

CS 5003: Parameterized Algorithms

Lectures 26-27

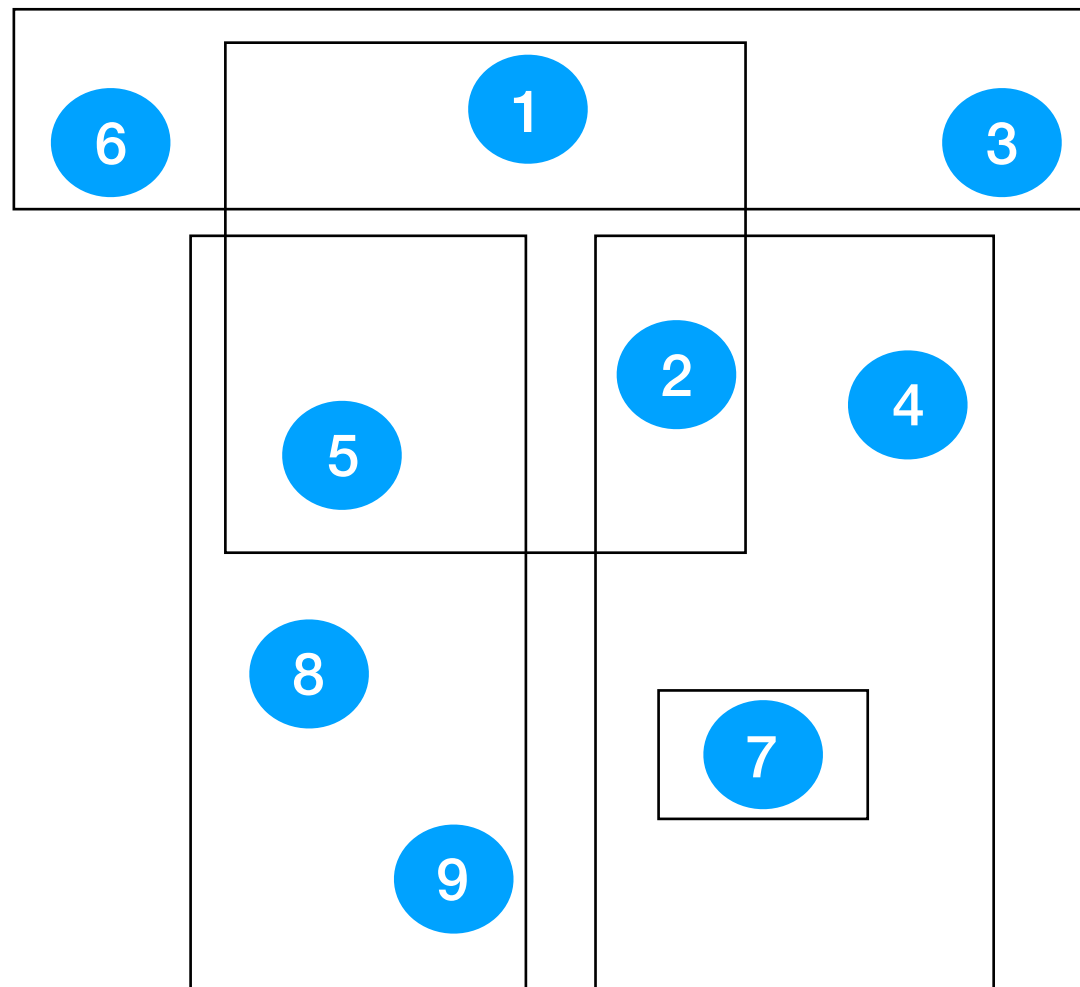
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Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Sunflower Lemma

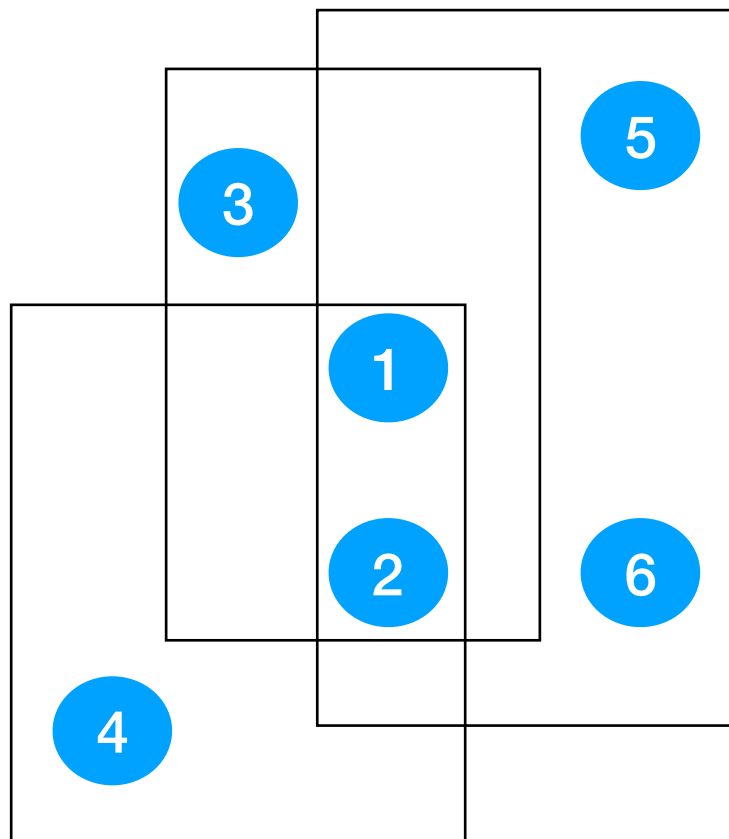
- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family of subsets of U : $\mathcal{F} = \{S_1, S_2, \dots, S_m\}$



Sunflower Lemma

- * Universe $U = \{1, 2, 3, \dots, n\}$, Family of subsets of U : $F = \{S_1, S_2, \dots, S_m\}$
- * A sunflower with k petals and core Y is a collection $C \subseteq F$ of sets s.t.
 - * $S \cap S' = Y$ for any two distinct S, S' in C
 - * $S \setminus Y$ is non-empty for every S in C

Sunflower

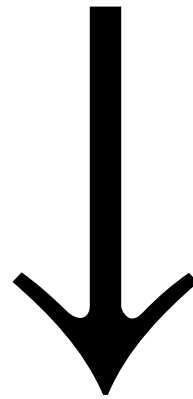


- * 3 Petals $\{3\}, \{4\}, \{5, 6\}$
 - * Each petal is non-empty
- * Core = $\{1, 2\}$

- * A set of pairwise disjoint sets is a sunflower with empty core

Sunflower Lemma

- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family of subsets of U : $F = \{S_1, S_2, \dots, S_m\}$ (no duplicates)
- * Each set in F has size d
- * $|F| > d! (k-1)^d$



- * F has a sunflower with k petals that can be obtained in polynomial time

Proof of Sunflower Lemma

Induction on d

- * $d = 1$ (singletons)

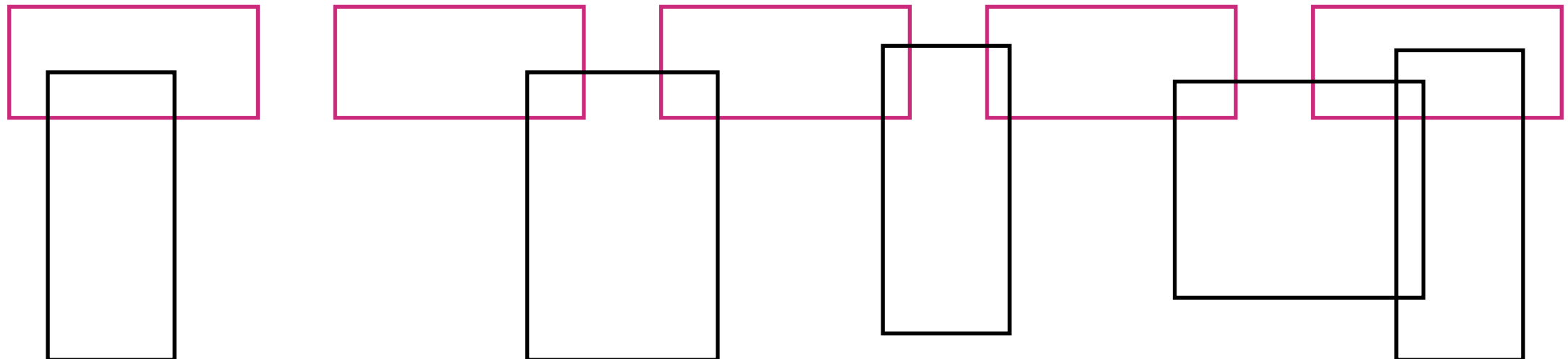
- * $d \geq 2$

- * $\{S_1, S_2, \dots, S_l\}$ maximal set of disjoint sets in F

- * If $l \geq k$, $\{S_1, S_2, \dots, S_l\}$ is the sunflower with k petals

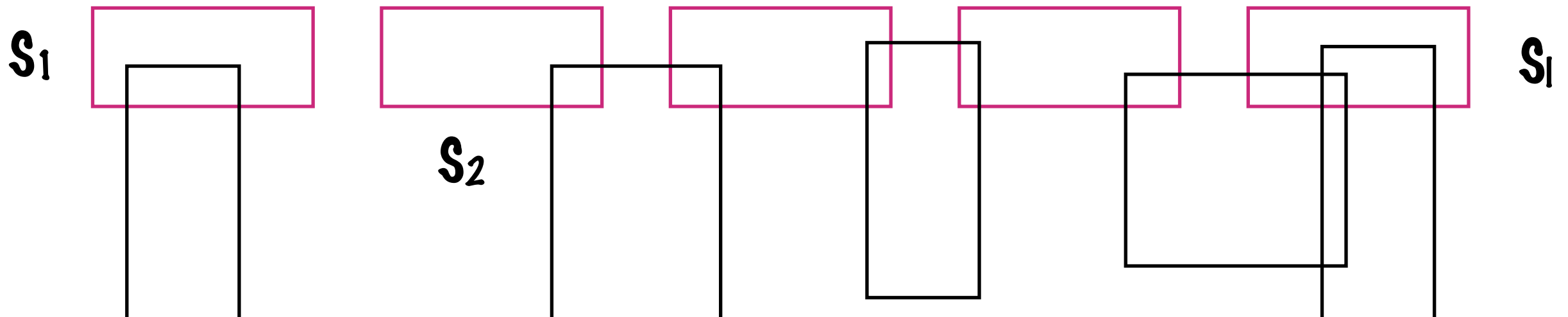
- * Otherwise, $l < k$

- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family of subsets of U : $F = \{S_1, S_2, \dots, S_m\}$
 - * No duplicates
- * Each set in F has size d , $|F| > d! (k-1)^d$



Proof of Sunflower Lemma

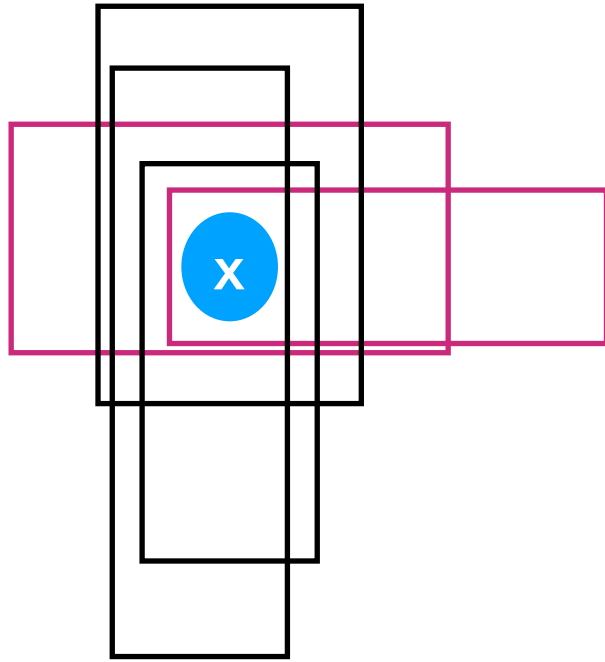
- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family of subsets of U : $F = \{S_1, S_2, \dots, S_m\}$
 - * No duplicates
- * Each set in F has size d , $|F| > d! (k-1)^d$



Suppose each element of S is in less than $|F|/|S|$ sets, then elements of S are in total $< |F|$ sets but every set in F has an element

- * $S = S_1 \cup S_2 \cup \dots \cup S_i$ and $|S| \leq d(k-1)$
- * Every set in F has an element in S
- * There is an element x that is in $\geq |F|/|S| > d! (k-1)^d / d(k-1)$ sets of F

Proof of Sunflower Lemma



- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family of subsets of U : $F = \{S_1, S_2, \dots, S_m\}$
 - * No duplicates
- * Each set in F has size d , $|F| > d! (k-1)^d$

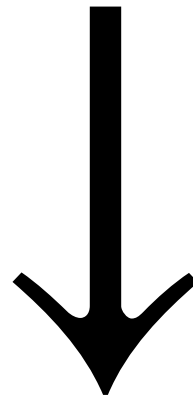
- * There is an element x that is in $\geq |F|/|S| > d! (k-1)^d / d(k-1)$ sets of F
- * $F' =$ sets in F containing x
- * F'' : obtained from F' by deleting x (no duplicates)
 - * $|F''| > (d-1)! (k-1)^{d-1}$
- * By induction hypothesis, F'' has a sunflower with k petals
 - * $\{S'_1, S'_2, \dots, S'_k\}$
- * $\{S'_1 \cup \{x\}, S'_2 \cup \{x\}, \dots, S'_k \cup \{x\}\}$ is a sunflower with k petals in F

Computing a Sunflower

- * Universe $U = \{1, 2, 3, \dots, n\}$
 - * Family of subsets of U : $F = \{S_1, S_2, \dots, S_m\}$
 - * No duplicates
 - * Each set in F has size d , $|F| > d! (k-1)^d$
-
- * Find a maximal set $\{S_1, S_2, \dots, S_l\}$ of disjoint sets in F
 - * Let x be an element that is in a maximum number of sets in F
 - * x is in $\geq |F|/|S|$ sets of F
 - * F' : sets in F containing x
 - * F'' : obtained from F' by deleting x (no duplicates)
 - * $|F''| > (d-1)! (k-1)^{d-1}$ as $|F|/|S| > d! (k-1)^d / d(k-1)$
 - * Recurse on F'' to find a sunflower $\{S'_1, S'_2, \dots, S'_k\}$ with k petals
 - * $\{S'_1 \cup \{x\}, S'_2 \cup \{x\}, \dots, S'_k \cup \{x\}\}$ is a sunflower with k petals in F

Sunflower Lemma (Variant)

- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family of subsets of U : $F = \{S_1, S_2, \dots, S_m\}$ (no duplicates)
- * Each set in F has size at most d
- * $|F| > d * d! (k-1)^d$



- * F has a sunflower with k petals that can be obtained in polynomial time

Hint: There exists $r \leq d$ s.t. $|F_r| > d! (k-1)^d$ where F_r is the subset of F containing sets of size $=r$

d-Hitting Set

Input:

- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family $F = \{S_1, S_2, \dots, S_m\}$ of $\leq d$ sized subsets of U
- * A non-negative integer k

Question: Does there exist $V \subseteq U$ with $|V| \leq k$ s.t for each S in F , $S \cap V \neq \emptyset$?

Some Common Hitting Sets

- * Vertex Cover
- * Feedback Vertex Set
- * Odd Cycle Transversal
- * Cluster Vertex Deletion

d-Hitting Set

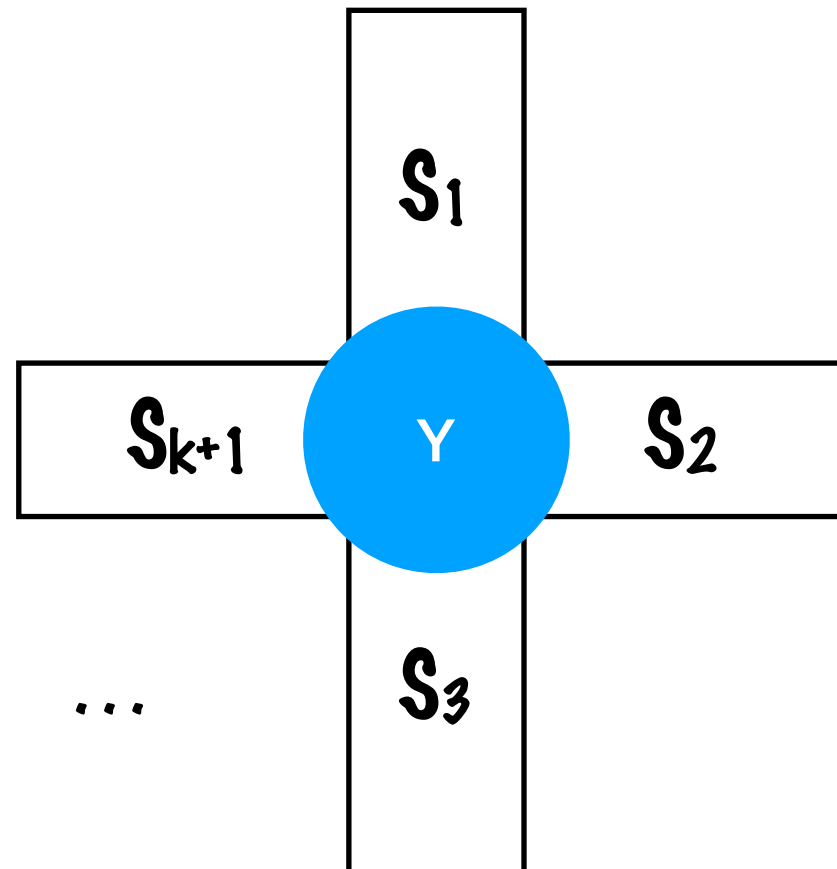
Input:

- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family $F = \{S_1, S_2, \dots, S_m\}$ of $\leq d$ sized subsets of U
- * A non-negative integer k

Question: Does there exist $V \subseteq U$ with $|V| \leq k$ s.t for each S in F , $S \cap V \neq \emptyset$?

- * If $|F| \leq d * d! k^d$ when something (d) is given as part of problem definition, treat it as constant
 - * Kernel with $d * d! k^d$ sets and $d^2 * d! k^d$ elements
- * Otherwise
 - * find a sunflower with $(k+1)$ petals

d-Hitting Set



forward dirn: Suppose there exist such a V ; a hitting set, then it clearly hits Y . reverse dirn: Hitting Y is equivalent to hitting all of

- * If Y is empty then there is no (U, F) has no hitting set of size $\leq k$
- * Otherwise,
 - * Any hitting set of size $\leq k$ has a non-empty intersection with Y
 - * Delete S_1, S_2, \dots, S_{k+1} from F and add Y to F to get resultant instance (U', F', k)

(U, F, k) is a yes-instance iff (U', F', k) is a yes-instance

d-Set Packing

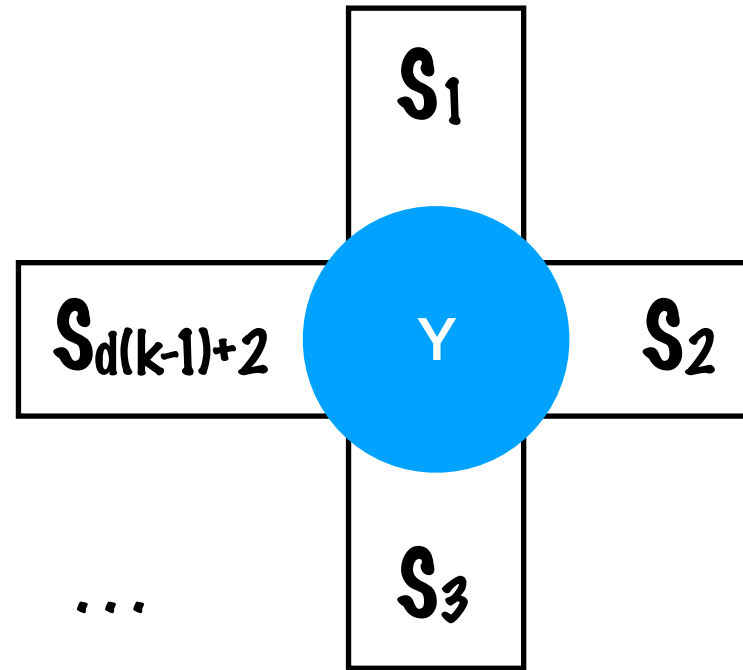
Input:

- * Universe $U = \{1, 2, 3, \dots, n\}$
- * Family $F = \{S_1, S_2, \dots, S_m\}$ of $\leq d$ sized subsets of U
- * A non-negative integer k

Question: Does there exist $F' \subseteq F$ of pairwise disjoint sets s.t. $|F'| = k$?

- * If $|F| \leq d * d! (d(k-1)+1)^d$
 - * Kernel with $d * d! (d(k-1)+1)^d$ sets and $d^2 * d! (d(k-1)+1)^d$ elements
- * Otherwise
 - * find a sunflower with $d(k-1)+2$ petals

d-Set Packing



- * Delete S_1 from F to get resultant instance (U', F', k)
- * (U, F, k) is a yes-instance iff (U', F', k) is a yes-instance

- * Reverse direction is easy

- * Forward direction

for forward dirn: Clearly if Y is empty then we already have our F' , o/w we can

- * Suppose P is a k -set packing containing S_1
- * $P \setminus \{S_1\}$ has $\leq d(k-1)$ elements (set X)
- * There is a set S_i that has no element from X for some $2 \leq i \leq d(k-1)+2$
- * $P \setminus \{S_1\} \cup \{S_i\}$ is a k -set packing not containing S_1