Here the test statistic is equal for the unseparated pairs of n_{11}. Since none of these orderings is identical, the p values are not generally equal. Note that the orderings for T_4 and T_6 differ only in the position of the table, with $n_{11} = 0$.

4. REMARKS

Another group of test statistics for testing independence is derivable from R_2 by dividing by an estimate of standard error other than the usual pooled estimate. If the unpooled estimate

$$S_1 = \{n_{11}n_{21}/n_{+1}^3 + n_{12}n_{22}/n_{+2}^3\}^{1/2}$$

is used, $P(R_2/S_1) \neq P(R_2)$. Dozzi and Riedwyl (1984) suggested that

$$S_2 = \{(n_{11}n_{21}/n_{+1} + n_{12}n_{22}/n_{+2}) \times (n_{+1}^{-1} + n_{+2}^{-1})/(N - 2)\}^{1/2}$$

so that R_2/S_2 is the one-sided t statistic for testing the equality of two means. It follows from results in Dozzi and Riedwyl (1984) that $P(R_2/S_3) = P(R_2)$. Similar comments are applicable to T_2, R_3, and T_3.

Finally, all of the test statistics proposed in the literature for testing independence of two dichotomous factors have not been discussed here. For example, see Dozzi and Riedwyl (1984), Garside and Mack (1976), and Upton (1982) for discussion of other tests.

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Prediction Limits for a Univariate Normal Observation

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Basic statistics textbooks invariably introduce the topic of prediction in the context of simple regression for the bivariate normal model. It seems more appropriate, however, to introduce the topic in conjunction with univariate normal inference. Prediction limits for the univariate case build on elementary theory that is familiar to the student. This approach provides an easier transition to prediction in the regression framework and exposes the student at an earlier stage to important applications and ideas connected with prediction. Adoption of the approach may give prediction the more prominent position in basic statistical education that it deserves.

KEY WORDS: Applications; Statistical education.

1. INTRODUCTION

An extensive survey of basic statistics textbooks shows that the topic of prediction limits, if covered at all, is first introduced in the context of simple regression for the bivariate normal model [see, for example, Neter, Wasserman, and Whitmore (1982) and Newbold (1984)]. In this setting, the $1 - \alpha$ prediction limits for a new independent observation of Y at level X are given by

$$\hat{Y} \pm t_{1-\alpha/2, n-2}\,(\mathrm{MSE})^{1/2}[1 + (1/n) + (X - \bar{X})^2/(n - 1)s_X^2]^{1/2}, \quad (1.1)$$

where $\hat{Y}$ is the fitted regression value at X, MSE is the mean squared error of the fitted regression model, n is the sample size, $\bar{X}$ and s_X^2 are the mean and variance of the sample X_i values, and $t_{1-\alpha/2,n-2}$ is the $1 - \alpha/2$ fractile of a t distribution with $n - 2$ df.

The theoretical derivation of the regression prediction limits in (1.1) is often neither readily grasped by students nor seen by them to follow from more elementary results presented earlier in their course. To provide a smoother transition to the prediction limits in the regression case and as a useful prediction method in its own right, it seems more desirable to introduce prediction limits in the context of inference methods for a univariate normal population.

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Concern about the omission of elementary predictive inference in basic statistical education was expressed years ago by Hahn (1969) who stated that

> . . . we note with surprise the extent to which the concept of a prediction interval, which is frequently what is required in practical applications, has been omitted in texts on elementary statistics, which provide very extensive coverage of confidence intervals for the mean, and, sometimes of tolerance intervals. As a result, the construction of a prediction interval is unfamiliar to almost all non-statistical users of statistics who frequently confuse it with the other two types of intervals. The only introductory textbook coverage of the concept seems to be with regard to regression models for the special case of a single future observation. . . . (p. 886)

From this author's experience, an elementary discussion of prediction limits fits well in a basic course immediately following the discussion of a confidence interval for the mean of a normal population. Students view the topic as a natural and straightforward extension of the subject of normal inference. Class time taken up by the topic is saved subsequently in the easier introduction of prediction in a regression context, which is made possible by the earlier presentation of the univariate case.

2. DEVELOPMENT OF THE PREDICTION LIMITS

The theory for predictive inference in the normal population case is fully developed in the literature (e.g., Hahn 1969, 1970) and also is covered extensively in a few standard reference textbooks (e.g., Cox and Hinkley 1974, pp. 242–245, and Nelson 1982, pp. 224–225). The requisite formulas for prediction limits for a single observation from a normal population are easily derived, as follows.

Let Y represent an observation drawn from the normal distribution $N(\mu, \sigma^2)$ for which a prediction interval is required. If μ and σ are known, then

$$\mu \pm z_{1-\alpha/2}\sigma \tag{2.1}$$

are $1 - \alpha$ prediction limits for Y, where $z_{1-\alpha/2}$ is the $1 - \alpha/2$ fractile of the standard normal distribution. This result will be apparent to the student. The quality control application described in Section 3 provides a setting in which knowledge of the parameters is a realistic claim.

Now proceed to the more common case in which both μ and σ are unknown. Let Y_i, $i = 1, \ldots, n$, be a random sample from $N(\mu, \sigma^2)$, which has been drawn independently of Y, the observation to be predicted. Denote the sample mean and standard deviation by $\bar{Y}$ and s, respectively. $\bar{Y}$ may be used as a point predictor of Y, and $Y - \bar{Y}$ will be the prediction error. Students must be familiar with the following results to understand the theoretical derivation.

1. $\bar{Y}$ is an unbiased predictor in the sense that $E\{Y - \bar{Y}\} = 0$.
2. The variance of $Y - \bar{Y}$ is $\sigma^2\{Y - \bar{Y}\} = \sigma^2 + (\sigma^2/n)$. It is revealing to point out to the student that the first variance term reflects the variation of Y about μ and the second variance term reflects the variation contributed because $\bar{Y}$ is used as an estimate of μ.
3. $Y - \bar{Y}$ is normally distributed because Y and $\bar{Y}$ are independent normal random variables. This fact and the two previous ones imply, therefore, that $Y - \bar{Y}$ is distributed as $N(0, \sigma^2(1 + 1/n))$.
4. $Y - \bar{Y}$ and s^2 are independent random variables.
5. $(n - 1)s^2/\sigma^2$ is distributed as χ^2_{n-1}.

The preceding facts, taken together, give the key distributional result

$$(Y - \bar{Y}) \Big/ s\left[1 + \frac{1}{n}\right]^{1/2} = t_{n-1}. \tag{2.2}$$

The desired $1 - \alpha$ prediction limits are, therefore,

$$\bar{Y} \pm t_{1-\alpha/2,n-1}s\left[1 + \frac{1}{n}\right]^{1/2}. \tag{2.3}$$

Should it happen that either μ or σ is known, then the limits in (2.3) become modified, as follows:

$$\mu \pm t_{1-\alpha/2,n-1}s \qquad \text{if } \mu \text{ is known} \tag{2.4}$$

and

$$\bar{Y} \pm z_{1-\alpha/2}\sigma\left[1 + \frac{1}{n}\right]^{1/2} \qquad \text{if } \sigma \text{ is known.} \tag{2.5}$$

The student should be informed that the relative frequency interpretation of the confidence level $1 - \alpha$ for the prediction limits in (2.3) is one based on repeated drawings of *both* the original sample Y_i, $i = 1, \ldots, n$, and the predicted observation Y (Cox and Hinkley 1974, p. 244). It is also important for the student to note that the preceding prediction intervals do not share the robustness property of the t-based confidence interval for a population mean. Any departure of the population from normality has a direct effect on the confidence level associated with these prediction limits (Hahn 1969, p. 886).

3. APPLICATIONS AND EXTENSIONS

Many practical applications of the prediction limits suggest themselves. One might surmise that the omission of these prediction limits in basic statistics textbooks is one reason why applications have not come to the fore. Their inclusion in the standard presentation of univariate normal inference would help to give predictive inference a more prominent place in applied statistics.

Nelson (1982, p. 225), for example, illustrated the application of prediction limits (2.3) to predicting the log-life of an electric-motor insulation specimen from experimental data. In quality control, prediction limits (2.1) may be used as control limits for an attribute of a production item for which the process parameters are specified to be μ and σ or as performance limits in quality assurance claims (Hahn 1969, pp. 880–881). Finally, forecasting the next observation in an independent and stationary normal process is a natural application of prediction limits (2.3). For instance, a city government may wish to predict total snow accumulation during the next winter from a historical record of annual accumulations to decide on the quantity of road salt to purchase (Dudewicz 1976, p. 339).

In presenting the prediction limits in a course, instructors may wish to extend the technical development along the lines of some of the following comments.

1. The prediction limits can be used in conjunction with a normalizing transformation, as illustrated by the logarithmic transformation in the preceding example from Nelson (1982). Transformations to achieve approximate normality are especially important in light of the lack of robustness of the prediction limits noted earlier.

2. Some instructors may wish to contrast prediction limits with tolerance limits; the former refers to limits for a single independent observation from a normal population, and the latter refers to limits within which a certain fraction of the entire normal population is claimed to lie.

3. The prediction limits (2.3) can be extended readily to include the case of predicting the mean of m new observations in an independent random sample from the same normal population. In this case the $1 - \alpha$ limits for the predicted mean have the form

$$\bar{Y} \pm t_{1-\alpha/2,n-1} s \left[\frac{1}{m} + \frac{1}{n} \right]^{1/2}. \tag{3.1}$$

In addition, prediction limits of the form $\bar{Y} \pm rs$ can be constructed that will contain *all m* new observations with a given level of confidence. Tables of the multiplier r may be found in Hahn (1969, 1970).

4. The instructor may wish to note that result (2.2) holds approximately for the standardized sample values

$$Z_i = (Y_i - \bar{Y})/s, \qquad i = 1, \ldots, n, \tag{3.2}$$

and this fact may be useful for outlier identification in normal samples. The class discussion of (3.2) provides a useful preparation for the subsequent discussion of standardized residuals in a regression framework.

5. For courses with a Bayesian orientation, the limits in (2.3) are the central $1 - \alpha$ posterior probability limits for a prediction of Y based on a normal sample $Y_1, \ldots, Y_n$ and an uninformative improper prior joint density function for μ and σ (e.g., see DeGroot 1970, chap. 10).

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Relationships Among Common Univariate Distributions

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Common univariate distributions are usually discussed separately in introductory probability textbooks, which makes it difficult for students to understand the relationships among these distributions. The purpose of this article is to present a figure that illustrates some of these relationships.

KEY WORDS: Limiting distributions; Transformations of random variables.

1. INTRODUCTION

Students in a first course in probability usually study common univariate distributions. Most introductory textbooks discuss each of the distributions in separate sections. One of the drawbacks of this approach is that students often do not grasp all of the interrelationships among the distributions. The purpose of this article is to present and discuss a figure that overcomes this shortfall.

There are several excellent sources for studying univariate distributions. Hastings and Peacock's (1975) handbook shows graphs of densities and variate relationships for several distributions. Hirano, Kuboki, Aki, and Kuribayashi (1983) gave graphs of univariate distributions for many combinations of parameter values. For more detail, Johnson and Kotz (1970) have done a four-volume series covering univariate and multivariate distributions. Recently, Patil, Boswell, Joshi, and Ratnaparkhi (1985) and Patil, Boswell, and Ratnaparkhi (1985) have also completed volumes on discrete and continuous distributions. Other books on distributions and modeling include Ord (1972), Patel, Kapadia, and Owen (1976), and Shapiro and Gross (1981). Diagrams that relate these distributions to one another may be found in Nakagawa and Yoda (1977), Taha (1982), and Marshall and Olkin (1985).

2. DISCUSSION

The diagram in Figure 1 shows some relationships among common univariate distributions that might be presented in

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