CS 5003: Parameterized Algorithms

Lectures 34-35

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Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Treewidth

- A tree decomposition of a graph G is a pair (T,B) where T is a tree and B: V(T) -> $2^{V(G)}$
  - For each vertex v in G, there is a node x in V(T) such that v is in B(x)
  - For each edge e={u, v} in G, there is a node x in V(T) such that u are v are in B(x)
  - For each vertex v in G, the set $\{x \in V(T) : v \in B(x)\}$ induces a connected graph
- Width of a tree decomposition T = w(T) = max $\{|B(x)| : x \in V(T)\}$ - 1
- Treewidth of G, tw(G) = min {w(T) : T is a tree decomposition of G}
- An optimal tree decomposition of G is a tree decomposition of G of width tw(G)

Treewidth

* A simple tree decomposition (T, B) is one where there is no pair of distinct nodes x and y in T such that B(x) $\subseteq$ B(y)

* Any simple tree decomposition (T, B) of G satisfies |V(T)| <= |V(G)|
* For any G, there is an opt tree decomposition that is simple
* Given G, k, there exists an algorithm running in $2^{O(k^3)}$ n time that returns a tree decomp of G of width <=k (if one exists)

Properties of Treewidth

- If H is a subgraph of G, then tw(H) <= tw(G)
- If tw(G) <=k, G has a vertex of degree at most k
  - Look at a simple opt tree decomposition
  - Leaf t has a vertex v that is not in any other bag
  - v has neighbours only in B(t) and |B(t)| <= k+1
- If tw(G) <=k, G has at most nk - (k+1 choose 2) edges
  - Induction on n (base: n=k+1)
  - Induction step: Let v be a vertex of deg <=k
  - tw(G-v) <= tw(G) and G-v has <= (n-1)k - (k+1 choose 2) edges
  - G has <= nk - (k+1 choose 2) edges

from optimum tree decomposition of G, remove

thus O(nk) edges.

Properties of Treewidth

- If G is a tree, then tw(G) <= 1
  - Induction on n: base case: n=1, 2
  - Let v be a leaf in G. By induction hypothesis, tw(G-v) <= 1
  - Look at an optimal tree decomposition of G-v
  - To this, add a leaf node with bag {v,u} adjacent to a node containing u

note that G \ v is a tree.

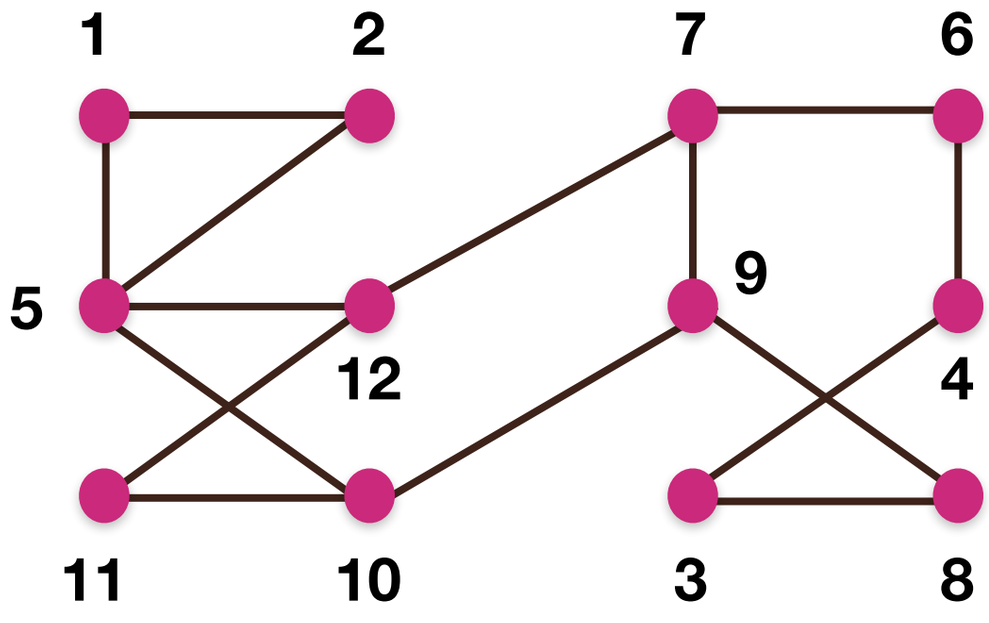
- If G is a cycle, then tw(G)<=2
  - Let v be a vertex in G
  - G-v is a path and hence tw(G-v)<=1
  - Take an opt tree decomposition of G-v and add v to every bag
- If G is a cycle, then tw(G)>=2
  - Suppose tw(G)=1
  - Look at a simple opt tree decomposition
  - Leaf t has a vertex v that is not in any other bag
  - v has neighbours only in B(t) i.e., v has degree 1

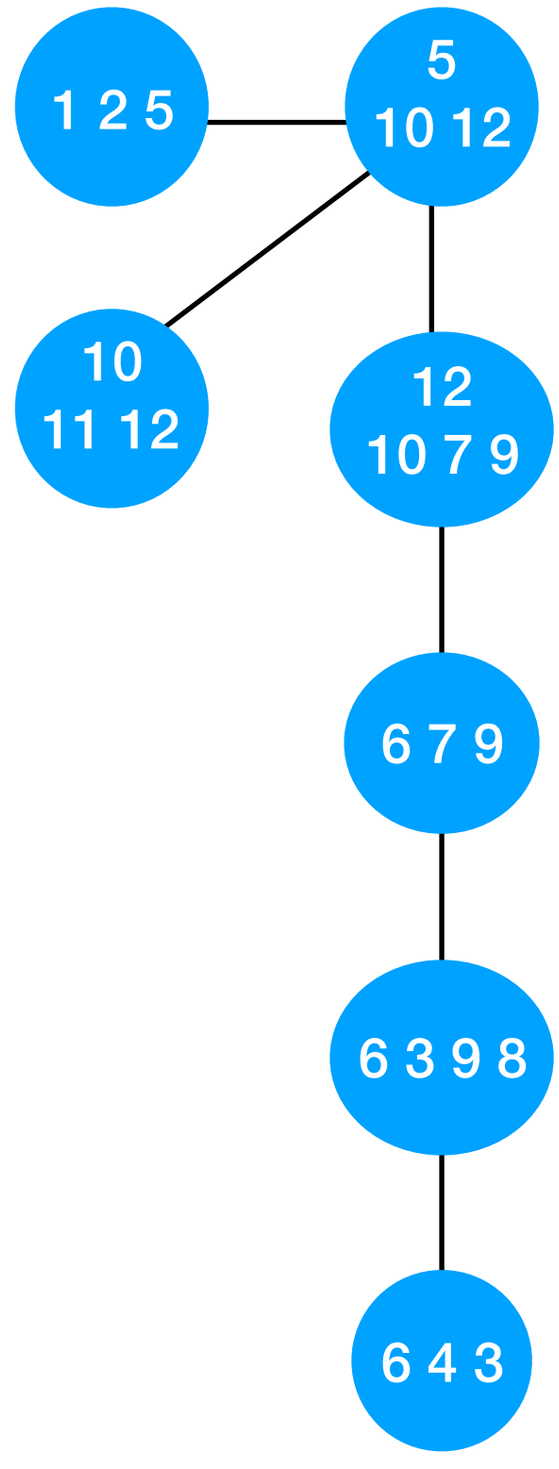
Properties of Treewidth

- A graph G on n vertices has tw <= 1 iff G is a forest
  - (⇐) If G is a forest, then tw(G) <= 1
  - (⇒) If G is a graph with tw(G) <=1, then
    - If tw=0, then G is a tree on single vertex
    - If tw=1 and G has a cycle C, then as tw(G(V(C),E(C)))=2 and tw(G(V(C),E(C))) <= tw(G), it follows that tw(G) >=2
- A graph G on n vertices has tw = n-1 iff G is a complete graph
  - Let G be a non-complete graph
    - Let u and v be non-adjacent
    - Then G has a tree decomposition consisting of 2 nodes with bags V(G)\{u} and V(G)\{v}
  - Suppose the complete graph on n vertices has tw <=n-2
    - There is a vertex v with degree at most n-2

Properties of Treewidth

* Every bag of a tree decomposition T is a separator
  * For any two adjacent nodes x and y in T,
    * $B(x) \cap B(y)$ is a separator of G
    * $B(x) \cap B(y)$ separates $V(T \setminus T_x)$ and $V(T_x)$





Properties of Treewidth

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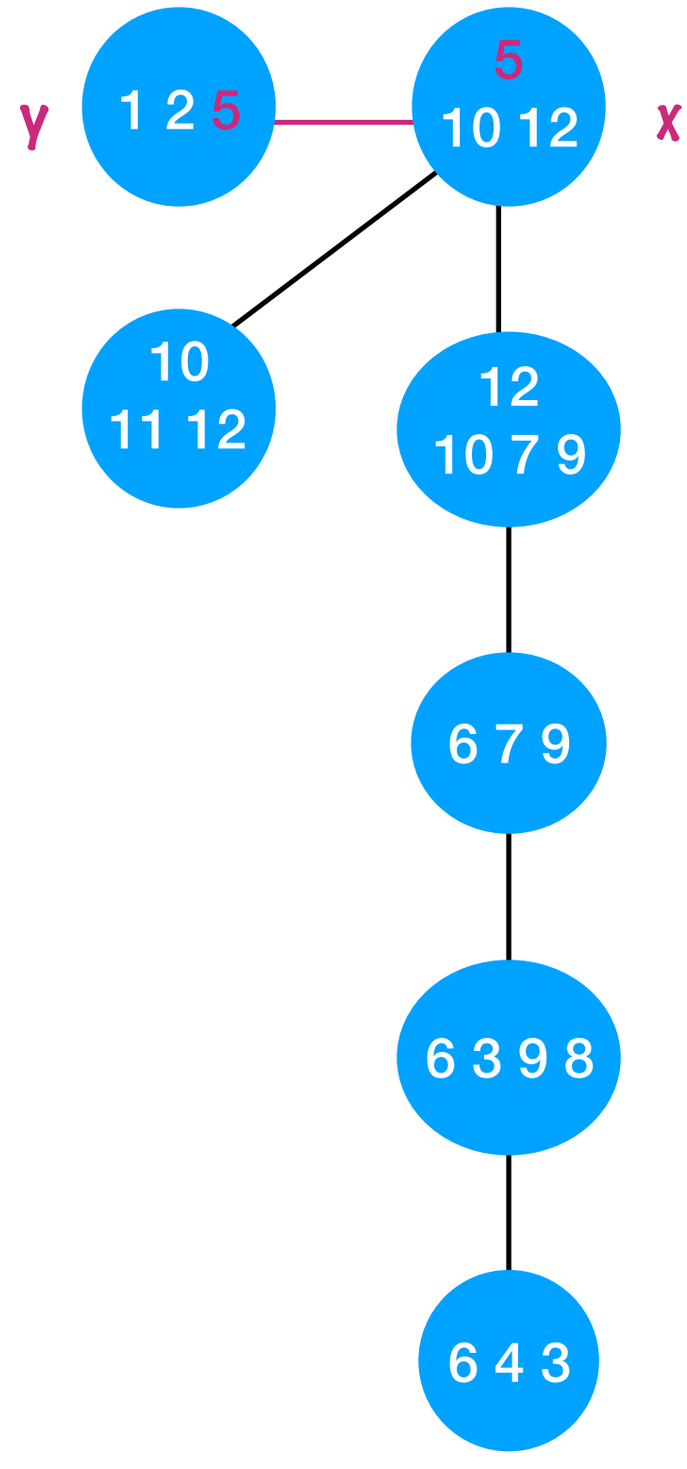

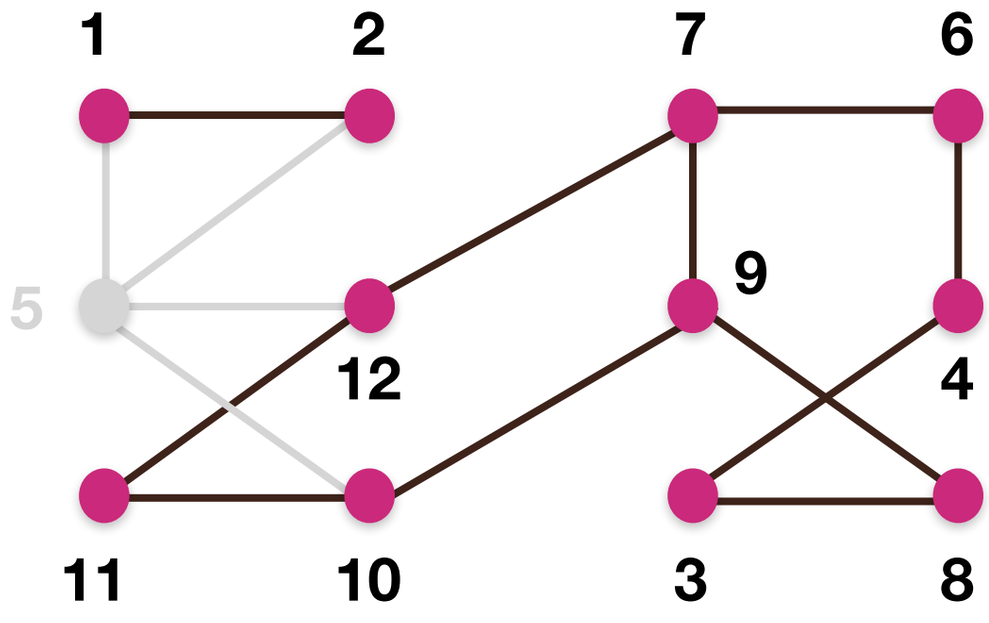

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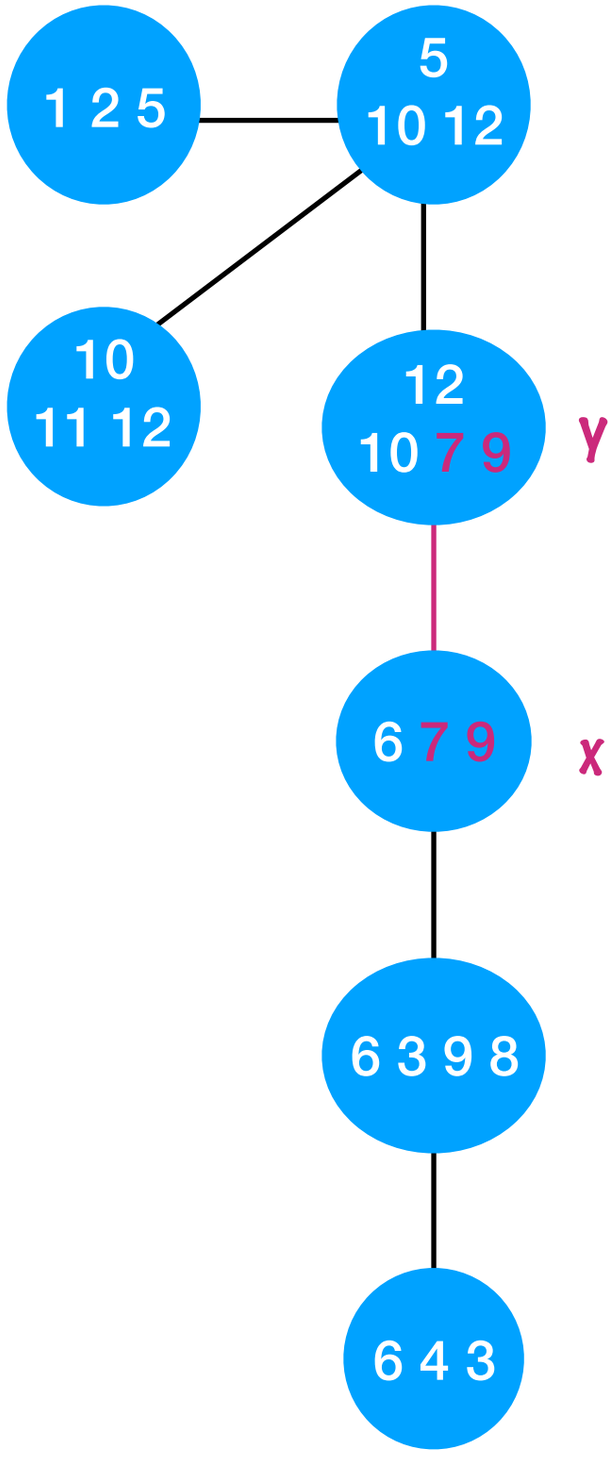

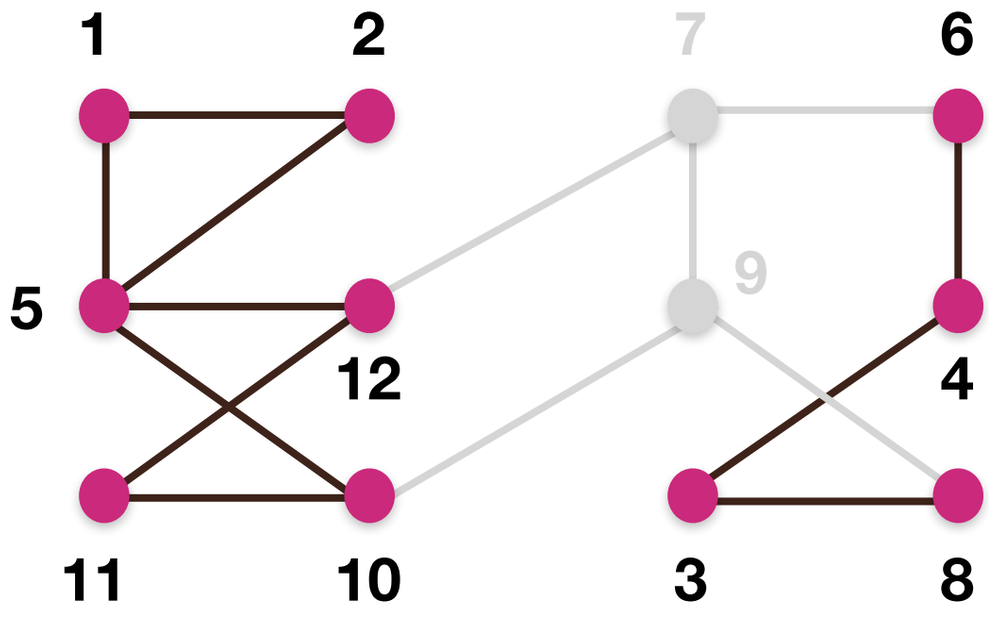

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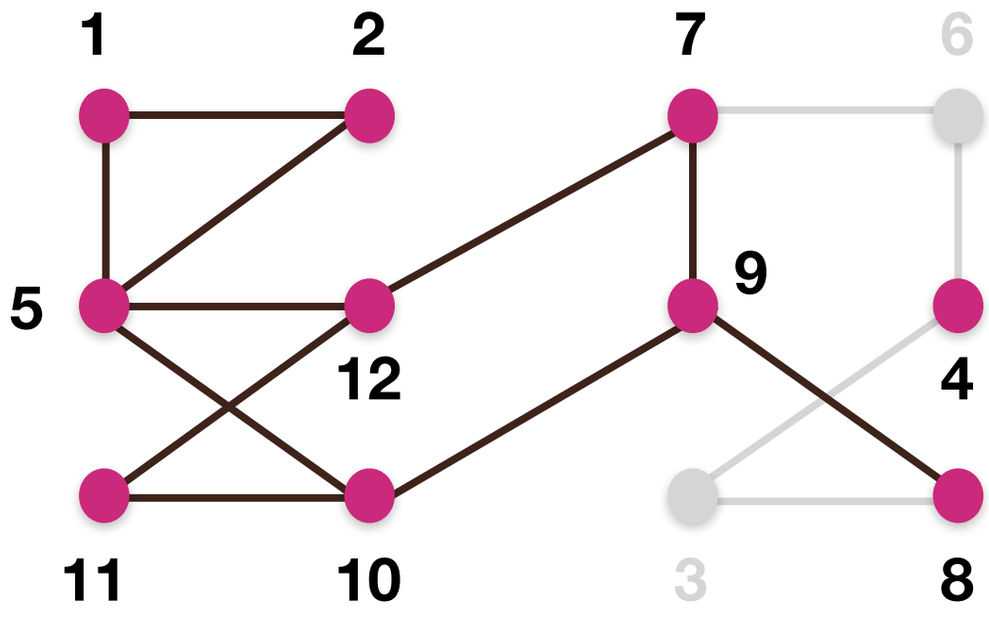
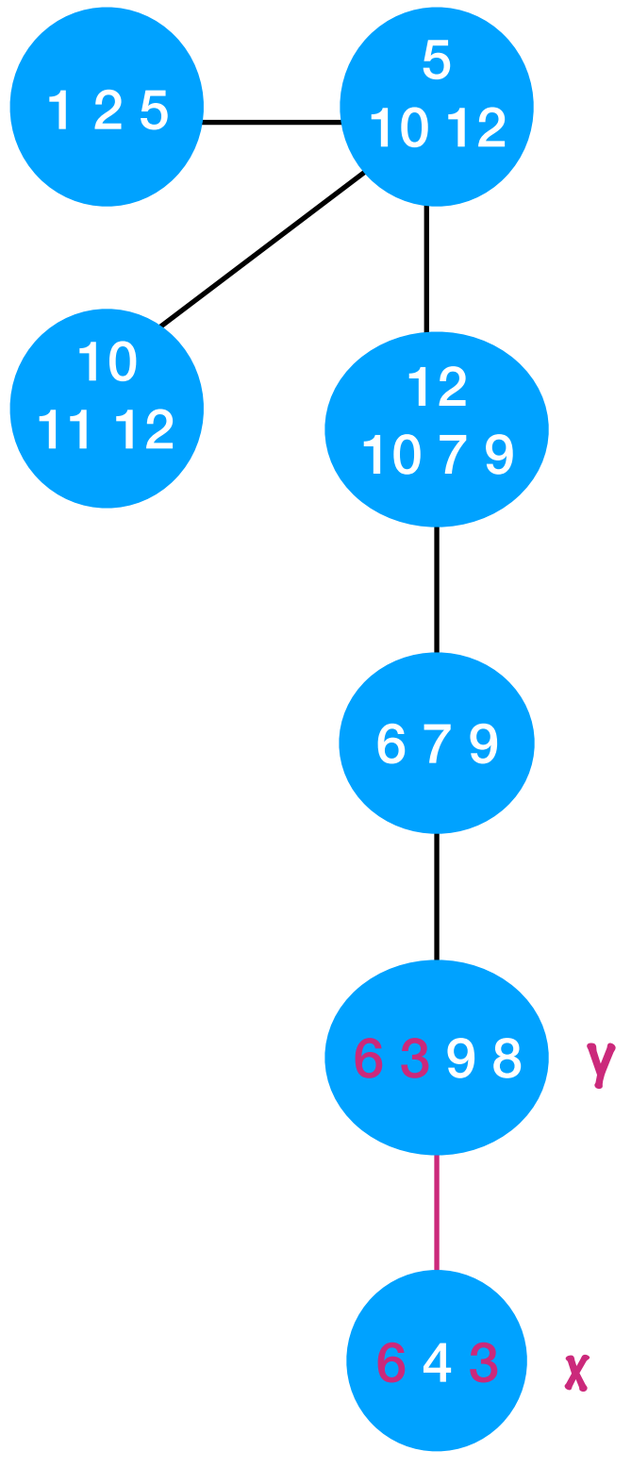
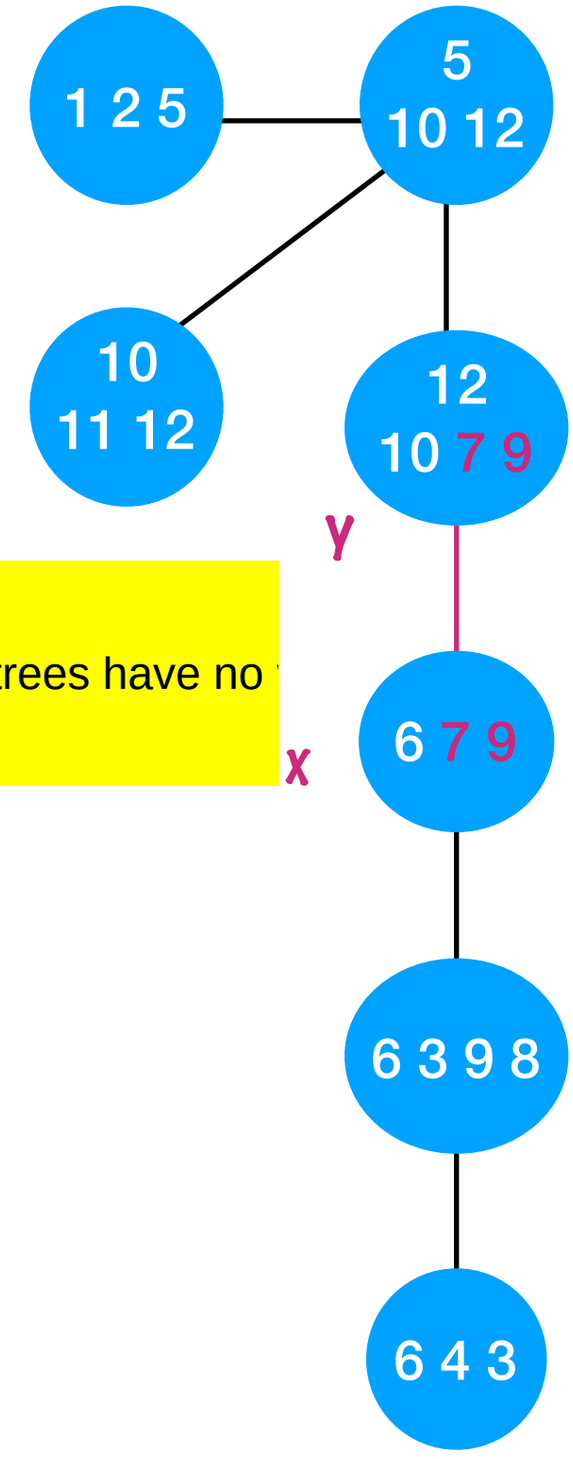




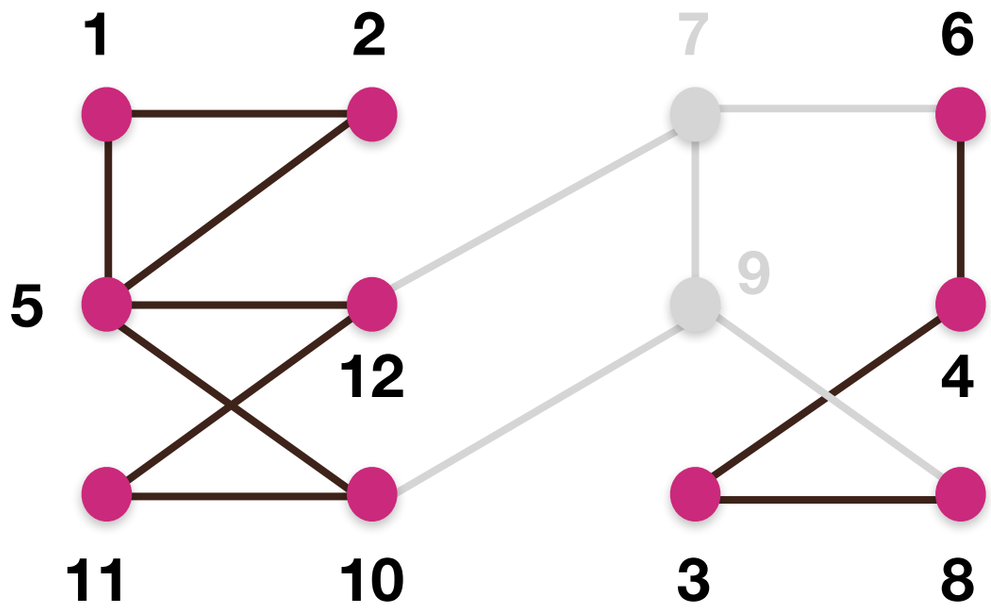
Properties of Treewidth

For any two adjacent nodes x, y in T, $B(x) \cap B(y)$ separates $V(T \setminus T_x)$ & $V(T_x)$

* Consider a path P between a in $V(T \setminus T_x)$ and b in $V(T_x)$
* $P = (a, v_1, v_2, \ldots, v_k, b)$
  * P has a pair of vertices v_i in $V(T \setminus T_x)$ and v_{i+1} in $V(T_x)$ (or)
  * P has a vertex v_i in $V(T \setminus T_x) \cap V(T_x)$
* Thus, there is i s.t v_i is in $B(x) \cap B(y)$



Note that V(Tx) and the other thing doesn't include vertices of B(x) intersection B(y). So, these two trees have no

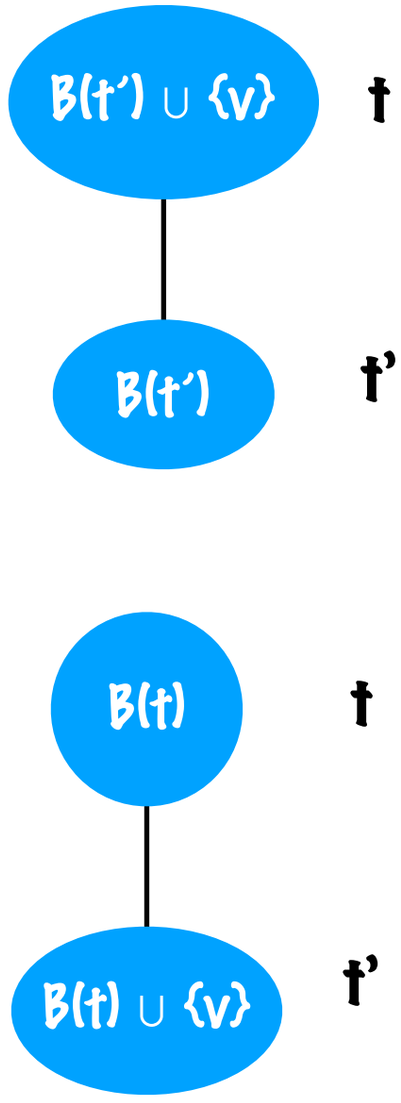


Nice Tree Decomposition

Definition: Nice Tree Decomposition (T, B)

- T is a rooted tree
- B(root) = ∅ and B(x) = ∅ for each leaf x in T
- Every non-leaf node of T is one of the following 3 types
  - Introduce node
  - Forget node
  - Join node

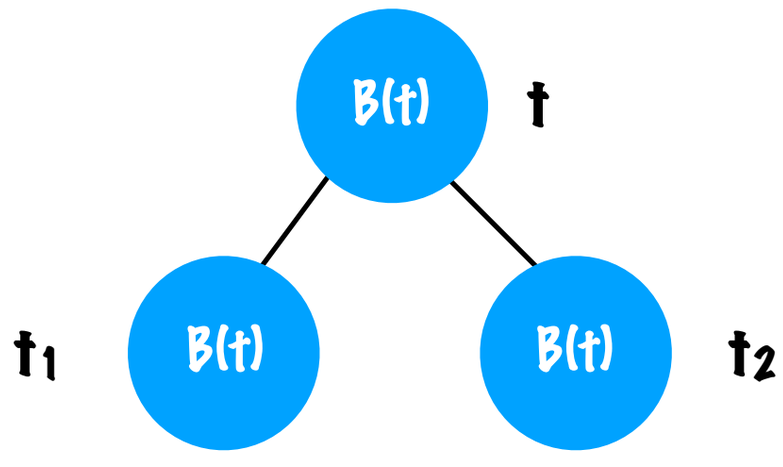
Nice Tree Decomposition



* **Introduce node:**
  * a node t with exactly one child t' such that B(t) = B(t') ∪ {v}

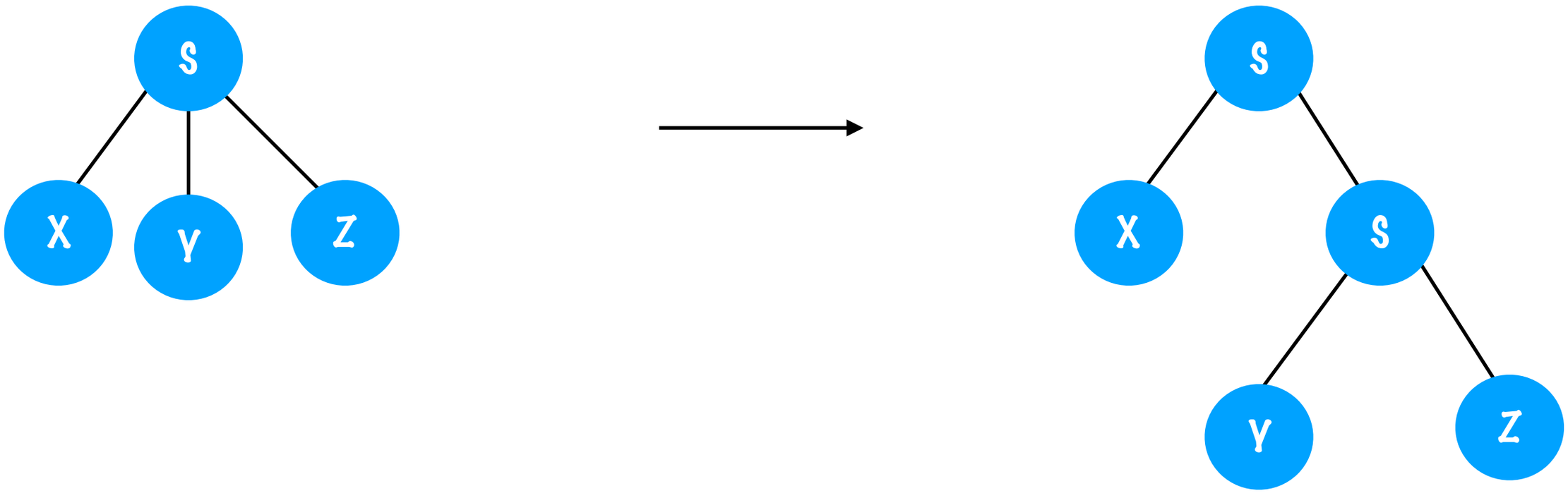
    for some vertex v not in B(t')
* **Forget node:**
  * a node t with exactly one child t' such that B(t') = B(t) ∪ {v}

    for some vertex v not in B(t)
* **Join node:**
  * a node t with 2 children t_1 and t_2 such that B(t) = B(t_1) = B(t_2)



Nice Tree Decomposition

Lemma: There is a poly-time algorithm that given a simple tree decomposition (T, B) of G, outputs a nice tree decomposition (T', B') s.t w(T') <= w(T) and |V(T')| is O(w(T)*|V(G)|).



binary tree

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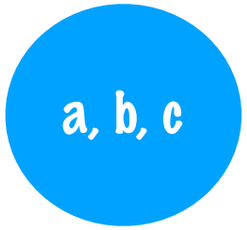
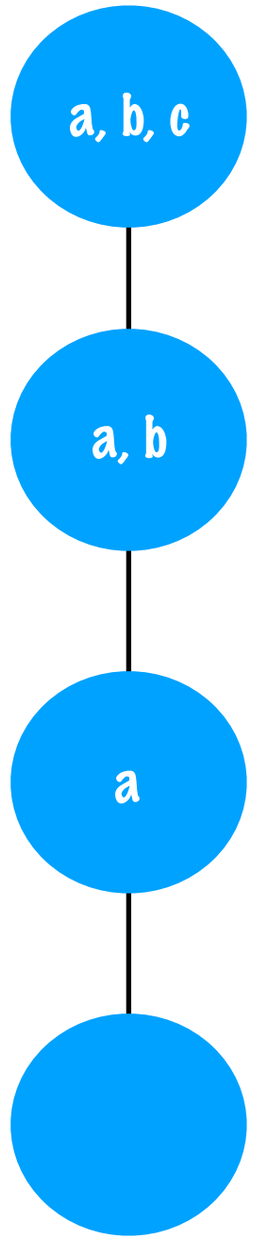


leaf



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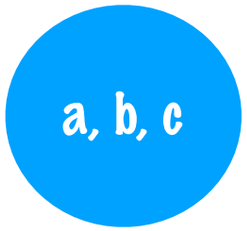
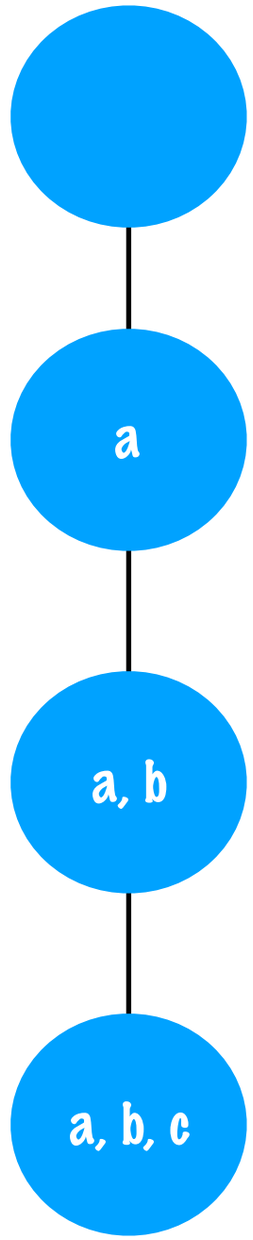


root



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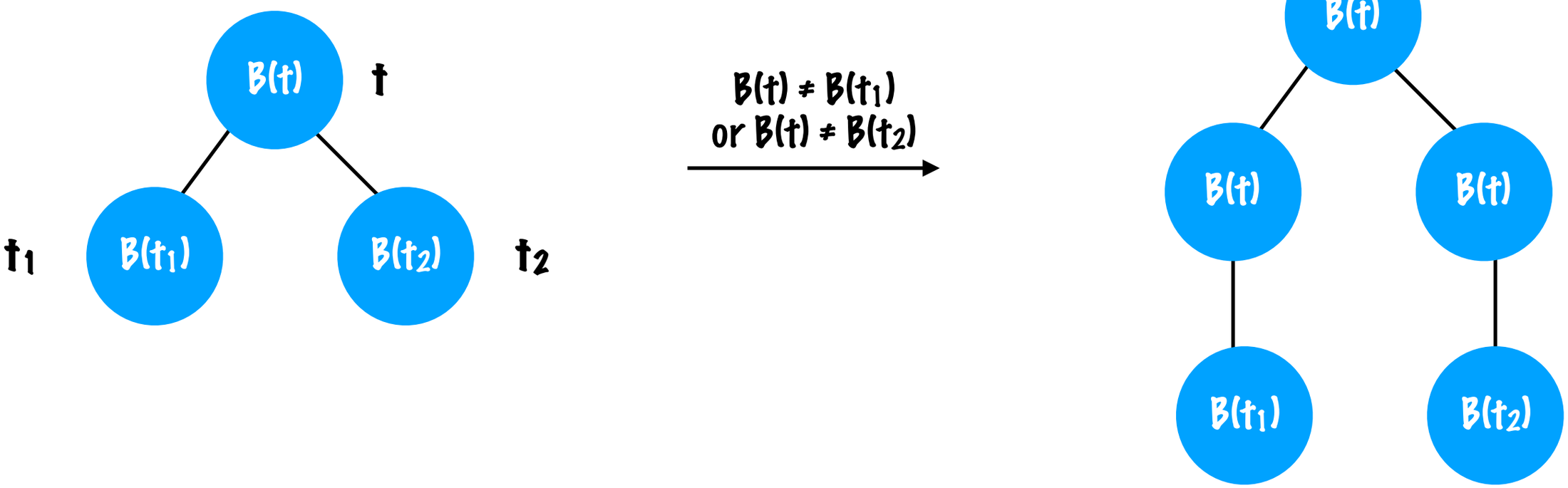

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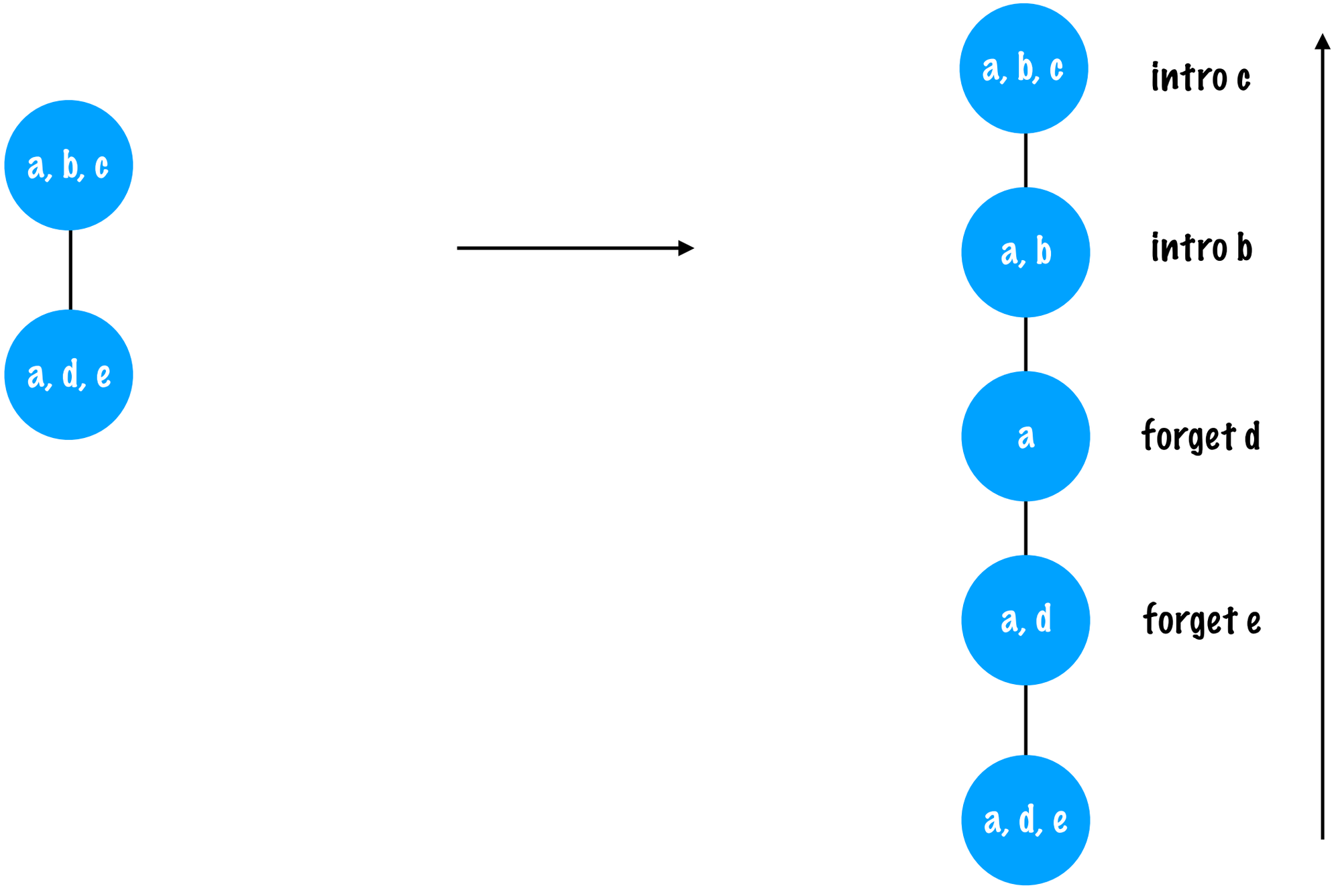

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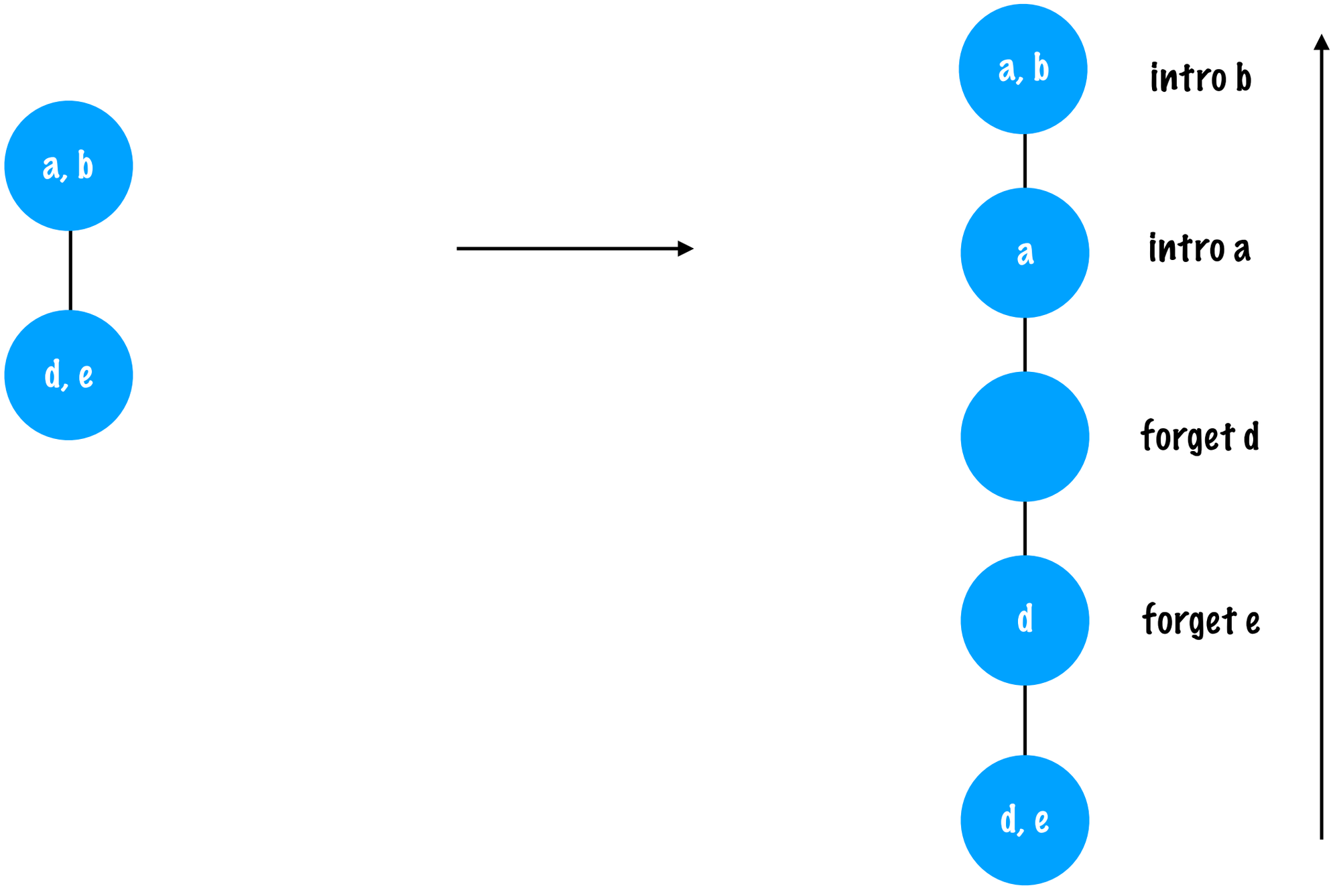


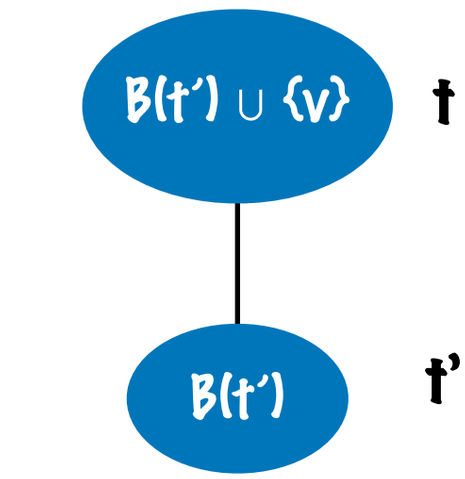
Nicer Tree Decomposition

Note: If tw(G) <=k, G has O(nk) edges

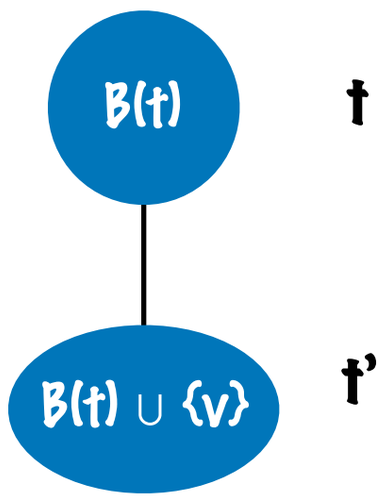
Lemma: There is a poly-time algorithm that given a nice tree decomposition (T, B) of G, outputs a nicer tree decomposition (T', B') s.t w(T') = w(T) and |V(T')| is O(|V(G)|*w(T)).

O(|V(G)|*w(T)) + O(|V(G)|*w(T)) = O(..)Also once we have introduced edge node, we can consider it for deciding closest to roo

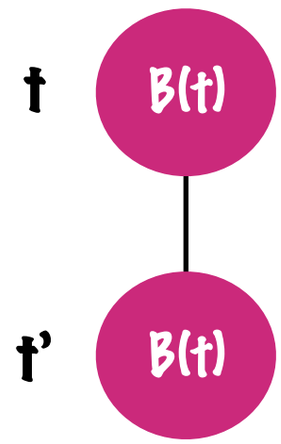
Introduce Node



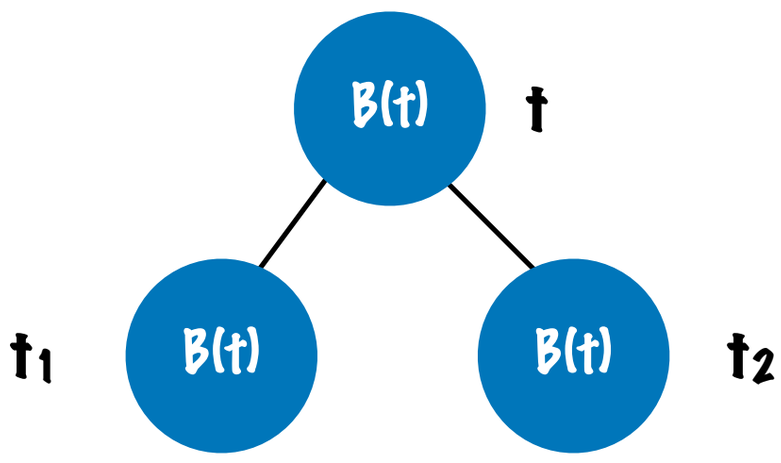
Forget Node



Introduce Edge Node



Join Node



- For every edge {u, v} in G, there is exactly one introduce edge node with label uv
  - Look at a node t' closest to root containing u,v in its bag
  - Add an introduce edge node t as the parent of t'