CS5003: Parameterized Algorithms — Given on: 6 Aug 14:00 hrs

Problem Set #1 — Due on : 18 Aug 23:55 hrs

- Submit your solutions (use LaTeX) electronically in the moodle page.
- Every algorithm must be accompanied by a proof of its correctness and an analysis of its running time.
- Collaboration is encouraged, but all write-ups must be done individually and independently. Please mention the name of your collaborator (if any).

1. Given an undirected simple graph G on n vertices and a non-negative integer k, the Vertex Cover problem is the task of determining if G has a vertex cover of size at most k and the Independent Set problem is the task of determining if G has an independent set of size at least k.
   (a) Describe an iterative compression algorithm for solving Vertex Cover in $2^k n^{\mathcal{O}(1)}$ time.
   (b) Give a polynomial-time reduction from Independent Set to Vertex Cover.
   (c) Does the above reduction and the iterative compression algorithm for solving Vertex Cover imply that Independent Set can be solved in $2^k n^{\mathcal{O}(1)}$ time where k is the size of the independent set that we are looking for. Why or why not?

2. A directed acyclic graph is a directed graph that has no directed cycle. A tournament is a directed graph in which there is exactly one arc between every pair of distinct vertices. A feedback arc set of a directed graph is a set of arcs whose deletion results in a directed acyclic graph.
   (a) Show that a directed graph is acyclic if and only if it has a topological ordering.
   (b) Show that a tournament is acyclic if and only if it has no directed triangles.
   (c) Show that an acyclic tournament has a unique topological ordering.
   (d) Show that a set of arcs is a minimal feedback arc set if and only if it is a minimal set of arcs whose reversal results in a directed acyclic graph.
   (e) Design an $3^k n^{\mathcal{O}(1)}$ time algorithm to determine if a tournament has a feedback arc set of size at most k or not.

3. In the Feedback Vertex Set problem, given an undirected graph G and a non-negative integer k, the task is to determine if G has a feedback vertex set of size at most k.
   (a) Show that if a graph on n vertices has minimum degree at least 3, then it contains a cycle of length at most $2\lceil \log n \rceil$.
   (b) Use the observation in the previous question to design an $(\log n)^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$-time algorithm for Feedback Vertex Set on undirected graphs. Is this an FPT algorithm for Feedback Vertex Set parameterized by the solution size k?
   (c) Show that Feedback Vertex Set on undirected regular graphs has a kernel with $\mathcal{O}(k)$ vertices.