Problem Sheet 1

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Unless mentioned, assume given graph to be $G = (V, E)$

1. 2

1.1. (a)

1.1.1. Condition is neccessary

Assume that graph (V, E) is acyclic.

Claim: $\exists v \in V; \operatorname{Indeg} v = 0$

Proof: Assume not, that implies for each vertex $\operatorname{Indeg} v > 0$. Thus start from any arbitrary vertex v and go to vertex u s.t. $(u, v) \in E$. Repeat the same procedure from u. At end we must repeat a already seen vertex by pigeon hole principle, thus giving us a cycle $\Rightarrow\Leftarrow$.

Thus pick that vertex with zero Indeg (remove it from the graph) and add it as a first vertex for our topological ordering. Repeat the same procedure with the resultant graph, and result will be a valid topological ordering as for each edge (u, v), u is coming before v.

1.1.2. Condition is sufficient

Assume that graph (V, E) has a topological ordering $(t_1, t_2, \ldots, t_{|V|})$.

For a cycle to exist there must be an edge (t_j, t_i) such that $j > i$ which is not possible.

1.2. (b)

First direction follows trivially.

For other direction, consider shortest cycle $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{|S|}}\}$ (obviously $|S| > 3$), then, either their is an edge (v_{i_1}, v_{i_3}) or (v_{i_3}, v_{i_1}). In former case we get a shorter cycle $v_{i_1}, v_{i_3}, \ldots, v_{i_{|S|}}$ whereas in later we get a triangle $v_{i_1}, v_{i_2}, v_{i_3}$ both of which are not possible.

1.3. (c)

Consider two topological orderings $v_{i_1}, v_{i_2}, \ldots, v_{i_{|V|}}$ and $v_{i'_1}, v_{i'_2}, \ldots, v_{i'_{|V|}}$

Let the first point of difference occur at index k, i.e. $v_{i_1} = v_{i'_1}, v_{i_2} = v_{i'_2}, \ldots, v_{i_{k-1}} = v_{i'_{k-1}}$ and $v_{i_k} \neq v_{i'_k}$. Thus for the first topological sorting, there is an edge $(v_{i_k}, v_{i'_k}) \in E$ whereas in second there is $(v_{k'}, v_k)$ which is absurd.

1.4. (d)

1.4.1. Condition is neccessary

Assume that the set of arcs form a minimal feedback arc set, that implies, removing them we get a DAG which has a topological ordering $v_{i_1}, v_{i_2}, \ldots, v_{i_{|V|}}$. Since our feedback arc set is minimal, that implies adding any edge of it we get a cycle and hence its orientation is backward that is it is of the form (v_{i_j}, v_{i_k}) s.t. $j > k$. Thus reversing these edges will give us a DAG as it will have the same topological ordering. To prove that it is minimal, .