CS 5003: Parameterized Algorithms

Lectures 14-15

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Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Basic Definitions in Parameterized Complexity

* Parameterized Problem Q
    * Each instance is associated with a non-negative integer called parameter
* The size n of an instance (x, k) of Q
    * $n = |(x, k)| = |x| + k$
* Q is fixed-parameter tractable if it can be solved in $f(k)\, n^{O(1)}$ time
* A kernelization algorithm for Q is a polynomial-time algorithm that given any instance (x, k) of Q returns an instance (x', k') such that
    * $|(x', k')| \le g(k)$ and
    * (x, k) is an yes-instance of Q iff (x', k') is an yes-instance of Q

Vertex Cover

Vertex Cover (parameterized by solution size)

Instance: A graph G on n vertices m edges and integer k
Question: Does G have a vertex cover of size at most k?
Parameter: k

$f(k)\ (n+m)^{O(1)}$

Vertex Cover Above LP

Instance: A graph G on n vertices m edges and integer k
Question: Does G have a vertex cover of size at most k?
Parameter: $\lceil k - lp(G) \rceil$

$g(k-lp)\ (n+m)^{O(1)}$

Vertex Cover LP

$$\textit{minimize} \sum_{v \in V(G)} x(v)$$

$$\textit{subject to } x(v) + x(u) \geq 1 \textit{ for each edge } \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1 \textit{ for each vertex } v \in V(G)$$

Optimum solution x^*

$$\sum_{v \in V(G)} x^*(v) > k \implies (G, k) \textit{ is no instance}$$

$\downarrow$

Computable in P-time

Vertex Cover Above LP

Instance: A graph G on n vertices m edges and integer k
Question: Does G have a vertex cover of size at most k?
Parameter: $\lceil$k-lp(G)$\rceil$

Optimum solution x^*

$< \frac{1}{2}$

$> \frac{1}{2}$

$= \frac{1}{2}$

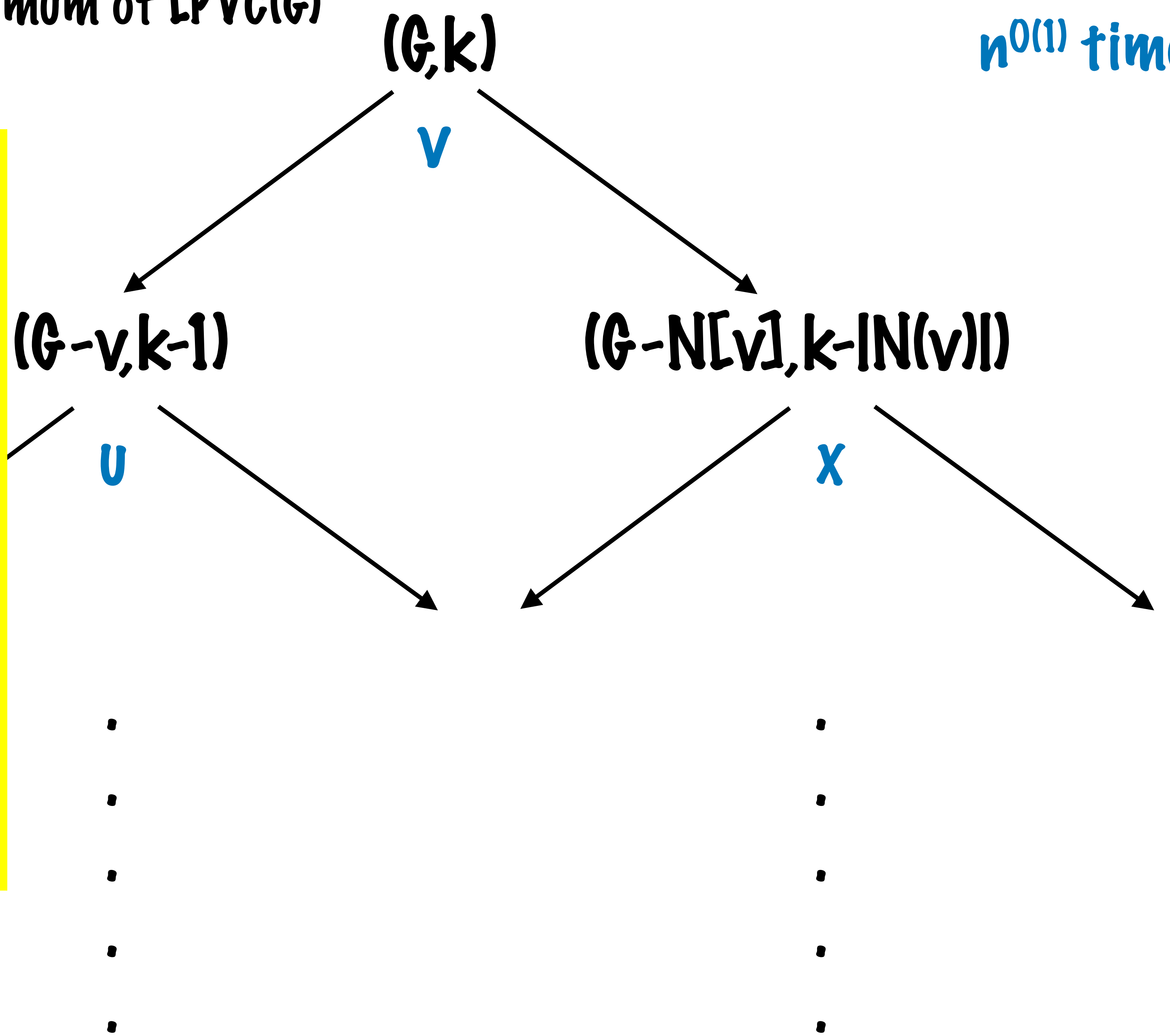
There is a min vertex cover including > 1/2 set and excluding <1/2 set

(G, k) is an yes-instance iff (G',k') is an yes-instance

Vertex Cover Above LP

* All 1/2 is the unique optimum of LPVC(G)
* k >= lpvc(G) = n/2

Clearly if all ones is the only optimum solution



$n^{O(1)}$ time

(G, k) is an yes-instance iff (G-v, k-1) or (G-N[v],k-|N(v)|) is an yes-instance

Vertex Cover Above LP

* Branch 1: v is in the vertex cover
  * $G' = G - v$
  * $k' = k - 1$
  * $lp(G') >= lp(G) - 1/2$

> * Suppose not
>   * $lp(G') < lp(G) - 1/2$
>   * $lp(G') <= lp(G) - 1$ (half-integrality)
>   * G has an optimum solution that is not all-1/2

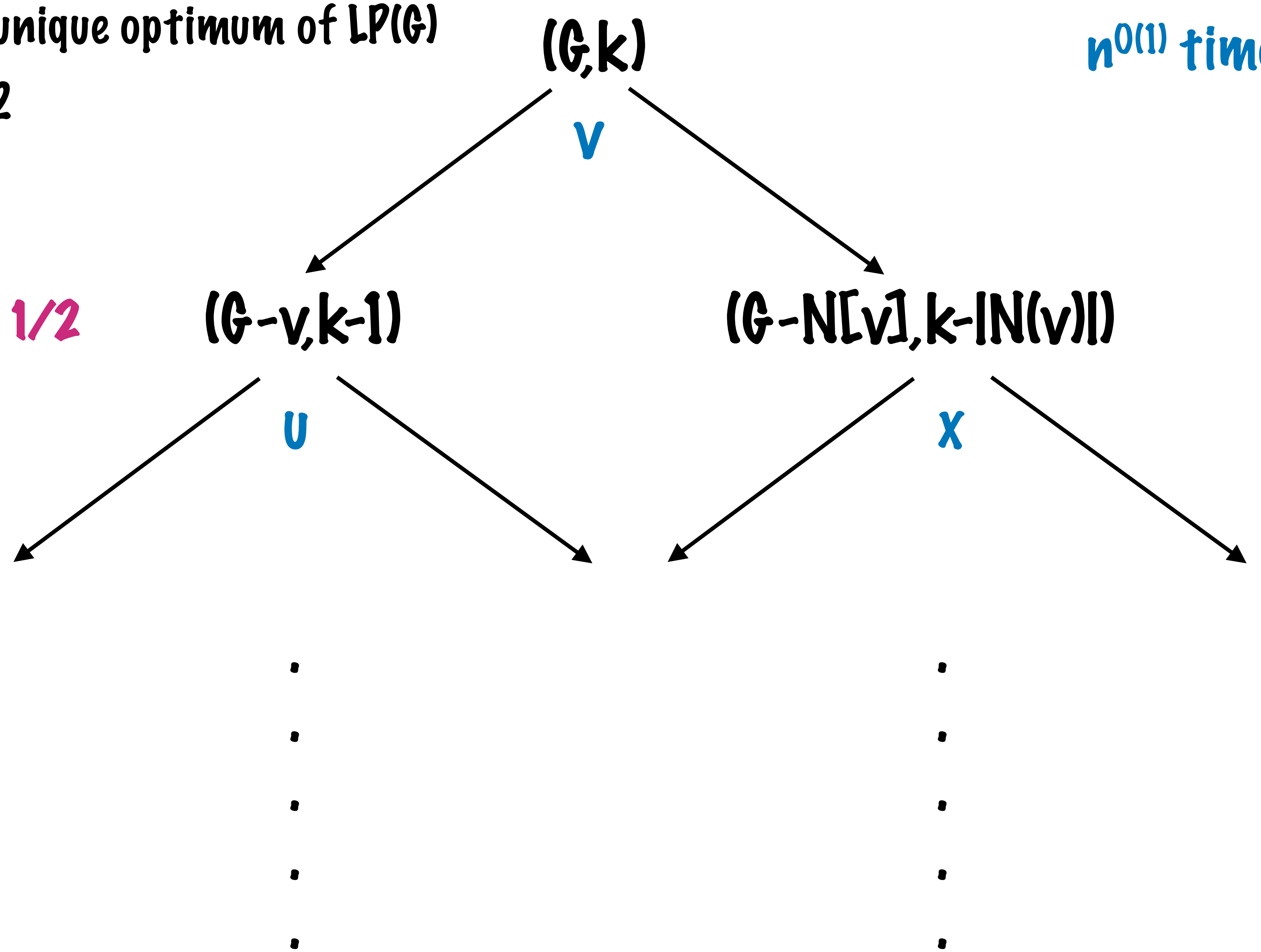
  * $k' - lp(G') <= k - 1 - lp(G) + 1/2 <= k - lp(G) - 1/2$

Vertex Cover Above LP

* All 1/2 is the unique optimum of LP(G)
* k >= lp(G) = n/2

$n^{O(1)}$ time

k-lp drops by 1/2



Apply preprocessing rules (reduction rules) at each node

Vertex Cover Above LP

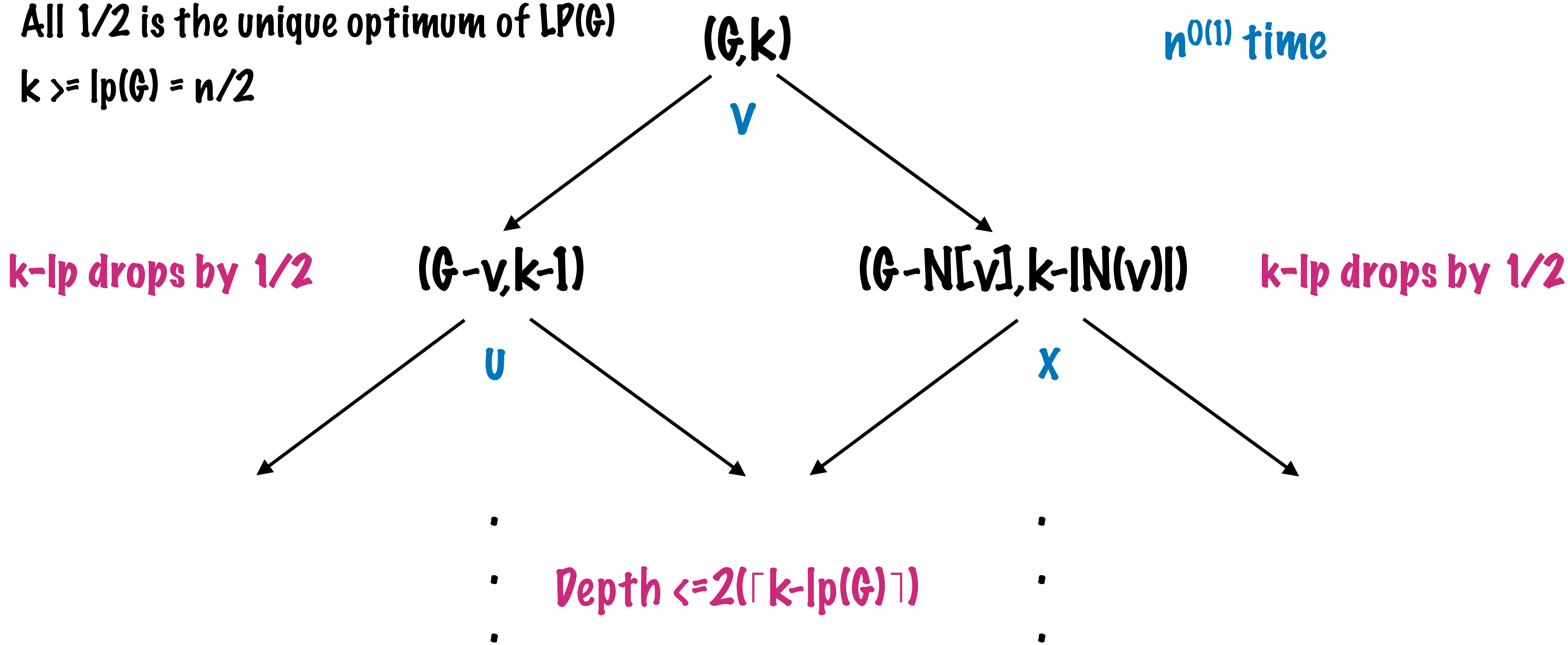
* Branch 2: v is not in the vertex cover
  * $G' = G - N[v]$
  * $k' = k - |N(v)|$
  * $lp(G') >= lp(G) - |N(v)| + 1/2$

    * Suppose not
      * $lp(G') < lp(G) - |N(v)| + 1/2$
      * $lp(G') <= lp(G) - |N(v)|$ (half-integrality)
      * G has an optimum solution that is not all-1/2

  * $k' - lp(G') <= k - |N(v)| - lp(G) + |N(v)| - 1/2 <= k - lp(G) - 1/2$

Vertex Cover Above LP

* All 1/2 is the unique optimum of LP(G)
* k >= lp(G) = n/2



* $4^{(\lceil k-lp(G)\rceil)} n^{O(1)}$ time algorithm
* Apply reduction rules at each node (Reduction Rules do not increase k-lp)
* At leaves, what is the computation?

Solving Vertex Cover LP

LPVC(G)

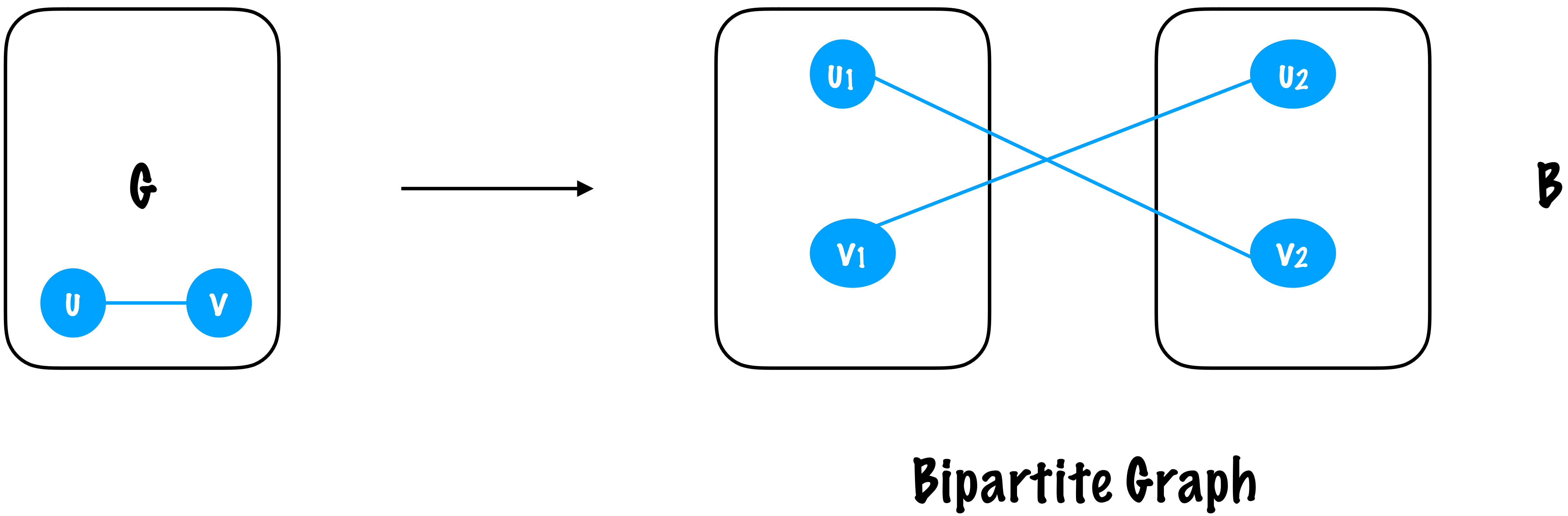
$$minimize \sum_{v \in V(G)} x(v)$$

$$subject\ to\ x(v) + x(u) \geq 1\ for\ each\ edge\ \{u, v\} \in E(G)$$

$$0 \leq x(v) \leq 1\ for\ each\ vertex\ v \in V(G)$$

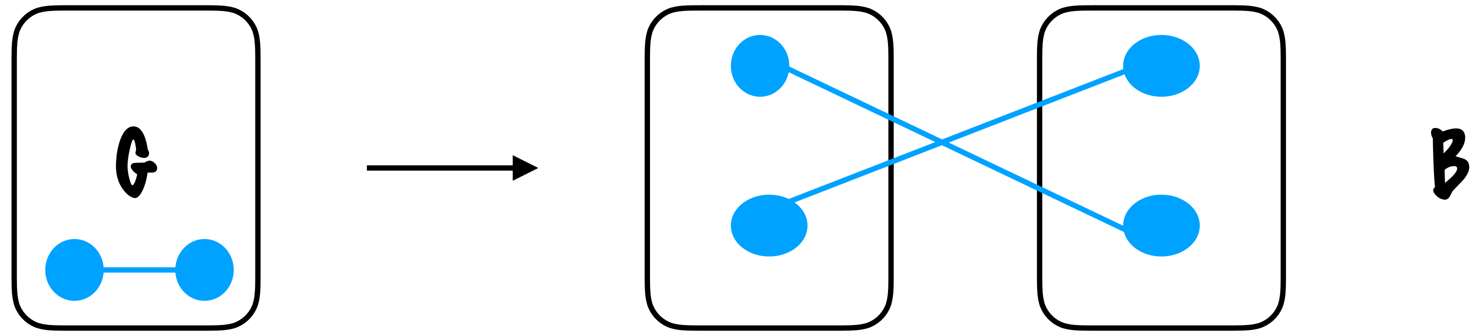
Theorem: There is an optimum solution to LPVC(G) that assigns 0, 1 or 1/2 to each of the variables

Solving Vertex Cover LP

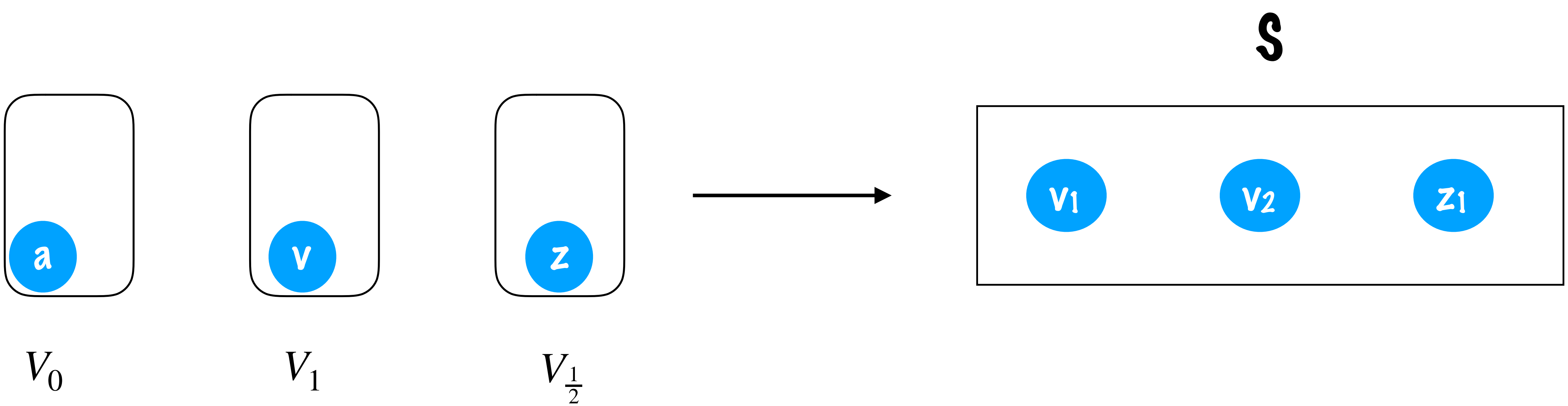


Theorem: Optimum solution to LPVC(G) = |Min Vertex Cover of B|

Solving Vertex Cover LP



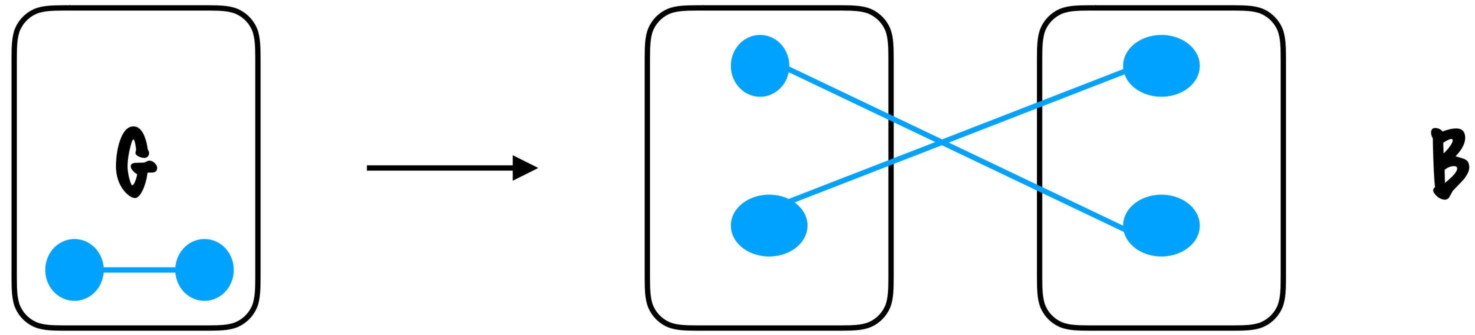
Feasible half integral solution x* to LPVC(G)



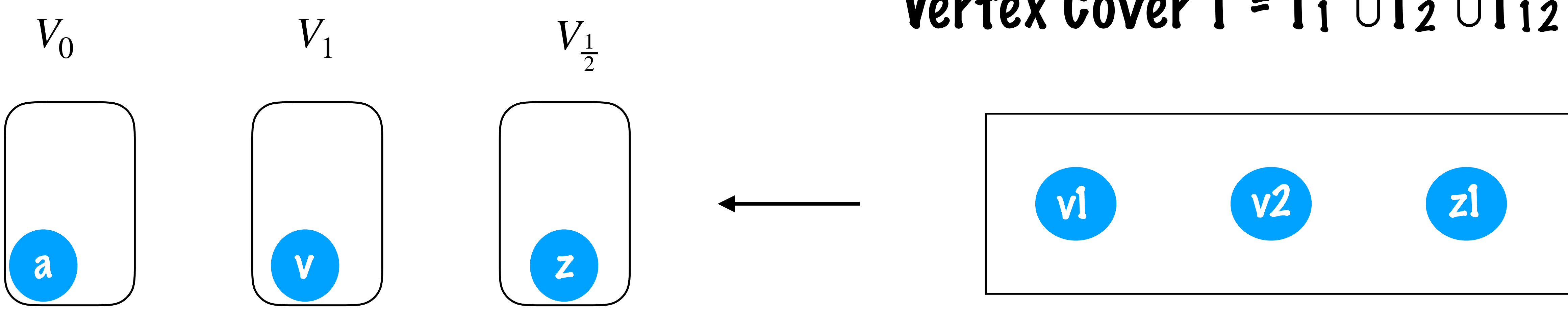
$$|S| = \sum_{v \in V_1} 2 + \sum_{v \in V_{\frac{1}{2}}} 1 = 2\Big(\sum_{v \in V_1} 1 + \sum_{v \in V_{\frac{1}{2}}} \frac{1}{2}\Big) = \boxed{2} \sum_{v \in V(G)} x^*(v)$$

And thus size of minimum VC is <= 2 * opt value.

Solving Vertex Cover LP



Vertex Cover $T = T_1 \cup T_2 \cup T_{12}$ of B



For any edge (u, v). Suppose both of them are not assigned value one and u_1 \in T_1. That means as there is as well an edge (u2, v2) we

$$\sum_{v \in V(G)} y^*(v) = \sum_{v: v_1, v_2 \in T} 1 + \sum_{v: |\{v_1, v_2\} \cap T| = 1} \frac{1}{2} = \frac{1}{2}|T_1| + \frac{1}{2}|T_2| + \frac{1}{2}|T_{12}| = \frac{1}{2}|T|$$

Thus 2 * opt is less than equal to |MVC|.