# Assignment 1

1. Let $H$ be a subgroup of a group $G$. Show that $H$ is a normal subgroup of $G$ if and only if it is a union of conjugacy classes in $G$.

2. Let $G$ be a group generated by 3 elements $x, y, z \in G$ which satisfies the relations

$$x^3 = e, \;\; y^2 = z^2 = e, \;\; yz = zy, \;\; yx = xy, \;\; zx = x^2 z.$$

   Show that $G$ is isomorphic to the dihedral group $D_6$.

3. List the set of finite groups upto isomorphism with atmost 3 conjugacy classes.

4. Let $S_n$ be the group of permutations of the set $\{1, 2, \ldots, n\}$. Let $e_i \in \mathbb{R}^n$ be the vector $(0, \ldots, \underbrace{1}_{i}, 0 \ldots, 0) \in \mathbb{R}^n$. For $\sigma \in S_n$, consider the map

$$T_\sigma : \mathbb{R}^n \to \mathbb{R}^n$$

   defined by

$$T_\sigma(a_1, \ldots, a_n) \;\; = \;\; (a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

   (a) Show that $T_\sigma$ is an invertible linear map for all $\sigma \in S_n$.

   (b) Let $A_\sigma$ be the matrix associated to $T_\sigma$ with respect to the basis $\{e_1, \ldots, e_n\}$. Show that the map

$$\mathcal{A} : S_n \to GL(n, \mathbb{R})$$

   given by

$$\mathcal{A}(\sigma) = A_\sigma.$$

   is a group homomorphism.

   (c) Show that $det(A_\sigma) \in \{1, -1\}$ for all $\sigma \in S_n$ and the map

$$\mathcal{S} : S_n \to \{1, -1\}, \;\; \mathcal{S}(\sigma) = det(A_\sigma).$$

   is a group homomorphism, where the group structure on $\{1, -1\}$ is given by multiplication.

   (d) Show that $\mathcal{S}$ agrees with the sign of a permutation which we had discussed in the class.

5. Classify groups of order 8 upto isomorphism.

6. Show that any group of order 99 is abelian.