CS 5003: Parameterized Algorithms
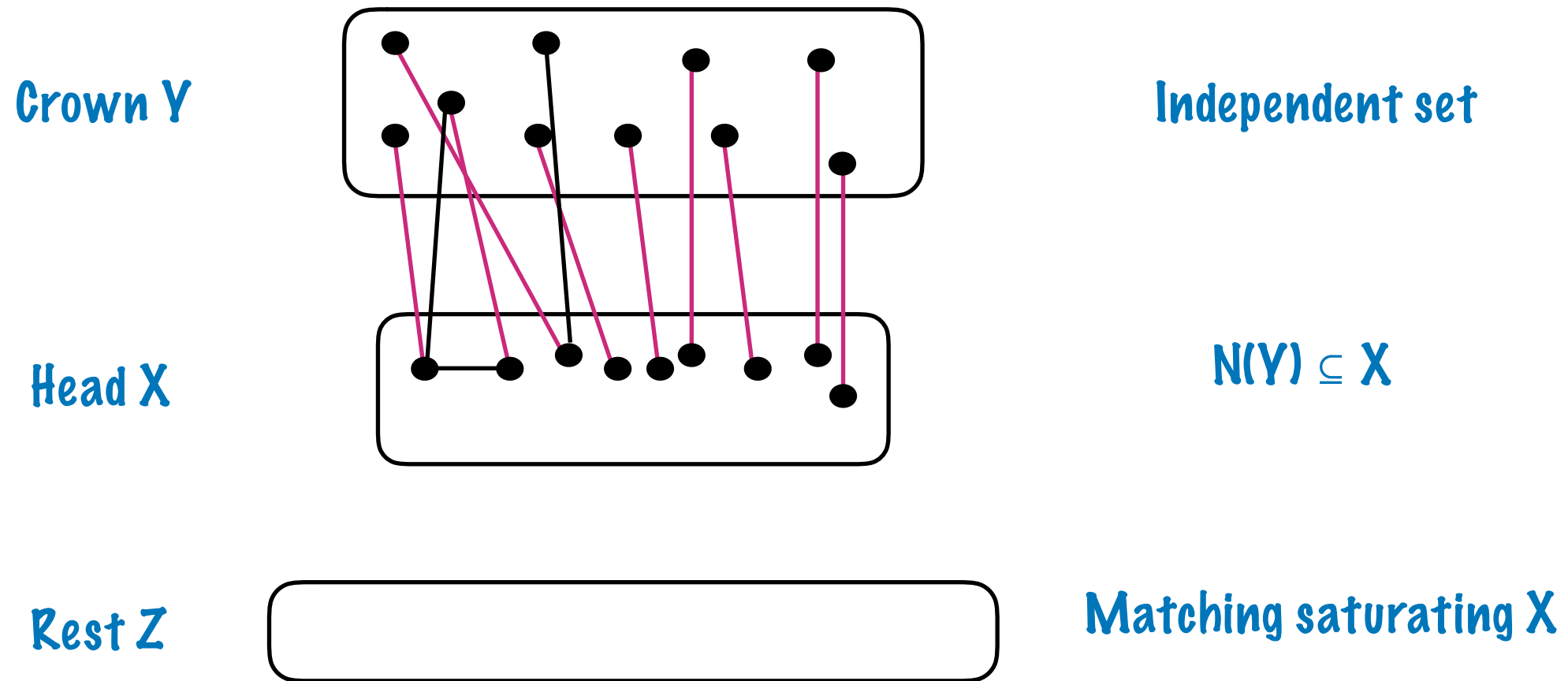
Lectures 28-31

Krithika Ramaswamy

## IIT Palakkad

Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Crown Lemma

Crown Y — Independent set

Head X — $N(Y) \subseteq X$

Rest Z — Matching saturating X

Crown Lemma: Let G be a graph without isolated vertices and with at least $3k + 1$ vertices. Then, there is a polynomial time algorithm that either finds a matching of size $k + 1$ in G, or finds a crown decomposition of G.
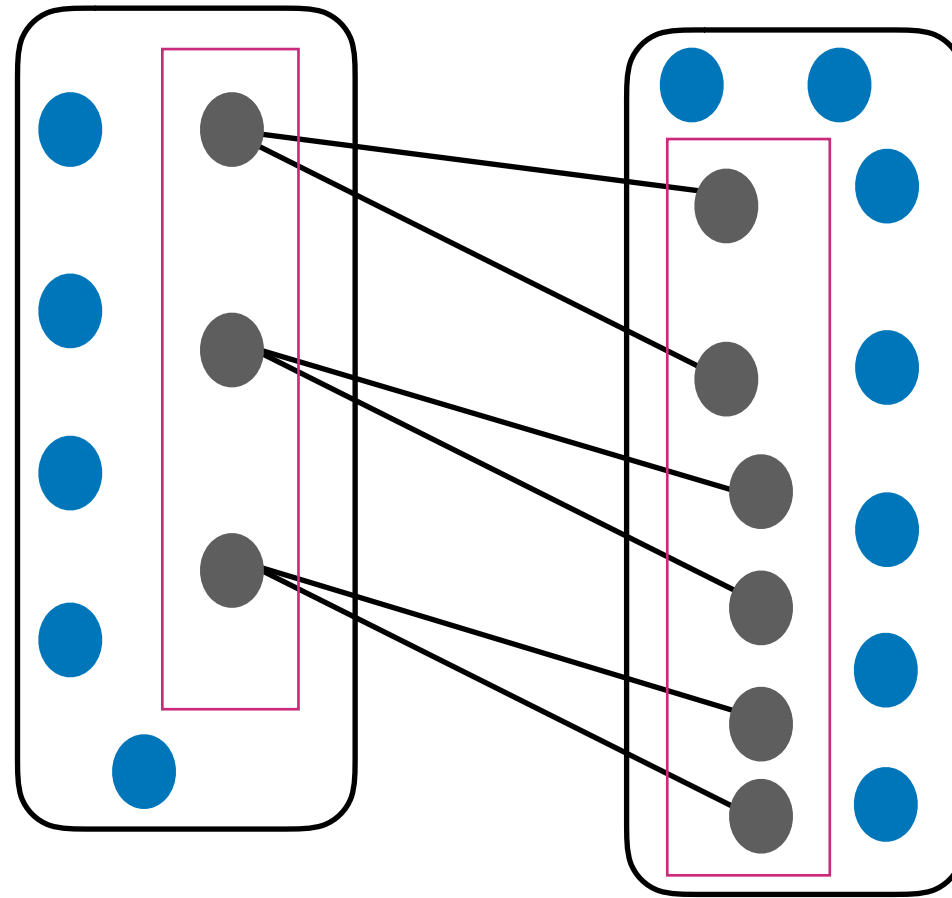
Towards a Generalization of Crown Lemma

Y

X $N(Y) \subseteq X$

Z   2-expansion from X to Y
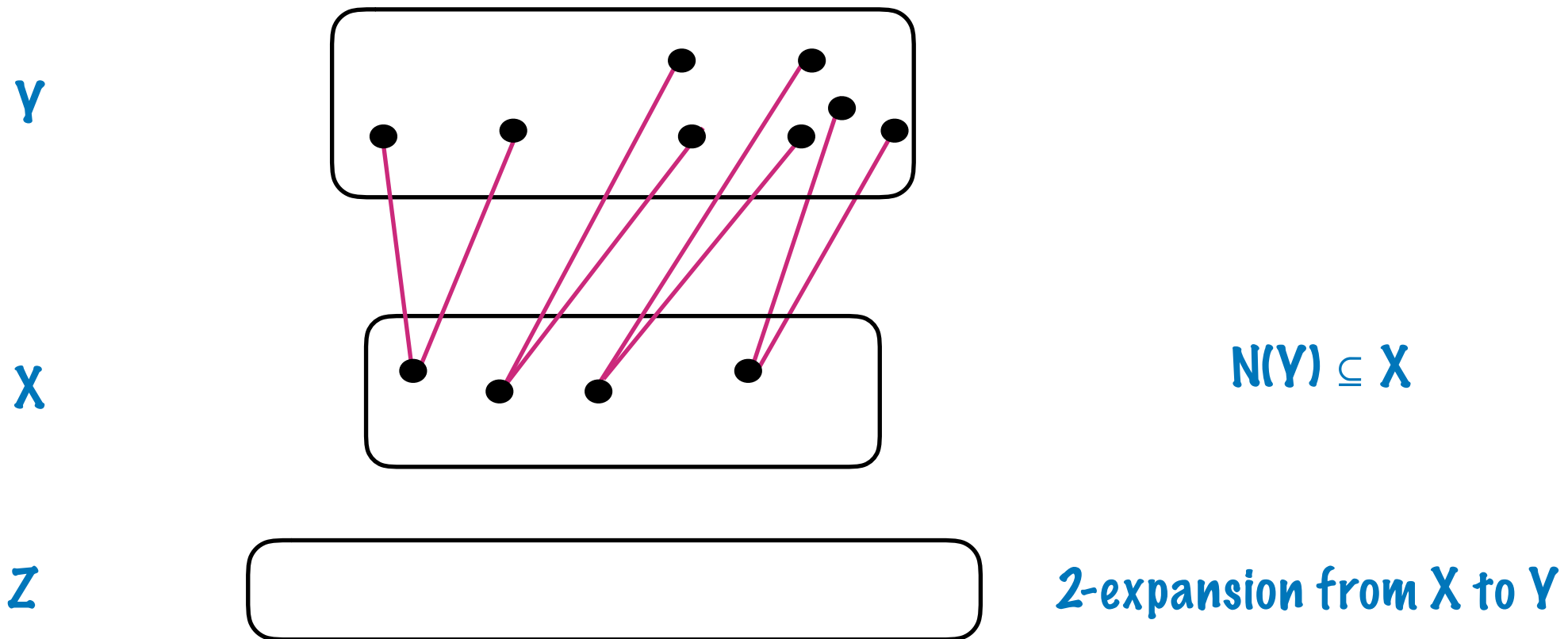
* **Definition:** In a bipartite graph with bipartition (A, B), a set M of edges is called a **2-expansion** from A to B if
  * Every vertex of A is incident with exactly 2 edges of M
  * M saturates exactly 2|A| vertices in B
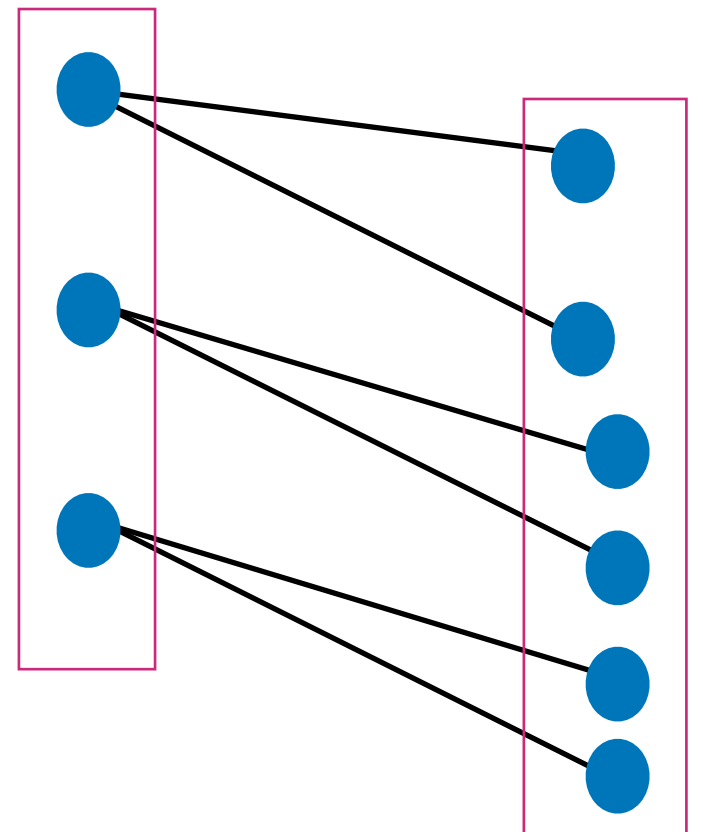
2-Expansion Lemma

2-Expansion Lemma: Let G be a bipartite graph with bipartition (A, B) s.t $|B| > 2\,|A|$ and there are no isolated vertices in B. Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a 2-expansion into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m\, n^{1/2})$ time.
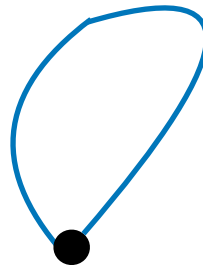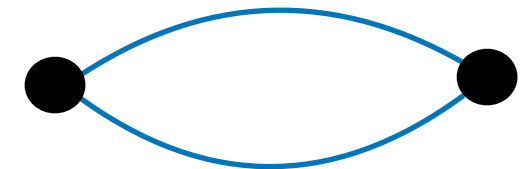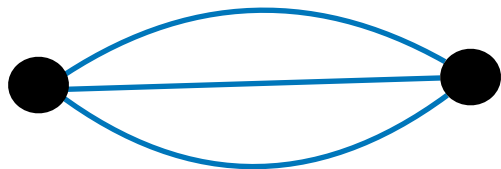
Feedback Vertex Set

Assume graph is a multigraph

Reduction Rule 1: Delete isolated vertices

Reduction Rule 2: Delete degree-1 vertices

Reduction Rule 3: If there is a loop at a vertex v, delete v from the graph and reduce the parameter by 1

Reduction Rule 4: If there is an edge with multiplicity > 2, reduce it to 2

Reduction Rule 5: Short circuit degree-2 vertices

Feedback Vertex Set - Towards a Quadratic Kernel

Claim: If min deg >=3 and max deg <= d and G has an FVS <=k then n < (d+1)k and m < 2dk.

Suppose X is FVS <=k and let Y=G-X

$$3|Y| \leq \sum_{v \in Y} deg(v) = \sum_{v \in Y} |N(v) \cap X| + \sum_{v \in Y} |N(v) \cap Y| \leq E(X,Y) + 2(|Y| - 1)$$
$$< E(X,Y) + 2|Y|$$
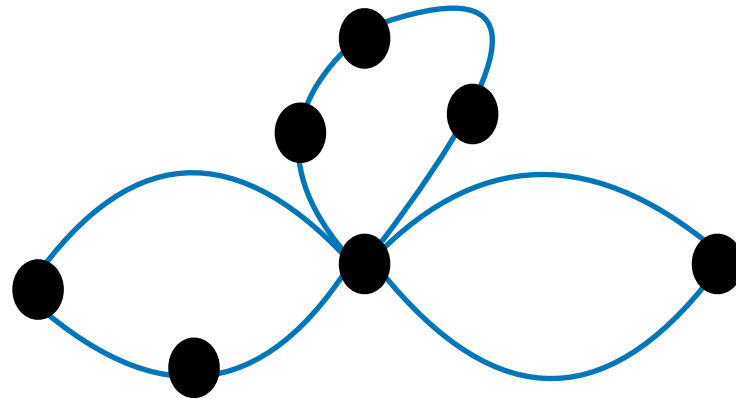$$\leq d|X| + 2|Y|$$

$$\therefore |Y| < d|X|$$

$$n = |X| + |Y| < (d+1)|X| \leq (d+1)k$$

$$m < d|X| + |Y| \leq 2d|X| \leq 2dk$$

Goal: Reduce max degree to <= 10k to get quadratic kernel

Feedback Vertex Set - Towards a Quadratic Kernel

Definition: A **v-flower** with r petals is a set of r cycles that pairwise intersect only at v

Flower Lemma: There is a polynomial time algorithm that given a graph G and a vertex v without a self-loop, satisfies one of the following:
* Declares (G,k) is a no-instance of Feedback Vertex Set
* Returns a v-flower with (k+1) petals
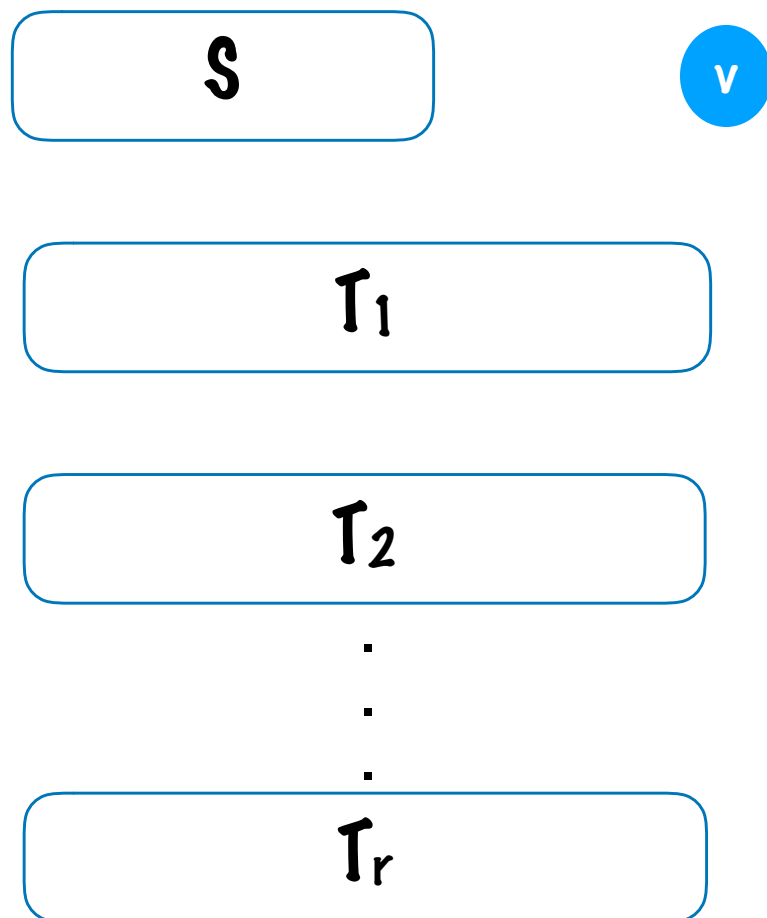* Finds FVS not containing v of size <= 3k

Reduction Rule 6: If there is a v-flower with k+1 petals, then delete v and reduce the parameter by 1

Feedback Vertex Set - Towards a Quadratic Kernel

* Suppose v has degree >10k and Flower Lemma returns FVS S not containing v of size <= 3k

S v

T_1

T_2

.

.

.

T_r

Trees

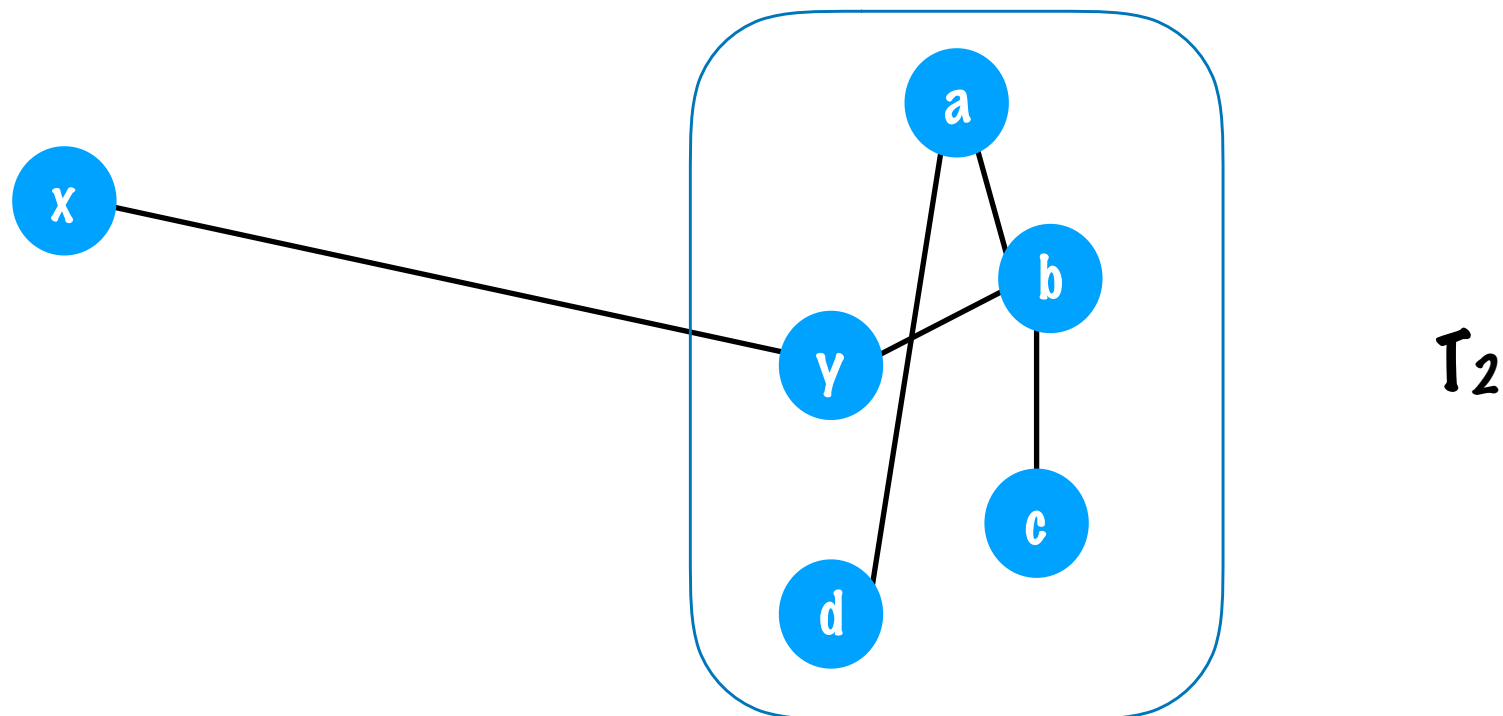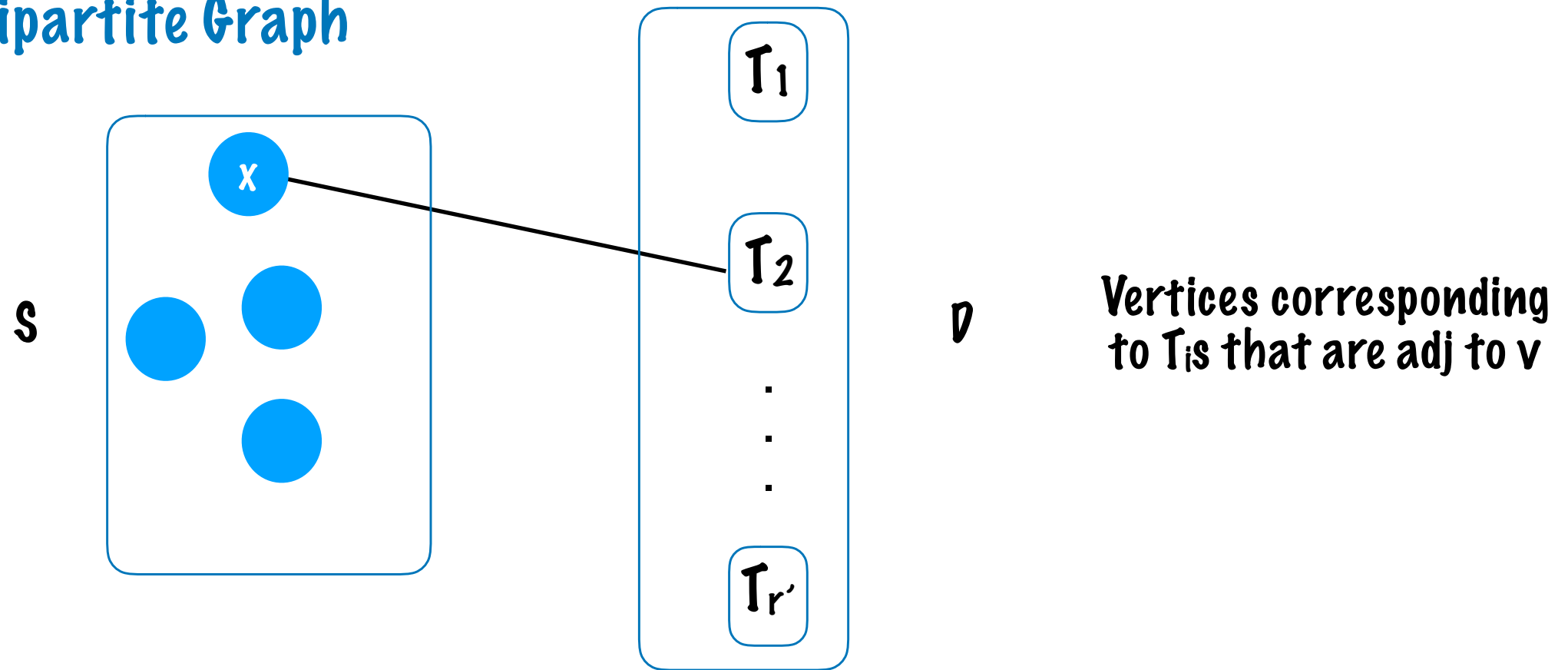# Feedback Vertex Set - Towards a Quadratic Kernel

S

v

T_1

T_2

.
.
.

T_r
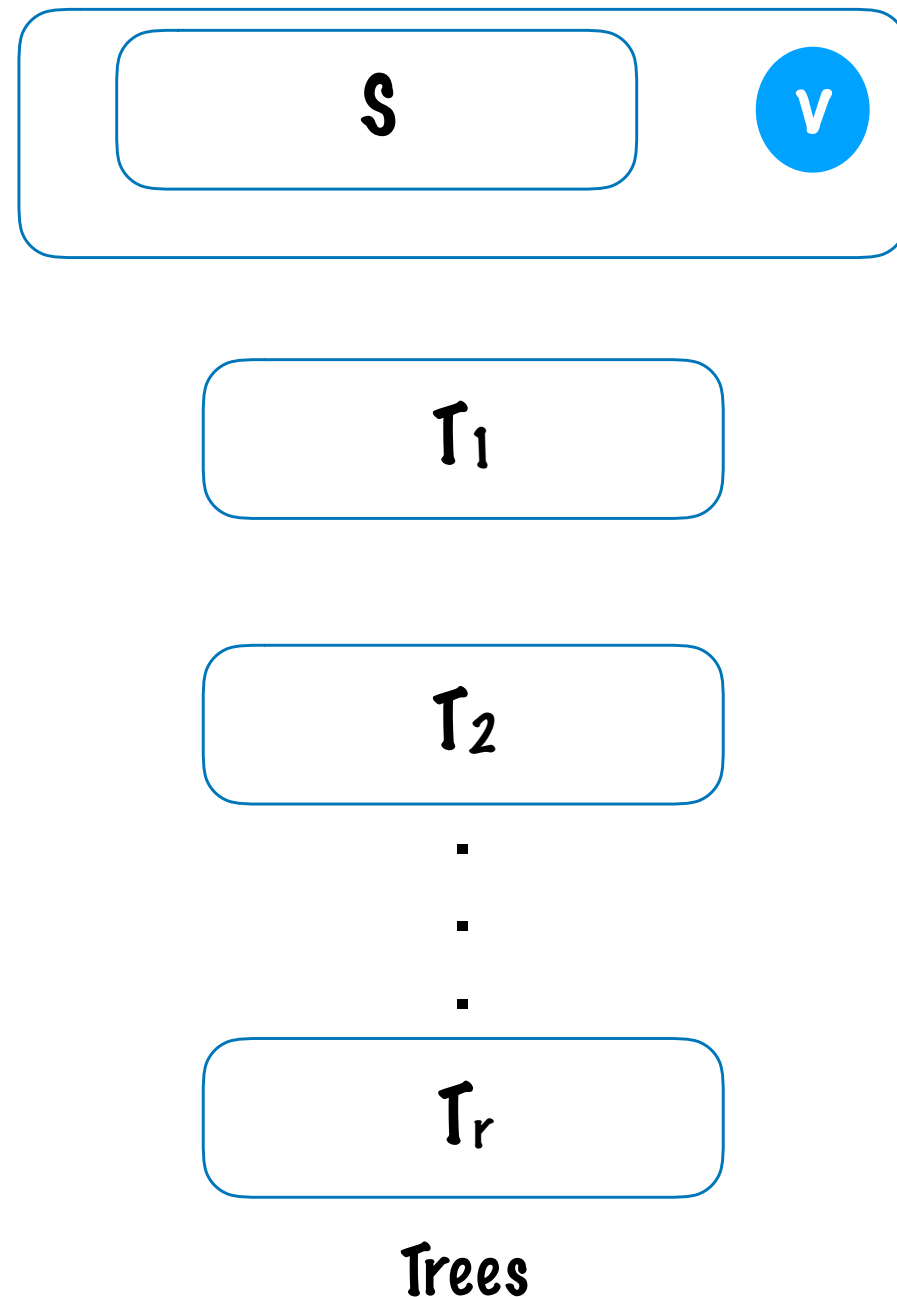
* v has deg $>10k$

* If there are $> 2k$ double edges incident on it, apply Reduction Rule 6

* Otherwise, there are $<=2k$ double edges incident on v

* There are $<= 4k$ edges between v and S

  * $<=2l + (3k-l) = 3k+l <=4k$
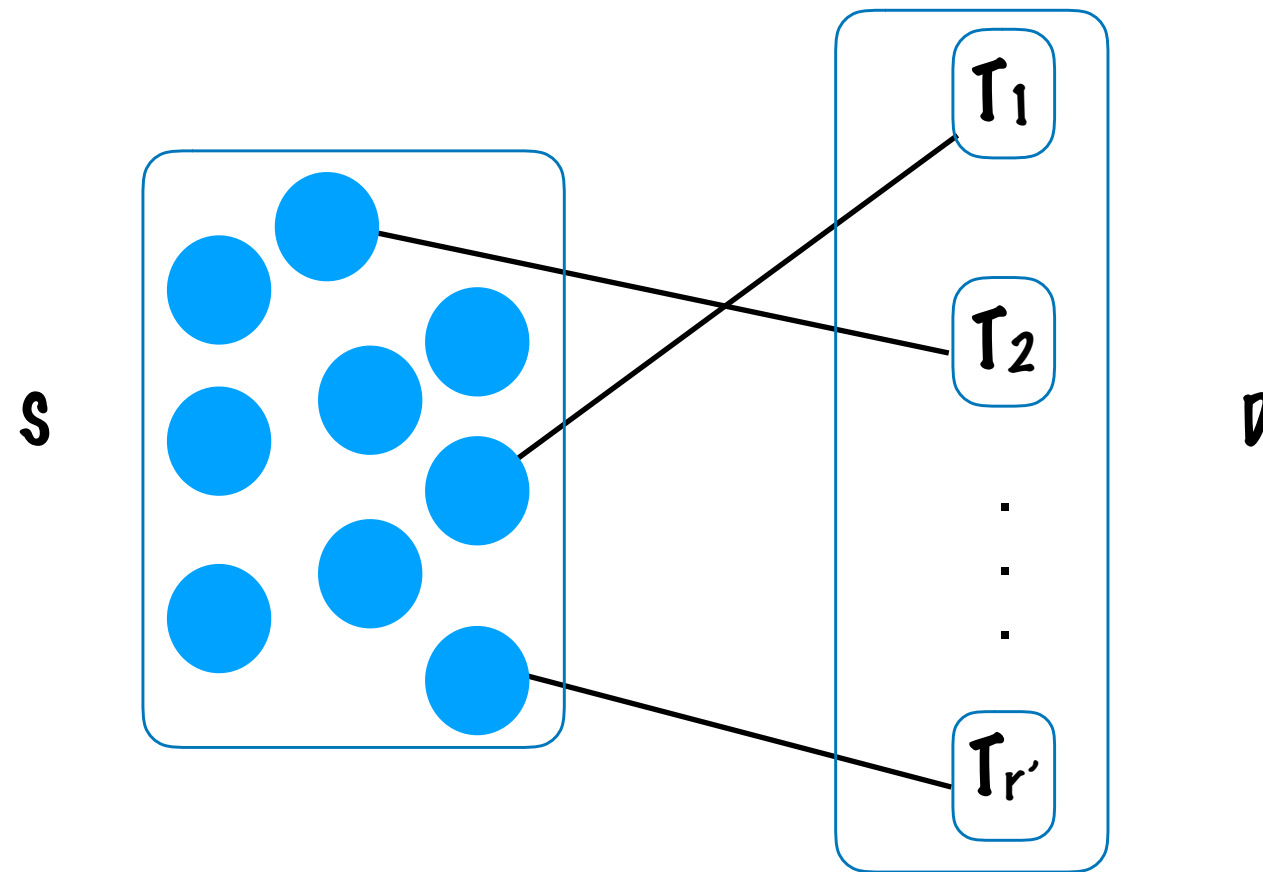
* There are $<=r$ edges between v and T_is

$\therefore$ there are $>6k$ edges between v and T_is. In particular, $r>6k$

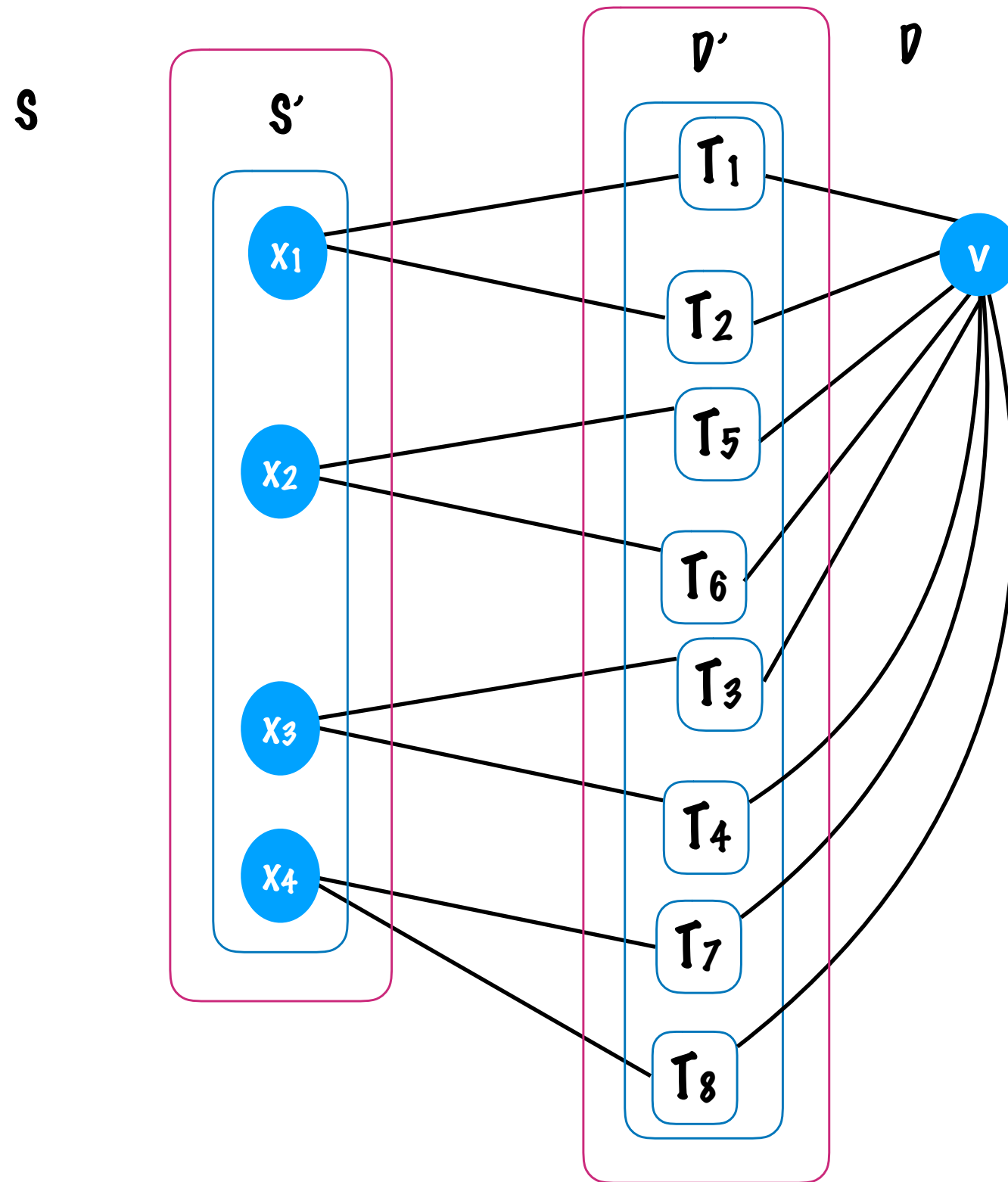# Feedback Vertex Set - Towards a Quadratic Kernel

Auxiliary Bipartite Graph

Vertices corresponding to T_is that are adj to v

Feedback Vertex Set - Towards a Quadratic Kernel

- No vertex in D is isolated as each T_i has a deg <= 1 vertex x_i which has deg >=3 in G. Even if x_i is adj to v, it has >=1 nbr in S

- $|D|$>6k and $|S|$<=3k i.e., $|D|$ >2$|S|$

    - There are non-empty sets $D' \subseteq D$ and $S' \subseteq S$ s.t

    - S' has a 2-expansion into D' and no vertex in D' has a neighbour outside S'

Feedback Vertex Set - Towards a Quadratic Kernel

Feedback Vertex Set - Towards a Quadratic Kernel
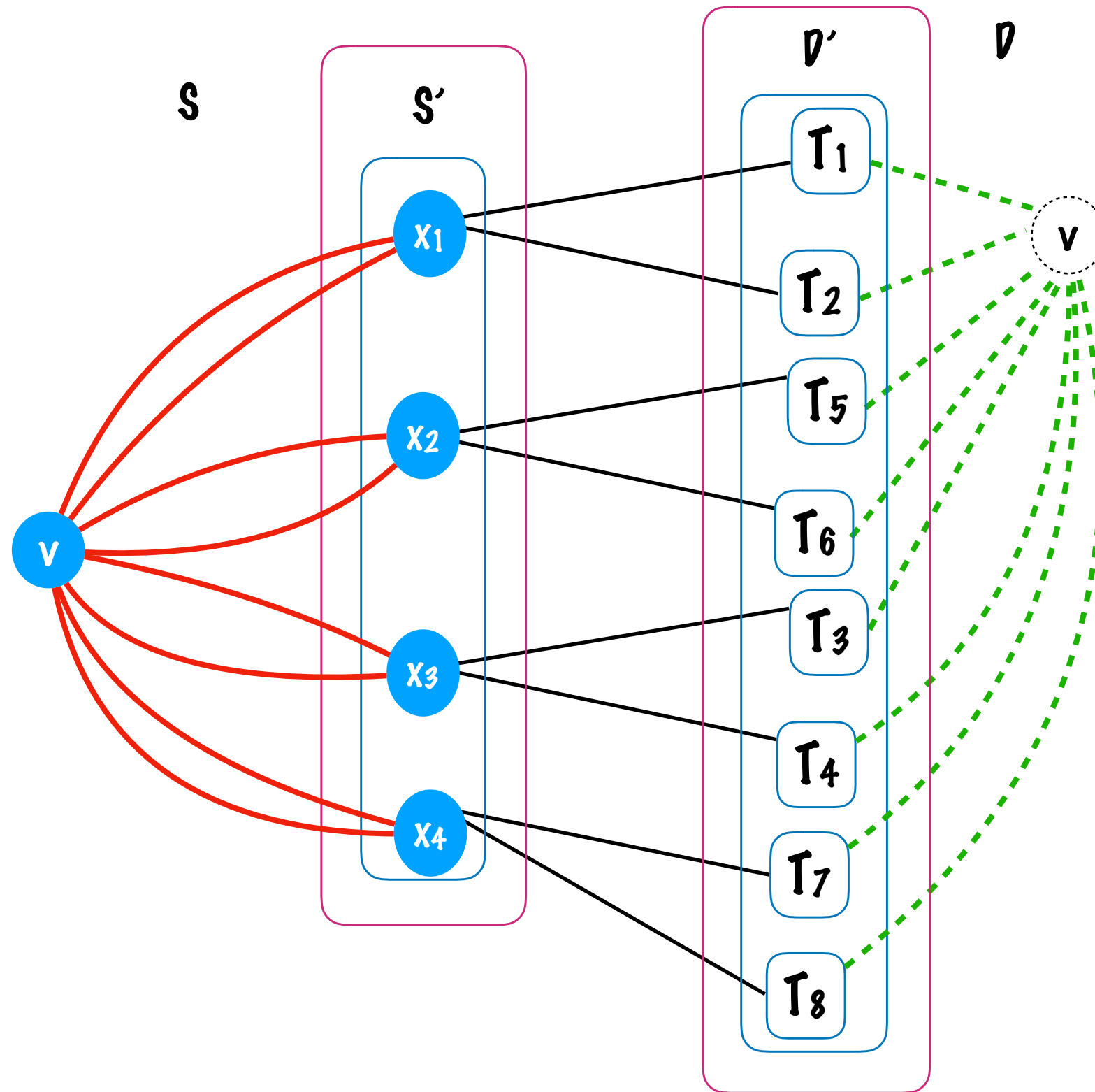
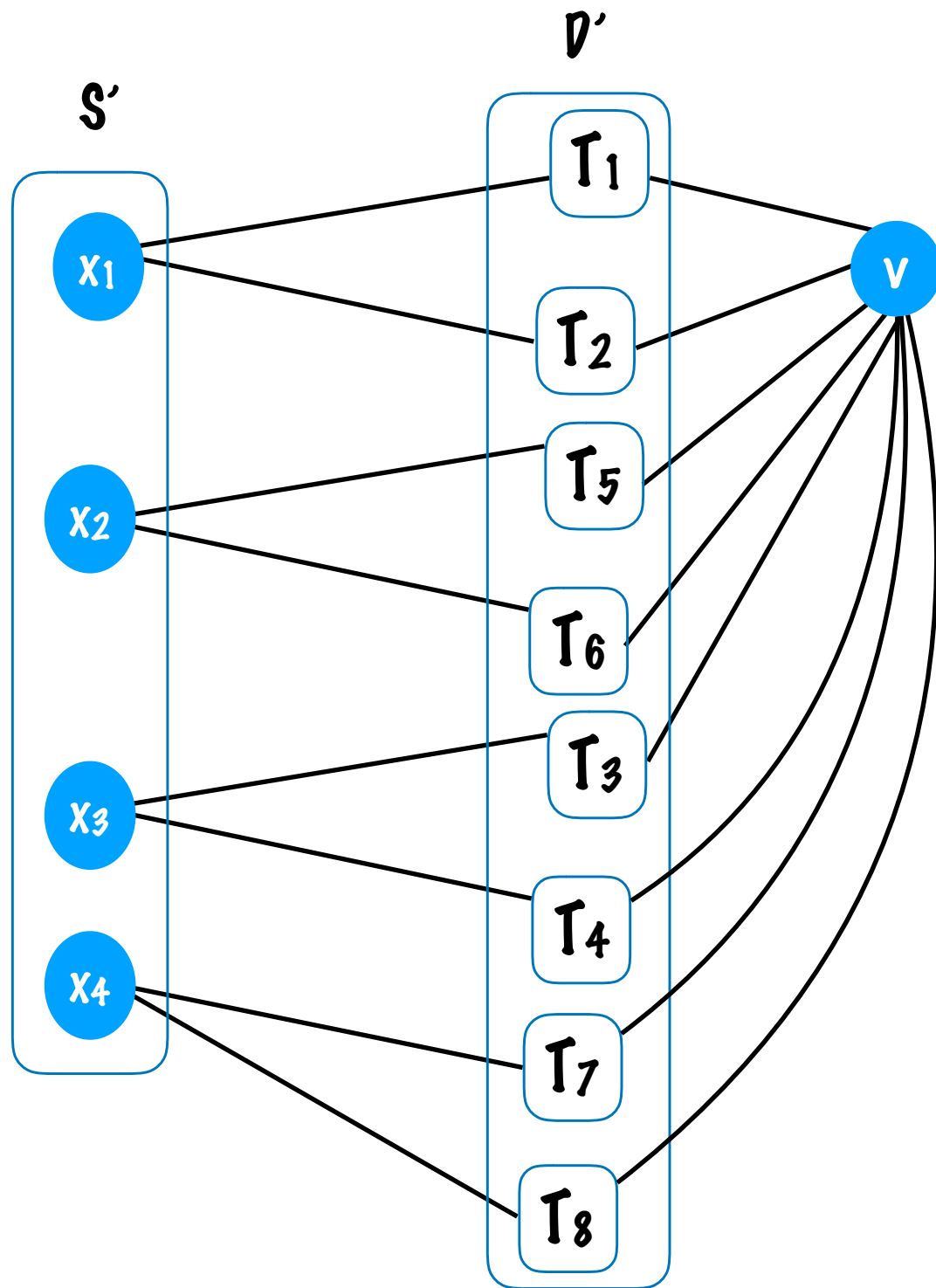**Reduction Rule 7:** Add double edges between v and every vertex in S'. Delete edges between D' to v.

Feedback Vertex Set - Towards a Quadratic Kernel
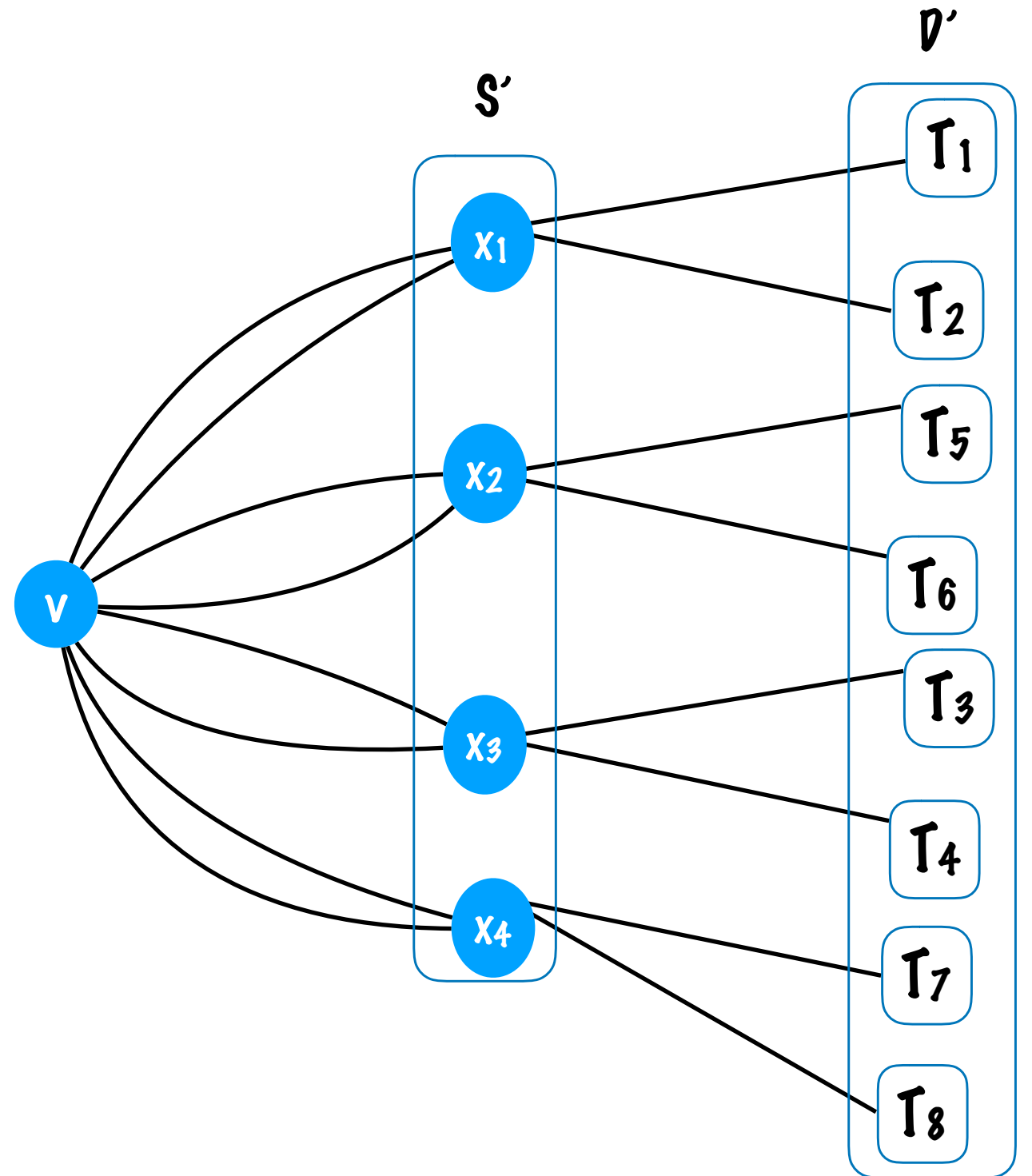
Feedback Vertex Set - Towards a Quadratic Kernel

Suppose G' has FVS W of $\leq k$. Then, v is in W or S' is in W

* Case: v is in W

    * $G-v = G'-v$ and W is FVS of G too

* Case: S' is in W

    * Any cycle in G passing through T_i in D' also passes through a vertex in S'

Feedback Vertex Set - Towards a Quadratic Kernel

Suppose G has FVS W of <=k. Then, G has <=k FVS that either has v or S'

Feedback Vertex Set - Towards a Quadratic Kernel

Claim: If G has <=k FVS, then G has <=k FVS that either has v or S'

S'

D'

T_1
T_2
T_5
T_6
T_3
T_4
T_7
T_8

x_1
x_2
x_3
x_4

v

* Any FVS Z not containing v has at least |S'| vertices

* If Z does not contain some x_i from S', then Z contains a vertex q_i from $V(T_{i1}) \cup V(T_{i2})$

* For example,
  * Suppose x_1 and x_3 are not in Z
  * Then, q_1 (from $V(T_1) \cup V(T_2)$) and q_3 ($V(T_3) \cup V(T_4)$) are in Z
  * Replace q_1 and q_3 by x_1 and x_3 to get Z'
  * If G-Z' is not a forest, then there is a cycle C containing say q_1
    * C must have a vertex from S' ($\Rightarrow\Leftarrow$)
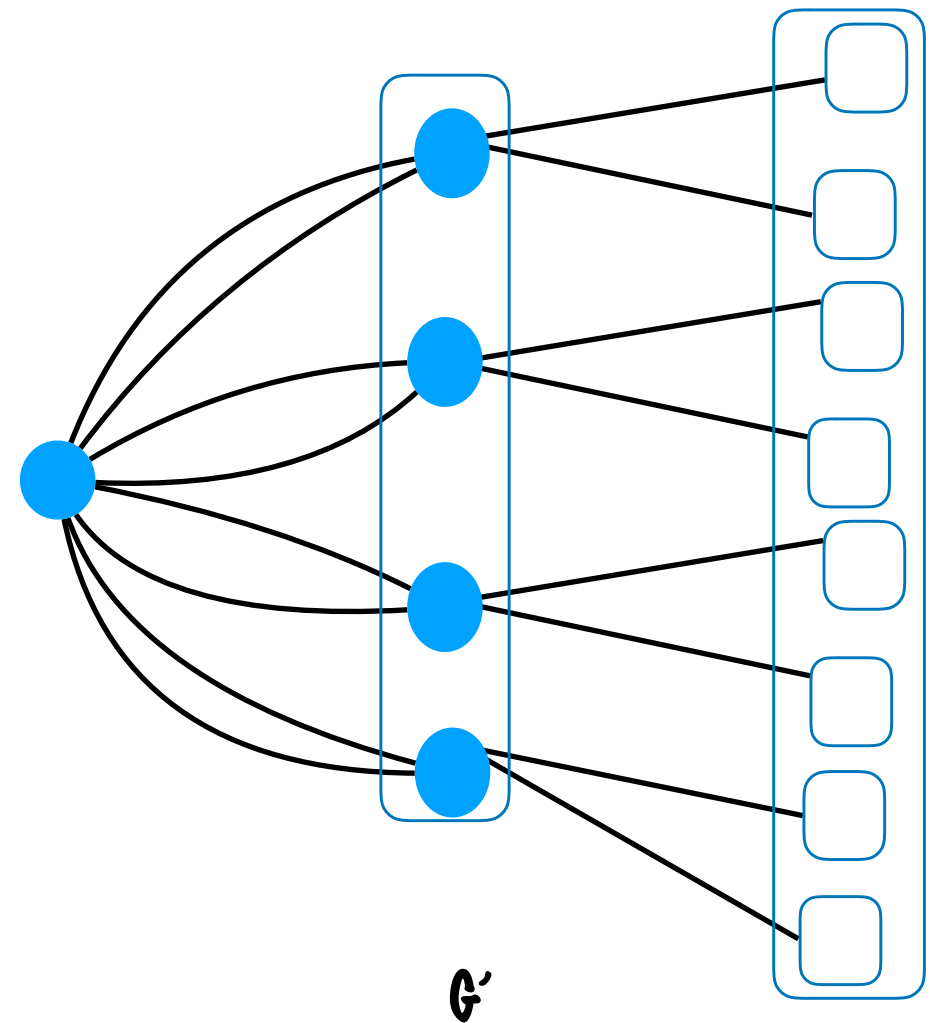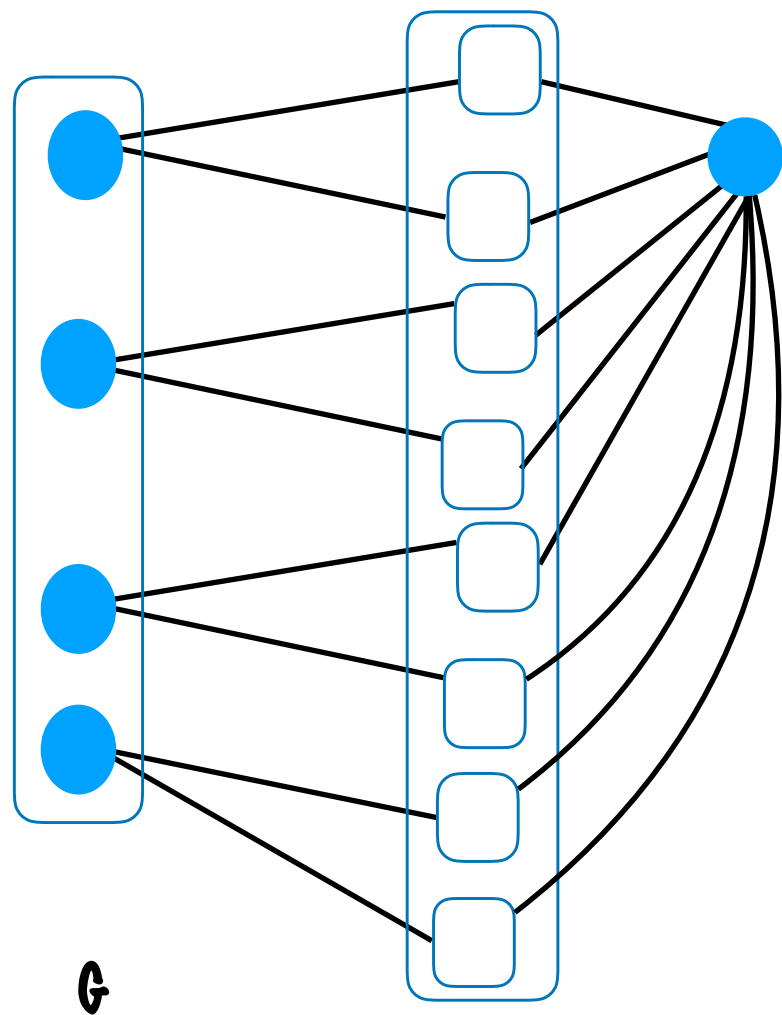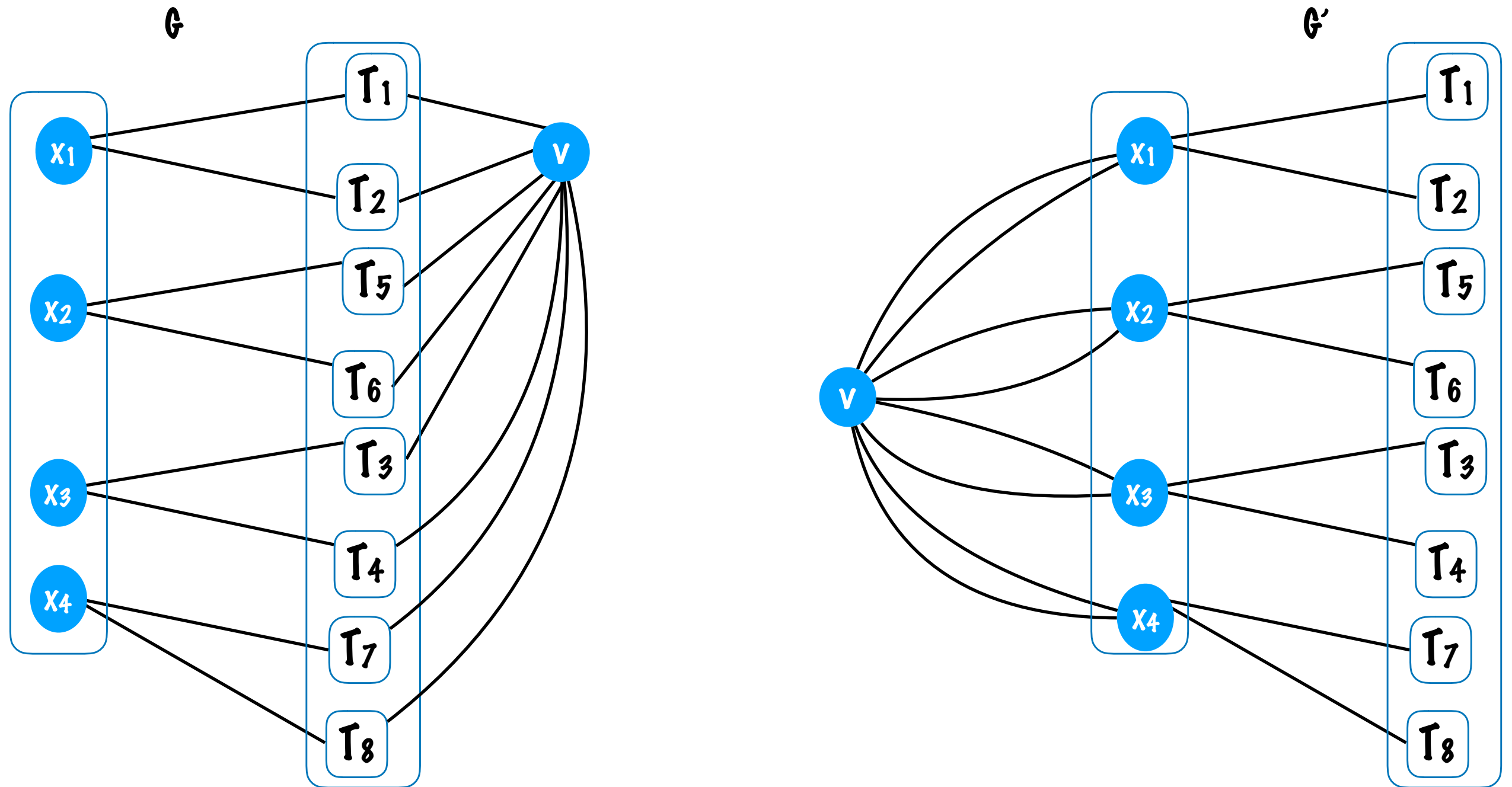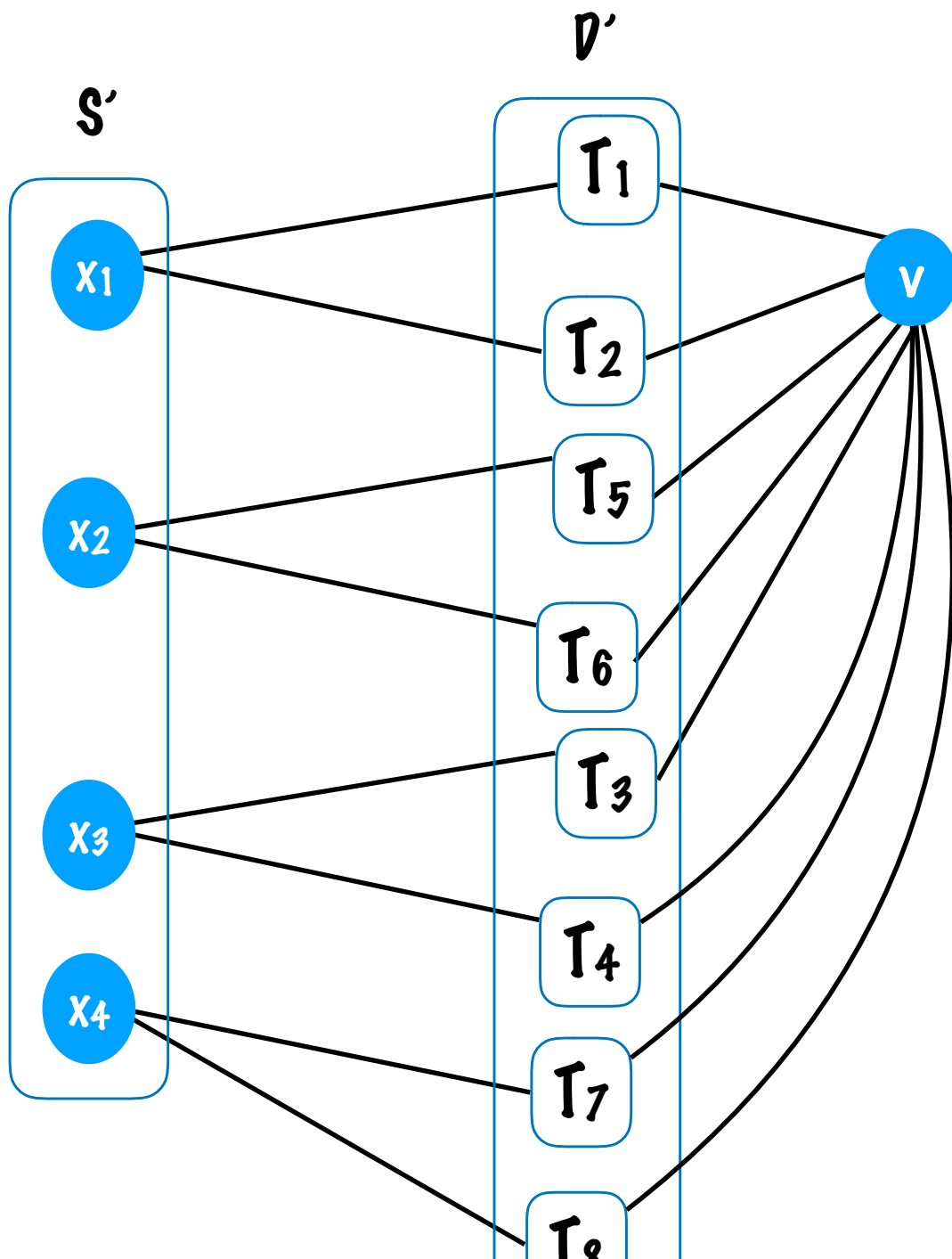
Feedback Vertex Set - Towards a Quadratic Kernel
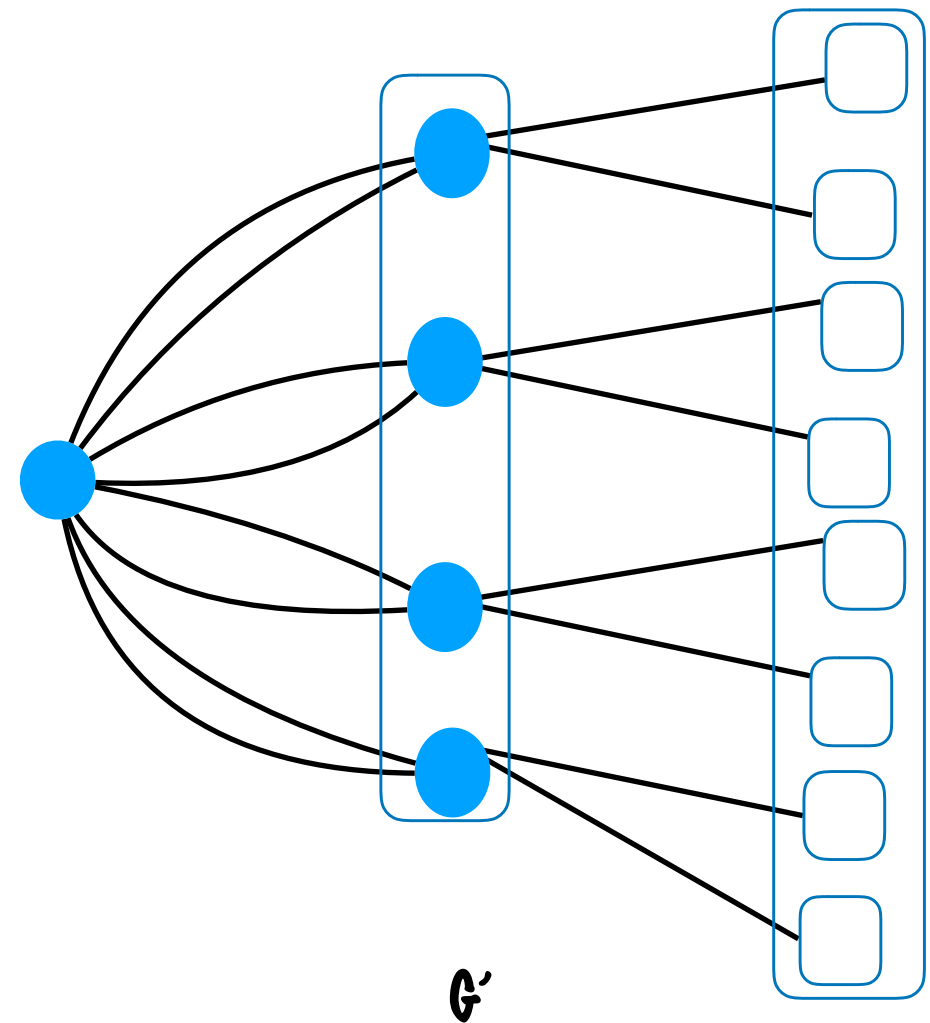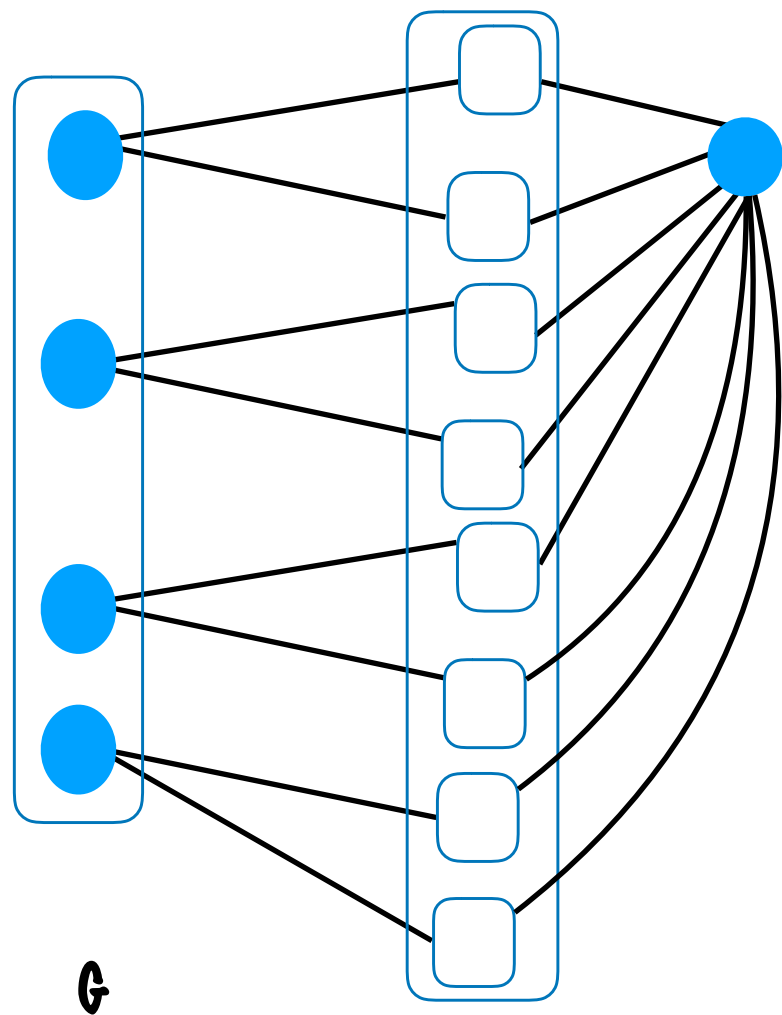
Suppose G has FVS W of <=k. Then, v is in W or S' is in W

* Case: v is in W

    * G-v = G'-v and W is FVS of G' too

* Case: S' is in W

    * Any cycle in G'-W implies a double edge between v and a vertex in T_i ($\Rightarrow\Leftarrow$)

Feedback Vertex Set - Towards a Quadratic Kernel

Reduction Rule 7: Add double edges between v and every vertex in S'. Delete edges between D' to v.
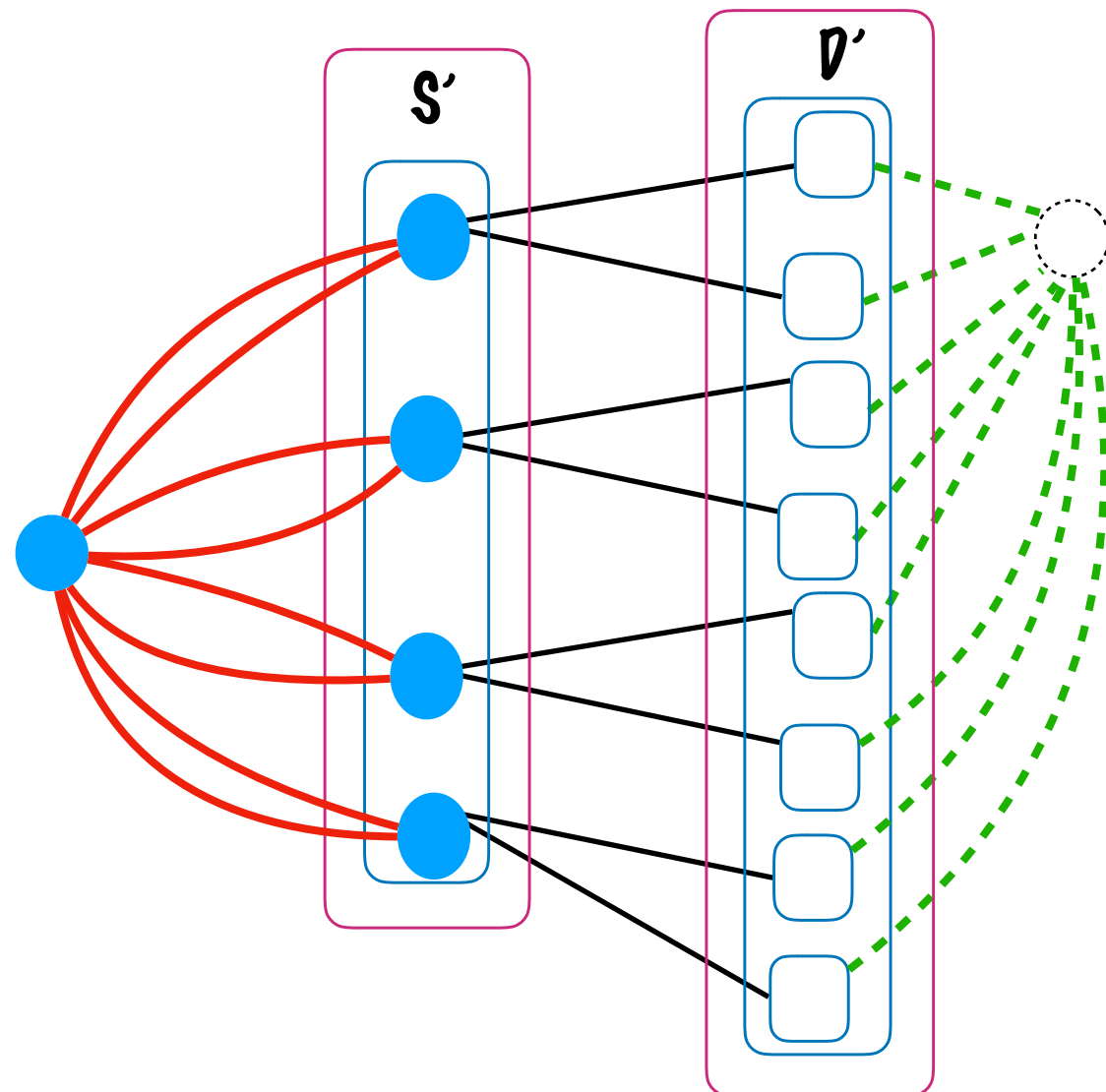
* Reduction Rule 7 can be applied <= m times
  * No. of single edges incident on v dec
* In polynomial time, either we will find a v-flower with k+1 petals, or the deg of v becomes <=10k or we will determine that (G,k) is a no-instance

Feedback Vertex Set - Quadratic Kernel

* **Reduction Rule 1:** Delete isolated vertices

* **Reduction Rule 2:** Delete degree-1 vertices

* **Reduction Rule 3:** If there is a loop at a vertex v, delete v and reduce param by 1

* **Reduction Rule 4:** If there is an edge with multiplicity > 2, reduce it to 2

* **Reduction Rule 5:** Short-circuit degree 2 vertices

* **Reduction Rule 6:** If v is a vertex of degree >10k and there is a v-flower with k+1 petals, delete v and reduce param by 1

* If v has degree >10k and has > 2k double edges, then apply **Reduction Rule 6**

* If v has degree >10k and Flower Lemma returns a FVS S of <=3k vertices s.t $v \notin S$

    * Use 2-expansion lemma to find $S' \subseteq S$ s.t if G has <=k FVS, then G has <=k FVS that either has v or S'

    * **Reduction Rule 7:** Add double edges between v and every vertex in S' and delete edges from D' to v where D' is the set of vertices in G-(S∪{v}) saturated by the 2-expansion

* When none of the reductions rules are applicable, every vertex has degree <=10k

    * $n=O(k^2)$ and $m=O(k^2)$

Flower Lemma

Flower Lemma: There is a polynomial time algorithm that given a graph G and a vertex v without a self-loop, satisfies one of the following:

* Declares (G,k) is a no-instance of Feedback Vertex Set

* Returns a v-flower with (k+1) petals

* Finds FVS not containing v of size <= 3k

Flower Lemma

* Let Z be a 2-approximation FVS of G

* If |Z| >2k, then declare that (G,k) is a no-instance of Feedback Vertex Set

* Otherwise, |Z|<=2k

* If v is not in Z, then Z is the required FVS

* Otherwise, v is in Z

T_1

T_2

Z ⓥ

. . . . .

Forest

T_r

Flower Lemma

T_1
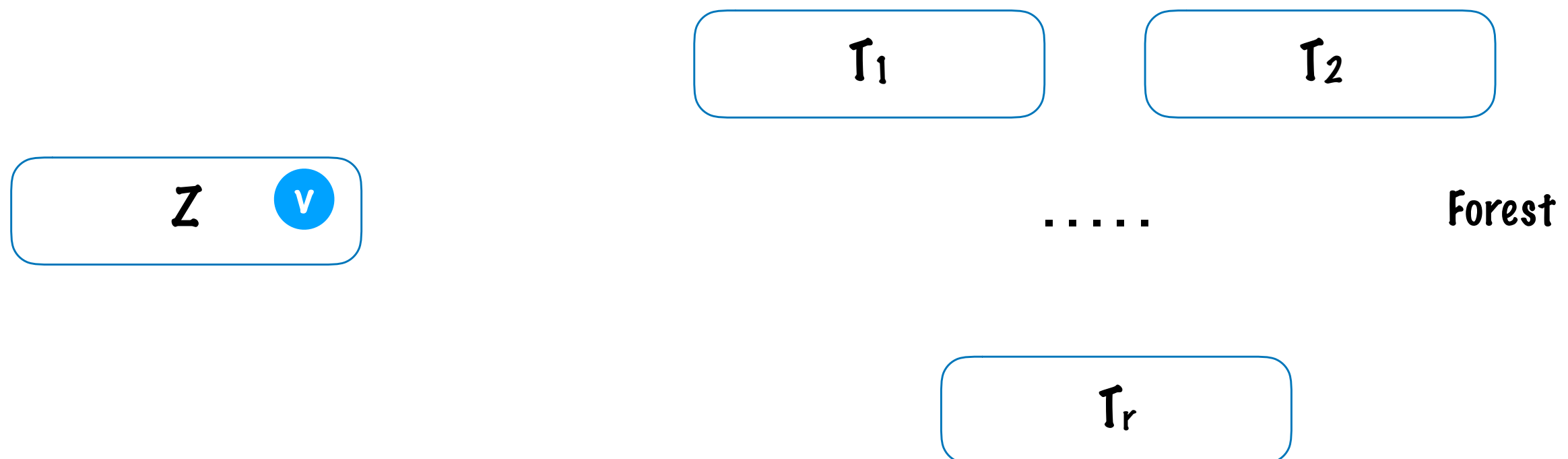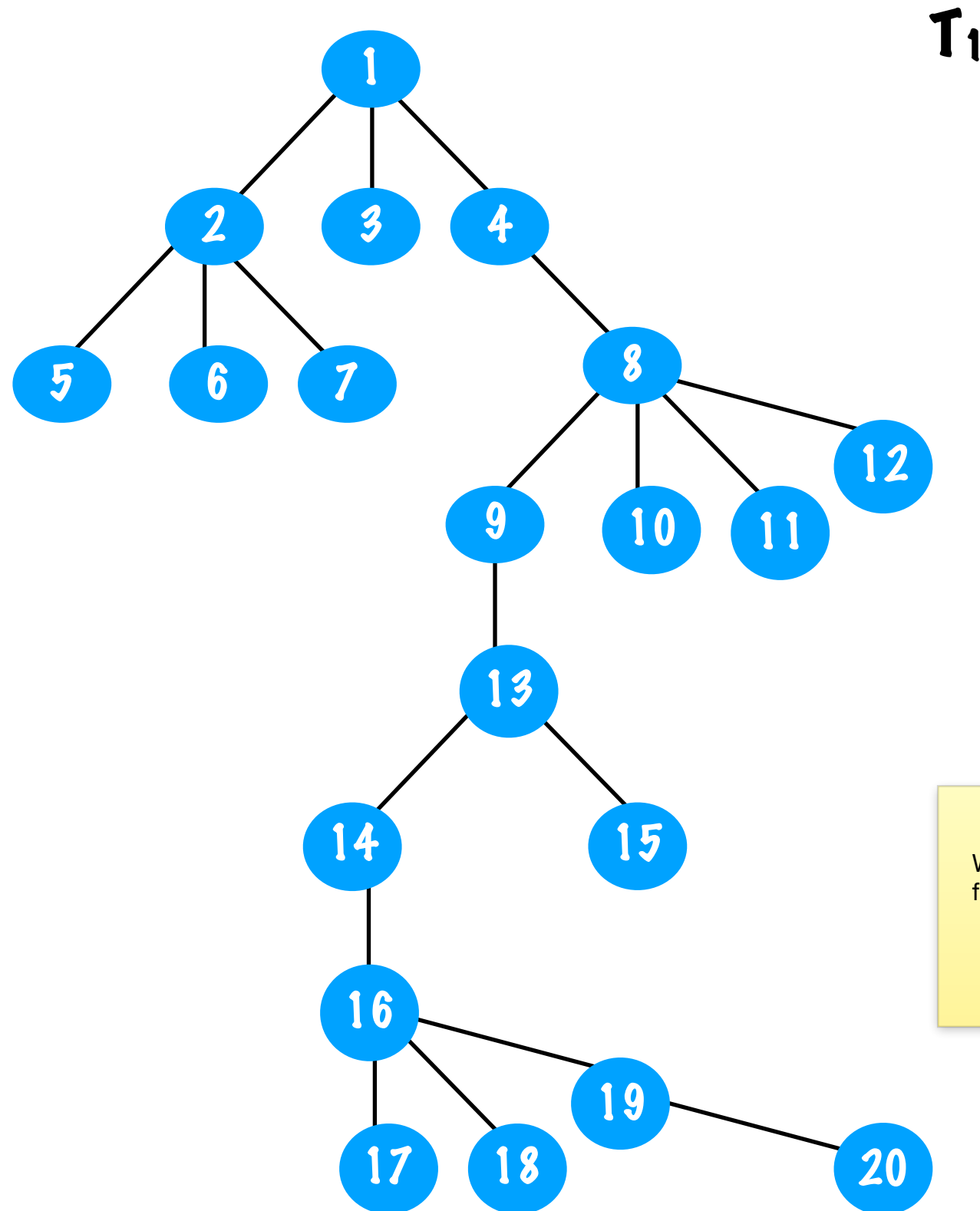
Why can't a v-flower have vertices from Z?

Flower Lemma
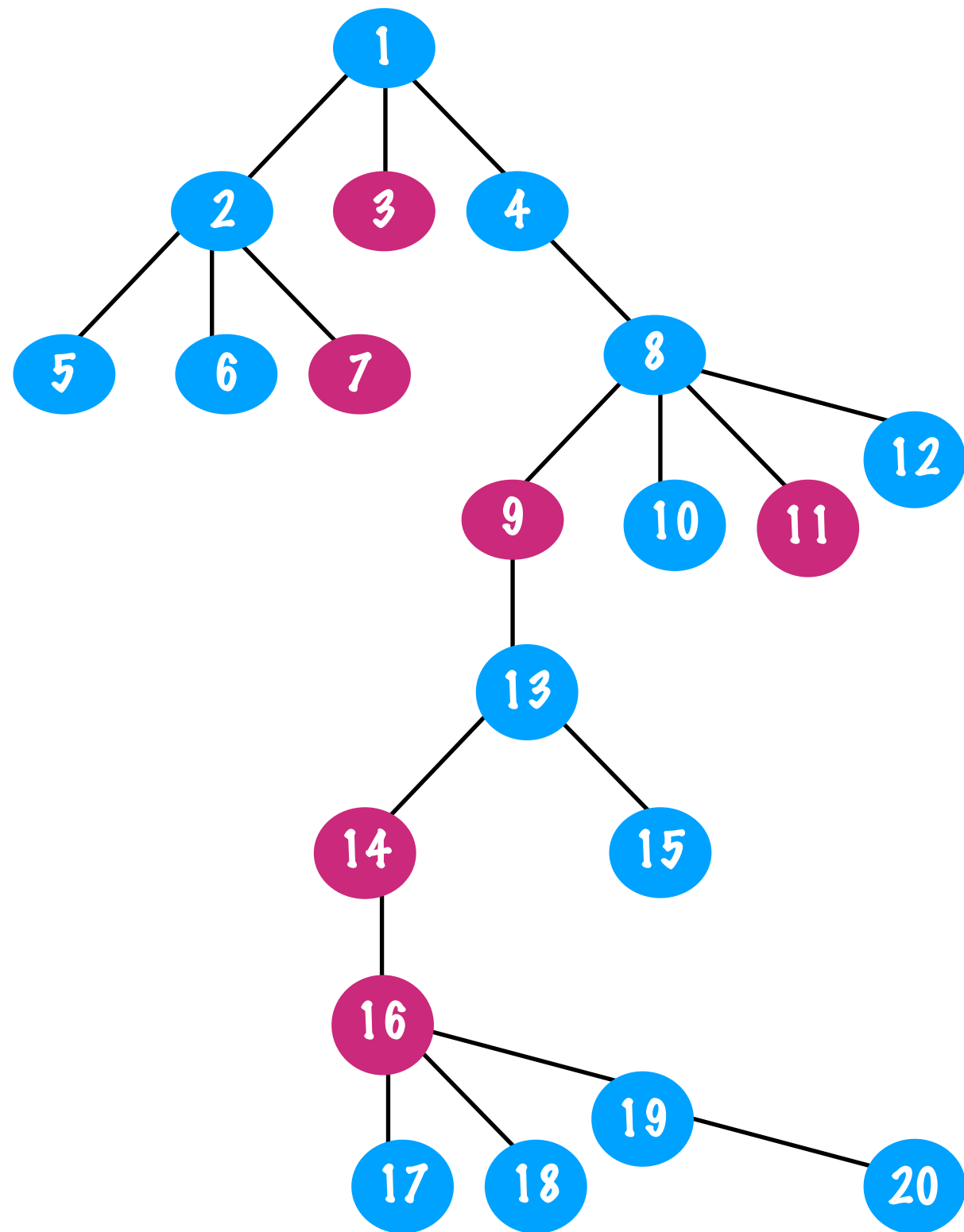
Neighbours of v

Flower Lemma
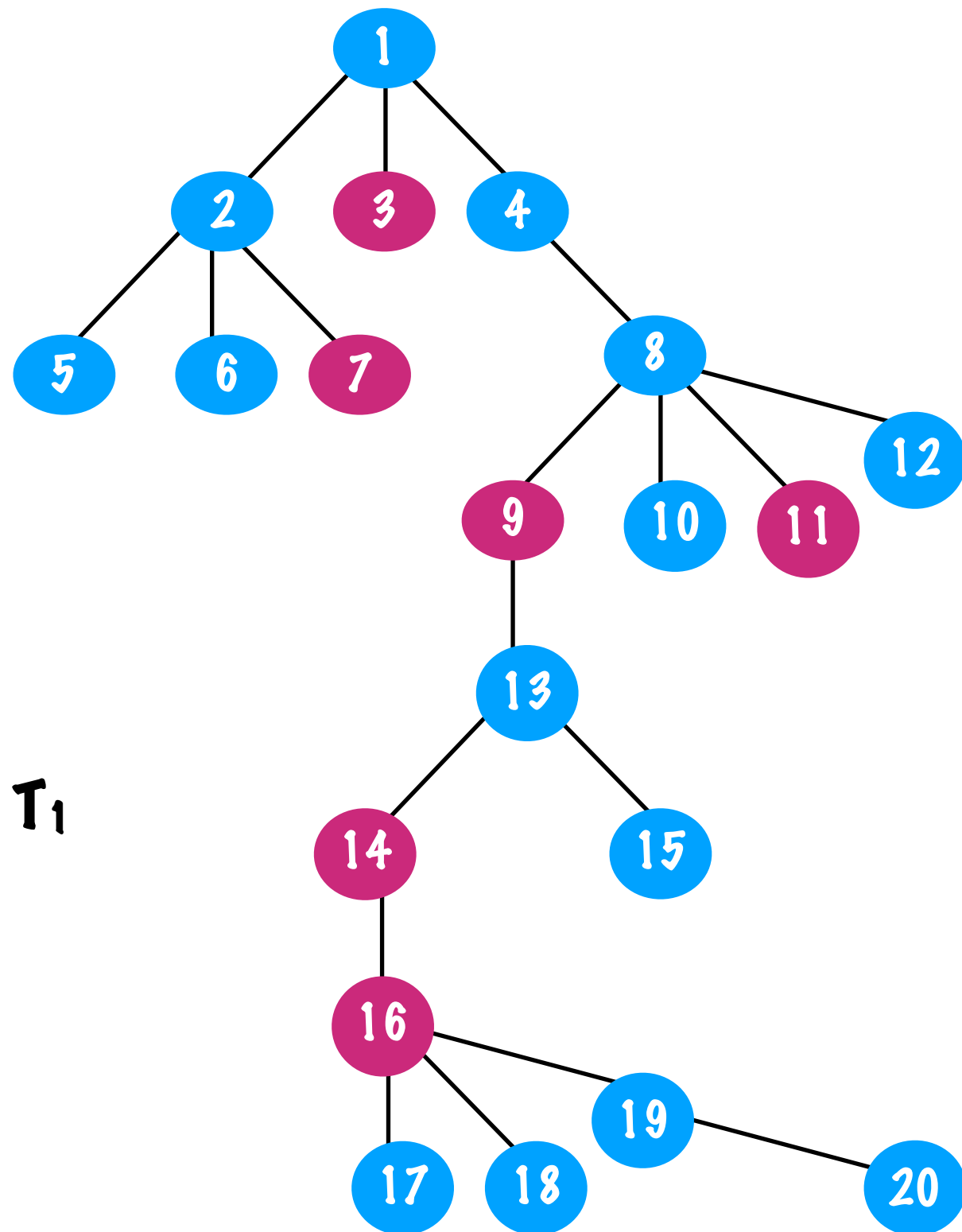
* lca(3,7)=1
* lca(14,16)=14
* lca(16,9)=9
* lca(9,11)=8
* . . . . .
* Find least common ancestor of every pair of neighbours of v

Flower Lemma

T_1
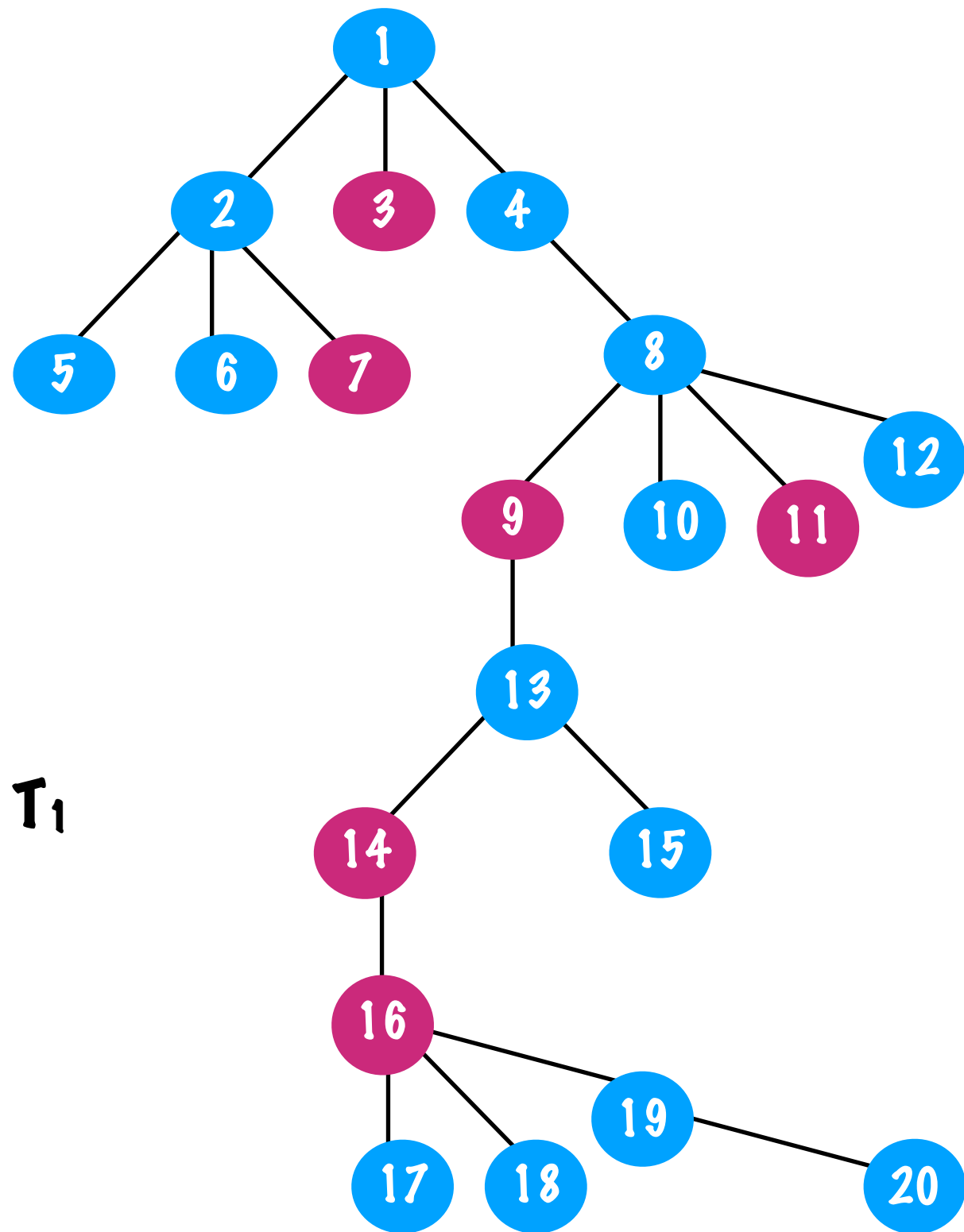
* lca(14,16)=14 is the deepest
* Add 14-16 path to P
* Add 14 to Y
* Delete subtree rooted at 14

Flower Lemma

T_1
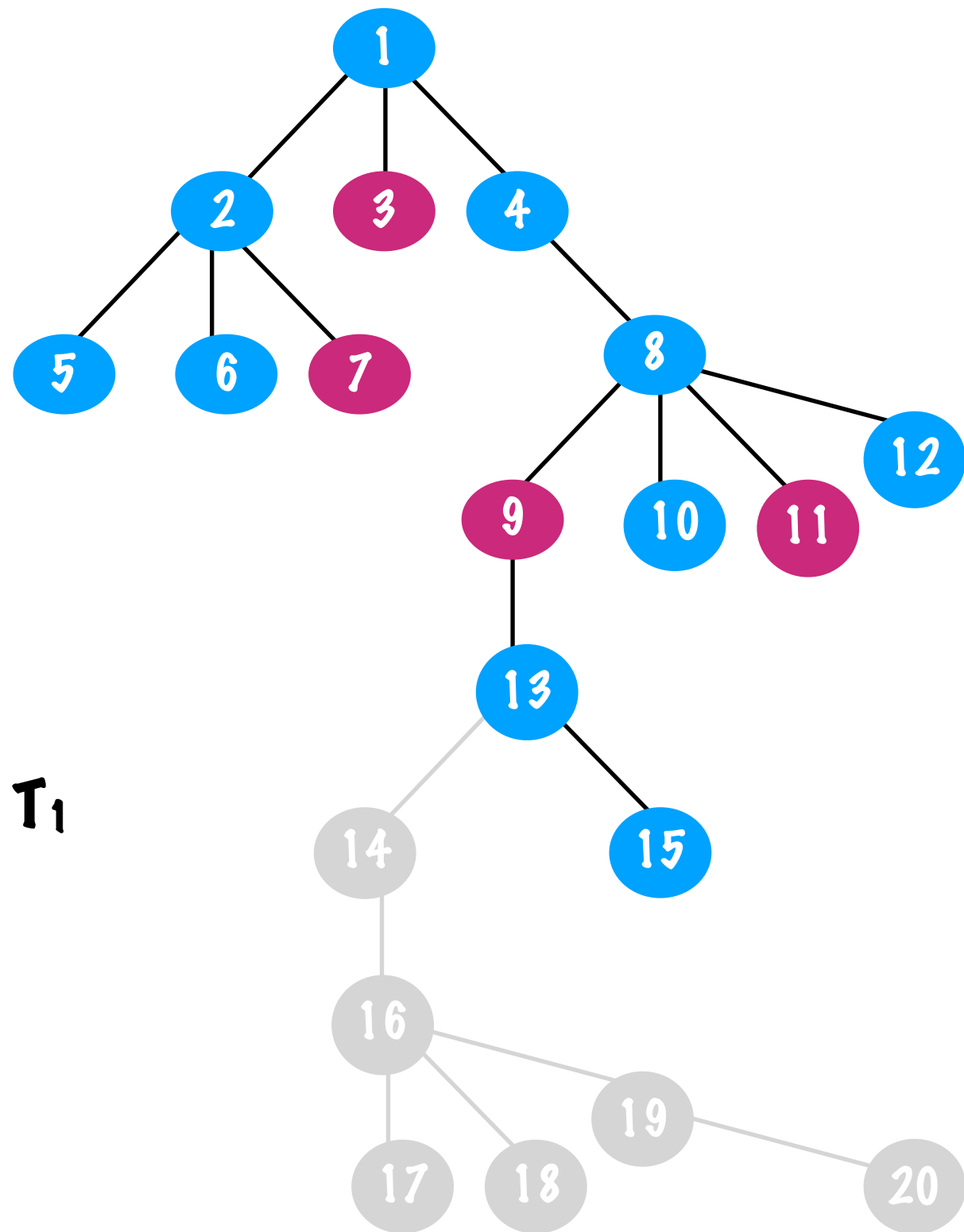
* lca(9,11)=8 is the deepest
* Add 9-8-11 path to P
* Add 8 to Y
* Delete subtree rooted at 8

Flower Lemma

T₁
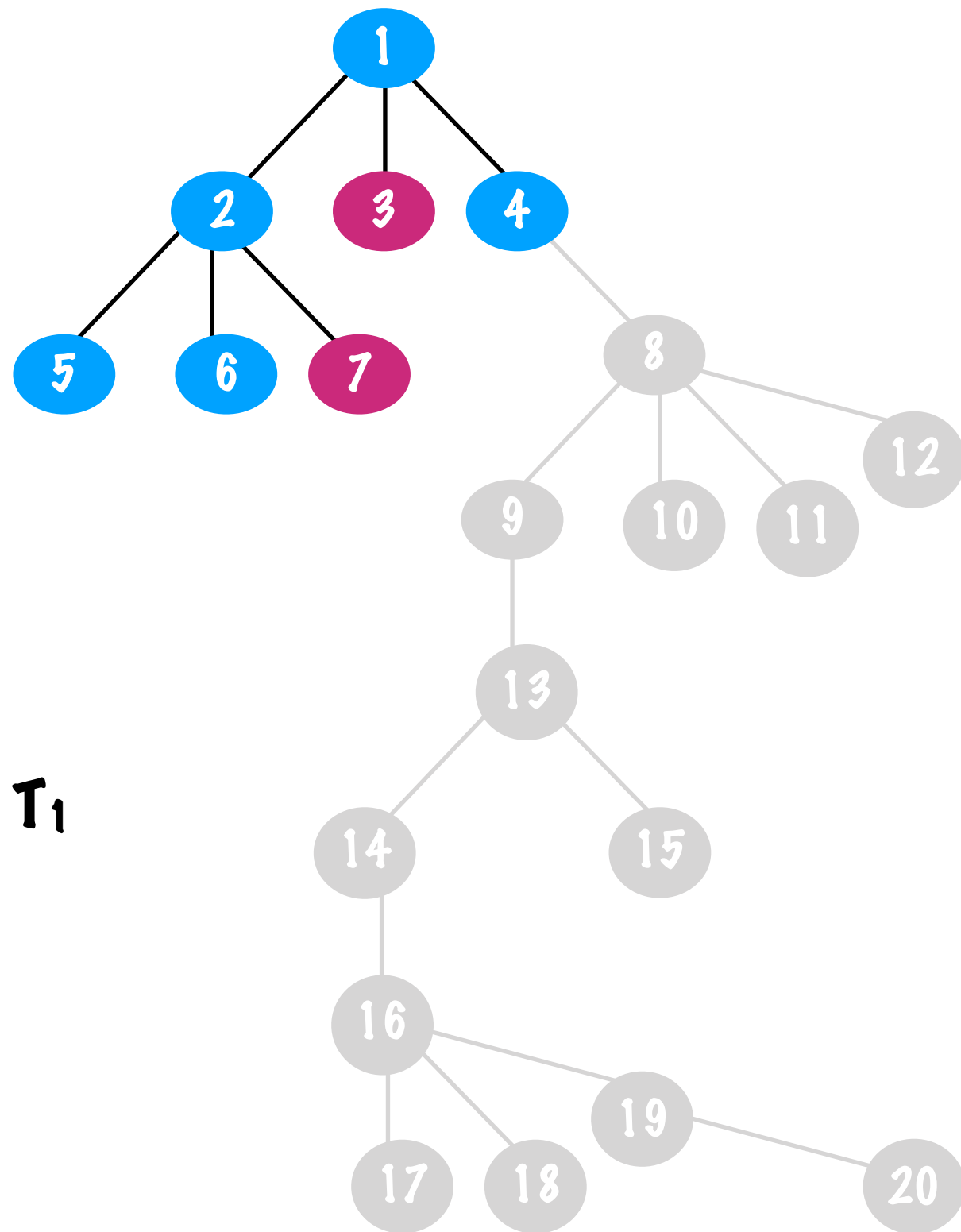
* lca(3,7)=1 is the deepest
* Add 7-2-1-3 path to P
* Add 1 to Y
* Delete subtree rooted at 1

Flower Lemma

* $P = \{7\text{-}2\text{-}1\text{-}3,\ 9\text{-}8\text{-}11,\ 14\text{-}16\}$
* $Y = \{1,\ 8,\ 14\ \}$

T_1
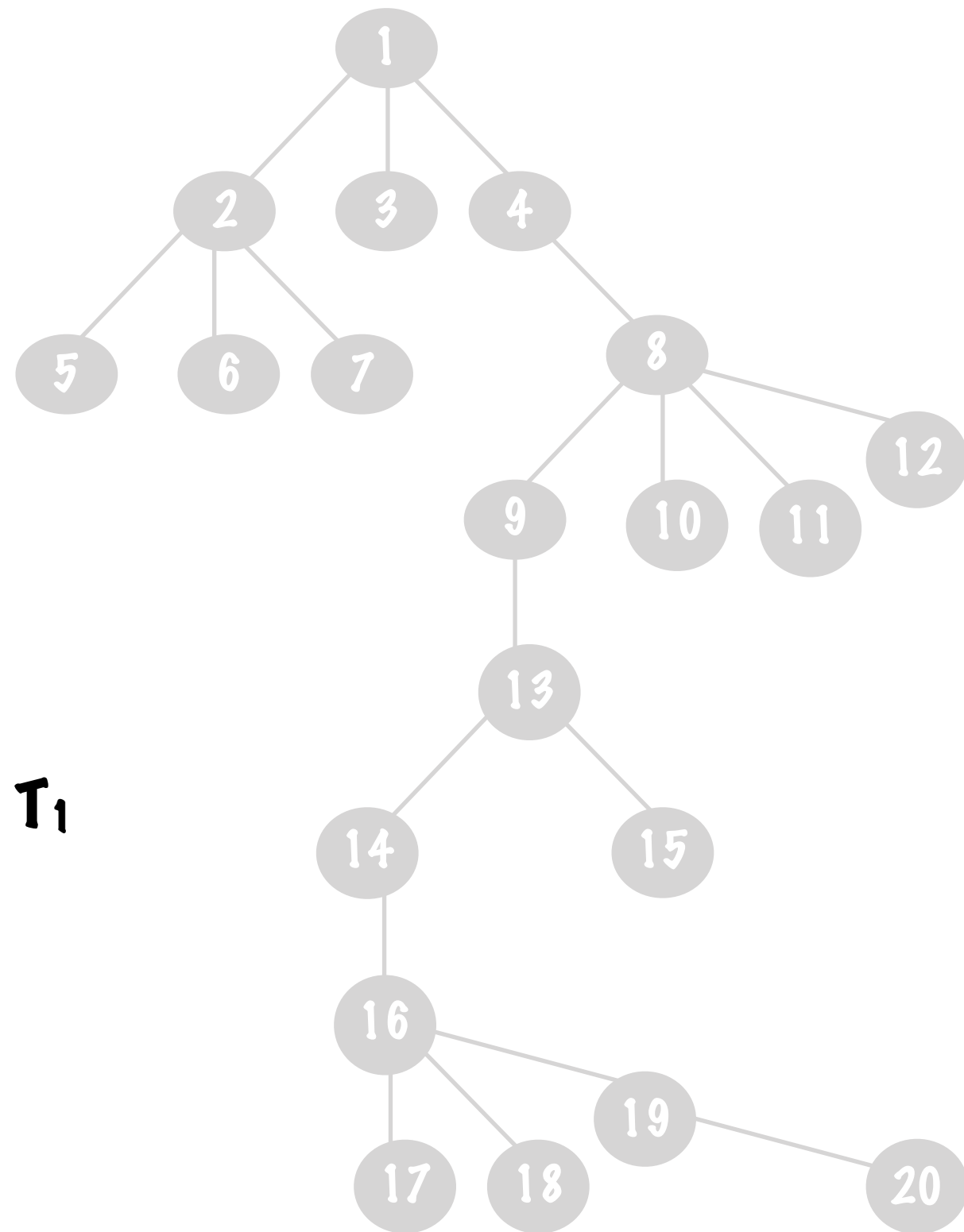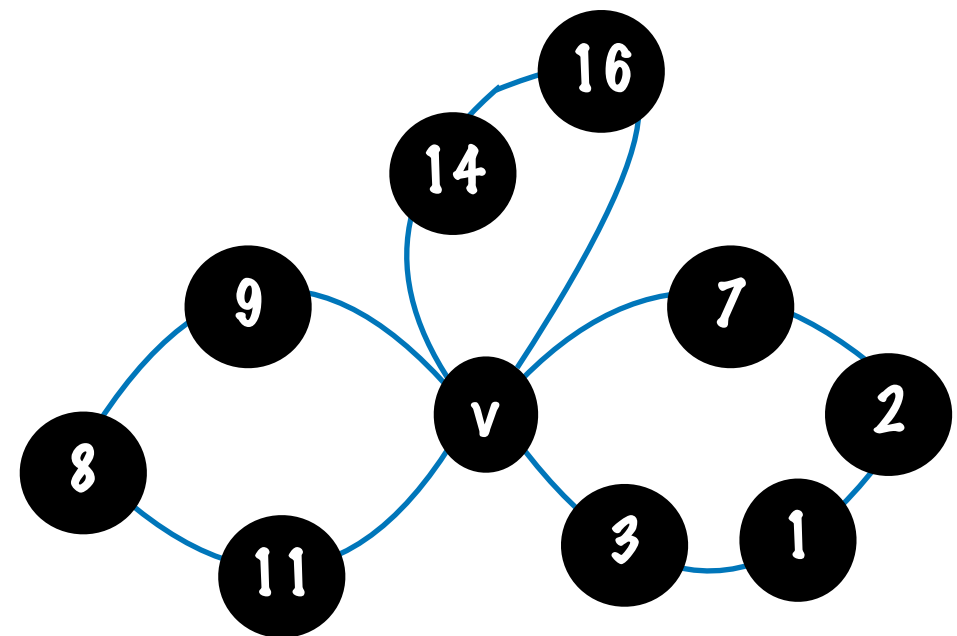
Repeat with T_2

Flower Lemma
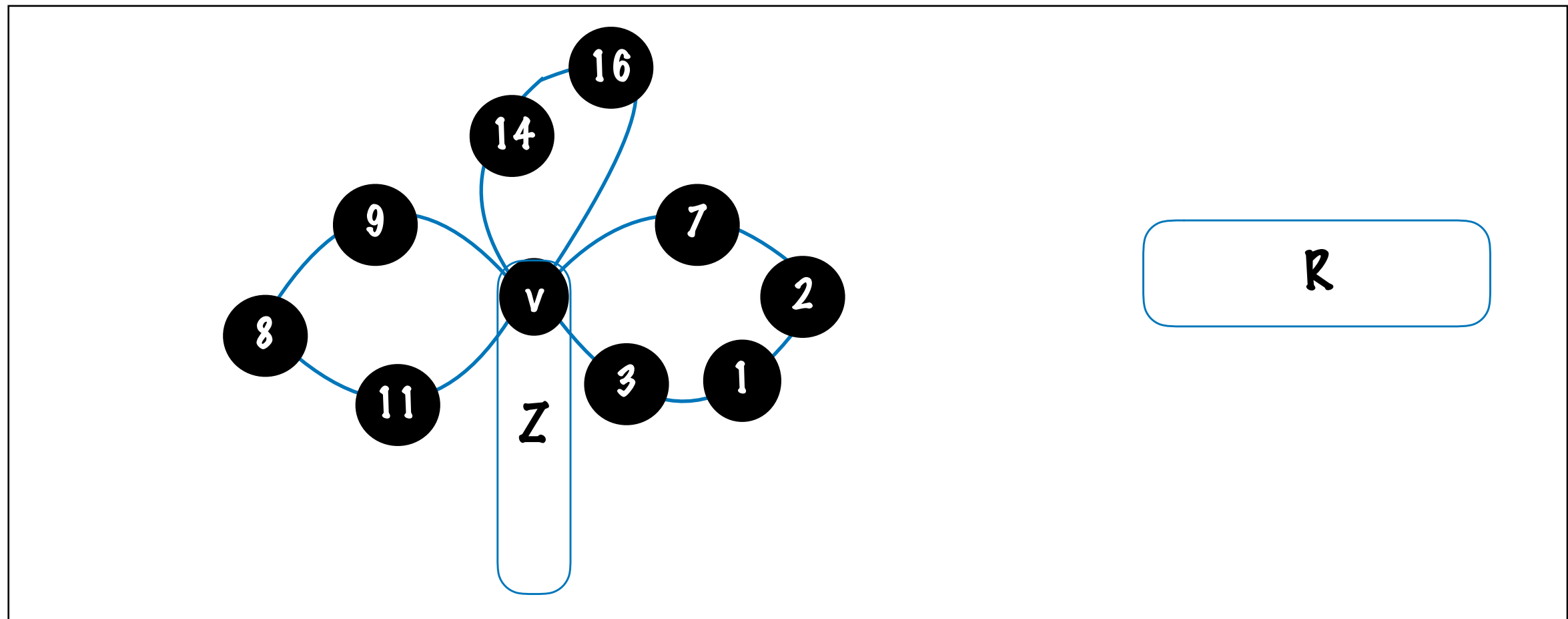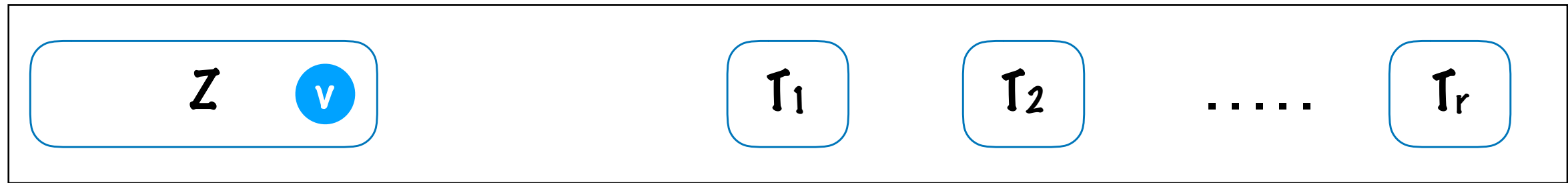
* P = {7-2-1-3, 9-8-11, 14-16}
* Y = {1, 8, 14}
* If |P| >k, then there is a v-flower with k+1 petals

Flower Lemma
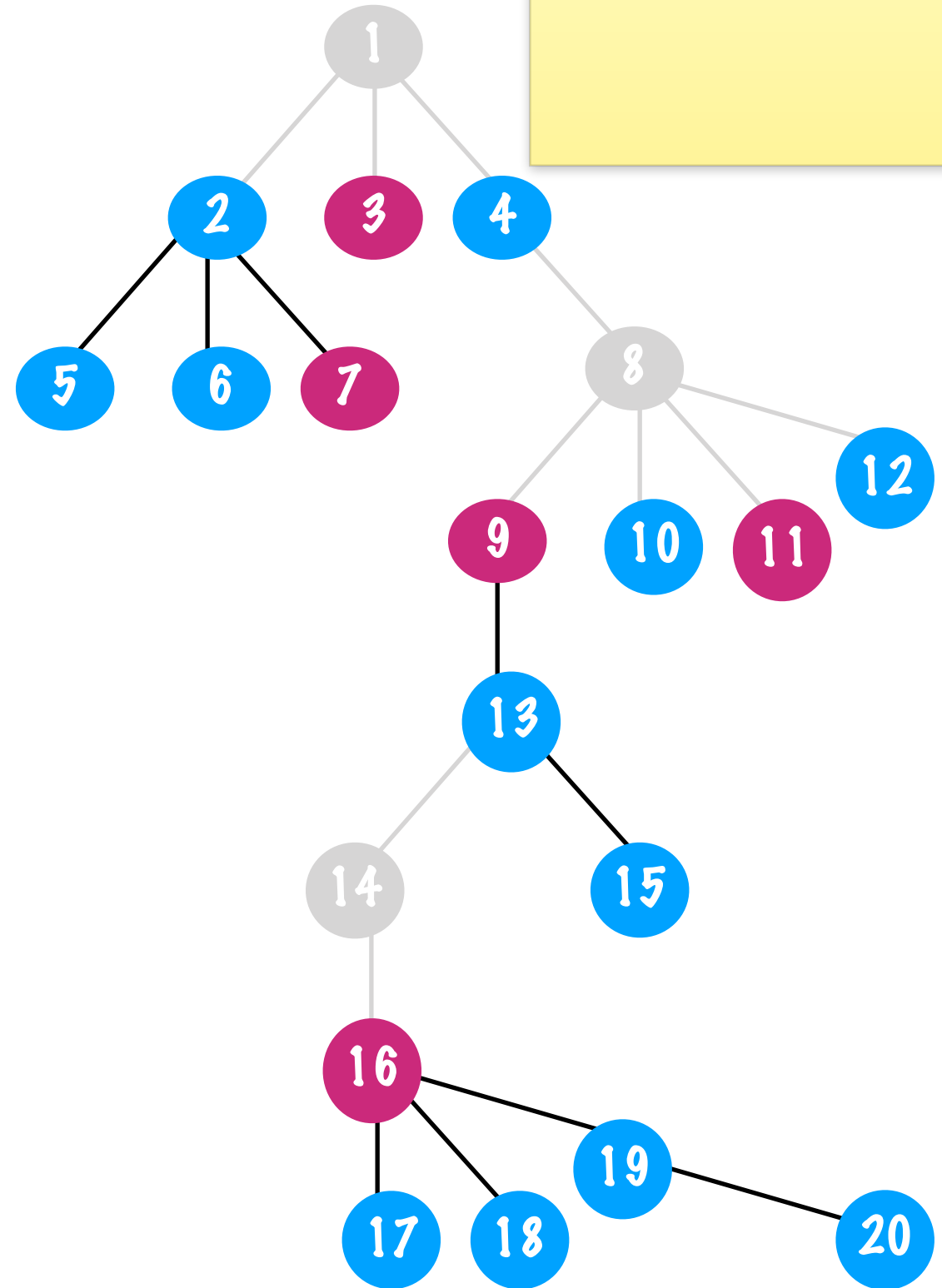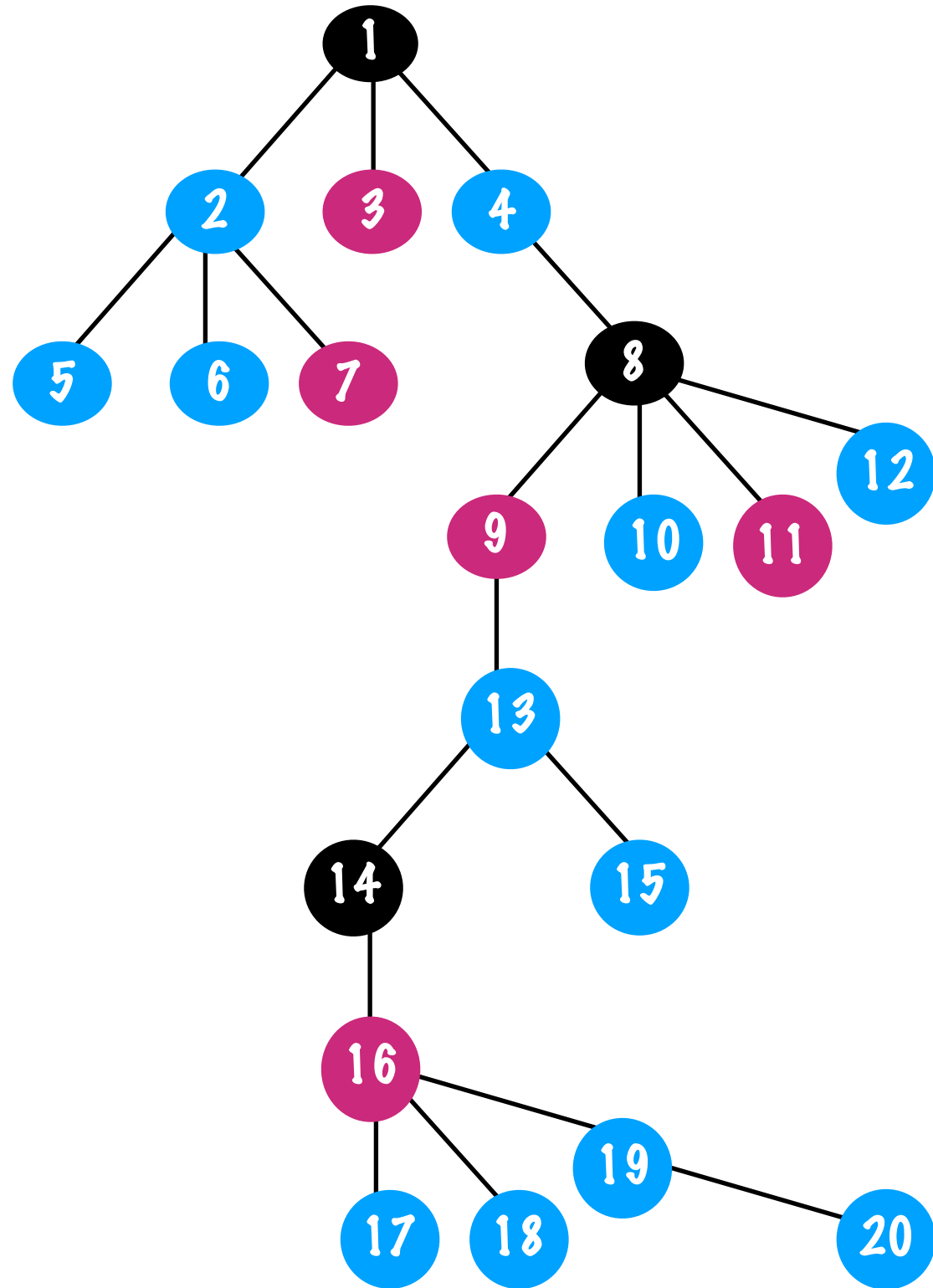
* Y = {1, 8, 14 }
* |Y| = |P| <= k and S=(Z∪Y)\{v} is the required FVS of size <=3k as no component of G-S has two neighbours of v

Matching and Vertex Cover in Bipartite Graphs

Kőnig's Theorem: For a bipartite graph, Max Mat = Min VC

Hall's Theorem: Let G be a bipartite graph with bipartition (A, B). Then, G has a matching saturating A if and only if $|N(X)| >= |X|$ for all $X \subseteq A$.

Hopcroft-Karp Algorithm: Let G be a bipartite graph with bipartition (A, B).
* Then, a max mat and a min vc of G can be obtained in $O(m\, n^{1/2})$ time.
* Further, in $O(m\, n^{1/2})$ time, we can either find a matching saturating A or an inclusion-minimal set $X \subseteq A$ such that $|N(X)| < |X|$.

Kőnig's Theorem

For a bipartite graph, |Max Mat| = |Min VC|

* Let M be a maximum matching of G(A,B)
* Let D be a digraph obtained from G by orienting edges of G as follows:
    * If edge {a,b} with a $\in$ A and b $\in$ B is in M, direct {a,b} as (b,a)
    * If edge {a,b} with a $\in$ A and b $\in$ B is not in M, direct {a,b} as (a,b)
* There is no M-alternating path in D from a free vertex in A to a free vertex in B
* If |M|=|A| then A is a min vertex cover
* L = set of vertices reachable from a free vertex in A

Claim: $(A \backslash L) \cup (B \cap L)$ is a min vertex cover of G

König's Theorem

Claim: $S = (A \setminus L) \cup (B \cap L)$ **is a min vertex cover of G**
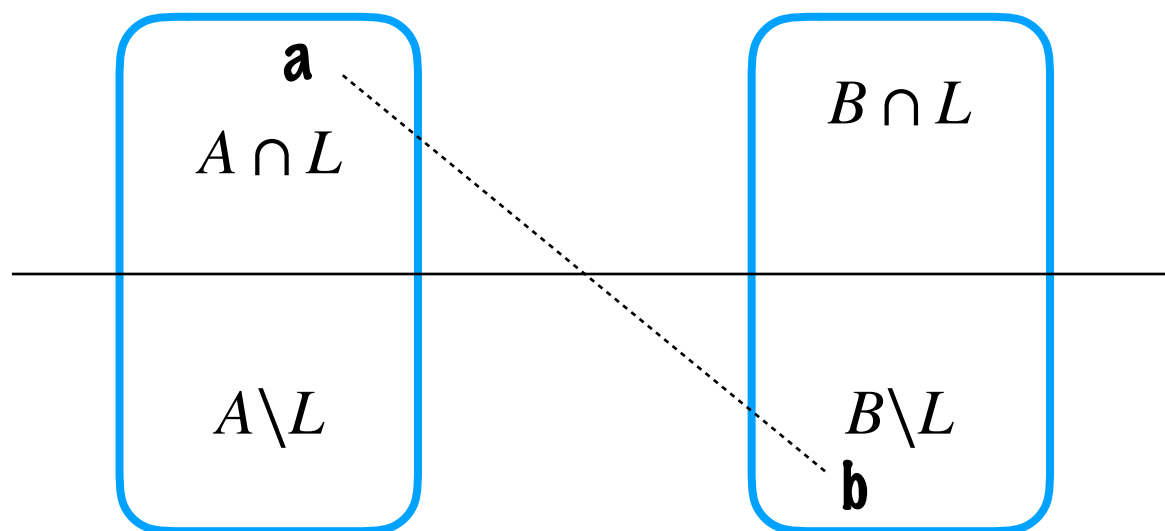
* If $\{a,b\} \in M$ with $b \in B \setminus L$ and $a \in A \cap L$
  * $(b,a) \in E(D)$
  * b is the only in-nbr of $a \in L$
  * $\therefore b \in L$
* If $\{a,b\} \notin M$
  * $(a,b) \in E(D)$
  * $\therefore b \in L$ as $a \in L$
* No vertex $a' \in A \setminus L$ is free by defn of L
* No vertex $b' \in B \cap L$ is free
  * o/w, M-alternating path from a free vertex in A to b' as $b' \in L$
* No edge $\{a',b'\}$ with $a' \in A \setminus L$ and $b' \in B \cap L$ is in M
  * o/w, $(b',a') \in E(D)$ and $a' \in L$
* $\therefore |S| = |M|$

Hall's Theorem

Let G be a bipartite graph with bipartition (A, B). Then, G has a matching saturating A if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$.

Sufficiency: Induction on $|A|$

* Base: $|A|=1$

* Induction Step: Let $a \in A$ and b be one of its neighbours

* If $|N(X)\backslash\{b\}| \geq |X|$ for all $X \subseteq A\backslash\{a\}$

    * By induction hypothesis, there is a matching M saturating $A\backslash\{a\}$ in $G-\{a,b\}$
    * Then, $M \cup \{\{a,b\}\}$ is a matching saturating A in G

* Otherwise, $|N(X)\backslash\{b\}| < |X|$ for some $X \subseteq A\backslash\{a\}$

    * As $|N(X)| \geq |X|$, it follows that $|N(X)| = |X|$
    * By induction hypothesis there is a matching M saturating X in $G(X, N(X))$
    * For any $Y \subseteq A\backslash X$, $|N(Y)\backslash N(X)| \geq |N(Y \cup X)| - |N(X)| \geq |Y|+ |X| - |N(X)| = |Y|$
    * By induction hypothesis there is a matching M' saturating $A\backslash X$ in $G(A\backslash X, B\backslash N(X))$
    * $M \cup M'$ is a matching saturating A

Hopcroft-Karp Algorithm

Let G be a bipartite graph with bipartition (A, B).

* Then, a max mat and a min vc of G can be obtained in $O(m\, n^{1/2})$ time.

* Further, in $O(m\, n^{1/2})$ time, we can either find a matching saturating A or an inclusion-minimal set $X \subseteq A$ such that $|N(X)| < |X|$.

Finding Minimal Hall Set

* Suppose M is a maximum matching of G(A,B) such that $|M| < |A|$

* Let $A=\{a_1, a_2, a_3, \ldots, a_r\}$ and let $X_1=\{a_1\}$ where a_1 is a vertex not saturated by M

  * Every vertex in $N(a_1)$ is saturated by M

* For i=1 to r-1

  * If $|N(X_i)| < |X_i|$, Return X_i

  * $X_{i+1} = X_i \cup$ M-partners($N(X_i)$) (Note: $|X_{i+1}| > |X_i|$)

* Return X_r

Extended Hall's Theorem

Let G be a bipartite graph with bipartition (A, B). Then, there is a 2-expansion of A into B iff $|N(X)| >= 2|X|$ for all $X \subseteq A$.

Sufficiency:

* To G(A,B), add a copy of A to get bipartite graph G'(A',B)

* If G'(A',B) has a matching saturating A', then this matching corresponds to a 2-exp of A into B

* Otherwise, by Hall's theorem, $|N_{G'}(X)| < |X|$ for some $X \subseteq A'$

    * w.l.o.g assume X has both copies or no copy of a vertex of A

    * $|N_{G'}(X)| = |N_{G'}(X \cap A)| = |N_G(X \cap A)| >= 2|X \cap A| = |X|$ (a contradiction)

2-Expansion Lemma

Let G be a bipartite graph with bipartition (A, B) s.t $|B| \geq 2 |A|$ and there are no isolated vertices in B. Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a 2-exp into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m \; n^{1/2})$ time.

* If A has a 2-expansion M into B, then $X = A$ and $Y = V(M) \cap B$

    * Clearly, $N(Y) \subseteq X$

* Otherwise, by Extended Hall's theorem, $|N_G(X)| < 2|X|$ for some $X \subset A$

* Let G'(A',B') denote the graph obtained from G by deleting X and $N_G(X)$

* $|B'| = |B| - |N_G(X)| > |B| - 2|X| \geq 2|A| - 2|X| = 2|A'|$

* In G, no vertex in B' has a neighbour in X. So, B' has no isolated vertices in G'

* By induction hypothesis, there exists non-empty sets $X' \subseteq A'$ and $Y' \subseteq B'$ such that X' has a 2-exp into Y' and $N_{G'}(Y') \subseteq X'$

* As $N_G(Y') = N_{G'}(Y')$, it follows that $N_G(Y') \subseteq X'$

Ex: Runtime analysis

q-Expansion Lemma

Definition: In a bipartite graph with bipartition (A, B), a set M of edges is called a **q-expansion** from A to B if
* Every vertex of A is incident with exactly q edges of M
* M saturates exactly q|A| vertices in B
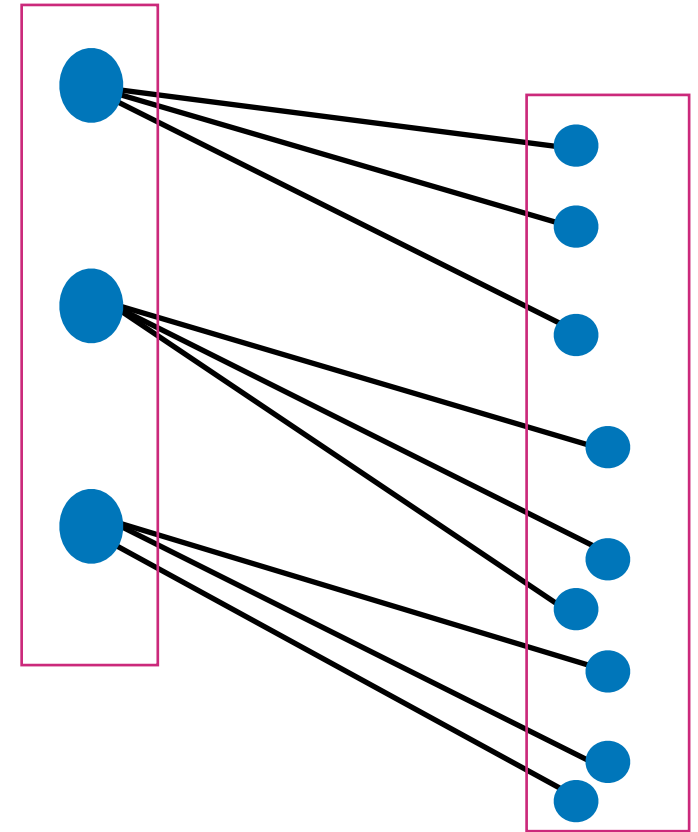
Extended Hall's Theorem: Let G be a bipartite graph with bipartition (A, B). Then, there is a q-expansion of A into B iff |N(X)| >= q|X| for all $X \subseteq A$.

q-Expansion Lemma: Let q be a positive integer and let G be a bipartite graph with bipartition (A, B) such that |B| > q |A| and there are no isolated vertices in B. Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a q-expansion into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m \, n^{1/2})$ time