3.1 Exponential Growth and Decay

Motivating Questions

- What does it mean to say that a function is “exponential”?
- How much data do we need to know in order to determine the formula for an exponential function?
- Are there important trends that all exponential functions exhibit?

Linear functions have constant average rate of change and model many important phenomena. In other settings, it is natural for a quantity to change at a rate that is proportional to the amount of the quantity present. For instance, whether you put \$100 or \$100000 or any other amount in a mutual fund, the investment's value changes at a rate proportional the amount present. We often measure that rate in terms of the annual percentage rate of return.

Suppose that a certain mutual fund has a 10% annual return. If we invest \$100, after 1 year we still have the original \$100, plus we gain 10% of \$100, so

$$100 \xrightarrow{\text{year}}{}^{1} 100 + 0.1(100) = 1.1(100).$$

If we instead invested \$100000, after 1 year we again have the original \$100000, but now we gain 10% of \$100000, and thus

$$100000 \xrightarrow{\text{year}}{}^{1} 100000 + 0.1(100000) = 1.1(100000).$$

We therefore see that regardless of the amount of money originally invested, say P, the amount of money we have after 1 year is $1.1P$.

If we repeat our computations for the second year, we observe that

$$1.1(100) \xrightarrow{\text{year}}{}^{2} 1.1(100) + 0.1(1.1(100) = 1.1(1.1(100)) = 1.1^2(100).$$

The ideas are identical with the larger dollar value, so

$$1.1(100000) \xrightarrow{\text{year}}{}^{2} 1.1(100000) + 0.1(1.1(100000) = 1.1(1.1(100000)) = 1.1^2(100000),$$

and we see that if we invest P dollars, in 2 years our investment will grow to 1.1^2P.

Of course, in 3 years at 10%, the original investment P will have grown to 1.1^3P. Here we see a new kind of pattern developing: annual growth of 10% is leading to *powers* of the base 1.1, where the power to which we raise 1.1 corresponds to the number of years the investment has grown. We often call this phenomenon *exponential growth*.

Preview Activity 3.1.1. Suppose that at age 20 you have \$20000 and you can choose between one of two ways to use the money: you can invest it in a mutual fund that will, on average, earn 8% interest annually, or you can purchase a new automobile that will, on average, depreciate 12% annually. Let's explore how the \$20000 changes over time.

Let $I(t)$ denote the value of the \$20000 after t years if it is invested in the mutual fund, and let $V(t)$ denote the value of the automobile t years after it is purchased.

a. Determine $I(0)$, $I(1)$, $I(2)$, and $I(3)$.

b. Note that if a quantity depreciates 12% annually, after a given year, 88% of the quantity remains. Compute $V(0)$, $V(1)$, $V(2)$, and $V(3)$.

c. Based on the patterns in your computations in (a) and (b), determine formulas for $I(t)$ and $V(t)$.

d. Use *Desmos* to define $I(t)$ and $V(t)$. Plot each function on the interval $0 \leq t \leq 20$ and record your results on the axes in Figure 3.1.1, being sure to label the scale on the axes. What trends do you observe in the graphs? How do $I(20)$ and $V(20)$ compare?

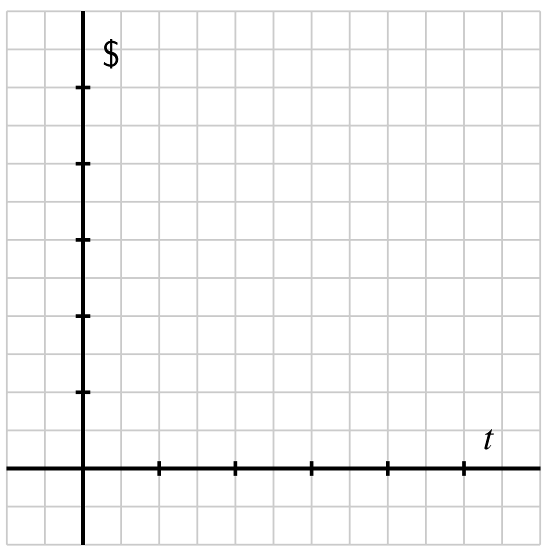

Figure 3.1.1. Blank axes for plotting I and V.

3.1.1 Exponential functions of form $f(t) = ab^t$

In Preview Activity 3.1.1, we encountered the functions $I(t)$ and $V(t)$ that had the same basic structure. Each can be written in the form $g(t) = ab^t$ where a and b are positive constants and $b \neq 1$. Based on our earlier work with transformations, we know that the constant a is a vertical scaling factor, and thus the main behavior of the function comes from b^t, which we call an "exponential function".

Definition 3.1.2. Let b be a real number such that $b > 0$ and $b \neq 1$. We call the function defined by

$$f(t) = b^t$$

an ***exponential function*** with ***base*** b.

For an exponential function $f(t) = b^t$, we note that $f(0) = b^0 = 1$, so an exponential function of this form always passes through $(0, 1)$. In addition, because a positive number raised to any power is always positive (for instance, $2^{10} = 1096$ and $2^{-10} = \frac{1}{2^{10}} = \frac{1}{2096}$), the output of an exponential function is also always positive. In particular, $f(t) = b^t$ is never zero and thus has no x-intercepts.

Because we will be frequently interested in functions such as $I(t)$ and $V(t)$ with the form ab^t, we will also refer to functions of this form as "exponential", understanding that technically these are vertical stretches of exponential functions according to Definition 3.1.2. In Preview Activity 3.1.1, we found that $I(t) = 20000(1.08)^t$ and $V(t) = 20000(0.88)^t$. It is natural to call 1.08 the "growth factor" of I and similarly 0.88 the growth factor of V. In addition, we note that these values stem from the actual growth rates: 0.08 for I and -0.12 for V, the latter being negative because value is depreciating. In general, for a function of form $f(t) = ab^t$, we call b the ***growth factor***. Moreover, if $b = 1 + r$, we call r the ***growth rate***. Whenever $b > 1$, we often say that the function f is exhibiting "exponential growth", wherease if $0 < b < 1$, we say f exhibits "exponential decay".

We explore the properties of functions of form $f(t) = ab^t$ further in Activity 3.1.2.

Activity 3.1.2. In *Desmos*, define the function $g(t) = ab^t$ and create sliders for both a and b when prompted. Click on the sliders to set the minimum value for each to 0.1 and the maximum value to 10. Note that for g to be an exponential function, we require $b \neq 1$, even though the slider for b will allow this value.

a. What is the domain of $g(t) = ab^t$?

b. What is the range of $g(t) = ab^t$?

c. What is the y-intercept of $g(t) = ab^t$?

d. How does changing the value of b affect the shape and behavior of the graph of $g(t) = ab^t$? Write several sentences to explain.

e. For what values of the growth factor b is the corresponding growth rate positive? For which b-values is the growth rate negative?

f. Consider the graphs of the exponential functions p and q provided in Figure 3.1.3. If $p(t) = ab^t$ and $q(t) = cd^t$, what can you say about the values a, b, c, and d (beyond the fact that all are positive and $b \neq 1$ and $d \neq 1$)? For instance, can you say a certain value is larger than another? Or that one of the values is less than 1?

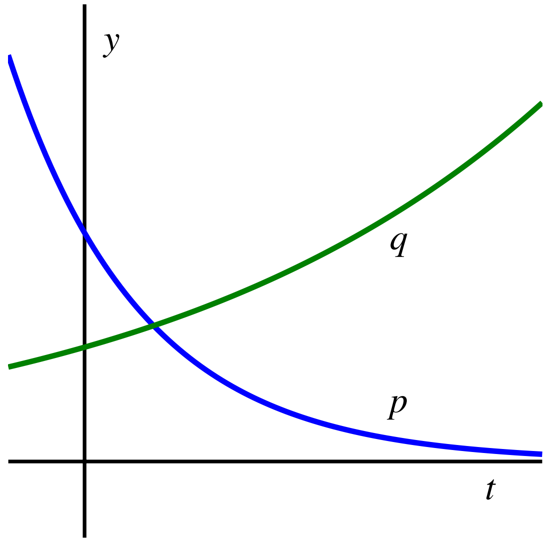

Figure 3.1.3. Graphs of exponential functions p and q.

3.1.2 Determining formulas for exponential functions

To better understand the roles that a and b play in an exponential function, let's compare exponential and linear functions. In Table 3.1.4 and Table 3.1.5, we see output for two different functions r and s that correspond to equally spaced input.

Table 3.1.4.

t	0	3	6	9
$r(t)$	12	10	8	6

Table 3.1.5.

t	0	3	6	9
$s(t)$	12	9	6.75	5.0625

In Table 3.1.4, we see a function that exhibits constant average rate of change since the change in output is always $\triangle r = -2$ for any change in input of $\triangle t = 3$. Said differently, r is a linear function with slope $m = -\frac{2}{3}$. Since its y-intercept is $(0, 12)$, the function's formula is $y = r(t) = 12 - \frac{2}{3}t$.

In contrast, the function s given by Table 3.1.5 does not exhibit constant average rate of change. Instead, another pattern is present. Observe that if we consider the ratios of consecutive outputs in the table, we see that

$$\frac{9}{12} = \frac{3}{4}, \frac{6.75}{9} = 0.75 = \frac{3}{4}, \text{ and } \frac{5.0625}{6.75} = 0.75 = \frac{3}{4}.$$

So, where the *differences* in the outputs in Table 3.1.4 are constant, the *ratios* in the outputs in Table 3.1.5 are constant. The latter is a hallmark of exponential functions and may be used to help us determine the formula of a function for which we have certain information.

If we know that a certain function is linear, it suffices to know two points that lie on the line to determine the function's formula. It turns out that exponential functions are similar: knowing two points on the graph of a function known to be exponential is enough information to determine the function's formula. In the following example, we show how knowing two values of an exponential function enables us to find both a and b exactly.

Example 3.1.6. Suppose that p is an exponential function and we know that $p(2) = 11$ and $p(5) = 18$. Determine the exact values of a and b for which $p(t) = ab^t$.

▼ Solution.

Since we know that $p(t) = ab^t$, the two data points give us two equations in the unknowns a and b. First, using $t = 2$,

$$ab^2 = 11, \tag{3.1.1}$$

and using $t = 5$ we also have

$$ab^5 = 18. \tag{3.1.2}$$

Because we know that the quotient of outputs of an exponential function corresponding to equally-spaced inputs must be constant, we thus naturally consider the quotient $\frac{18}{11}$. Using Equation (3.1.1) and Equation (3.1.2), it follows that

$$\frac{18}{11} = \frac{ab^5}{ab^2}.$$

Simplifying the fraction on the right, we see that $\frac{18}{11} = b^3$. Solving for b, we find that $b = \sqrt[3]{\frac{18}{11}}$ is the exact value of b. Substituting this value for b in Equation (3.1.1), it then follows that $a\left(\sqrt[3]{\frac{18}{11}}\right)^2 = 11$, so $a = \frac{11}{\left(\frac{18}{11}\right)^{2/3}}$.

Therefore,

$$p(t) = \frac{11}{\left(\frac{18}{11}\right)^{2/3}}\left(\sqrt[3]{\frac{18}{11}}\right)^t \approx 7.9215 \cdot 1.1784^t,$$

and a plot of $y = p(t)$ confirms that the function indeed passes through $(2, 11)$ and $(5, 18)$ as shown in Figure 3.1.7.

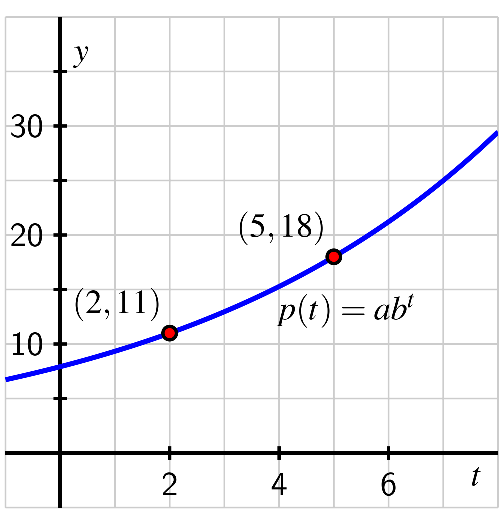

Figure 3.1.7. Plot of $p(t) = ab^t$ that passes through $(2, 11)$ and $(5, 18)$.

Activity 3.1.3. The value of an automobile is depreciating. When the car is 3 years old, its value is $\$12500$; when the car is 7 years old, its value is $\$6500$.

a. Suppose the car's value t years after its purchase is given by the function $V(t)$ and that V is exponential with form $V(t) = ab^t$, what are the exact values of a and b?

b. Using the exponential model determined in (a), determine the purchase value of the car and estimate when the car will be worth less than $\$1000$.

c. Suppose instead that the car's value is modeled by a linear function L and satisfies the values stated at the outset of this activity. Find a formula for $L(t)$ and determine both the purchase value of the car and when the car will be worth $\$1000$.

d. Which model do you think is more realistic? Why?

3.1.3 Trends in the behavior of exponential functions

Recall that a function is increasing on an interval if its value always increases as we move from left to right. Similarly, a function is decreasing on an interval provided that its value always decreases as we move from left to right.

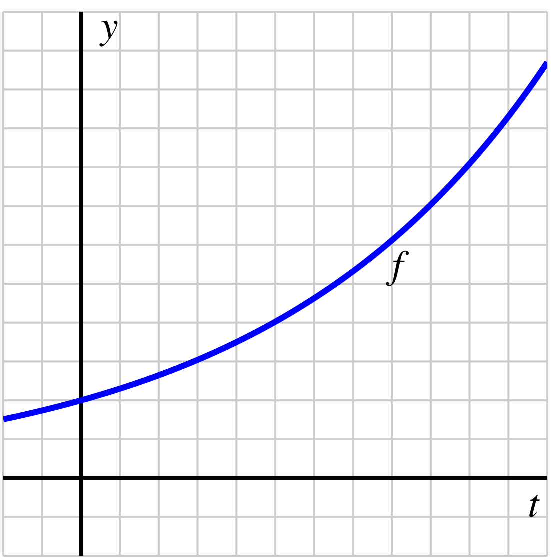

Figure 3.1.8. The exponential function f.

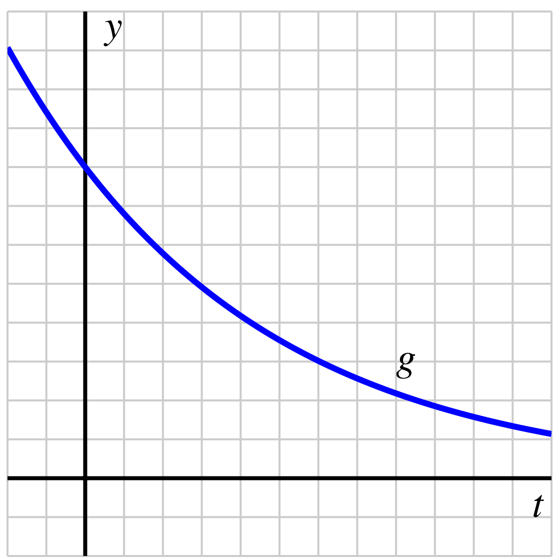

Figure 3.1.9. The exponential function g.

If we consider an exponential function f with a growth factor $b > 1$, such as the function pictured in Figure 3.1.8, then the function is always increasing because higher powers of b are greater than lesser powers (for example, $(1.2)^3 > (1.2)^2$). On the other hand, if $0 < b < 1$, then the exponential function will be decreasing because higher powers of positive numbers less than 1 get smaller (e.g., $(0.9)^3 < (0.9)^2$), as seen for the exponential function in Figure 3.1.9.

An additional trend is apparent in the graphs in Figure 3.1.8 and Figure 3.1.9. Each graph bends upward and is therefore concave up. We can better understand why this is so by considering the average rate of change of both f and g on consecutive intervals of the same width. We choose adjacent intervals of length 1 and note particularly that as we compute the average rate of change of each function on such intervals,

$$AV_{[t,t+1]} = \frac{f(t+1) - f(t)}{t+1-t} = f(t+1) - f(t).$$

Thus, these average rates of change are also measuring the total change in the function across an interval that is 1-unit wide. We now assume that $f(t) = 2(1.25)^t$ and $g(t) = 8(0.75)^t$ and compute the rate of change of each function on several consecutive intervals.

Table 3.1.10.

t	$f(t)$	$AV_{[t,t+1]}$
0	2	0.5
1	2.5	0.625
2	3.125	0.78215
3	3.90625	0.97656

Table 3.1.11.

t	$g(t)$	$AV_{[t,t+1]}$
0	8	-2
1	6	-1.5
2	4.5	-1.125
3	3.375	-0.84375

From the data in Table 3.1.10, we see that the average rate of change is increasing as we increase the value of t. We naturally say that f appears to be "increasing at an increasing rate". For the function g, we first notice that its average rate of change is always negative, but also that the average rate of change gets *less negative* as we increase the value of t. Said differently, the average rate of change of g is also increasing as we increase the value of t. Since g is always decreasing but its average rate of change is increasing, we say that g appears to be "decreasing at an increasing rate". These trends hold for exponential functions generally[1] according to the following conditions.

Trends in exponential function behavior.

For an exponential function of the form $f(t) = ab^t$ where a and b are both positive with $b \neq 1$,

- if $b > 1$, then f is always increasing and always increases at an increasing rate;
- if $0 < b < 1$, then f is always decreasing and always decreases at an increasing rate.

Observe how a function's average rate of change helps us classify the function's behavior on an interval: whether the average rate of change is always positive or always negative on the interval enables us to say if the function is always increasing or always decreasing, and then how the average rate of change itself changes enables us to potentially say *how* the function is increasing or decreasing through phrases such as "decreasing at an increasing rate".

Activity 3.1.4. For each of the following prompts, give an example of a function that satisfies the stated characteristics by both providing a formula and sketching a graph.

a. A function p that is always decreasing and decreases at a constant rate.

b. A function q that is always increasing and increases at an increasing rate.

c. A function r that is always increasing for $t < 2$, always decreasing for $t > 2$, and is always changing at a decreasing rate.

d. A function s that is always increasing and increases at a decreasing rate. (Hint: to find a formula, think about how you might use a transformation of a familiar function.)

e. A function u that is always decreasing and decreases at a decreasing rate.

3.1.4 Summary

- We say that a function is exponential whenever its algebraic form is $f(t) = ab^t$ for some positive constants a and b where $b \neq 1$. (Technically, the formal definition of an exponential function is one of form $f(t) = b^t$, but in our everyday usage of the term "exponential" we include vertical stretches of these functions and thus allow a to be any positive constant, not just $a = 1$.)
- To determine the formula for an exponential function of form $f(t) = ab^t$, we need to know two pieces of information. Typically this information is presented in one of two ways.
  - If we know the amount, a, of a quantity at time $t = 0$ and the rate, r, at which the quantity grows or decays per unit time, then it follows $f(t) = a(1 + r)^t$. In this setting, r is often given as a percentage that we convert to a decimal (e.g., if the quantity grows at a rate of 7% per year, we set $r = 0.07$, so $b = 1.07$).
  - If we know any two points on the exponential function's graph, then we can set up a system of two equations in two unknowns and solve for both a and b exactly. In this situation, it is useful to consider the quotient of the two known outputs, as demonstrated in Example 3.1.6.
- Exponential functions of the form $f(t) = ab^t$ (where a and b are both positive and $b \neq 1$) exhibit the following important characteristics:
  - The domain of any exponential function is the set of all real numbers and the range of any exponential function is the set of all positive real numbers.
  - The y-intercept of the exponential function $f(t) = ab^t$ is $(0, a)$ and the function has no x-intercepts.

- If $b > 1$, then the exponential function is always increasing and always increases at an increasing rate. If $0 < b < 1$, then the exponential function is always decreasing and always decreases at an increasing rate.

3.1.5 Exercises

1. Suppose $Q = 30.8(0.751)^t$. Give the starting value a, the growth *factor* b, and the growth *rate* r if $Q = a \cdot b^t = a(1+r)^t$.

$a =$ ________

$b =$ ________

$r =$ ________ %

Activate

2. Find a formula for $P = f(t)$, the size of the population that begins in year $t = 0$ with 2090 members and decreases at a 3.7 % annual rate. Assume that time is measured in years.

$P = f(t) =$ ________________

Activate

3. (a) The annual inflation rate is 3.8% per year. If a movie ticket costs \$9.00 today, find a formula for p, the price of a movie ticket t years from today, assuming that movie tickets keep up with inflation.

$P = f(t) =$ ________________

(b) According to your formula, how much will a movie ticket cost in 30 years?

Activate

4. In the year 2003, a total of 7.2 million passengers took a cruise vacation. The global cruise industry has been growing at 9% per year for the last decade. Assume that this growth rate continues.

(a) Write a formula for to approximate the number, N, of cruise passengers (in millions) t years after 2003.

$N =$ ____________________

(b) How many cruise passengers (in millions) are predicted in the year 2011?

$N =$ ____________________

(c) How many cruise passengers (in millions) were there in the year 2000?

$N =$ ____________________

Activate

5. The populations, P, of six towns with time t in years are given by

1	$P = 800(0.78)^t$
2	$P = 900(1.06)^t$
3	$P = 1600(0.96)^t$
4	$P = 1400(1.187)^t$
5	$P = 500(1.14)^t$
6	$P = 2800(0.8)^t$

Answer the following questions regarding the populations of the six towns above. Whenever you need to enter several towns in one answer, enter your answer as a comma separated list of numbers. For example if town *1*, town *2*, town *3*, and town *4*, are all growing you could enter ***1, 2, 3, 4***; or ***2, 4, 1, 3***; or any other order of these four numerals separated by commas.

(a) Which of the towns are growing? ____________________

(b) Which of the towns are shrinking? ____________________

(c) Which town is growing the fastest? _____

What is the annual percentage growth RATE of that town? __________ %

(d) Which town is shrinking the fastest? _____

What is the annual percentage decay RATE of that town? __________ %

(e) Which town has the largest initial population? ____________________

(f) Which town has the smallest initial population? ____________________

Activate

6. (a) Determine whether function whose values are given in the table below could be linear or exponential.

- linear
- exponential

$x =$	0	1	2	3	4
$h(x) =$	14	8	2	-4	-10

Find a possible formula for this function.

$h(x) =$ ____________________

(b) Determine whether function whose values are given in the table below could be linear or exponential.

- linear
- exponential

$x =$	0	1	2	3	4
$i(x) =$	14	12.6	11.34	10.206	9.1854

Find a possible formula for this function.

$i(x) =$ ____________________

Activate

7. A population has size 8000 at time $t = 0$, with t in years.

(a) If the population decreases by 125 people per year, find a formula for the population, P, at time t.

$P(t) =$ ______________

(b) If the population decreases by 6% per year, find a formula for the population, P, at time t.

$P(t) =$ ______________

Activate

8. Grinnell Glacier in Glacier National Park in Montana covered about 142 acres in 2007 and was found to be shrinking at about 4.4% per year.[2]

a. Let $G(t)$ denote the area of Grinnell Glacier in acres in year t, where t is the number of years since 2007. Find a formula for $G(t)$ and define the function in *Desmos*.

b. How many acres of ice were in the glacier in 1997? In 2012? What does the model predict for 2022?

c. How many total acres of ice were lost from 2007 to 2012?

d. What was the average rate of change of G from 2007 to 2012? Write a sentence to explain the meaning of this number and include units on your answer. In addition, how does this compare to the average rate of change of G from 2012 to 2017?

e. How would you you describe the overall behavior of G, and thus what is happening to the Grinnell Glacier?

9. Consider the exponential function f whose graph is given by Figure 3.1.12. Note that f passes through the two noted points exactly.

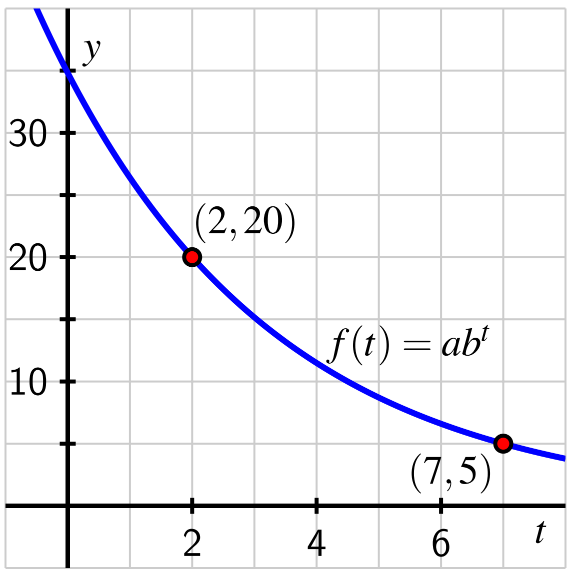

Figure 3.1.12. A plot of the exponential function f.

a. Determine the values of a and b exactly.

b. Determine the average rate of change of f on the intervals $[2, 7]$ and $[7, 12]$. Which average rate is greater?

c. Find the equation of the linear function L that passes through the points $(2, 20)$ and $(7, 5)$.

d. Which average rate of change is greater? The average rate of change of f on $[0, 2]$ or the average rate of change of L on $[0, 2]$?

10. A cup of hot coffee is brought outside on a cold winter morning in Winnipeg, Manitoba, where the surrounding temperature is 0 degrees Fahrenheit. A temperature probe records the coffee's temperature (in degrees Fahrenheit) every minute and generates the data shown in Table 3.1.13.

Table 3.1.13.

t	0	2	4	6	8	10
$F(t)$	175	129.64	96.04	71.15	52.71	39.05

a. Assume that the data in the table represents the overall trend of the behavior of F. Is F linear, exponential, or neither? Why?

b. Is it possible to determine an exact formula for F? If yes, do so and justify your formula; if not, explain why not.

c. What is the average rate of change of F on $[4, 6]$? Write a sentence that explains the practical meaning of this value in the context of the overall exercise.

d. How do you think the data would appear if instead of being in a regular coffee cup, the coffee was contained in an insulated mug?

11. The amount (in milligrams) of a drug in a person's body following one dose is given by an exponential decay function. Let $A(t)$ denote the amount of drug in the body at time t in hours after the dose was taken. In addition, suppose you know that $A(3) = 22.7$ and $A(6) = 15.2$.

a. Find a formula for A in the form $A(t) = ab^t$, where you determine the values of a and b exactly.

b. What is the size of the initial dose the person was given?

c. How much of the drug remains in the person's body 8 hours after the dose was taken?

d. Estimate how long it will take until there is less than 1 mg of the drug remaining in the body.

e. Compute the average rate of change of A on the intervals $[3, 5]$, $[5, 7]$, and $[7, 9]$. Write at least one careful sentence to explain the meaning of the values you found, including appropriate units. Then write at least one additional sentence to explain any overall trend(s) you observe in the average rate of change.

f. Plot $A(t)$ on an appropriate interval and label important points and features of the graph to highlight graphical interpretations of your answers in (b), (c), (d), and (e).