CS 5003: Parameterized Algorithms

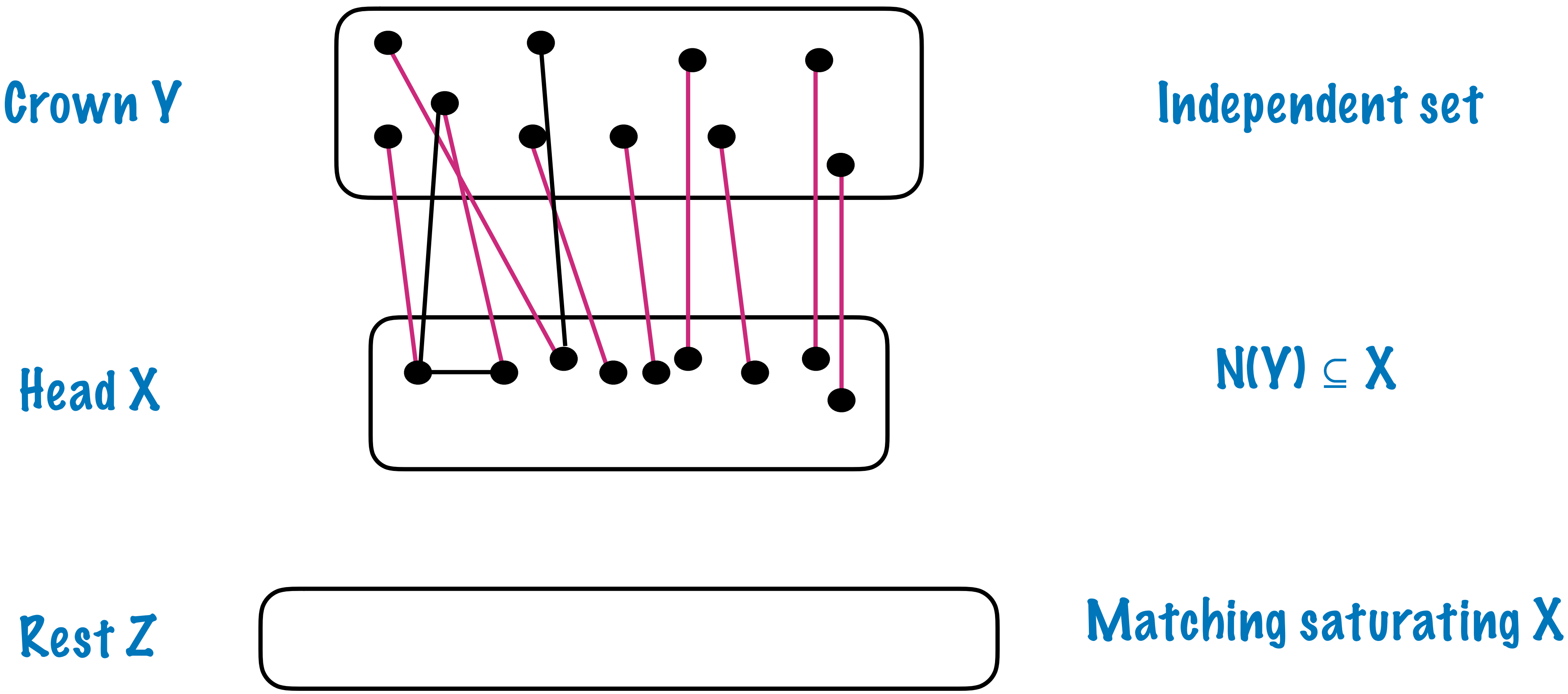
Lectures 28-31

Krithika Ramaswamy

IIT Palakkad

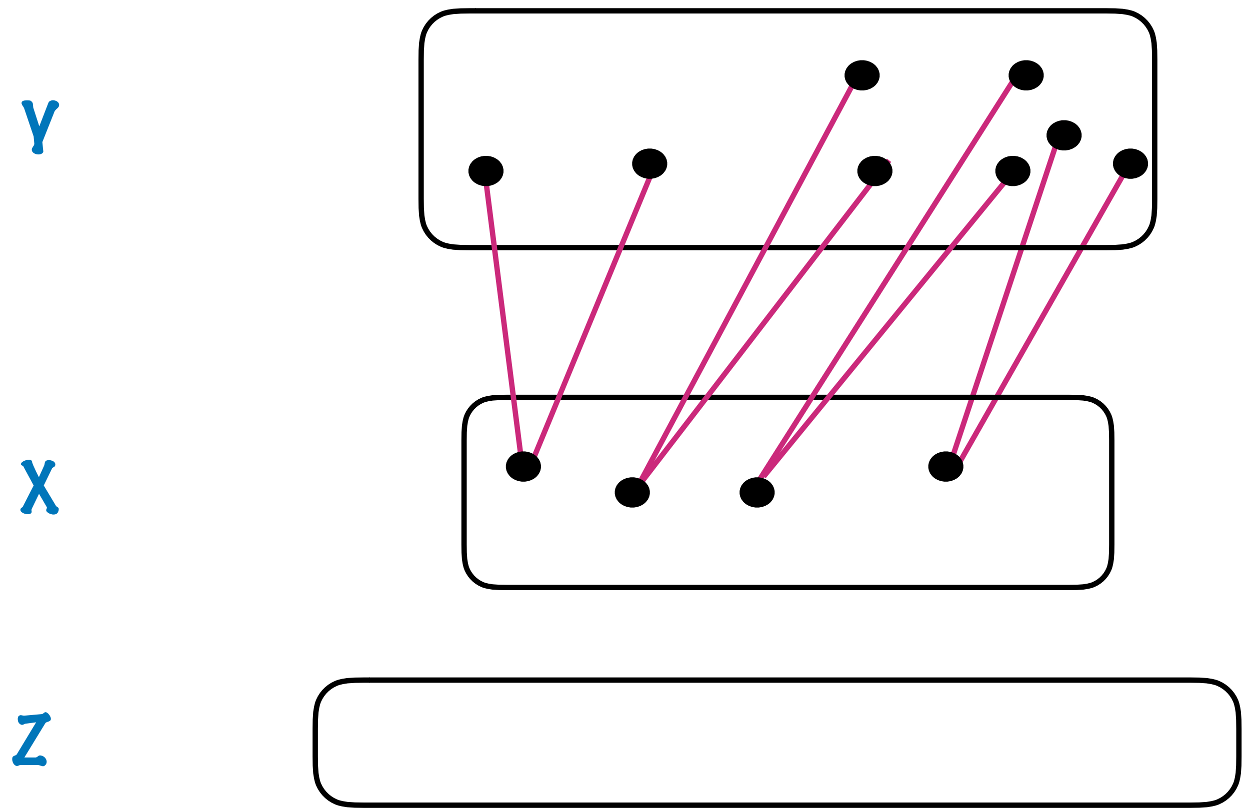
Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Crown Lemma



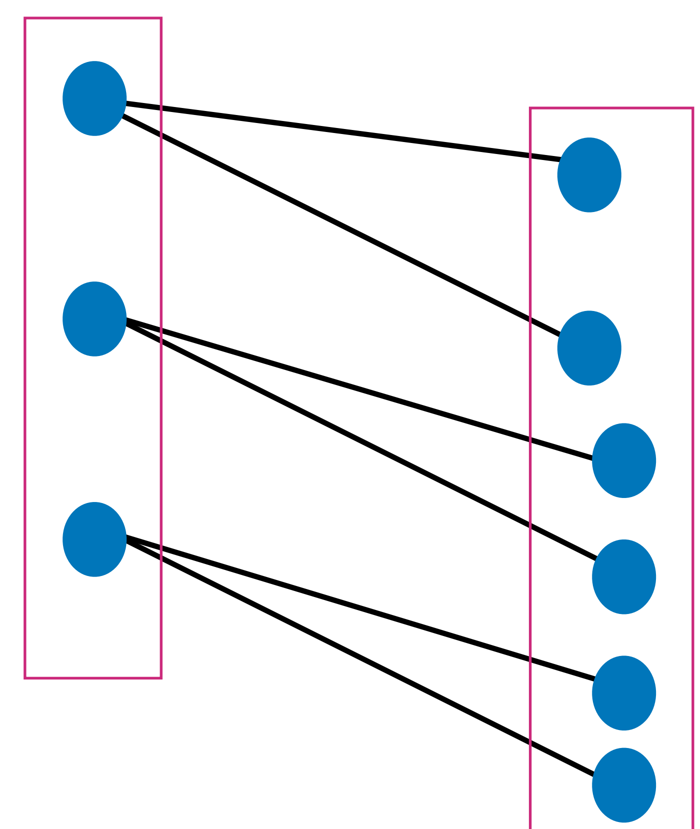
Crown Lemma: Let G be a graph without isolated vertices and with at least $3k + 1$ vertices. Then, there is a polynomial time algorithm that either finds a matching of size $k + 1$ in G, or finds a crown decomposition of G.

Towards a Generalization of Crown Lemma



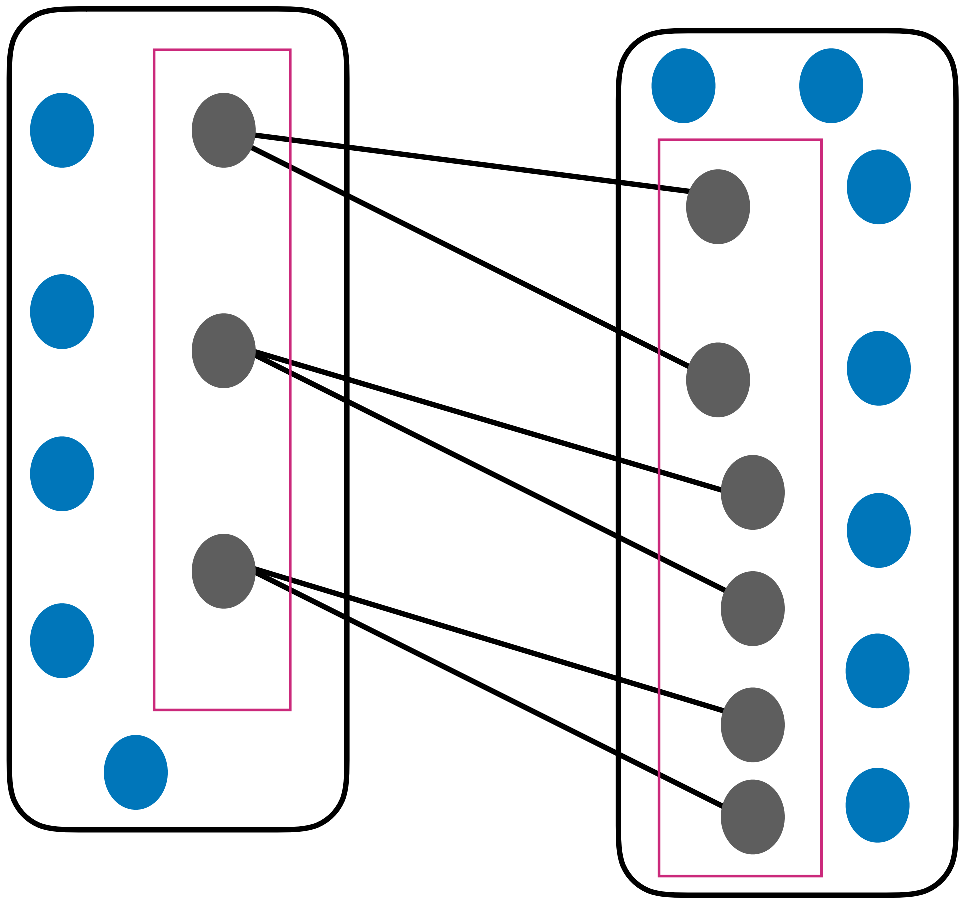
$N(Y) \subseteq X$

2-expansion from X to Y



* **Definition:** In a bipartite graph with bipartition (A, B), a set M of edges is called a 2-expansion from A to B if
  * Every vertex of A is incident with exactly 2 edges of M
  * M saturates exactly $2|A|$ vertices in B

2-Expansion Lemma



2-Expansion Lemma: Let G be a bipartite graph with bipartition (A, B) s.t $|B| > 2|A|$ and there are no isolated vertices in B. Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a 2-expansion into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m\, n^{1/2})$ time.

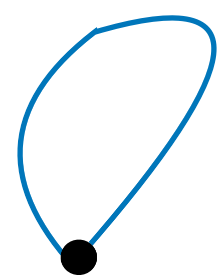
Feedback Vertex Set

Assume graph is a multigraph

Reduction Rule 1: Delete isolated vertices

Reduction Rule 2: Delete degree-1 vertices

Reduction Rule 3: If there is a loop at a vertex v, delete v from the graph and reduce the parameter by 1



Reduction Rule 4: If there is an edge with multiplicity > 2, reduce it to 2

Reduction Rule 5: Short circuit degree-2 vertices

Feedback Vertex Set - Towards a Quadratic Kernel

Claim: If min deg >=3 and max deg <= d and G has an FVS <=k then n < (d+1)k and m < 2dk.

Suppose X is FVS <=k and let Y=G-X

$$\begin{aligned} 3|Y| \le \sum_{v \in Y} deg(v) = \sum_{v \in Y} |N(v) \cap X| + \sum_{v \in Y} |N(v) \cap Y| &\le E(X,Y) + 2(|Y| - 1) \\ &< E(X,Y) + 2|Y| \\ &\le d|X| + 2|Y| \end{aligned}$$

$$\therefore |Y| < d|X|$$

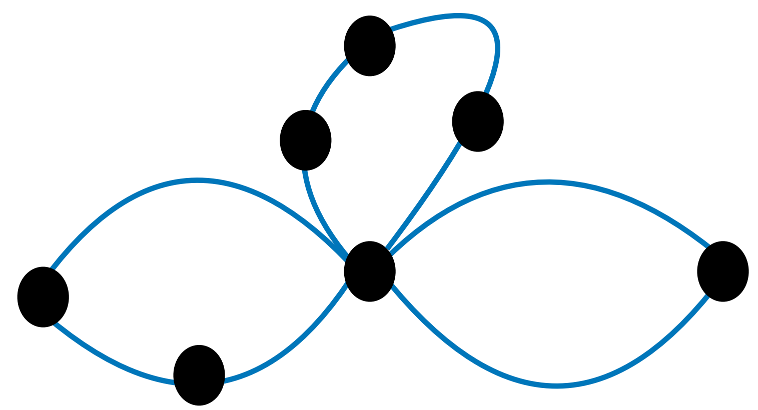
$$n = |X| + |Y| < (d+1)|X| \le (d+1)k$$

$$m < d|X| + |Y| \le 2d|X| \le 2dk$$

Goal: Reduce max degree to <= 10k to get quadratic kernel

Feedback Vertex Set - Towards a Quadratic Kernel

Definition: A **v-flower** with r petals is a set of r cycles that pairwise intersect only at v



Flower Lemma: There is a polynomial time algorithm that given a graph G and a vertex v without a self-loop, satisfies one of the following:

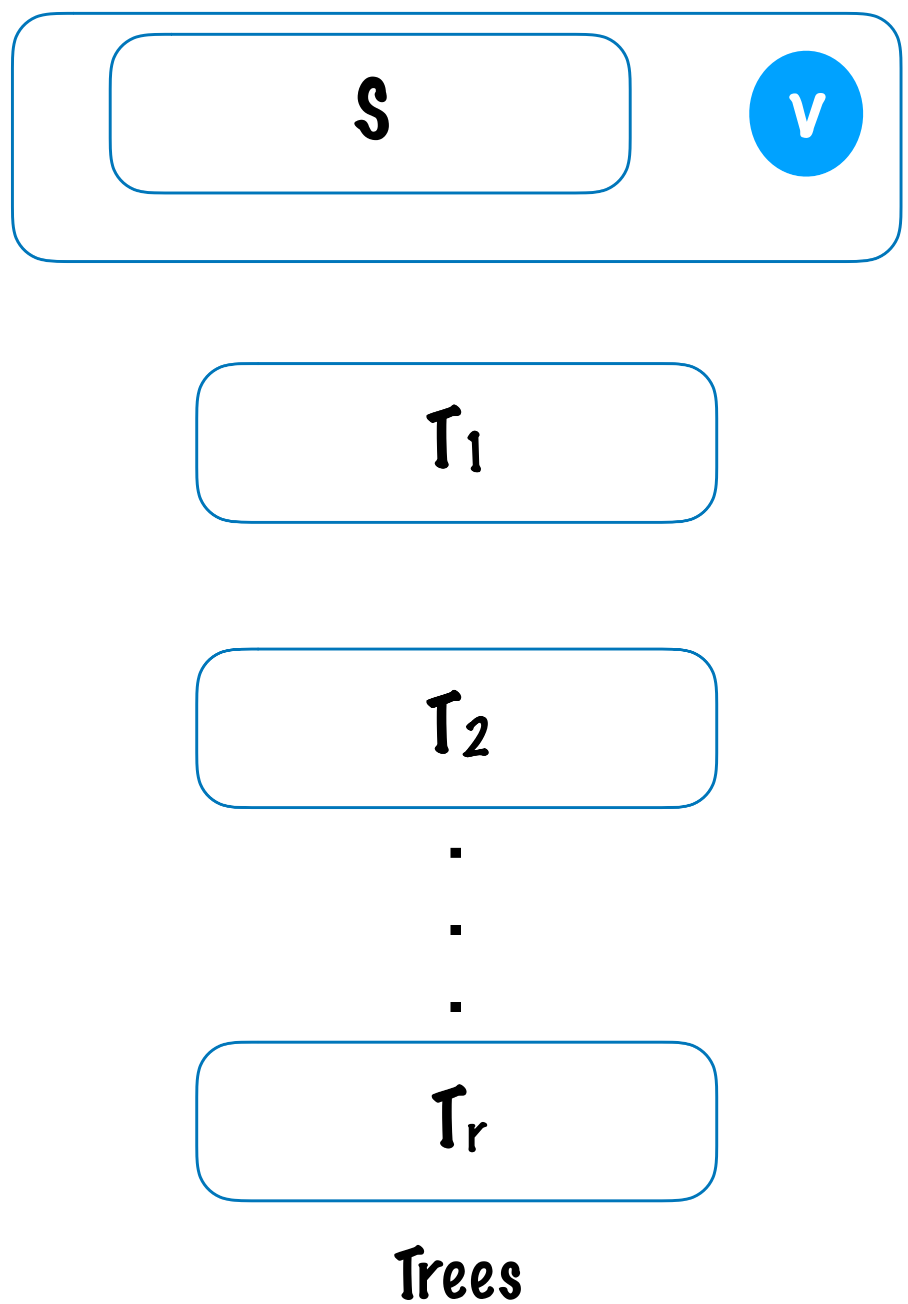
* Declares (G,k) is a no-instance of Feedback Vertex Set
* Returns a v-flower with (k+1) petals
* Finds FVS not containing v of size <= 3k

Note: If algorithm selects option

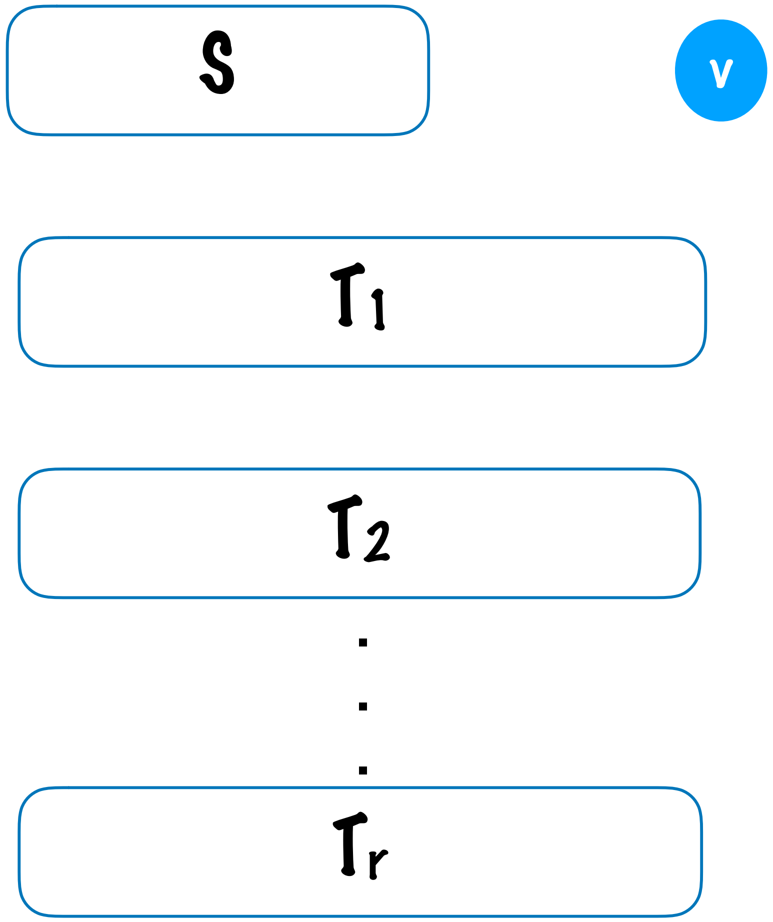
Reduction Rule 6: If there is a v-flower with k+1 petals, then delete v and reduce the parameter by 1

Feedback Vertex Set - Towards a Quadratic Kernel

* Suppose v has degree >10k and Flower Lemma returns FVS S not containing v of size <= 3k



Feedback Vertex Set - Towards a Quadratic Kernel



- v has deg >10k
- If there are > 2k double edges incident on it, apply Reduction Rule 6
- Otherwise, there are <=2k double edges incident on v
- There are <= 4k edges between v and S
  - <=2l + (3k-l) = 3k+l <=4k

Note: 2k + (3k - k) =

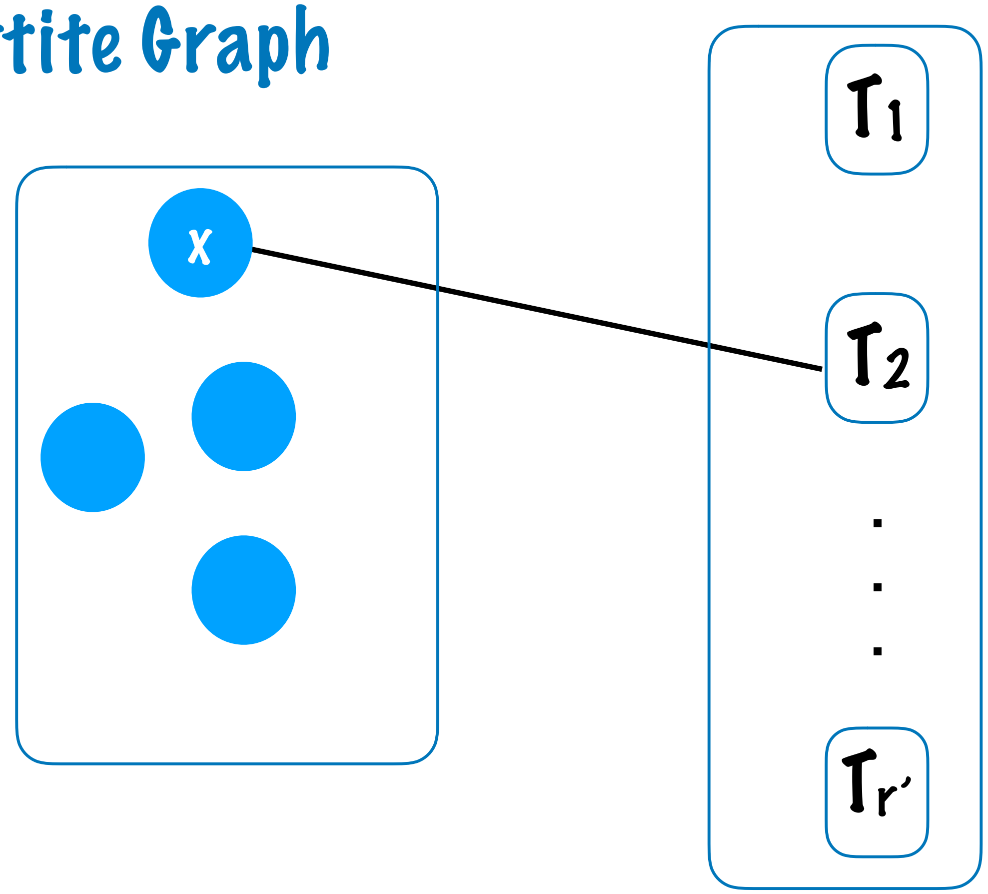
- There are <=r edges between v and T_is

∴ there are >6k edges between v and T_is. In particular, r>6k

Feedback Vertex Set - Towards a Quadratic Kernel

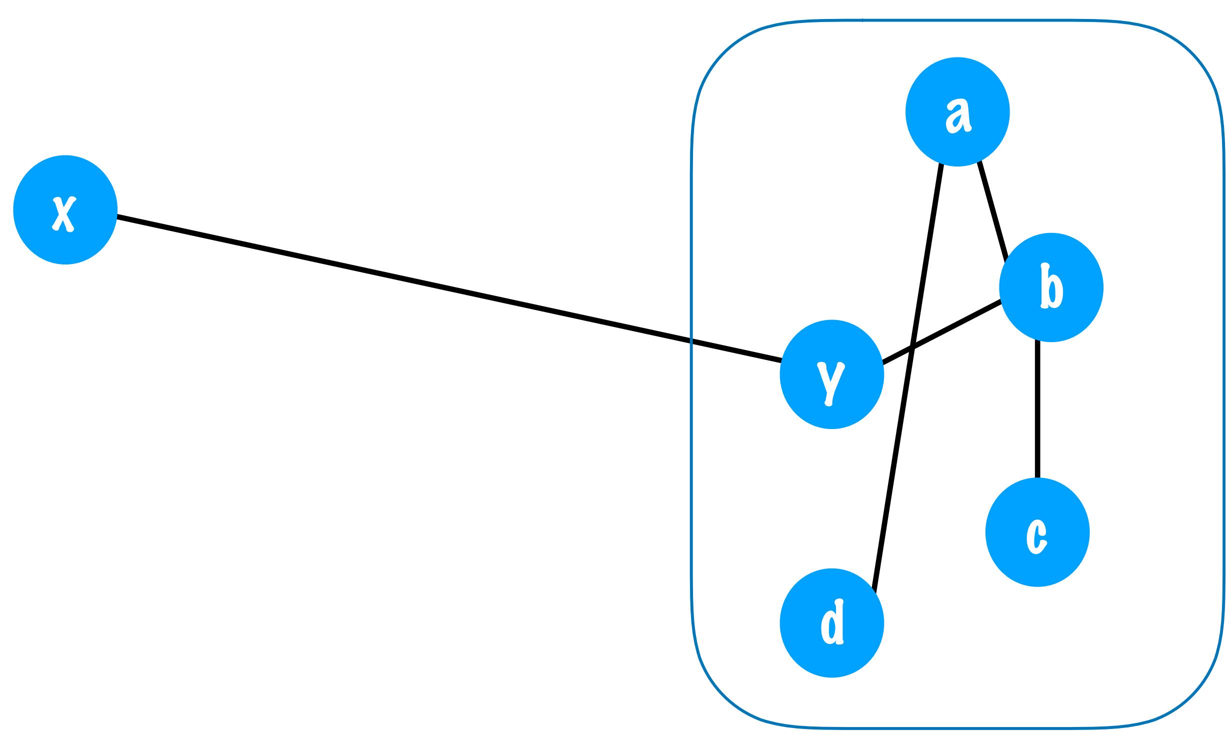
Auxiliary Bipartite Graph

S



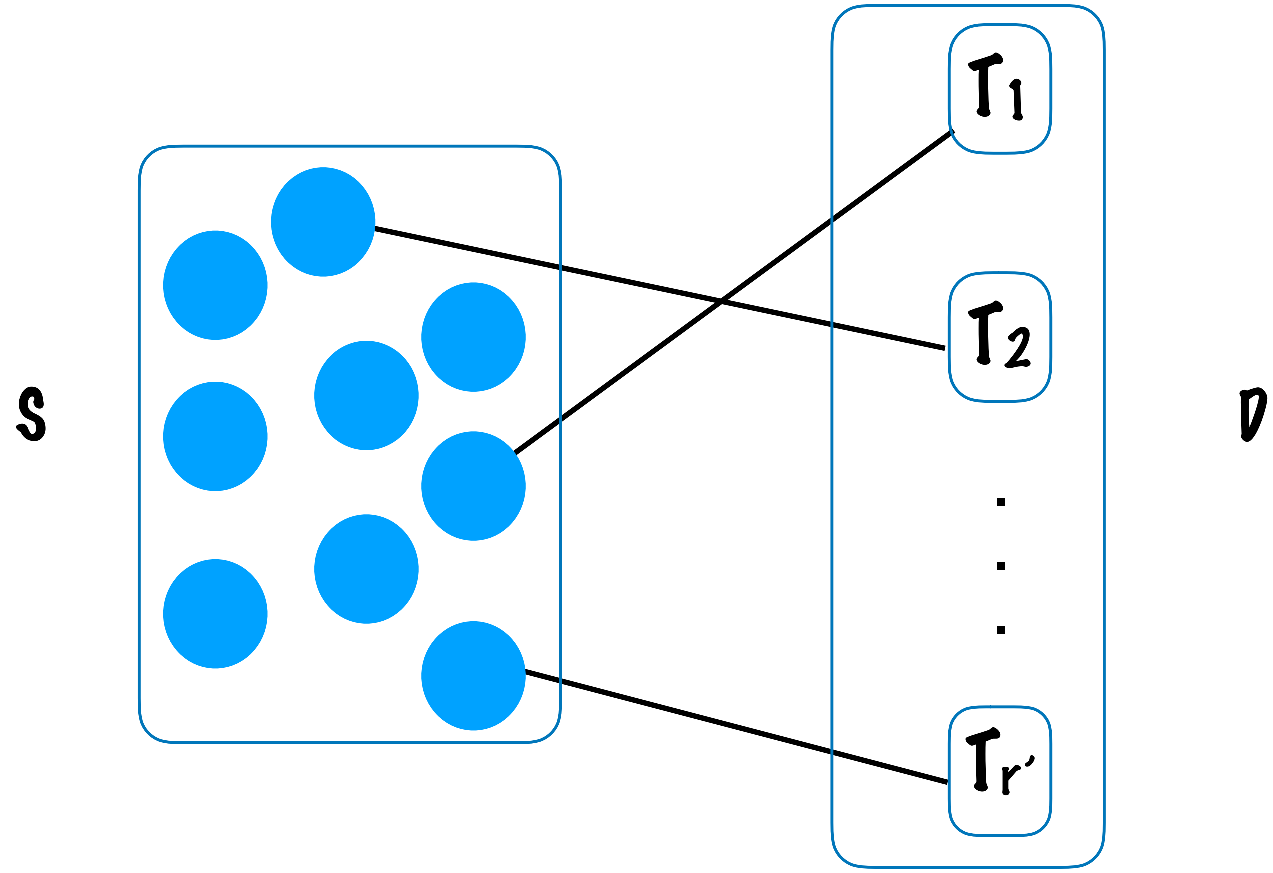
D

Vertices corresponding to T_is that are adj to v



T_2

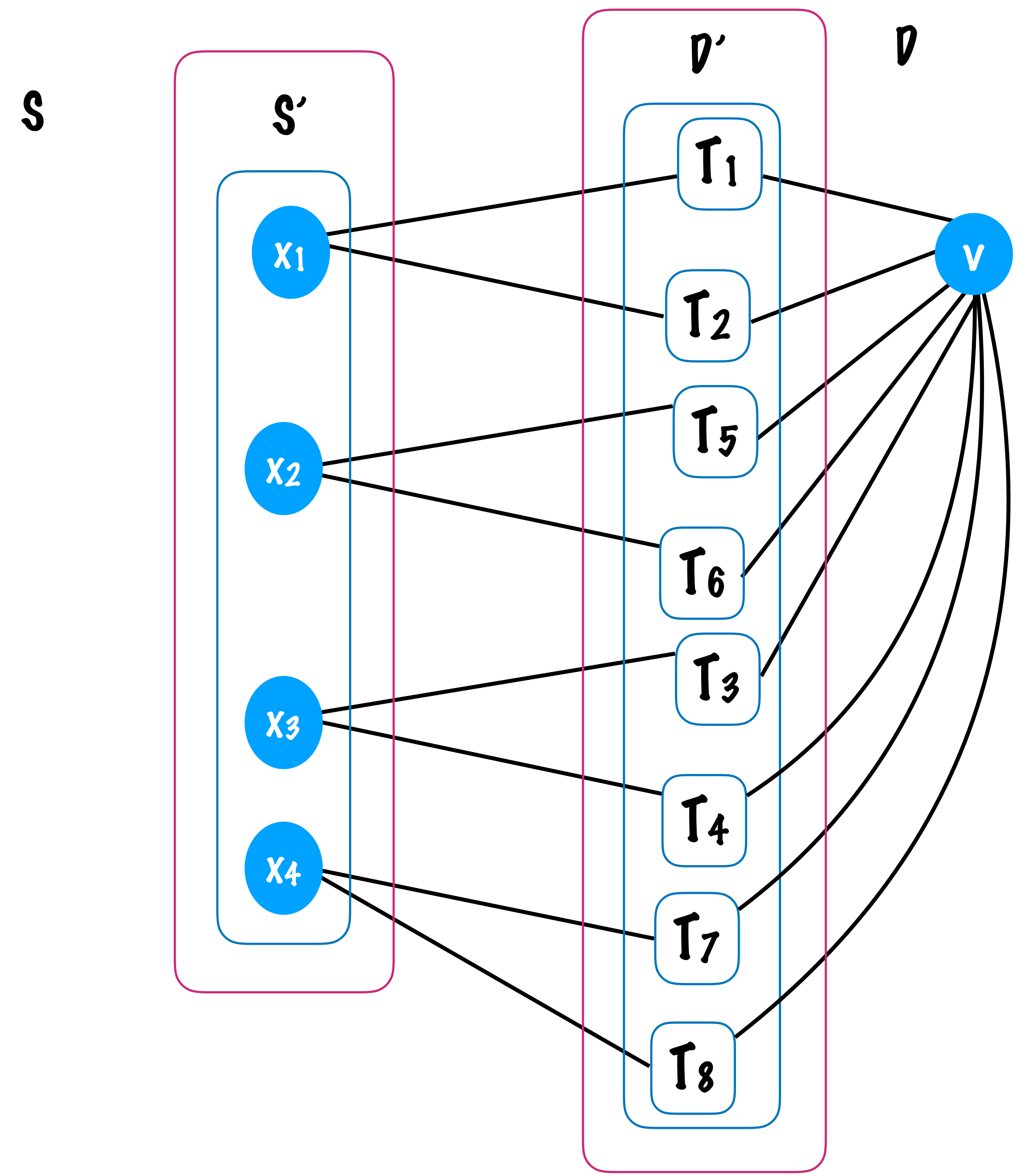
Feedback Vertex Set - Towards a Quadratic Kernel



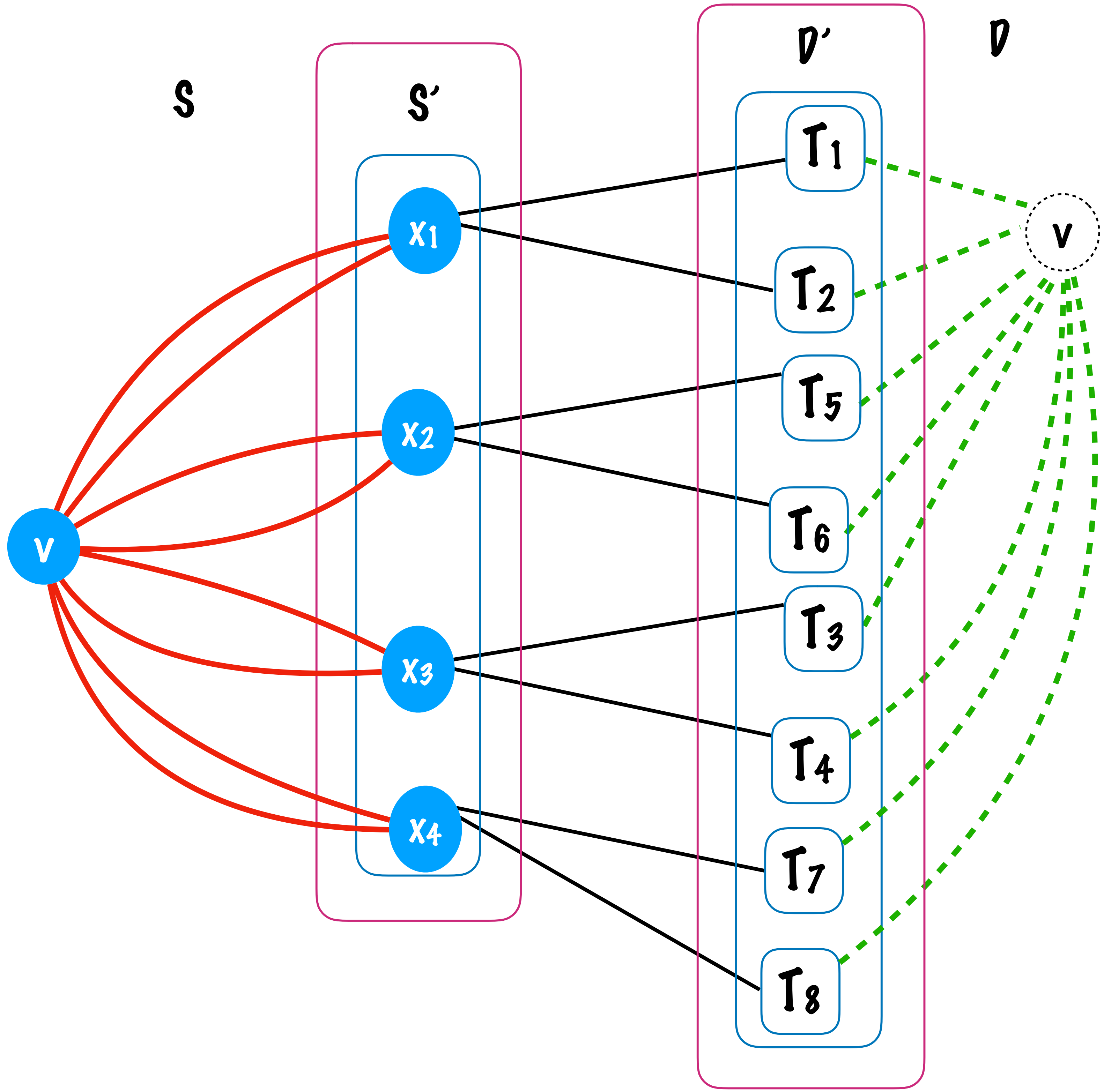
xi is just any leaf to Ti

* No vertex in D is isolated as each T_i has a deg <= 1 vertex x_i which has deg >=3 in G. Even if x_i is adj to v, it has >=1 nbr in S
* |D|>6k and |S|<=3k i.e., |D| >2|S|
  * There are non-empty sets $D' \subseteq D$ and $S' \subseteq S$ s.t
  * S' has a 2-expansion into D' and no vertex in D' has a neighbour outside S'

Feedback Vertex Set - Towards a Quadratic Kernel

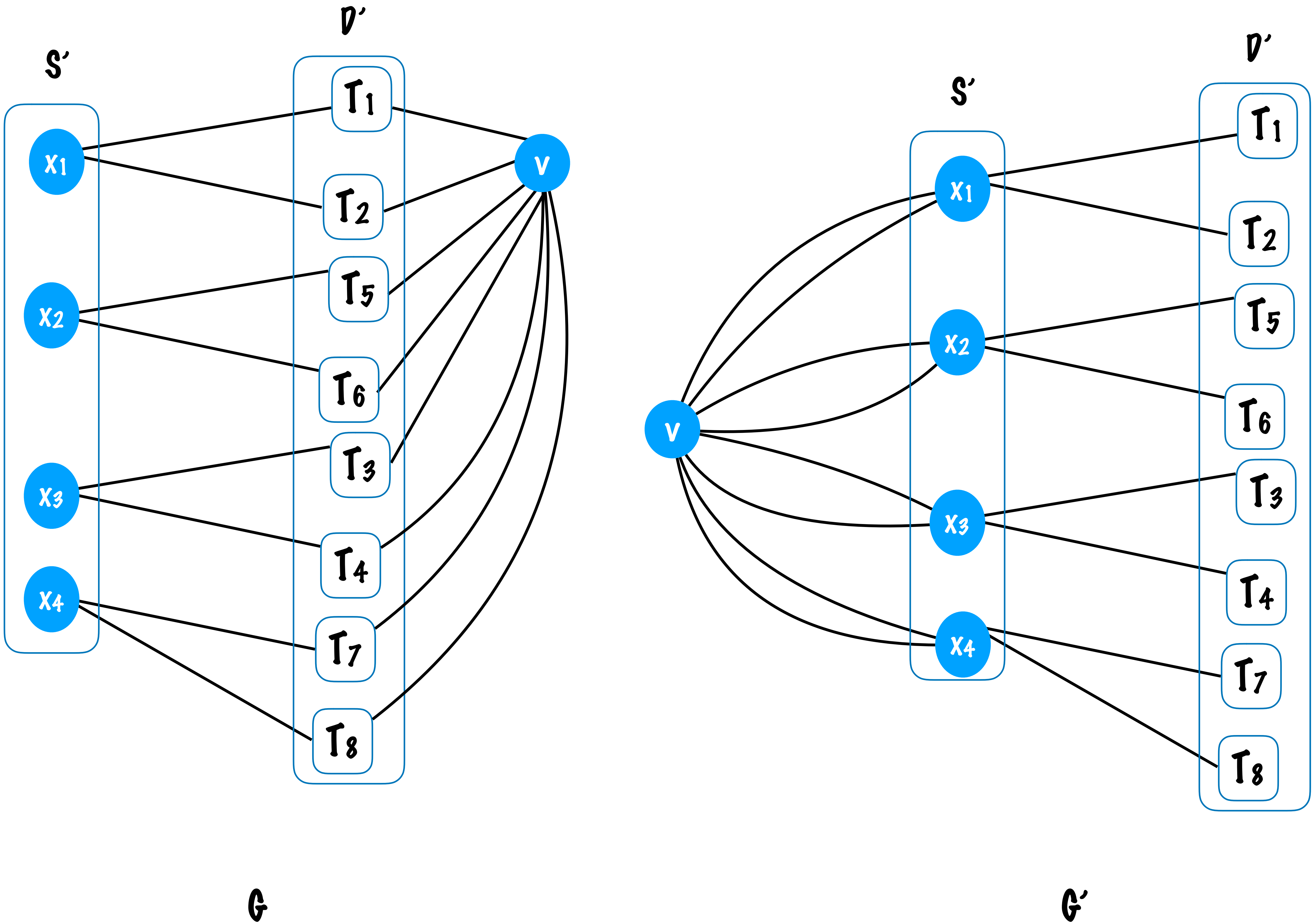


Feedback Vertex Set - Towards a Quadratic Kernel

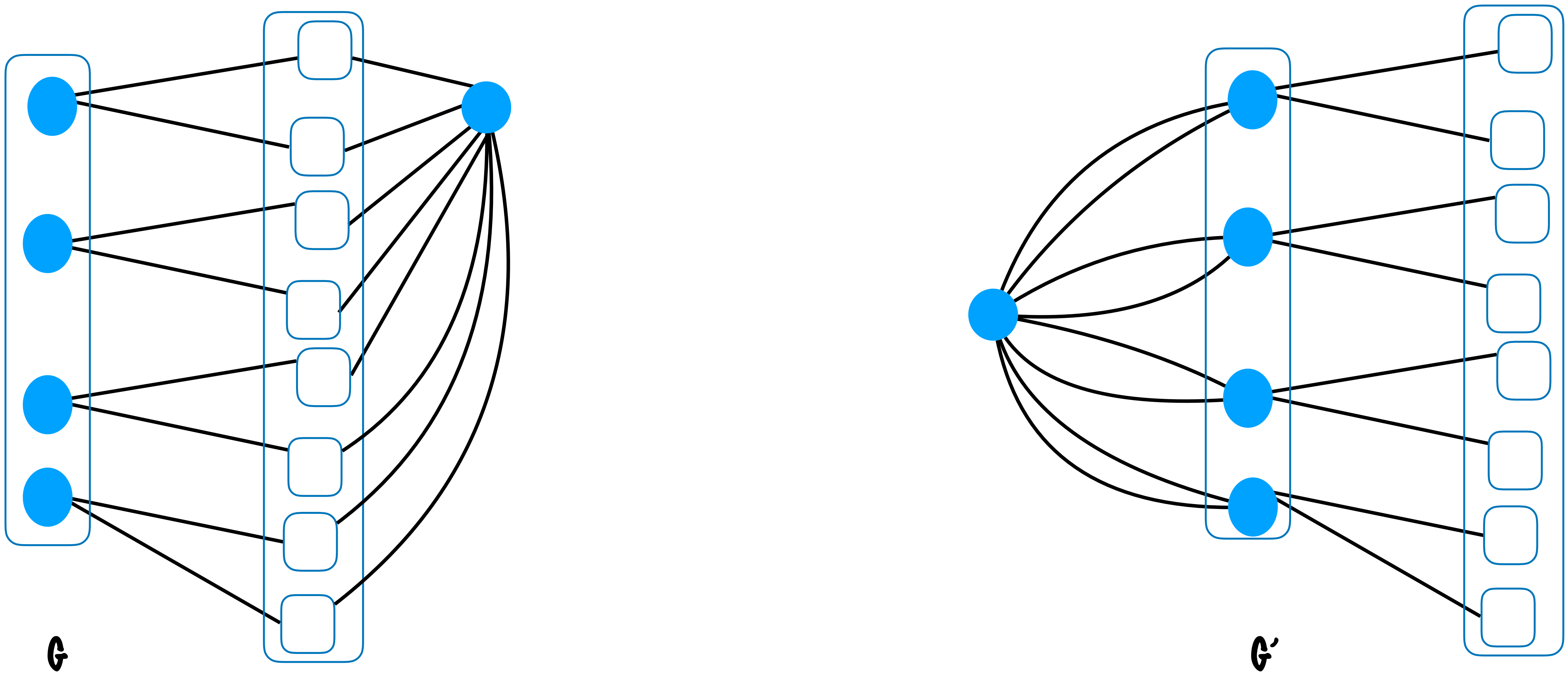


Reduction Rule 7: Add double edges between v and every vertex in S′. Delete edges between D′ to v.

Feedback Vertex Set - Towards a Quadratic Kernel



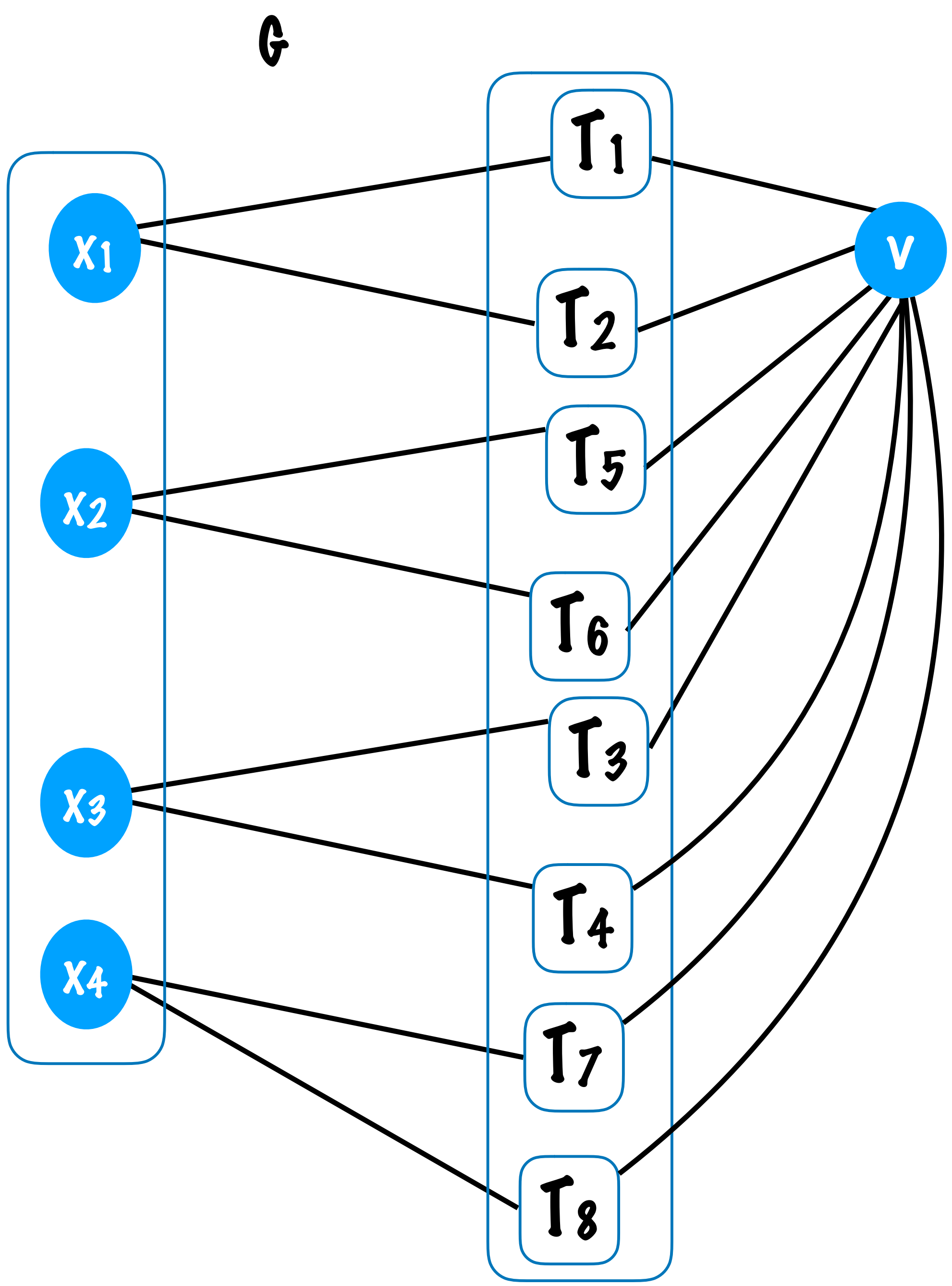
Feedback Vertex Set - Towards a Quadratic Kernel

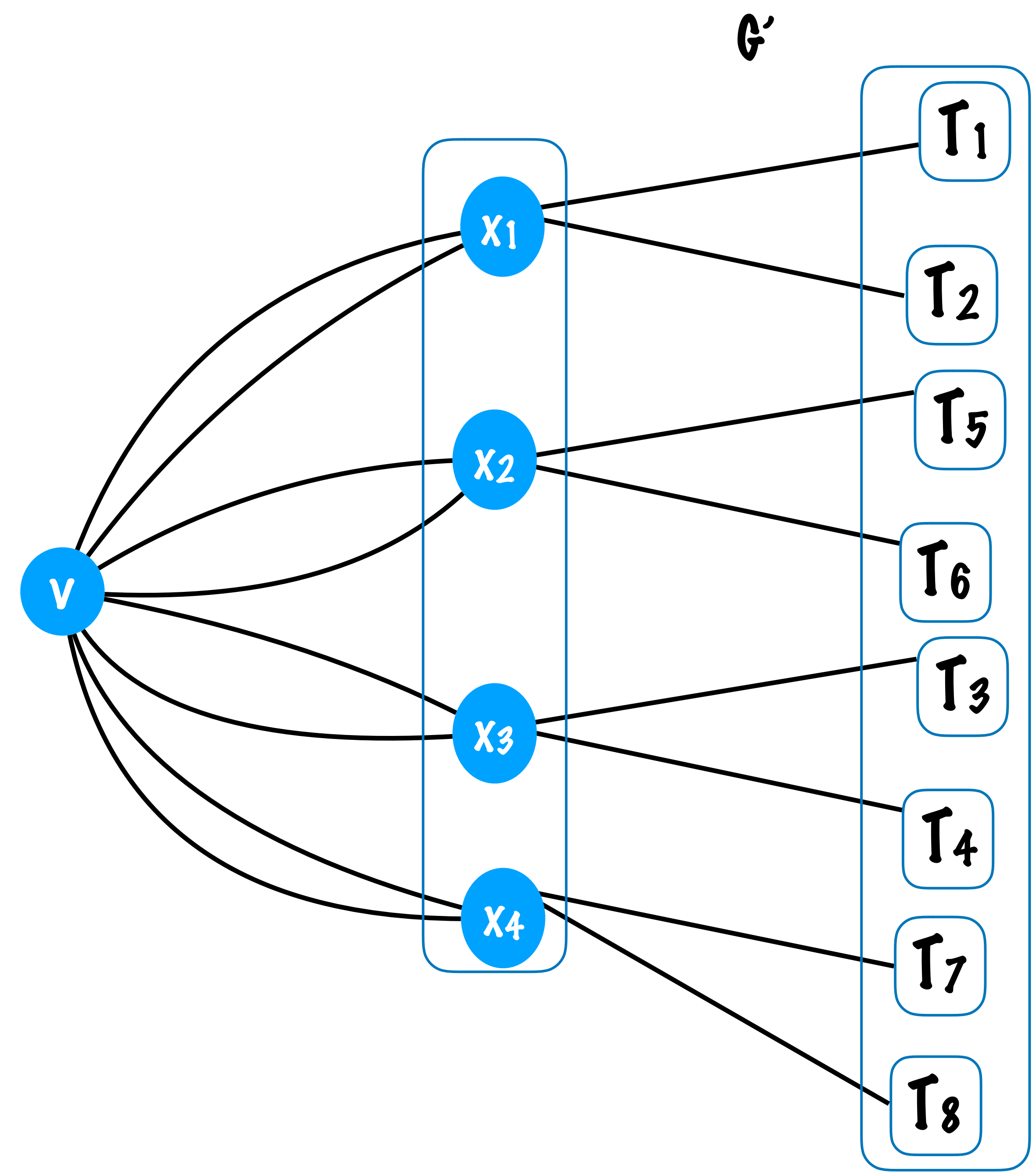


Suppose G' has FVS W of <=k. Then, v is in W or S' is in W

- Case: v is in W
  - G-v = G'-v and W is FVS of G too
- Case: S' is in W
  - Any cycle in G passing through T_i in D' also passes through a vertex in S'

Feedback Vertex Set - Towards a Quadratic Kernel

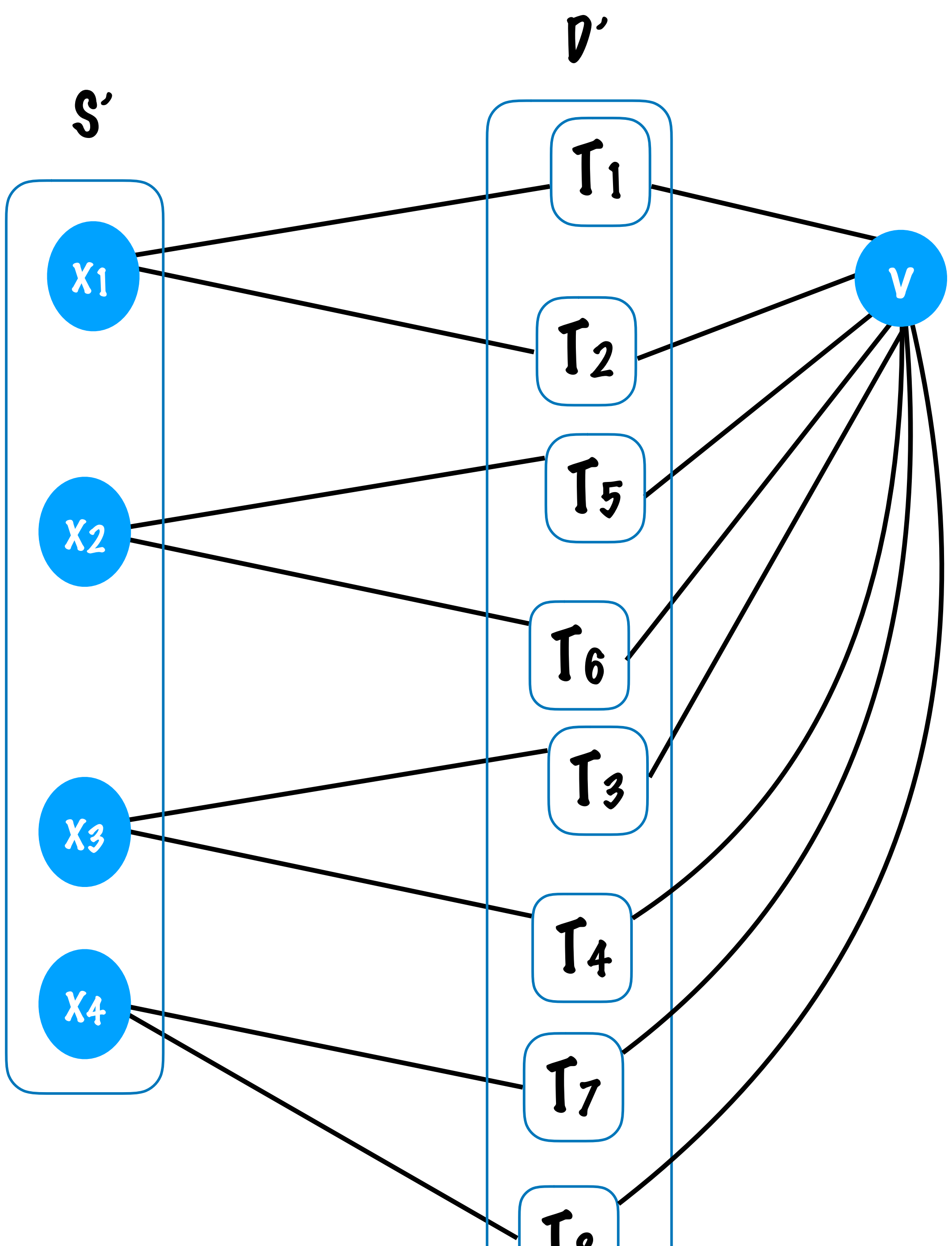




Suppose G has FVS W of <=k. Then, G has <=k FVS that either has v or S′

Feedback Vertex Set - Towards a Quadratic Kernel

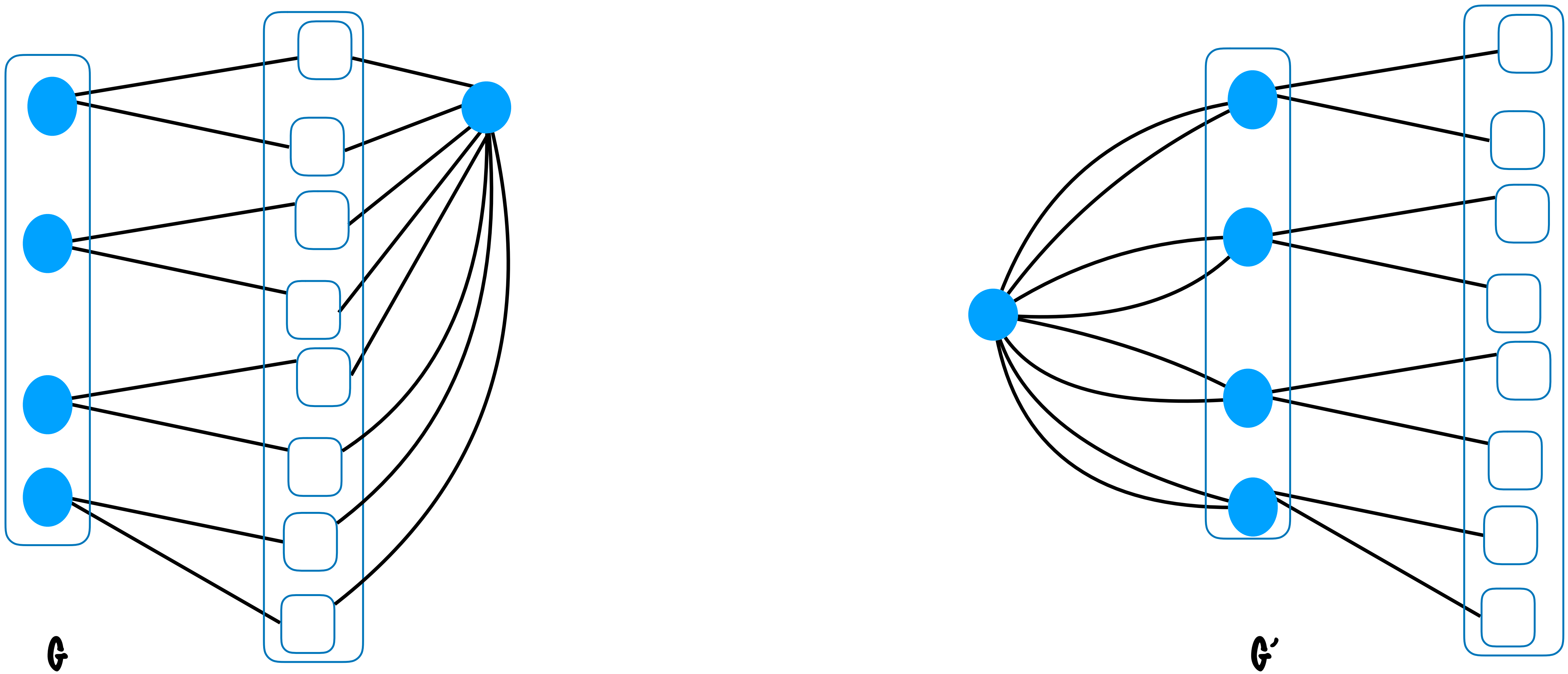
Claim: If G has <=k FVS, then G has <=k FVS that either has v or S'



neighbourhood of D' is in S' and therefore vertex in Ti cannot b

- Any FVS Z not containing v has at least |S'| vertices
- If Z does not contain some x_i from S', then Z contains a vertex q_i from $V(T_{i1}) \cup V(T_{i2})$
- For example,
  - Suppose x_1 and x_3 are not in Z
  - Then, q_1 (from $V(T_1) \cup V(T_2)$) and q_3 ($V(T_3) \cup V(T_4)$) are in Z
  - Replace q_1 and q_3 by x_1 and x_3 to get Z'
  - If G-Z' is not a forest, then there is a cycle C containing say q_1
    - C must have a vertex from S' ($\Rightarrow\Leftarrow$)

Feedback Vertex Set - Towards a Quadratic Kernel

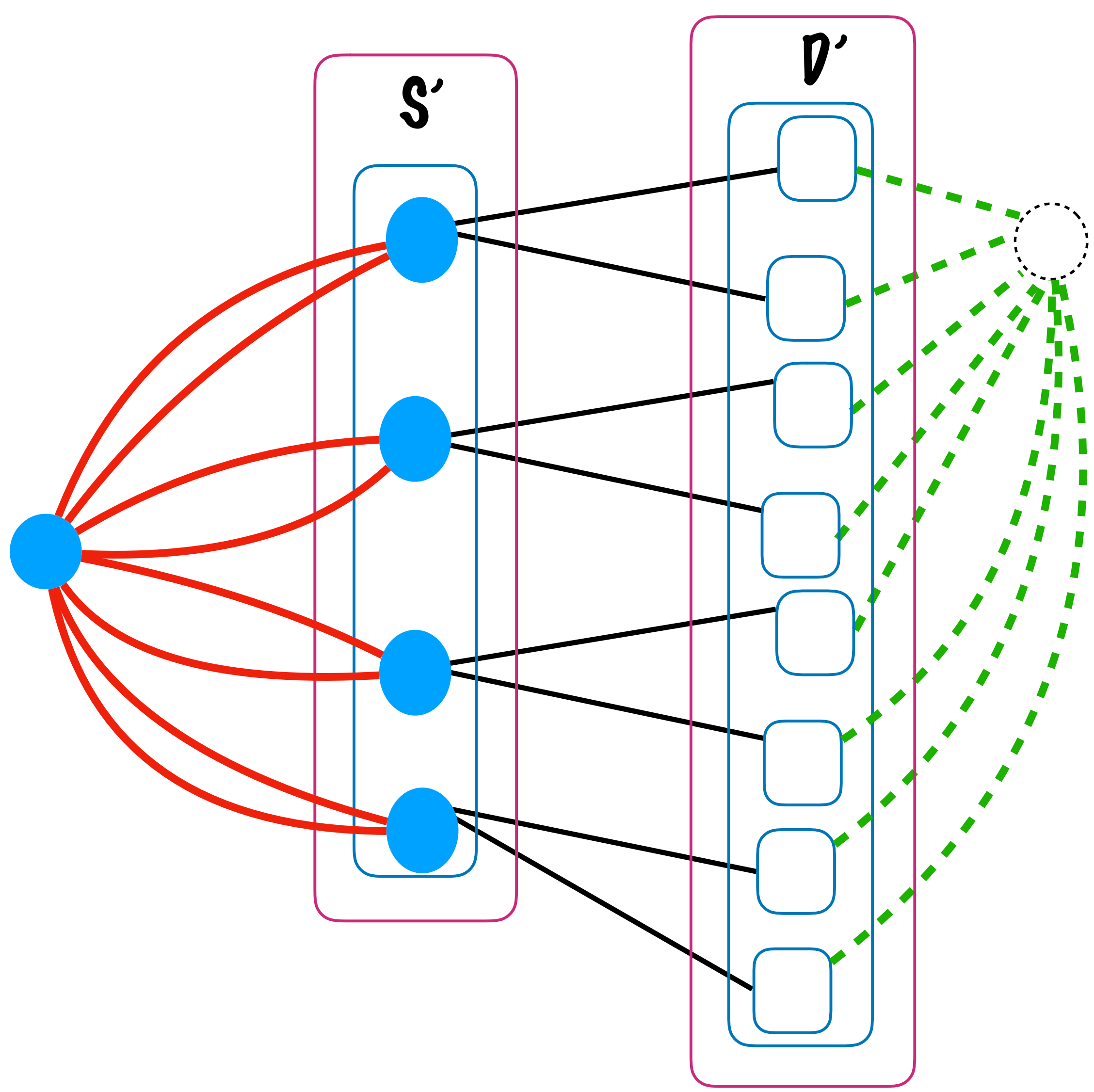


Suppose G has FVS W of <=k. Then, v is in W or S' is in W

* Case: v is in W
  * G-v = G'-v and W is FVS of G' too
* Case: S' is in W
  * Any cycle in G'-W implies a double edge between v and a vertex in T_i ($\Rightarrow\Leftarrow$)

Feedback Vertex Set - Towards a Quadratic Kernel

Reduction Rule 7: Add double edges between v and every vertex in S'. Delete edges between D' to v.



* Reduction Rule 7 can be applied <= m times
    * No. of single edges incident on v dec
* In polynomial time, either we will find a v-flower with k+1 petals, or the deg of v becomes <=10k or we will determine that (G,k) is a no-instance

Feedback Vertex Set - Quadratic Kernel

- Reduction Rule 1: Delete isolated vertices
- Reduction Rule 2: Delete degree-1 vertices
- Reduction Rule 3: If there is a loop at a vertex v, delete v and reduce param by 1
- Reduction Rule 4: If there is an edge with multiplicity > 2, reduce it to 2
- Reduction Rule 5: Short-circuit degree 2 vertices
- Reduction Rule 6: If v is a vertex of degree >10k and there is a v-flower with k+1 petals, delete v and reduce param by 1
- If v has degree >10k and has > 2k double edges, then apply Reduction Rule 6
- If v has degree >10k and Flower Lemma returns a FVS S of <=3k vertices s.t $v \notin S$
  - Use 2-expansion lemma to find $S' \subseteq S$ s.t if G has <=k FVS, then G has <=k FVS that either has v or S'
  - Reduction Rule 7: Add double edges between v and every vertex in S' and delete edges from D' to v where D' is the set of vertices in $G-(S \cup \{v\})$ saturated by the 2-expansion
- When none of the reductions rules are applicable, every vertex has degree <=10k
  - $n=O(k^2)$ and $m=O(k^2)$

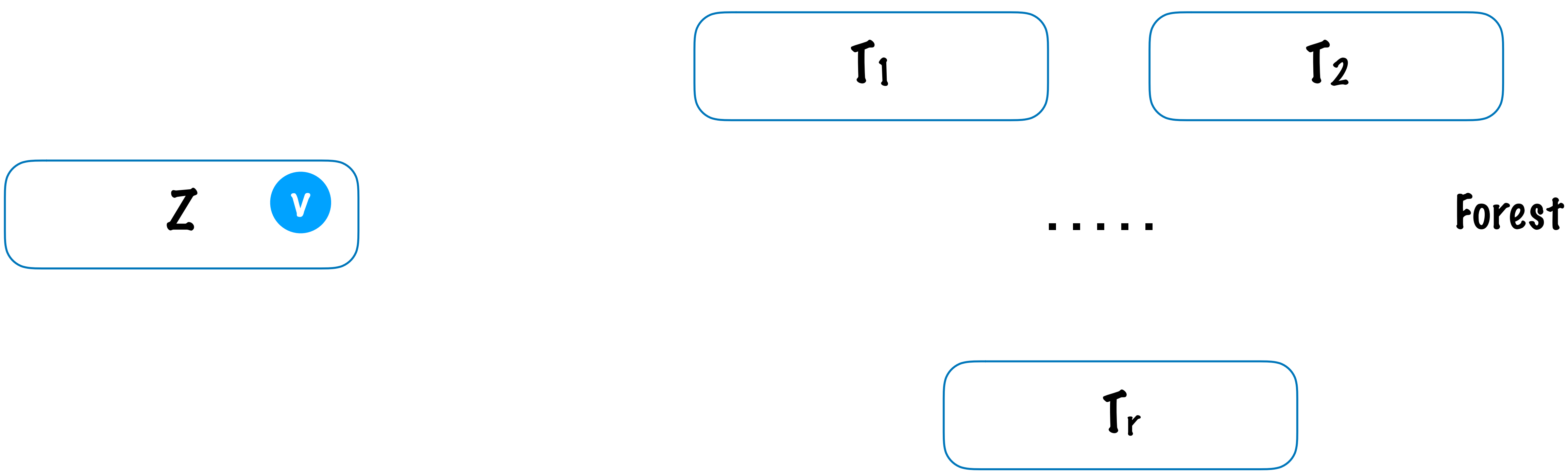
Flower Lemma

Flower Lemma: There is a polynomial time algorithm that given a graph G and a vertex v without a self-loop, satisfies one of the following:

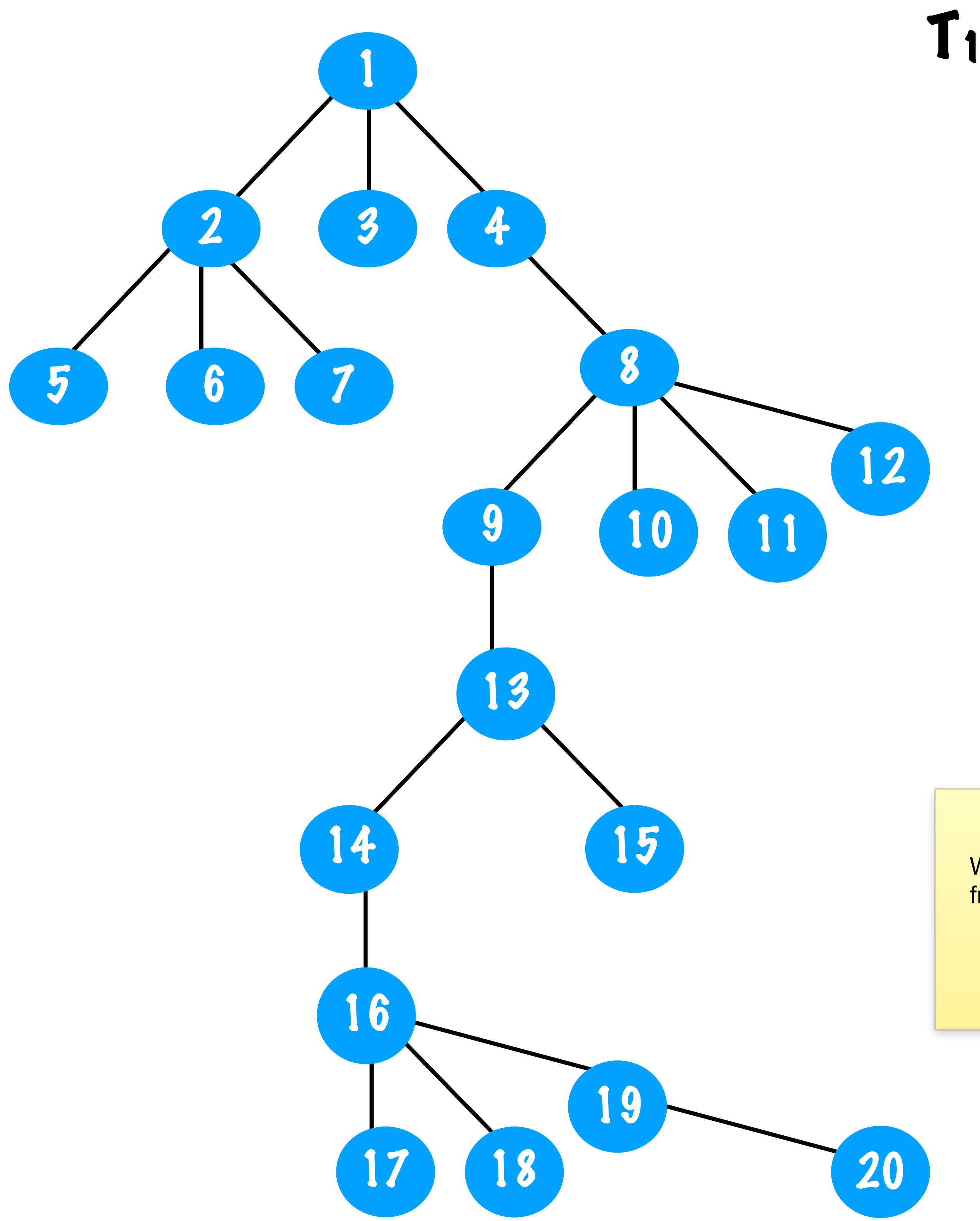
* Declares (G,k) is a no-instance of Feedback Vertex Set
* Returns a v-flower with (k+1) petals
* Finds FVS not containing v of size <= 3k

Flower Lemma

* Let Z be a 2-approximation FVS of G
* If |Z| >2k, then declare that (G,k) is a no-instance of Feedback Vertex Set
* Otherwise, |Z|<=2k
* If v is not in Z, then Z is the required FVS
* Otherwise, v is in Z

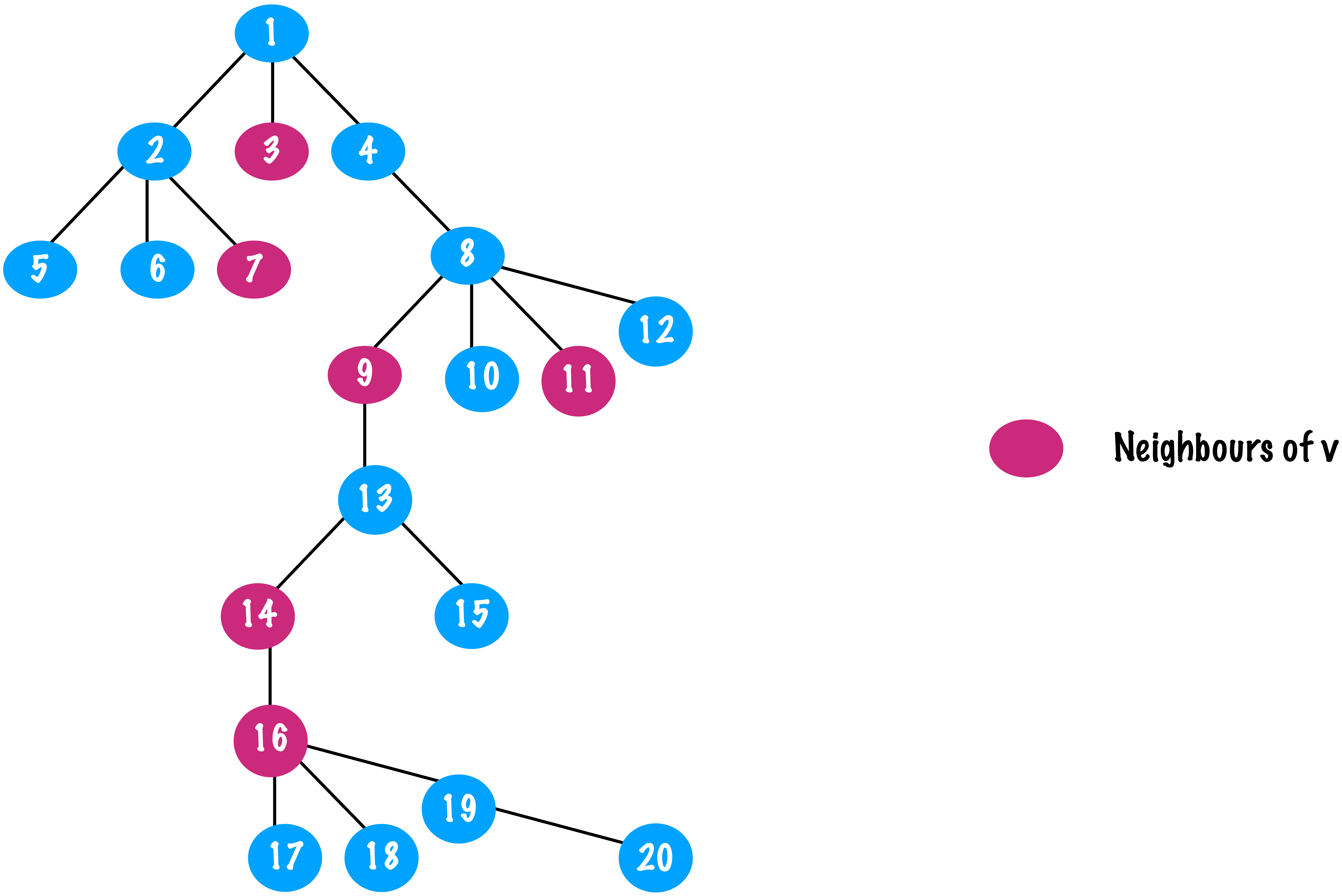


Flower Lemma

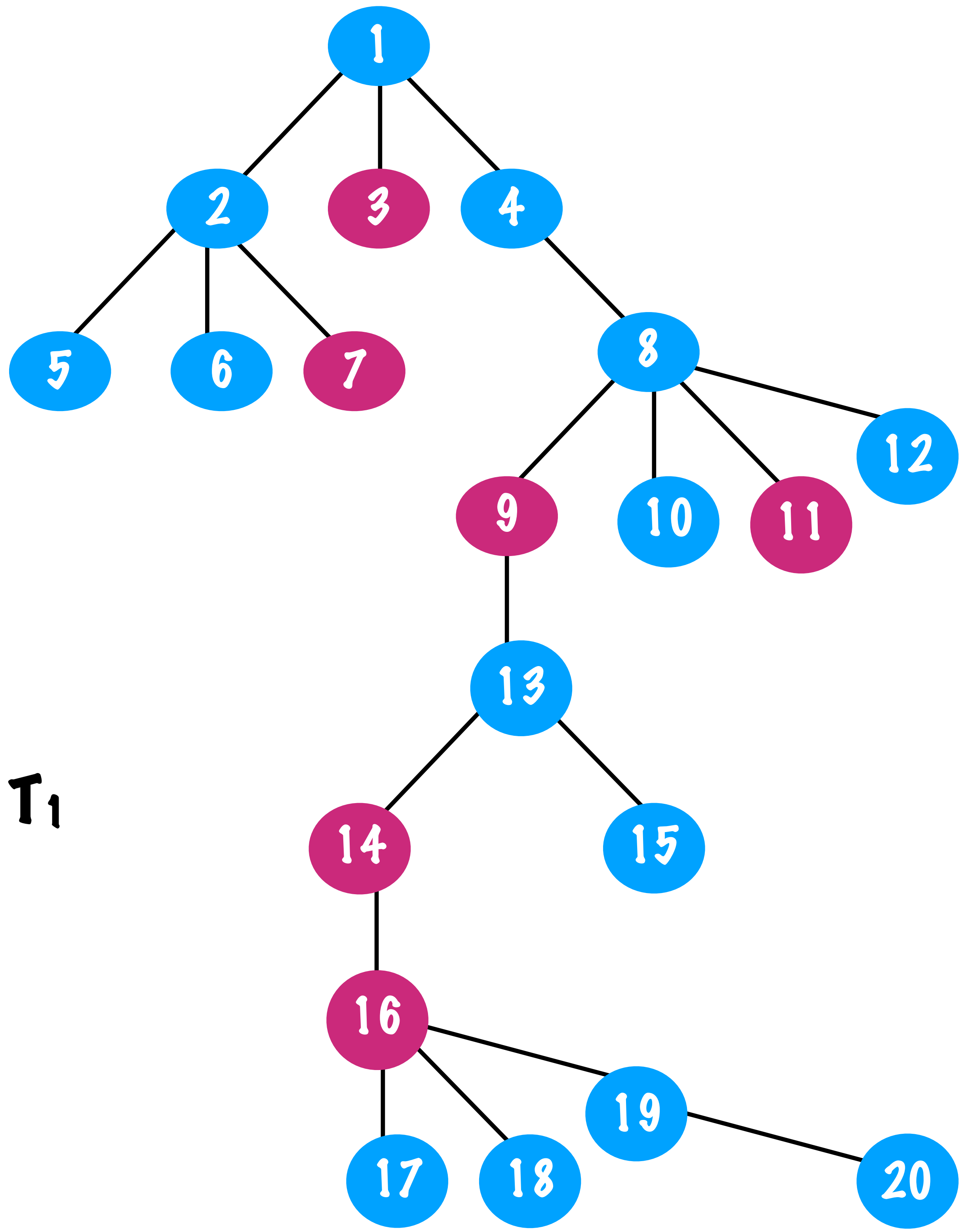


Why can't a v-flower have vertices from Z?

Flower Lemma

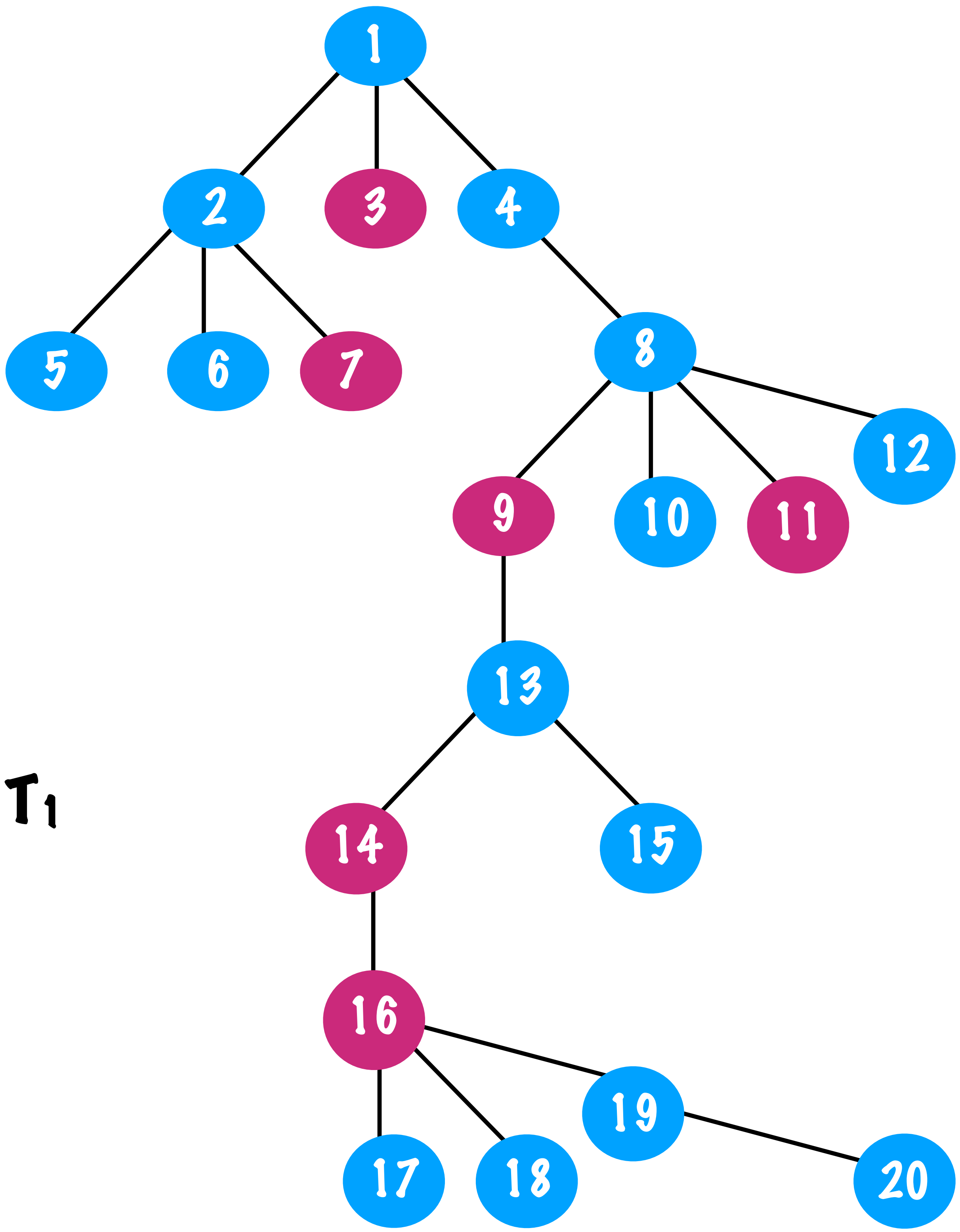


Flower Lemma



- lca(3,7)=1
- lca(14,16)=14
- lca(16,9)=9
- lca(9,11)=8
- .....
- Find least common ancestor of every pair of neighbours of v

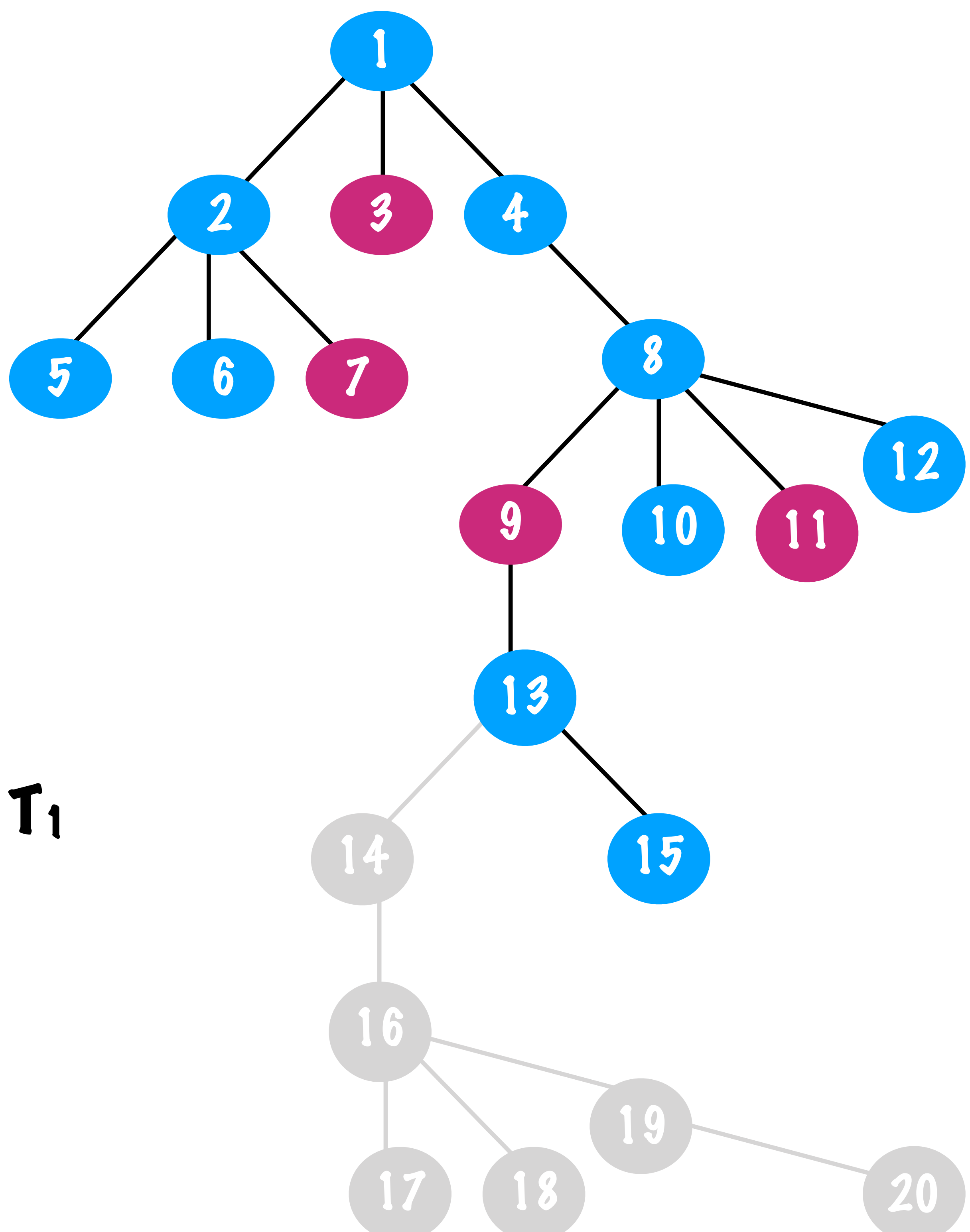
Flower Lemma



T_1

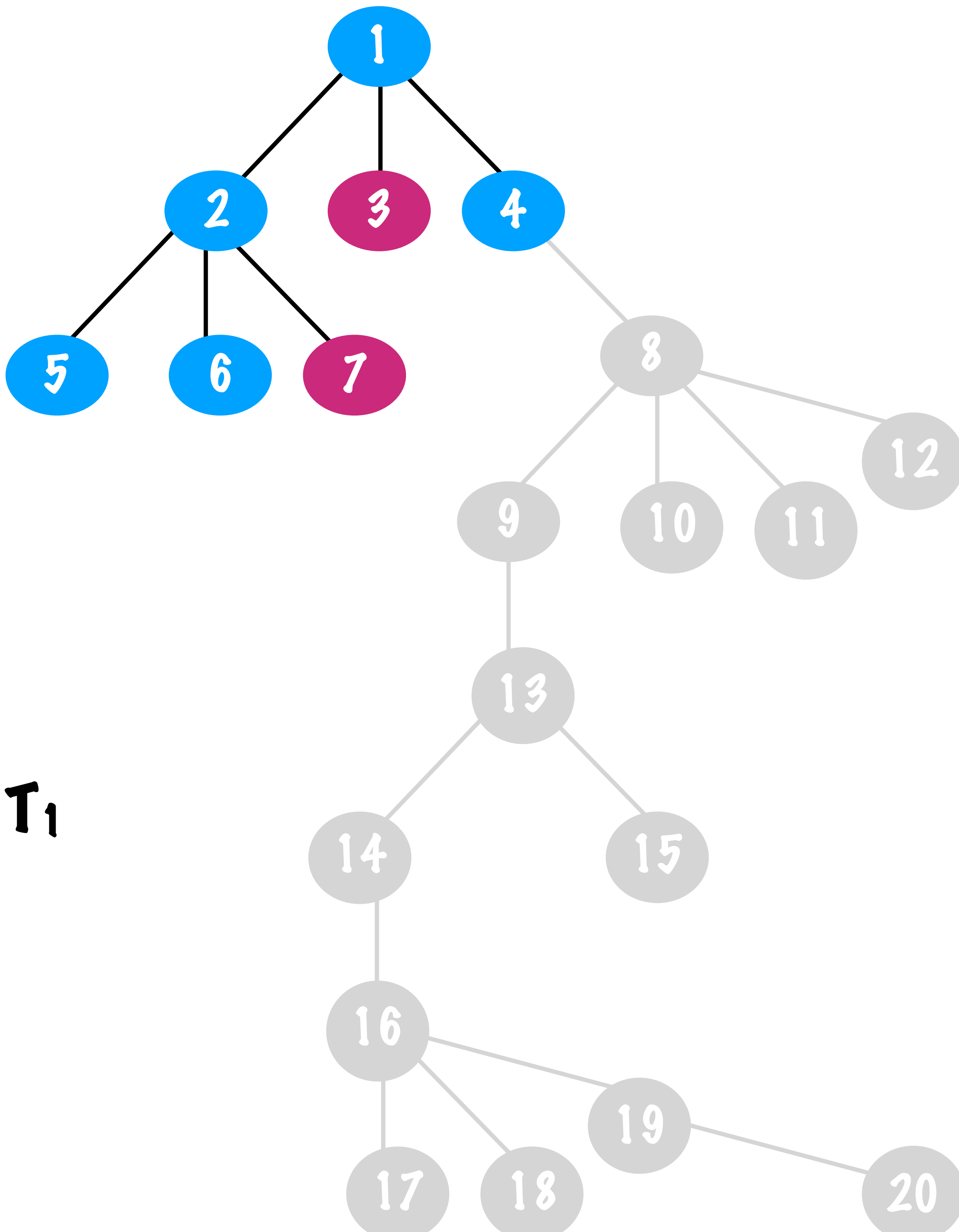
* lca(14,16)=14 is the deepest
* Add 14-16 path to P
* Add 14 to Y
* Delete subtree rooted at 14

Flower Lemma



- lca(9,11)=8 is the deepest
- Add 9-8-11 path to P
- Add 8 to Y
- Delete subtree rooted at 8

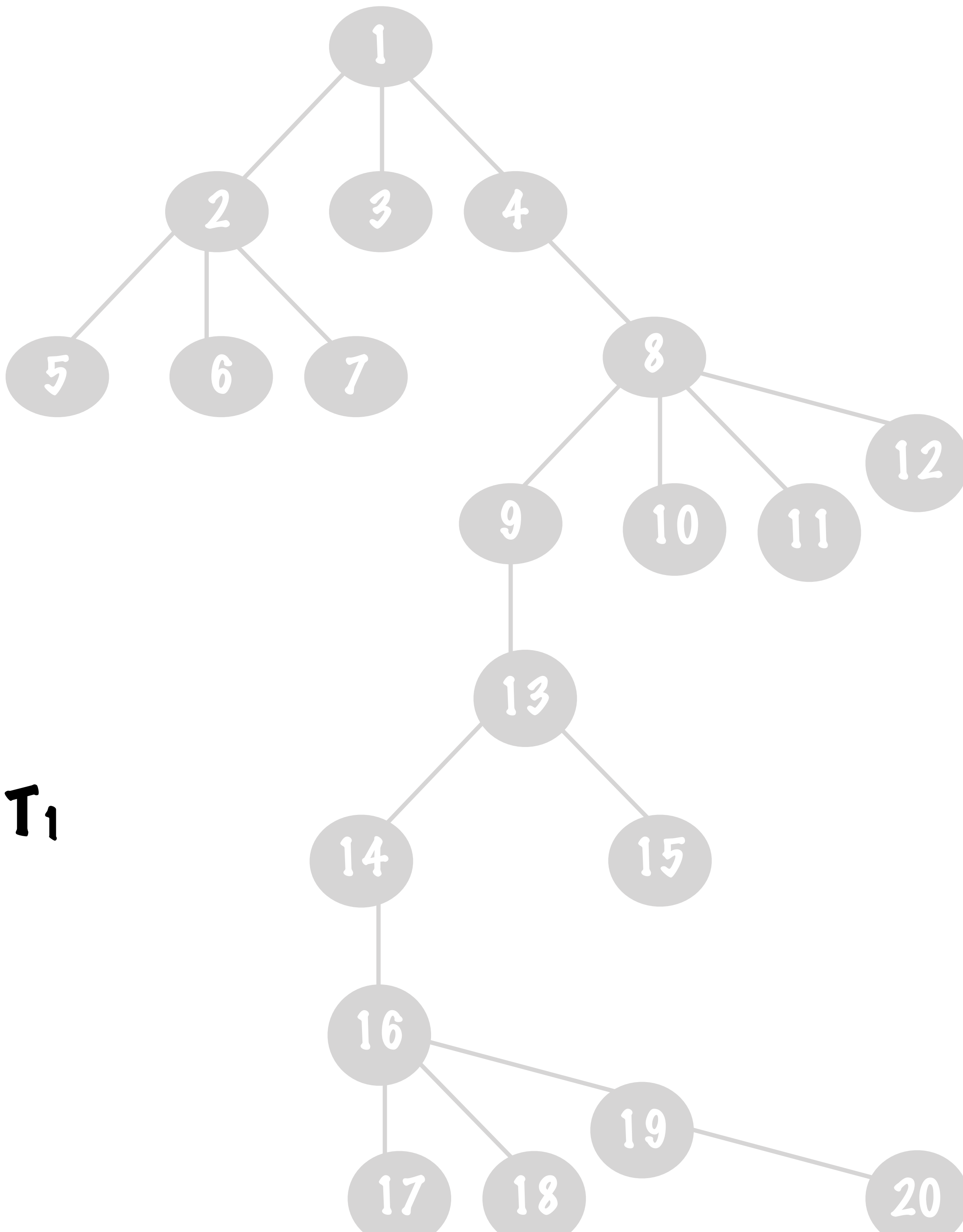
Flower Lemma



T_1

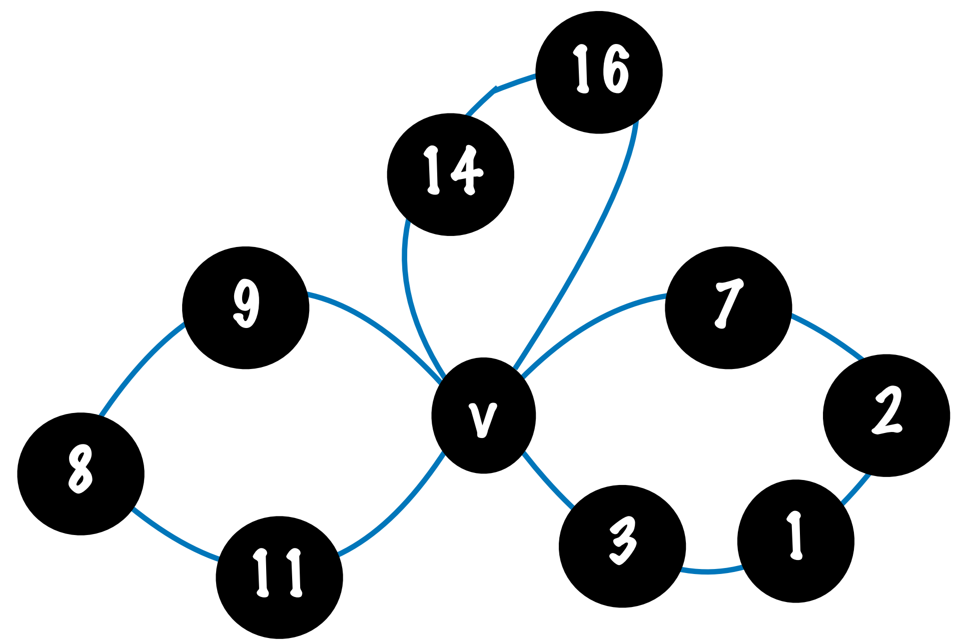
* lca(3,7)=1 is the deepest
* Add 7-2-1-3 path to P
* Add 1 to Y
* Delete subtree rooted at 1

Flower Lemma



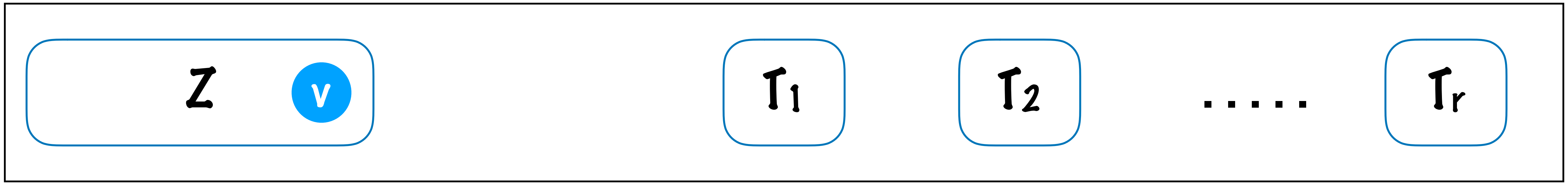
T_1

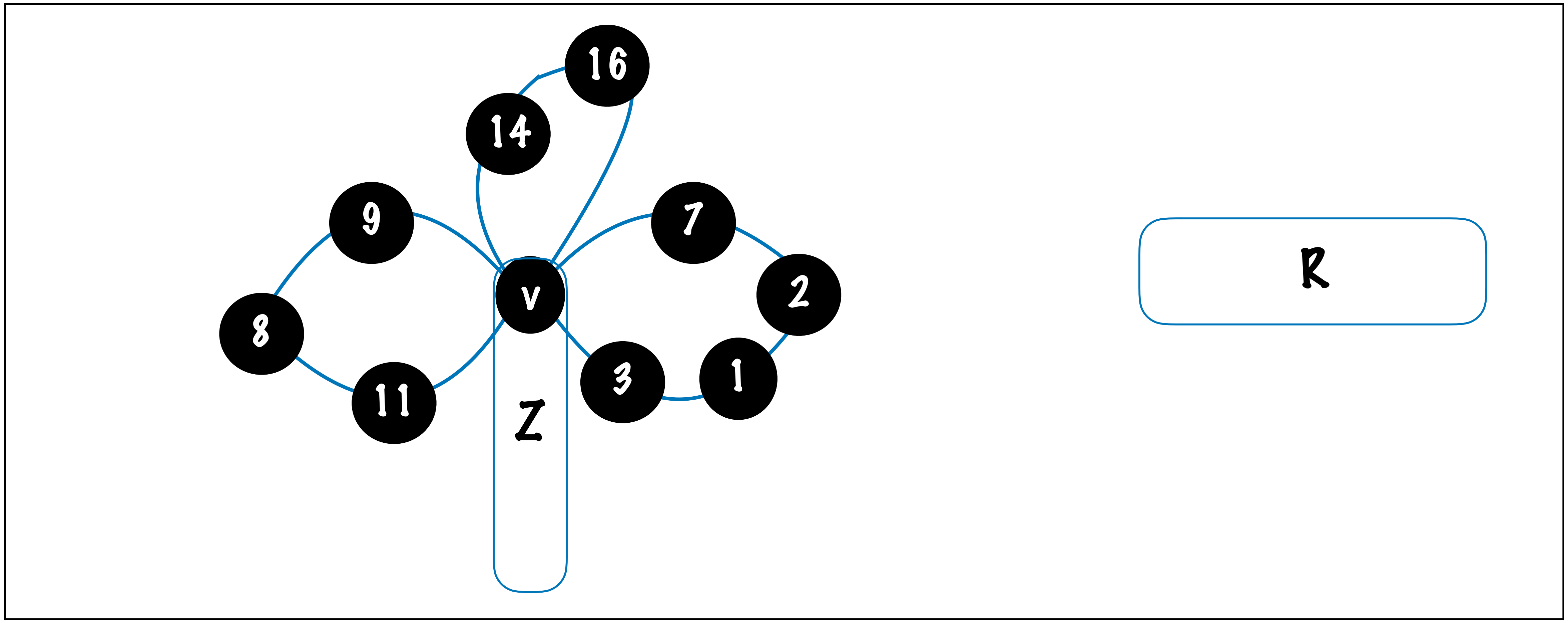
* P = {7-2-1-3, 9-8-11, 14-16}
* V = {1, 8, 14 }



Repeat with T_2

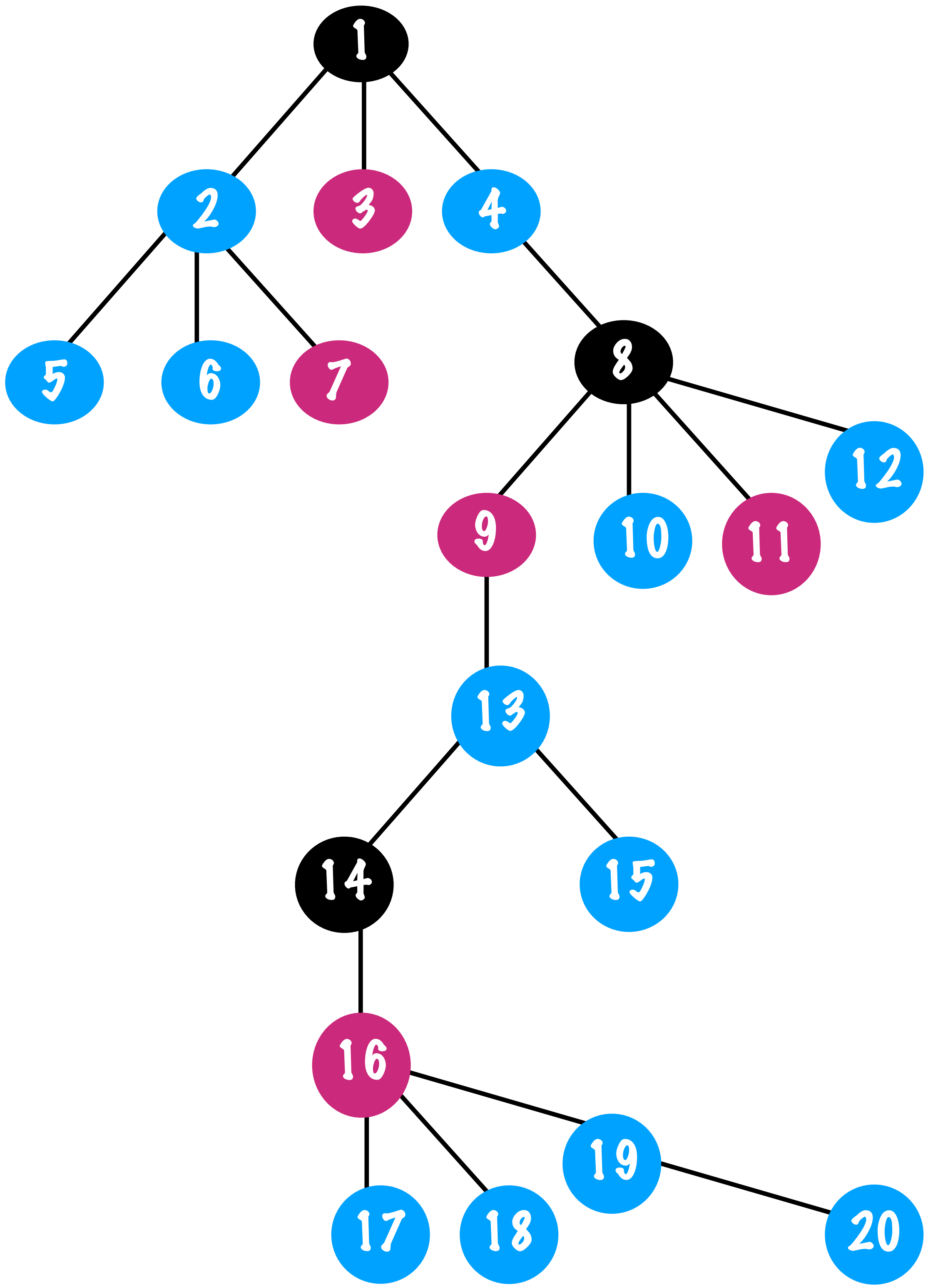
Flower Lemma

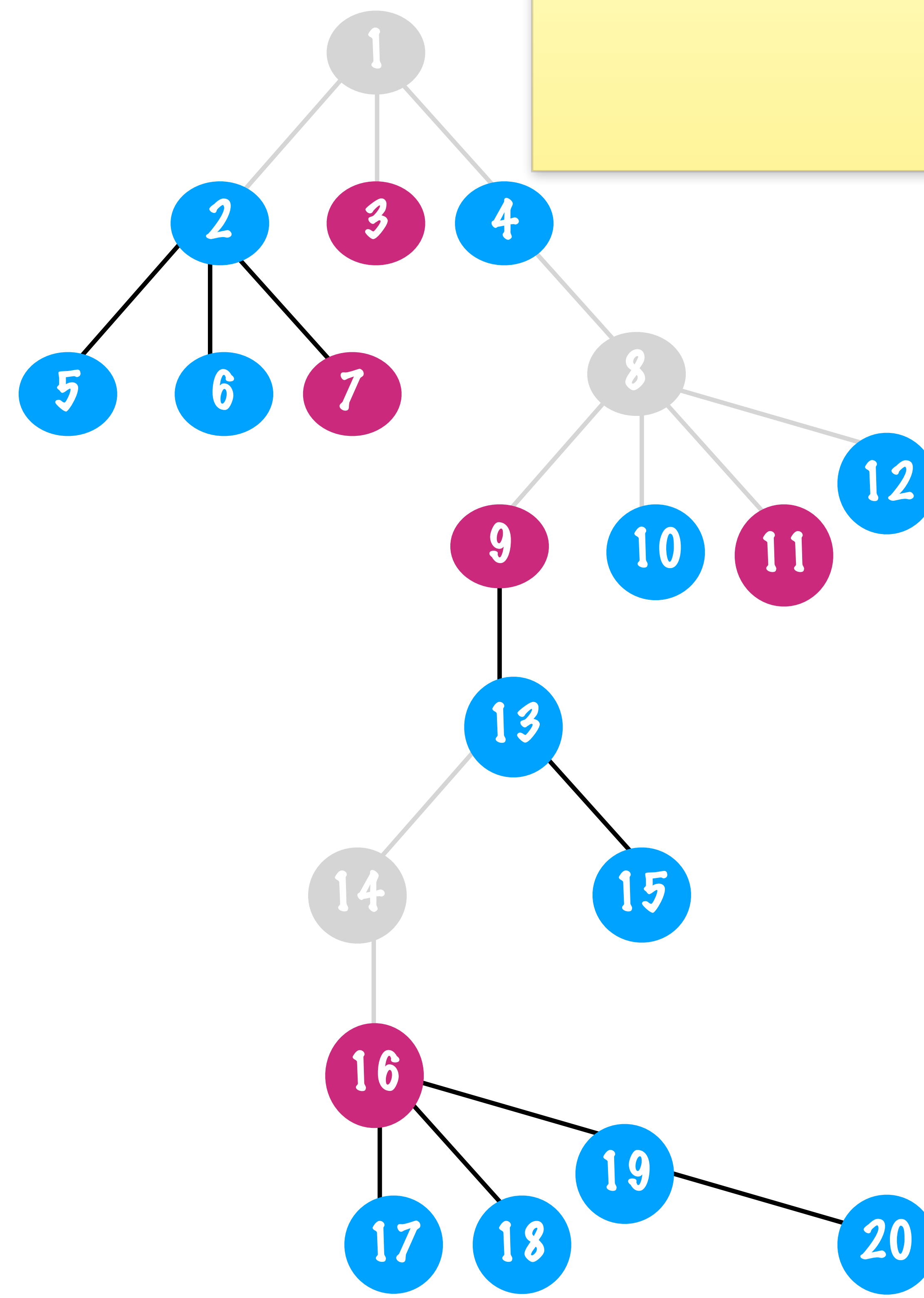




* P = {7-2-1-3, 9-8-11, 14-16}
* Y = {1, 8, 14 }
* If |P| >k, then there is a v-flower with k+1 petals

Flower Lemma





Z \cup Y \minus v is FVS <=3k not containing v

- Y = {1, 8, 14 }
- |Y| = |P| <= k and S=(Z∪Y)\{v} is the required FVS of size <=3k as no component of G-S has two neighbours of v

Matching and Vertex Cover in Bipartite Graphs

Kőnig's Theorem: For a bipartite graph, Max Mat = Min VC

Hall's Theorem: Let G be a bipartite graph with bipartition (A, B). Then, G has a matching saturating A if and only if $|N(X)| >= |X|$ for all $X \subseteq A$.

Hopcroft-Karp Algorithm: Let G be a bipartite graph with bipartition (A, B).

* Then, a max mat and a min vc of G can be obtained in $O(m\, n^{1/2})$ time.
* Further, in $O(m\, n^{1/2})$ time, we can either find a matching saturating A or an inclusion-minimal set $X \subseteq A$ such that $|N(X)| < |X|$.

Kőnig's Theorem

For a bipartite graph, |Max Mat| = |Min VC|

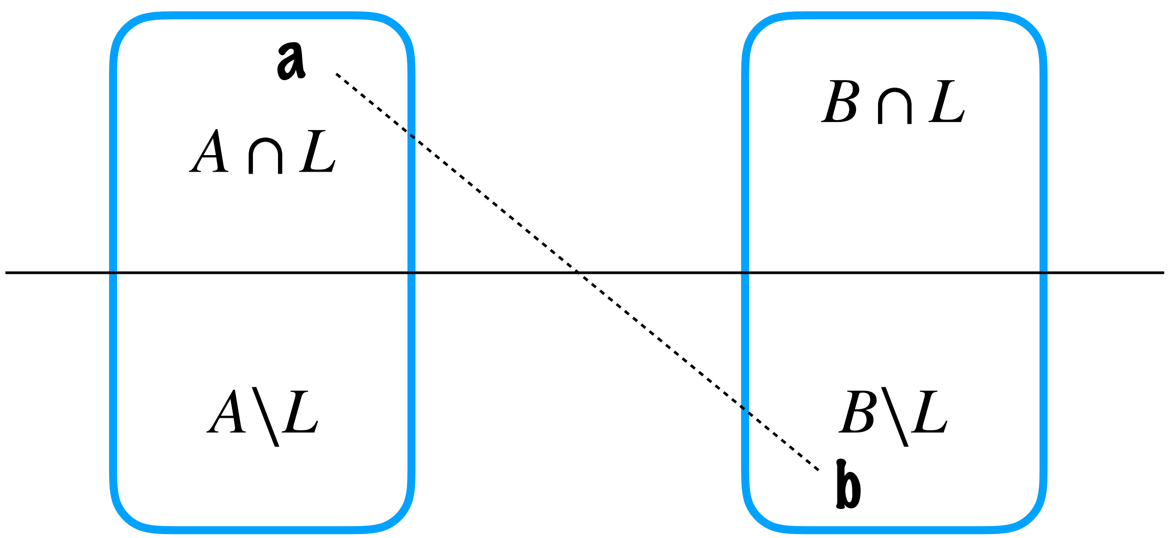
* Let M be a maximum matching of G(A,B)
* Let D be a digraph obtained from G by orienting edges of G as follows:
  * If edge {a,b} with $a \in A$ and $b \in B$ is in M, direct {a,b} as (b,a)
  * If edge {a,b} with $a \in A$ and $b \in B$ is not in M, direct {a,b} as (a,b)
* There is no M-alternating path in D from a free vertex in A to a free vertex in B
* If |M|=|A| then A is a min vertex cover
* L = set of vertices reachable from a free vertex in A

Claim: $(A \setminus L) \cup (B \cap L)$ is a min vertex cover of G

Kőnig's Theorem

Claim: $S = (A \backslash L) \cup (B \cap L)$ is a min vertex cover of G

only in-nbr as for any other undirected edge to a, if 'c'



- If $\{a,b\} \in M$ with $b \in B \backslash L$ and $a \in A \cap L$
  - $(b,a) \in E(D)$
  - b is the only in-nbr of $a \in L$
  - $\therefore b \in L$
- If $\{a,b\} \notin M$
  - $(a,b) \in E(D)$
  - $\therefore b \in L$ as $a \in L$
- No vertex $a' \in A \backslash L$ is free by defn of L
- No vertex $b' \in B \cap L$ is free
  - o/w, M-alternating path from a free vertex in A to b' as $b' \in L$
- No edge $\{a',b'\}$ with $a' \in A \backslash L$ and $b' \in B \cap L$ is in M
  - o/w, $(b',a') \in E(D)$ and $a' \in L$
- $\therefore |S| = |M|$

Hall's Theorem

Let G be a bipartite graph with bipartition (A, B). Then, G has a matching saturating A if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$.

Sufficiency: Induction on $|A|$

* Base: $|A|=1$
* Induction Step: Let $a \in A$ and b be one of its neighbours
* If $|N(X)\setminus\{b\}| \geq |X|$ for all $X \subseteq A\setminus\{a\}$
  * By induction hypothesis, there is a matching M saturating $A\setminus\{a\}$ in $G-\{a,b\}$
  * Then, $M \cup \{\{a,b\}\}$ is a matching saturating A in G
* Otherwise, $|N(X)\setminus\{b\}| < |X|$ for some $X \subseteq A\setminus\{a\}$
  * As $|N(X)| \geq |X|$, it follows that $|N(X)| = |X|$
  * By induction hypothesis there is a matching M saturating X in $G(X, N(X))$
  * For any $Y \subseteq A\setminus X$, $|N(Y)\setminus N(X)| \geq |N(Y \cup X)| - |N(X)| \geq |Y| + |X| - |N(X)| = |Y|$
  * By induction hypothesis there is a matching M' saturating $A\setminus X$ in $G(A\setminus X, B\setminus N(X))$
  * $M \cup M'$ is a matching saturating A

Hopcroft-Karp Algorithm

Let G be a bipartite graph with bipartition (A, B).

* Then, a max mat and a min vc of G can be obtained in $O(m\, n^{1/2})$ time.
* Further, in $O(m\, n^{1/2})$ time, we can either find a matching saturating A or an inclusion-minimal set $X \subseteq A$ such that $|N(X)| < |X|$.

Finding Minimal Hall Set

* Suppose M is a maximum matching of $G(A,B)$ such that $|M| < |A|$
* Let $A=\{a_1, a_2, a_3, \ldots, a_r\}$ and let $X_1=\{a_1\}$ where a_1 is a vertex not saturated by M
  * Every vertex in $N(a_1)$ is saturated by M
* For $i=1$ to $r-1$
  * If $|N(X_i)| < |X_i|$, Return X_i
  * $X_{i+1} = X_i \cup$ M-partners$(N(X_i))$ (Note: $|X_{i+1}| > |X_i|$)
* Return X_r

Each vertex in N(Xi) is a part of matching as suppose for the first time in the jth iteration N(Xj) contain

Extended Hall's Theorem

Let G be a bipartite graph with bipartition (A, B). Then, there is a 2-expansion of A into B iff $|N(X)| >= 2|X|$ for all $X \subseteq A$.

Sufficiency:

* To $G(A,B)$, add a copy of A to get bipartite graph $G'(A',B)$
* If $G'(A',B)$ has a matching saturating A', then this matching corresponds to a 2-exp of A into B
* Otherwise, by Hall's theorem, $|N_{G'}(X)| < |X|$ for some $X \subseteq A'$
  * w.l.o.g assume X has both copies or no copy of a vertex of A
  * $|N_{G'}(X)| = |N_{G'}(X \cap A)| = |N_G(X \cap A)| >= 2|X \cap A| = |X|$ (a contradiction)

If there is single copy jus

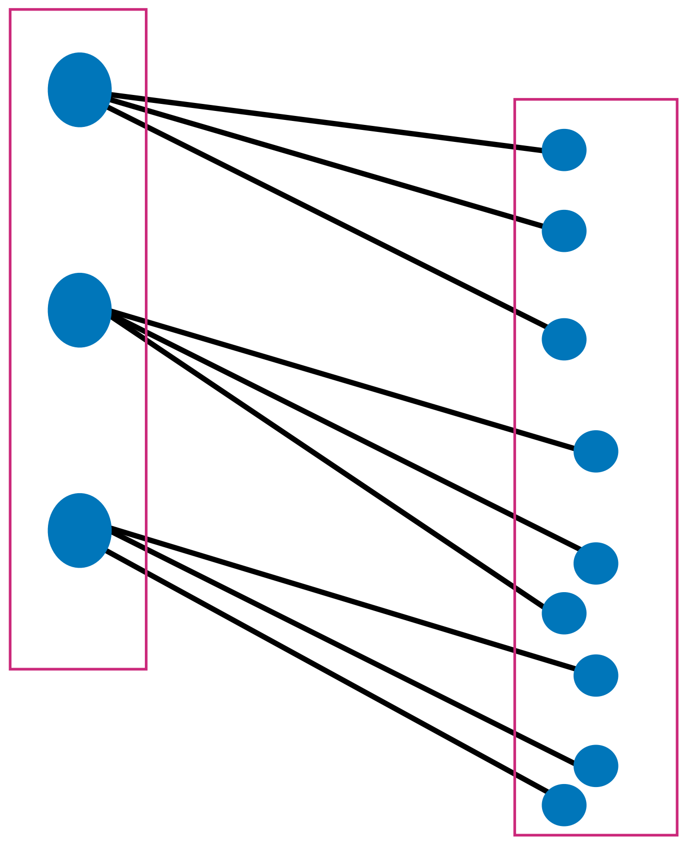
2-Expansion Lemma

Let G be a bipartite graph with bipartition (A, B) s.t $|B| >= 2\ |A|$ and there are no isolated vertices in B. Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a 2-exp into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m\ n^{1/2})$ time.

* If A has a 2-expansion M into B, then $X=A$ and $Y=V(M) \cap B$
  * Clearly, $N(Y) \subseteq X$
* Otherwise, by Extended Hall's theorem, $|N_G(X)| < 2|X|$ for some $X \subset A$
* Let $G'(A',B')$ denote the graph obtained from G by deleting X and $N_G(X)$
* $|B'| = |B| - |N_G(X)| > |B| - 2|X| >= 2|A| - 2|X| = 2|A'|$
* In G, no vertex in B' has a neighbour in X. So, B' has no isolated vertices in G'
* By induction hypothesis, there exists non-empty sets $X' \subseteq A'$ and $Y' \subseteq B'$ such that X' has a 2-exp into Y' and $N_{G'}(Y') \subseteq X'$
* As $N_G(Y') = N_{G'}(Y')$, it follows that $N_G(Y') \subseteq X'$

Ex: Runtime analysis

q-Expansion Lemma



Definition: In a bipartite graph with bipartition (A, B), a set M of edges is called a q-expansion from A to B if

* Every vertex of A is incident with exactly q edges of M
* M saturates exactly q|A| vertices in B

Extended Hall's Theorem: Let G be a bipartite graph with bipartition (A, B). Then, there is a q-expansion of A into B iff $|N(X)| >= q|X|$ for all $X \subseteq A$.

q-Expansion Lemma: Let q be a positive integer and let G be a bipartite graph with bipartition (A, B) such that $|B| > q\,|A|$ and there are no isolated vertices in B. Then, there exists non-empty sets $X \subseteq A$ and $Y \subseteq B$ such that X has a q-expansion into Y and $N(Y) \subseteq X$. Further, the sets X and Y can be found in $O(m\, n^{1/2})$ time