CS 5003: Parameterized Algorithms

Lecture 4

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References: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Branching or Depth-Bounded Search Trees

Vertex Cover

(G,k)

v

max deg

k drops by 1 (G-v,k-1)

(G-N[v],k-|N(v)|) k drops by 3

....

Let T(k) denote the no. of leaves in the tree rooted at instance with parameter k

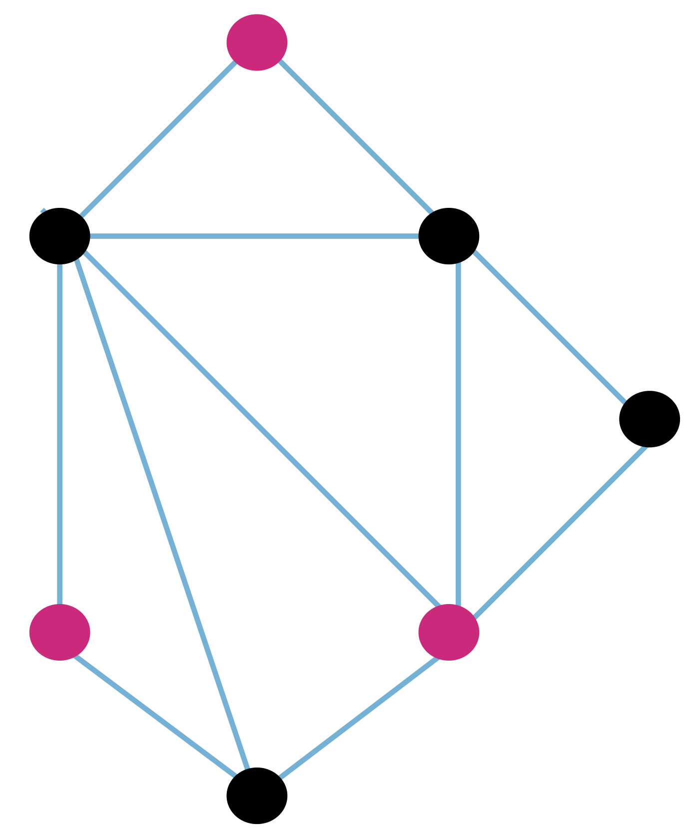
$$T(k) \le \begin{cases} T(k-1)+T(k-3) & \text{if } k \ge 3 \\ 1 & \text{otherwise} \end{cases}$$

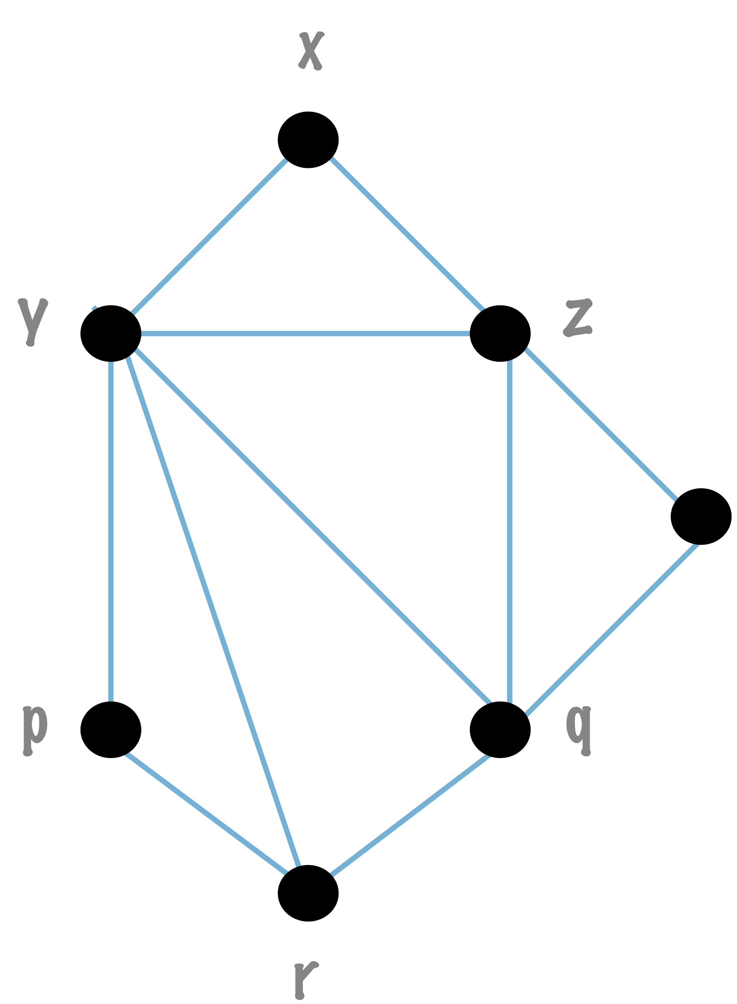
$T(k) \le 1.4656^k$

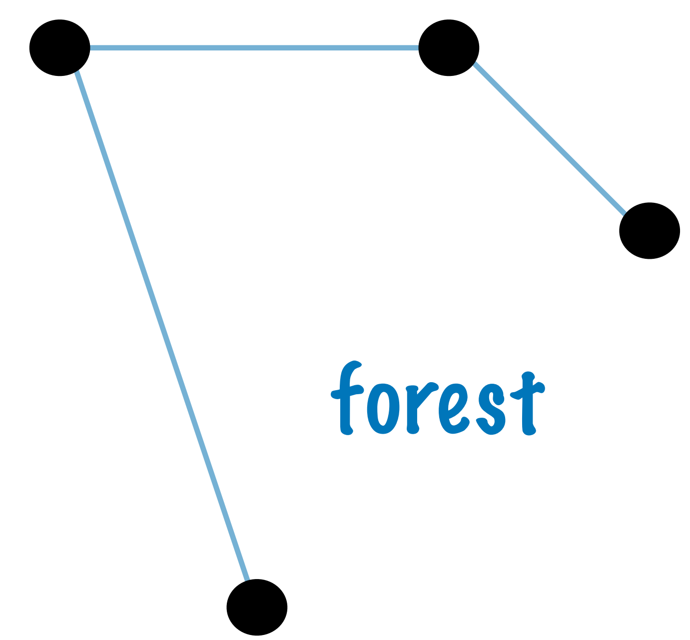
$O^*(1.4656^k)$ time algorithm

Feedback Vertex Set

FVS - set of vertices that has at least one vertex of every cycle







forest

Feedback Vertex Set

Feedback Vertex set

Instance: An undirected graph G and an integer k
Question: Does there exist a feedback vertex set of G of size at most k?
Parameter: k

Can we try branching?

(G,k) cycle $C = \{v_1, v_2, \ldots, v_r\}$

$(G-v_1, k-1) \quad \ldots \quad (G-v_r, k-1)$

$\vdots$

Longest cycle length = r

$$T(k) \le \begin{cases} rT(k-1) & \text{if } k \ge 1 \\ 1 & \text{otherwise} \end{cases}$$

$T(k) \le r^k$

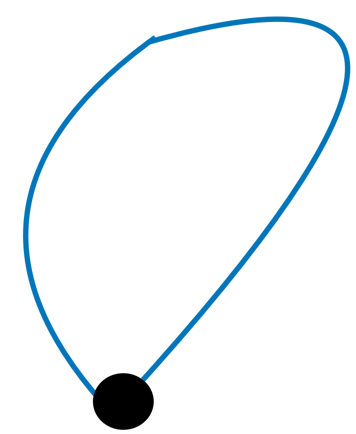
$O^*(r^k)$ time algorithm

Theorem: FVS is FPT with respect to k+r as parameter

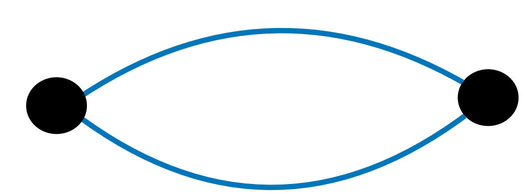
Feedback Vertex Set

Assume graph is a multigraph

* Reduction Rule 1: Delete isolated vertices
* Reduction Rule 2: Delete degree-1 vertices
* Reduction Rule 3: If there is a loop at a vertex v, delete v from the graph and reduce the parameter by 1

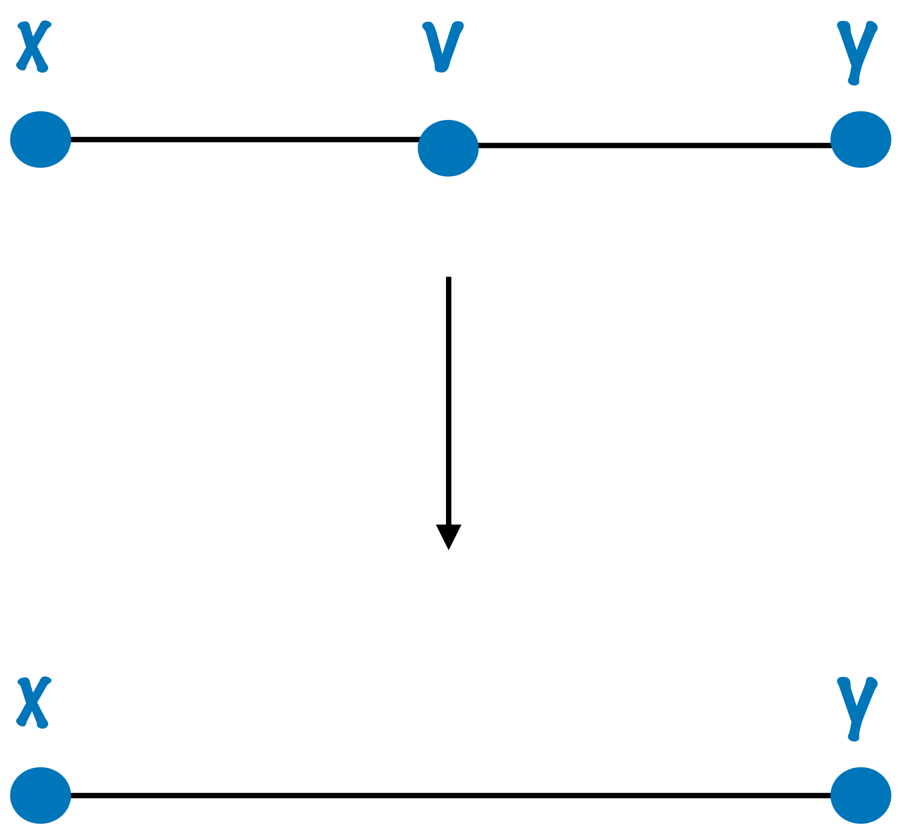


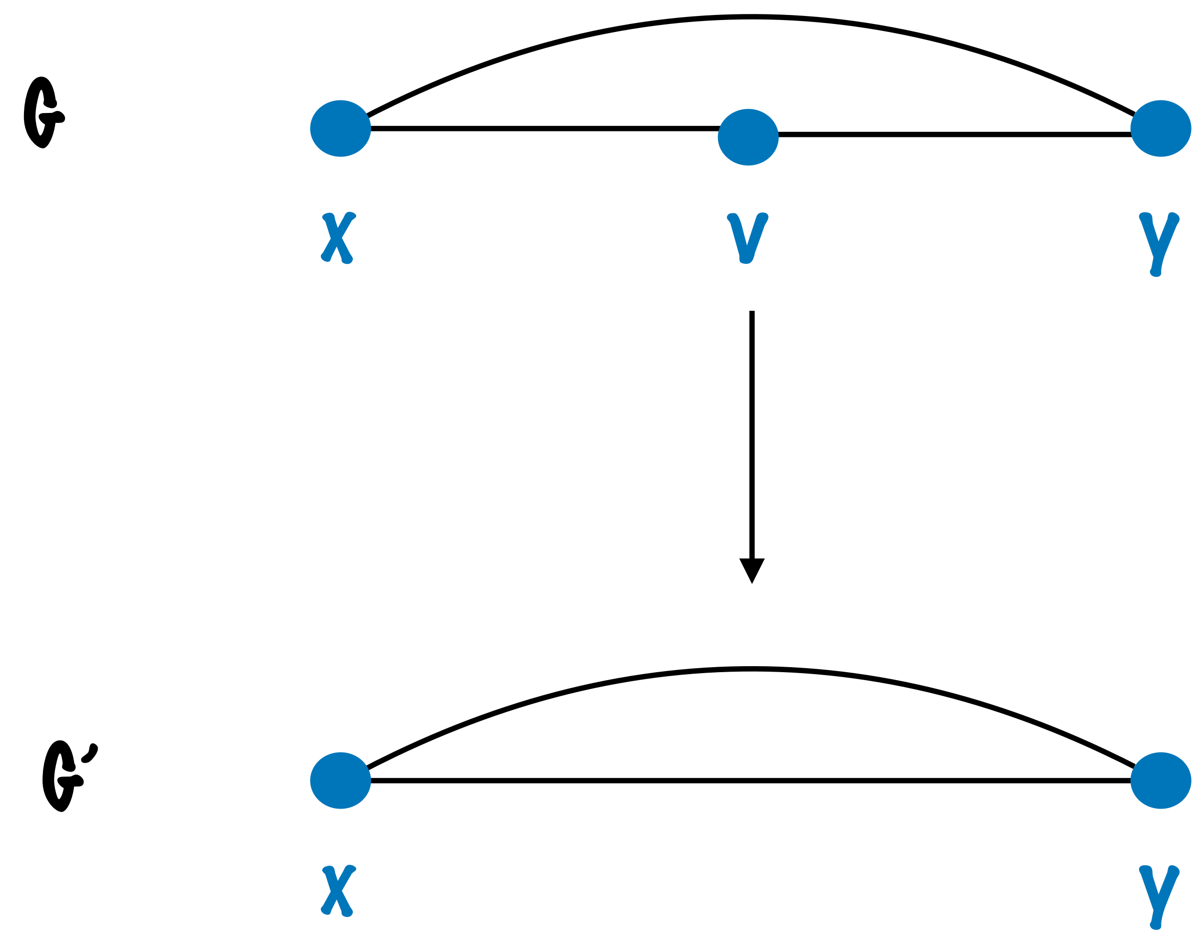
* Reduction Rule 4: If there is an edge with multiplicity > 2, reduce it to 2



Feedback Vertex Set

* Reduction Rule 5: Short circuit degree-2 vertices





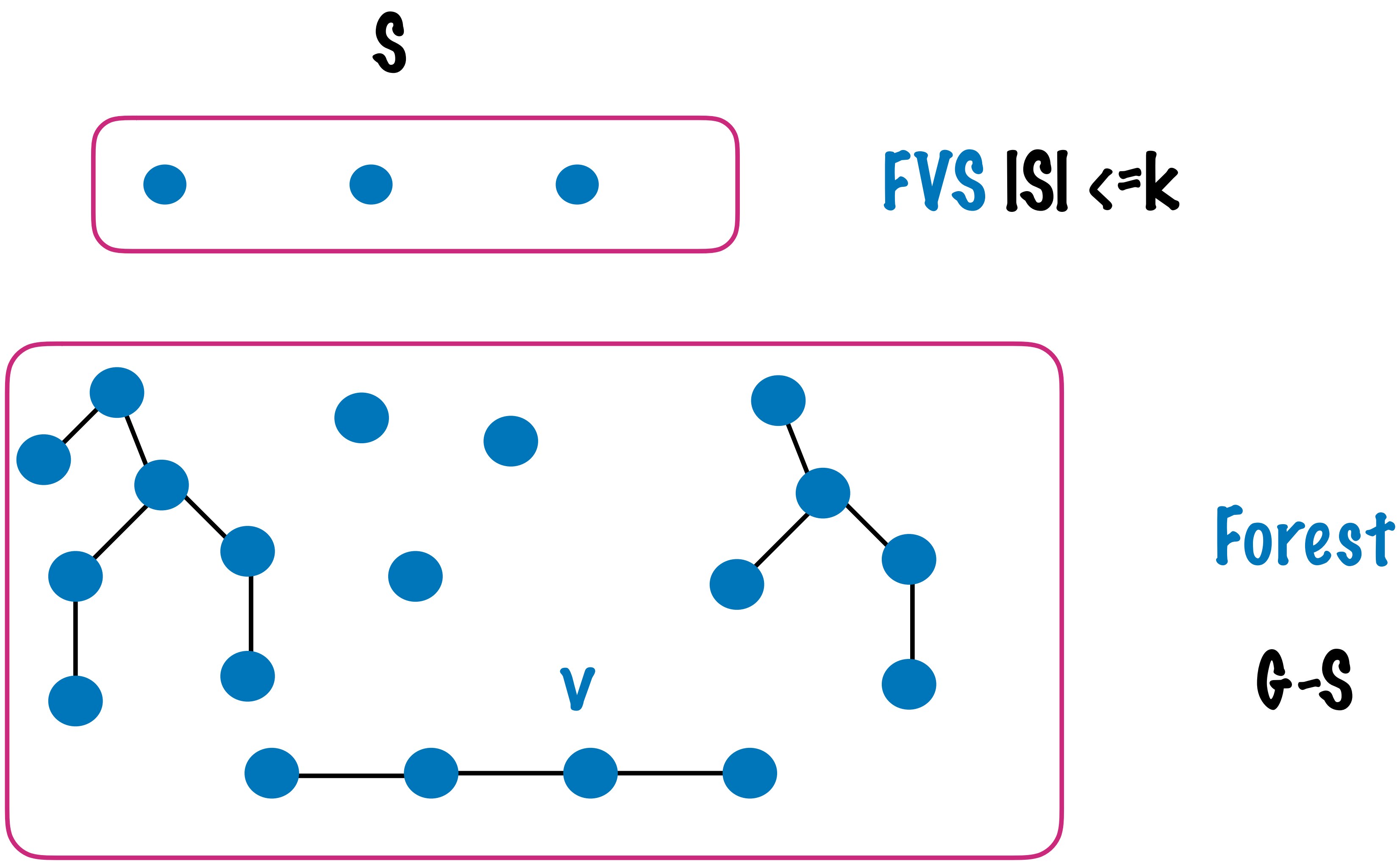
Any cycle through v also goes through x and y

(G,k) is an yes-instance iff (G′,k) is an yes-instance

Feedback Vertex Set

* Min degree >=3

Suppose (G,k) is an yes-instance



* Forest has "very less" edges
* Not many "high-degree" vertices can be in G-S

Feedback Vertex Set

$$v_1 \; v_2 \; v_3 \; v_4 \; v_5 \; . \; . \; . \; . \; . \; . \; . \; v_n$$

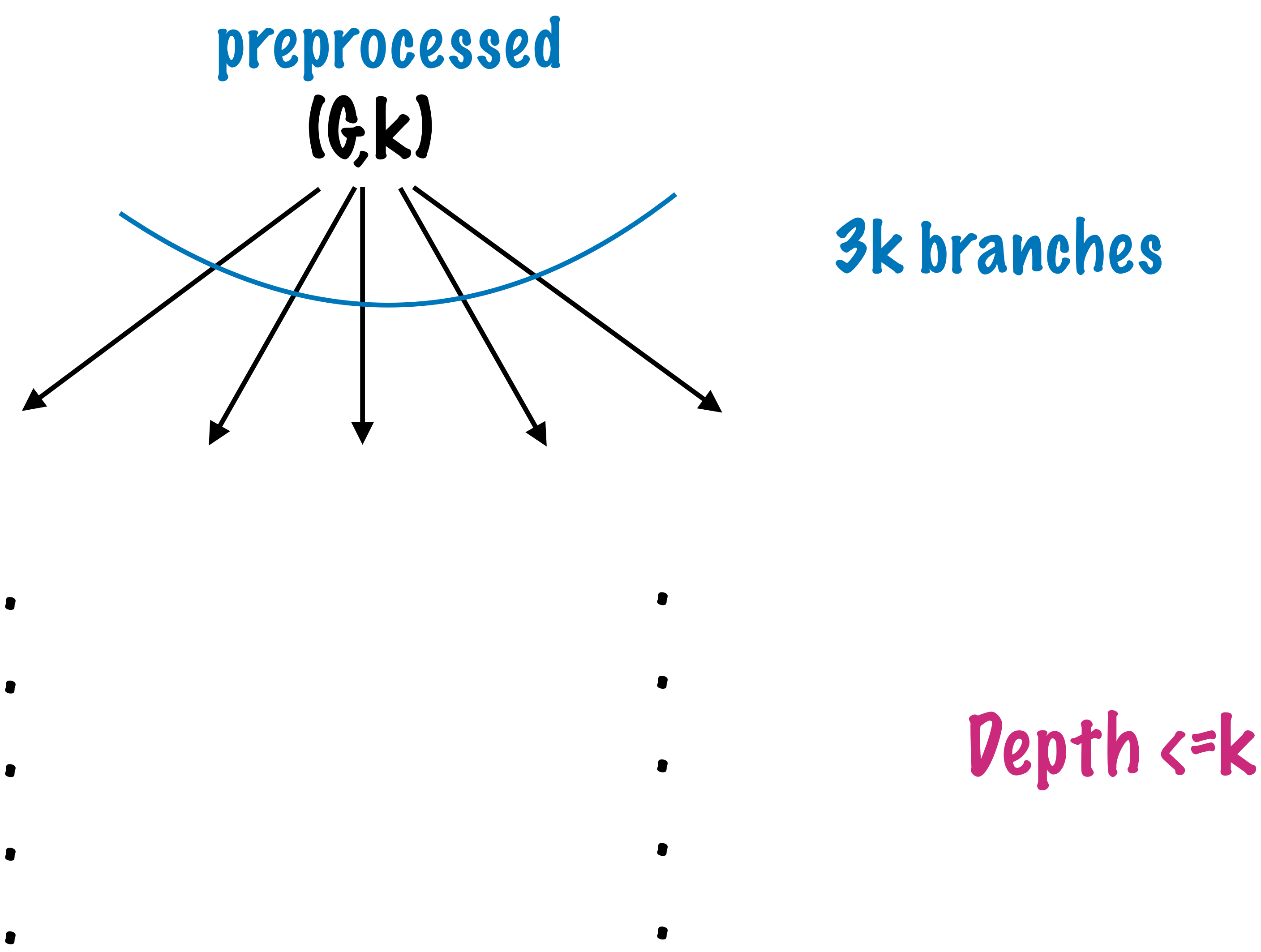
$$\deg(v_1) >= \deg(v_2) >= \deg(v_3) >= \deg(v_4) >= \deg(v_5) >= \; . \; . \; . \; . \; . \; >= \deg(v_n)$$

$$V_h = \{v_1, v_2, v_3, \; . \; . \; . \; . \; . \; v_{3k}\}$$

Lemma: Every FVS of size <= k contains at least one vertex from V_h.

Feedback Vertex Set

Every FVS <= k contains at least one vertex from V_h

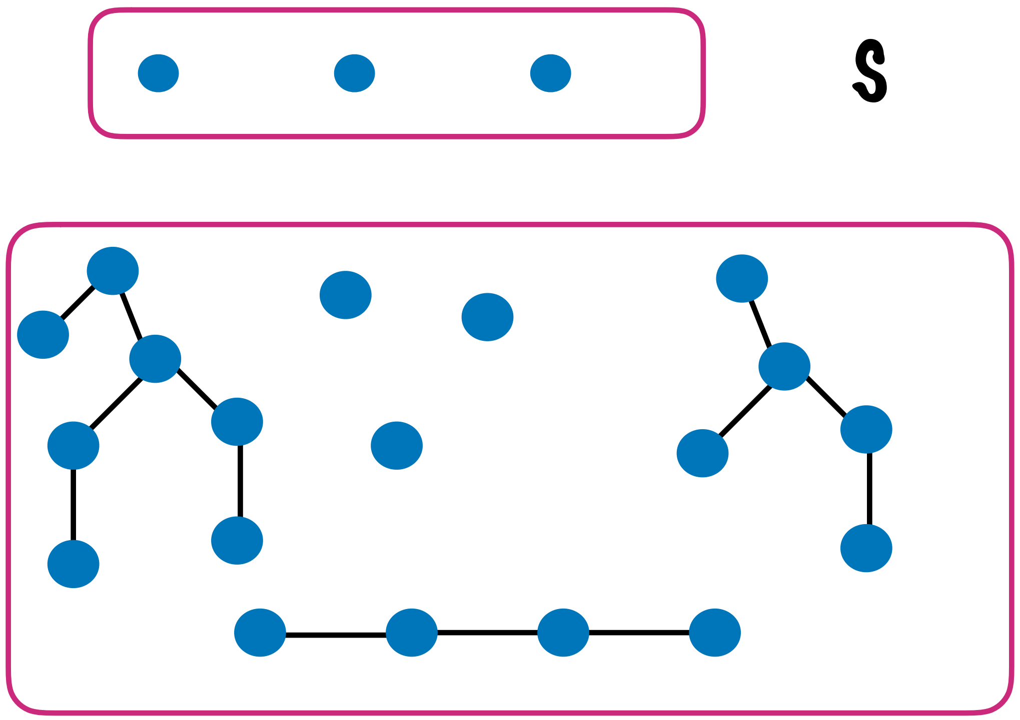


An $O((3k)^k n^c)$ algorithm

Theorem: FVS is FPT with respect to the solution size as parameter

Feedback Vertex Set

Lemma: Every FVS of size <= k contains at least one vertex from V_h



<= n-|S|-1 edges

$$m \le n - |S| - 1 + \sum_{v \in S} \deg(v)$$

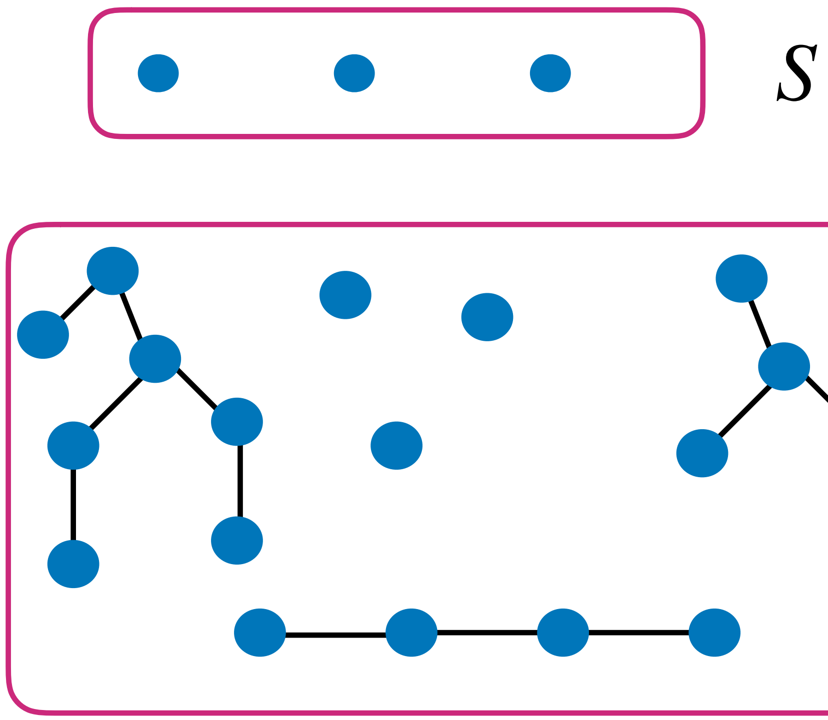
$$\sum_{v \in S} (\deg(v) - 1) \ge m - n + 1$$

Feedback Vertex Set

$$\sum_{v \in S} (deg(v) - 1)) \geq m - n + 1$$

Lemma: Every FVS of size <= k contains at least one vertex from V_h

Suppose not



$$3(\sum_{v \in S} (deg(v) - 1)) \leq \sum_{i=1}^{3k} (deg(v_i) - 1)$$

$$\sum_{v \in S} (deg(v) - 1) \leq \sum_{i > 3k} (deg(v_i) - 1)$$

$\leq n - |S| - 1 \ edges$

$$2m - n = \sum_{i=1}^{n} (deg(v_i) - 1) \geq 4(m - n + 1)$$

$2m < 3n$

A contradiction