CS 5003: Parameterized Algorithms

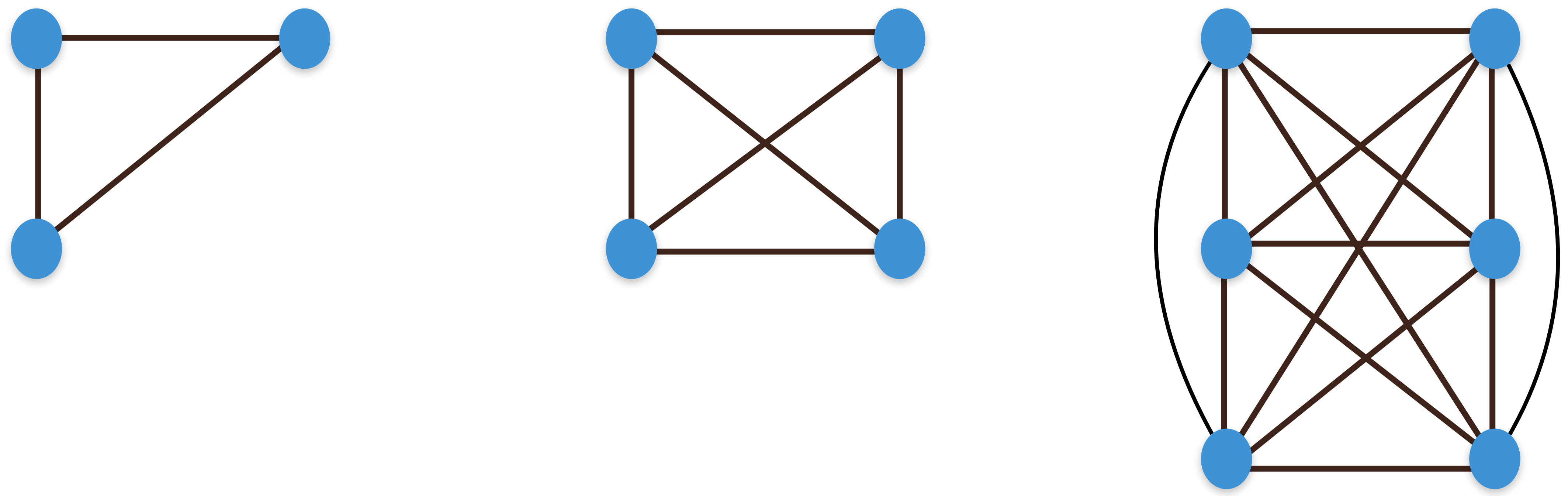
Lectures 24-25

Krithika Ramaswamy

IIT Palakkad

References: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

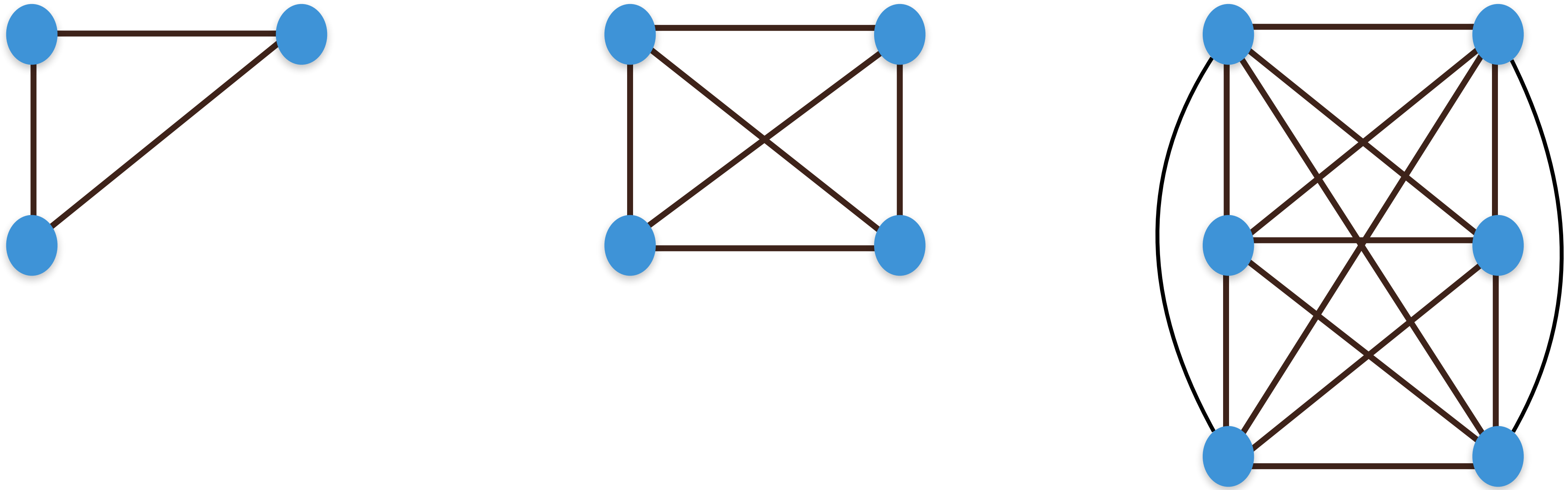
Cluster Graphs



A graph in which each component is a complete graph

d-Cluster Graphs

d-clustering is NP hard as finding maximum clique can be reduced

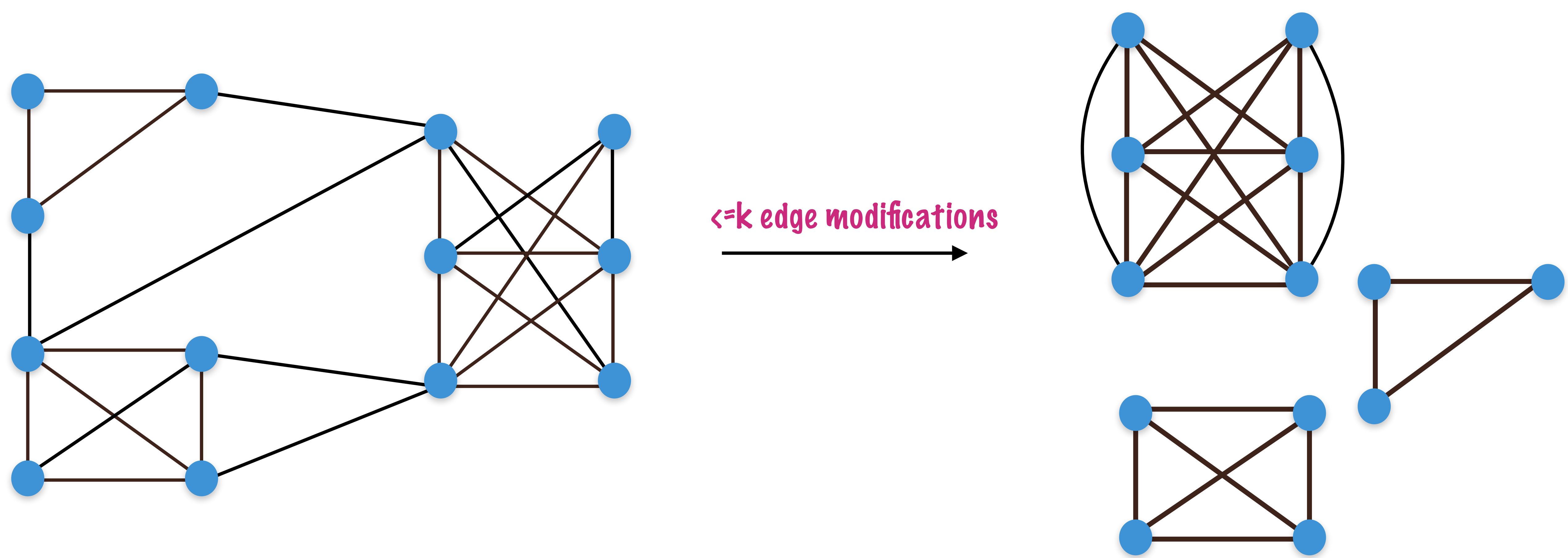


A graph with d components each of which is a complete graph

Clustering

Cluster Editing

<u>Instance:</u> An undirected graph G and an integer k
<u>Question:</u> Does there exist a set of at most k edge additions and deletions to G so that the resulting graph is a cluster graph?
<u>Parameter:</u> k



cluster graph

Cluster Editing

Induced P3 = u-v-w, st edge u-w is not there.Forward dirn is easy.Reverse dirn: Assume for the

Lemma: A graph is a cluster graph iff it has no P_3 as an induced subgraph

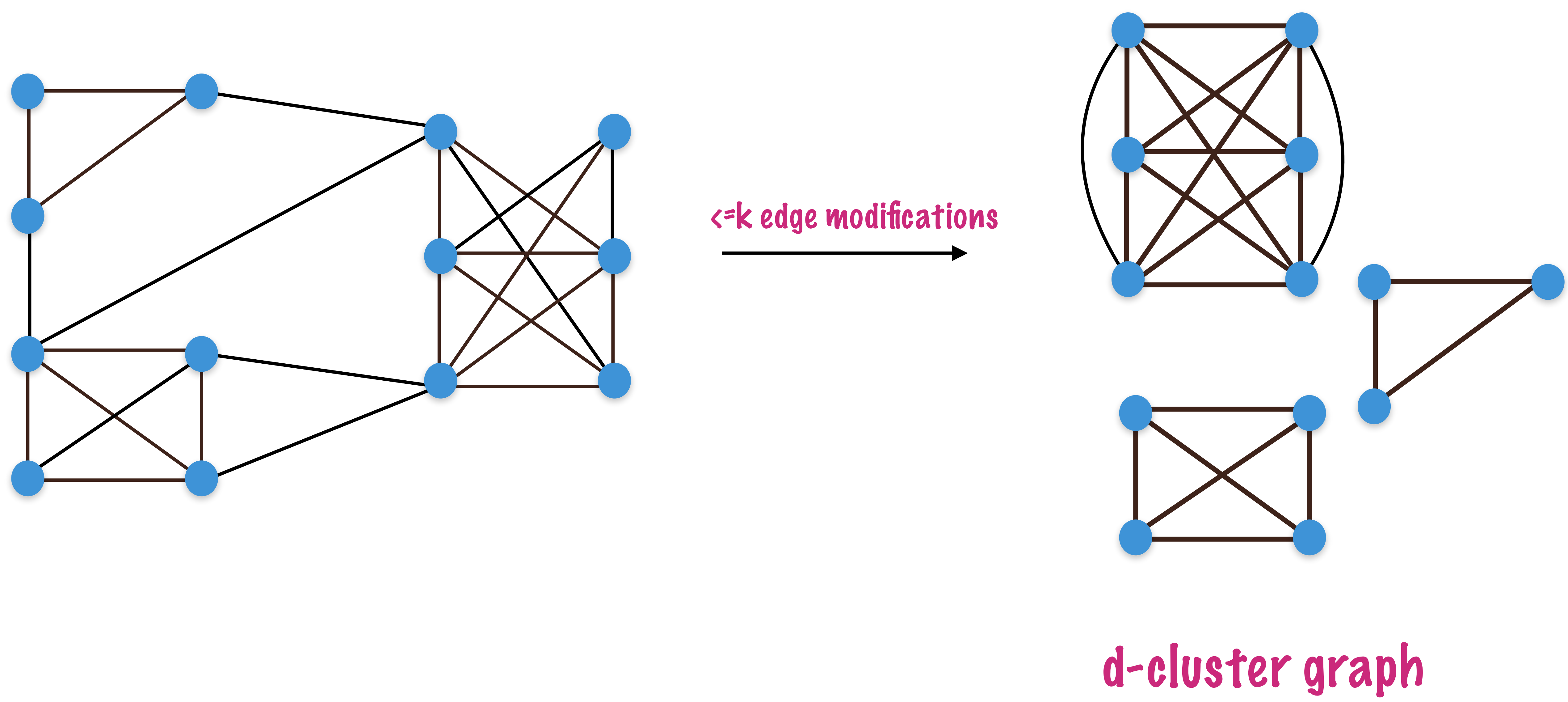
Algorithm_Cluster_Editing(G,k)

* If G is a cluster graph declare that (G,k) is yes-instance
* If k<=0 return, otherwise, find a P_3 = (w, u, v)
* **Branch 1:** Delete {u, v}, decrease k by 1, set {u, v} as forbidden. Recurse on resultant instance.
* **Branch 2:** Delete {u, w}, decrease k by 1, set {u, w} as forbidden, {v, w} as forbidden, {u, v} as permanent. Recurse on resultant instance.
* **Branch 3:** Add {w, v}, decrease k by 1, set {u, v}, {u, w} and {v, w} as permanent. Recurse on resultant instance.

$3^k n^{O(1)}$ algorithm

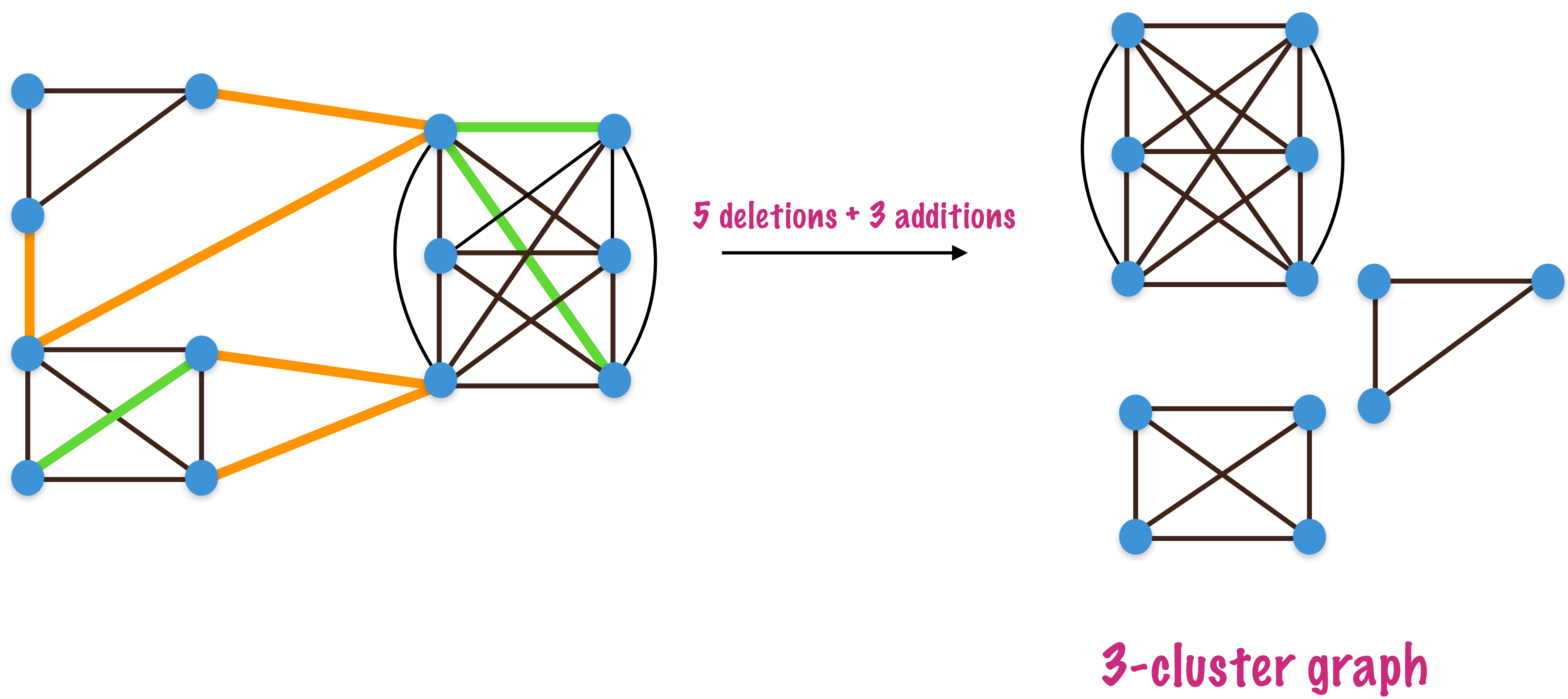
d-Clustering

<u>Instance:</u> An undirected graph G and an integer k
<u>Question:</u> Does there exist a set of at most k edge additions and deletions to G so that the resulting graph is a d-cluster graph?
<u>Parameter:</u> k



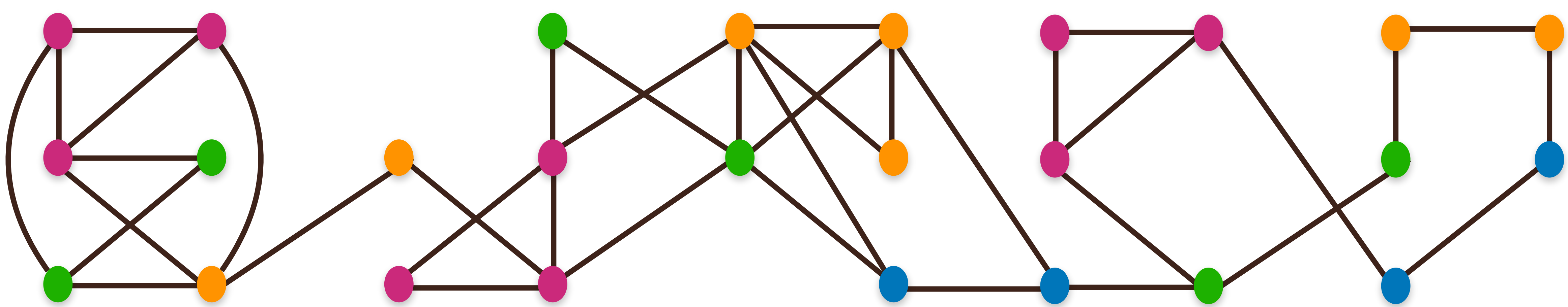
3-Clustering

Instance: An undirected graph G and an integer k
Question: Does there exist a set of at most k edge additions and deletions to G so that the resulting graph is a 3-cluster graph?
Parameter: k



Chromatic Coding Algorithm

* Randomly color the vertices of G using $q = \lceil (8k)^{.5} \rceil$ colours

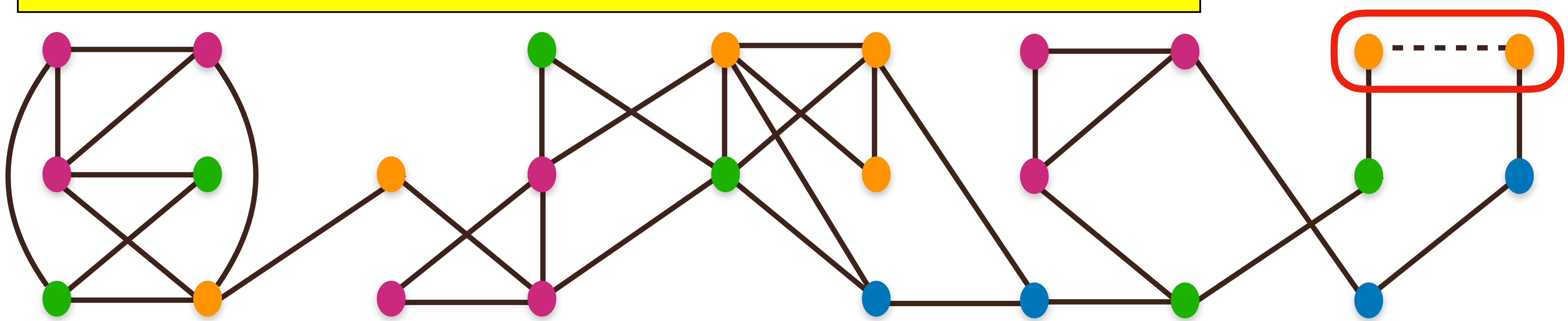


* Focus on finding a solution in which each edge is properly colored

Chromatic Coding Algorithm

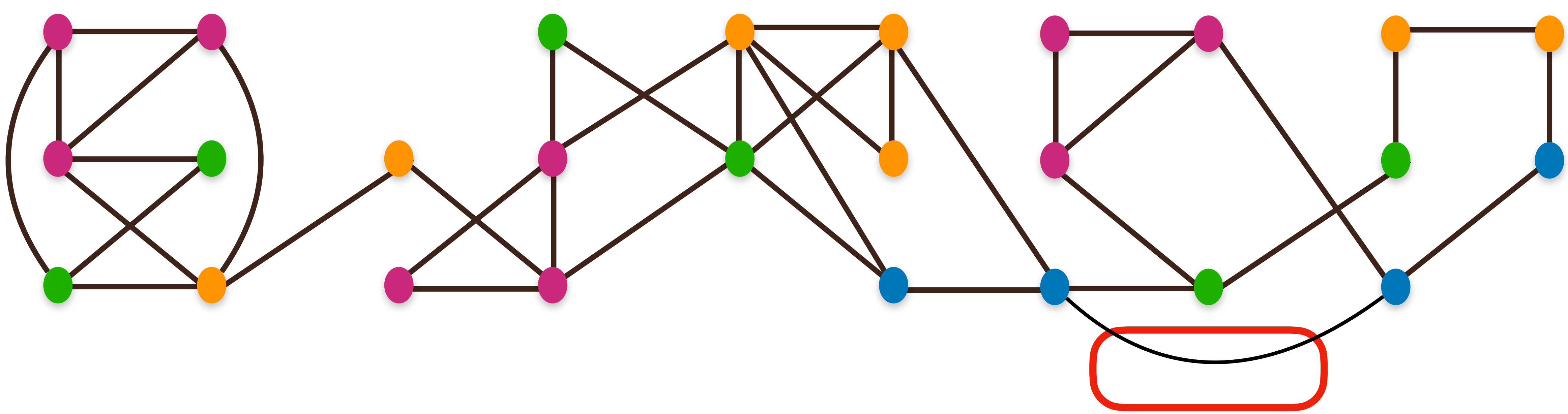
- Randomly color the vertices of G using $q = \lceil (8k)^{.5} \rceil$ colours
- Focus on finding a solution in which each edge is properly colored

edge is properly colored if its end points have different colors.



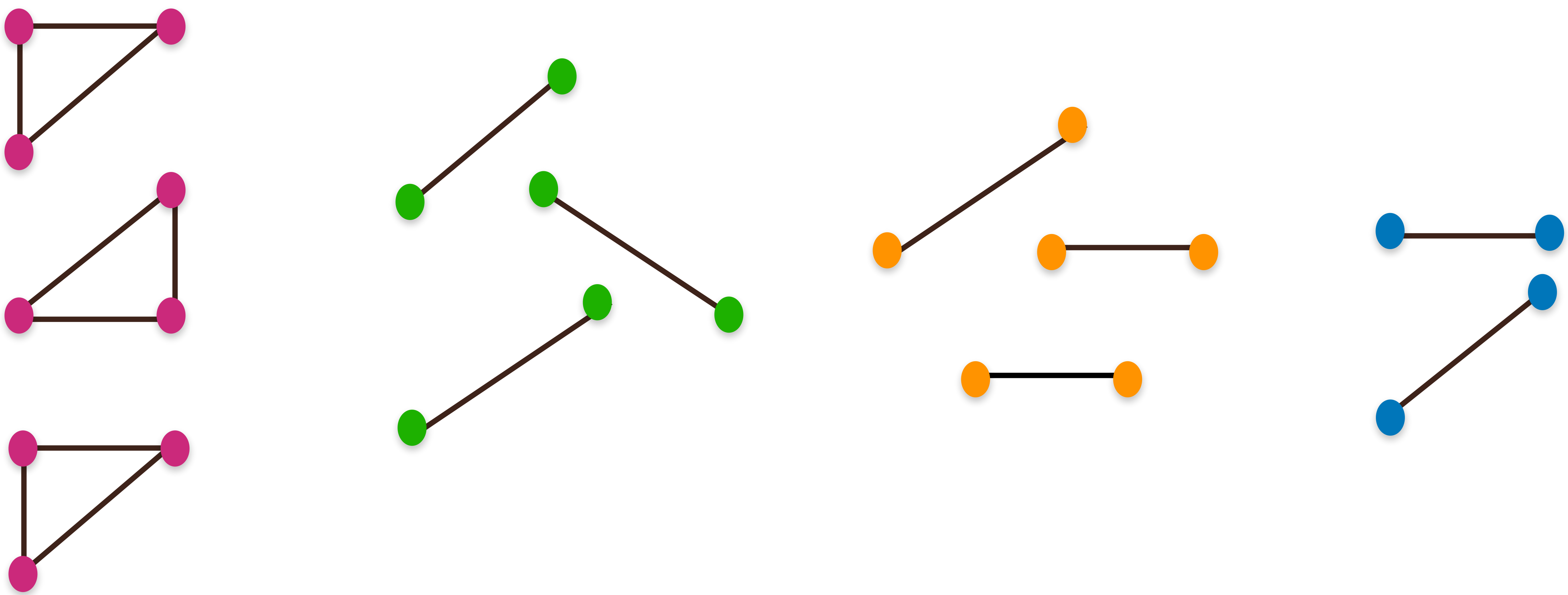
Chromatic Coding Algorithm

* Randomly color the vertices of G using $q = \lceil (8k)^{.5} \rceil$ colours
* Focus on finding a solution in which each edge is properly colored



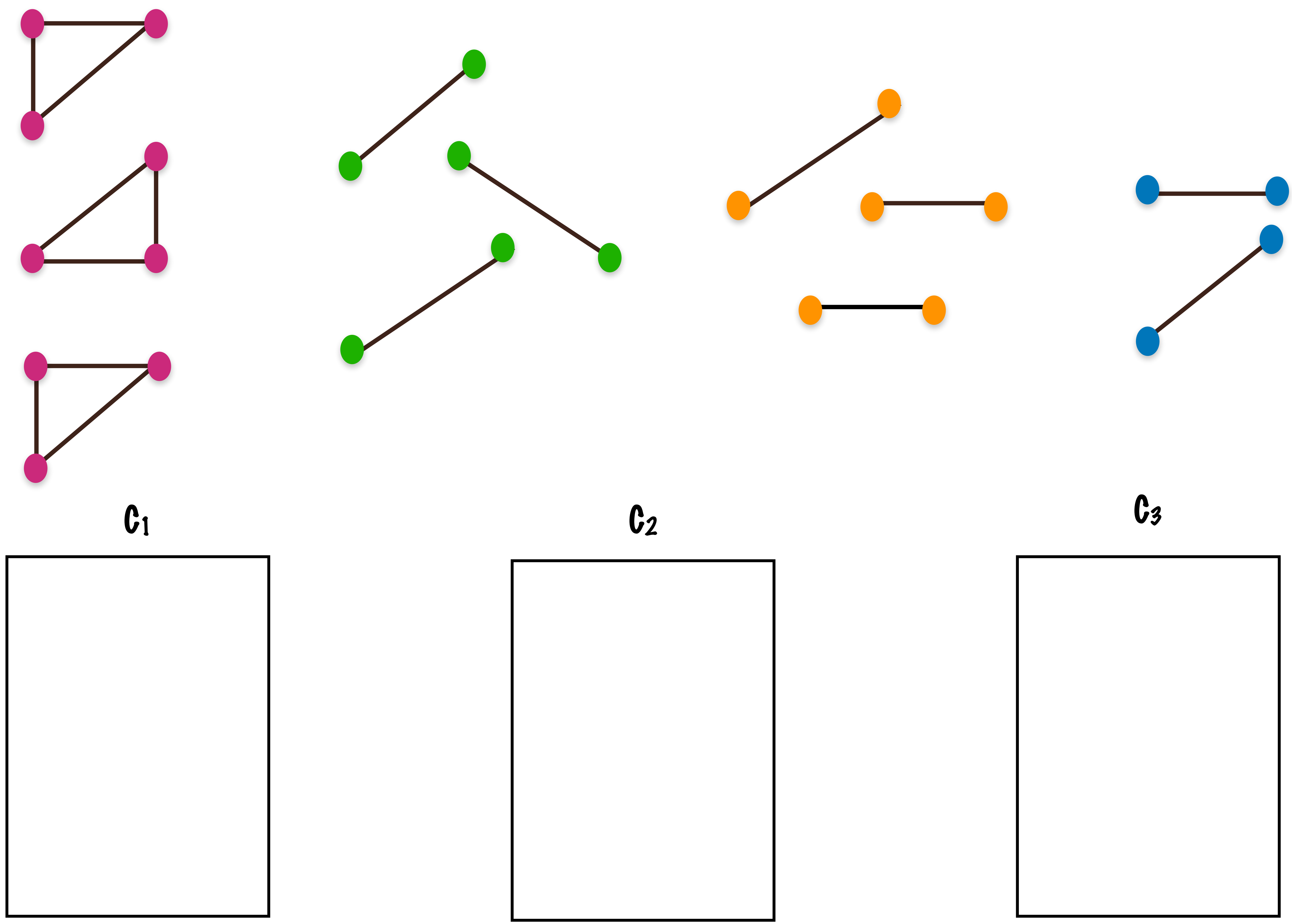
Chromatic Coding Algorithm

* Randomly color the vertices of G using $q = \lceil (8k)^{.5} \rceil$ colours
* Focus on finding a solution in which each edge is properly colored
  * Colorful solution

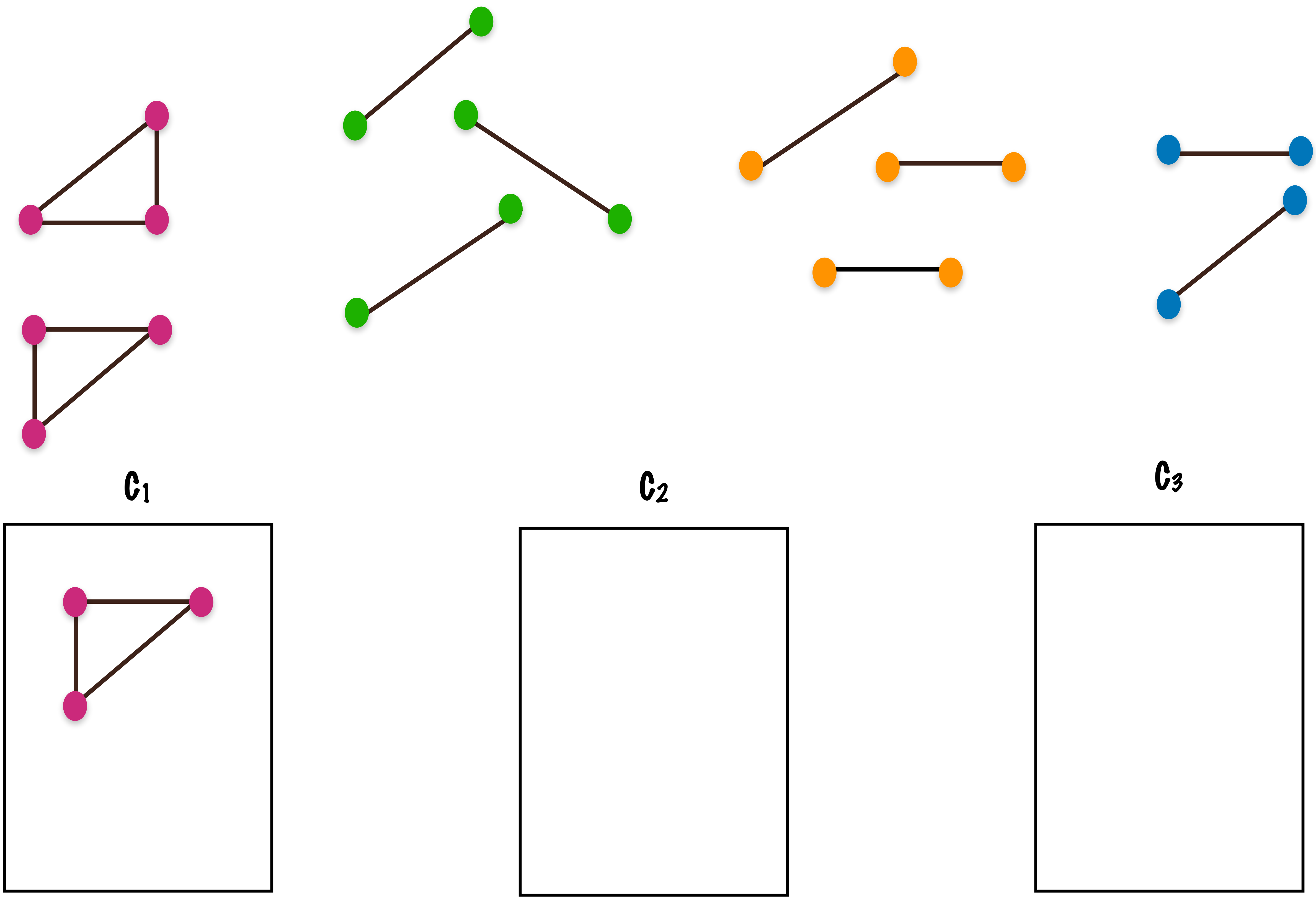


* Each colour class is a cluster graph with at most d components
  * Otherwise, no colorful solution

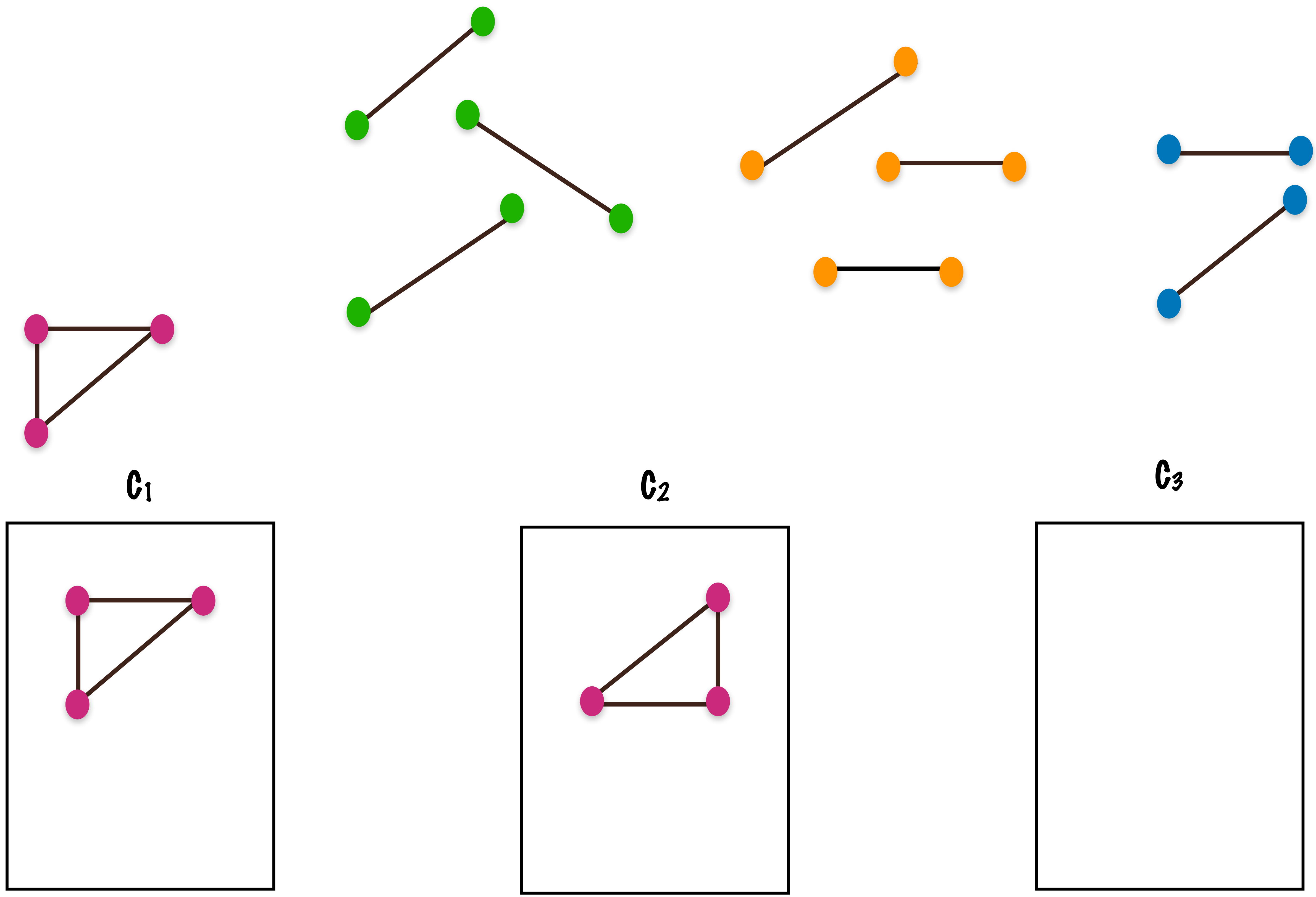
Chromatic Coding Algorithm



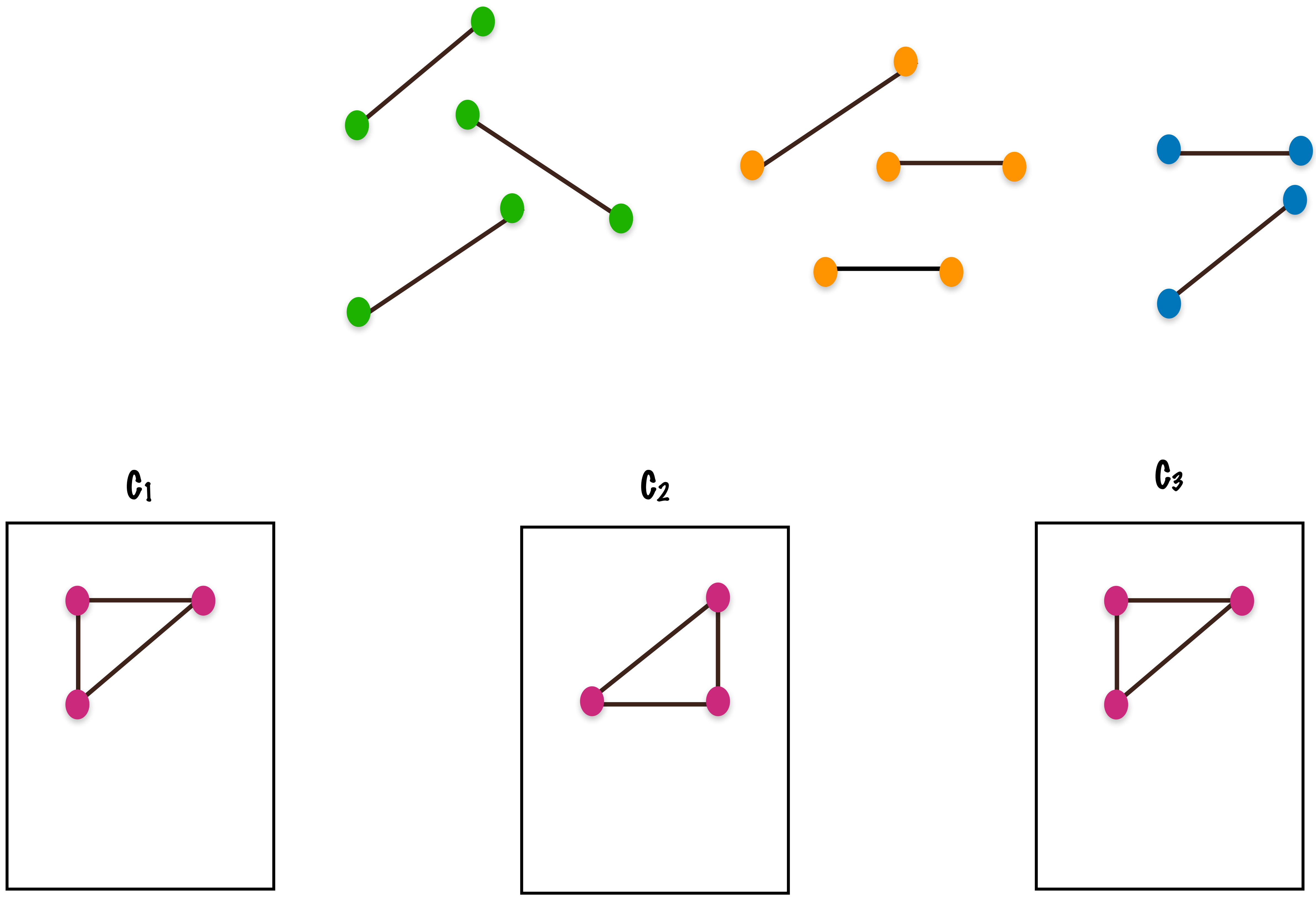
Chromatic Coding Algorithm



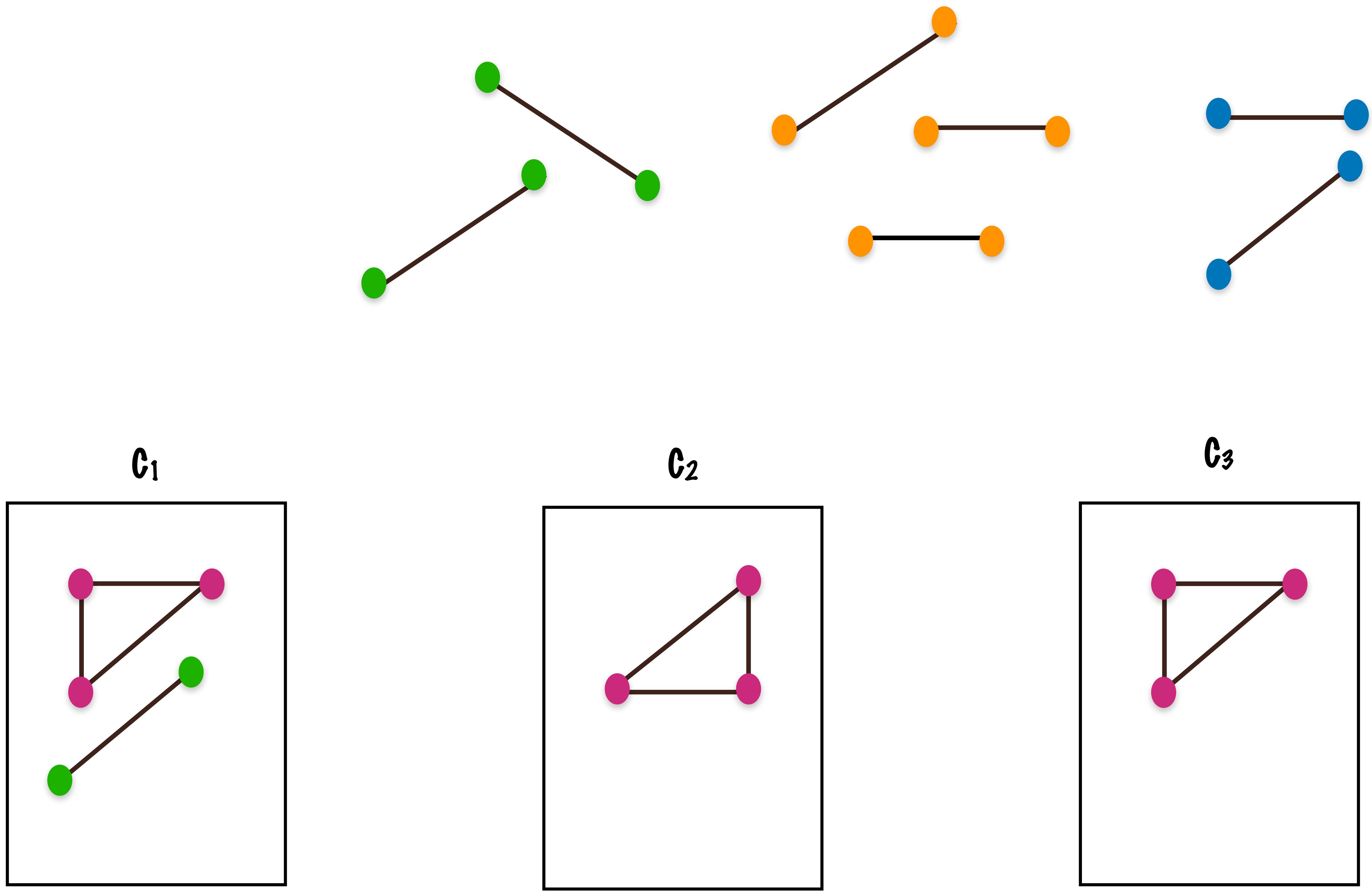
Chromatic Coding Algorithm



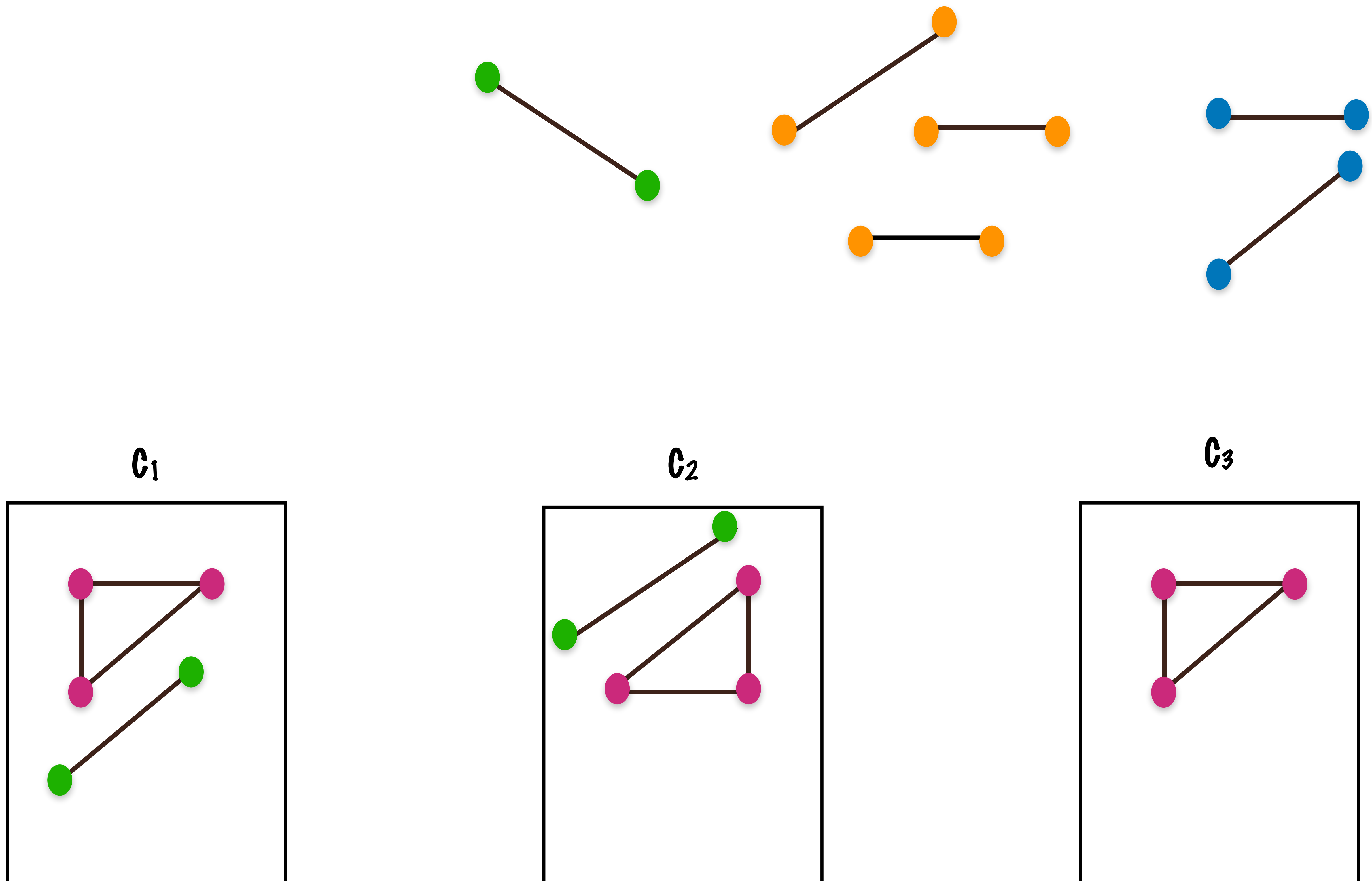
Chromatic Coding Algorithm



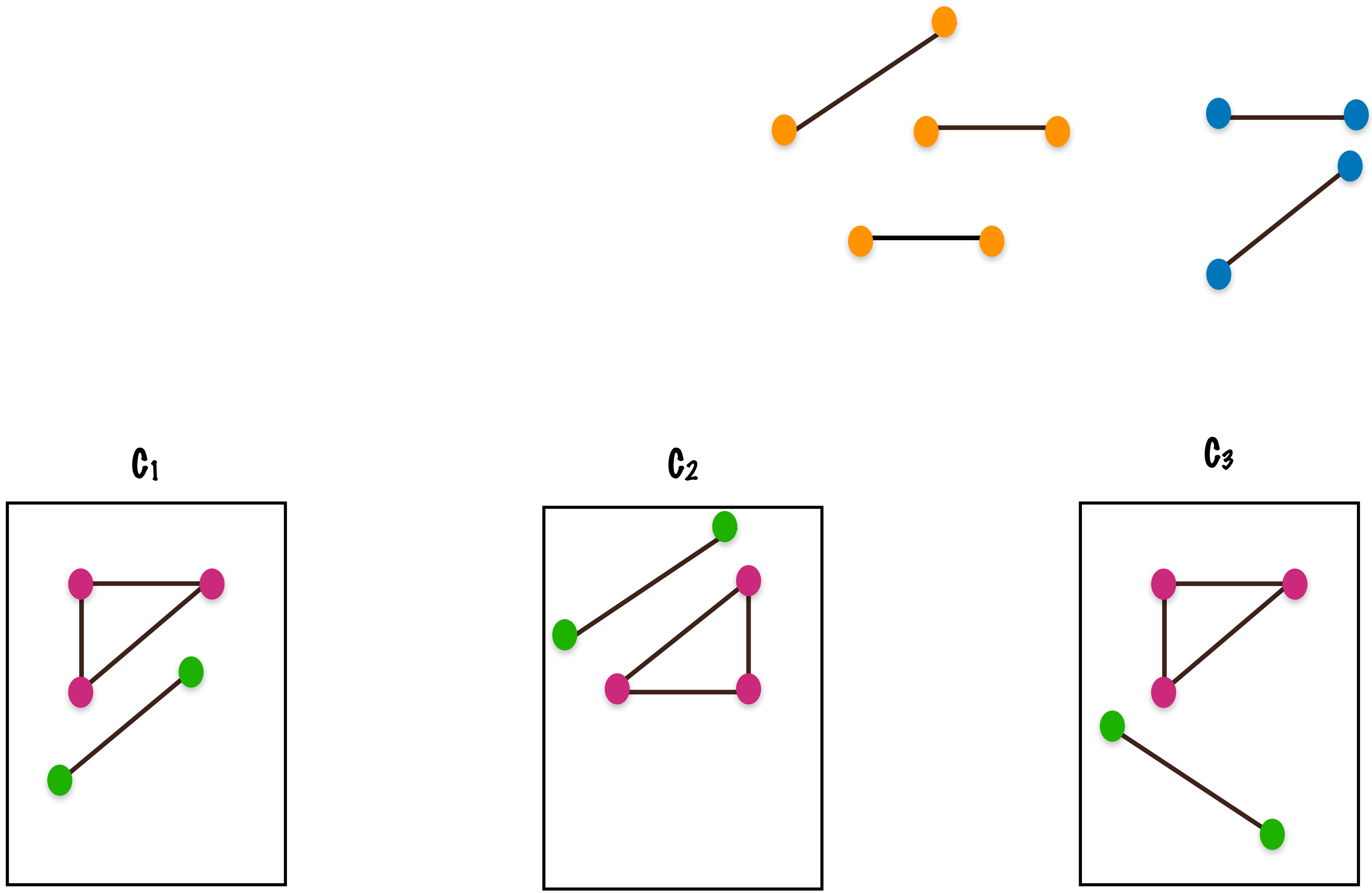
Chromatic Coding Algorithm



Chromatic Coding Algorithm



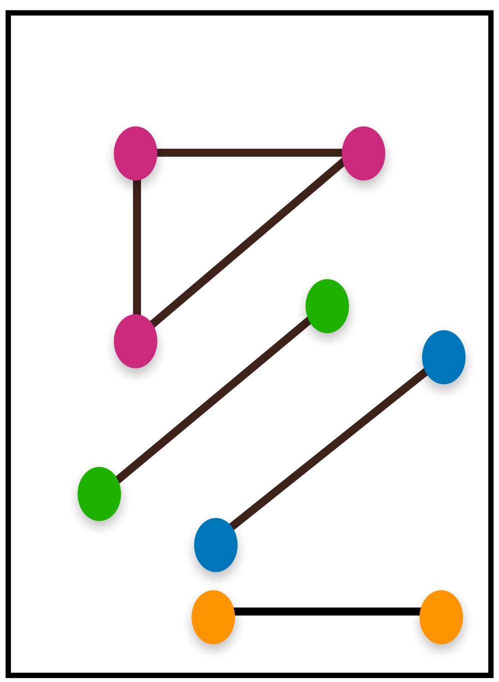
Chromatic Coding Algorithm



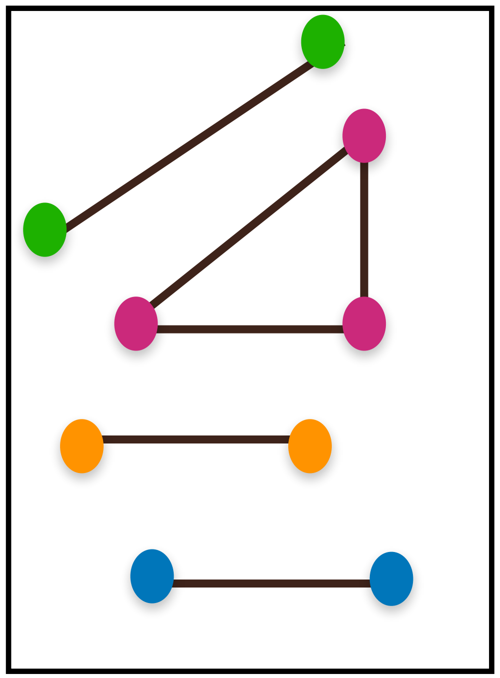
Chromatic Coding Algorithm

- At most d^{dq} choices

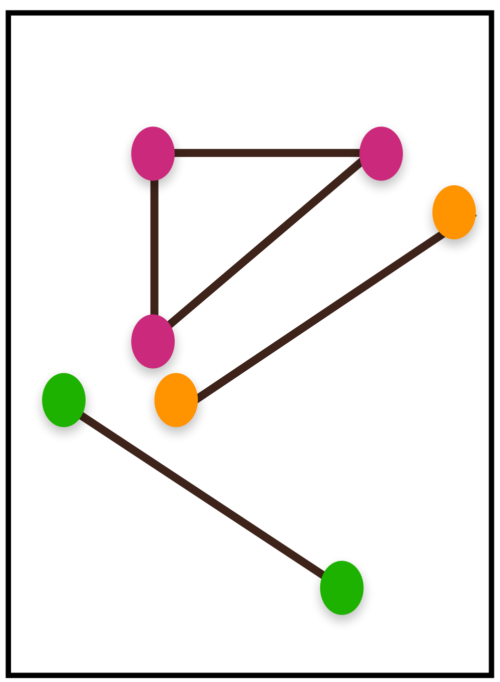
C_1



C_2

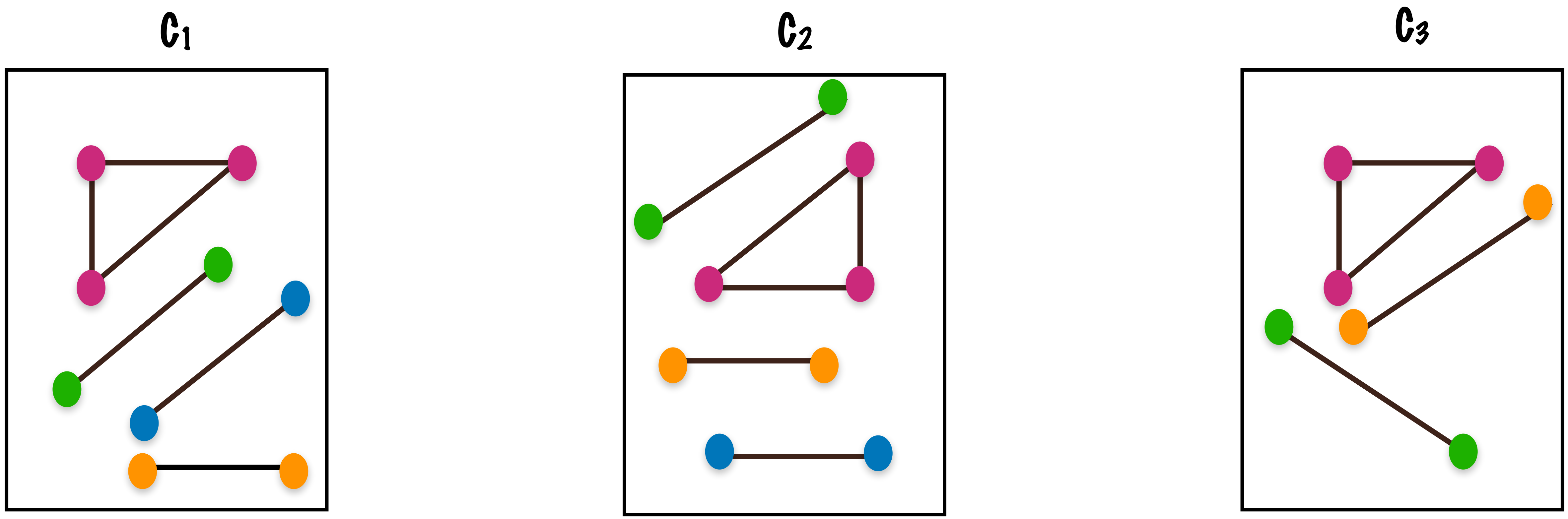


C_3



- For each choice, find a solution that respects that choice

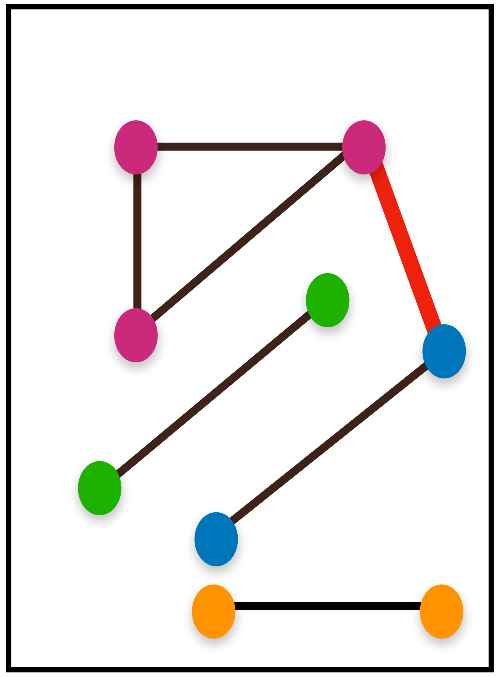
Chromatic Coding Algorithm



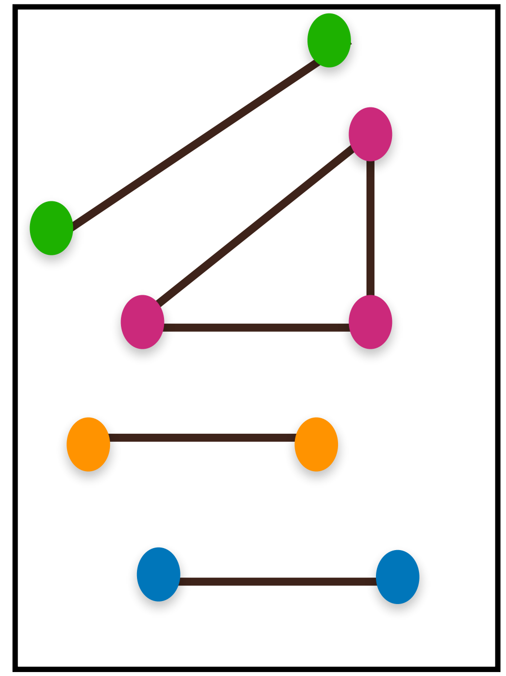
- Add an edge between every pair of non-adjacent vertices that are guessed to be in the same component
- Delete the edge between every pair of adjacent vertices that are guessed to be in different components

Chromatic Coding Algorithm

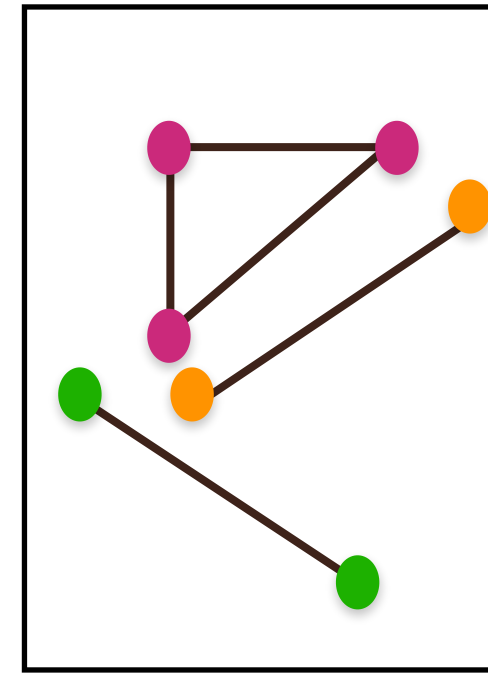
C_1



C_2



C_3



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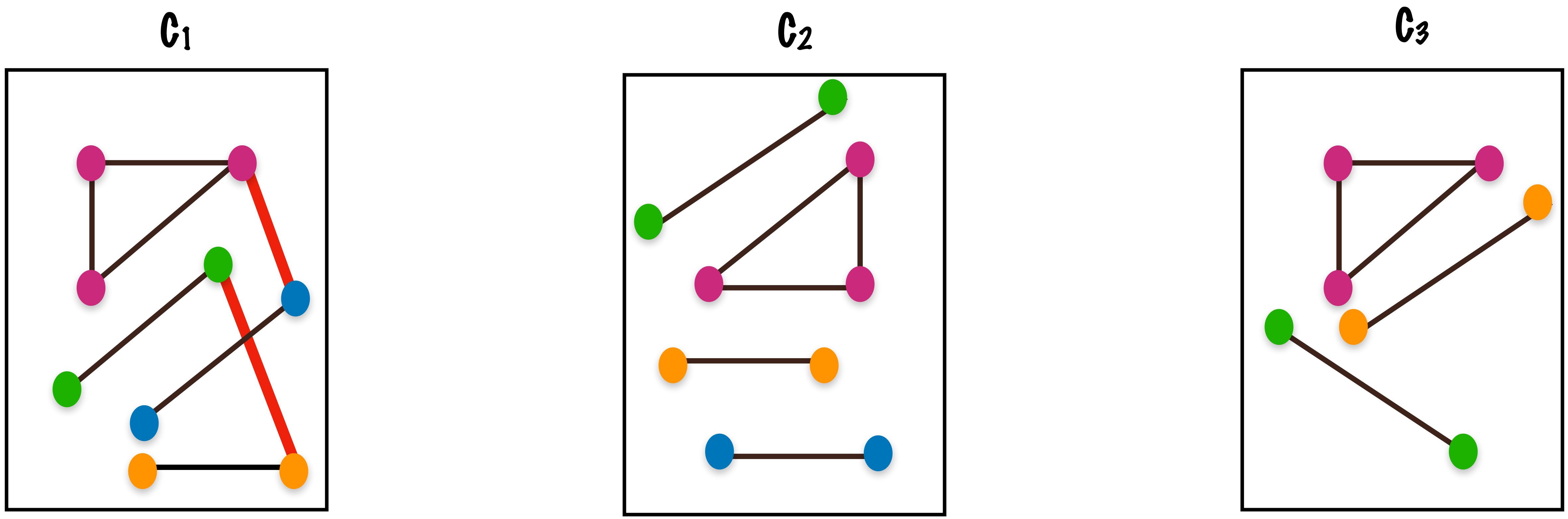
Chromatic Coding Algorithm



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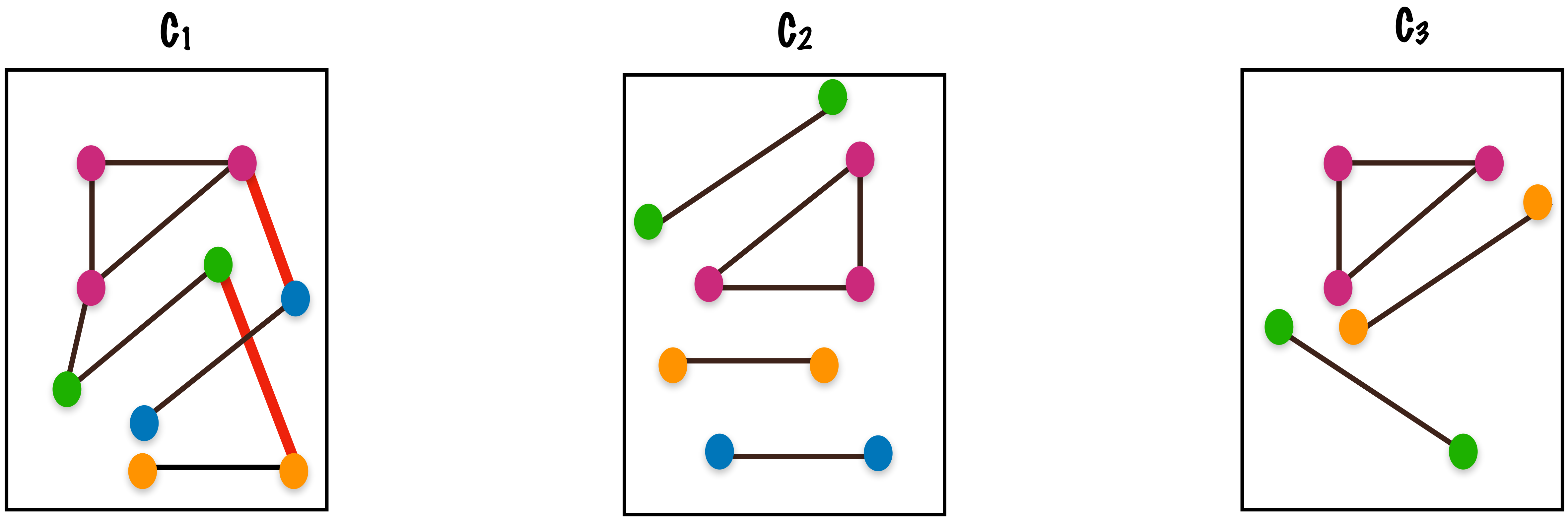
Chromatic Coding Algorithm



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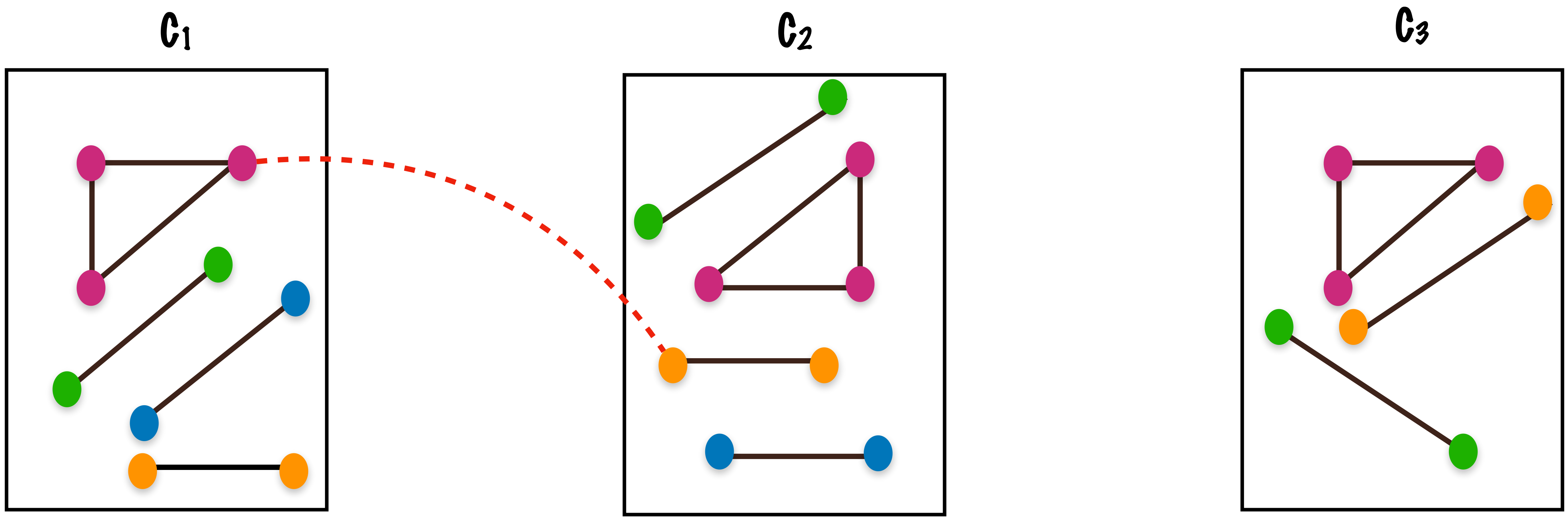
Chromatic Coding Algorithm



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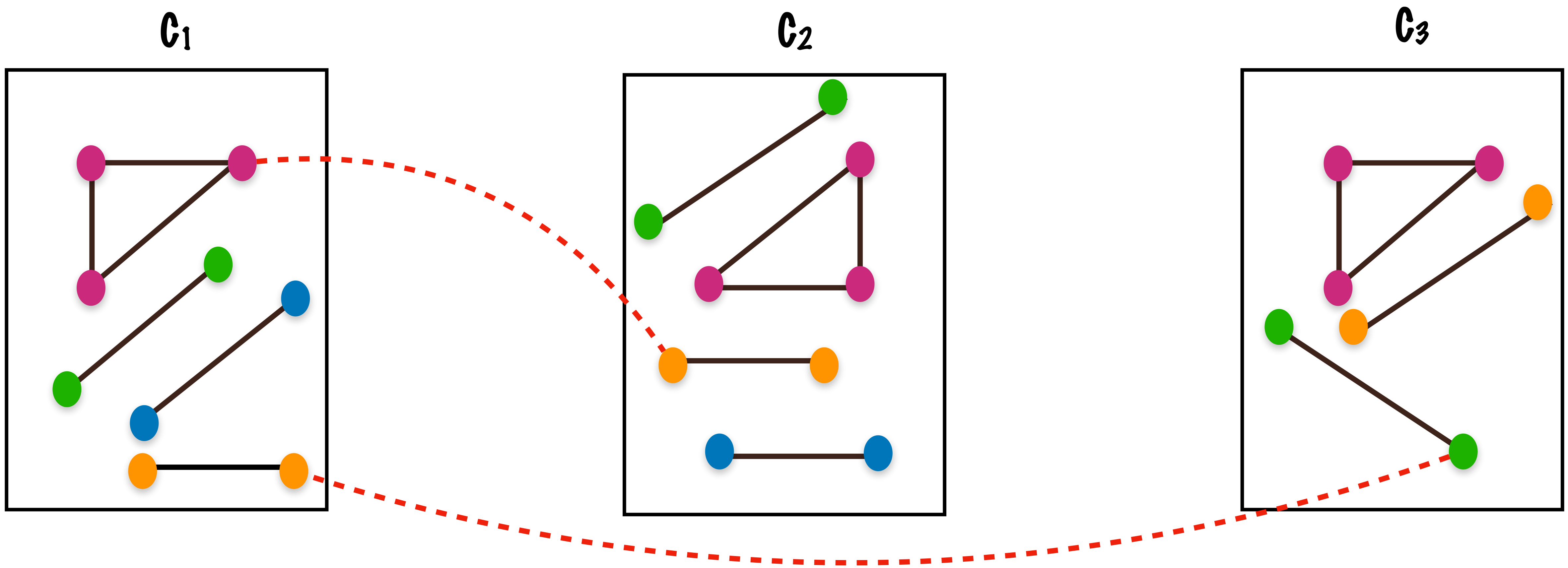
Chromatic Coding Algorithm



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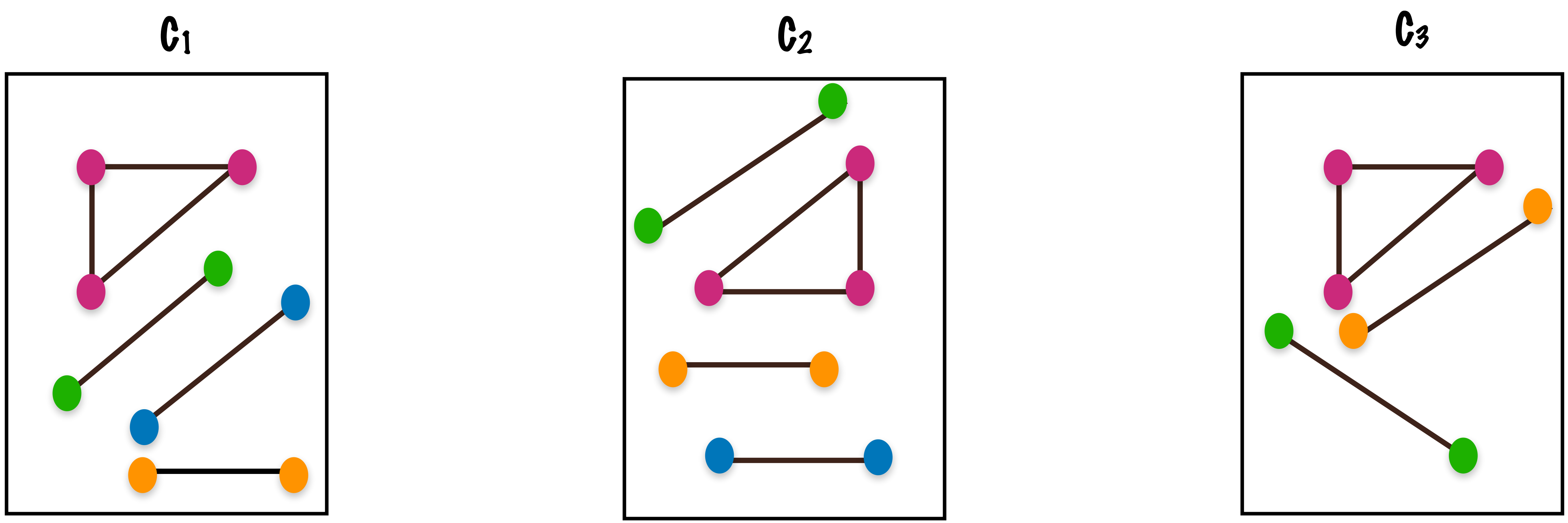
Chromatic Coding Algorithm



* Add an edge between every pair of non-adjacent vertices that are guessed to be in the same component
* Delete the edge between every pair of adjacent vertices that are guessed to be in different components
* If total no. of edge modifications is <= k, then we have a colorful solution

d^{qd} = d^{sqrt(k)d} = 2^{log(d^{sqrt(k)d})} = ...

$2^{O(\mathrm{sqrt}(k)\, d \log d)}\, n^{O(1)}$ randomized algorithm

Analysis

* Running Time: $2^{O(\mathrm{sqrt}(k)\, d \log d)}\, n^{O(1)}$
* Correctness:
    * If (G,k) is a no-instance then Algorithm is correct
    * If (G,k) is a yes-instance
        * The colouring need not color the edges of any solution properly
        * Success probability >= $2^{-\mathrm{sqrt}(k/2)}$

Theorem: d-Clustering can be solved in randomized $2^{O(\mathrm{sqrt}(k)\, d \log d)}\, n^{O(1)}$ time, with success probability at least $2^{-\mathrm{sqrt}(k/2)}$.

* By repeating the algorithm $2^{O(\mathrm{sqrt}(k/2))}$ times,

Theorem: d-Clustering can be solved in randomized $2^{O(\mathrm{sqrt}(k)\, d \log d)}\, n^{O(1)}$ time, with constant success probability.

Analysis

Lemma: If the vertices of a simple graph G on k edges are coloured independently and uniformly at random with $\lceil (8k)^{.5} \rceil$ colors, then the probability that E(G) is properly colored is >= at least $2^{-\mathrm{sqrt}(k/2)}$.

* Let v_1 be a vertex of min degree in $G_0 = G$
* Let v_2 be a vertex of min degree in $G_1 = G - v_1$
* Let v_3 be a vertex of min degree in $G_2 = G - \{v_1 \cup v_2\}$
* Let v_4 be a vertex of min degree in $G_3 = G - \{v_1 \cup v_2 \cup v_3\}$
* . . .
* Let v_n be a vertex of min degree in $G_{n-1} = G - \{v_1 \cup v_2 \cup v_3 \cup \ldots \cup v_{n-1}\}$
* G_n is the empty graph

Analysis

Lemma: If the vertices of a simple graph G on k edges are coloured independently and uniformly at random with $\lceil (8k)^{.5} \rceil$ colors, then the probability that E(G) is properly colored is >= at least $2^{-sqrt(k/2)}$.

* v_i is a vertex of min degree in $G_{i-1} = G - \{v_1 \cup v_2 \cup v_3 \cup \ldots \cup v_{i-1}\}$
* d_i is the degree of v_i in G_{i-1}
* $d_i <= |V(G_{i-1})| - 1$ as graph is simple
* $2k = 2|E(G)| >= 2|E(G_{i-1})| >= d_i |V(G_{i-1})|$ >= di(di + 1) >= di^2
* $d_i <= sqrt(2k)$

Analysis

Lemma: If the vertices of a simple graph G on k edges are coloured independently and uniformly at random with $\lceil (8k)^{.5} \rceil$ colors, then the probability that E(G) is properly colored is >= at least $2^{-\mathrm{sqrt}(k/2)}$.

* v_i is a vertex of min degree in $G_{i-1} = G - \{v_1 \cup v_2 \cup v_3 \cup \ldots \cup v_{i-1}\}$
* d_i is the degree of v_i in G_{i-1} and $d_i <= \mathrm{sqrt}(2k)$

Consider the process of randomly coloring V(G) in the order $v_n, v_{n-1}, v_{n-2}, \ldots, v_1$

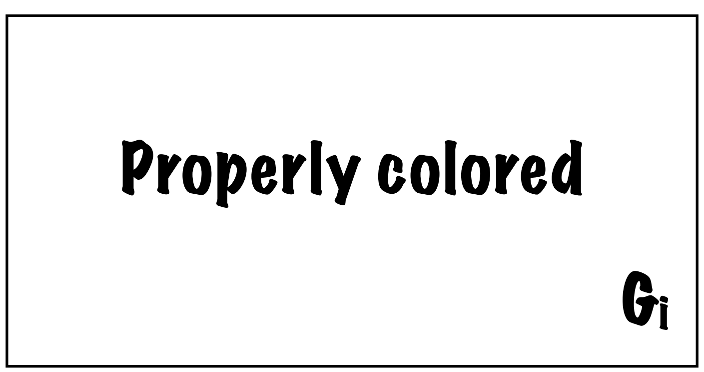
* P_i :event that $E(G_i)$ is properly colored
  * $Pr(P_n) = 1$

Analysis

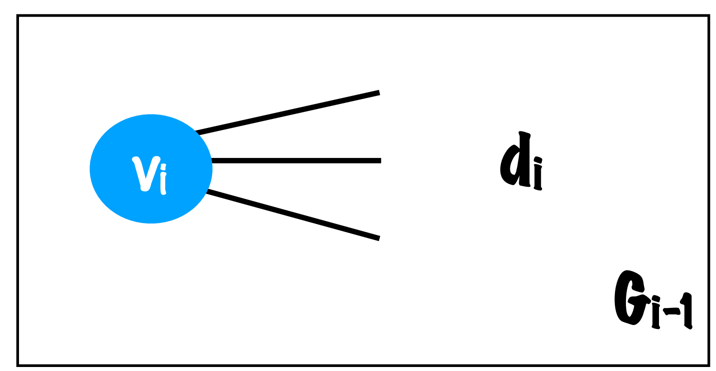
Identity: 1-x >= 2^{-2x} for 0<= x <= 1/2

> **Lemma:** If the vertices of a simple graph G on k edges are coloured independently and uniformly at random with $\lceil (8k)^{.5} \rceil$ colors, then the probability that E(G) is properly colored is >= at least $2^{-\mathrm{sqrt}(k/2)}$.

- P_i :event that $E(G_i)$ is properly colored
  - $Pr(P_n) = 1$
- $Pr(P_{i-1} \mid P_i) >= (q-d_i)/q > = 2\hat{}\{-2d_i/q\}$



$V_n, V_{n-1}, V_{n-2}, \ldots, V_{i+1}$



$V_n, V_{n-1}, V_{n-2}, \ldots, V_{i+1}, V_i$

Analysis

Lemma: If the vertices of a simple graph G on k edges are coloured independently and uniformly at random with $\lceil (8k)^{.5} \rceil$ colors, then the probability that E(G) is properly colored is >= at least $2^{-\mathrm{sqrt}(k/2)}$.

$$Pr(P_0) = Pr(P_0 \mid P_1)\, Pr(P_1) = Pr(P_0 \mid P_1)\, Pr(P_1 \mid P_2)\, Pr(P_2)$$

$$= \ldots = Pr(P_n) \prod Pr(P_{i-1} \mid P_i)$$

$$\geq \prod 2\hat{}\{-2d_i/q\} = 2\hat{}\{-2\textstyle\sum d_i/q\} = 2\hat{}\{-2k/\lceil \mathrm{sqrt}(8k) \rceil\} = 2^{-\mathrm{sqrt}(k/2)}$$

Derandomization

Definition: A (n,k,q)-coloring family F is a family of functions from [n] to [q] such that for every graph G on vertex set [n] with <= k edges, there is a function f in F that properly colors E(G).

Theorem: For any n,k>=1, there is a $(n, k, \mathcal{O}(\sqrt{k}))$-coloring family of size $2^{O(\sqrt{k}\log k)}\log n$ that can be constructed in $2^{O(\sqrt{k}\log k)}n\log n$ time.

Theorem: d-Clustering can be solved in $2^{O(\sqrt{k}(d+\log k))}n^{O(1)}$ time.