QUESTION 4

Is the following proof of the Sandwich Theorem correct? Grade it according to the course rubric.

Theorem (Sandwich Theorem)

Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are sequences such that, from some point n_0 onwards,

$$a_n \leq b_n \leq c_n.$$

Suppose further that

$$\lim_{n\to\infty} a_n = L \ , \ \lim_{n\to\infty} c_n = L.$$

Then $\{b_n\}_{n=1}^{\infty}$ is convergent and

$$\lim_{n\to\infty} b_n = L.$$

Proof: Since $\lim_{n\to\infty} a_n = L$, we can find an integer n_1 such that

$$n \geq n_1 \Rightarrow |a_n - L| < \epsilon$$

Since $\lim_{n\to\infty} c_n = L$, we can find an integer n_2 such that

$$n \geq n_2 \Rightarrow |c_n - L| < \epsilon$$

Let $M = \max\{n_0, n_1, n_2\}$. Then

$$\begin{aligned} n \geq M \quad &\Rightarrow \quad (-\epsilon < a_n - L < \epsilon) \ \wedge \ (-\epsilon < c_n - L < \epsilon) \\ &\Rightarrow \quad -\epsilon < a_n - L \leq b_n - L \leq c_n - L < \epsilon \ \ (\text{using } a_n \leq b_n \leq c_n) \\ &\Rightarrow \quad -\epsilon < b_n - L < \epsilon \\ &\Rightarrow \quad |b_n - L| < \epsilon \end{aligned}$$

By the definition of a limit, this proves that $\{b_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} b_n = L$, as required.