CS 5003: Parameterized Algorithms

Lectures 26-27

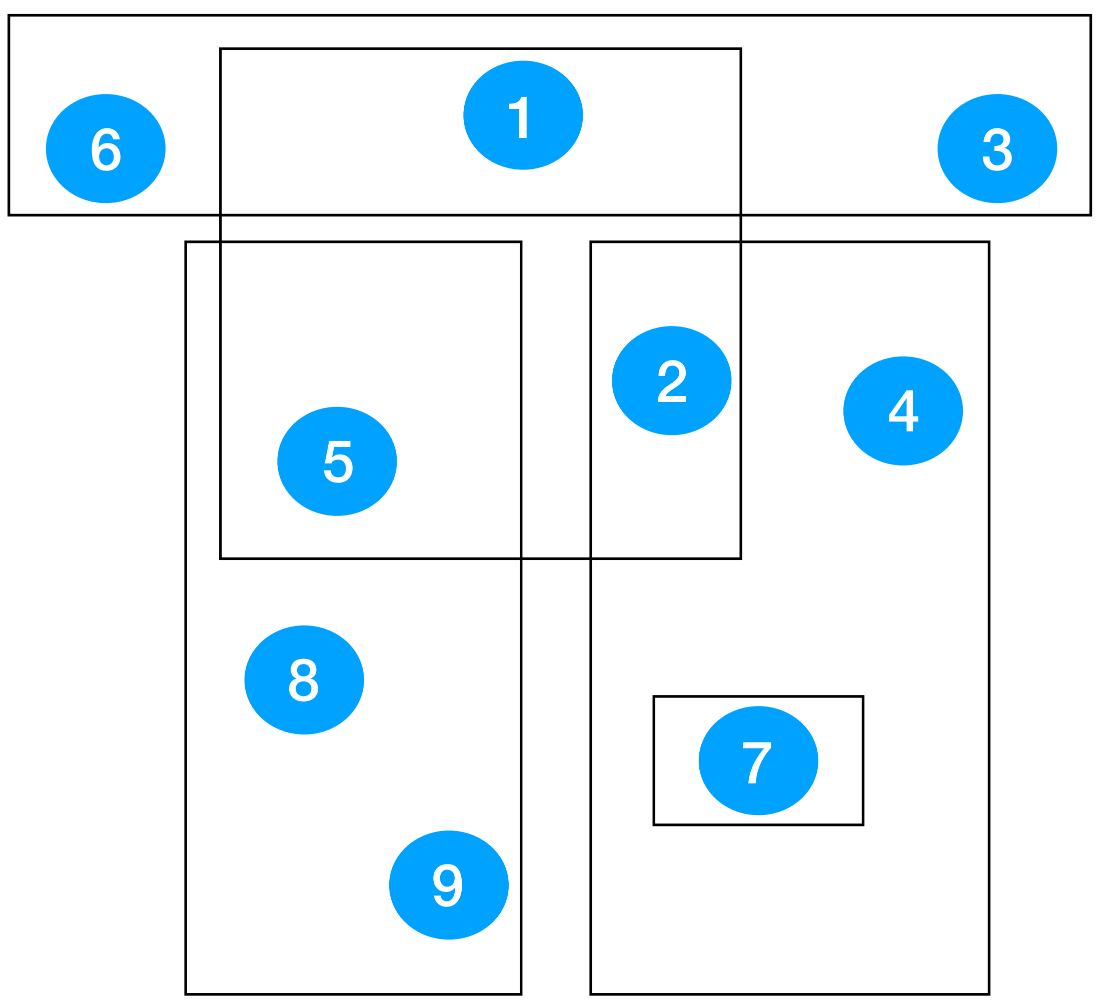
Krithika Ramaswamy

IIT Palakkad

Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Sunflower Lemma

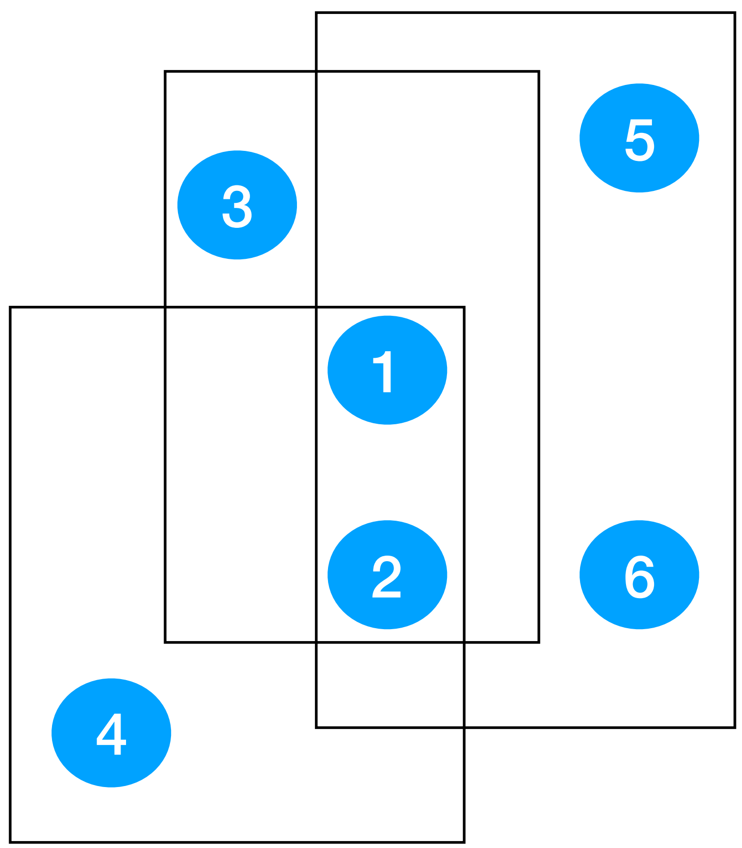
* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$



Sunflower Lemma

- Universe $U = \{1, 2, 3, \ldots, n\}$, Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
- A sunflower with k petals and core Y is a collection $C \subseteq F$ of sets s.t
  - $S \cap S' = Y$ for any two distinct S, S' in C
  - $S \setminus Y$ is non-empty for every S in C

Sunflower



- 3 Petals $\{3\}, \{4\}, \{5, 6\}$
  - Each petal is non-empty
- Core = $\{1, 2\}$

- A set of pairwise disjoint sets is a sunflower with empty core

Sunflower Lemma

* Universe $U = \{1, 2, 3, \dots, n\}$
* Family of subsets of U: $F = \{S_1, S_2, \dots, S_m\}$ (no duplicates)
* Each set in F has size d
* $|F| > d!\,(k-1)^d$

↓

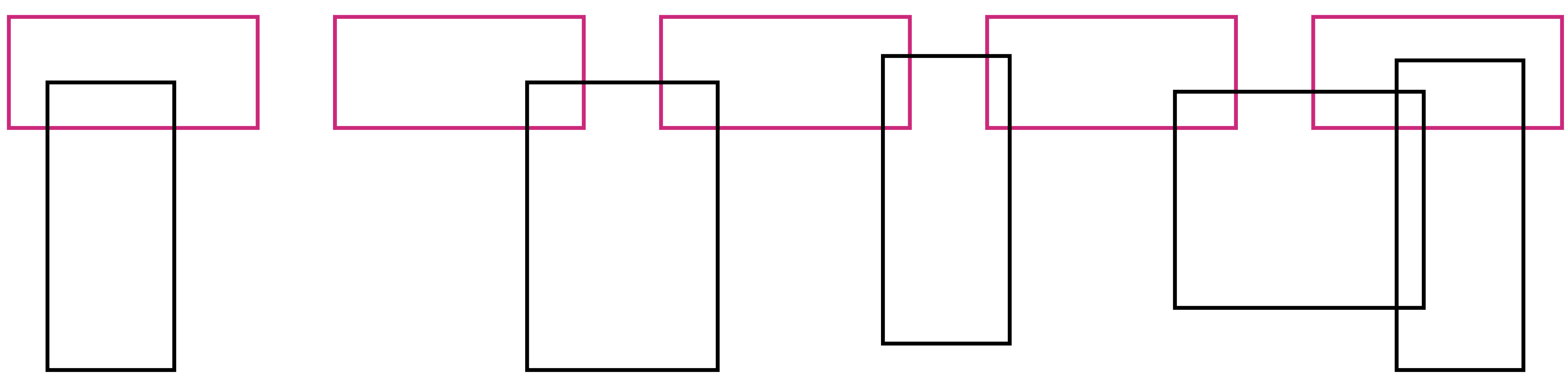
* F has a sunflower with k petals that can be obtained in polynomial time

Proof of Sunflower Lemma

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
  - No duplicates
- Each set in F has size d, $|F| > d!\,(k-1)^d$

Induction on d

- d = 1 (singletons)
- d>=2
  - $\{S_1, S_2, \ldots, S_l\}$ maximal set of disjoint sets in F
  - If $l \geq k$, $\{S_1, S_2, \ldots, S_l\}$ is the sunflower with k petals
  - Otherwise, $l < k$



Proof of Sunflower Lemma

- Universe $U = \{1, 2, 3, \dots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \dots, S_m\}$
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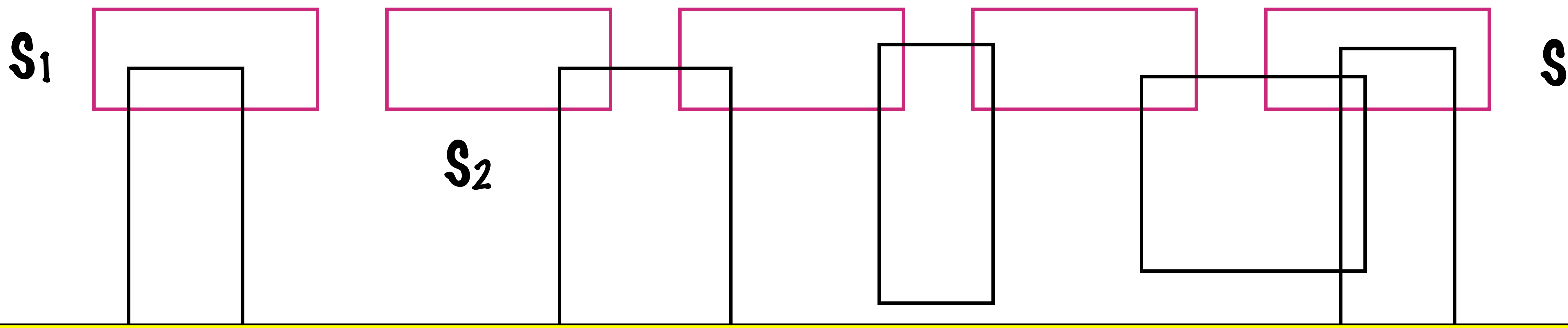
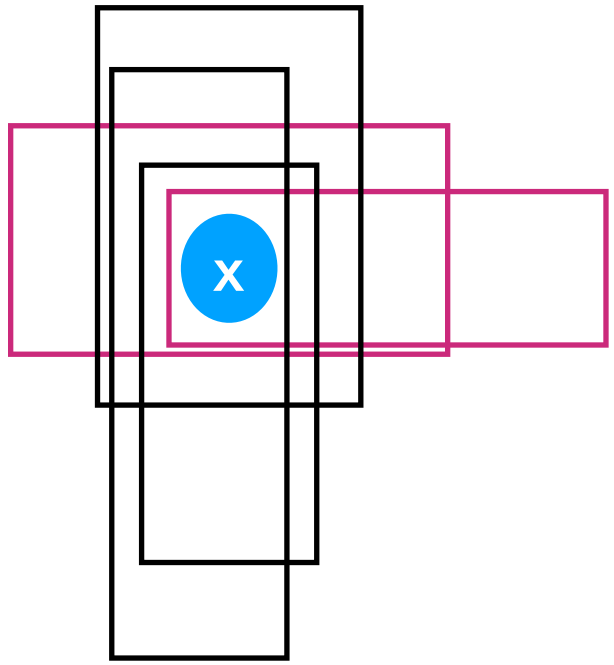


Suppose each element of S is in less than |F|/|S| sets, then elements of S are in total < |F| sets but every set in F has an element

- $S = S_1 \cup S_2 \cup \dots \cup S_l$ and $|S| \le d(k-1)$
- Every set in F has an element in S
- There is an element x that is in $\ge |F|/|S| > d!\,(k-1)^d / d(k-1)$ sets of F

Proof of Sunflower Lemma

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
  - No duplicates
- Each set in F has size d, $|F| > d!\,(k-1)^d$



- There is an element x that is in $\geq |F|/|S| > d!\,(k-1)^d / d(k-1)$ sets of F
- F' = sets in F containing x
- F'' : obtained from F' by deleting x (no duplicates)
  - $|F''| > (d-1)!\,(k-1)^{d-1}$
- By induction hypothesis, F'' has a sunflower with k petals
  - $\{S'_1, S'_2, \ldots, S'_k\}$
- $\{S'_1 \cup \{x\}, S'_2 \cup \{x\}, \ldots, S'_k \cup \{x\}\}$ is a sunflower with k petals in F

Computing a Sunflower

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
  - No duplicates
- Each set in F has size d, $|F| > d!\,(k-1)^d$

- Find a maximal set $\{S_1, S_2, \ldots, S_l\}$ of disjoint sets in F
- Let x be an element that is in a maximum number of sets in F
  - x is in $>= |F|/|S|$ sets of F
- F' : sets in F containing x
- F'' : obtained from F' by deleting x (no duplicates)
  - $|F''| > (d-1)!\,(k-1)^{d-1}$ as $|F|/|S| > d!\,(k-1)^d / d(k-1)$
- Recurse on F'' to find a sunflower $\{S'_1, S'_2, \ldots, S'_k\}$ with k petals
- $\{S'_1 \cup \{x\}, S'_2 \cup \{x\}, \ldots, S'_k \cup \{x\}\}$ is a sunflower with k petals in F

Sunflower Lemma (Variant)

* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$ (no duplicates)
* Each set in F has size at most d
* $|F| > d * d!\,(k-1)^d$

↓

* F has a sunflower with k petals that can be obtained in polynomial time

Hint: There exists $r <= d$ s.t $|F_r| > d!\,(k-1)^d$ where F_r is the subset of F containing sets of size $=r$

d-Hitting Set

Input:

* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family $F = \{S_1, S_2, \ldots, S_m\}$ of <= d sized subsets of U
* A non-negative integer k

Question: Does there exist $V \subseteq U$ with $|V| \le k$ s.t for each S in F, $S \cap V \neq \emptyset$?

Some Common Hitting Sets

* Vertex Cover
* Feedback Vertex Set
* Odd Cycle Transversal
* Cluster Vertex Deletion

d-Hitting Set

Input:

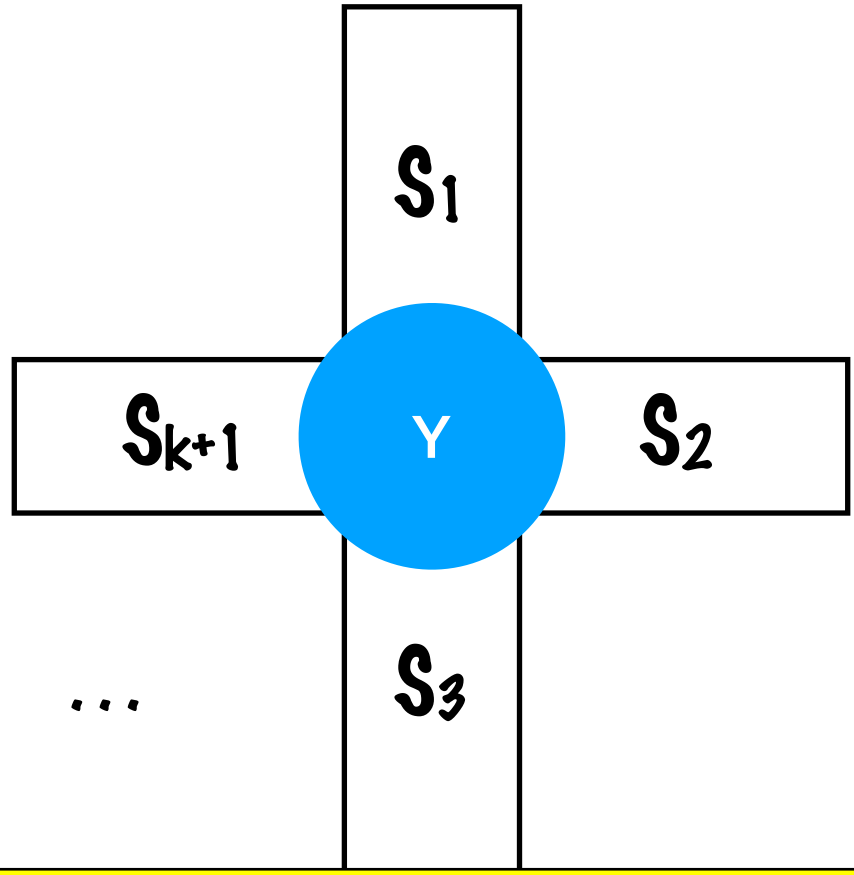
* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family $F = \{S_1, S_2, \ldots, S_m\}$ of <= d sized subsets of U
* A non-negative integer k

Question: Does there exist $V \subseteq U$ with $|V| <= k$ s.t for each S in F, $S \cap V \neq \emptyset$?

* If $|F| <= d * d!\, k^d$
  * Kernel with $d * d!\, k^d$ sets and $d^2 * d!\, k^d$ elements
* Otherwise
  * find a sunflower with (k+1) petals

when something (d) is given as part of problem definition, treat it as c

d-Hitting Set



forward dirn: Suppose there exist such a V; a hitting set, then it clearly hits Y.reverse dirn: Hitting Y is equivalent to hitting all o

- If Y is empty then (U,F) has no hitting set of size <=k
- Otherwise,
  - Any hitting set of size <= k has a non-empty intersection with Y
  - Delete $S_1, S_2, \ldots, S_{k+1}$ from F and add Y to F to get resultant instance (U',F',k)

(U,F,k) is a yes-instance iff (U',F',k) is a yes-instance

d-Set Packing

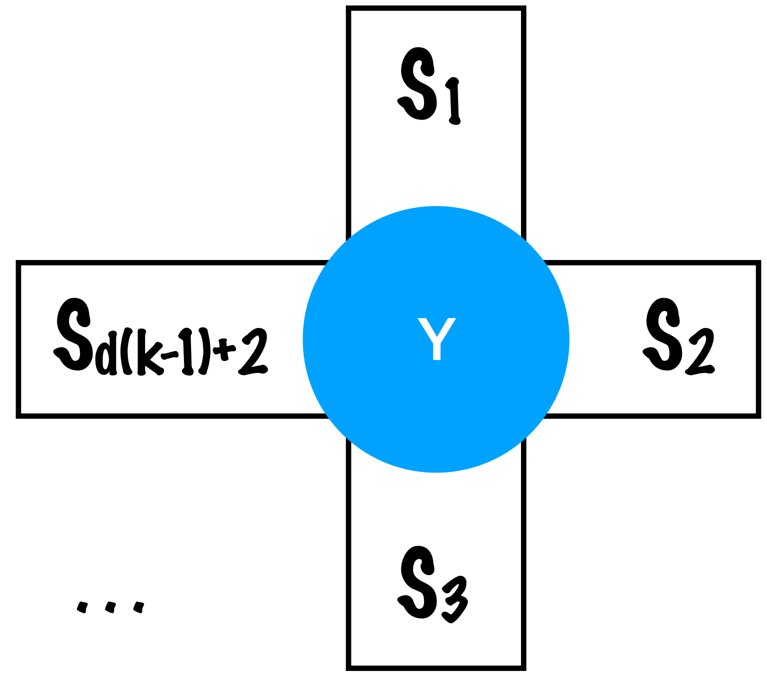
Input:

* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family $F = \{S_1, S_2, \ldots, S_m\}$ of <= d sized subsets of U
* A non-negative integer k

Question: Does there exist $F' \subseteq F$ of pairwise disjoint sets s.t. |F'| >= k?

* If $|F| <= d * d! (d(k-1)+1)^d$
    * Kernel with $d * d! (d(k-1)+1)^d$ sets and $d^2 * d! (d(k-1)+1)^d$ elements
* Otherwise
    * find a sunflower with $d(k-1)+2$ petals

d-Set Packing



* Delete S_1 from F to get resultant instance (U',F',k)
* (U,F,k) is a yes-instance iff (U',F',k) is a yes-instance
  * Reverse direction is easy
  * Forward direction
    * Suppose P is a k-set packing containing S_1
    * P\{S_1} has <= d(k-1) elements (set X)
    * There is a set S_i that has no element from X for some 2 <= i <= d(k-1)+2
    * P\{S_1}∪{S_i} is a k-set packing not containing S_1

for forward dirn: Clearly if Y is empty then we already have our F', o/w we can