

CS 5003: Parameterized Algorithms

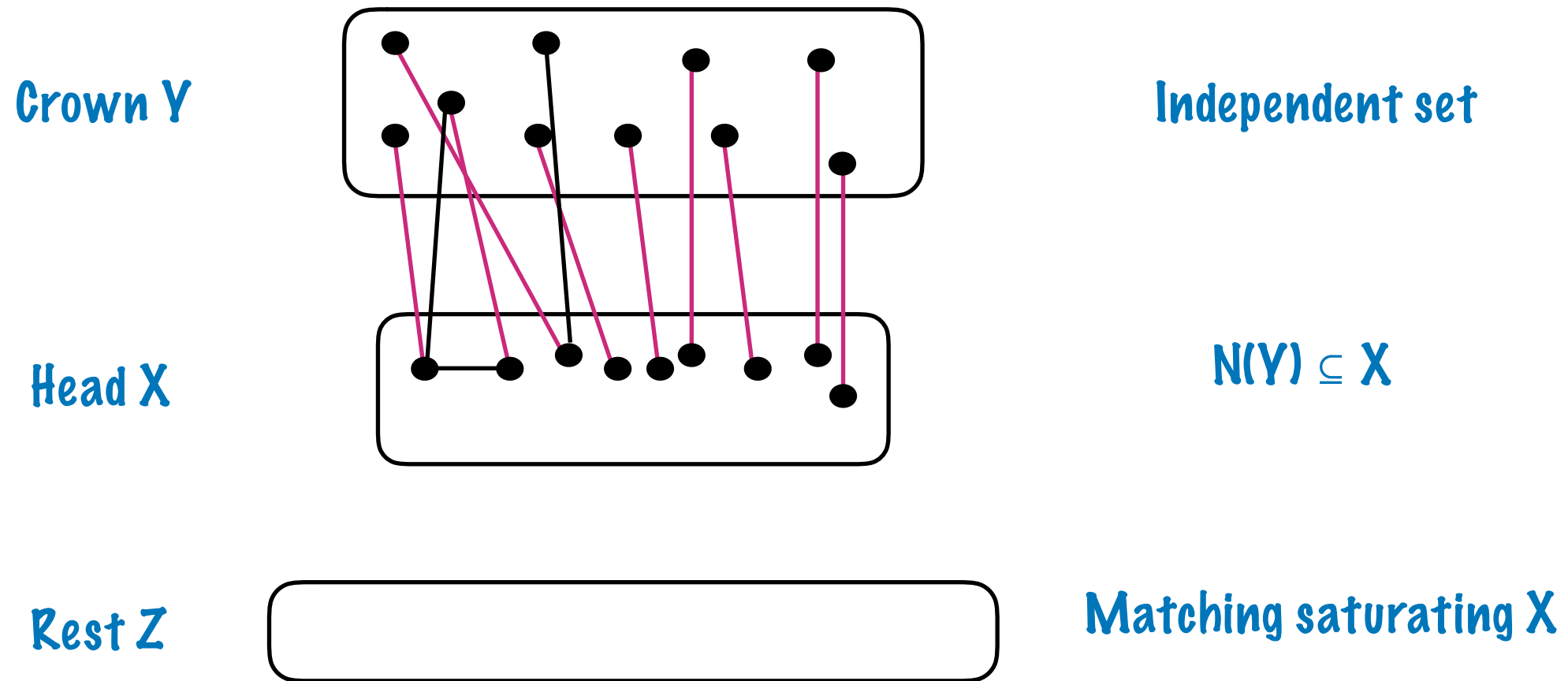
Lectures 28-31

Krithika Ramaswamy

IIT Palakkad

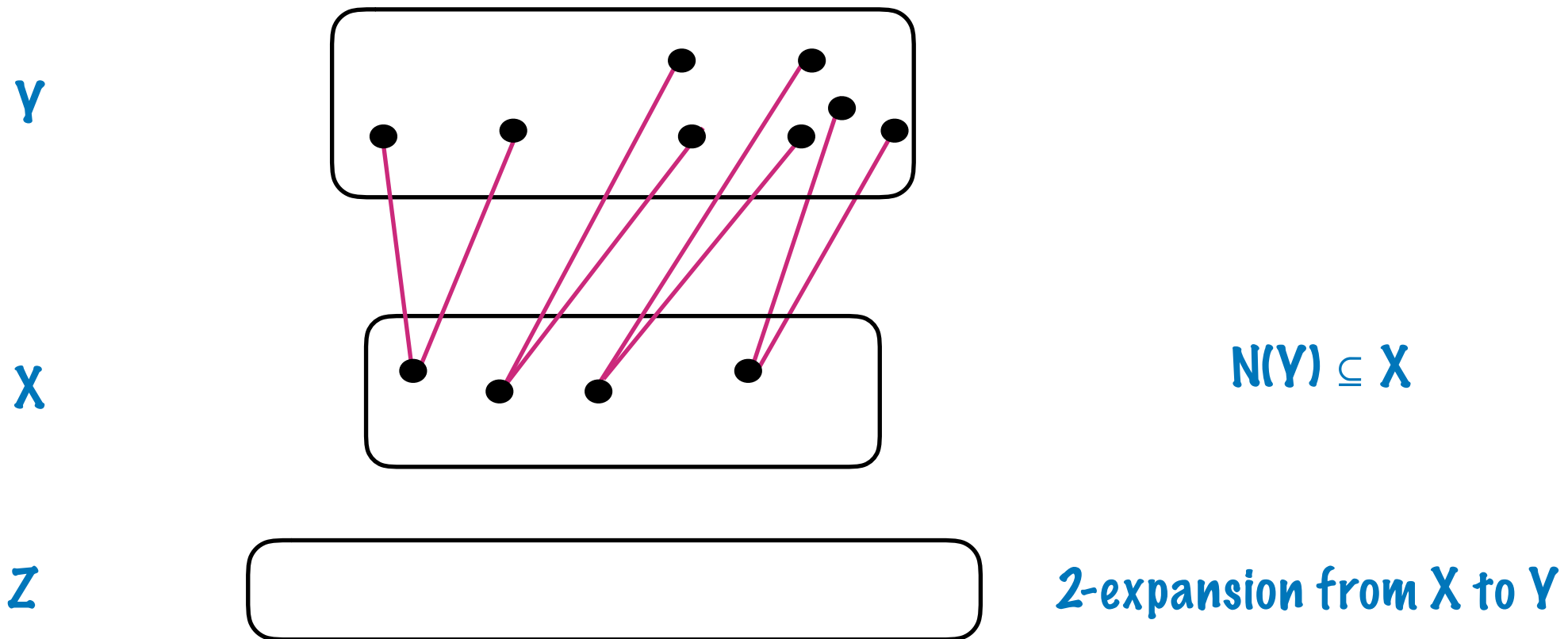
Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Crown Lemma

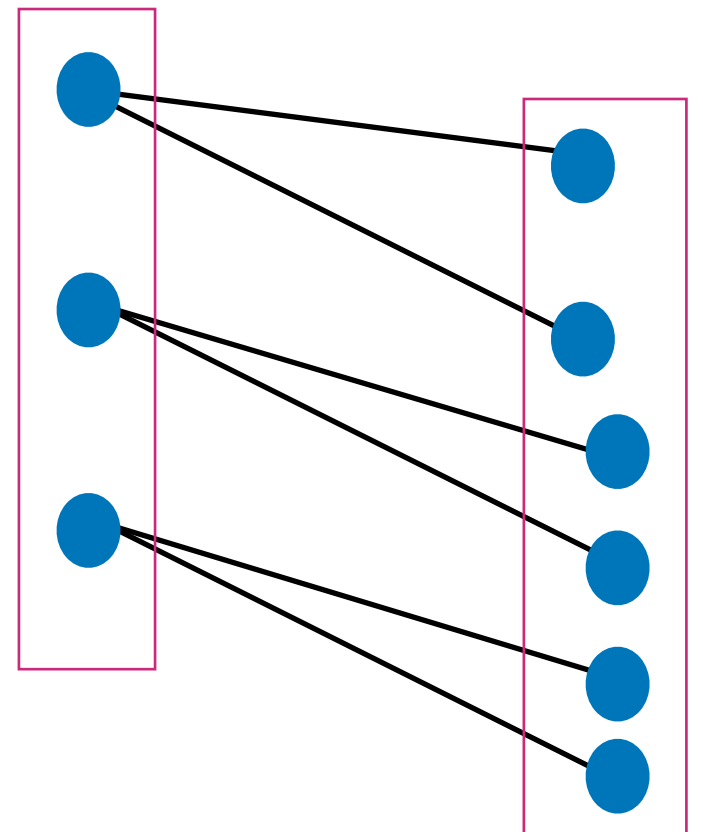


Crown Lemma: Let G be a graph without isolated vertices and with at least $3k + 1$ vertices. Then, there is a polynomial time algorithm that either finds a matching of size $k + 1$ in G , or finds a crown decomposition of G .

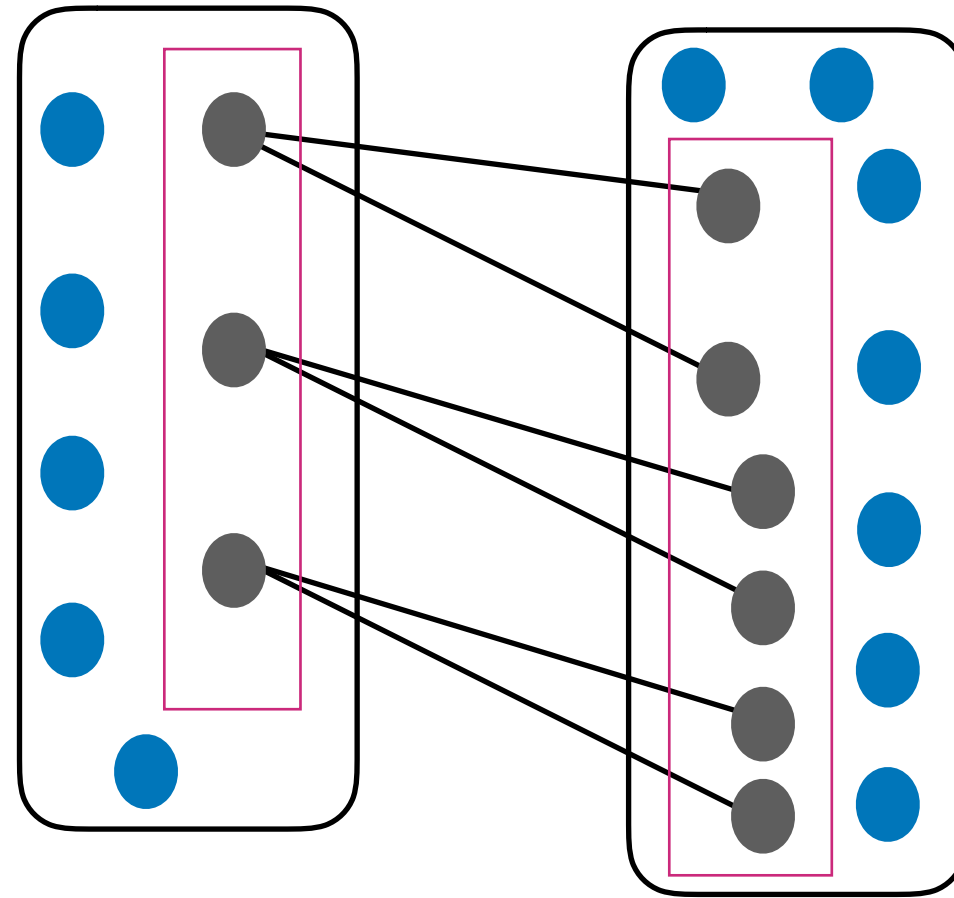
Towards a Generalization of Crown Lemma



- * **Definition:** In a bipartite graph with bipartition (A, B) , a set M of edges is called a **2-expansion** from A to B if
 - * Every vertex of A is incident with exactly 2 edges of M
 - * M saturates exactly $2|A|$ vertices in B



2-Expansion Lemma



2-Expansion Lemma: Let G be a bipartite graph with bipartition (A, B) s.t. $|B| > 2|A|$ and there are no isolated vertices in B . Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a 2-expansion into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m n^{1/2})$ time.

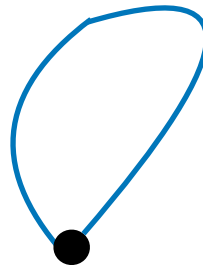
Feedback Vertex Set

Assume graph is a multigraph

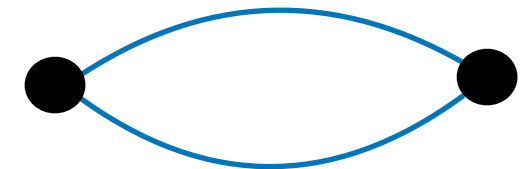
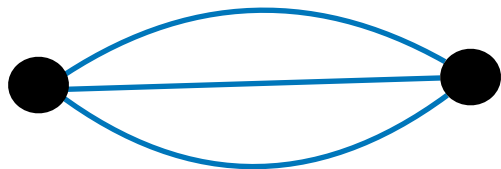
Reduction Rule 1: Delete isolated vertices

Reduction Rule 2: Delete degree-1 vertices

Reduction Rule 3: If there is a loop at a vertex v , delete v from the graph and reduce the parameter by 1



Reduction Rule 4: If there is an edge with multiplicity > 2 , reduce it to 2



Reduction Rule 5: Short circuit degree-2 vertices



Feedback Vertex Set - Towards a Quadratic Kernel

Claim: If $\min \deg \geq 3$ and $\max \deg \leq d$ and G has an FVS $\leq k$ then $n < (d+1)k$ and $m < 2dk$.

Suppose X is FVS $\leq k$ and let $Y = G - X$

$$\begin{aligned} 3|Y| \leq \sum_{v \in Y} \deg(v) &= \sum_{v \in Y} |N(v) \cap X| + \sum_{v \in Y} |N(v) \cap Y| \leq E(X, Y) + 2(|Y| - 1) \\ &< E(X, Y) + 2|Y| \\ &\leq d|X| + 2|Y| \end{aligned}$$

$$\therefore |Y| < d|X|$$

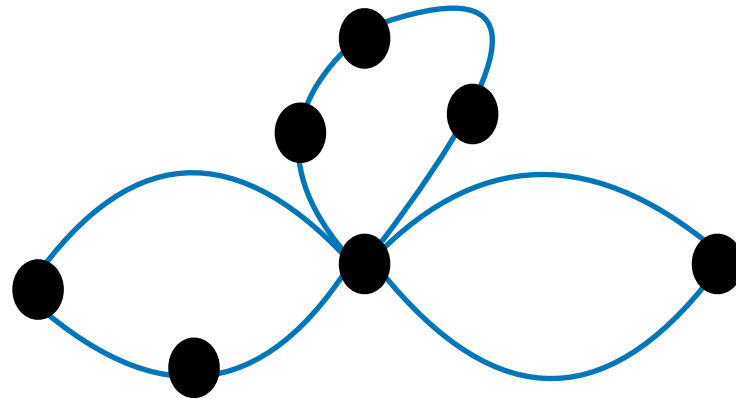
$$n = |X| + |Y| < (d+1)|X| \leq (d+1)k$$

$$m < d|X| + |Y| \leq 2d|X| \leq 2dk$$

Goal: Reduce max degree to $\leq 10k$ to get quadratic kernel

Feedback Vertex Set - Towards a Quadratic Kernel

Definition: A **v-flower** with r petals is a set of r cycles that pairwise intersect only at v



Flower Lemma: There is a polynomial time algorithm that given a graph G and a vertex v without a self-loop, satisfies one of the following:

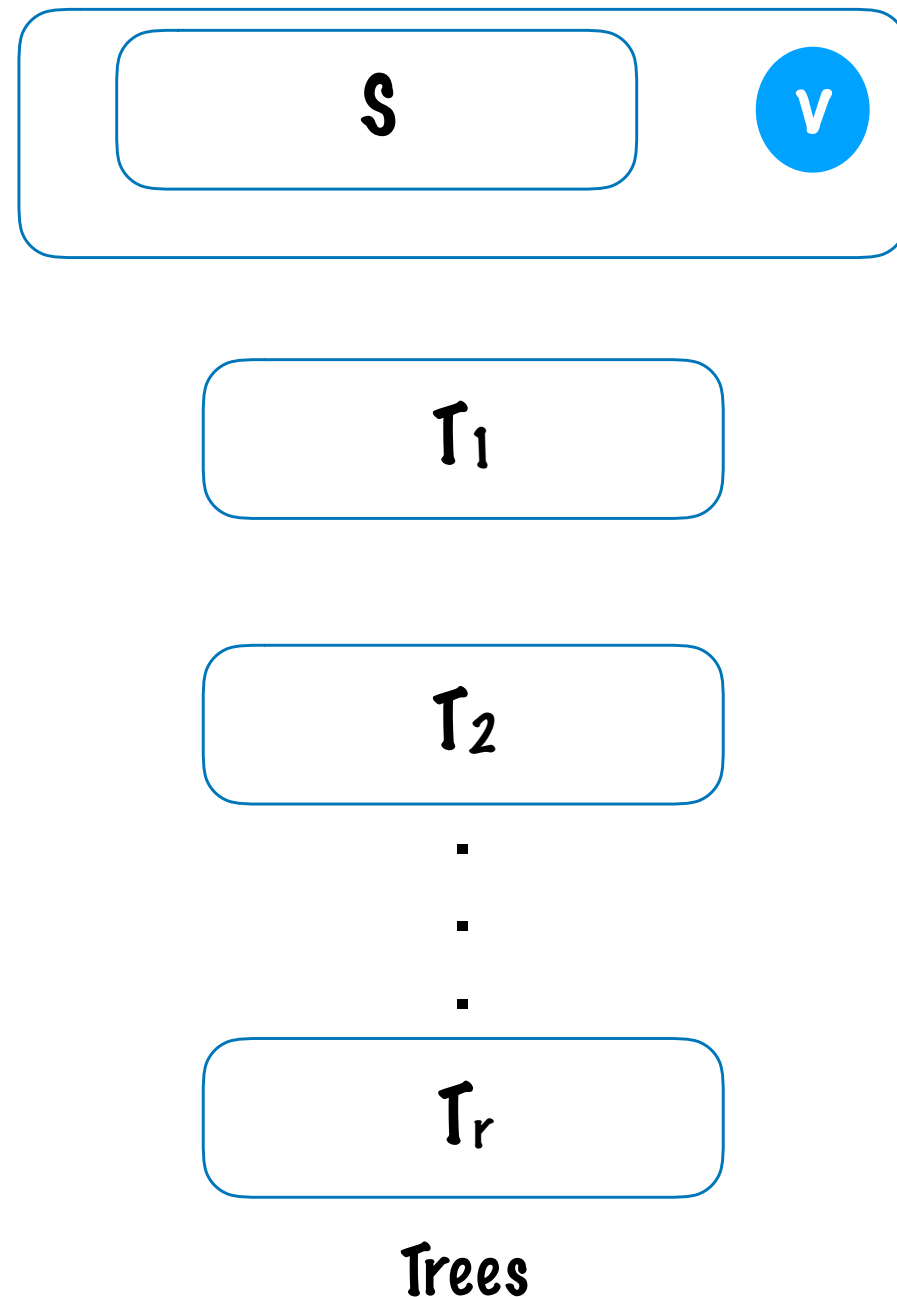
- * Declares (G, k) is a no-instance of Feedback Vertex Set
- * Returns a v -flower with $(k+1)$ petals
- * Finds FVS not containing v of size $\leq 3k$

Note: If algorithm selects option

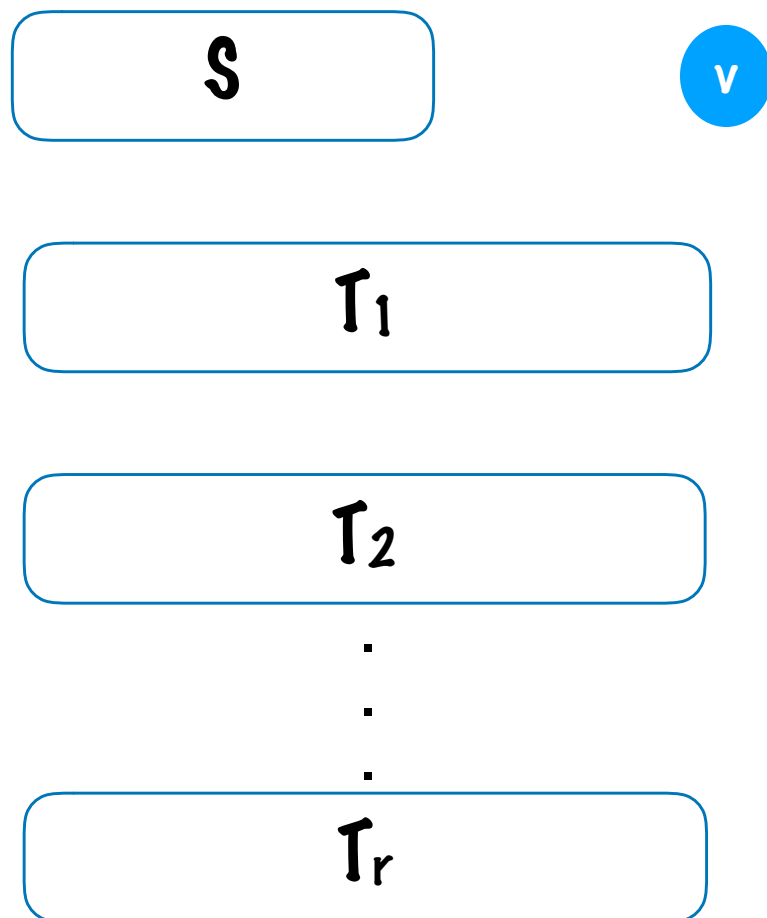
Reduction Rule 6: If there is a v -flower with $k+1$ petals, then delete v and reduce the parameter by 1

Feedback Vertex Set - Towards a Quadratic Kernel

- * Suppose v has degree $> 10k$ and Flower Lemma returns FVS S not containing v of size $\leq 3k$



Feedback Vertex Set - Towards a Quadratic Kernel

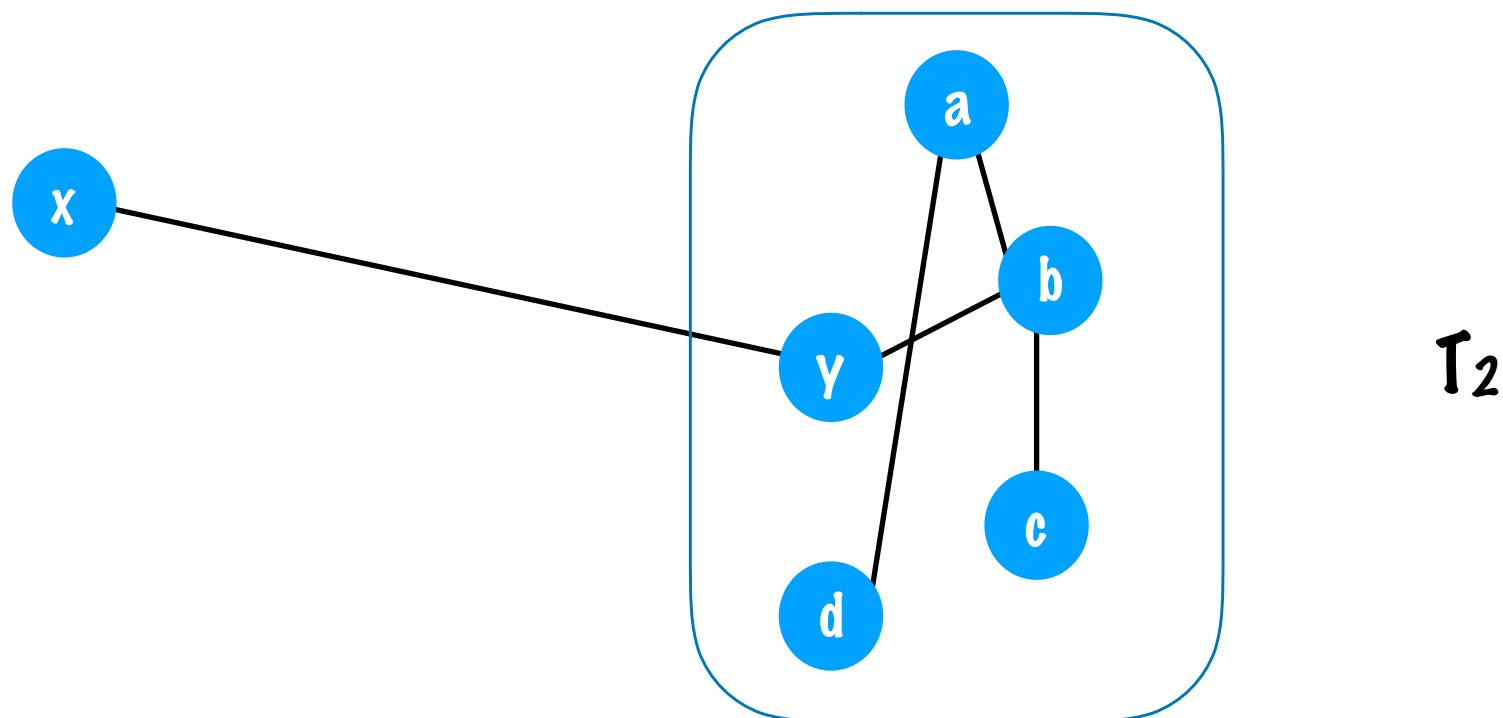
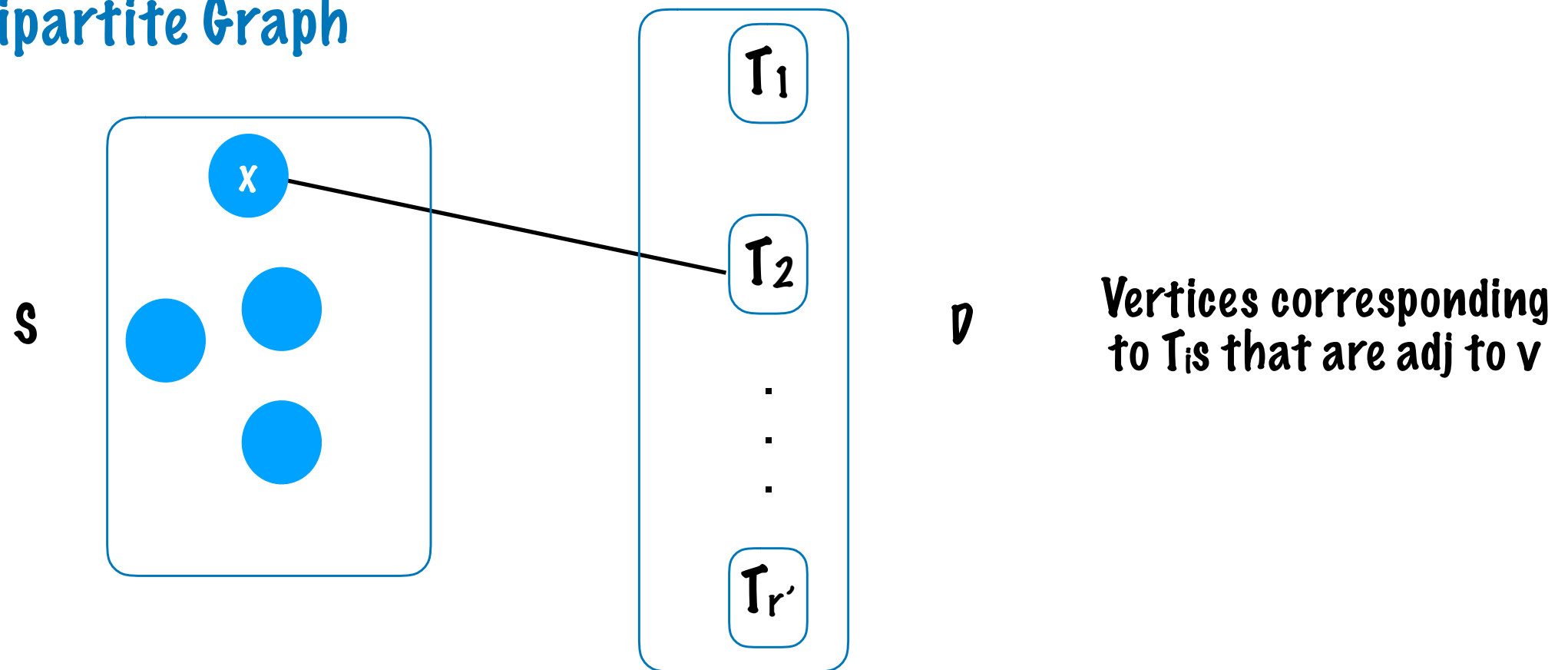


- * v has $\deg > 10k$
- * If there are $> 2k$ double edges incident on it, apply Reduction Rule 6
- * Otherwise, there are $\leq 2k$ double edges incident on v
- * There are $\leq 4k$ edges between v and S
 - * $\leq 2k + (3k - k) = 3k + k \leq 4k$ Note: $2k + (3k - k) =$
- * There are $\leq r$ edges between v and T_i

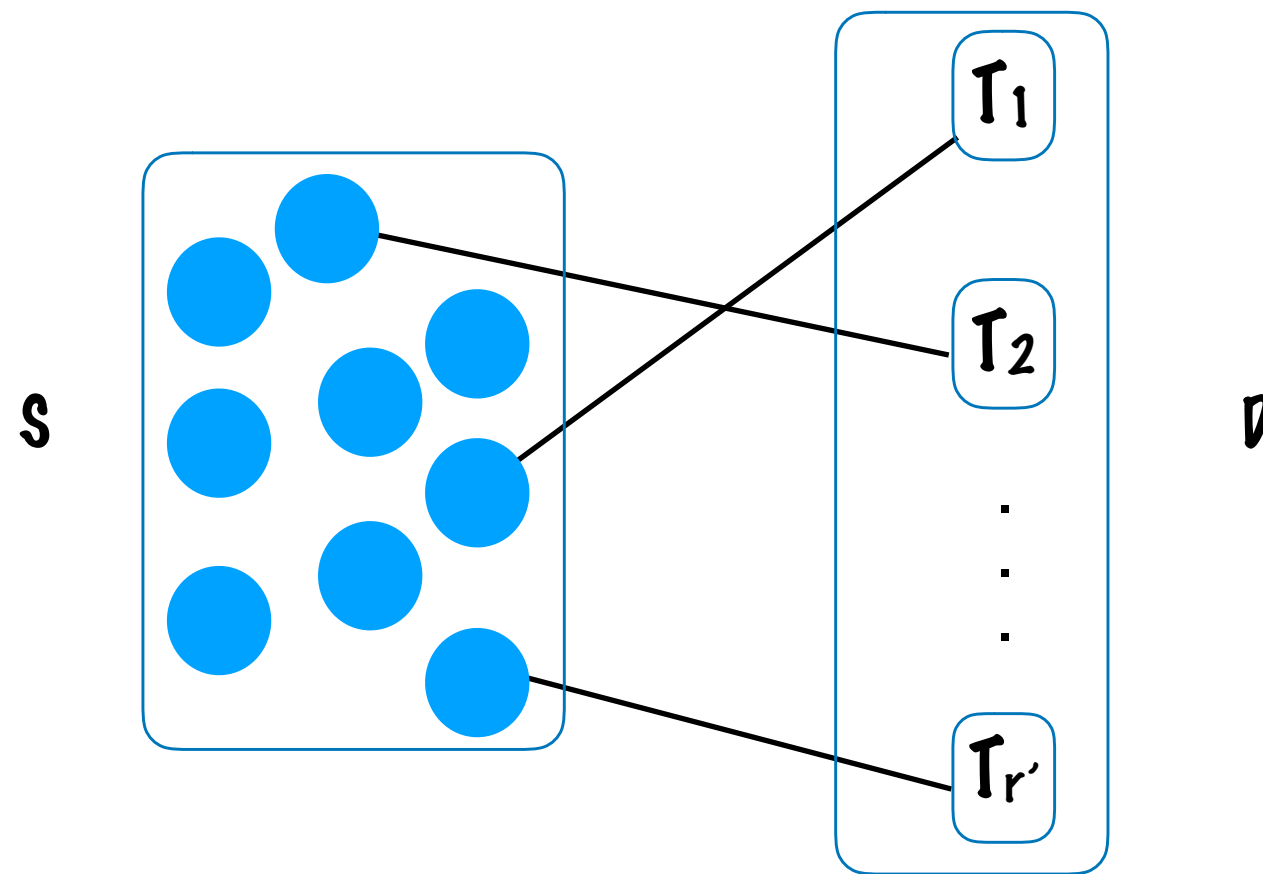
\therefore there are $> 6k$ edges between v and T_i s. In particular, $r > 6k$

Feedback Vertex Set - Towards a Quadratic Kernel

Auxiliary Bipartite Graph



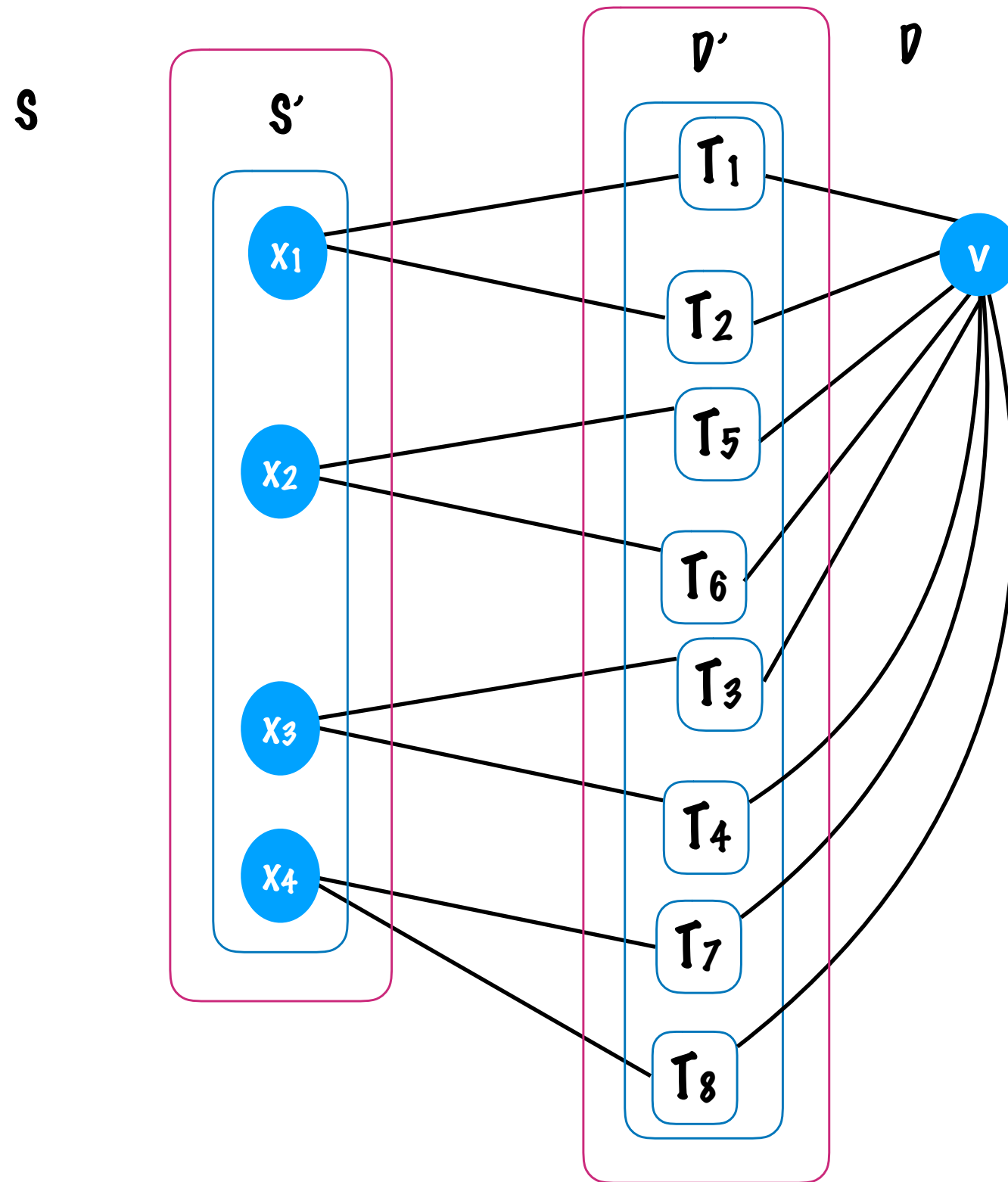
Feedback Vertex Set - Towards a Quadratic Kernel



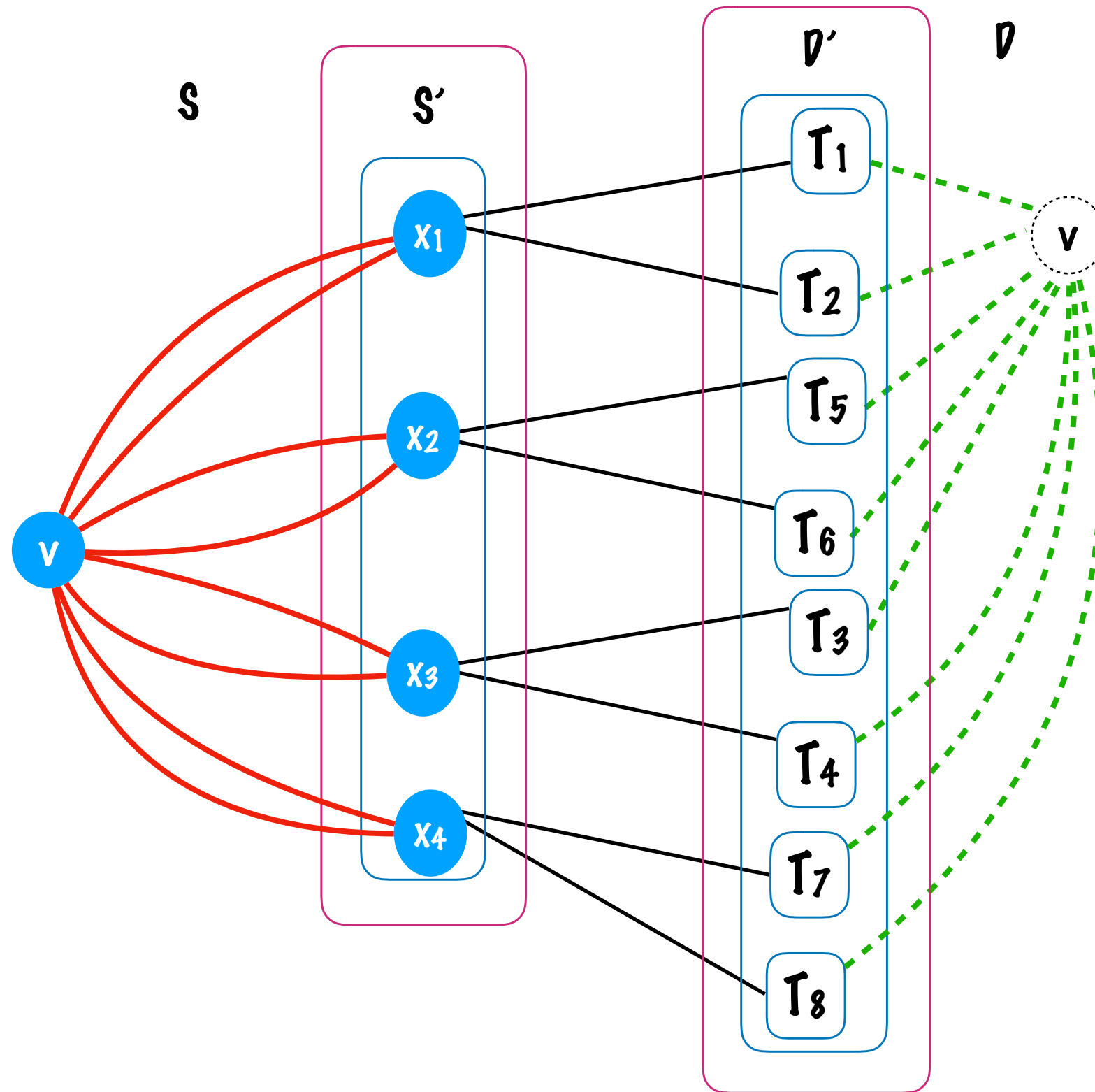
x_i is just any leaf to T_i

- * No vertex in \mathcal{D} is isolated as each T_i has a $\deg \leq 1$ vertex x_i which has $\deg \geq 3$ in G .
Even if x_i is adj to v , it has ≥ 1 nbr in S
- * $|\mathcal{D}| > 6k$ and $|S| \leq 3k$ i.e., $|\mathcal{D}| > 2|S|$
 - * There are non-empty sets $\mathcal{D}' \subseteq \mathcal{D}$ and $S' \subseteq S$ s.t
 - * S' has a 2-expansion into \mathcal{D}' and no vertex in \mathcal{D}' has a neighbour outside S'

Feedback Vertex Set - Towards a Quadratic Kernel

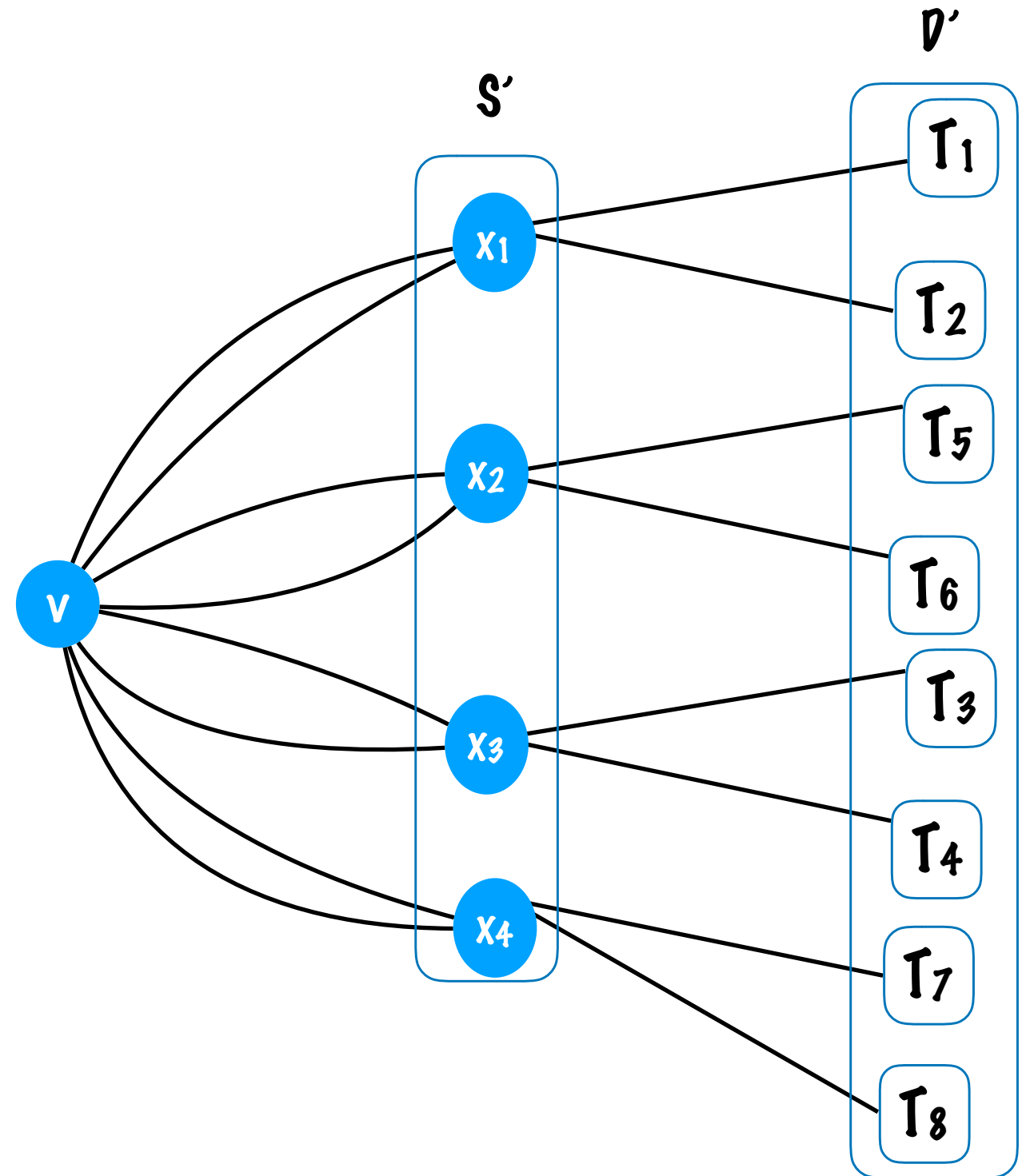
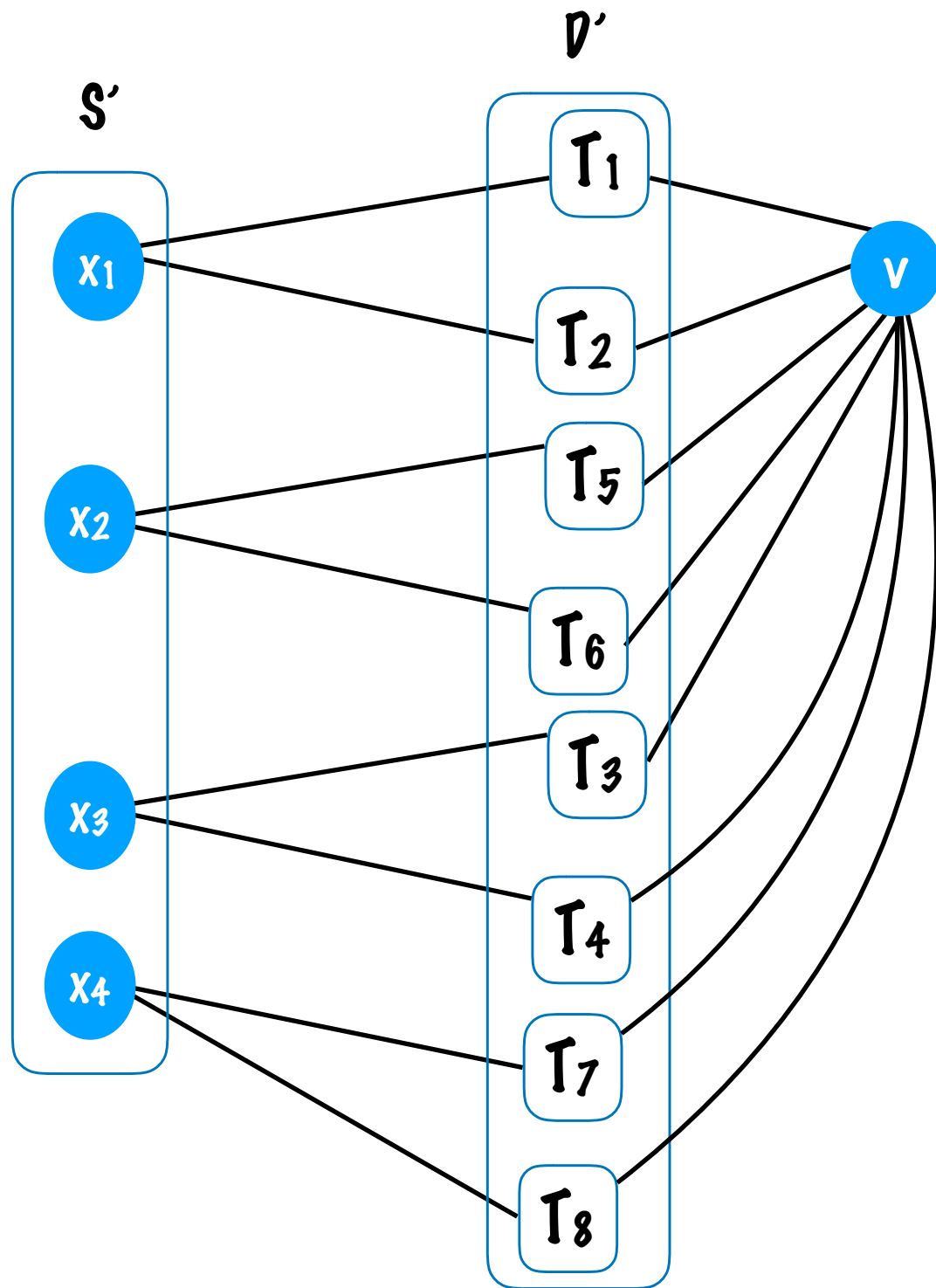


Feedback Vertex Set - Towards a Quadratic Kernel

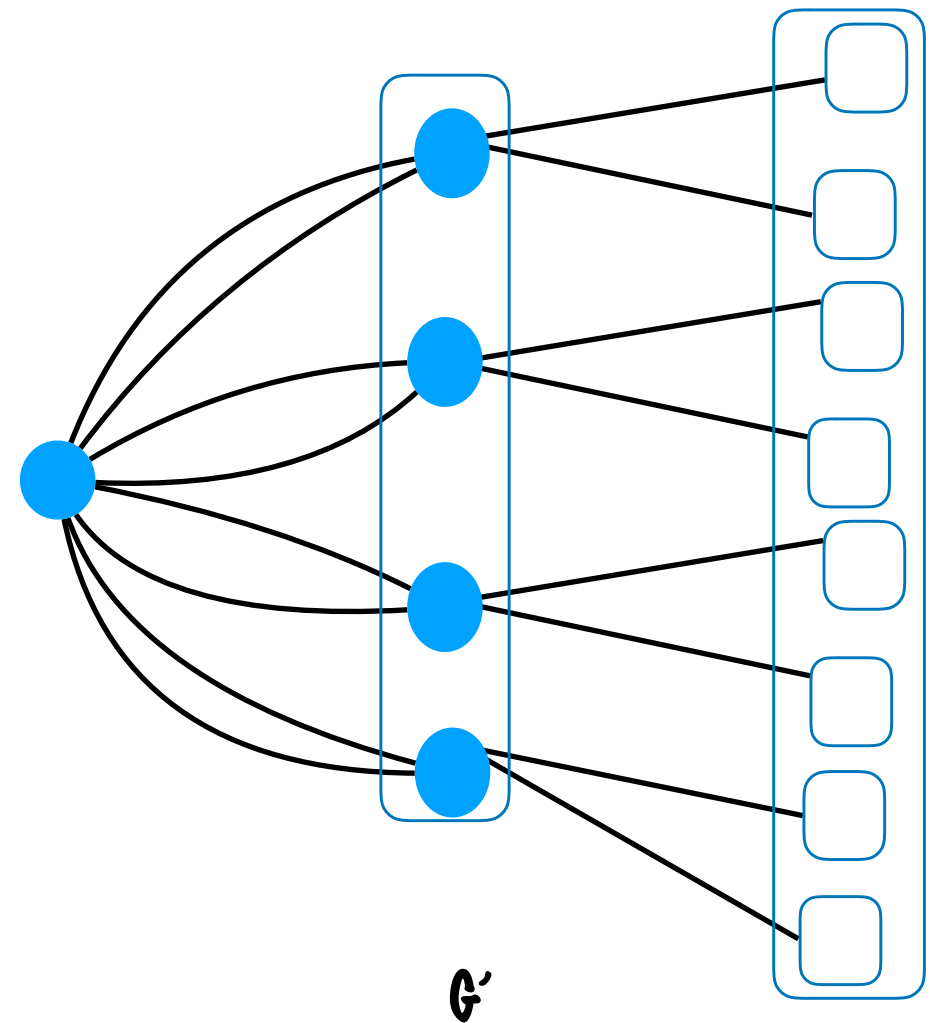
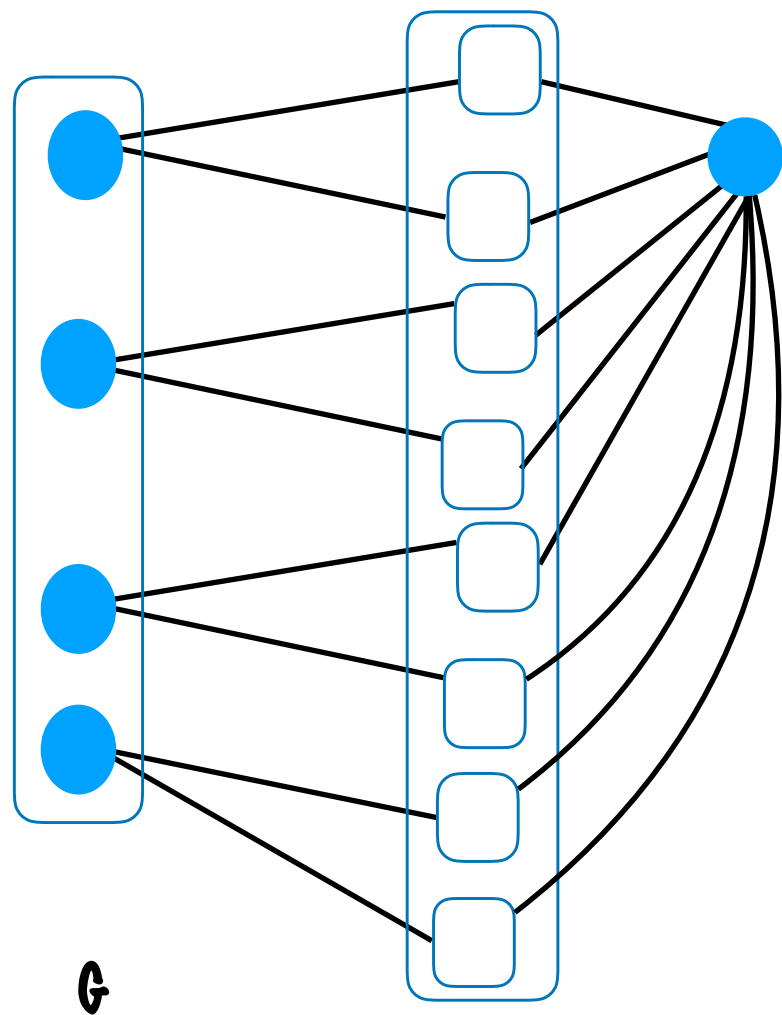


Reduction Rule 7: Add double edges between v and every vertex in S' . Delete edges between D' to v .

Feedback Vertex Set - Towards a Quadratic Kernel



Feedback Vertex Set - Towards a Quadratic Kernel



Suppose G' has FVS W of $\leq k$. Then, v is in W or S' is in W

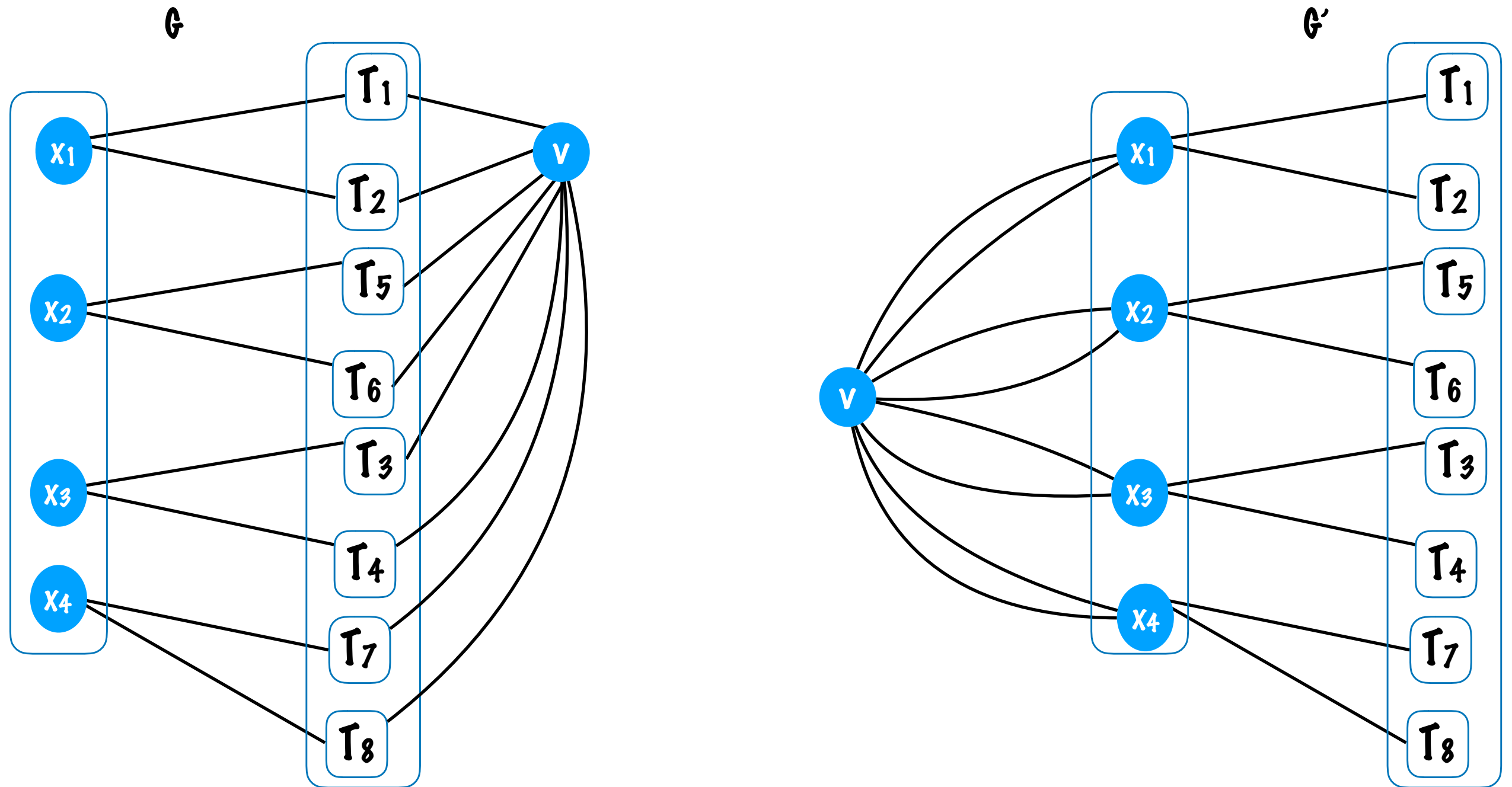
- * Case: v is in W

- * $G - v = G' - v$ and W is FVS of G too

- * Case: S' is in W

- * Any cycle in G passing through T_i in \mathcal{D}' also passes through a vertex in S'

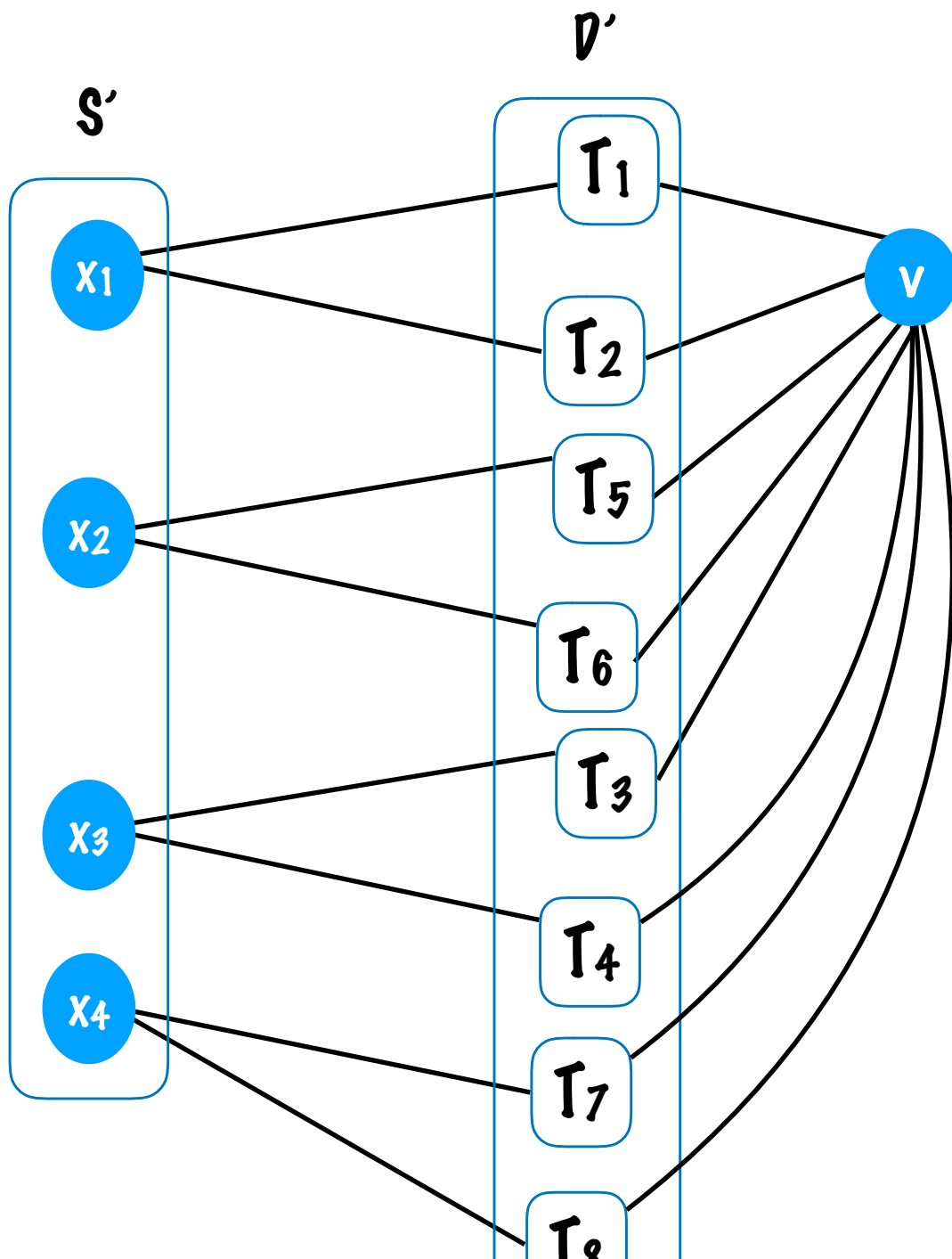
Feedback Vertex Set - Towards a Quadratic Kernel



Suppose G has FVS W of $\leq k$. Then, G has $\leq k$ FVS that either has v or S'

Feedback Vertex Set - Towards a Quadratic Kernel

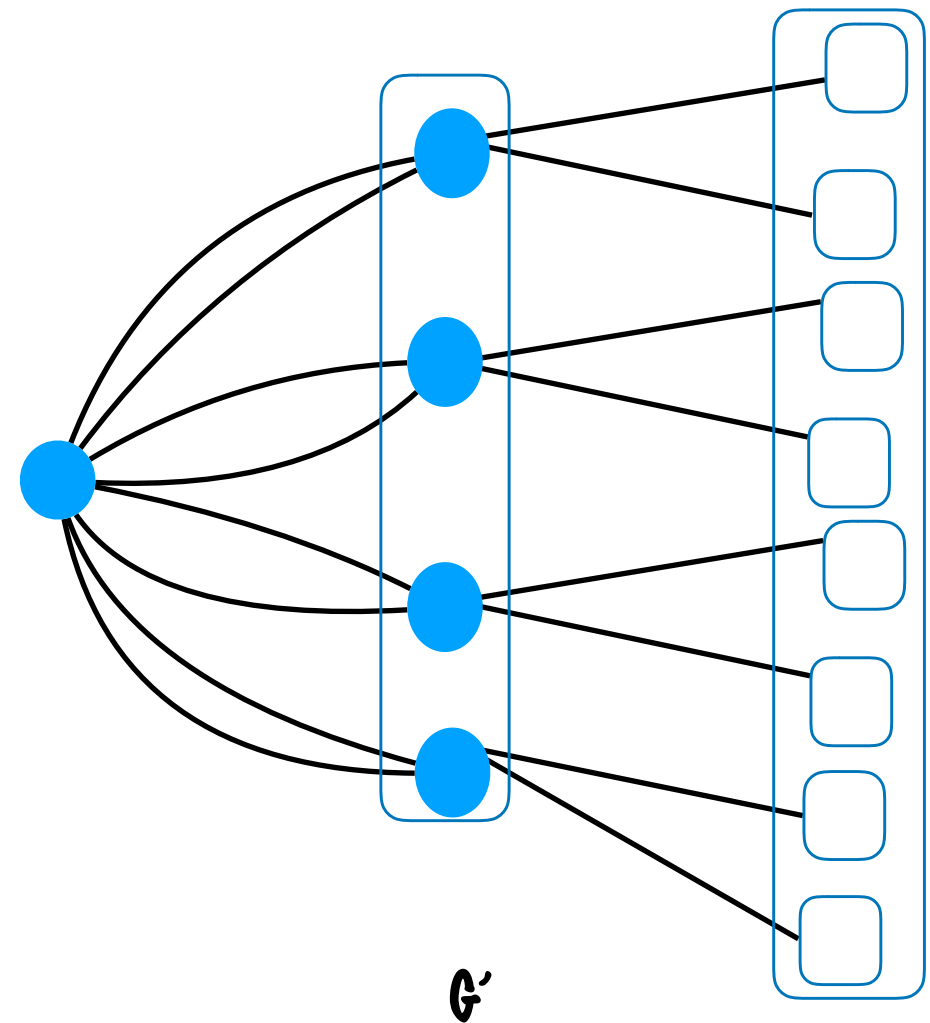
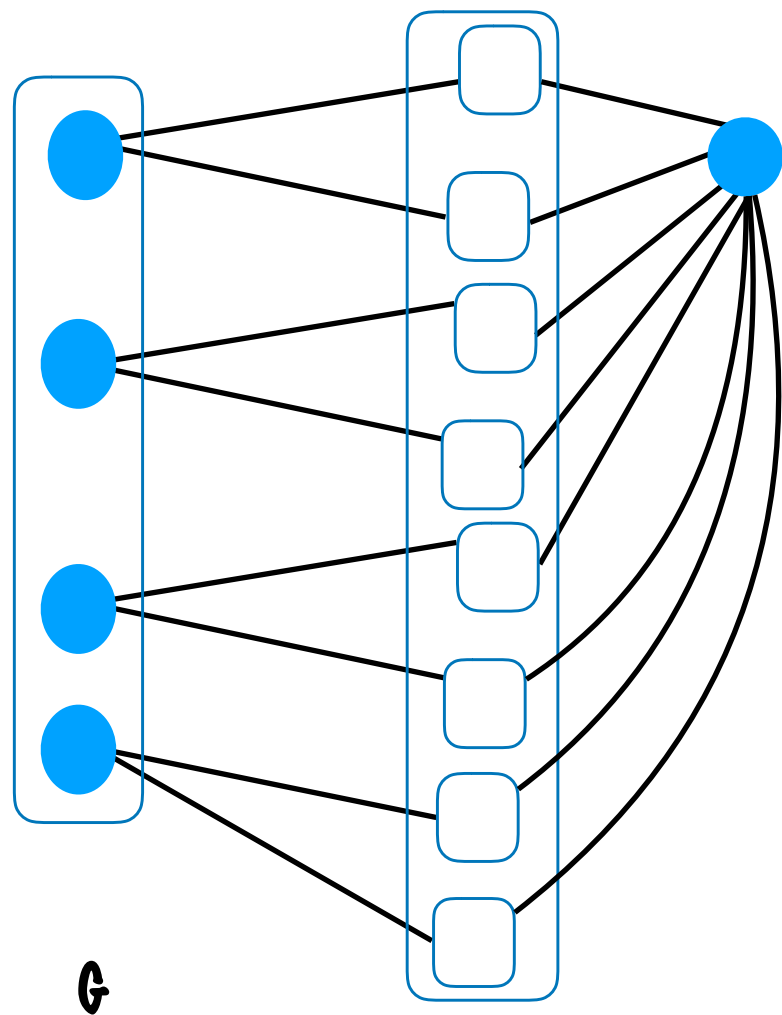
Claim: If G has $\leq k$ FVS, then G has $\leq k$ FVS that either has v or S'



neighbourhood of D' is in S' and therefore vertex in T_i cannot be

- * Any FVS Z not containing v has at least $|S'|$ vertices
- * If Z does not contain some x_i from S' , then Z contains a vertex q_i from $V(T_{i1}) \cup V(T_{i2})$
- * For example,
 - * Suppose x_1 and x_3 are not in Z
 - * Then, q_1 (from $V(T_1) \cup V(T_2)$) and q_3 ($V(T_3) \cup V(T_4)$) are in Z
 - * Replace q_1 and q_3 by x_1 and x_3 to get Z'
 - * If $G - Z'$ is not a forest, then there is a cycle C containing say q_1
 - * C must have a vertex from S' ($\Rightarrow \Leftarrow$)

Feedback Vertex Set - Towards a Quadratic Kernel



Suppose G has FVS W of $\leq k$. Then, v is in W or S' is in W

- * Case: v is in W

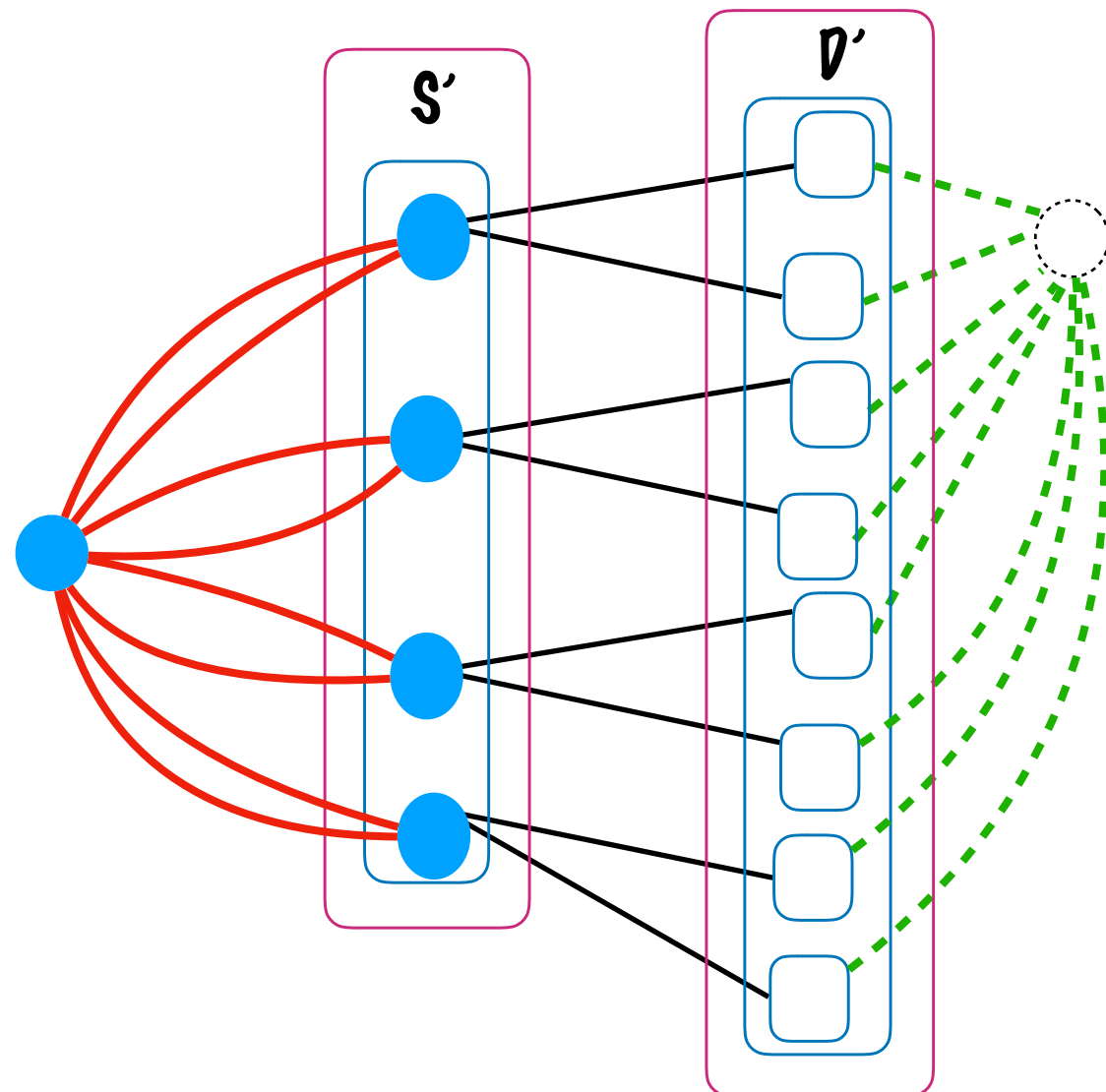
- * $G - v = G' - v$ and W is FVS of G' too

- * Case: S' is in W

- * Any cycle in $G' - W$ implies a double edge between v and a vertex in $T_i (\Rightarrow \Leftarrow)$

Feedback Vertex Set - Towards a Quadratic Kernel

Reduction Rule 7: Add double edges between v and every vertex in S' . Delete edges between D' to v .



- * Reduction Rule 7 can be applied $\leq m$ times
- * No. of single edges incident on v dec
- * In polynomial time, either we will find a v -flower with $k+1$ petals, or the deg of v becomes $\leq 10k$ or we will determine that (G, k) is a no-instance

Feedback Vertex Set - Quadratic Kernel

- * **Reduction Rule 1:** Delete isolated vertices
- * **Reduction Rule 2:** Delete degree-1 vertices
- * **Reduction Rule 3:** If there is a loop at a vertex v , delete v and reduce param by 1
- * **Reduction Rule 4:** If there is an edge with multiplicity > 2 , reduce it to 2
- * **Reduction Rule 5:** Short-circuit degree 2 vertices
- * **Reduction Rule 6:** If v is a vertex of degree $> 10k$ and there is a v -flower with $k+1$ petals, delete v and reduce param by 1
- * If v has degree $> 10k$ and has $> 2k$ double edges, then apply **Reduction Rule 6**
- * If v has degree $> 10k$ and Flower Lemma returns a FVS S of $\leq 3k$ vertices s.t $v \notin S$
 - * Use 2-expansion lemma to find $S' \subseteq S$ s.t if G has $\leq k$ FVS, then G has $\leq k$ FVS that either has v or S'
 - * **Reduction Rule 7:** Add double edges between v and every vertex in S' and delete edges from D' to v where D' is the set of vertices in $G - (S \cup \{v\})$ saturated by the 2-expansion
- * When none of the reductions rules are applicable, every vertex has degree $\leq 10k$
 - * $n = O(k^2)$ and $m = O(k^2)$

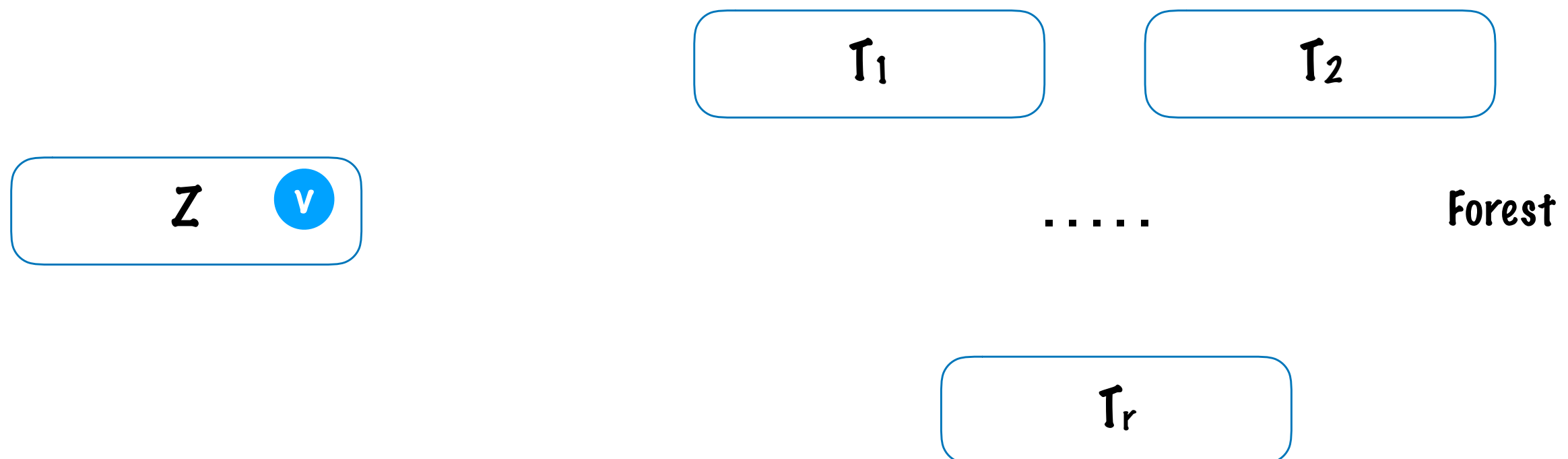
Flower Lemma

Flower Lemma: There is a polynomial time algorithm that given a graph G and a vertex v without a self-loop, satisfies one of the following:

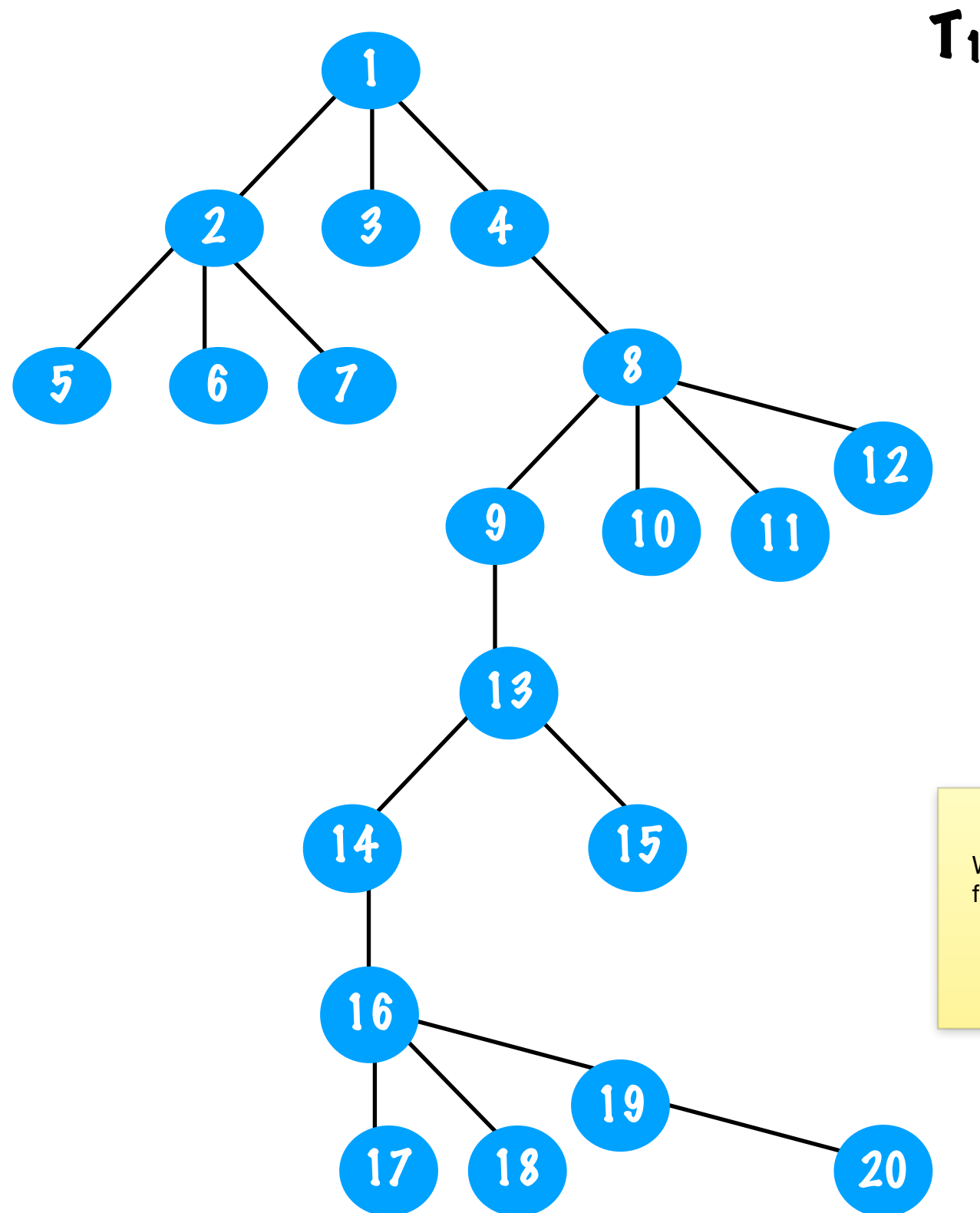
- * Declares (G,k) is a no-instance of Feedback Vertex Set
- * Returns a v -flower with $(k+1)$ petals
- * Finds FVS not containing v of size $\leq 3k$

Flower Lemma

- * Let Z be a 2-approximation FVS of G
- * If $|Z| > 2k$, then declare that (G, k) is a no-instance of Feedback Vertex Set
- * Otherwise, $|Z| \leq 2k$
- * If v is not in Z , then Z is the required FVS
- * Otherwise, v is in Z

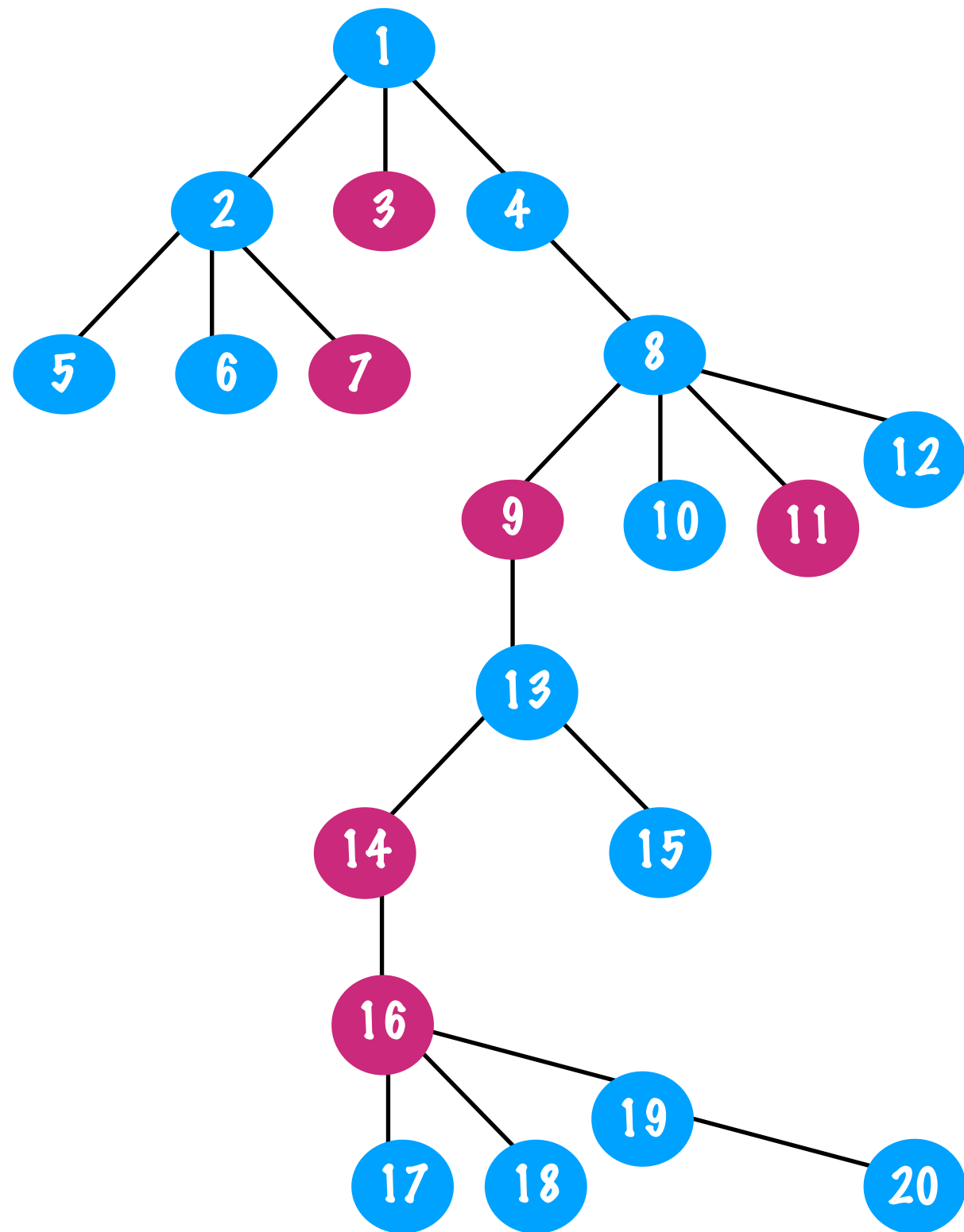


Flower Lemma



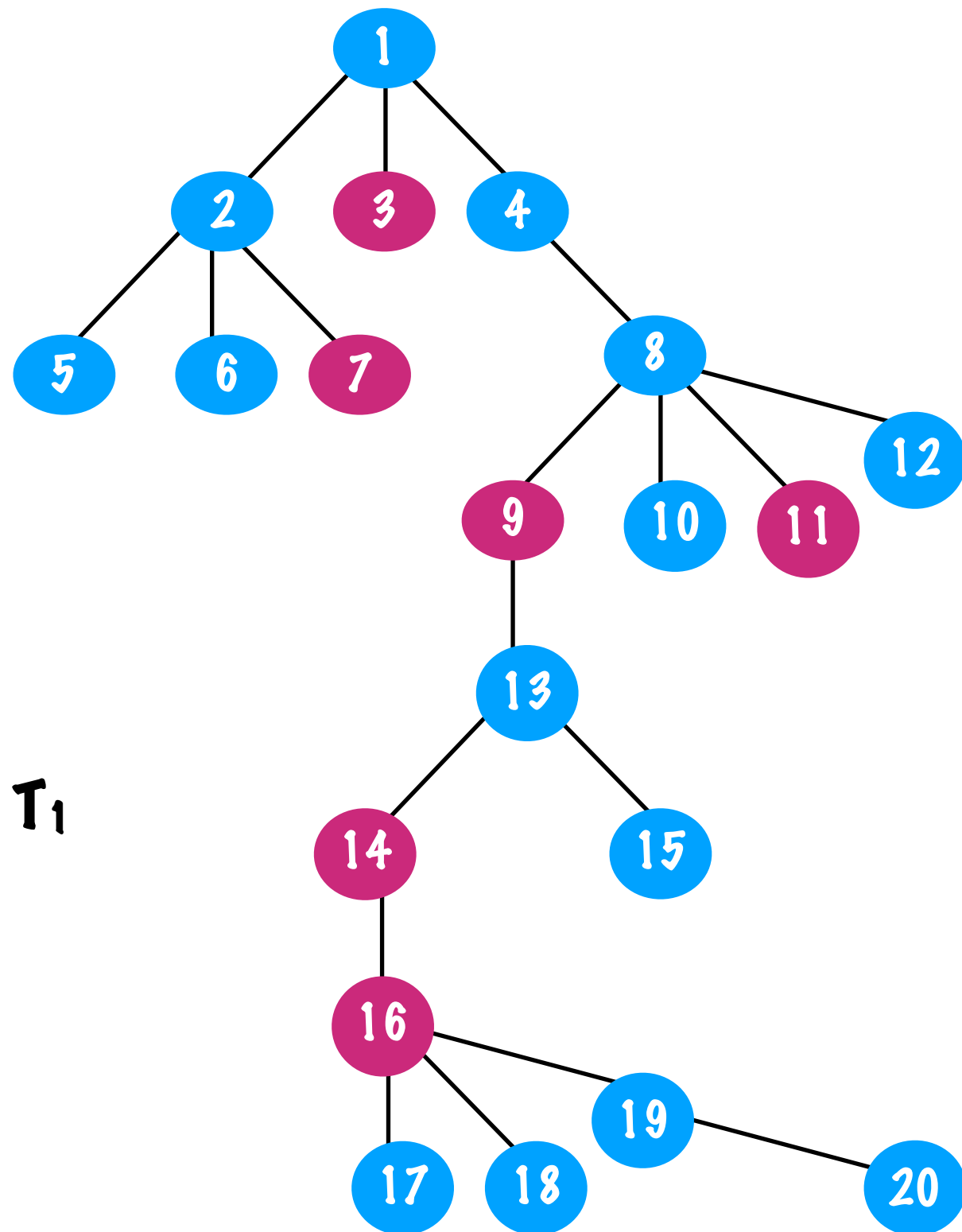
Why can't a v-flower have vertices from Z ?

Flower Lemma



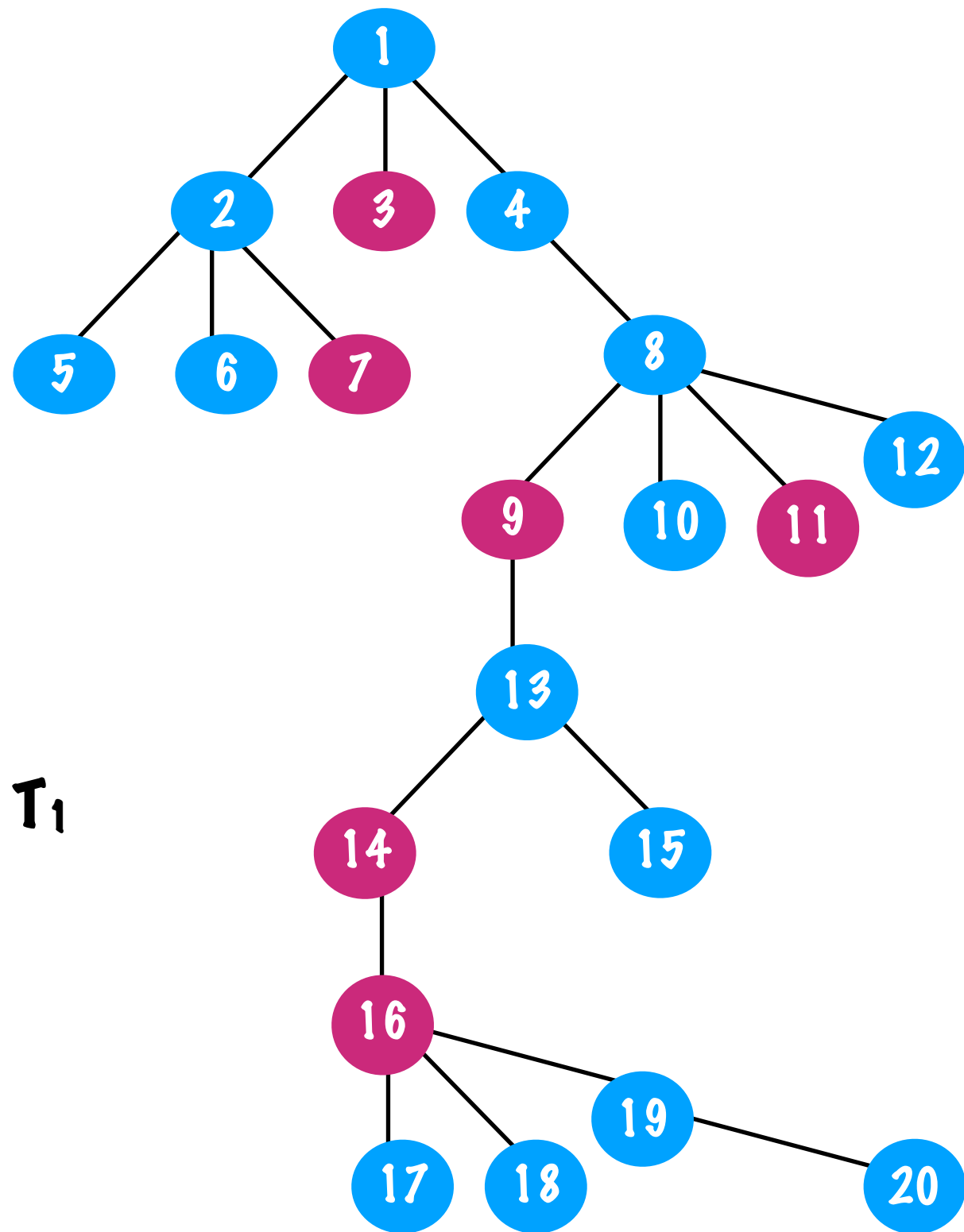
 Neighbours of v

Flower Lemma



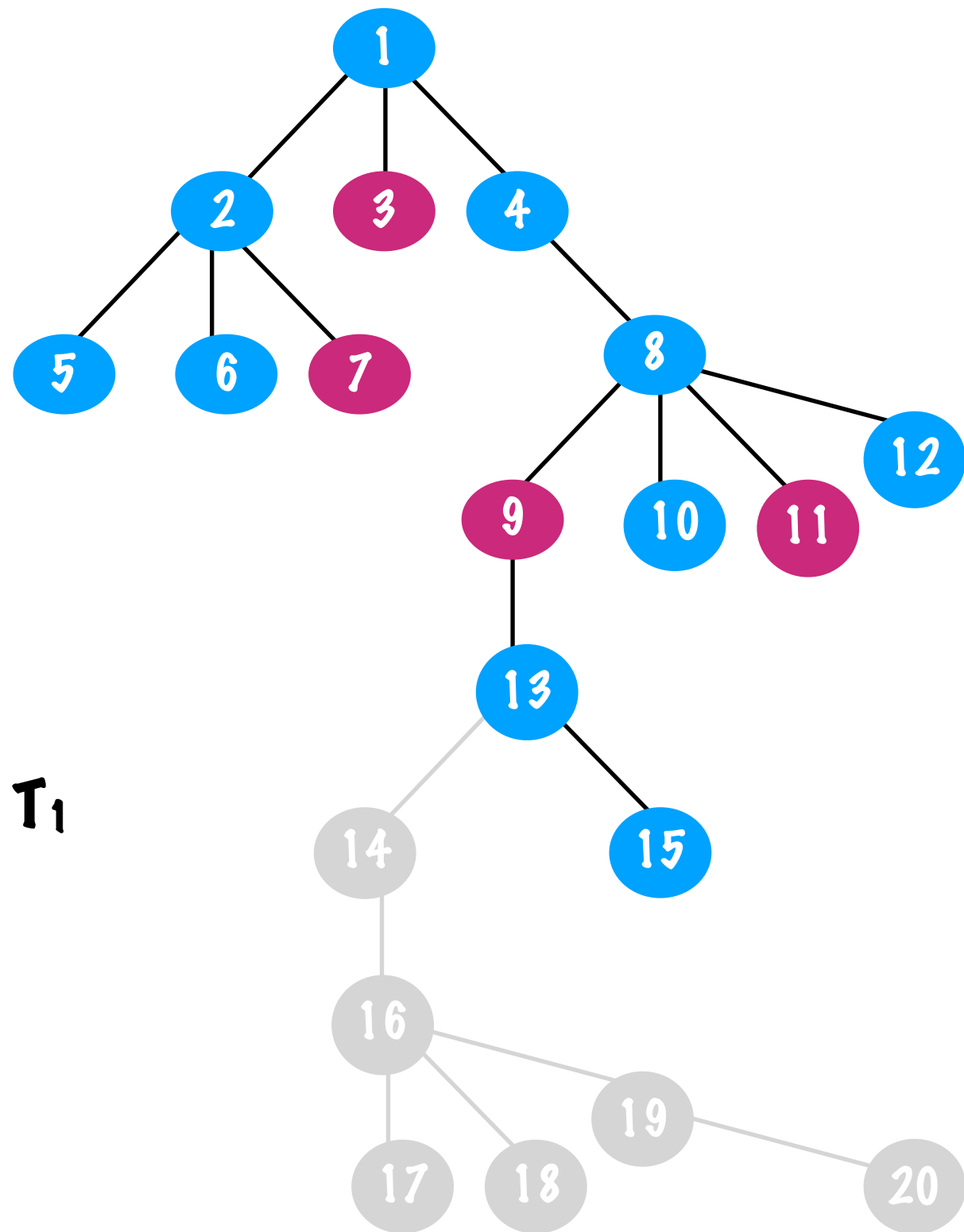
- * $\text{lca}(3,7)=1$
- * $\text{lca}(14,16)=14$
- * $\text{lca}(16,9)=9$
- * $\text{lca}(9,11)=8$
- *
- * Find least common ancestor of every pair of neighbours of v

Flower Lemma



- * $\text{lca}(14, 16) = 14$ is the deepest
- * Add 14-16 path to P
- * Add 14 to Y
- * Delete subtree rooted at 14

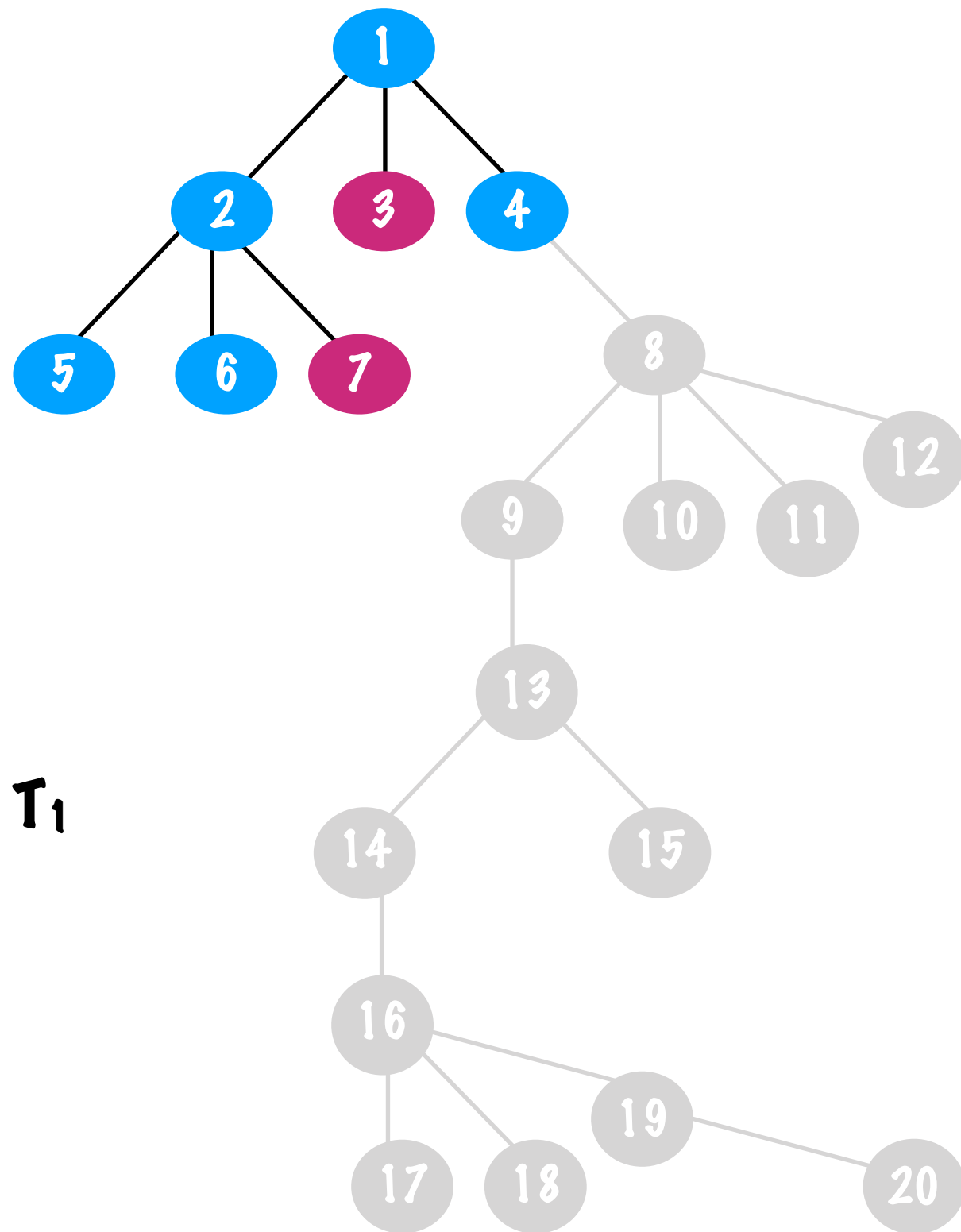
Flower Lemma



T₁

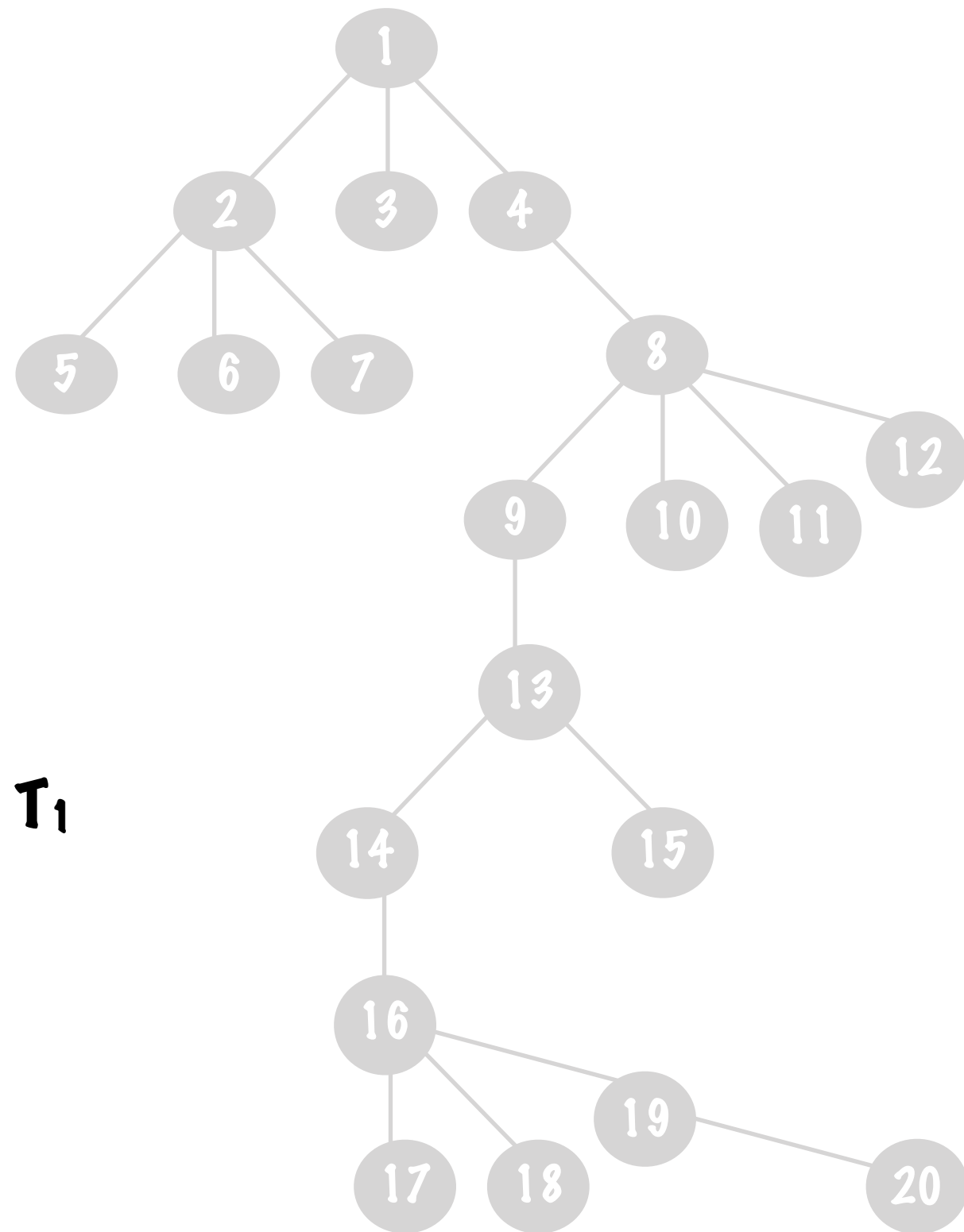
- * $\text{lca}(9, 1) = 8$ is the deepest
- * Add 9-8-1 path to P
- * Add 8 to Y
- * Delete subtree rooted at 8

Flower Lemma



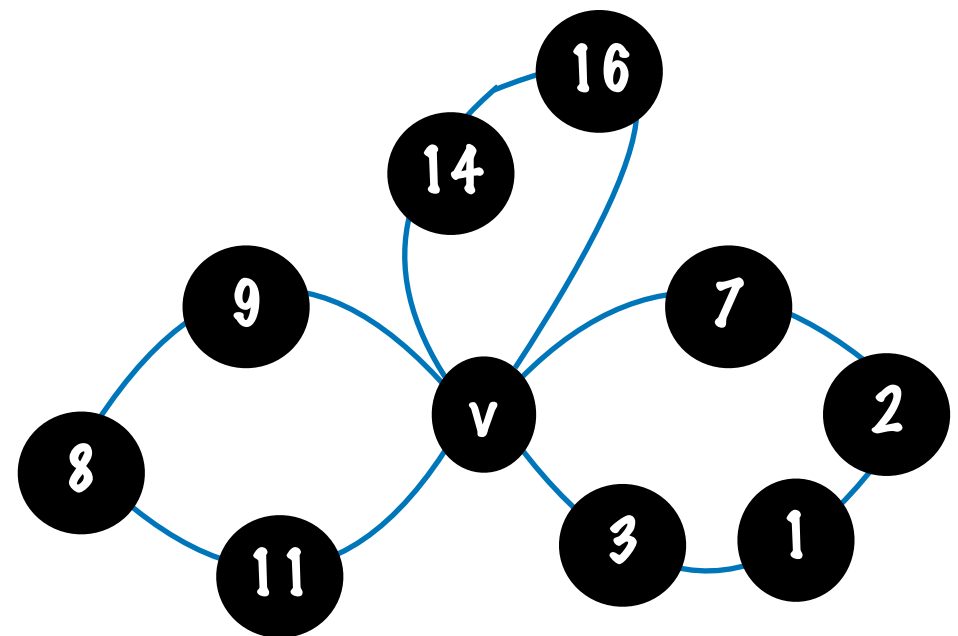
- * $\text{lca}(3,7)=1$ is the deepest
- * Add 7-2-1-3 path to \mathcal{P}
- * Add 1 to \mathcal{Y}
- * Delete subtree rooted at 1

Flower Lemma



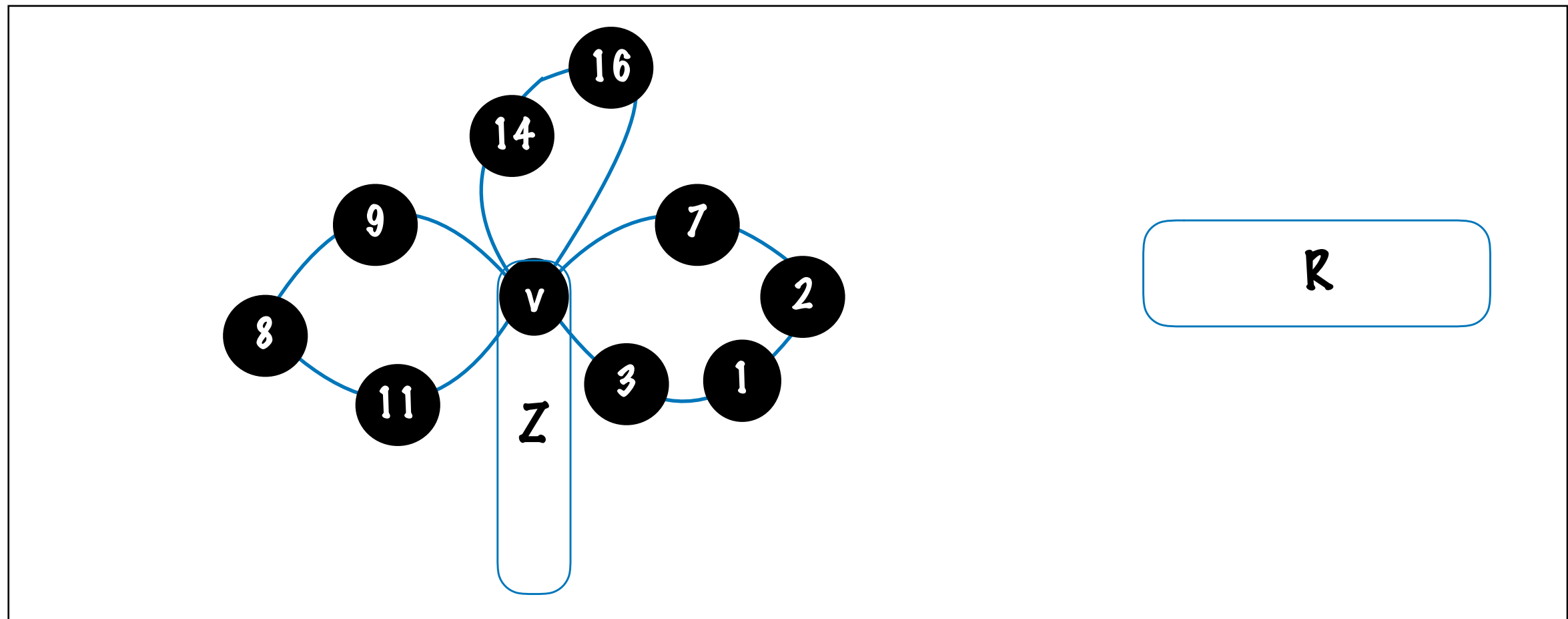
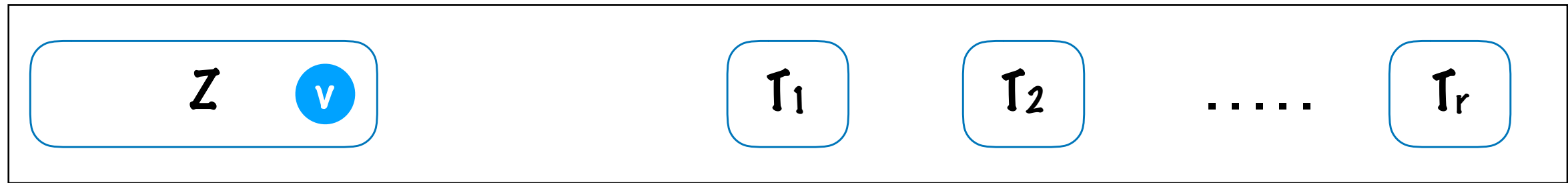
* $P = \{7-2-1-3, 9-8-11, 14-16\}$

* $Y = \{1, 8, 14\}$



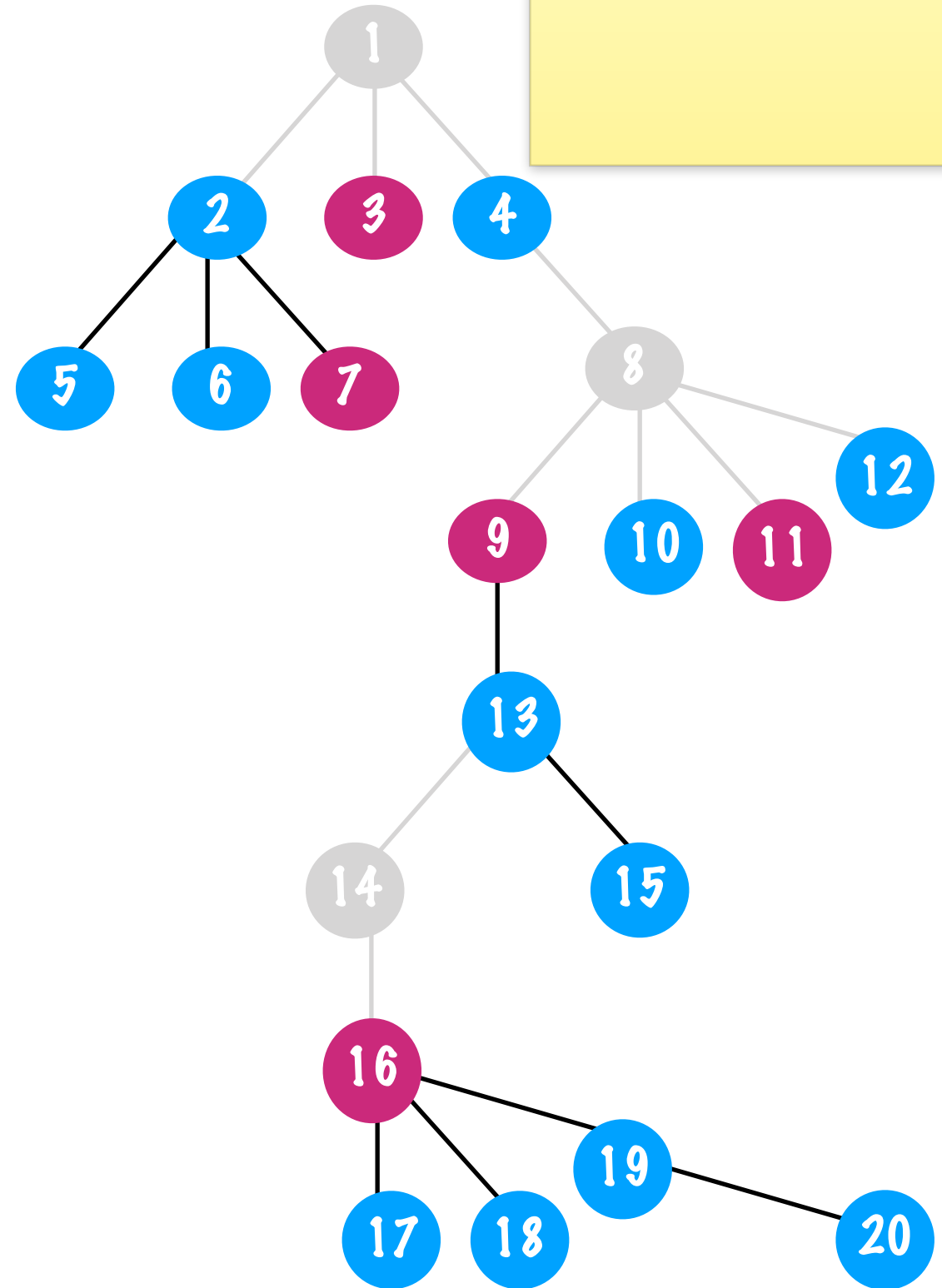
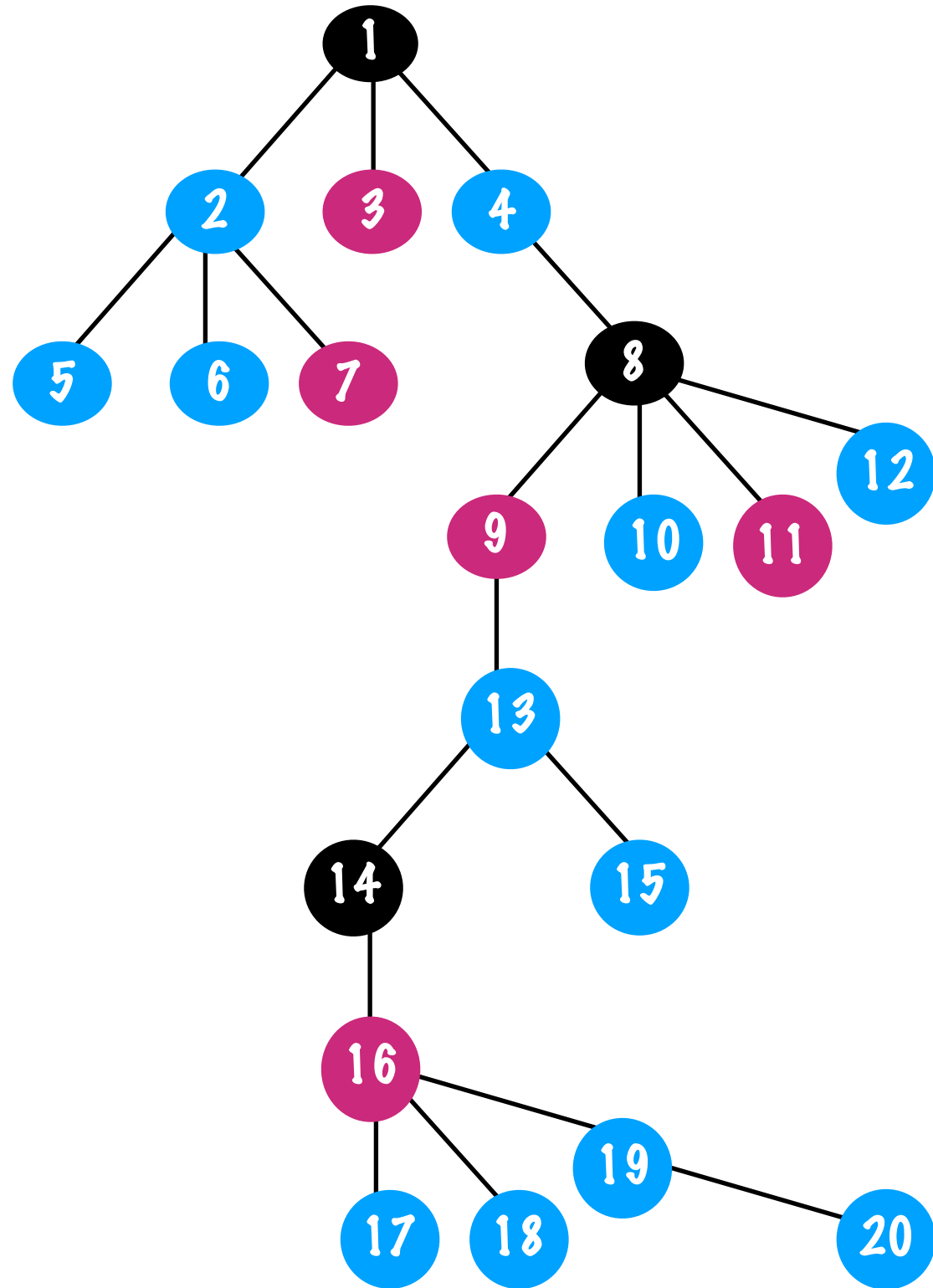
Repeat with T_2

Flower Lemma



- * $P = \{7-2-1-3, 9-8-11, 14-16\}$
- * $Y = \{1, 8, 14\}$
- * If $|P| > k$, then there is a v -flower with $k+1$ petals

Flower Lemma



$Z \setminus \cup Y \setminus \text{minus } v$ is FVS $\leq 3k$ not containing v

* $Y = \{1, 8, 14\}$

* $|Y| = |P| \leq k$ and $S = (Z \cup Y) \setminus \{v\}$ is the required FVS of size $\leq 3k$ as no component of $G - S$ has two neighbours of v

Matching and Vertex Cover in Bipartite Graphs

König's Theorem: For a bipartite graph, $\text{Max Mat} = \text{Min VC}$

Hall's Theorem: Let G be a bipartite graph with bipartition (A, B) . Then, G has a matching saturating A if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$.

Hopcroft-Karp Algorithm: Let G be a bipartite graph with bipartition (A, B) .

- * Then, a max mat and a min vc of G can be obtained in $O(m n^{1/2})$ time.
- * Further, in $O(m n^{1/2})$ time, we can either find a matching saturating A or an inclusion-minimal set $X \subseteq A$ such that $|N(X)| < |X|$.

Kőnig's Theorem

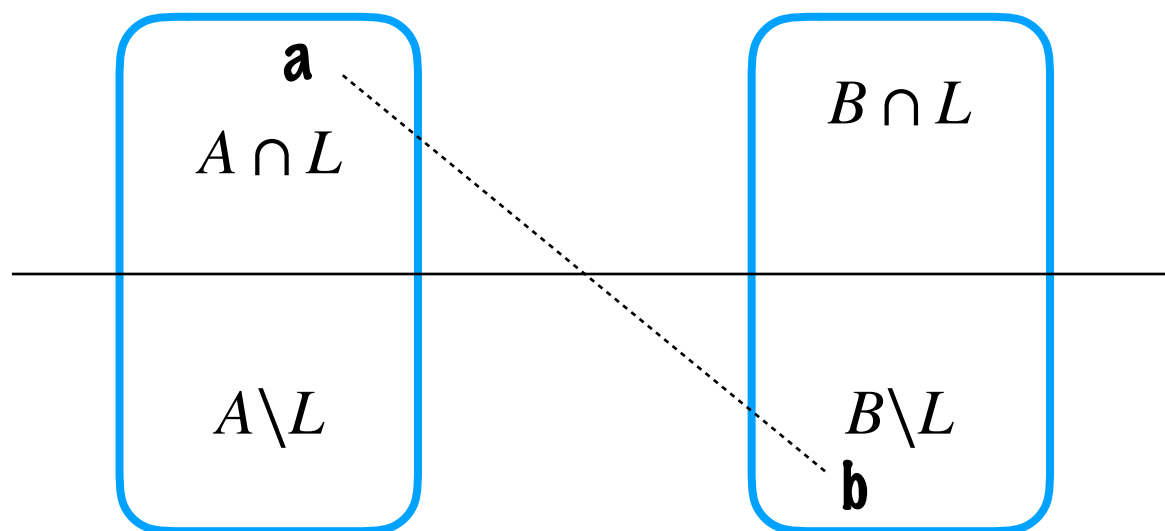
For a bipartite graph, $|Max\ Mat| = |Min\ VCI|$

- * Let M be a maximum matching of $G(A,B)$
- * Let D be a digraph obtained from G by orienting edges of G as follows:
 - * If edge $\{a,b\}$ with $a \in A$ and $b \in B$ is in M , direct $\{a,b\}$ as (b,a)
 - * If edge $\{a,b\}$ with $a \in A$ and $b \in B$ is not in M , direct $\{a,b\}$ as (a,b)
- * There is no M -alternating path in D from a free vertex in A to a free vertex in B
- * If $|M|=|A|$ then A is a min vertex cover
- * L = set of vertices reachable from a free vertex in A

Claim: $(A \setminus L) \cup (B \cap L)$ is a min vertex cover of G

König's Theorem

Claim: $S = (A \setminus L) \cup (B \cap L)$ is a min vertex cover of G



- * If $\{a, b\} \in M$ with $b \in B \setminus L$ and $a \in A \cap L$
 - * $(b, a) \in E(D)$
 - * b is the only in-nbr of $a \in L$
 - * $\therefore b \in L$
- * If $\{a, b\} \notin M$
 - * $(a, b) \in E(D)$
 - * $\therefore b \in L$ as $a \in L$
- * No vertex $a' \in A \setminus L$ is free by defn of L
- * No vertex $b' \in B \cap L$ is free
 - * o/w, M -alternating path from a free vertex in A to b' as $b' \in L$
- * No edge $\{a', b'\}$ with $a' \in A \setminus L$ and $b' \in B \cap L$ is in M
 - * o/w, $(b', a') \in E(D)$ and $a' \in L$
- * $\therefore |S| = |M|$

Hall's Theorem

Let G be a bipartite graph with bipartition (A, B) . Then, G has a matching saturating A if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$.

Sufficiency: Induction on $|A|$

- * Base: $|A|=1$
- * Induction Step: Let $a \in A$ and b be one of its neighbours
- * If $|N(X) \setminus \{b\}| \geq |X|$ for all $X \subseteq A \setminus \{a\}$
 - * By induction hypothesis, there is a matching M saturating $A \setminus \{a\}$ in $G - \{a, b\}$
 - * Then, $M \cup \{a, b\}$ is a matching saturating A in G
- * Otherwise, $|N(X) \setminus \{b\}| < |X|$ for some $X \subseteq A \setminus \{a\}$
 - * As $|N(X)| \geq |X|$, it follows that $|N(X)| = |X|$
 - * By induction hypothesis there is a matching M saturating X in $G(X, N(X))$
 - * For any $Y \subseteq A \setminus X$, $|N(Y) \setminus N(X)| \geq |N(Y \cup X)| - |N(X)| \geq |Y| + |X| - |N(X)| = |Y|$
 - * By induction hypothesis there is a matching M' saturating $A \setminus X$ in $G(A \setminus X, B \setminus N(X))$
 - * $M \cup M'$ is a matching saturating A

Hopcroft-Karp Algorithm

Let G be a bipartite graph with bipartition (A, B) .

- * Then, a max mat and a min vc of G can be obtained in $O(m n^{1/2})$ time.
- * Further, in $O(m n^{1/2})$ time, we can either find a matching saturating A or an inclusion-minimal set $X \subseteq A$ such that $|N(X)| < |X|$.

Finding Minimal Hall Set

- * Suppose M is a maximum matching of $G(A, B)$ such that $|M| < |A|$
- * Let $A = \{a_1, a_2, a_3, \dots, a_r\}$ and let $X_1 = \{a_1\}$ where a_1 is a vertex not saturated by M
 - * Every vertex in $N(a_1)$ is saturated by M
- * For $i=1$ to $r-1$ Each vertex in $N(X_i)$ is a part of matching as suppose for the first time in the j th iteration $N(X_j)$ contains
 - * If $|N(X_i)| < |X_i|$, Return X_i
 - * $X_{i+1} = X_i \cup M\text{-partners}(N(X_i))$ (Note: $|X_{i+1}| > |X_i|$)
- * Return X_r

Extended Hall's Theorem

Let G be a bipartite graph with bipartition (A, B) . Then, there is a 2-expansion of A into B iff $|N(X)| \geq 2|X|$ for all $X \subseteq A$.

Sufficiency:

- * To $G(A, B)$, add a copy of A to get bipartite graph $G'(A', B)$
- * If $G'(A', B)$ has a matching saturating A' , then this matching corresponds to a 2-exp of A into B
- * Otherwise, by Hall's theorem, $|N_{G'}(X)| < |X|$ for some $X \subseteq A'$
 - * w.l.o.g assume X has both copies or no copy of a vertex of A
 - * $|N_{G'}(X)| = |N_G(X \cap A)| = |N_G(X \cap A)| \geq 2|X \cap A| = |X|$ (a contradiction)

If there is single copy just

2-Expansion Lemma

Let G be a bipartite graph with bipartition (A, B) s.t. $|B| \geq 2|A|$ and there are no isolated vertices in B . Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a 2-exp into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m n^{1/2})$ time.

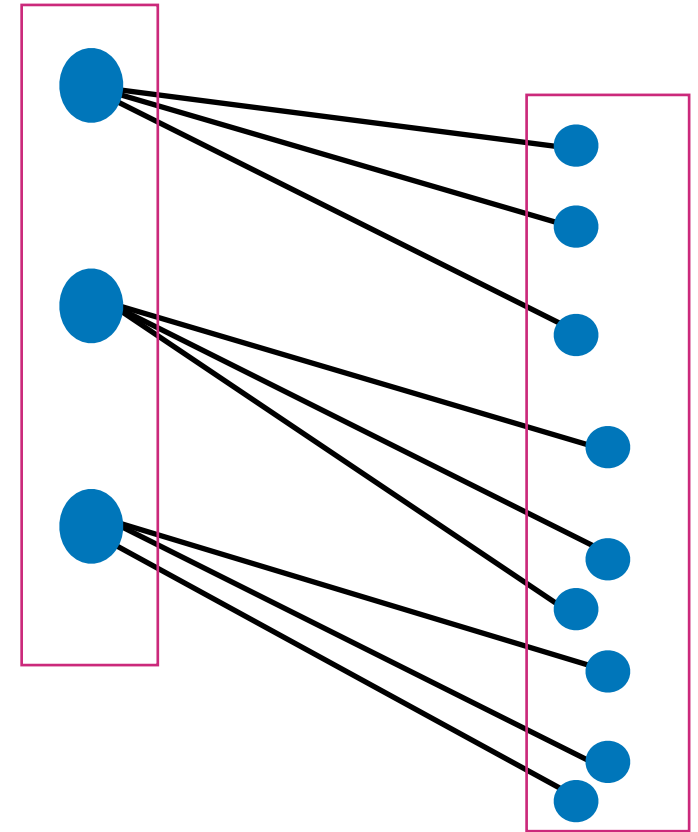
- * If A has a 2-expansion M into B , then $X=A$ and $Y=V(M) \cap B$
 - * Clearly, $N(Y) \subseteq X$
- * Otherwise, by Extended Hall's theorem, $|N_G(X)| < 2|X|$ for some $X \subset A$
- * Let $G'(A', B')$ denote the graph obtained from G by deleting X and $N_G(X)$
- * $|B'| = |B| - |N_G(X)| > |B| - 2|X| \geq 2|A| - 2|X| = 2|A'|$
- * In G , no vertex in B' has a neighbour in X . So, B' has no isolated vertices in G'
- * By induction hypothesis, there exists non-empty sets $X' \subseteq A'$ and $Y' \subseteq B'$ such that X' has a 2-exp into Y' and $N_{G'}(Y') \subseteq X'$
- * As $N_G(Y') = N_{G'}(Y')$, it follows that $N_G(Y') \subseteq X'$

Ex: Runtime analysis

q-Expansion Lemma

Definition: In a bipartite graph with bipartition (A, B) , a set M of edges is called a **q-expansion** from A to B if

- * Every vertex of A is incident with exactly q edges of M
- * M saturates exactly $q|A|$ vertices in B



Extended Hall's Theorem: Let G be a bipartite graph with bipartition (A, B) . Then, there is a q -expansion of A into B iff $|N(X)| \geq q|X|$ for all $X \subseteq A$.

q-Expansion Lemma: Let q be a positive integer and let G be a bipartite graph with bipartition (A, B) such that $|B| > q|A|$ and there are no isolated vertices in B . Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a q -expansion into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m n^{1/2})$ time