CS 5003: Parameterized Algorithms

Lectures 10-11

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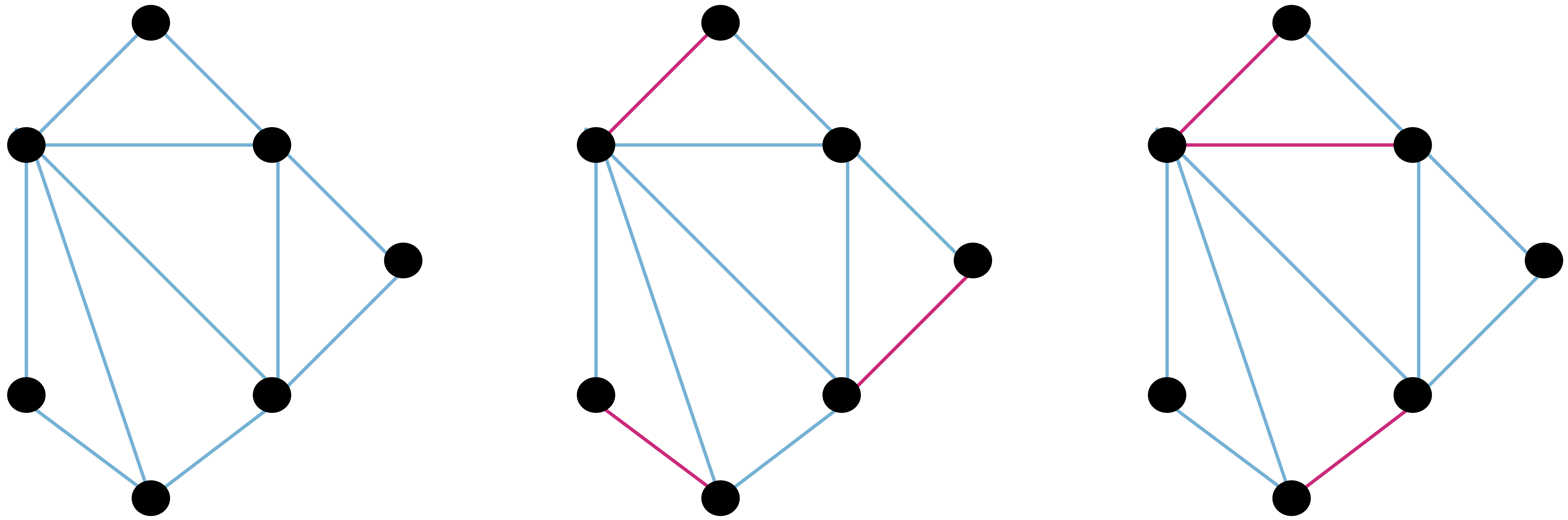
Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

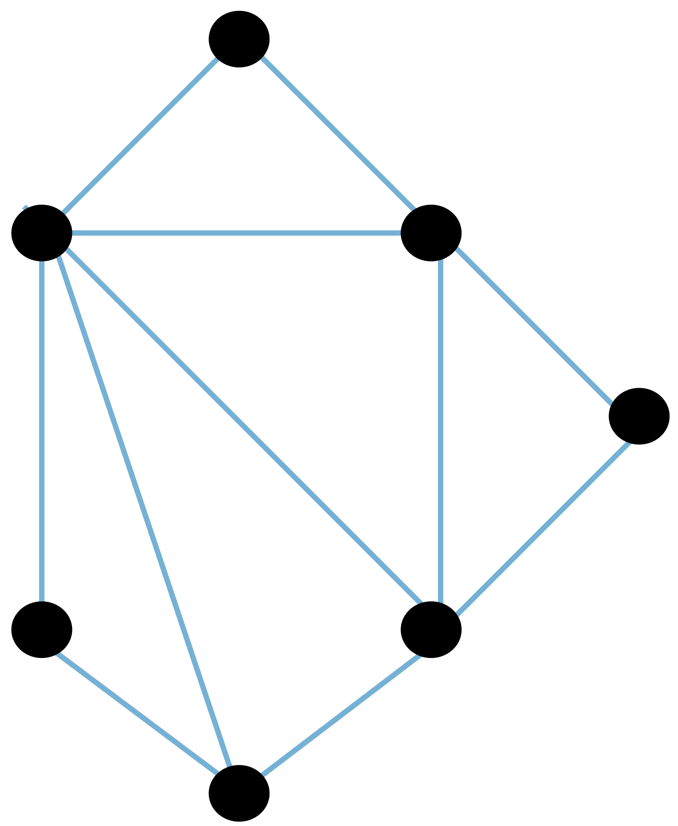
Matching

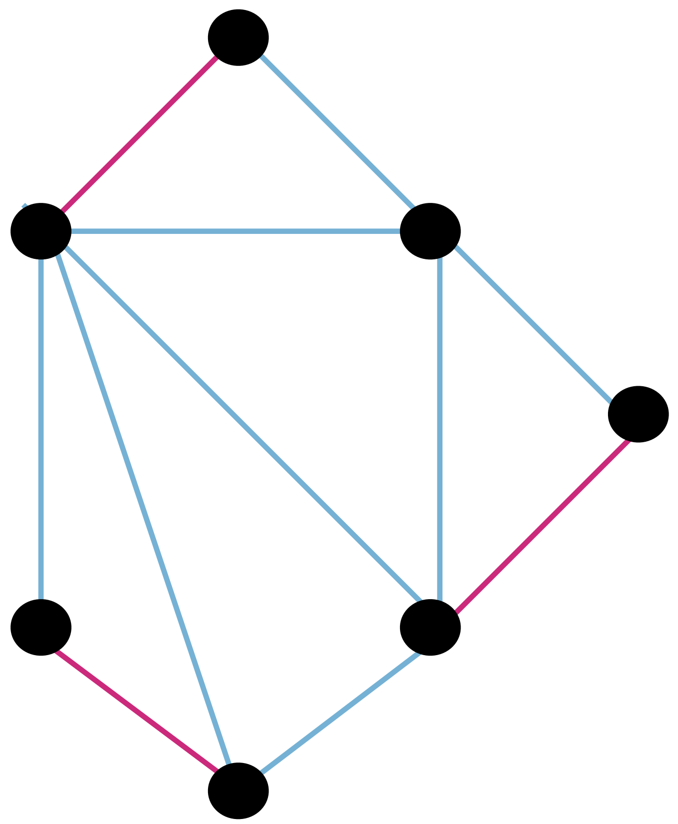
Matching - set of edges that do not share any endpoint



Matching and Vertex Cover

Matching - set of edges that do not share any endpoint

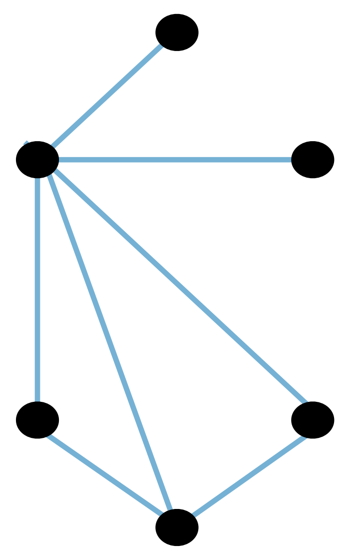


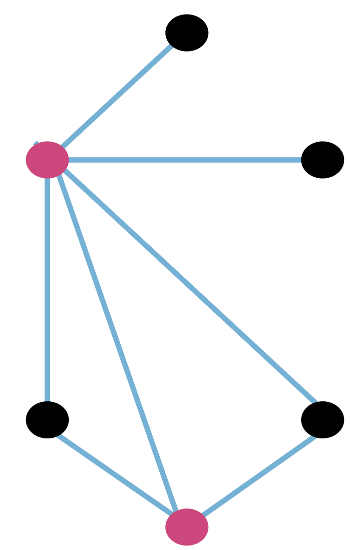


Matching of size x => Any vertex cover has size >= x

Vertex Cover

Vertex cover - set of vertices that has at least one endpoint of each edge





Instance: A graph G on n vertices m edges and integer k
Question: Does G have a vertex cover of size at most k?
Parameter: k

- Kernel with k^2 edges and $2k^2/3$ vertices
- $O(n^3+1.4656^k k^3)$ time algorithm

Vertex Cover: 3k vertex kernel

Crown Decomposition

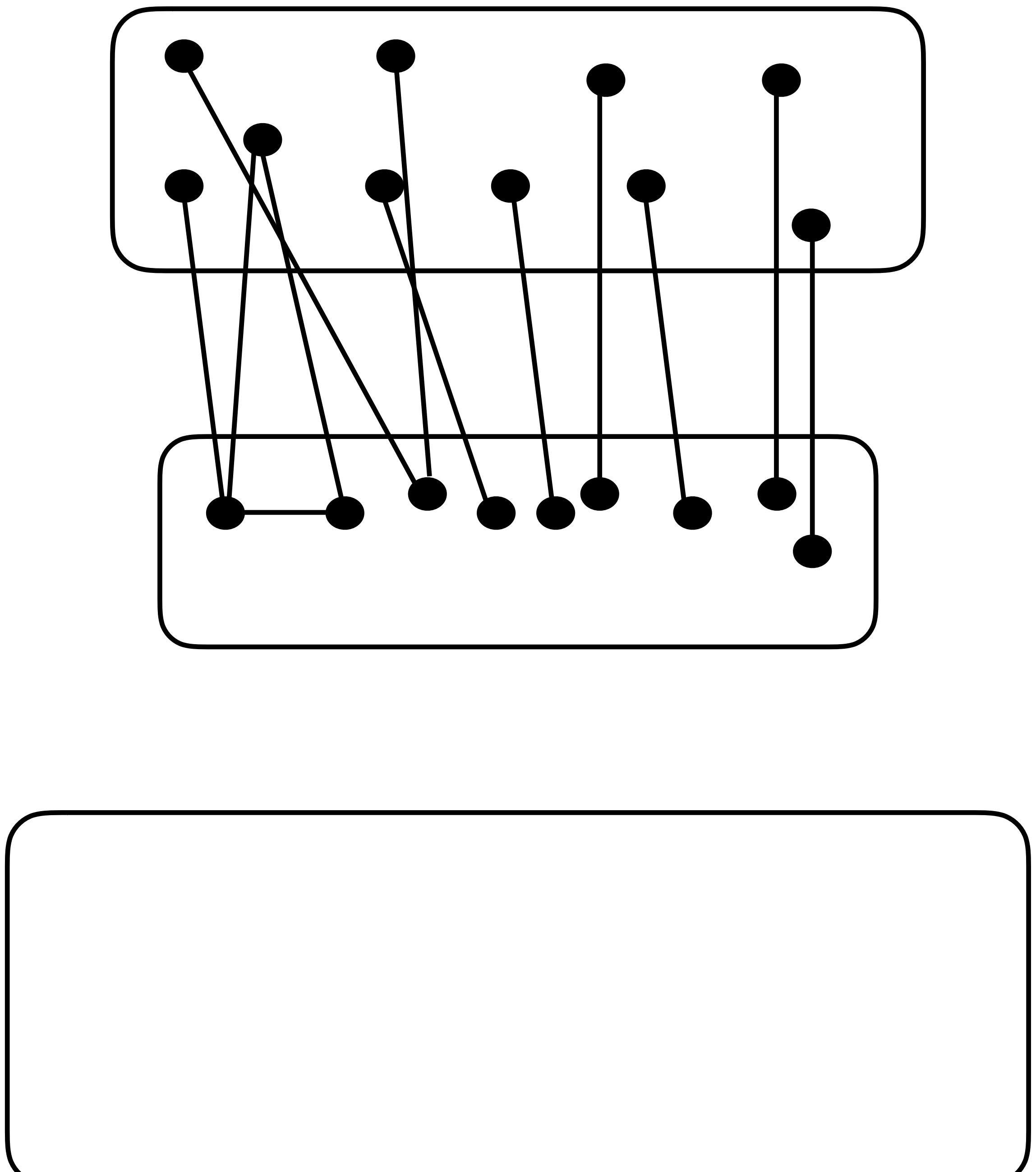
Crown C

Independent set

Head H

$N(C) \subseteq H$

Rest R

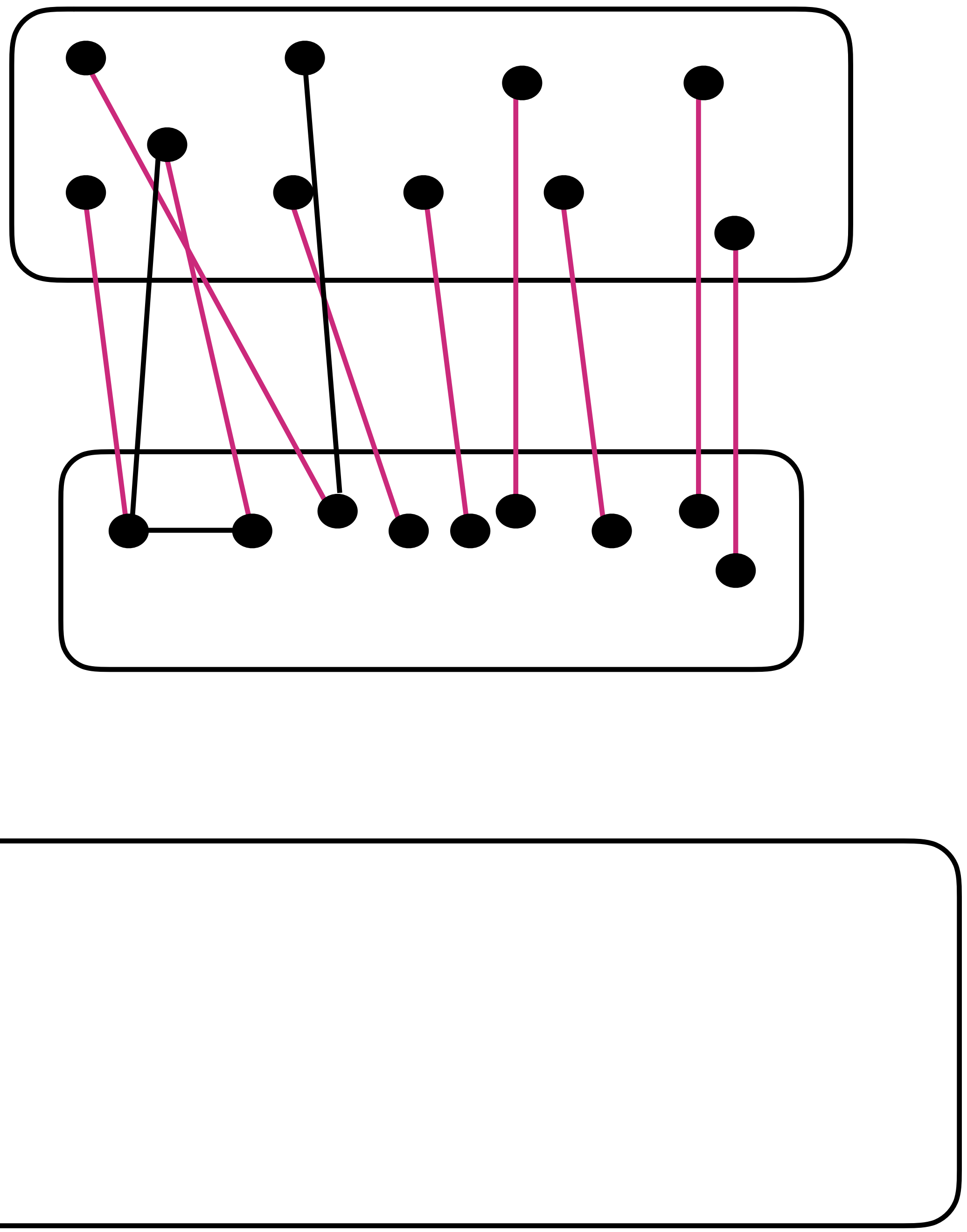


Vertex Cover: 3k vertex kernel

Crown Decomposition

Crown C

Independent set



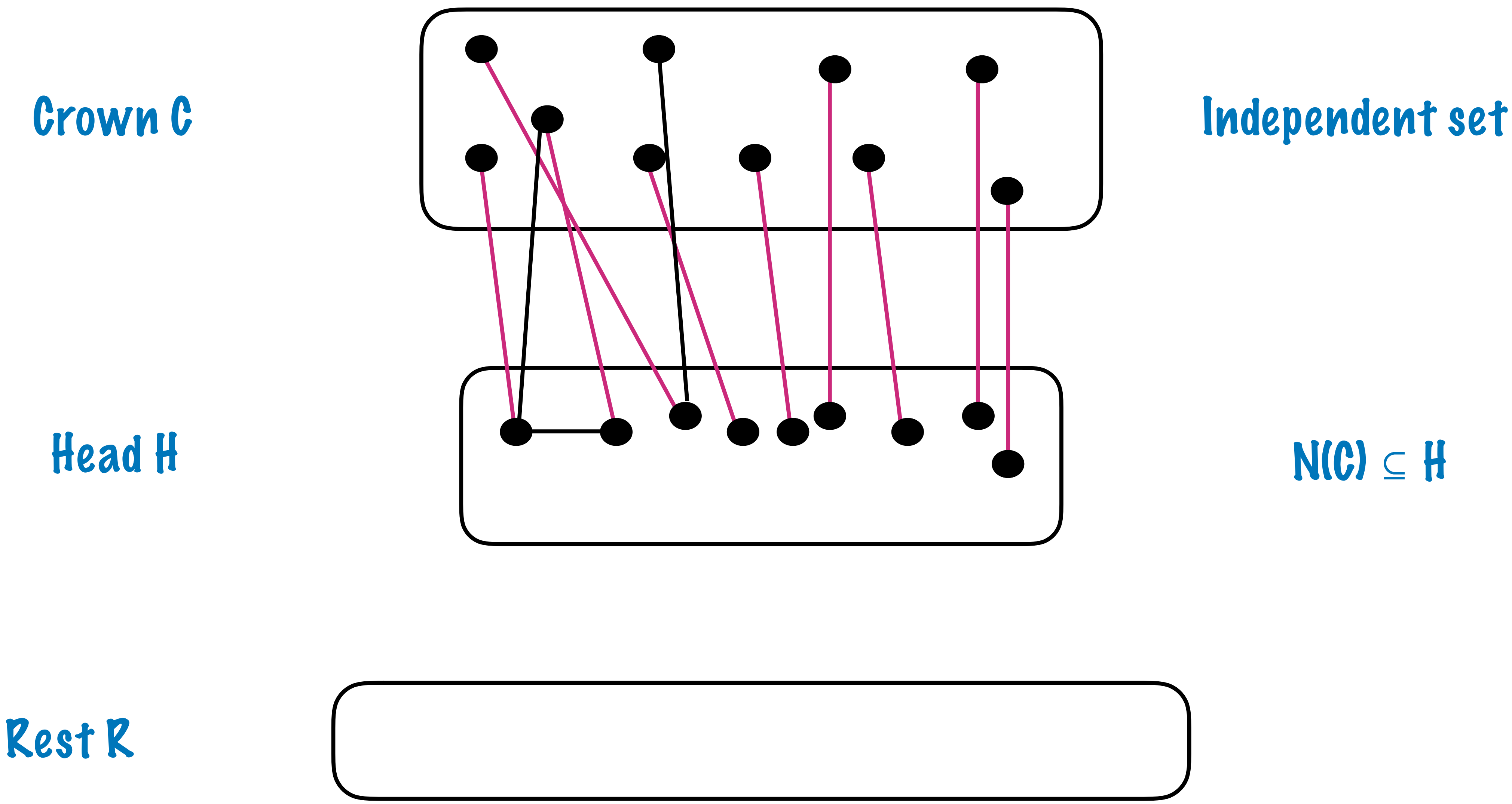
Head H

$N(C) \subseteq H$

Rest R

Matching saturating H

Vertex Cover: 3k vertex kernel



(G,k) is an yes-instance iff (G-(H∪C), k-|H|) is an yes-instance

To show forward direction, consider VC S. Define S' = S \ C U H which clearly is a VC. Now, |S \cap (H U C)| >= |H| and |S' \c

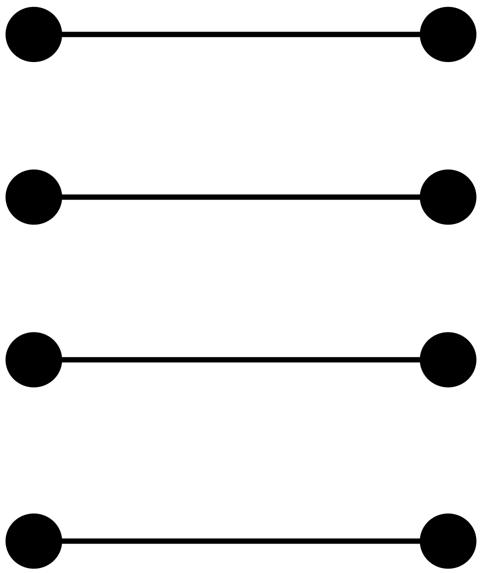
Vertex Cover: 3k vertex kernel

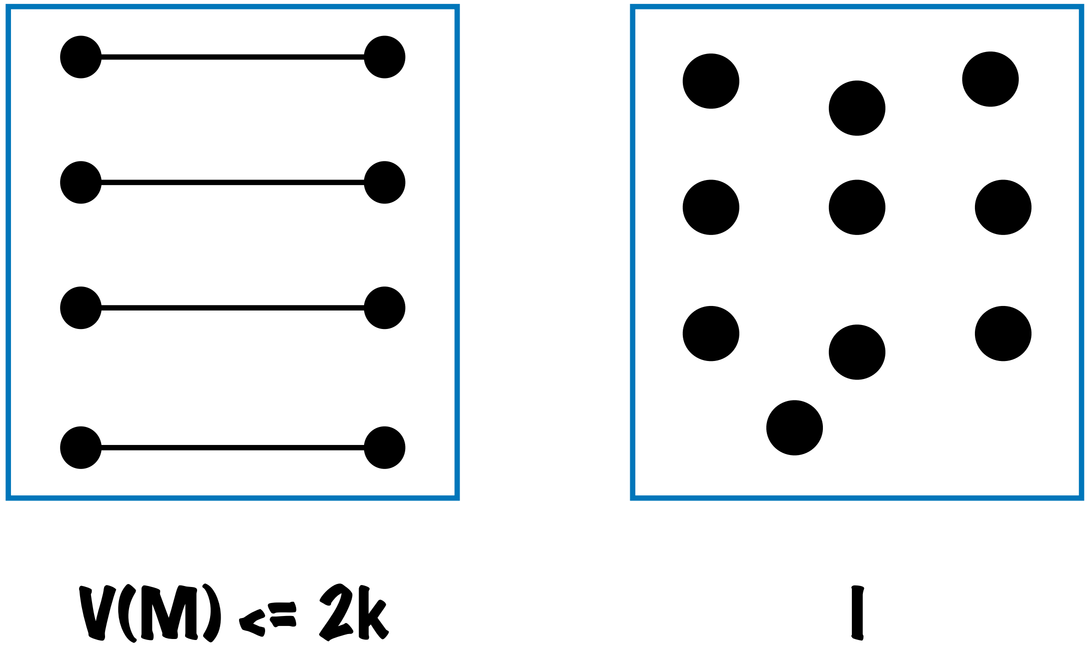
Crown Lemma: Let G be a graph without isolated vertices and with at least 3k + 1 vertices. Then, there is a polynomial time algorithm that either finds a matching of size k + 1 in G, or finds a crown decomposition of G.

* **Reduction Rule 1:** Delete isolated vertices
* **Reduction Rule 2:**
  * If Crown Lemma finds a mat of size k+1, then (G,k) is a no-instance
  * Otherwise, (C,H,R) is a crown
    * Add H into the solution, delete $H \cup C$, reduce k by |H|
* If Reduction Rules 1 & 2 can't be applied, then G has at most 3k vertices

Proof of Crown Lemma

* G is a graph without isolated vertices and with >= 3k+1 vertices
* Find a maximal matching M in G
* If |M| >= k+1, then (G,k) is a no-instance
* Otherwise,

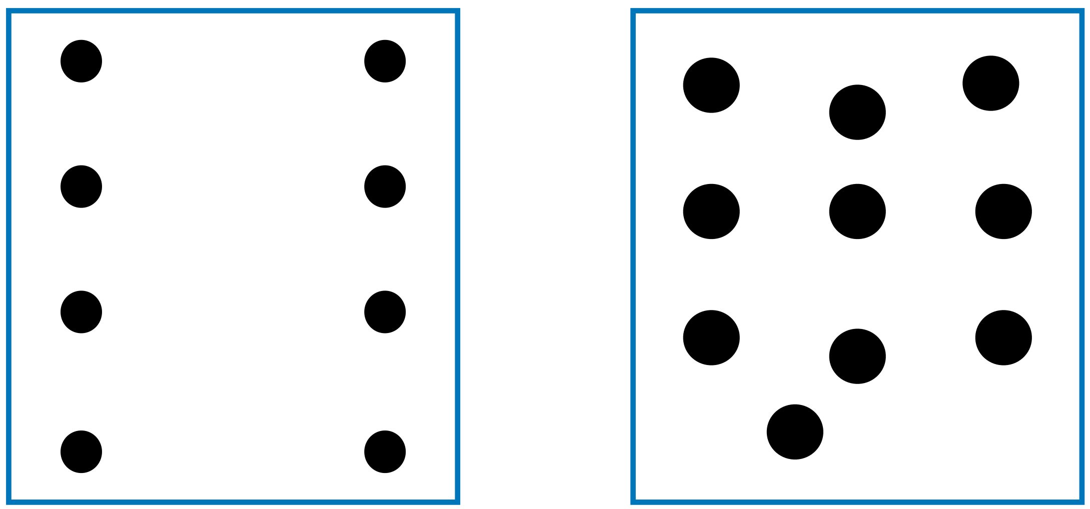




Proof of Crown Lemma

* Bipartite graph B

> **König's Theorem:** For a bipartite graph, |Max Mat| = |Min VC|



V(M) I

X: min vertex cover

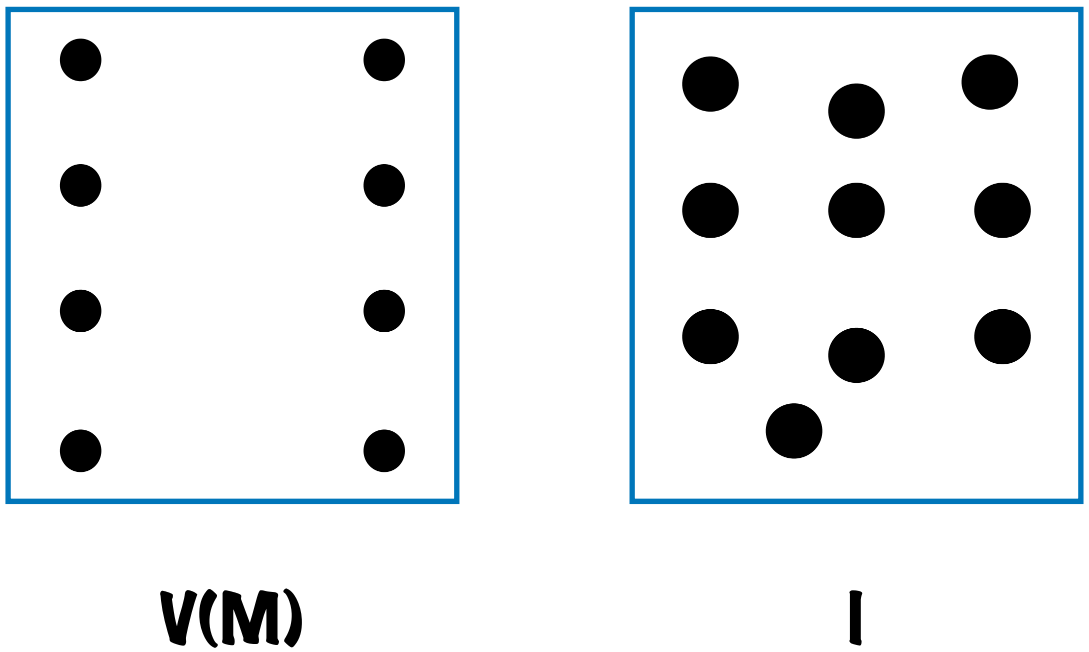
M': max matching

* Find a maximum matching M' and minimum vertex cover of B
* If |M'| >= k+1, then (G,k) is a no-instance

Proof of Crown Lemma

* Otherwise, |M'| <= k

König's Theorem: For a bipartite graph, |Max Mat| = |Min VC|



X: min vertex cover

M': max matching

| X | = | M' | <= k

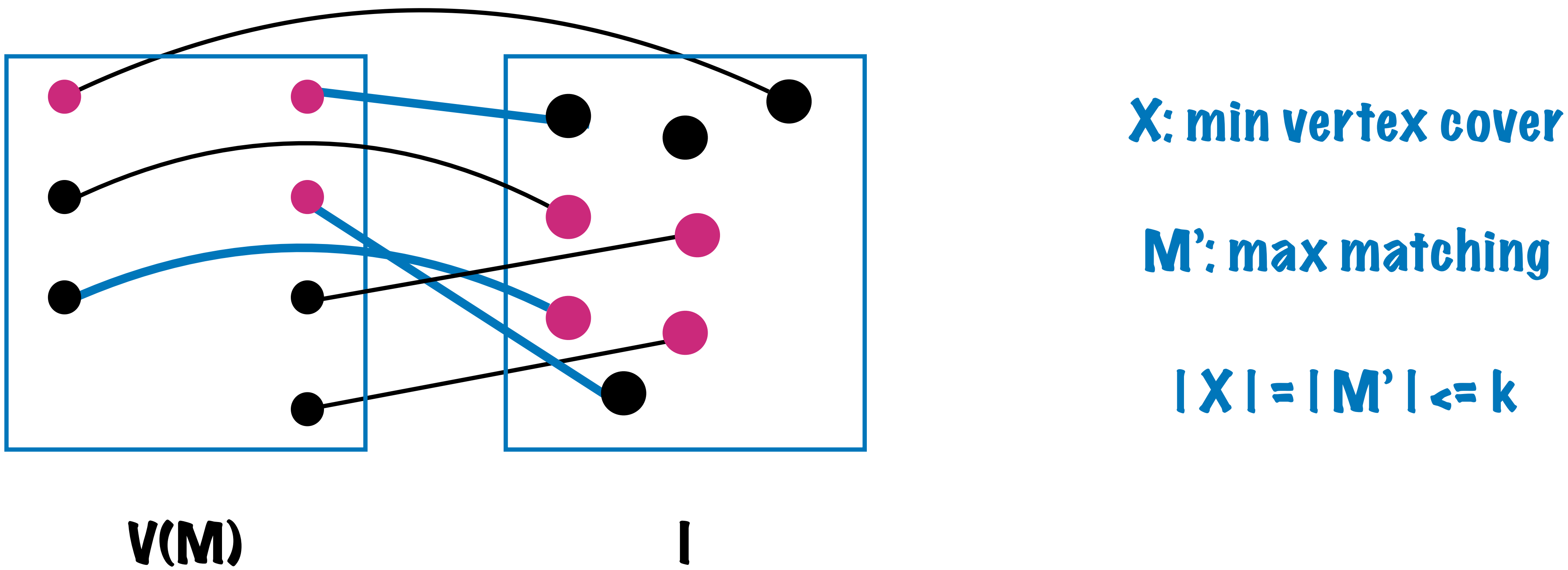
Claim: X has at least one vertex from V(M)

* If $X \subseteq I$, then X = I. Then, | V(M) | + | I | <= 2k + k. A contradiction!

X = I because, o/w since we have no isolated vertices, there exist a vertex v in I which is not in X but has it

Proof of Crown Lemma

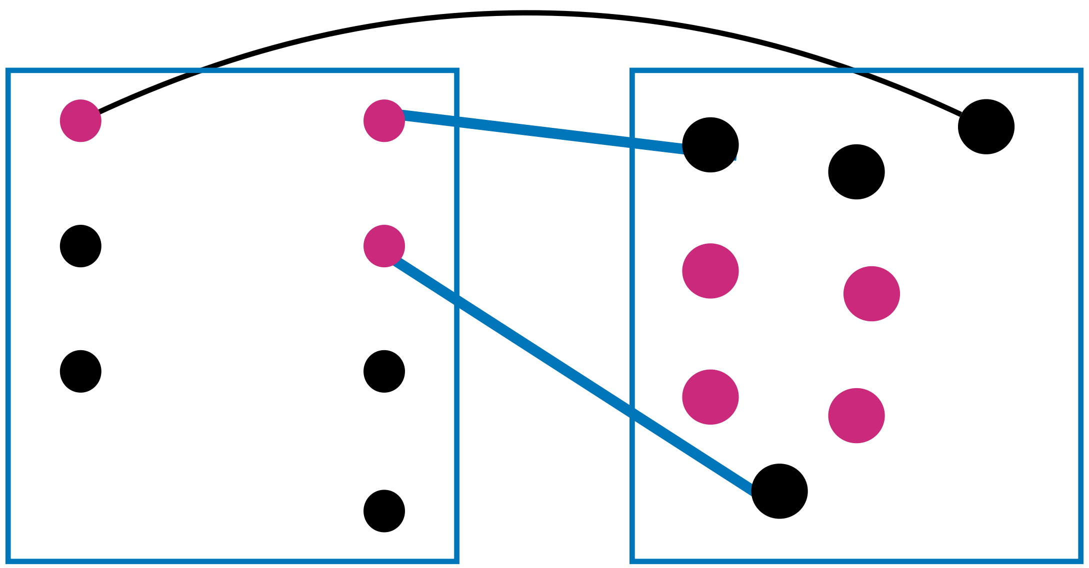
* X has at least one vertex from V(M)



* Every edge of M′ has exactly one endpoint in X
* M′′ ⊆ M′ such that each edge in M′′ has an endpoint in X ∩ V(M)

Proof of Crown Lemma

* M'' ⊆ M' such that each edge in M'' has an endpoint in X ∩ V(M)



V(M) I

X: min vertex cover

M': max matching

| X | = | M' | <= k

From our C, there won't be an edge toward V(m) of rest as o/w this edge is not having any

V(M'') ∩ I
C

V(M'') ∩ X
H

Rest