceased. Furthermore, I have not received outside assistance in the midst of nor prior to taking this exam.*

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| Problem Number | Problem Points | Points Earned |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 20 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| Total | 80 | |

**1** (15 points) Evaluate each statement as true or false (write the whole word "true" or "false" on the line provided). Provide a brief (it need not be a formal proof) justification for your answer.

We define

$$B = \{X \subseteq \mathbb{Z} : |X| = 5\}$$

that is, subsets of the integers of size 5.

(a) False

$$(1,2,3,4,5) \in B.$$

(1,2,3,4,5) is a list.

B contains sets. Therefore, a list is not an element of B.

(b) False

$$\{3,3,3,4,5\} \in B.$$

$|\{3,3,3,4,5\}| = |\{3,4,5\}| = 3.$

B contains sets of cardinality 5, so $\{3,3,4,5\}$ is not an element of B.

(c) True

$$\emptyset \subseteq B.$$

The empty set is a subset of any set.

(d) True

$$\forall X \in B, \forall Y \in B,\ X \cap Y \in B \iff X = Y.$$

"for all X in B and for all y in B, $x \cap y \in B$ iff $x = y$."

X, y each contains 5 distinct elements.
Their intersection is in B if it also contains 5 distinct element, which could only be the case if X, y contains exactly the same 5 elements. This means that X, y must be equal.

(e) False

$$\exists X \in B,\ \prod_{x \in X} x = 2.$$

"There exists a set X, an element of B, such that the product of the elements of X is equal to 2"

If $X \in B$, then $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1, x_2, x_3, x_4, x_5$ are distinct integers.

There are no 5 distinct integers whose product is 2, because the distinct factors of 2 are: $\pm 1, \pm 2$.

**2** (10 points) Let $B$ be defined as in question 1. Consider the subsets of $B$

$$B_n = \{X \in B : \forall x \in X, 0 \leq x \leq n\}$$

where $n$ is a positive integer.

(a) List out the elements of $B_3$, $B_4$, and $B_5$.

$B_3 = \{\} = \phi$ (there is no set $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1, x_2, x_3, x_4, x_5$ are distinct integers between 0 and 3 inclusive)

$B_4 = \{\{0,1,2,3,4\}\}$

$B_5 = \{\{0,1,2,3,4\}, \{0,1,2,4,5\}, \{0,1,3,4,5\}, \{0,2,3,4,5\}, \{1,2,3,4,5\}\}$

One missing in $B_5$: {0, 1, 2, 3, 5}. This is the only one since the formula below for $|B_n|$ is correct.

(b) Find a formula for $|B_n|$.

$|B_n|$ = The number of subsets of $\{0, 1, \ldots, n\}$ containing five distinct integers

= The number of ways to choose 5 elements out of a set of n+1 elements

$= \binom{n+1}{5}$

**3** (10 points each) Prove or disprove each statement.

(a) Here $x$ and $y$ are Boolean expressions.

$$(x \vee y) \to \neg y = (\neg x) \wedge (\neg y)$$

Claim: The two expressions are not logically equivalent.

Disprove using a Truth Table:

| $x$ | $y$ | $x \vee y$ | $\neg y$ | $(x \vee y \to \neg y)$ | $\neg x \wedge \neg y$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | T | T | T | F |
| F | T | T | F | F | F |
| F | F | F | T | T | T |

Since the two expressions do not always evaluate to the same value for each possible inputs, then they are not logically equivalent.

(b) Here $A$, $B$, and $C$ are sets.

$$C \cap (A - B) = (C \cap A) - (C \cap B)$$

Claim: True.

Proof: First, we show that $C \cap (A - B) \subseteq (C \cap A) - (C \cap B)$.

Suppose $x \in C \cap (A-B)$, an arbitrary element.
Then $x \in C$, $x \in A - B$. So, $x \in C$, $x \in A$, and $x \notin B$.
Therefore, $x \in C \cap A$ but $x \notin C \cap B$.
By the defn of set difference, $x \in (C \cap A) - (C \cap B)$
Since $x$ denotes an arbitrary element of $C \cap (A - B)$, this shows that $C \cap (A-B) \subseteq (C \cap A) - (C \cap B)$

Next, we show that $(C \cap A) - (C \cap B) \subseteq C \cap (A - B)$.

Suppose $x \in (C \cap A) - (C \cap B)$, arbitrary element.
Then, $x \in C \cap A$ but $x \notin C \cap B$.
Since $x \in C \cap A$, then $x \in C$ and $x \in A$.
Since $x \notin C \cap B$, then $x \notin C$ <u>or</u> $x \in B$
We know that $x \in C$, so it must be the case that $x \notin B$.
So, $x \in C$, $x \in A$, $x \notin B$.
This means that $x \in C$, $x \in A - B$
Therefore, $x \in C \cap (A - B)$.
Hence, we have shown that $(C \cap A) - (C \cap B) \subseteq C \cap (A - B)$

Since the two sets are subsets of one another, then they are equal: $(C \cap A) - (C \cap B) = C \cap (A - B)$.

$\square$

**4** (10 points) Let $A$ be a finite set. Explain, using English sentences, what is meant by, and the main idea behind, the proof of the statement

$$|2^A| = 2^{|A|}.$$

(We wish to show that the cardinality of the power set of a finite set is $2^{|A|}$.)

- Suppose $A$ has $n$ elements (that is, let $n = |A|$), $n \in \mathbb{N}$.
- The power set of $A$ contains all subsets of $A$. So, we want to count the number of subsets of $A$.
- For a given subset of $A$, each of the $n$ elements of $A$ is either in this set or not in this set.

  So, there are 2 possibilities for each element of $A$.

  By the multiplication principle, the total number of possibilities is $\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{n \text{ times}} = 2^n$.

  Therefore, there are $2^n$ subsets of $A$.

In class, this argument was made more rigorously by explicitly relating each subset of $A = \{1, 2, \ldots, n\}$ to a list of length $n$, where each element of the list is either 0 or 1, indicating whether the corresponding element of $A$ is in the subset.

So, $|2^A| = 2^{|A|}$.

**5** (10 points)

Consider the sets

$$X = \{(x-1, x-x^2) : x \in \mathbb{Z}\} \text{ and } Y = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x+y \leq 0\}.$$

Show that $X \subseteq Y$ but $X \neq Y$.

Proof that $X \subseteq Y$:

Consider an arbitrary element of $X$, call it $(a,b)$.
Since $(a,b) \in X$, this means that $a = x-1$ and $b = x-x^2$ for some integer $x$.

This means that $a$ and $b$ are both integers as well, so $(a,b) \in \mathbb{Z} \times \mathbb{Z}$.

Note that
$$\begin{aligned} a+b &= x-1+x-x^2 \\ &= 2x-1-x^2 \\ &= (-1)(x^2-2x+1) \\ &= (-1)(x-1)^2 \end{aligned}$$

Since $(x-1)^2 \geq 0$ for any integer $x$, then
$$a+b = (-1)(x-1)^2 \leq 0.$$

Since $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ and $a+b \leq 0$, then $(a,b) \in Y$.

Therefore, $X \subseteq Y$, as desired. $\square$

To show $X \neq Y$, we need to find an element of $Y$ that is not contained in $X$.

For example, consider $(0,-1)$. Since $(0,-1) \in \mathbb{Z} \times \mathbb{Z}$ and $0+(-1) \leq 0$, then $(0,-1) \in Y$. However, $(0,-1) \notin X$.

Because there exists no $x \in \mathbb{Z}$ such that $(0,-1) = (x-1, x-x^2)$. (i.e. for all $x \in \mathbb{Z}$, $(0,-1) \neq (x-1, x-x^2)$.)

If $(0,-1)$ were in $X$, then $0 = x-1 \Rightarrow x = 1$.
But if $x = 1$, then $x - x^2 = 0 \neq -1$. So, $(0,-1) \notin X$

**6** (15 points) Let $R$ be the relation "is divisible by" on the set of integers $\mathbb{Z}$. That is to say,

$$(a,b) \in R \iff a|b.$$

Determine if this relation is each of reflexive, symmetric, and transitive. Write a formal proof or disproof for each. (Feel free to use more than one page.)

Reflexive: Yes.

Consider any integer $a$. $a = 1 \cdot a$. So $a|a$.
Therefore, $(a,a) \in R$ for each $a \in \mathbb{Z}$.
So, $R$ is reflexive.

Symmetric: No

Counterexample: $(1,3) \in R$ since $1|3$

However, $(3,1) \in R$ since 3 does not divide 1.

minor error: (3, 1) is not in R.

Transitive: Yes.

Suppose $(a,b) \in R$ and $(b,c) \in R$.
Then, $a|b$ and $b|c$.
So, there are integers $x, y$ such that
$b = x \cdot a$ and $c = y \cdot b$.
This means that $c = y \cdot (x \cdot a)$
$= (y \cdot x) \cdot a$,
where $x \cdot y \in \mathbb{Z}$.
So, $a|c$. Therefore $(a,c) \in R$.

This shows that $R$ is transitive. $\square$

(Use this space for extra work as needed.)