Math 120 - Spring 2018 - Final Exam - Version A

You have 110 minutes to complete this final exam. Books, notes and electronic devices are not permitted. Read and follow directions carefully. Show and check all work. If applicable, draw a box around your answers, use standard notation and write in complete sentences. If a problem is not clear, please ask for clarification.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
Total	/100

I pledge that I have completed this final exam in compliance with the NYU CAS Honor Code. In particular, I have neither given nor received unauthorized assistance during this exam.

Name Solutions

Signature

Date

Section

1 Short Answer/Computations

1. (10 points) Let $A = \{1, 2, 3, 4\}$. Write a set B and a mapping $f : A \to B$ such that f is

(a) surjective but not injective.

$B = \{1, 2, 3\}$

$$f(a) = \begin{cases} a & \text{if } 1 \leq a \leq 3 \\ 3 & \text{if } a = 4 \end{cases}$$

(b) injective but not surjective.

$B = \{1, 2, 3, 4, 5\}$

$f(a) = a \quad \forall a \in A$

(c) bijective.

$B = \{1, 2, 3, 4\}$

$f(a) = a \quad \forall a \in A$

(d) neither injective nor surjective.

$B = \{1, 2\}$

$f(a) = 1 \quad \forall a \in A$

2. (10 points) Define functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ -x & \text{if } -1 \le x \le 1 \\ x-2 & \text{if } x > 1 \end{cases}$$

$$g(x) = \begin{cases} x-2 & \text{if } x < -1 \\ -x & \text{if } -1 \le x \le 1 \\ x+2 & \text{if } x > 1 \end{cases}$$

(a) Compute the function $f \circ g$.

$(f \circ g)(x) = x$

(b) Complete the function $g \circ f$.

$$g \circ f = g(f(x)) = \begin{cases} \boxed{x} & \text{if } x < -3 \\ -x-2 & \text{if } -3 \le x < -1 \\ \boxed{x} & \text{if } -1 \le x \le 1 \\ -x+2 & \text{if } 1 < x \le 3 \\ \boxed{x} & \text{if } x > 3 \end{cases}$$

(c) Is g the inverse of f? Explain.

No, because $f(0) = f(2) = 0$, so f is not injective and hence not invertible.

(d) Is f injective, surjective, both or neither?

f is not injective but is surjective.

(e) Is g injective, surjective, both or neither?

g is injective but not surjective.

3. (10 points) Suppose that $\rho = (1,6,3)(2,7,4,5)$ and $\sigma = (1,4,2)(3,6)$ are permutations in S_7.

(a) Compute $\sigma \circ \rho$.

$\sigma \circ \rho = (1,3,4,5)(2,7)$

(b) Write ρ as a composition of transpositions.

$\rho = (1,3)(1,6)(2,5)(2,4)(2,7)$

(c) Determine $\mathrm{sgn}(\rho)$.

$\mathrm{Sgn}(\rho) = -1$

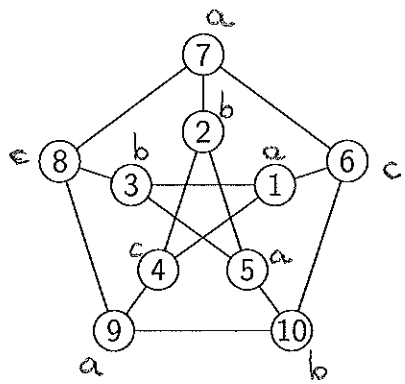
(d) Compute the inverse of σ.

$\sigma^{-1} = (2,4,1)(6,3)$

(e) Find the smallest possible k for which σ^k is the identity function.

$k = 2 \times 3 = 6$

4. (10 points) The 3-regular graph G below is known as the *Peterson* graph.



(a) How many induced subgraphs does G have?

$2^{10} - 1$

no vertices

(b) Find $\alpha(G)$ (independence number).

4 (e.g. $\{1, 2, 8, 10\}$)

~~(c)~~ Find diam(G).
(Recall: the *diameter* of G is the maximum of the shortest distance between every pair of vertices in G.)

2

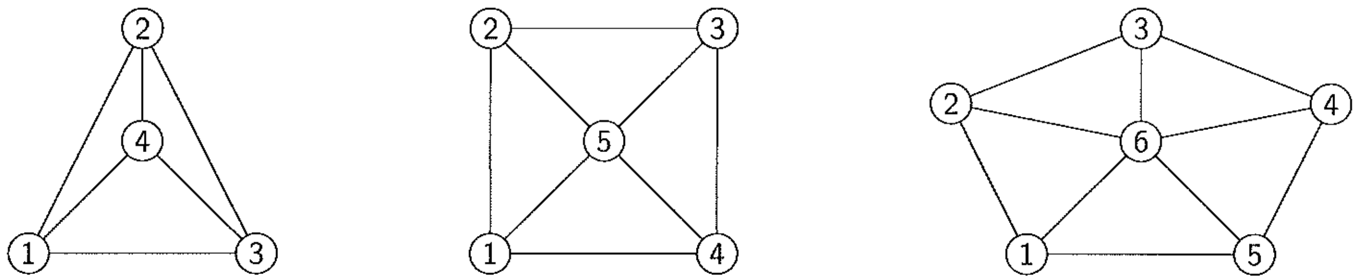
(d) Find a Hamiltonian path in G.
(Recall: a path in G that contains all the vertices in G is called a *Hamiltonian* path.)

1, 3, 5, 2, 4, 9, 8, 7, 6, 10

~~(e)~~ The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices in G share the same color. Find $\chi(G)$.

3 (see diagram above)

5. (10 points) A *wheel* graph W_n is a connected graph on n vertices that contains a cycle C_{n-1}, and for which every vertex in the cycle C_{n-1} is joined to the n^{th} vertex. Wheel graphs for $n = 4, 5, 6$ are shown below.

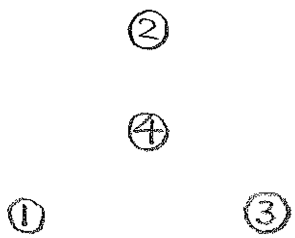


W_4 $\qquad$ W_5 $\qquad$ W_6

(a) How many cycle subgraphs are in W_4?

7 (1~2~3~1, 1~2~4~1, 1~3~4~1, 2~3~4~2, 1~2~3~4~1, 1~2~4~3~1, 1~3~2~4~1)

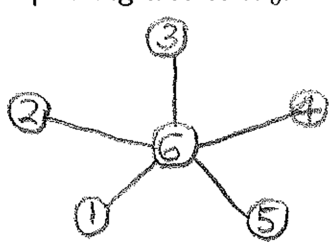
(b) Draw $\overline{W_4}$.



(c) Find a cut vertex of W_5, if any.

None

(d) Find a spanning tree of W_6.



(e) Compute ΔW_n (maximum degree).

n − 1

2 Long Answer/Proofs

6. (10 points) Let $G = (V, E)$ be a connected graph. Define a relation R on V as follows:

For any vertices $a, b \in V$, $a\ R\ b$ provided that there is an (a, b)-walk in G with an even number of edges.

(a) Prove that R is an equivalence relation on V.

– Reflexivity: $\forall\ v \in V$, (v) is a (v,v)-walk with 0 (hence an even number) of edges, so vRv.

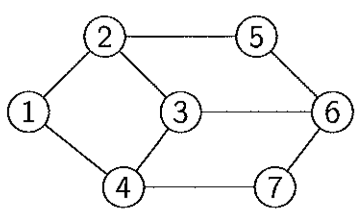
– Symmetry: $\forall\ u, v \in V$, if $W = (u, w_1, w_2, \ldots, w_{n-1}, v)$ is a (u,v)-walk with an even number of edges, then $W^{-1} = (v, w_{n-1}, w_{n-2}, \ldots, w_1, u)$ is a (v,u)-walk with an even number of edges,

so $uRv \Rightarrow vRu$

– Transitivity: $\forall\ u, v, w \in V$, if W_1 is a (u,v)-walk with an even number of edges (say $2k$, $k \in \mathbb{Z}$) and W_2 is a (v,w)-walk with an even number of edges (say 2ℓ, $\ell \in \mathbb{Z}$), then the concatenation $W_1 + W_2$ is a (u,w)-walk with $2k + 2\ell = 2(\underbrace{k+\ell}_{\in \mathbb{Z}})$ edges, so it also has an even number of edges.

So $uRv \wedge vRw \Rightarrow uRw$

(b) Consider the graph below. What are the equivalence classes of R for this graph?



$\{1, 3, 5, 7\}$, $\{2, 4, 6\}$

7. (10 points) Prove the following statements by well-ordering principle or pigeonhole principle.

(a) Every natural number $n > 1$ has a prime factor.

Suppose for the sake of contradiction that some natural nos > 1 do not have prime factors. Let X be the set of such natural nos.

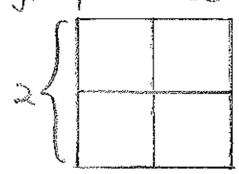
By assumption, X is nonempty. Since $X \subseteq \mathbb{N}$, by the well-ordering principle, it has a smallest element, call it k. Since $k > 1$, it is either prime or composite.

- If k is prime, then k is a prime factor of k.
- If k is composite, then $k = ab$ where $1 < a, b < k$. By the minimality of k, a and b have prime factors. Suppose p is a prime factor of a, then $a = cp$ for some $c \in \mathbb{Z}$, so $k = ab = bcp$. Then p is a prime factor of k.

In either case, we reach a contradiction, hence $X = \phi$ and all natural nos. > 1 have prime factors.

(b) Given any five points inside of a square of side length 2, two of the points have distance no more than $\sqrt{2}$.

Chop up the square into 4 squares of side length 1 as in the diagram below (Assign possession of the shared sides to one of the small squares it borders.)



By the pigeonhole principle, since there are more points (5 pigeons) than small squares (4 holes), at least one small square must contains 2 of the points. Then the distance between these two points is at most $\sqrt{1^2+1^2}$ (by Pythagoras' Theorem) $= \sqrt{2}$.

8. (10 points) Suppose $x \neq 1$ and $n \in \mathbb{N}$. Prove the following algebraic identity by induction on $n \geq 1$.

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

(a) Base case: When $x = 1$, LHS $= 1 + x = \dfrac{x^2 - 1}{x - 1} =$ RHS.

(b) Inductive hypothesis: Suppose that when $n = k \geq 1$,

$$1 + x + x^2 + \ldots + x^k = \frac{x^{k+1} - 1}{x - 1}.$$

(c) Inductive step: We want to show the identity is true when $n = k+1$.

$$1 + x + x^2 + \ldots + x^k + x^{k+1}$$

$$\overset{\text{by inductive hyp}}{=} \frac{x^{k+1} - 1}{x - 1} + x^{k+1}$$

$$= \frac{x^{k+1} - 1}{x - 1} + \frac{x^{k+2} - x^{k+1}}{x - 1}$$

$$= \frac{x^{(k+1)+1} - 1}{x - 1}$$

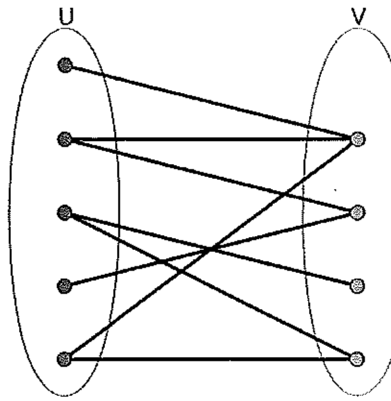
$\therefore$ the identity is true for all $n \geq 1$ by induction.

9. (10 points) Let $f\colon A \to B$, $g\colon B \to C$ and $h\colon B \to C$ be functions such that f is bijective. Prove that if $g \circ f = h \circ f$, then $g = h$.

Since f is bijective, f^{-1} exists, so

$$g = g \circ f \circ f^{-1} = h \circ f \circ f^{-1} = h.$$

10. (10 points) A graph $G = (U \cup V, E)$ is *bipartite* if the vertices of G can be divided into two disjoint and independent sets U and V such that every edge joins a vertex in U to one in V. Vertex sets U and V are called the *parts* of the graph. Below is an example of a bipartite graph:



Prove (by induction on the number of vertices) or disprove:

(a) Every tree is bipartite.

We prove the statement by induction on the number of vertices n of a tree.

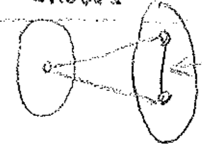
Base case : $n=1$. The trivial graph is bipartite because it has no edges, take $U=\{v\}, V=\emptyset$.

Induction hypothesis: When $n=k\geq 1$, Suppose the statement is true for all trees on k vertices.

Induction step: Consider $n=k+1$. Let T be any tree on $k+1$ vertices. Since $k+1\geq 2$, T has a leaf v. Suppose the unique vertex incident to v is u. By the induction hypothesis, $T-v$ is bipartite. Suppose it has parts U, V. If $u\in U$, let $v\in V$, then the edge uv joins a vertex in U to a vertex in V and V is still an independent set. Since the rest of the graph is bipartite, so is T. If $u\in V$, let $v\in U$, then a similar argument shows T is bipartite.

(b) Cycles with an odd number of vertices are bipartite.

False. C_3 is not bipartite. WLOG



not independent set