QUESTION 4

The *Fibonacci sequence* is defined iteratively by setting $F_1 = F_2 = 1$ and thereafter letting $F_{n+2} = F_n + F_{n+1}$. This fascinating sequence has been known for at least 1500 years. It has several connections to the natural world, some of which are described in the second lecture of Devlin's mathematics survey course on iTunes University, listed as recommended supplementary viewing to this course. It also has a number of pleasing mathematical connections. Here is one:

Theorem For any natural number n,

$$\sum_{k=1}^{n} {F_k}^2 = F_n F_{n+1}$$

Proof: By induction.

For $n = 1$, the left-hand side is ${F_1}^2 = 1^2 = 1$ and the right-hand side is $F_1 F_2 = 1.1 = 1$, so the identity is valid for $n = 1$.

Assume the identity holds for n. Then:

$$\begin{aligned}
\sum_{k=1}^{n+1} {F_k}^2 &= \sum_{k=1}^{n} {F_k}^2 \; + \; {F_{n+1}}^2 \\
&= F_n F_{n+1} \; + \; {F_{n+1}}^2, \text{ by the induction hypothesis} \\
&= F_{n+1}(F_n + F_{n+1}), \text{ by algebra} \\
&= F_{n+1} F_{n+2}, \text{ by the definition of } F_{n+2}
\end{aligned}$$

which is the identity for $n + 1$. The proof is complete.