CS 5003: Parameterized Algorithms

Lectures 12-13

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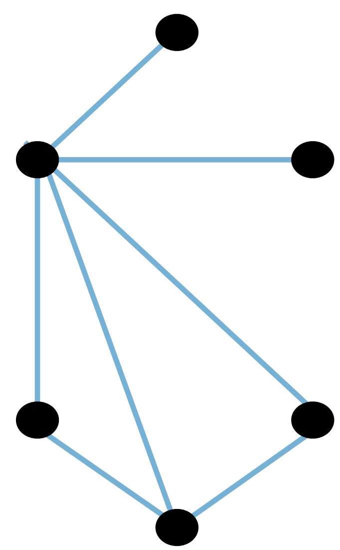
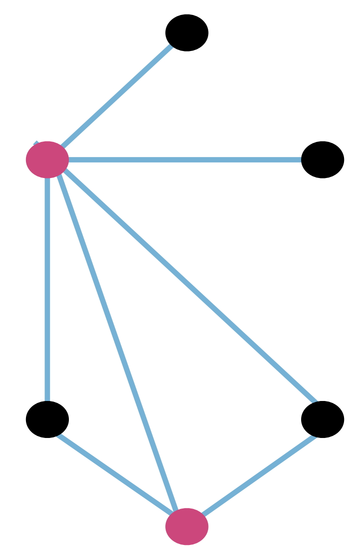
Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Vertex Cover

Vertex cover - set of vertices that has at least one endpoint of each edge





Instance: A graph G on n vertices m edges and integer k
Question: Does G have a vertex cover of size at most k?
Parameter: k

* Kernel with k^2 edges and $2k^2/3$ vertices
* $O(n^3 + 1.4656^k k^3)$ time algorithm
* 3k vertex kernel

Vertex Cover: 2k vertex kernel

Integer Linear Programming

* Given
  * A set of int-valued variables
  * A set of linear inequalities (constraints)
  * A linear cost function
* Objective is to find an assignment to the variables satisfying all constraints and maximizes/minimizes the cost function

Vertex Cover: 2k vertex kernel

Integer Linear Programming

$$\textit{minimize} \sum_{v \in V(G)} x(v)$$

$$\textit{subject to } x(v) + x(u) \geq 1 \textit{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \textit{ for each vertex } v \in V(G)$$

$$x(v) \in \mathbb{Z} \textit{ for each vertex } v \in V(G)$$

Claim: Optimum value <=k iff G has a vertex cover of size at most k

Vertex Cover: 2k vertex kernel

Integer Linear Programming

$$\textit{minimize} \sum_{v \in V(G)} x(v)$$

$$\textit{subject to } x(v) + x(u) \geq 1 \textit{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \textit{ for each vertex } v \in V(G)$$

$$x(v) \in \mathbb{Z} \textit{ for each vertex } v \in V(G)$$

Theorem: Integer Linear Programming is NP-hard

Vertex Cover: 2k vertex kernel

Linear Programming

$$\text{minimize} \sum_{v \in V(G)} x(v)$$

$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

$$\cancel{x(v) \in \mathbb{Z} \text{ for each vertex } v \in V(G)}$$

Theorem: Linear Programming is in P

Vertex Cover: 2k vertex kernel

$$\text{minimize} \sum_{v \in V(G)} x(v)$$

Linear Programming

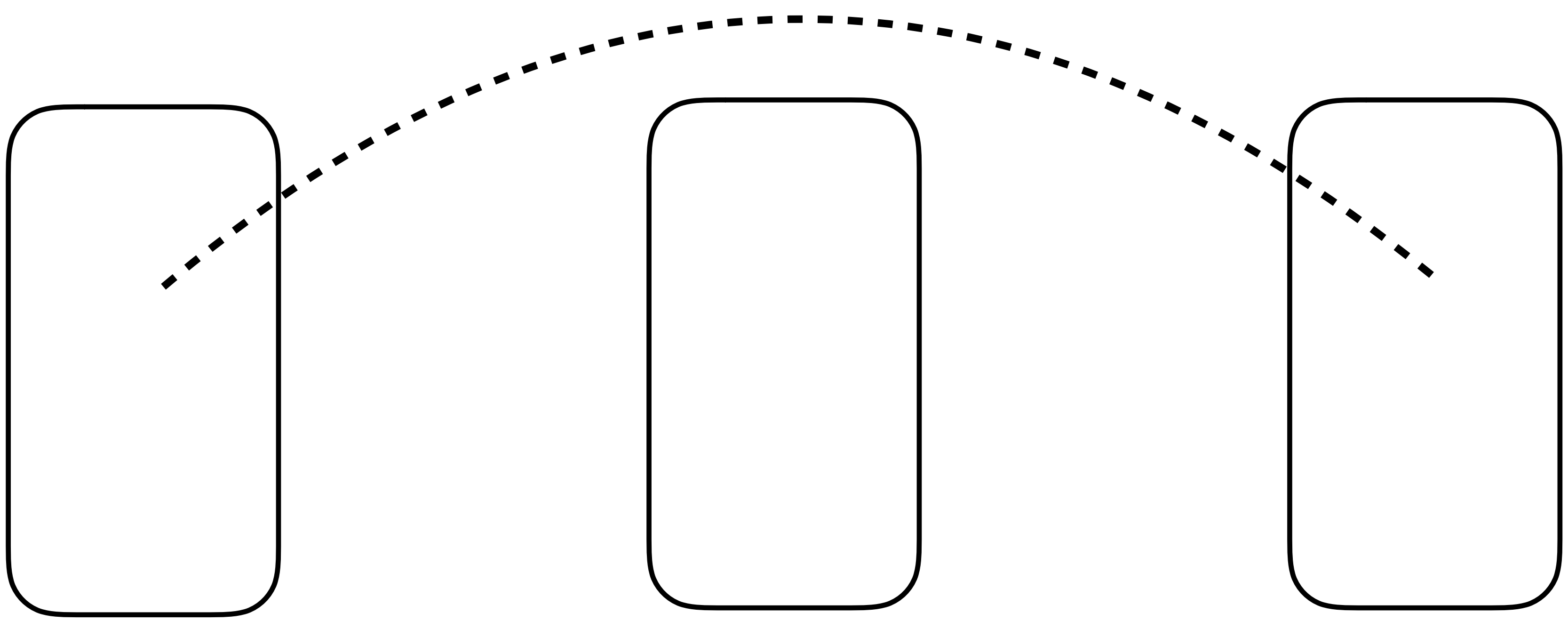
$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Optimum solution x^*

$$\sum_{v \in V(G)} x^*(v) > k \implies (G, k) \text{ is no instance}$$

Crown?



$< \frac{1}{2}$ $> \frac{1}{2}$ $= \frac{1}{2}$

Independent Set

Vertex Cover: 2k vertex kernel

Linear Programming

$$\text{minimize} \sum_{v \in V(G)} x(v)$$

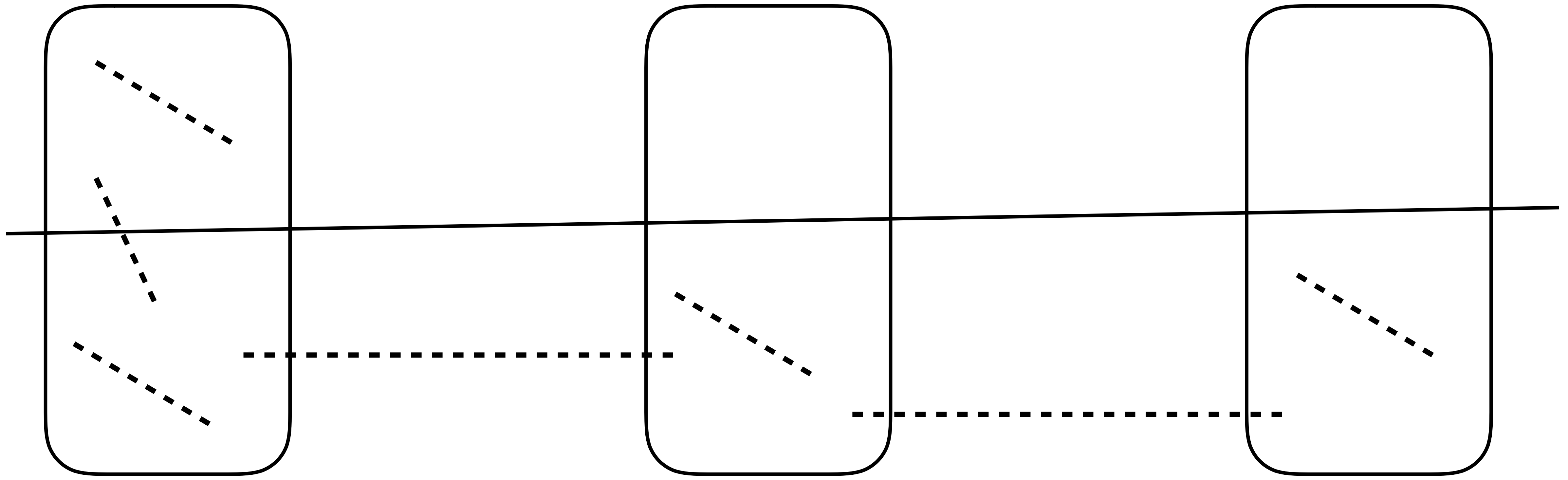
$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Optimum solution x^*

min vertex cover X

Region above the line belongs to VC, also dotted lines shown are not possib



$< \frac{1}{2}$ $> \frac{1}{2}$ $= \frac{1}{2}$

Vertex Cover: 2k vertex kernel

$$\text{minimize} \sum_{v \in V(G)} x(v)$$

Linear Programming

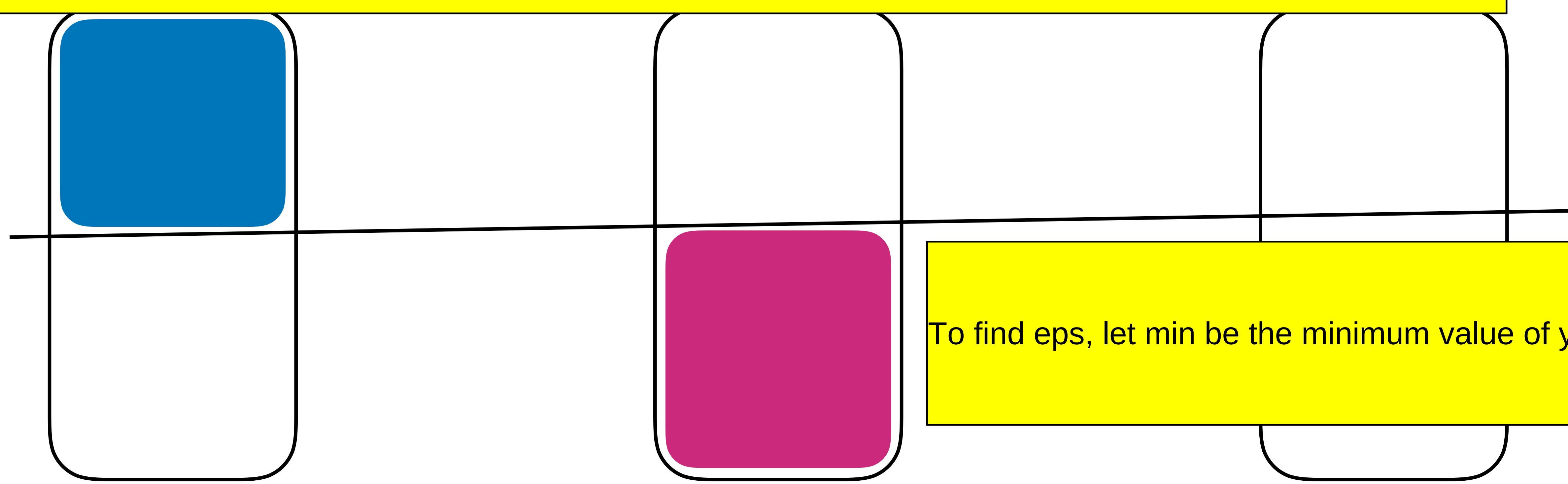
$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Optimum solution x^*

min vertex cover X

Let B = Blue, P = Pink, suppose |P| <= |B| then X' = (X \ B) U P is of size less than or equal to |X|.Now suppose



To find eps, let min be the minimum value of y(v) for

$< \frac{1}{2}$ $> \frac{1}{2}$ $= \frac{1}{2}$

Vertex Cover: 2k vertex kernel

Linear Programming

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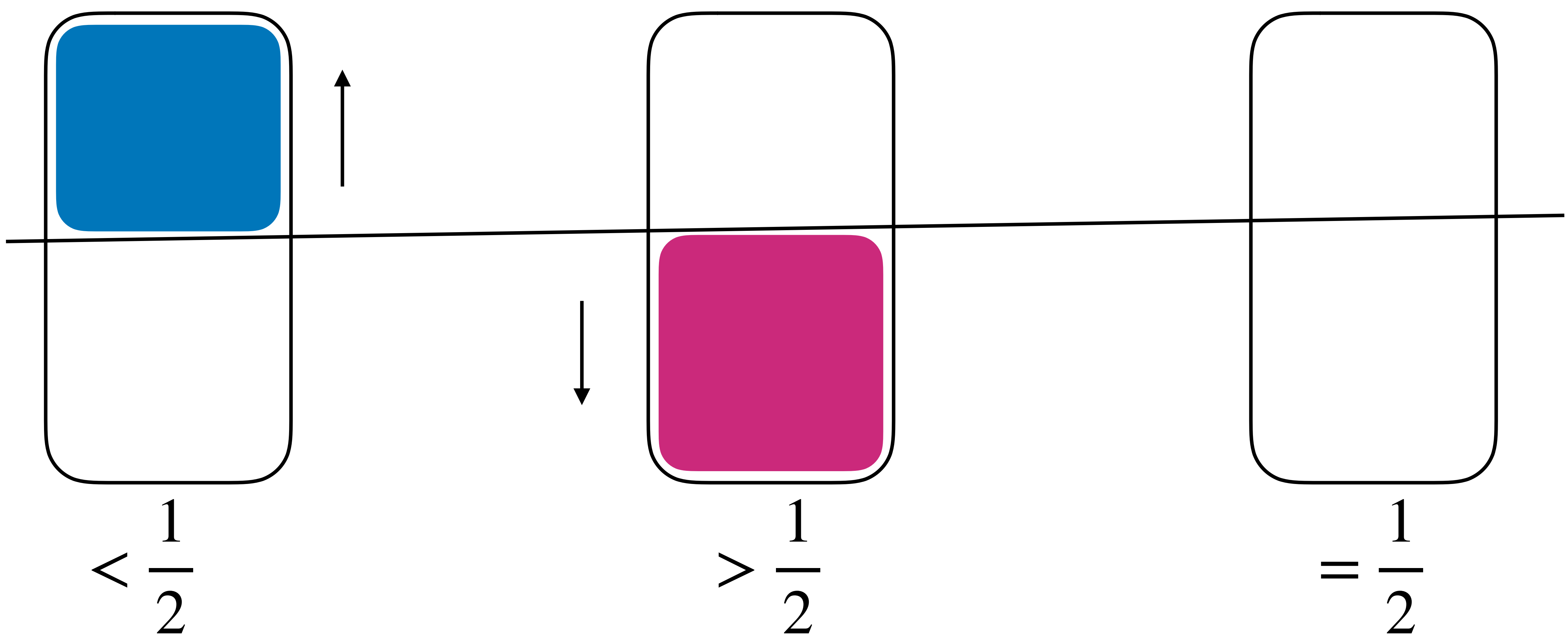
$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Optimum solution x^*

min vertex cover X



A feasible solution better than x*

Vertex Cover: 2k vertex kernel

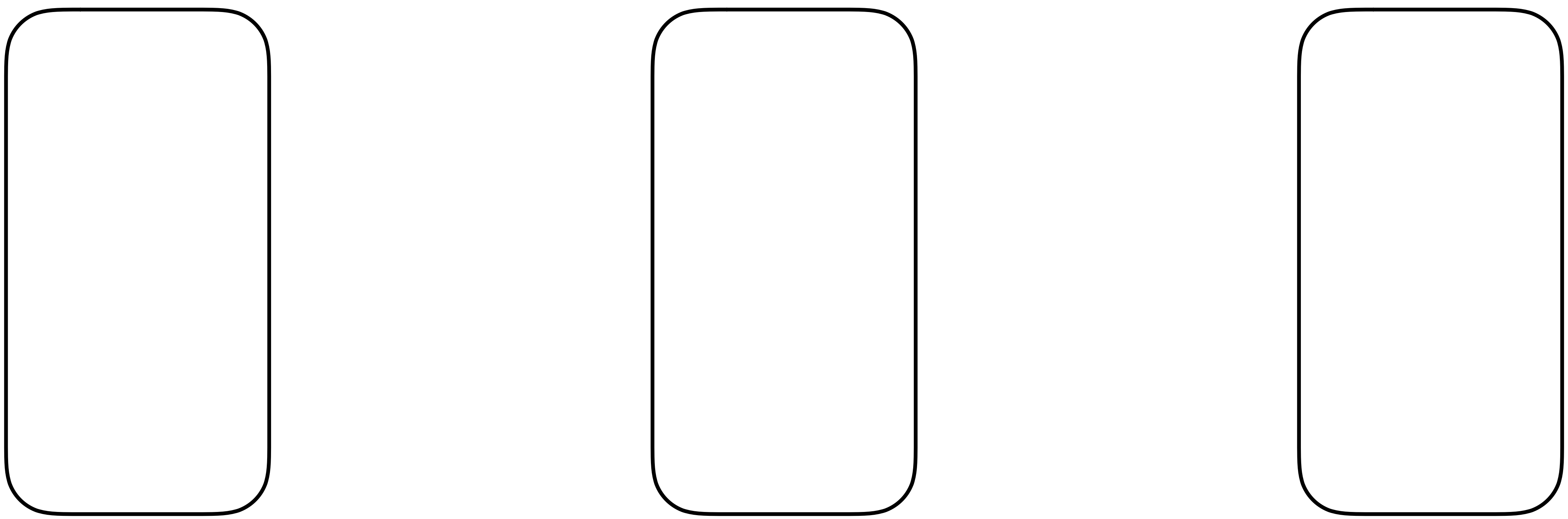
$$\text{minimize} \sum_{v \in V(G)} x(v)$$

Linear Programming

$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Optimum solution x^*



$< \frac{1}{2}$ $> \frac{1}{2}$ $= \frac{1}{2}$

There is a min vertex cover including > 1/2 set and excluding < 1/2 set

Vertex Cover: 2k vertex kernel

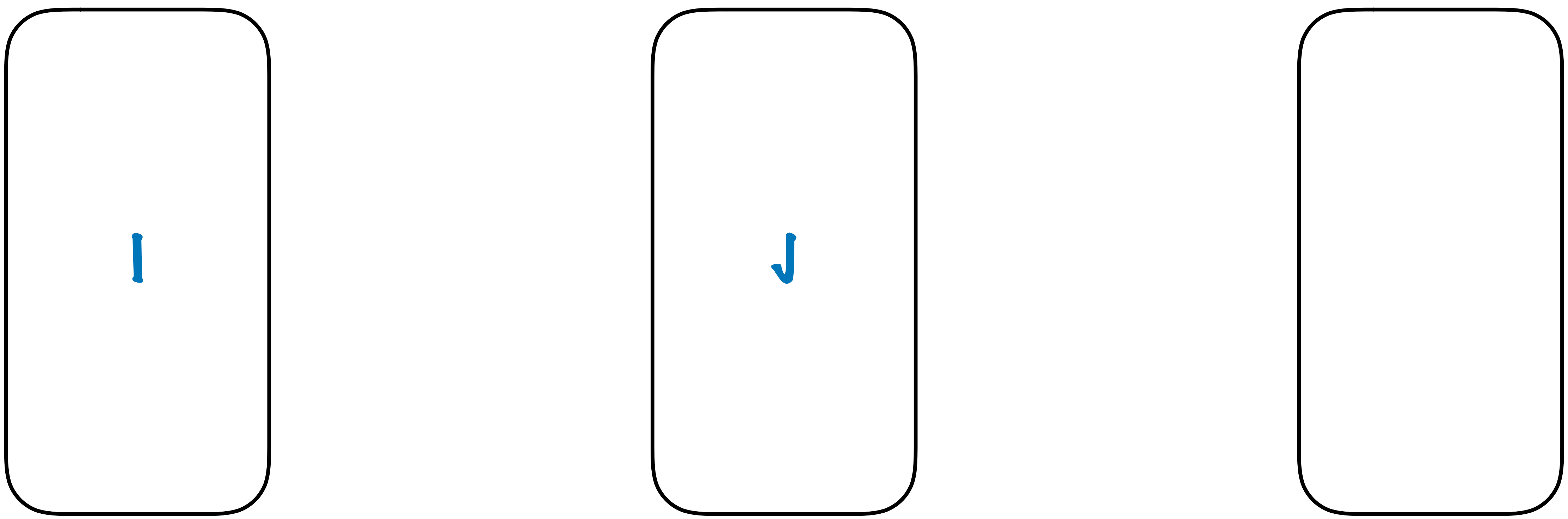
Linear Programming

$$\text{minimize} \sum_{v \in V(G)} x(v)$$

$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Optimum solution x^*



$< \frac{1}{2}$ $> \frac{1}{2}$ $= \frac{1}{2}$

(G, k) is yes-instance iff (G-(I ∪J), k-|J|) is yes-instance

Vertex Cover: 2k vertex kernel

Linear Programming

$$\text{minimize} \sum_{v \in V(G)} x(v)$$

$$\text{subject to } x(v) + x(u) \geq 1 \text{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Optimum solution x^*

$$\boxed{= \frac{1}{2}}$$

$$k \geq \sum_{v \in V(G)} x^*(v) = \frac{n}{2} \implies n \leq 2k$$