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**THERMAL PHYSICS**

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# ONE-DIMENSIONAL HEAT TRANSFER BY CONDUCTION THROUGH COMPOSITE MATERIALS

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[**DOWNLOAD DIRECTORY FOR MATLAB SCRIPTS**](http://www.physics.usyd.edu.au/teach_res/mp/mscripts)

It is necessary to modify the mscripts and comment or uncomment lines of code to run the simulations with different input and output parameters.

**op\_rs\_fb\_xy.m**

Calculation of the radiant flux density (irradiance) in a plane perpendicular to the optical axis for the radiant flux of convergent beam emitted from a circular aperture. It uses Method 3: one-dimensional form of Simpson’s rule for the integration of the diffraction integral. This mscript is only valid for irradiance distributions that are circularly symmetrical about the optical axis (Z axis).

**op\_rs\_fb\_z.m**

Calculation of the radiant flux density (irradiance) along the optical axis for the radiant flux of convergent beams emitted from a circular aperture. It uses Method 3: one-dimensional form of Simpson’s rule for the integration of the diffraction integral.

**op\_rs\_fb\_xz.m**

Calculation of the irradiance in the meridional - XZ plane for the radiant flux of convergent beam emitted from a circular aperture. It uses Method 3: one-dimensional form of Simpson’s rule for the integration of the diffraction integral.

The transfer of heat is very important in industry, in the home and in biological systems. At times it is desirable to obtain the maximum heat transfer, at other times to reduce it. It is essential that the thermal properties of materials from the lowest to the highest temperatures are known. Then systems can be designed to meet the requirements of maximum or minimum transfer of energy. For example, what is the purpose of the large cooling towers in a power station? How can we keep our home warm in winter? How does our body try to keep us cool during a hot summer day? On a very hot day you pick up two bars lying on the ground in the sun, one is made of wood, the other metal, what would you feel and why? In answering these questions you have to consider, what produces a transfer of energy, and what causes a change

in temperature. Your answers may include the terms hot, cold, temperature, heat, energy, energy transfer, power, energy flux, and energy flux density but what do these terms mean? Can you clearly differentiate the concepts associated with each of these terms?

You can gain a deeper and better understanding of these concepts, by modelling the transfer of energy along a rod by running the simulations with a wide range of input parameters.

***Heat is a misleading term***. A better alternative is to simply use the term **energy** and so on most occasions the word energy will be used and not the term heat. The **rate of energy transfer** through the a can also be referred to as the **power** or **energy flux**. The **energy flux density** *J* is a very useful term because it is independent of the cross-sectional area *A* of the rod. The energy flux density is defined as

(1) 

We will model the one-dimensional heat transfer along a cylindrical rod. The model can be used to compute both the temperature and the transfer of energy as functions of time and position along a rod.

The rate of energy transfer through a rod is dependent upon the thermal conductivity *k* of the material, the cross-sectional area *A* of the rod and the temperature gradient as described by the differential equation

(2) 

Using equations (1) and (2), the energy flux density is

(3) 

When energy is transferred to or from an object its temperature changes. This change in temperature *dT* depends upon the objects mass *m*, its specific heat capacity c and the amount of energy *Qnet* gained or lost by the object. The change in temperature is given by

(3) 

and the time rate of change of temperature with time is

(4) 

For an element of density and volume , the mass of the element is

(5) 

Combing equations (1), (4) and (5), the time rate of change of the temperature is

(6) 

where net energy flux density is given by

(7) 

*J* is the contribution to the energy flux density due to the conduction of energy along the rod, *Jint* is the internal heating of the rod (for example, an electric heating element) and *Jenv* is the due to the transfer of energy to and from an element to the surrounding environment (for example, by thermal radiation or convection).

We can find the temperature and energy flux densities as functions of time and positions by solving the two differential equations (3) and (6) using a finite difference approach.

Consider a rod of radius *r* and of length *L* divided into *N* elements and the width of each element is. It is assumed that all the properties of an element are uniform throughout the volume of the element. Each element is characterised by its thermal conductivity *k*, specific heat capacity *c* and its density *ρ*. The temperatures are calculated at the faces of each element. At time *t*, for the *n*th element, the face temperatures at positions *xn* and *xn*+1 are *T*(*t*, *xn*) and *T*(*t*, *xN*+1) where *n* = 1, 2, 3, … , N and .



Fig. 1. Schematic diagram for modelling the heat conduction through a cylindrical rod.

The differential equation (3) for the energy flux density can be expressed as a finite difference equation for the *n*th element at time *t* as

(8) 

The differential equation (6) for the temperature can be expressed as a finite difference equation for the *n*th element at time  as

(9) 

At time *t* = 0, the temperatures at the faces of each element is specified as well as the boundary conditions at the ends and along the length of the rod. From the initial conditions and the boundary conditions we first calculate the energy flux density through each element using equation (8). Then at a time we calculate the temperature at the faces of each element with equation (9) and the new values for the energy flux density using equation (8). We then continue updating the energy flux densities then the temperatures at successive time steps to compute the time evolution of the temperature and energy flux density.

For a perfecting insulated rod (*J*env = 0) and with no internal heating (*J*int = 0) and for the boundary conditions of fixed temperatures at the ends of the rod, a sample mscript to calculate the temperature and energy flux density is

K1 = -( k ./ dx ); % constant

K2 = dt ./ ( rho .\* c .\* dx ); % constant

for ct = 1 : nt-1 % time loop

for cx = 1 : nx % position loop

J(ct,cx) = K1(cx) \* ( T(ct,cx+1) - T(ct,cx) );

end

for cx = 2 : nx

T(ct+1,cx) = T(ct,cx) + K2(cx) \* ( J(ct,cx-1) - J(ct,cx));

end

end

J(end,:) = J(end-1,:);

The mscript **tp\_rod\_001.m** gives the results of a simulation in a Figure Window as shown in figure (2). The Figure Window shows:

* A summary of the numerical values for the input and calculated parameters.
* Plots of the temperature profile along the rod at four successive time intervals.
* Plots of the energy flux density through the rod at four successive time intervals.
* Plots of the time evolution of the temperature at five positions along the rod.
* Plots of the time evolution of the energy flux density at five positions along the rod.
* A visual plot of the temperature along the length of the rod.

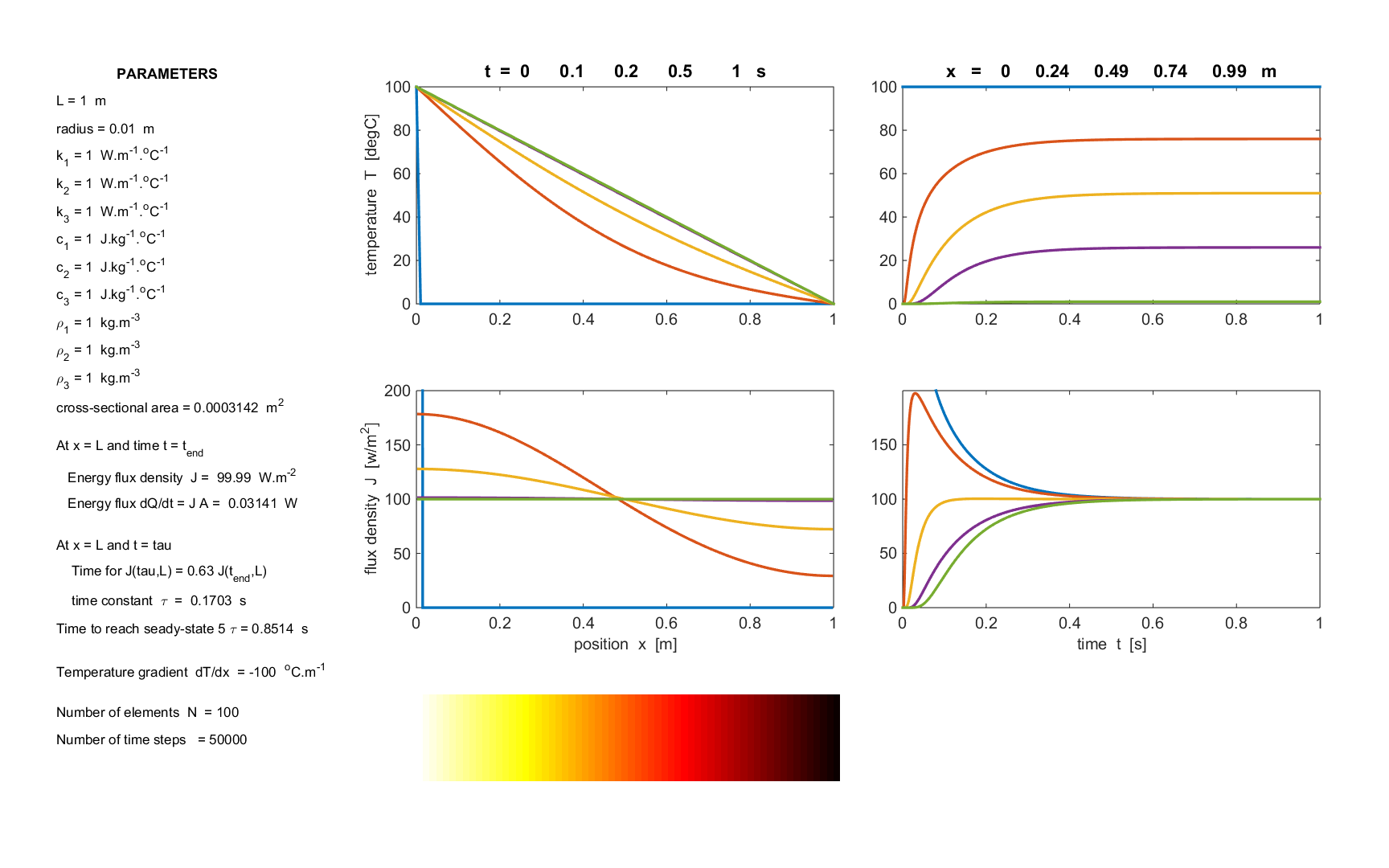


Fig. 2. The Matlab Figure Window for a simulation for a uniform rod with the temperatures fixed at each end using the mscript **tp\_rod\_001.m**.

**SINGLE SEGMENT ROD – single segment**

**ENDS OF ROD MAINTAINED AT FIXED TEMPERATURES**

We will first consider a rod composited of only a single uniform material that is insulated along its length with zero internal heating (*Jint* = 0) and zero energy transfer to the surrounding environment through the sides of the rod for the internal elements (*Jenv\_n* = 0 *n* = 2, 3, … *N*-1). To maintain a constant temperature at the ends of the rod an energy exchange must occur between the end elements and the surrounding environment. For fixed temperatures *T*1 and *T*N+1 then if *T*1 > *T*N+1

*J*1 = *Jenv*\_1 energy is transferred from a hot reservoir to element 1

*JN* = *Jenv*\_*N* energy is transferred from element *N* to a cold reservoir

A schematic diagram of the rod is shown in figure 3.

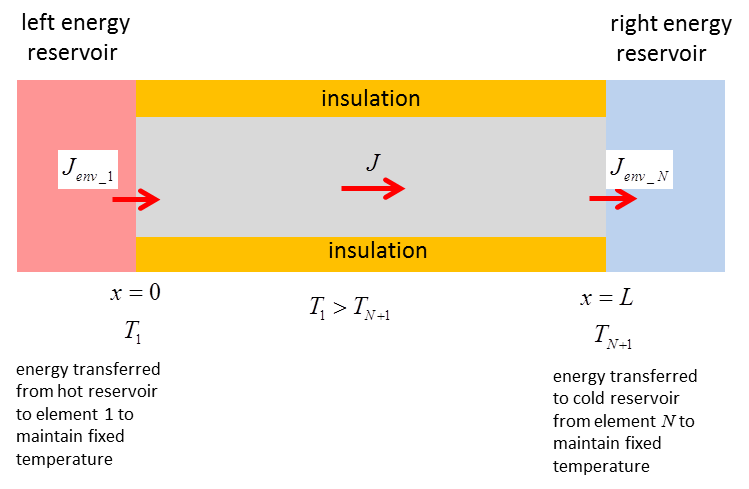


Fig. 3. Energy transfer through a rod with fixed temperatures maintained at the ends.

Consider the input parameters:

length *L* = 1.00 m

radius *r* = 0.10 m

thermal conductivity *k* = 1.00 W.m-1.oC-1

specific heat capacity *c* = 1.00 J.kg-1.oC-1

density ρ = 1 kg.m-3

*x* = 0 *T*1 = 100 oC *x* = *L* = *xN*+1 *TN*+1 = 0 oC

From theoretical consideration as described by equations (2) and (3), the temperature should be a linear function of position along the rod and the temperature gradient should be



In the equilibrium state, the energy flux density through the rod should be uniform with a value

*J* = + 100 W.m‑2

and the energy flux

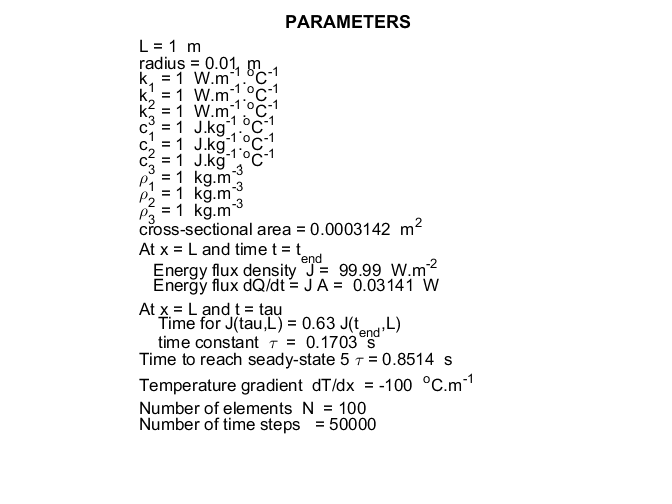


Running the numerical simulation using the mscript **tp\_rod\_001.m** with these input parameters with 100 elements and 50 000 time steps gives results that are in almost perfect agreement with the theoretical predictions

Numerical results:

 *J* = + 99.99 W.m‑2 

Figures 4 and 5 display the graphical results of the simulation. Also, summary of the parameters for a simulation are shown in a Figure Window:



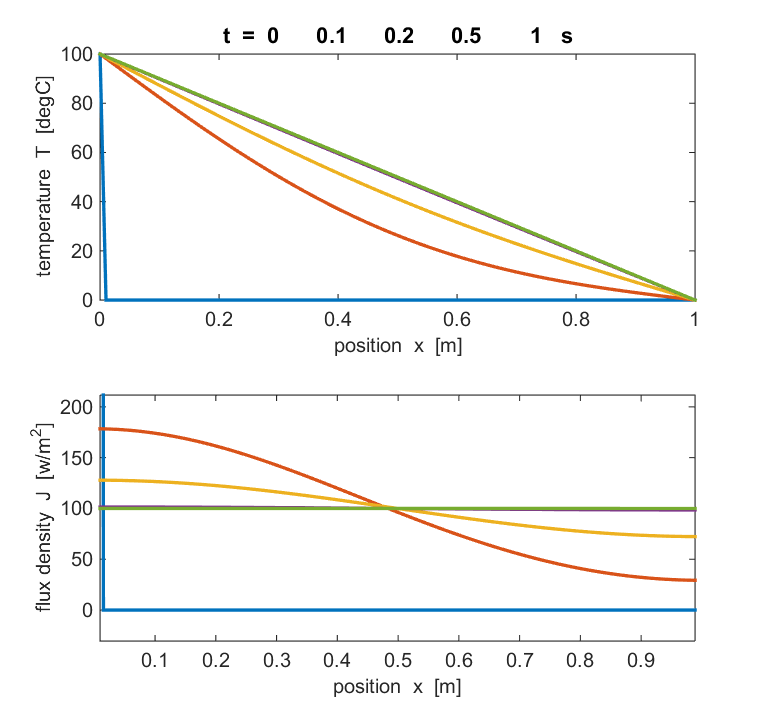




Fig. 4. The temperature profile along the rod and the energy flux density at times 0, (1/10) *tmax*, (2/10)*tmax*, (5/10)*tmax* and *tmax* where *tmax* is the simulation time.

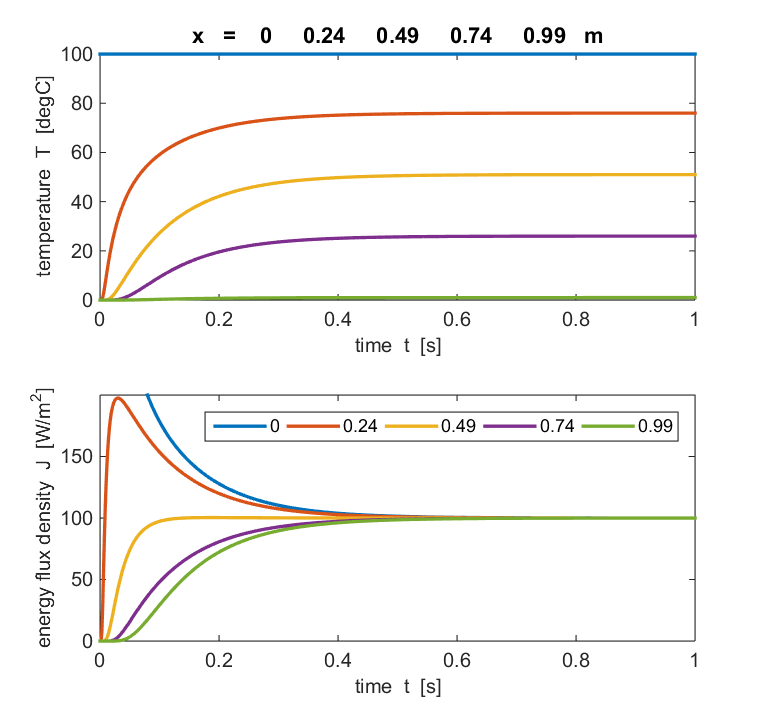


Fig. 5. The temperature and energy flux density variation with time at fixed positions along the rod: 0, (1/4) *L*, (1/2)*L*, (3/4)*L.*

Energy is transferred through the rod from the hot reservoir to the cold reservoir.

Initially a large amount of energy is transferred through the rod in the direction of decreasing temperature, heating the rod which results in the rise in temperature along its length. Finally a steady-state situation is established with a linear variation in temperature along the rod and a constant energy flux density through it.

It is difficult to determine the time to reach the steady state situation from the graphs. To have a consistent means to estimate this time, we can calculate the time interval for the energy flux density to reach 63% of its final value. This time interval is called the time constant *τ*. The time to reach the steady-state is then defined to be equal to 5*τ*. for our simulation

time constant *τ* = 0.170 s

time to reach steady-state 5*τ* = 0.851 s

We can investigate the physics of heat conduction by running the simulation for a wide range of input parameters. The best way to start is simply to change only one of the input parameters at a time. For each simulation, predict the changes that would occur then check your predictions by observing the results of the simulation.

**Changing one parameter at a time**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| ***L* = 1 m *r* = 0.01 m *k* = 1 W.m-1.oC-1 *c* = 1 J.kg-1.oC-1 *ρ* = 1 kg.m-3** | | | | |
| **dT/dx**  **[oC.m-1]**  **-100** | **A**  **[m2]**  **3.14×10-4** | **J**  **[W.m-2]**  **100** | **dQ/dt**  **[W]**  **3.14×10-2** | **5*τ***  **[s]**  **0.851** |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| ***L* = 2 m** *r* = 0.01 m *k* = 1 W.m-1.oC-1 *c* = 1 J.kg-1.oC-1 *ρ* = 1 kg.m-3 | | | | |
| dT/dx  [oC.m-1]  -50 | A  [m2]  3.14×10-4 | J  [W.m-2]  50 | dQ/dt  [W]  1.57×10-2 | 5*τ*  [s]  3.41 |
| *change* ×  (1/2) | 1 | (1/2) | (1/2) | 4 |

Increasing the length: decreases the temperature gradient; decreases the energy transfer through the rod; takes much longer to reach steady state.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *L* = 1 m ***r* = 0.02 m** *k* = 1 W.m-1.oC-1 *c* = 1 J.kg-1.oC-1 *ρ* = 1 kg.m-3 | | | | |
| dT/dx  [oC.m-1]  -100 | A  [m2]  12.6×10-4 | J  [W.m-2]  100 | dQ/dt  [W]  12.6×10-2 | 5*τ*  [s]  8.51 |
| *change* ×  1 | 4 | 1 | 4 | 1 |

Increasing the radius: no changes in temperature gradient, energy flux density and time to reach steady-state; increase in area and energy flux.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *L* = 1 m *r* = 0.01 m ***k* = 2 W.m-1.oC-1**  *c* = 1 J.kg-1.oC-1 *ρ* = 1 kg.m-3 | | | | |
| dT/dx  [oC.m-1]  -100 | A  [m2]  3.14×10-4 | J  [W.m-2]  200 | dQ/dt  [W]  6.28×10-2 | 5*τ*  [s]  0.426 |
| *change* ×  1 | 1 | 2 | 2 | (1/2) |

Increasing the thermal conductivity: no changes in temperature gradient and area; increase in energy flux density and energy flux; decrease in time to reach steady-state.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *L* = 1 m *r* = 0.01 m *k* = 2 W.m-1.oC-1 ***c* = 1 J.kg-1.oC-1** *ρ* = 1 kg.m-3 | | | | |
| dT/dx  [oC.m-1]  -100 | A  [m2]  3.14×10-4 | J  [W.m-2]  100 | dQ/dt  [W]  3.14×10-2 | 5*τ*  [s]  1.066 |
| *change* ×  1 | 1 | 1 | 1 | 2 |

Increasing the thermal conductivity: no changes in temperature gradient, area, energy flux density and energy flux; increase in time to reach steady-state.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *L* = 1 m *r* = 0.01 m *k* = 2 W.m-1.oC-1 *c* = 1 J.kg-1.oC-1 ***ρ* = 1 kg.m-3** | | | | |
| dT/dx  [oC.m-1]  -100 | A  [m2]  3.14×10-4 | J  [W.m-2]  100 | dQ/dt  [W]  3.14×10-2 | 5*τ*  [s]  1.066 |
| *change* ×  1 | 1 | 1 | 1 | 2 |

Increasing the thermal conductivity: no changes in temperature gradient, area, energy flux density and energy flux; increase in time to reach steady-state.

In equation(2)

(2) 

the significance of the minus sign ( **-** ) is that the energy transfer is in the direction of decreasing temperature, i.e., heat always flows spontaneously from the location of higher temperature to locations of lower temperature. In our simulation *T*1 = 100 oC and *TN*+1 = 0 oC hence the energy flow is from the left to right and the *J* > 0. If we reverse the temperatures *T*1 = 0 oC and *T*N+1 = 100 oC then *J* < 0 and the energy flow is from the right to the left as shown in figure (6).

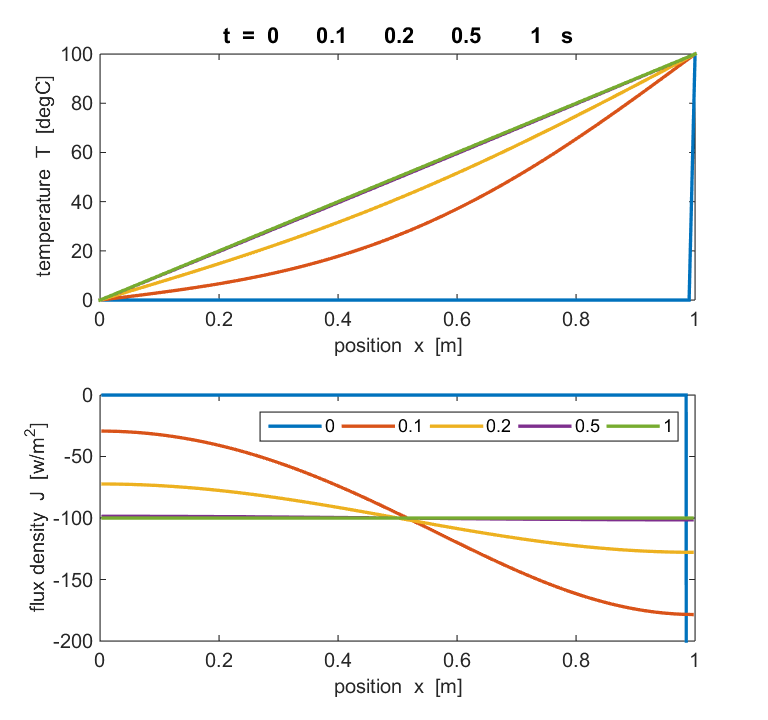


Fig. (6). Heat always flows in a direction from hot (high temperature) to cold (low temperature).

**Which is better – a copper based saucepan or an aluminium based saucepan ?**

Consider two saucepans of the same dimensions filled with equal quantities of boiling water sitting on a hot plate. We can model the heat transfer through the base using the mscript **tp\_rod\_001.m**.

Base

thickness *L* = 0.003 m

radius *r* = 0.050 m

Copper: *k* = 400 W.m-1.oC-1 *c* = 380 J.kg-1.oC-1 *ρ* = 8900 kg.m-3

*ρ* *c* = 3.38×106 (kg.m-3) (W.m-1.oC-1)

Aluminium: *k* = 240 W.m-1.oC-1 *c* = 900 J.kg-1.oC-1 *ρ* = 2700 kg.m-3

*ρ* *c* = 2.43×106 (kg.m-3) (W.m-1.oC-1)

Boundary conditions

hot plate *T*1 = 250 oC

boiling water *TN*+1 = 100 oC