DOING PHYSICS WITH MATLAB

COMPUTATIONAL OPTICS

RAYLEIGH-SOMMERFELD DIFFRACTION RECTANGULAR APERTURES

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op_rs_rxy_01.m

Calculation of the irradiance in a plane perpendicular to the optical axis for a uniformly illuminated rectangular aperture.

simpsonxy_coeff.m

Function to calculate Simpson's two-dimensional coefficients

fn_distancePQ.m

Function to calculate distance between two points

turningPoints.m

Function to find the zero crossings of a function and its maxima and minima

op_rs_rxy_02.m

Updated version of **op_rs_rxy_01.m**. Calculation of the irradiance in a plane perpendicular to the optical axis for a rectangular aperture. The irradiance is not scaled but calculated in W.m^{-2}. The energy per second from the aperture to the observation screen is calculated and the energy per second within a circle of radius equal to the position of the first minima in the X direction in the observation screen is also calculated.

simpson2d.m

Function to calculate the value of a two-dimensional integral using the Simpson's [2D] method.

fn_distancePQ.m

Function to calculate distance between two points

FRAUNHOFER DIFFRACTION – RECTANGULAR APERTURE

We will consider the Fraunhofer diffraction patterns for a single rectangular aperture that is uniformly illuminated. The X and Y widths the rectangular aperture are a_x and a_y respectively. The irradiance distribution I is given by the product of two single slit functions

$$(1) \qquad I = I_o \left(\frac{\sin(v_{Px})}{v_{Px}} \right)^2 \left(\frac{\sin(v_{Py})}{v_{Py}} \right)^2 \qquad \textbf{\textcolor{purple}{Fraunhofer diffraction}}$$

where I_0 is a normalizing constant and the optical coordinates v_{Px} and v_{Py} are

$$(2) \qquad v_{Px} = \tfrac{1}{2} k\, a_x \sin\theta_x = \left(\frac{\pi}{\lambda} \right) a_x \sin\theta_x \qquad v_{Py} = \tfrac{1}{2} k\, a_y \sin\theta_y = \left(\frac{\pi}{\lambda} \right) a_y \sin\theta_y$$

and the angles θ_x and θ_y define the direction of the diffracted ray

$$(3) \qquad \sin\theta_x = \frac{x_P}{\sqrt{x_P^2 + z_P^2}} \qquad \sin\theta_y = \frac{y_P}{\sqrt{y_P^2 + z_P^2}}$$

The geometry for the diffraction pattern from a rectangular aperture is shown in figure (1).
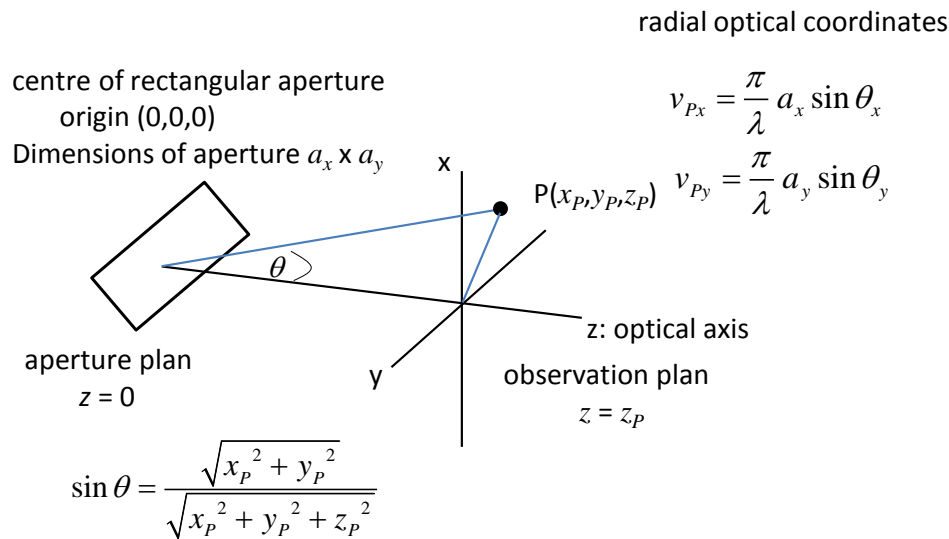
radial optical coordinates

centre of rectangular aperture
origin (0,0,0)
Dimensions of aperture a_x x a_y

$$v_{Px} = \frac{\pi}{\lambda} a_x \sin\theta_x$$

$$v_{Py} = \frac{\pi}{\lambda} a_y \sin\theta_y$$

$P(x_P, y_P, z_P)$

θ

z: optical axis

aperture plan
z = 0

observation plan
z = z_P

$$\sin\theta = \frac{\sqrt{x_P^2 + y_P^2}}{\sqrt{x_P^2 + y_P^2 + z_P^2}}$$

Fig. 1. Geometry for a rectangular aperture with dimensions a_x x a_y. The radial coordinates are scaled perpendicular distance from the optical axis.

The resulting diffraction pattern for the irradiance has lines of zeros when

(3a) $v_{Px} = m_x\,\pi \quad m_x = \pm 1, \pm 2, \pm 3, \ldots$ and $v_{Py} = m_y\,\pi \quad m_y = \pm 1, \pm 2, \pm 3, \ldots$

or

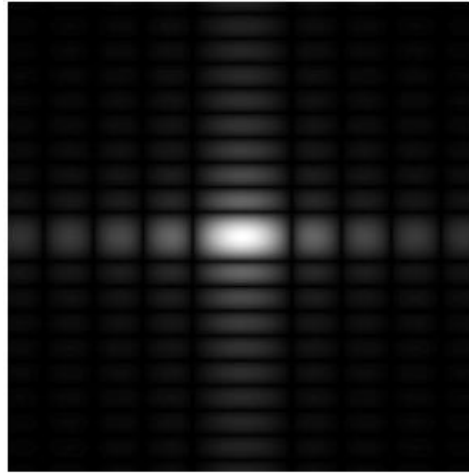(3b) $m_x\,\lambda = a_x \sin\theta_x \qquad m_y\,\lambda = a_y \sin\theta_y$

Fig. 1.   Photograph like image for the Fraunhofer diffraction pattern of a rectangular aperture $a_y = 2\,a_x$.

However, a more versatile approach to study the diffraction from an aperture is to evaluate the **Rayleigh-Sommerfeld diffraction integral of the first kind**. No approximations need to be made and the irradiance can be calculated in the near and far fields in an XY plane and along the Z axis right. There is excellent agreement between the far field predictions of the Fraunhofer theory and the results of evaluating the Rayleigh-Sommerfeld diffraction integral.

RAYLEIGH DIFFRACTION INTEGRAL OF THE FIRST KIND

The **Rayleigh-Sommerfeld region** includes the entire space to the right of the aperture. It is assumed that the Rayleigh-Sommerfeld diffraction integral of the first kind is valid throughout this space, right down to the aperture. There are no limitations on the maximum size of either the aperture or observation region, relative to the observation distance, because **no approximations have been made**.

The Rayleigh-Sommerfeld diffraction integral of the first kind (RS1) can be expressed as

$$(4) \qquad E_P = \frac{1}{2\pi} \iint_{S_A} E_Q \, \frac{e^{jk\,r_{PQ}}}{r_{PQ}^3} \, z_p \, (jk\,r_{PQ} - 1)\,\mathrm{d}S$$

where E_P is the electric field at the observation point P, E_Q is the electric field within the aperture and r_{PQ} is the distance from an aperture point Q to the point P. The double integral is over the area of the aperture S_A.

Approach 1 to evaluating The Diffraction Integral
op_rs_rxy_01.m

The double integral can be estimate numerically by a [two-dimensional form of Simpson's 1/3 rule](#). The electric field E_P at the point P is computed by

$$(5) \qquad E_P = E_0 \sum_{m=1}^{N} \sum_{n=1}^{N} \left(S_{mn} \, E_{Qmn} \, \frac{e^{jk\,r_{PQmn}}}{r_{PQmn}^3} \, z_{Pmn} \left(jk\,r_{PQmn} - 1 \right) \right)$$

where S_{mn} are the Simpson's two-dimensional coefficients and E_0 is a normalizing constant. Each term in equation (2) can be expressed as a matrix of size $N \times N$ and the matrices can be manipulated very easily in Matlab to give the estimate of the integral. The irradiance is proportional to the square of the magnitude of the electric field, hence the irradiance in the space beyond the aperture can be calculated by

$$(6) \qquad I = I_0 \left| E^* E \right|$$

where I_0 is a normalizing constant and E^* is the complex conjugate of E.

Approach 2 to evaluating The Diffraction Integral
op_rs_rxy_02.m

Instead of using arbitrary units for electric field and irradiance we can perform all calculations in S.I. units.

The **irradiance** or more generally the term **intensity** has S.I. units of $W.m^{-2}$. Another way of thinking about the irradiance is to use the term **energy density** as an alternative. The use of the letter I can be misleading, therefore, we will use the symbol u to represent the irradiance or energy density.

The irradiance or energy density u of a monochromatic light wave in matter is given in terms of its electric field E by

(7) $\qquad u = \dfrac{c\,n\,\varepsilon_0}{2}|E|^2$

n is the refractive index of the medium, c is the speed of light in vacuum and ε_0 is the permittivity of free space. This formula assumes that the magnetic susceptibility is negligible, i.e. $\mu_r \approx 1$ where μ_r is the magnetic permeability of the light transmitting media. This assumption is typically valid in transparent media in the optical frequency range.

At a point Q in the aperture plane, the energy density is given by the symbol u_Q and in the observation plane at a point P the symbol used is u_P. The energy transferred U from the aperture plane to observation plane per second is found by integrating the energy density with respect to an area

(8a) $\qquad U = \int_S u\,dS \qquad\qquad$ [W or J/s]

The energy per second radiated from the aperture is

(8b) $\qquad U_Q = \int_{S_Q} u_Q\,dS \qquad\qquad$ integration over the area of the aperture

The energy per second reaching an area S_P of the observation plane is

(8c) $\qquad U_P = \int_{S_P} u_P\,dS \qquad\qquad$ integration over the area element of the observation plane

In the mscript **op_rs_rxy_02.m** for the aperture, the dimensions a_x and a_y are specified [m] and the energy density u_Q [$W.m^{-2}$]. Hence, the electric field E_Q within the aperture is

$$(9) \qquad E_Q = \sqrt{\left(\dfrac{2\,u_Q}{c\,n\,\varepsilon_0} \right)}$$

The integral to find the electric E_P at a point P in the observation plane is

$$(4) \qquad E_P = \frac{1}{2\pi} \iint\limits_{S_A} E_Q \, \frac{e^{j k\, r_{PQ}}}{r_{PQ}{}^3} \, z_p \,(j k\, r_{PQ} - 1)\ \mathrm{d}S$$

where E_Q is given by equation (9) and the integral is computed using a two-dimensional form of Simposn's 1/3 rule.

The integrals given in equations (8) are also computed using the two-dimensional form of Simpson's 1/3 rule.

Once E_P is known, the energy density u_P is calculated from equation (7) and energy U_P from equation (8c).

SIMULATIONS   op_rs_rxy_01.m

Rectangular aperture (far field)

Input parameters

```
nP = 495;            % observation points for P  must be ODD
nQ = 147;            % aperture points for Q  must be ODD

wL = 632.8e-9;       % wavelength [m]

aQx = 1e-4;          % defines aperture size  [m]
aQy = 2e-4;

aP = 30e-3;          % defines size of observation space  [m]

zQ = 0; zP = 1;      % z distance between aperture plan and
                         observation plane [m]
```
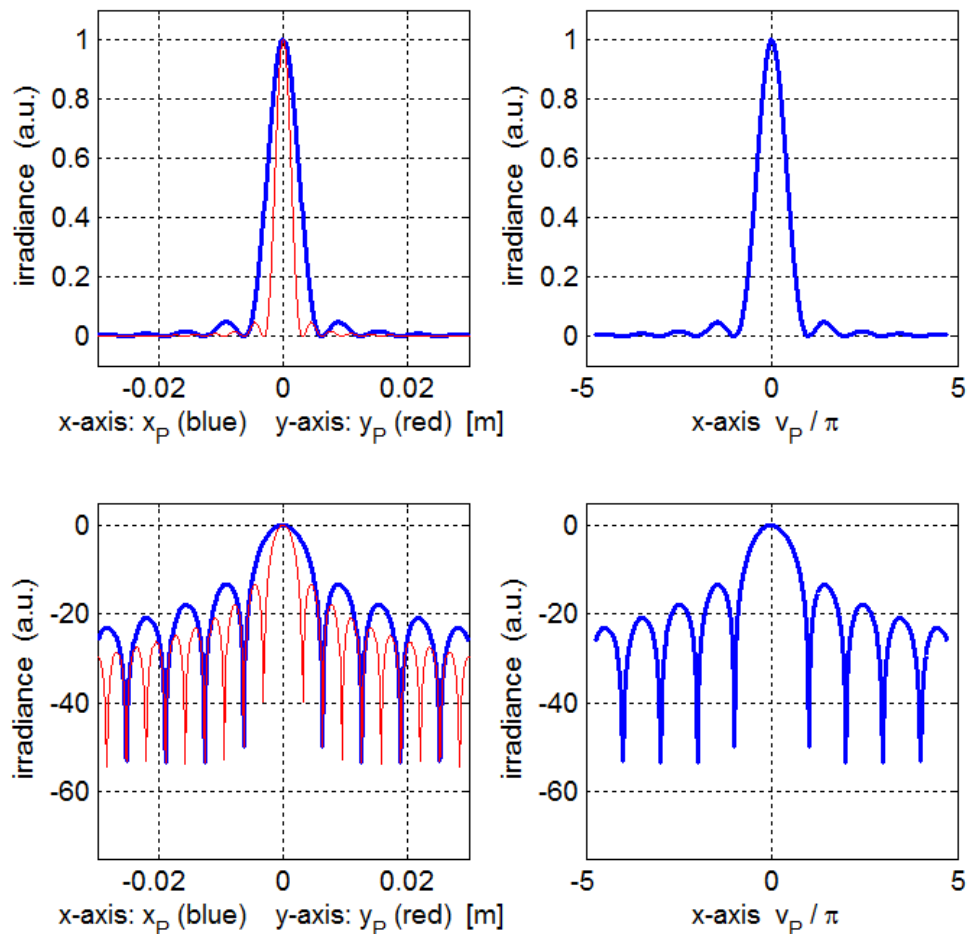
Fig. 2. Irradiance patterns in the along the *X* and *Y* axes. The maximum irradiance has been normalized to 1. The bottom curves use a log scale for the irradiance $I_{dB} = 10\log_{10}(I)$.

Figures (2) and (3) shows the irradiance for a rectangular aperture with $a_y = 2\,a_x$. The diffraction pattern is characterized by a strong central maximum and very weak peaks of decreasing magnitude away from the optical axis. The separation of the dark bands along the Y axes is narrower than along the X axis. Figure (3) shows photograph like images of the diffraction pattern. The irradiance is scaled to represent the different exposure times.
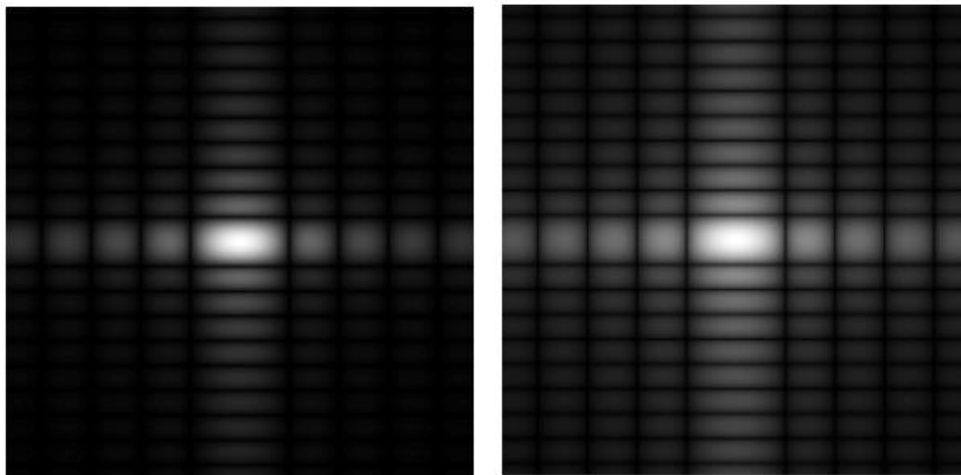

Fig. 3. Photograph like images for the diffraction patterns for a rectangular aperture. The image on the right is scaled to represent a longer exposure time.

In Tables 2 and 3 are listed values of the first few maxima and zeros of the diffraction pattern that were calculated using **op_rs_rxy_01.m** and the function **turningPoints.m**. The function returns the values of indexMax and indexMin. Then in the Command Window the values of the radial coordinates for the peaks and zeros can be displayed.

View document of Turning points of a function

The theoretical values are for the zero crossing and minima and maxima of the function $y_m{}^2 = \left(\sin x / x \right)^2$ which gives the Fraunhofer diffraction pattern for a single slit.

View document of the sinc function

Table 2.   Relative irradiances of the maxima of the diffraction pattern.

peaks	central	1st	2nd	3rd	4th
theory	**0**	± 1.429	± 2.462	± 3.470	± 4.478
vP / π	0	± 1.440	± 2.457	± 3.473	± 4.470
theory	1.0000	0.0472	0.0165	0.0083	0.0050
I/I_o	1.0000	0.0471	0.0165	0.0083	0.0050

Table 3. Zeros in the irradiance for the diffraction pattern

zeros	1st	2nd	3rd	4th
theory	± 1	± 2	± 3	± 4
v_P / π	± 0.9981	± 1.9960	± 2.9937	± 3.9910

How finding the zero crossings (indexMin) and maxima (indexMax) in the **Command Window** where indexMin gives the position in an array for the zeros and indexMax gives the position in arrays for the maxima. Note the Matlab variable for the irradiance is WP.

Type in the Command Window:

```
yData = WP(indexyP,:);
xData = xP(indexyP,:) ;
[indexMin indexMax] = turningPoints(xData,yData);

indexMin  →   24   55   86   117   179   210   241   272

vPx(indexyP, indexMin)./pi  →
```

 -3.9910 -2.9937 -1.9960 -0.9981   0.9981   1.9960   2.9937   3.9910

```
    indexMax  →   9   40   72   104   148   192   224   256   287

    vPx(indexyP, indexMax)./pi  →
```

 -4.470 -3.473 -2.457 -1.4440   0.0000   1.4440   2.457   3.473   4.470

```
    WP(indexyP, indexMax)  →
```

  0.0050   0.0083   0.0165   0.0471   1.0000   0.0471   0.0165   0.0083   0.0050

In a figure Window you can use the Data Cursor to find values for the zero crossings and maxima.

There is excellent agreement between the predictions of the Fraunhofer approximation and the calculations performed by evaluating the Rayleigh-Sommerfeld diffraction integral.

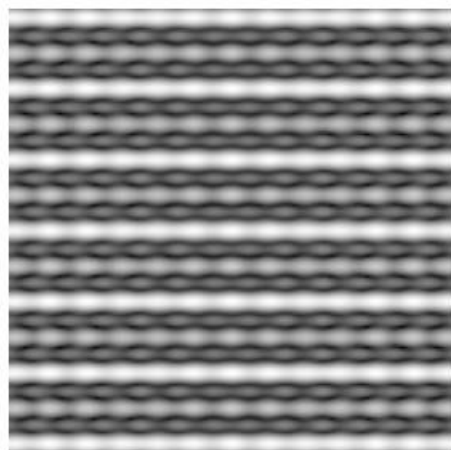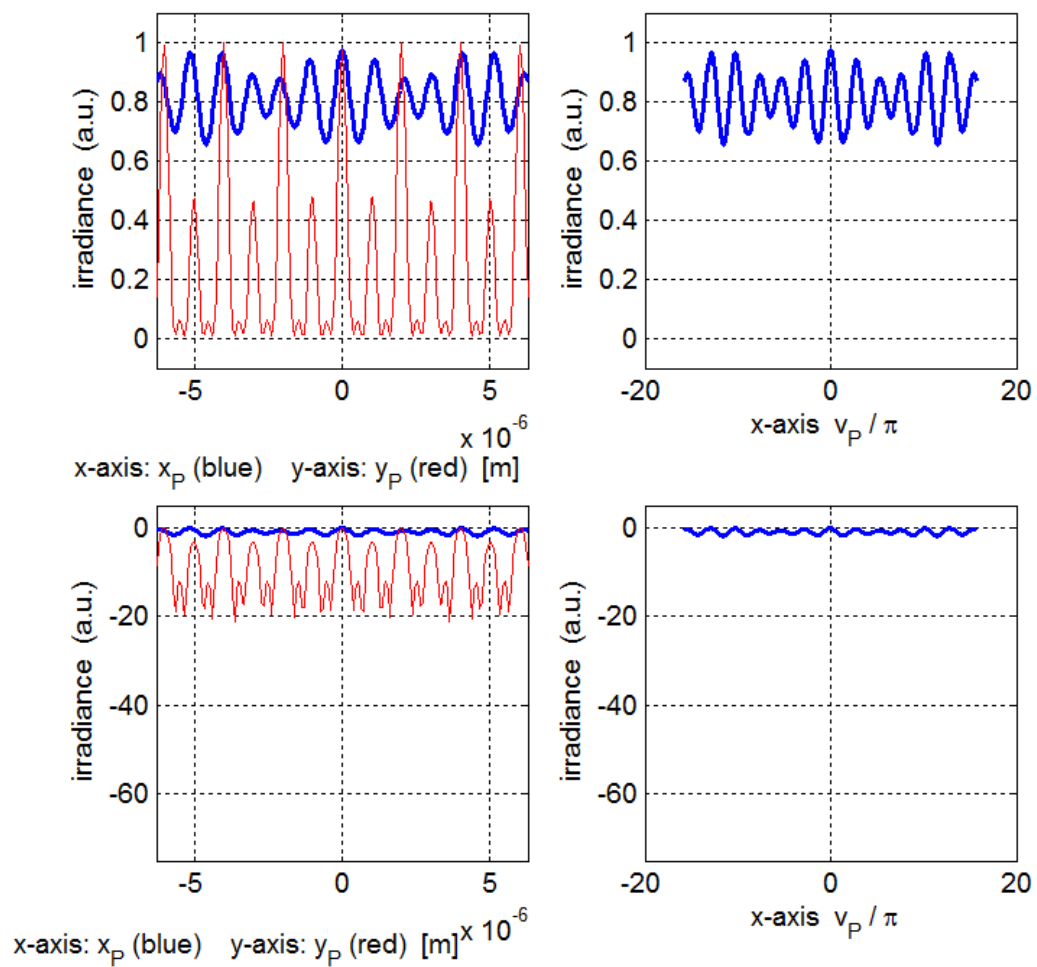## Diffraction pattern in near field

Fig. 4.   Diffraction pattern from a rectangular aperture. Observation plane close to aperture plane $z_P = 100\lambda$ and observation plane dimensions 10λx10λ.

Diffraction – single slit

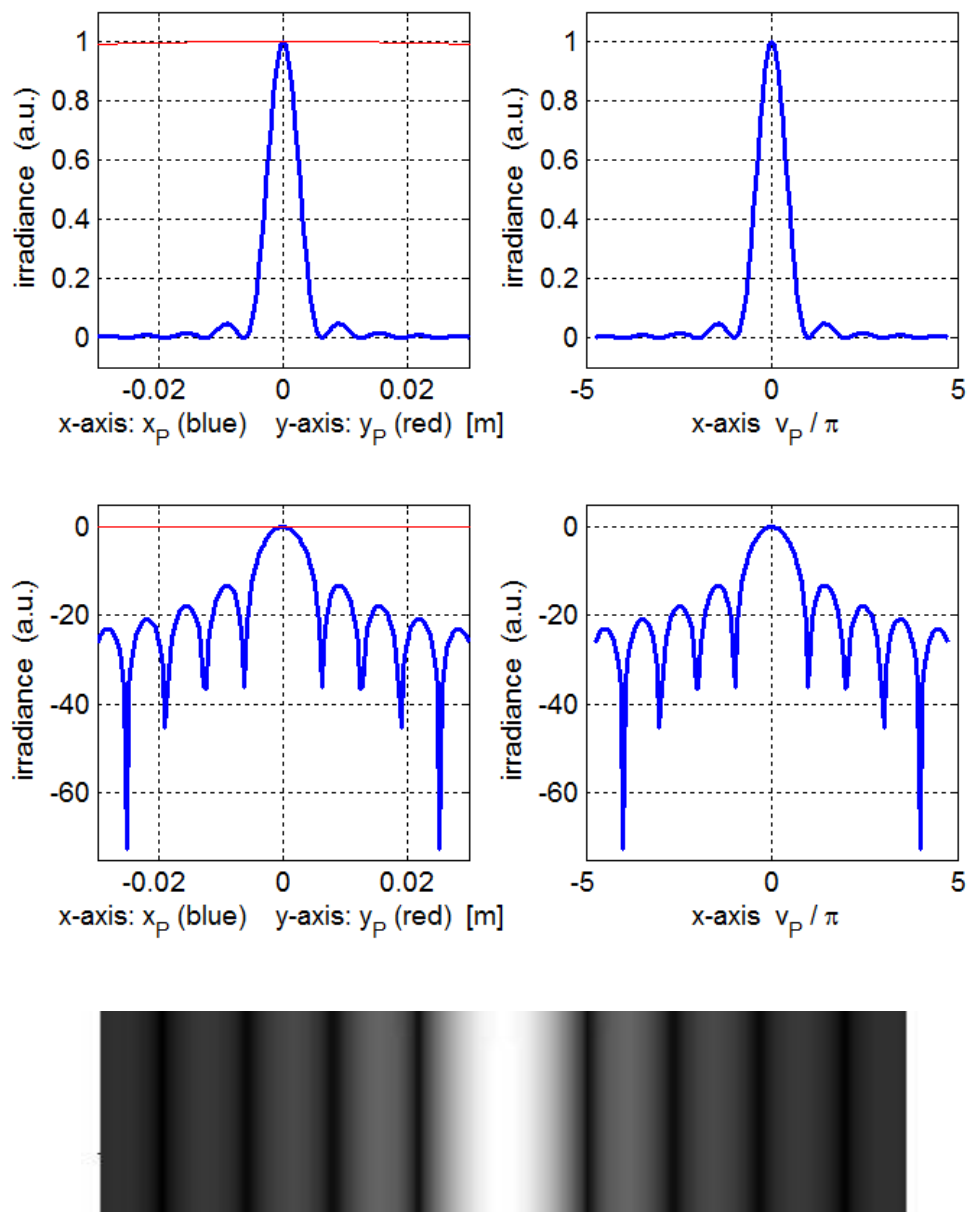slit dimensions $a_x = 1 \times 10^{-4}$ m $a_y = 1 \times 10^{-6}$ m

Fig. 5.   Diffraction pattern from a single slit.

Diffraction – square aperture
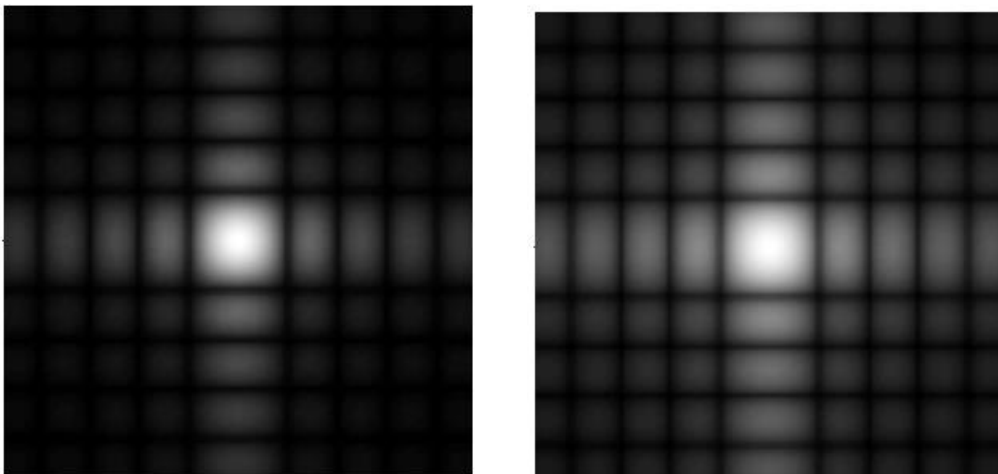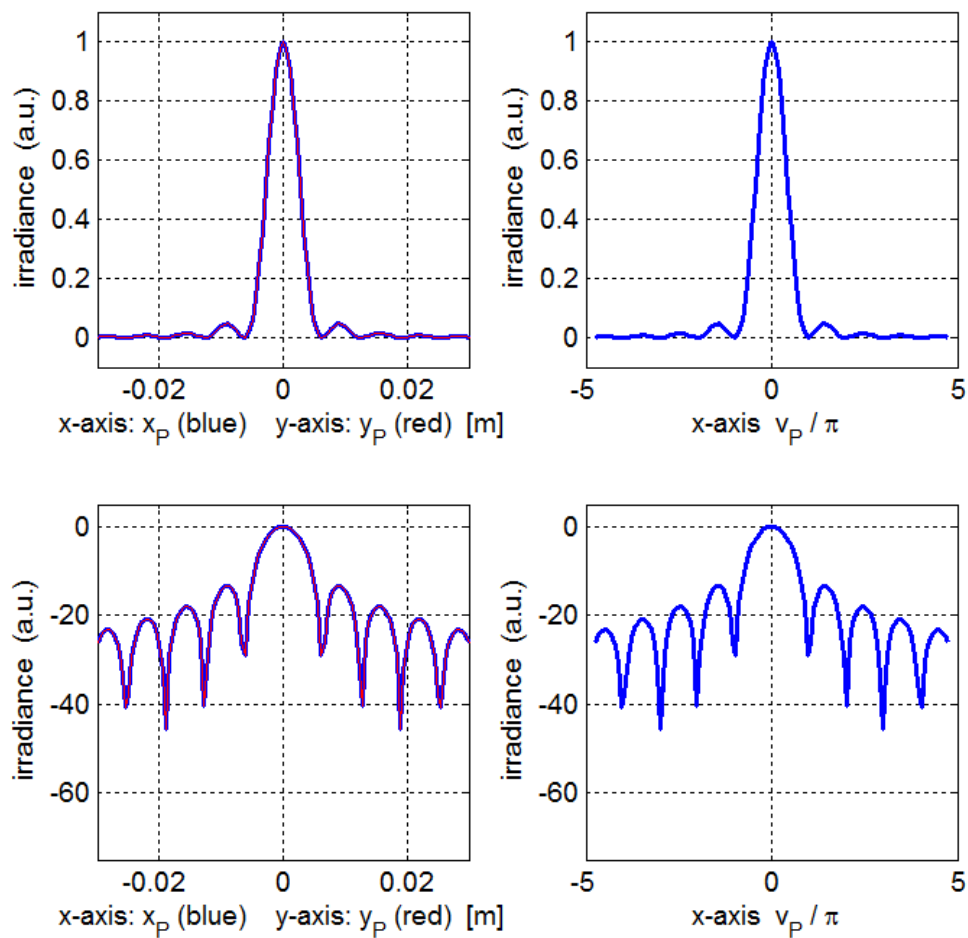
slit dimensions $a_x = 1\text{x}10^{-4}$ m $a_y = 1\text{x}10^{-4}$ m

Fig. 6. Diffraction pattern from a square aperture with different scaling for the irradiance.

SIMULATIONS   op_rs_rxy_02.m

The parameters for the aperture and observation planes are all changed within the mscript. After the program is run, a summary of the input and output parameters is shown in the Command Window.

Table 1: Parameter summary [SI units]

wavelength [m] = 6.33e-07
nQ = 099
nP = 201

Aperature Space
X width [m] = 1.000e-04
Y width [m] = 2.000e-04
energy density [W/m2] uQmax = 1.000e-03
energy from aperture [J/s]   UQ(theory) = 2.000e-11
energy from aperture [J/s]   UQ(calculated) = 2.000e-11

Observation Space
X width [m] = 4.000e-02
Y width [m] = 4.000e-02
distance aperture to observation plane [m]   zP = 1.000e+00
Fraunhofer: position of 1st min in X direction [m] = 6.328e-03
Fraunhofer: position of 1st min in Y direction [m] = 3.164e-03

max energy density [W./m2]   uPmax = 9.987e-07
energy to aperture [J/s]   UP = 1.902e-11

percentage energy enclosed within circle of radius (vPx = 3.14159) = 99.5
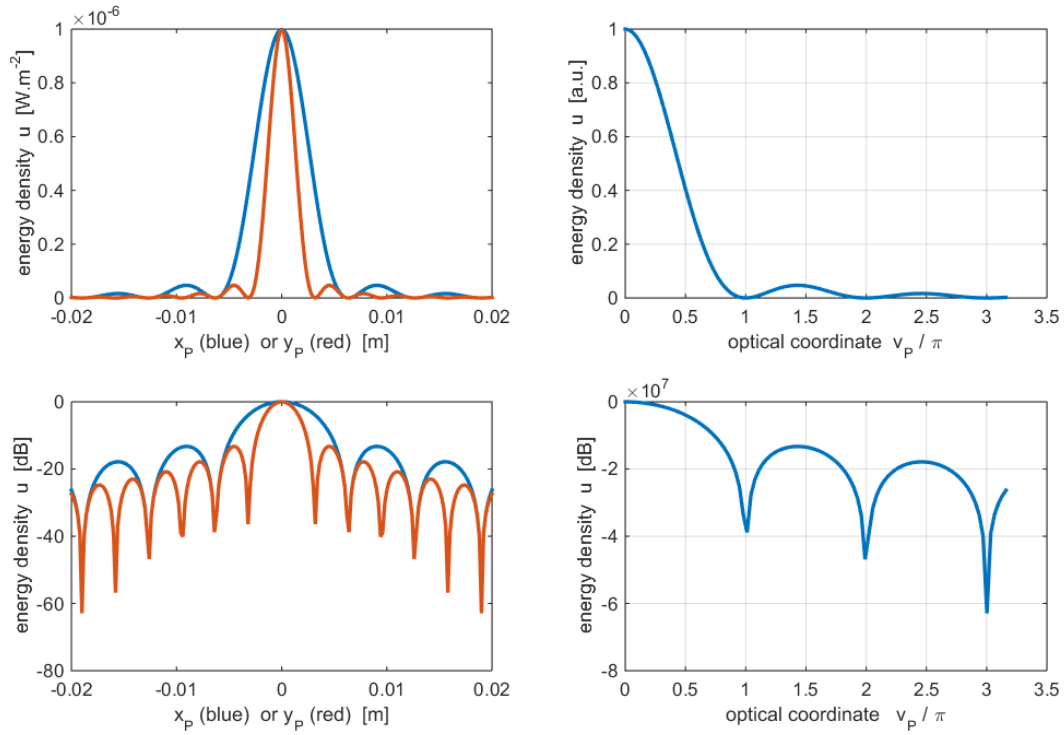

Elapsed time is 40.470176 seconds.

Fig. 7.   The variation on the energy density along the X and Y directions. The plot was created in Matlab version 2014b which has improved graphics compared to earlier versions.
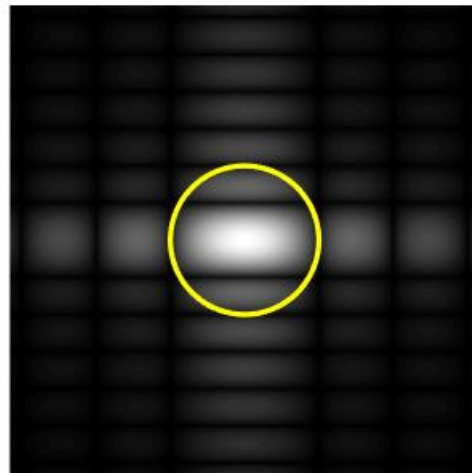

Fig. 8.   Black and white photograph like time exposure of the diffraction pattern. The yellow circle has a radius equal to the position of the first zero along the X axis. 99.5% of the energy incident upon the observation screen is concentrated within this circle.
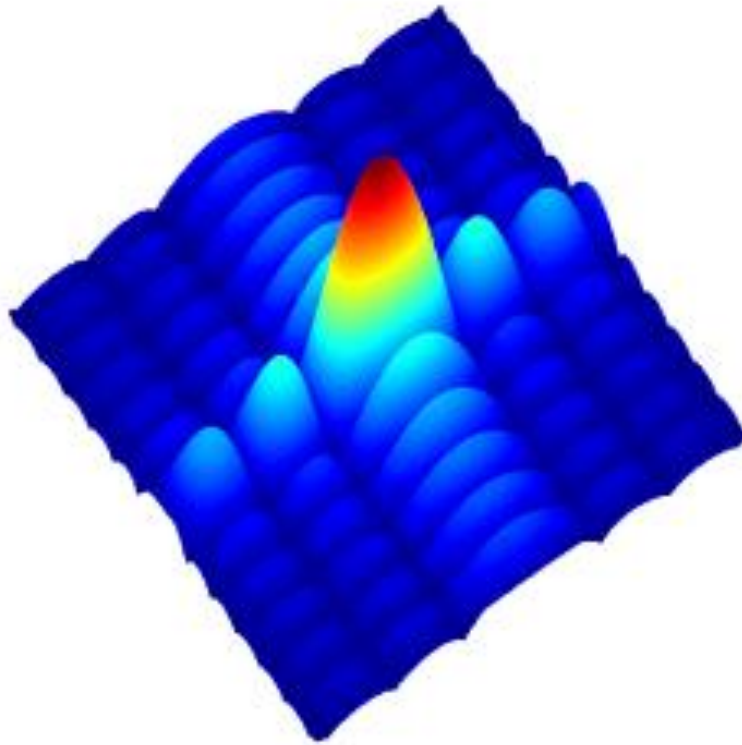
Fig. 9.   A Matlab surf style plot showing the energy density in the XY observation plane.

Investigations

In the Fraunhofer region (far field) the **inverse square law** applies. This can be tested by doubling the aperture plane to the observation plane distance and noting that the maximum energy density at a point on the Z axis decreases by a factor of 4. For example using the data as shown in Table 1 and only varying the distance z_P:

$z_{P1} = 1$

energy density $u_{Pmax1} = 9.987 \times 10^{-7}$ W.m^{-2}

1st min $x_{1min} = 6.328 \times 10^{-3}$ m $y_{1min} = 3.164 \times 10^{-3}$ m

$z_{P2} = 2$

energy density $u_{Pmax2} = 2.3497 \times 10^{-7}$ W.m^{-2}

1st min $x_{2min} = 12.66 \times 10^{-3}$ m $y_{2min} = 6.328 \times 10^{-3}$ m

$z_{P2} / z_{P1} = $ **2**

$u_{Pmax2} / u_{Pmax1} = $ **1/4**   **inverse square law satisfied**

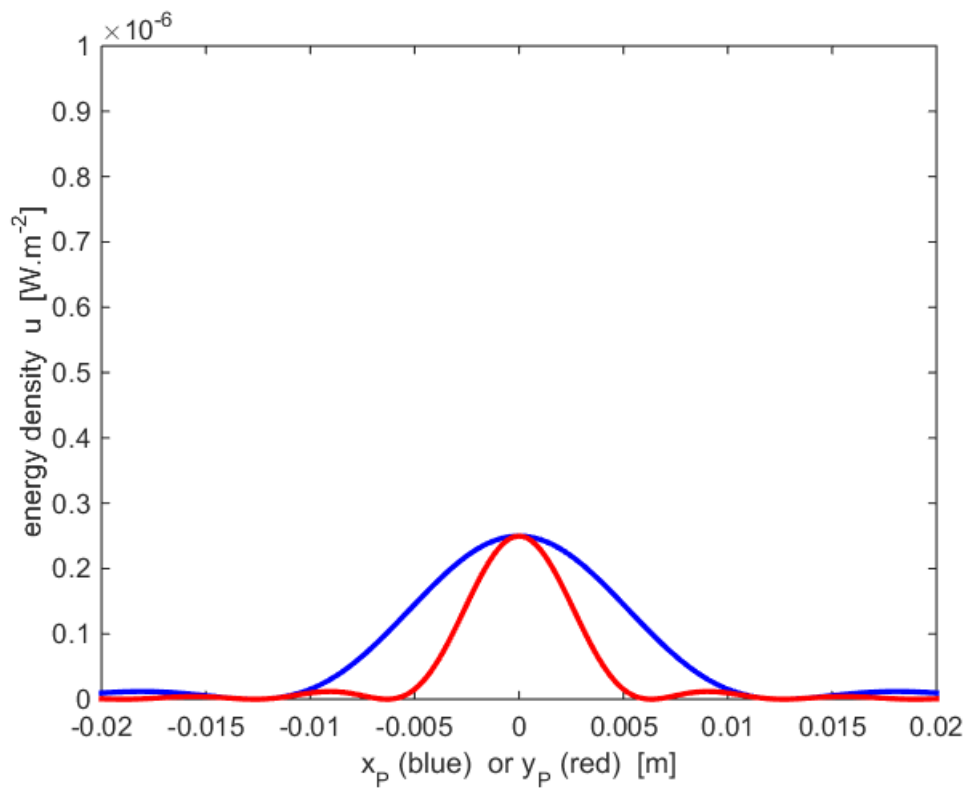$x_{2min} / x_{1min} = $ **2.0** $y_{2min} / y_{1min} = $ **2.0**   **pattern is wider**
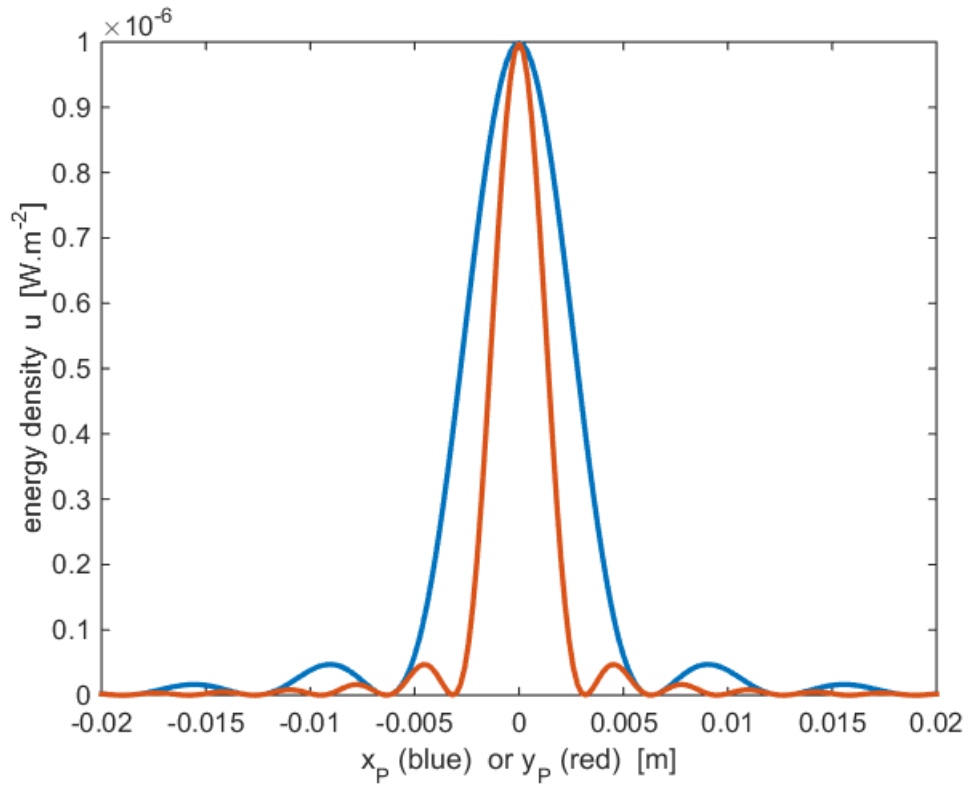
Fig. 10. Fraunhofer diffraction: upper plot z_P = 1.0 m and lower plot z_P = 2.0 m. As the aperture to observation plane distance increases the energy density distribution shape does not change but becomes wider and flatter.

Next, we can consider the effect of halving the area of the aperture by changing the width a_y in the Y direction from 2×10^{-4} m to 1.0×10^{-4} m.

Width $a_{y1} = 2 \times 10^{-4}$ m $\quad a_{y2} = 1 \times 10^{-2}$ m

Energy transferred from aperture to observation plane

$U_{Q1} = 2.0 \times 10^{-11}$ J.s^{-1} $\quad U_{Q2} = 1.0 \times 10^{-11}$ J.s^{-1}

$U_{Q2} / U_{Q1} = 0.50$

Maximum energy density in observation plane

$u_{Pmax1} = 9.987 \times 10^{-7}$ W.m^{-2} $\quad u_{Pmax2} = 2.497 \times 10^{-7}$ W.m^{-2}

$u_{Pmax2} / u_{Pmax1} = 0.25$

The effect of doubling the area of the aperture results in a doubling of the energy transferred from the aperture to the observation plane and a four-fold increase in the energy density.

Fresnel Diffraction $\Leftrightarrow$ Fraunhofer Diffraction

Fraunhofer diffraction is only valid in the far field. There is a transition from Fraunhofer diffraction to Fresnel diffraction as the distance between the aperture and observation planes decreases. The distance dividing the two regimes is known as the **Rayleigh distance** d_{RL}

$$d_{RL} = \frac{a^2}{\lambda} \qquad \text{where } a \text{ is the maximum of } a_x \text{ and } a_y$$

Fraunhofer diffraction (far field) $\quad z_P > d_{RL}$

Fresnel diffraction (near field) $\quad z_P < d_{RL}$

For an aperture $a_x = a_y = 1.0 \times 10^{-4}$ m and wavelength $\lambda = 6.33 \times 10^{-7}$ m, the Rayleigh distance is

$$d_{RL} = 1.58 \times 10^{-2} \text{ m}$$

The mscript **op_rs_rxy_02.m** can be run a few times to investigate the transition regime when $z_P \sim d_{RL}$ by changing the values for z_P, x_P and y_P in the mscript.

Rayleigh distance $d_{RL} = 0.0158$ m