

CS 5003: Parameterized Algorithms

Lectures 14-15

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Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Basic Definitions in Parameterized Complexity

- * Parameterized Problem Q
 - * Each instance is associated with a non-negative integer called **parameter**
- * The **size** n of an instance (x, k) of Q
 - * $n = |(x, k)| = |x| + k$
- * Q is **fixed-parameter tractable** if it can be solved in $f(k) n^{O(1)}$ time
- * A **kernelization algorithm** for Q is a **polynomial-time algorithm** that given any instance (x, k) of Q returns an instance (x', k') such that
 - * $|(x', k')| \leq g(k)$ and
 - * (x, k) is a yes-instance of Q iff (x', k') is a yes-instance of Q

Vertex Cover

Vertex Cover (parameterized by solution size)

Instance: A graph G on n vertices m edges and integer k

Question: Does G have a vertex cover of size at most k ?

Parameter: k

$$f(k) (n+m)^{O(1)}$$

Vertex Cover Above LP

Instance: A graph G on n vertices m edges and integer k

Question: Does G have a vertex cover of size at most k ?

Parameter: $\lceil k - \text{lp}(G) \rceil$

$$g(k - \text{lp}) (n+m)^{O(1)}$$

Vertex Cover LP

$$\text{minimize } \sum_{v \in V(G)} x(v)$$

subject to $x(v) + x(u) \geq 1$ for each edge $\{u, v\} \in E(G)$

$0 \leq x(v) \leq 1$ for each vertex $v \in V(G)$

Optimum solution x^*



Computable in P-time

$\sum_{v \in V(G)} x^*(v) > k \implies (G, k) \text{ is no instance}$

Vertex Cover Above LP

Instance: A graph G on n vertices m edges and integer k

Question: Does G have a vertex cover of size at most k ?

Parameter: $\lceil k - \text{lp}(G) \rceil$

*Optimum solution x^**

$$< \frac{1}{2}$$

$$> \frac{1}{2}$$

$$= \frac{1}{2}$$

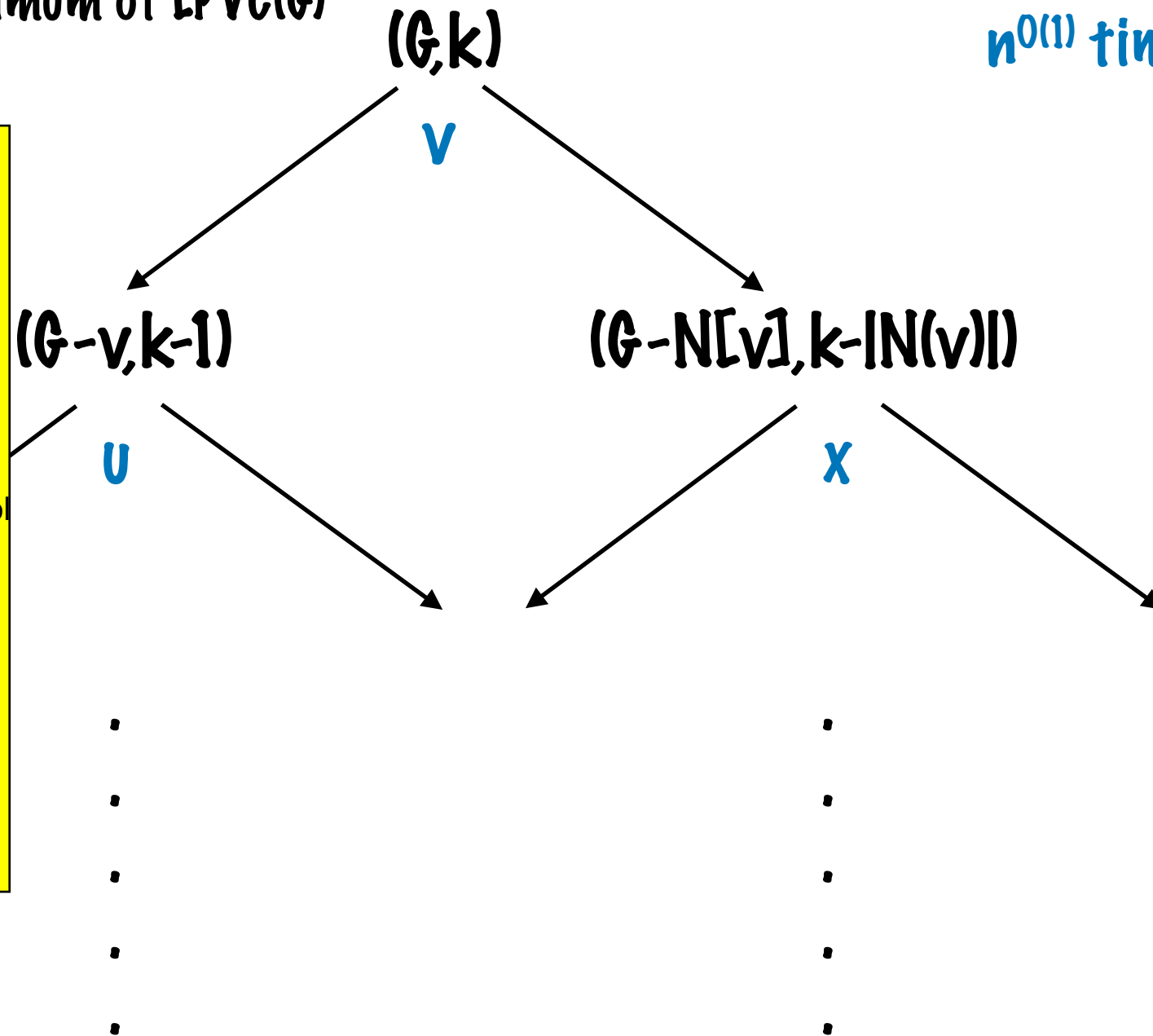
There is a min vertex cover including $> 1/2$ set and excluding $< 1/2$ set

(G, k) is a yes-instance iff (G', k') is a yes-instance

Vertex Cover Above LP

- * All $1/2$ is the unique optimum of $LPVC(G)$
- * $k \geq lpvc(G) = n/2$

$n^{O(1)}$ time



Clearly if all halves is not the only optimum sol

(G, k) is an yes-instance iff $(G-v, k-1)$ or $(G-N[v], k-|N(v)|)$ is an yes-instance

Vertex Cover Above LP

- * Branch 1: v is in the vertex cover

- * $G' = G - v$

- * $k' = k - 1$

- * $lp(G') \geq lp(G) - 1/2$

- * Suppose not

- * $lp(G') < lp(G) - 1/2$

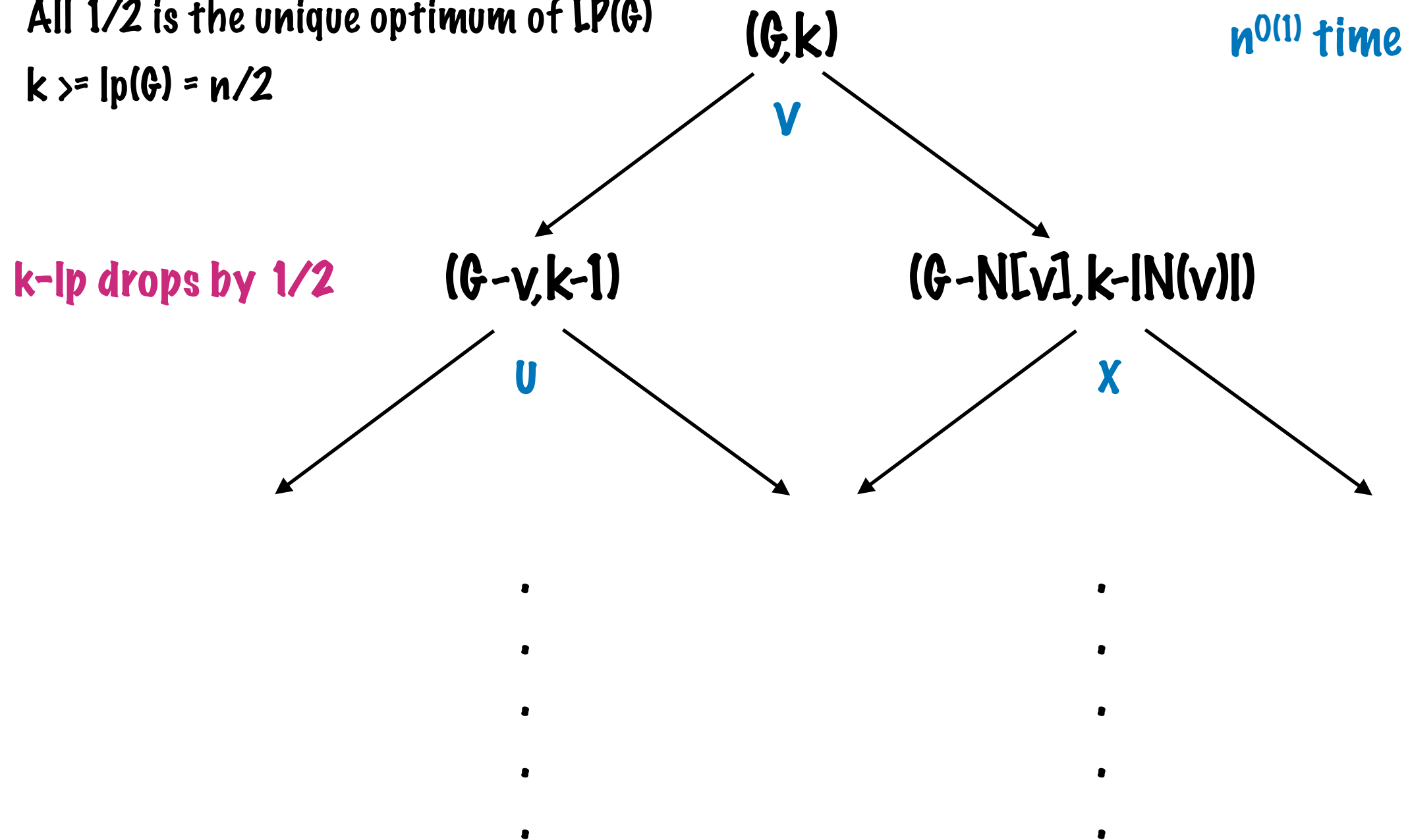
- * $lp(G') \leq lp(G) - 1$ (half-integrality)

- * G has an optimum solution that is not all-1/2

- * $k' - lp(G') \leq k - 1 - lp(G) + 1/2 \leq k - lp(G) - 1/2$

Vertex Cover Above LP

- * All $1/2$ is the unique optimum of $LP(G)$
- * $k \geq lp(G) = n/2$



Apply preprocessing rules (reduction rules) at each node

Vertex Cover Above LP

- * Branch 2: v is not in the vertex cover

- * $G' = G - N[v]$

- * $k' = k - |N(v)|$

- * $lp(G') \geq lp(G) - |N(v)| + 1/2$

- * Suppose not

- * $lp(G') < lp(G) - |N(v)| + 1/2$

- * $lp(G') \leq lp(G) - |N(v)|$ (half-integrality)

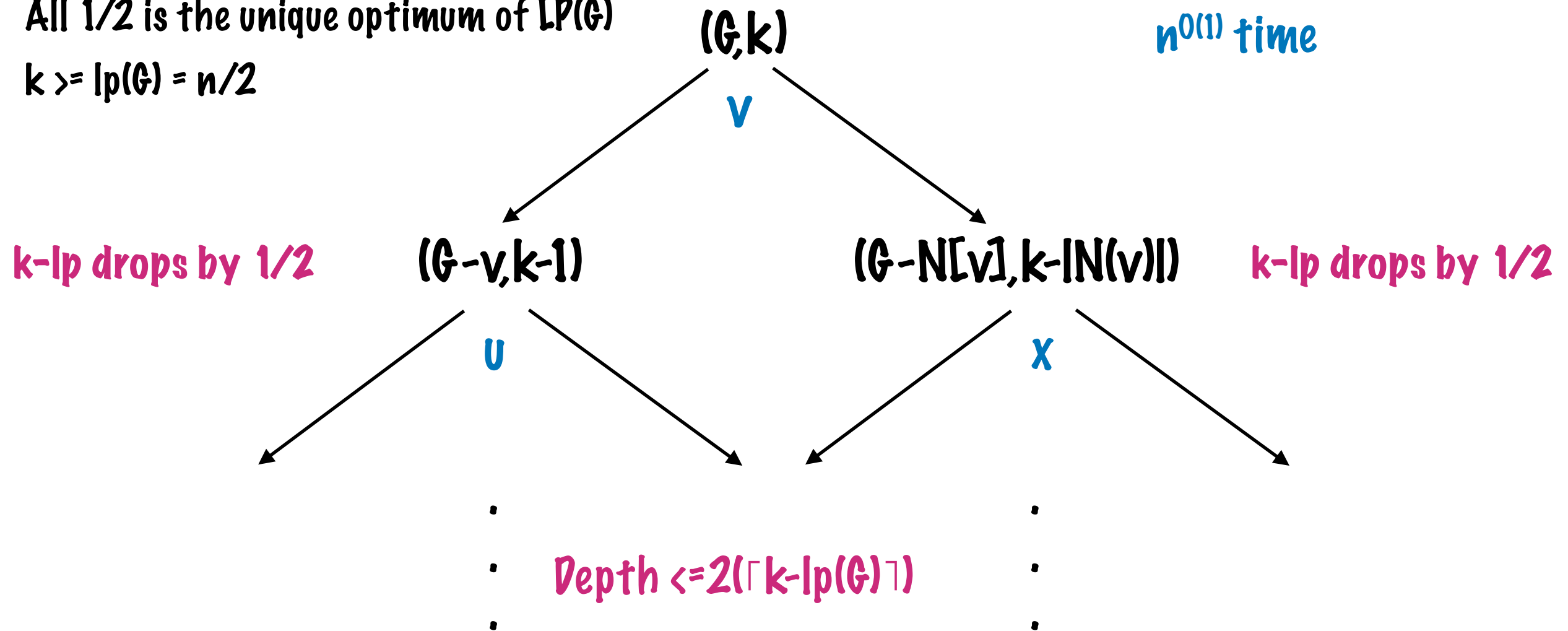
- * G has an optimum solution that is not all-1/2

- * $k' - lp(G') \leq k - |N(v)| - lp(G) + |N(v)| - 1/2 \leq k - lp(G) - 1/2$

Vertex Cover Above LP

* All 1/2 is the unique optimum of $LP(G)$

* $k \geq lp(G) = n/2$



* $4^{\lceil k-lp(G) \rceil} n^{O(1)}$ time algorithm

* Apply reduction rules at each node (Reduction Rules do not increase $k-lp$)

* At leaves, what is the computation?

Solving Vertex Cover LP

LPVC(G)

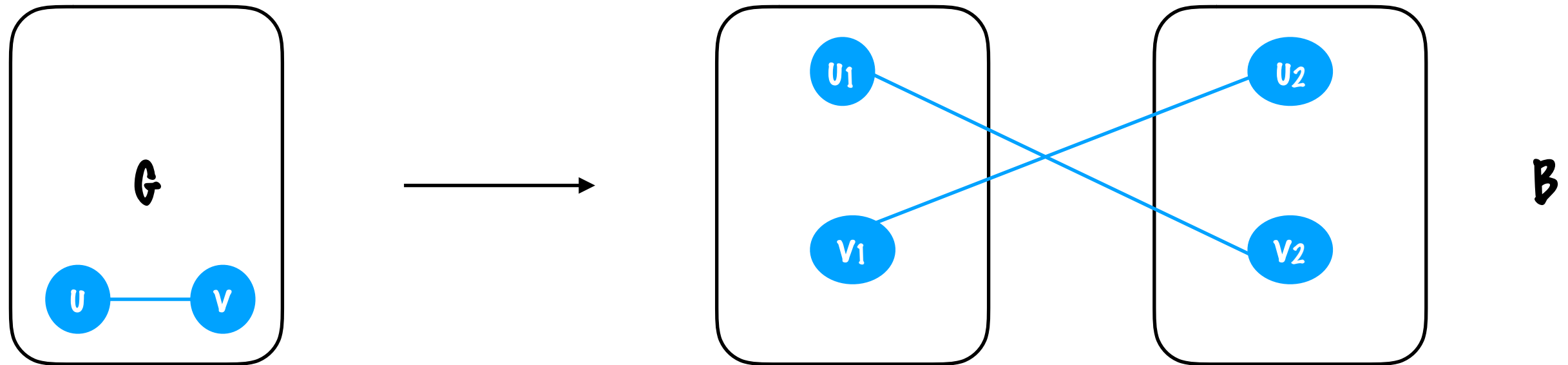
$$\text{minimize } \sum_{v \in V(G)} x(v)$$

subject to $x(v) + x(u) \geq 1$ for each edge $\{u, v\} \in E(G)$

$$0 \leq x(v) \leq 1 \text{ for each vertex } v \in V(G)$$

Theorem: There is an optimum solution to LPVC(G) that assigns 0, 1 or 1/2 to each of the variables

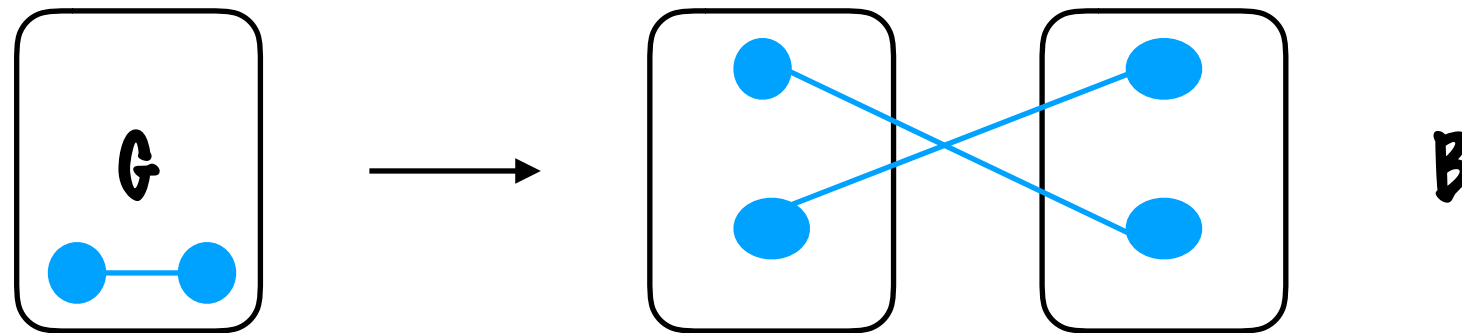
Solving Vertex Cover LP



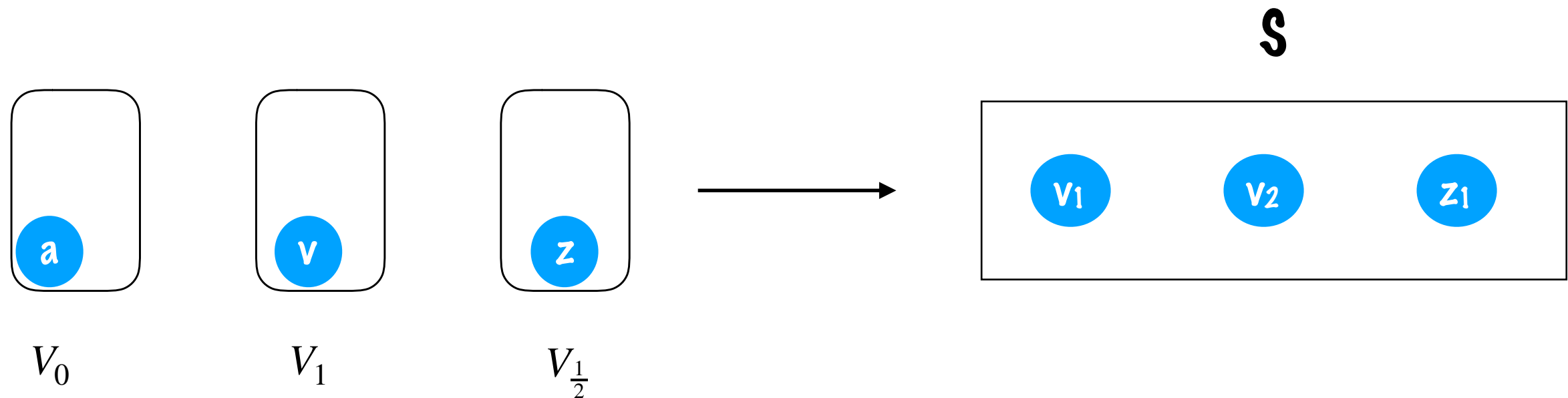
Bipartite Graph

Theorem² Optimum solution to $LPVC(G) = |Min\ Vertex\ Cover\ of\ B|$

Solving Vertex Cover LP



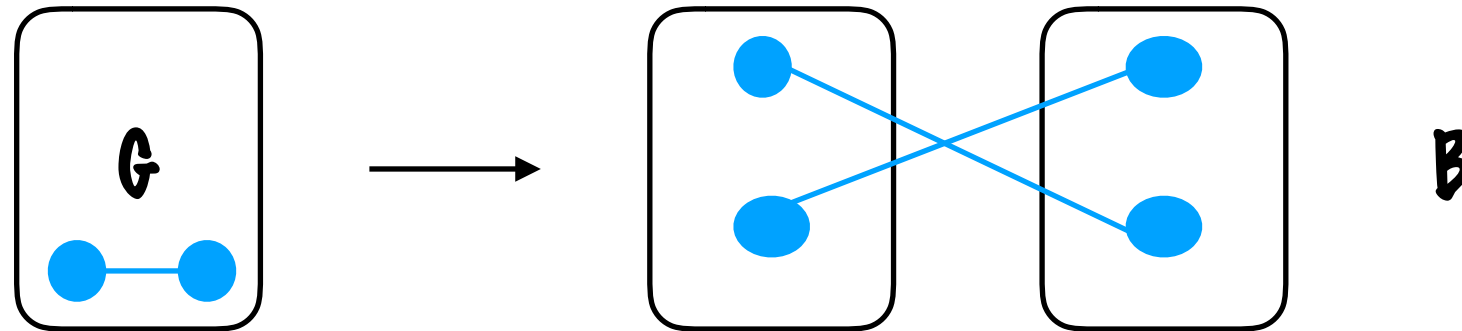
Feasible half integral solution x^* to $LPVC(G)$



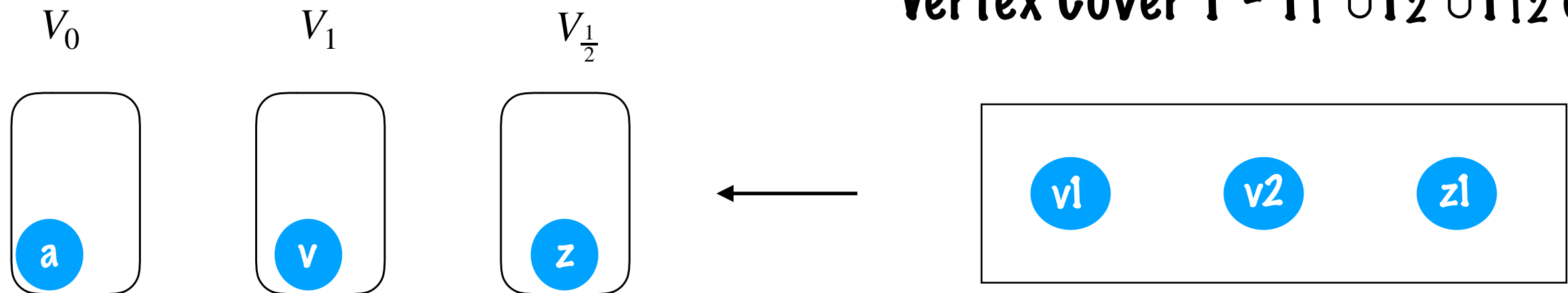
$$|S| = \sum_{v \in V_1} 2 + \sum_{v \in V_{\frac{1}{2}}} 1 = 2 \left(\sum_{v \in V_1} 1 + \sum_{v \in V_{\frac{1}{2}}} \frac{1}{2} \right) = 2 \sum_{v \in V(G)} x^*(v)$$

And thus size of minimum VC is $\leq 2 \cdot \text{opt value}$.

Solving Vertex Cover LP



Vertex Cover $T = T_1 \cup T_2 \cup T_{12}$ of B



For any edge (u, v) . Suppose both of them are not assigned value one and $u_1 \notin T_1$. That means as there is as well an edge (u_2, v_2) we

$$\sum_{v \in V(G)} y^*(v) = \sum_{v: v_1, v_2 \in T} 1 + \sum_{v: |\{v_1, v_2\} \cap T| = 1} \frac{1}{2} = \frac{1}{2} |T_1| + \frac{1}{2} |T_2| + \frac{1}{2} |T_{12}| = \frac{1}{2} |T|$$

Thus $2 * \text{opt}$ is less than equal to $|MVC|$.