On monotonicity of some functionals under rearrangements

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February 18, 2014

1 Introduction

First, we recall the layer cake representation for a measurable function $u : [-1,1] \to \mathbb{R}_+$ (here and elsewhere $\mathbb{R}_+ = [0,\infty)$). Namely, if we set $\mathcal{A}_t := \{x \in [-1,1] : \ u(x) > t\}$ then $u(x) = \int_0^\infty \chi_{\mathcal{A}_t}\,dt$.

We define the monotone rearrangement of a measurable set $E \subset [-1,1]$ and the monotone rearrangement of a nonnegative function $u \in W_1^1(-1,1)$ as follows:

$$E^* := [1 - |E|\,, 1]; \qquad u^*(x) := \int\limits_0^\infty \chi_{\mathcal{A}_t^*}\,dt.$$

Under the same conditions we define the symmetric rearrangement (symmetrization) for sets and functions:

$$\overline{E} := [-\frac{|E|}{2}, \frac{|E|}{2}]; \qquad \overline{u}(x) := \int\limits_0^\infty \chi_{\overline{\mathcal{A}_t}}\,dt.$$

We denote by $\mathfrak{F}$ the set of continuous functions $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which are convex and increasing with respect to the second argument.

Let us consider a functional

$$I(\mathfrak{a}, u) = \int\limits_{-1}^{1} F\big(u(x), \mathfrak{a}(x, u(x))\,|u'(x)|\,\big)\,dx, \tag{1}$$

where $\mathfrak{a} : [-1,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, $F \in \mathfrak{F}$.

It is well known that if $\mathfrak{a} \equiv const$, then the Pólya–Szegö type inequalities

$$I(\mathfrak{a}, u^*) \leqslant I(\mathfrak{a}, u), \qquad u \in W_1^1(-1,1); \tag{2}$$

$$I(\mathfrak{a}, \overline{u}) \leqslant I(\mathfrak{a}, u), \qquad u \in \overset{o}{W}{}_1^1(-1,1) \tag{3}$$

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hold, see for example [3] and references therein.

Inequality (3) and its multi-dimensional analogue are proved in [2], provided that the function $\mathfrak{a}$ is even and convex with respect to x. However, the proof contains a gap, and in fact this inequality was proved in [2] only for the Lipschitz functions u.

Namely, while proving inequality (3) for a natural class of functions, the author of [2] approximates $u \in \overset{o}{W}{}_1^1$ with finite integral (1) using piecewise linear functions u_k, claiming that $I(\mathfrak{a}, u_k) \to I(\mathfrak{a}, u)$. However, this assertion is not justified, and generally speaking, is not true. In 1926, M.A. Lavrentiev proposed the first example of an integral functional for which the infimum over the domain is strictly less than the infimum over the set of Lipschitz functions. A historical overview and simple examples of "one-dimensional" functionals for which the Lavrentiev phenomenon takes place can be found e.g. in [6]. Note that a thorough investigation of the Lavrentiev phenomenon for some classes of multidimensional functionals was carried out by V.V. Zhikov (see, e.g., [7], [8]).

In the paper [1] the absence of the Lavrentiev phenomenon was proved for the functionals $I(\mathfrak{a}, u) = \int_{-1}^{1} F(u, u')$. Moreover, it was shown that for every $u \in W_1^1(-1,1)$ there exists a sequence of the Lipschitz functions u_k, such that

$$u_k \to u \text{ in } W_1^1(-1,1) \quad \text{and} \quad I(\mathfrak{a}, u_k) \to I(\mathfrak{a}, u). \tag{4}$$

We modify the proof from [1] and prove the absence of the Lavrentiev phenomenon for the functionals of the form (1). This allows us to fill the gap in the proof from [2] in a one-dimensional case. In addition, we prove that evenness and convexity of the weight is a necessary condition for inequality (3) to hold.

The bulk of our paper is devoted to inequality (2). We find necessary and sufficient conditions on the weight $\mathfrak{a}$ for inequality (2) to hold[1]. Under certain additional assumptions this result was announced in [5].

We note also that inequality (2) was considered in [4] for functionals similar to (1) under additional constraint $u(-1) = 0$. We obtain necessary and sufficient conditions for (2) under this constraint. (The author of [4] assumed the weight $\mathfrak{a}$ decreasing in x.)

The article is divided into 8 sections. In Section 2 we deduce the assumptions on the weight function $\mathfrak{a}$ which are necessary for inequality (2). Auxiliary statements for weights satisfying necessary conditions are established in Section 3. In Section 4, inequality (2) is proved for piecewise linear functions u. In Section 5, we present the scheme for proving inequality (2) for a wider class of functions u. In Section 6 we prove inequality (2), provided that the weight $\mathfrak{a}$ first increases, then decreases. Section 7 is devoted to the proof of (2) under necessary conditions only. Finally, in Section 8, we deal with symmetric rearrangement. There we obtain necessary conditions on the weight and complete the proof of (3).

2 The conditions necessary for inequality (2)

Theorem 1. 1. *Let inequality (2) hold for some $F \in \mathfrak{F}$ and an arbitrary piecewise linear u. Then the weight function $\mathfrak{a}$ is even with respect to the first argument, that is $\mathfrak{a}(x, v) \equiv \mathfrak{a}(-x, v)$.*

[1]In particular, the inequality is satisfied if the weight function $\mathfrak{a}$ is even and concave in x.

2. *Let inequality (2) hold for arbitrary $F \in \mathfrak{F}$ and arbitrary piecewise linear u. Then the weight function $\mathfrak{a}$ satisfies*

$$\mathfrak{a}(s,v) + \mathfrak{a}(t,v) \geqslant \mathfrak{a}(1-t+s,v), \qquad -1 \leqslant s \leqslant t \leqslant 1, v \in \mathbb{R}_+. \tag{5}$$

Proof. **1.** Suppose that $\mathfrak{a}(x,v) \not\equiv \mathfrak{a}(-x,v)$. Then there is $\bar{x} \in (-1,1)$ and $\bar{v} \in \mathbb{R}_+$ such that

$$\mathfrak{a}(\bar{x},\bar{v}) < \mathfrak{a}(-\bar{x},\bar{v}).$$

Therefore, there is $\varepsilon > 0$ such that

$$\bar{x} - \varepsilon \leqslant x \leqslant \bar{x}, \bar{v} \leqslant v \leqslant \bar{v} + \varepsilon \quad \Longrightarrow \quad \mathfrak{a}(x,v) < \mathfrak{a}(-x,v).$$

Now we introduce the following function:

$$\begin{cases} u(x) = \bar{v} + \varepsilon, & x \in [-1, \bar{x} - \varepsilon] \\ u(x) = \bar{v} + \bar{x} - x, & x \in (\bar{x} - \varepsilon, \bar{x}) \\ u(x) = \bar{v}, & x \in [\bar{x}, 1] \end{cases}$$

Then $u^*(x,v) = u(-x,v)$ and

$$\begin{aligned} & I(\mathfrak{a},u) - I(\mathfrak{a},u^*) \\ &= \int\limits_{\bar{x}-\varepsilon}^{\bar{x}} F\big(\bar{v}+\bar{x}-x, \mathfrak{a}(x, \bar{v}+\bar{x}-x)\big)\,dx - \int\limits_{-\bar{x}}^{-\bar{x}+\varepsilon} F\big(\bar{v}+\bar{x}+x, \mathfrak{a}(x, \bar{v}+\bar{x}+x)\big)\,dx \\ &= \int\limits_{\bar{x}-\varepsilon}^{\bar{x}} \Big(F\big(\bar{v}+\bar{x}-x, \mathfrak{a}(x,\bar{v}+\bar{x}-x)\big) - F\big(\bar{v}+\bar{x}-x, \mathfrak{a}(-x,\bar{v}+\bar{x}-x)\big)\Big)\,dx < 0, \end{aligned}$$

which contradicts the assumption. Thus, the first statement is proved.

2. Suppose that assumption (5) is not satisfied. Then, by continuity of $\mathfrak{a}$, there exist $-1 \leqslant s \leqslant t \leqslant 1$, $\varepsilon, \delta > 0$ and $\bar{v} \in \mathbb{R}_+$, such that for any $0 \leqslant y \leqslant \varepsilon$ and $\bar{v} \leqslant v \leqslant \bar{v} + \varepsilon$ the following inequality holds:

$$\mathfrak{a}(s+y,v) + \mathfrak{a}(t-y,v) + \delta < \mathfrak{a}(1-t+s+2y,v).$$

Consider the function u (see Fig. 1):

$$\begin{cases} u(x) = \bar{v}, & x \in [-1,s] \cup [t,1] \\ u(x) = \bar{v} + x - s, & x \in [s, s+\varepsilon] \\ u(x) = \bar{v} + \varepsilon, & x \in [s+\varepsilon, t-\varepsilon] \\ u(x) = \bar{v} + t - x, & x \in [t-\varepsilon, t] \end{cases} \tag{6}$$

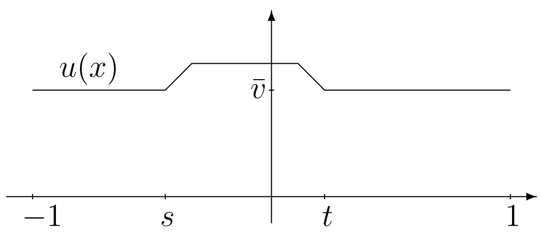

Fig. 1

Then

$$\begin{cases} u^*(x) = \bar{v}, & x \in [-1, 1-t+s] \\ u^*(x) = \bar{v} + \dfrac{x-(1-t+s)}{2}, & x \in [1-t+s, 1-t+s+2\varepsilon] \\ u^*(x) = \bar{v} + \varepsilon, & x \in [1-t+s+2\varepsilon, 1] \end{cases}$$

(see Fig. 2).

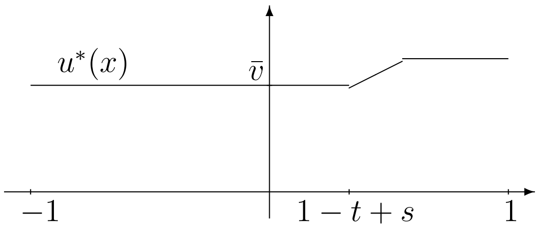

Fig. 2

We have

$$I(\mathfrak{a}, u^*) = \int\limits_0^{2\varepsilon} F\big(u(1-t+s+z), \frac{\mathfrak{a}(1-t+s+z, u(1-t+s+z))}{2}\big)\, dz$$

$$= \int\limits_0^{\varepsilon} 2F\big(\bar{v}+y, \frac{\mathfrak{a}(1-t+s+2y, \bar{v}+y)}{2}\big)\, dy$$

$$0 \leqslant I(\mathfrak{a}, u) - I(\mathfrak{a}, u^*) = \int\limits_0^{\varepsilon} \Big(F\big(\bar{v}+y, \mathfrak{a}(s+y, \bar{v}+y)\big) + F\big(\bar{v}+y, \mathfrak{a}(t-y, \bar{v}+y)\big)$$

$$- 2F\big(\bar{v}+y, \frac{\mathfrak{a}(1-t+s+2y, \bar{v}+y)}{2}\big)\Big)\, dy$$

$$< \int\limits_0^{\varepsilon} \Big(F\big(\bar{v}+y, \mathfrak{a}(s+y, \bar{v}+y)\big) + F\big(\bar{v}+y, \mathfrak{a}(t-y, \bar{v}+y)\big)$$

$$- 2F\big(\bar{v}+y, \frac{\mathfrak{a}(s+y, \bar{v}+y) + \mathfrak{a}(t-y, \bar{v}+y) + \delta}{2}\big)\Big)\, dy =: J.$$

Let us consider the function $F(v, p) = p^\alpha$. For $\alpha = 1$, the following inequality trivially holds:

$$\frac{F(v,p) + F(v,q)}{2} - F\big(v, \frac{p+q}{2} + \frac{\delta}{2}\big) < 0. \tag{7}$$

We are interested in p, q from the compact $[0, A]$, where

$$A = \max_{(x,v)} \mathfrak{a}, \qquad (x, v) \in [-1, 1] \times u([-1, 1]). \tag{8}$$

Therefore, there is an $\alpha > 1$, for which inequality (7) still holds. For example, any $1 < \alpha < (\log_2 \frac{2A}{A+\delta})^{-1}$ is suitable.

Thus, we obtain a function F strictly convex with respect to the second argument for which $J \leqslant 0$. This contradiction proves the second statement. $\square$

Remark 1. *It can be seen that, proving the second statement of Theorem 1, one can replace the function u on the interval $[-1, s]$ by any increasing function. Thus, in the case where u is pinned at the left end ($u(-1) = 0$), assumption (5) is also necessary for inequality (2) to hold.*

Remark 2. *Let $\mathfrak{a}(\cdot, v)$ be even. Then assumption (5) is equivalent to subadditivity of the function $\mathfrak{a}(1 - \cdot, v)$. In particular, if a nonnegative function $\mathfrak{a}$ is even and concave with respect to the first argument, then it satisfies assumption (5).*

3 Properties of the weight function

For brevity, in this section we omit the second argument of the function $\mathfrak{a}$. Thus, we assume, that $\mathfrak{a} \in C[-1, 1]$ and $\mathfrak{a} \geqslant 0$.

Lemma 1. *Let $\mathfrak{a}$ satisfy (5).*

1. *For any $-1 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant 1$ the following inequalities hold*

$$\sum_{k=1}^{n} \mathfrak{a}(t_k) \geqslant \mathfrak{a}(1 - \sum_{k=1}^{n} (-1)^k t_k), \qquad \text{for even } n,$$

$$\sum_{k=1}^{n} \mathfrak{a}(t_k) \geqslant \mathfrak{a}(-\sum_{k=1}^{n} (-1)^k t_k), \qquad \text{for odd } n.$$

2. *Assume that, in addition, the function $\mathfrak{a}$ is even. Then the following inequalities also hold:*

$$\sum_{k=1}^{n} \mathfrak{a}(t_k) \geqslant \mathfrak{a}(-1 + \sum_{k=1}^{n} (-1)^k t_k), \qquad \text{for even } n,$$

$$\sum_{k=1}^{n} \mathfrak{a}(t_k) \geqslant \mathfrak{a}(\sum_{k=1}^{n} (-1)^k t_k), \qquad \text{for odd } n.$$

Proof. **1.** We prove the lemma by induction. For $n = 1$ the assertion is trivial. Now let n be even. Then, by the induction hypothesis,

$$\sum_{k=1}^{n-1} \mathfrak{a}(t_k) \geqslant \mathfrak{a}(-\sum_{k=1}^{n-1} (-1)^k t_k).$$

Then

$$\sum_{k=1}^{n-1} \mathfrak{a}(t_k) + \mathfrak{a}(t_n) \geqslant \mathfrak{a}(-\sum_{k=1}^{n-1} (-1)^k t_k) + \mathfrak{a}(t_n) \geqslant \mathfrak{a}(1 - \sum_{k=1}^{n} (-1)^k t_k).$$

In the case of odd n we have the following induction hypothesis:

$$\sum_{k=2}^{n} \mathfrak{a}(t_k) \geqslant \mathfrak{a}(1 + \sum_{k=2}^{n} (-1)^k t_k).$$

Then

$$\mathfrak{a}(t_1)+\sum_{k=2}^{n}\mathfrak{a}(t_k)\geqslant \mathfrak{a}(t_1)+\mathfrak{a}(1+\sum_{k=2}^{n}(-1)^k t_k)\geqslant \mathfrak{a}(-\sum_{k=2}^{n}(-1)^k t_k+t_1)=\mathfrak{a}(-\sum_{k=1}^{n}(-1)^k t_k).$$

2. The proof of this part is trivial. □

Lemma 2. **1.** *Let $\mathfrak{a}$ satisfy* (5)*. If there is $x_0\in[-1,1]$, such that $\mathfrak{a}(x_0)=0$, then either $\mathfrak{a}\Big|_{[x_0,1]}\equiv 0$ or the set of zeros of $\mathfrak{a}$ is periodic on $[x_0,1]$ and the period is a divisor of $1-x_0$.*

2. *Let $\mathfrak{a}$ be even and satisfy* (5)*. If there is $x_0\in[-1,1]$, such that $\mathfrak{a}(x_0)=0$, then either $\mathfrak{a}\equiv 0$ or the function $\mathfrak{a}$ is periodic on $[-1,1]$ and the period is a divisor of $1-x_0$.*

Proof. **1.** Note that if $\mathfrak{a}(s)=\mathfrak{a}(t)=0$ for some $s\leqslant t$, then inequality (5) implies

$$0=\mathfrak{a}(s)+\mathfrak{a}(t)\geqslant \mathfrak{a}(1-(t-s))\geqslant 0$$

i.e. $\mathfrak{a}(1-(t-s))=0$. Substituting $s=t=x_0$, we obtain $\mathfrak{a}(1)=0$.

Similarly, if $s\leqslant 1-t$ and $\mathfrak{a}(s)=\mathfrak{a}(1-t)=0$, then $\mathfrak{a}(s+t)=0$.

Thus, the set of roots of $\mathfrak{a}$ is symmetric on the segment $[x_0,1]$ and whenever s and $s+\Delta$ $(\Delta\geqslant 0)$ are roots of $\mathfrak{a}$, values $s+k\Delta$ are roots of $\mathfrak{a}$ too, provided $s+k\Delta\leqslant 1$. This implies that the set of roots of $\mathfrak{a}$ is periodic on $[x_0,1]$ or coincides with it.

2. The periodicity of zeros of the function $\mathfrak{a}$ follows from its evenness and from the first assertion of the lemma. Denote the distance between consecutive zeros by Δ.

Then for $-1\leqslant x\leqslant 1-\Delta$ the following holds

$$\mathfrak{a}(x)=\mathfrak{a}(x)+\mathfrak{a}(1-\Delta)\geqslant \mathfrak{a}(x+\Delta).$$

On the other hand, $-1\leqslant -(x+\Delta)\leqslant 1-\Delta$, and

$$\mathfrak{a}(x+\Delta)=\mathfrak{a}(-(x+\Delta))+\mathfrak{a}(1-\Delta)\geqslant \mathfrak{a}(-x)=\mathfrak{a}(x).$$

Thus, $\mathfrak{a}(x)=\mathfrak{a}(x+\Delta)$. □

Lemma 3. *Suppose that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ satisfy* (5)*. Then the functions $\max(\mathfrak{a}_1(x),\mathfrak{a}_2(x))$ and $\mathfrak{a}_1(x)+\mathfrak{a}_2(x)$ also satisfy* (5)*.*

Proof. Set $\mathfrak{a}(x)=\max(\mathfrak{a}_1(x),\mathfrak{a}_2(x))$. Then

$$\begin{aligned}\mathfrak{a}(1-t+s)=\max(\mathfrak{a}_1(1-t+s),\mathfrak{a}_2(1-t+s))&\leqslant \max(\mathfrak{a}_1(s)+\mathfrak{a}_1(t),\mathfrak{a}_2(s)+\mathfrak{a}_2(t))\\ &\leqslant \max(\mathfrak{a}_1(s),\mathfrak{a}_2(s))+\max(\mathfrak{a}_1(t),\mathfrak{a}_2(t))=\mathfrak{a}(s)+\mathfrak{a}(t).\end{aligned}$$

The second part is obvious. □

Lemma 4. *Let the function $\mathfrak{a}$ satisfy* (5)*, $k\in\mathbb{N}$. Then a piecewise linear function $\mathfrak{a}_k$, interpolating $\mathfrak{a}$ using the nodes $(-1+\frac{2i}{k})$, $i=0,1,\dots,k$, also satisfies* (5)*.*

Proof. **1.** Let $s = -1 + \frac{2i}{k}$, $t = -1 + \frac{2j}{k}$. Then inequality (5) holds for $\mathfrak{a}_k$, because it does for $\mathfrak{a}$, and their values at these points coincide.

2. Now let $s = -1 + \frac{2i}{k}$ and $t \in [-1 + \frac{2j}{k}, -1 + \frac{2(j+1)}{k}]$.

Consider the linear function $h_1(t) = \mathfrak{a}_k(1 - t + s) - \mathfrak{a}_k(t) - \mathfrak{a}_k(s)$. It follows from part 1 that $h_1(-1 + \frac{2j}{k}) \leqslant 0$ and $h_1(-1 + \frac{2(j+1)}{k}) \leqslant 0$. Since h_1 is linear, $h_1(t) \leqslant 0$. Thus, the inequality holds for every $s = -1 + \frac{2i}{k}$ and $t \in [-1, 1]$.

3. Let s and t satisfy $1 - t + s = \frac{2j}{k}$.

Consider the function $h_2(y) = \mathfrak{a}_k(\frac{2j}{k}) - \mathfrak{a}_k(s + y) - \mathfrak{a}_k(t + y)$. If we choose y_0 such that $s + y_0$ is one of the nodes, then $t + y_0$ is also a node. Therefore, $h_2(y_0) = \mathfrak{a}(\frac{2j}{k}) - \mathfrak{a}(s + y_0) - \mathfrak{a}(t + y_0) \leqslant 0$. Since h_2 is linear between such y_0's, we obtain $h_2(y) \leqslant 0$ for all admissible y.

4. Finally, consider $h_3(s) = \mathfrak{a}_k(1 - t + s) - \mathfrak{a}_k(t) - \mathfrak{a}_k(s)$ for an arbitrary given $t \in [-1, 1]$. Note that the parts 2 and 3 above imply $h_3(s) \leqslant 0$ for any s such that either s or $1 - t + s$ is a node. Since h_3 is linear between these points, $h_3(s) \leqslant 0$ for all admissible s, and the statement follows therefrom. □

4 The result for piecewise linear functions

In this section we prove inequality (2) for piecewise linear functions. Without loss of generality, we assume that $F(\cdot, 0) \equiv 0$.

Theorem 2. *Let the function* $\mathfrak{a}$ *be even and satisfy the condition* (5)*. If* u *is a nonnegative piecewise linear function, then* $I(\mathfrak{a}, u) \geqslant I(\mathfrak{a}, u^*)$*.*

Proof. Let $-1 = x_1 < x_2 < \cdots < x_K = 1$ be the nodes of u. Consider the set U equal to the range of u with images of endpoints of linear pieces excluded: $U := u([-1, 1]) \setminus \{u(x_1), \ldots, u(x_K)\}$. It is obvious that the set U is the union of a finite number of intervals $U = \cup_{j=1}^{N} G_j$.

We denote by m_j the number of preimages for $u_0 \in G_j$, i.e. the number of solutions of the equation $u(y) = u_0$ (obviously, m_j does not depend on $u_0 \in G_j$). It is easy to see that the preimages are linear functions of u_0: $y = y_k^j(u_0)$, $k = 1, \ldots, m_j$, and $y_k^{j\prime}(u(y)) = \frac{1}{u'(y)}$. We assume that $y_1^j(u_0) < y_2^j(u_0) < \cdots < y_{m_j}^j(u_0)$.

The solution of the equation $u^*(y^*) = u_0$ ($u_0 \in U$) can be expressed in terms of y_k^j:

$u(-1) < u_0$	m_j is even	$y^* = 1 - \sum_{k=1}^{m_j} (-1)^k y_k^j$
	m_j is odd	$y^* = -\sum_{k=1}^{m_j} (-1)^k y_k^j$
$u(-1) > u_0$	m_j is even	$y^* = -1 + \sum_{k=1}^{m_j} (-1)^k y_k^j$
	m_j is odd	$y^* = \sum_{k=1}^{m_j} (-1)^k y_k^j$

Let $y^*(v) = (u^*)^{-1}(v)$. Then $y^{*\prime}(v) = \sum_{k=1}^{m_j} \left| y_k^{j\prime}(v) \right|$ for $v \in G_j$, as the signs in the expression for y^* and signs of $y_k^{j\prime}$ alternate, and $y^{*\prime}(v) \geqslant 0$.

The sets of zeros of $u'(x)$ and $u^{*\prime}(x)$ can have a nonzero measure. However, they do not contribute to the integral, since $F\big(u(x), 0\big) = 0$.

Consider the remaining parts of the integrals :

$$I(\mathfrak{a}, u) = \sum_{j=1}^{N} \int\limits_{u^{-1}(G_j)} F\big(u(x), \mathfrak{a}(x, u(x))\, |u'(x)|\,\big)\, dx$$

$$= \sum_{j=1}^{N} \int\limits_{G_j} \sum_{k=1}^{m_j} F\left(v, \frac{\mathfrak{a}(y_k^j(v), v)}{\left|y_k^{j\prime}(v)\right|}\right) \left|y_k^{j\prime}(v)\right| dv,$$

$$I(\mathfrak{a}, u^*) = \sum_{j=1}^{N} \int\limits_{(u^*)^{-1}(G_j)} F\big(u^*(x), \mathfrak{a}(x, u(x)) \left|u^{*\prime}(x)\right|\big)\, dx$$

$$= \sum_{j=1}^{N} \int\limits_{G_j} F\left(v, \frac{\mathfrak{a}(y^*(v), v)}{\sum_{k=1}^{m_j} \left|y_k^{j\prime}(v)\right|}\right) \sum_{k=1}^{m_j} \left|y_k^{j\prime}(v)\right| dv.$$

We fix j and v in the right parts and prove the inequality for the integrands. We denote $b_k := |y_k^{j\prime}(v)|$, $y_k := y_k^j(v)$, $y^* := y^*(v)$, $m := m_j$. Then the assertion takes the form:

$$T := \sum_{k=1}^{m} b_k F\left(v, \frac{\mathfrak{a}(y_k, v)}{b_k}\right) \geqslant F\left(v, \frac{\mathfrak{a}(y^*, v)}{\sum_{k=1}^{m} b_k}\right) \sum_{k=1}^{m} b_k.$$

By Jensen's inequality for the function $F(v, \cdot)$, we obtain

$$T \geqslant F\left(v, \frac{\sum_{k=1}^{m} \mathfrak{a}(y_k, v)}{\sum_{k=1}^{m} b_k}\right) \sum_{k=1}^{m} b_k.$$

Then it is sufficient to prove $\sum_{k=1}^{m} \mathfrak{a}(y_k, v) \geqslant \mathfrak{a}(y^*, v)$, which is true due to Lemma 1. $\square$

Remark 3. *In the paper [4] inequality* (2) *is proved under the additional assumption $u(-1) = 0$ for the weight functions $\mathfrak{a}$, decreasing in x. It is easy to see that under this assumption, the proof of Theorem* 2 *works for weights satisfying* (5) *without the evenness assumption, since in this case $u(-1) < u_0$, and we only need two of the four inequalities, given by the first part of Lemma* 1. *It is also obvious that assumption* (5) *is weaker than the assumption of $\mathfrak{a}$ decreasing in x.*

5 Extension of class of functions for which inequality (2) holds

The next statement is rather standard. However, we give a full proof for the reader's convenience.

Lemma 5. *Let the function $\mathfrak{a}$ be continuous. Then the functional $I(\mathfrak{a}, u)$ is weakly lower semicontinuous in $W_1^1(-1,1)$.*

Proof. Let $u_m \rightharpoonup u$ in $W_1^1(-1,1)$. Let us denote $A = \underline{\lim}\, I(\mathfrak{a}, u_m) \geqslant 0$. We are going to prove $I(\mathfrak{a}, u) \leqslant A$. In the case $A = \infty$ the assertion is trivial, so we can assume $A < \infty$. Switching to a subsequence, we obtain $A = \lim I(\mathfrak{a}, u_m)$.

The weak convergence implies, that there exists R_0 such that $\|u_m\|_{W_1^1(-1,1)} \leqslant R_0$. Moreover, switching to a subsequence, we can assume that $u_m \to u$ in $L_1(-1,1)$ and $u_m(x) \to u(x)$ almost everywhere. Then, by Egorov's theorem, for any ε there exists a set G_ε^1 such that $|G_\varepsilon^1| < \varepsilon$ and $u_m \rightrightarrows u$ in $[-1,1] \setminus G_\varepsilon^1$.

The uniform convergence of u_m implies that there exists K such that for each $m > K$ the inequality $|u_m| \leqslant |u| + \varepsilon$ holds in $[-1,1] \setminus G_\varepsilon^1$. Let $G_\varepsilon^2 = \{x \in [-1,1] \setminus G_\varepsilon^1 : |u(x)| \geqslant \frac{R_0+\varepsilon}{\varepsilon}\}$. Then

$$R_0 \geqslant \int\limits_{-1}^{1} |u(x)|\, dx \geqslant \int\limits_{G_\varepsilon^2} |u(x)|\, dx \geqslant \int\limits_{G_\varepsilon^2} \frac{R_0+\varepsilon}{\varepsilon}\, dx = \left|G_\varepsilon^2\right| \frac{R_0+\varepsilon}{\varepsilon}$$

That is, $|G_\varepsilon^2| \leqslant \varepsilon \frac{R_0}{R_0+\varepsilon} < \varepsilon$. Thus, the functions u_m converge uniformly and are uniformly bounded outside the set $G_\varepsilon := G_\varepsilon^1 \cup G_\varepsilon^2$.

The continuity of F and $\mathfrak{a}$ implies that for any ε and R, there exists $N(\varepsilon, R)$, such that if $x \in [-1,1] \setminus G_\varepsilon$, $|M| \leqslant R$ and $m > N(\varepsilon, R)$, then

$$|F\big(u_m(x), \mathfrak{a}(x, u_m(x))M\big) - F\big(u(x), \mathfrak{a}(x, u(x))M\big)| < \varepsilon.$$

Let $E_{m,\varepsilon} := \{x \in [-1,1] : |u_m'(x)| \geqslant \frac{R_0}{\varepsilon}\}$. Then

$$R_0 \geqslant \int\limits_{-1}^{1} |u_m'(x)|\, dx \geqslant \int\limits_{E_{m,\varepsilon}} |u_m'(x)|\, dx \geqslant \int\limits_{E_{m,\varepsilon}} \frac{R_0}{\varepsilon}\, dx = \frac{R_0}{\varepsilon} \left|E_{m,\varepsilon}\right|.$$

Therefore $|E_{m,\varepsilon}| \leqslant \varepsilon$.

Finally we set $L_{m,\varepsilon} := [-1,1] \setminus (E_{m,\varepsilon} \cup G_\varepsilon)$. Note, that $|L_{m,\varepsilon}| \geqslant 2 - 3\varepsilon$.

We put $R := \frac{R_0}{\varepsilon}$, $N(\varepsilon) := N(\varepsilon, \frac{R_0}{\varepsilon})$. For any $\varepsilon > 0$, $x \in L_{m,\varepsilon}$ and $m > N(\varepsilon)$ we have

$$\Big|F\big(u_m(x), \mathfrak{a}(x, u_m(x))\,|u_m'(x)|\,\big) - F\big(u(x), \mathfrak{a}(x, u(x))\,|u_m'(x)|\,\big)\Big| < \varepsilon,$$

thus

$$\int\limits_{L_{m,\varepsilon}} \Big|F\big(u_m(x), \mathfrak{a}(x, u_m(x))\,|u_m'(x)|\,\big) - F\big(u(x), \mathfrak{a}(x, u(x))\,|u_m'(x)|\,\big)\Big|\, dx < 2\varepsilon. \qquad (9)$$

We put $\varepsilon_j = \frac{\varepsilon}{2^j}$ $(j \geqslant 1)$, $m_j = N(\varepsilon_j) + j \to \infty$ and $L_\varepsilon = \bigcap L_{m_j,\varepsilon_j}$. Then $\sum \varepsilon_j = \varepsilon$ and therefore $|[-1,1] \setminus L_\varepsilon| < 3\varepsilon$. Since (9) implies

$$\int\limits_{L_\varepsilon} \Big|F\big(u_{m_j}(x), \mathfrak{a}(x, u_{m_j}(x))|u_{m_j}'(x)|\big) - F\big(u(x), \mathfrak{a}(x, u(x))|u_{m_j}'(x)|\big)\Big|\, dx < 2\varepsilon_j,$$

we obtain

$$A = \lim I(\mathfrak{a}, u_{m_j}) = \lim \int\limits_{-1}^{1} F\big(u_{m_j}(x), \mathfrak{a}(x, u_{m_j}(x))|u'_{m_j}(x)|\big)\, dx$$

$$\geqslant \varliminf \int\limits_{-1}^{1} \chi_{L_\varepsilon}(x) F\big(u(x), \mathfrak{a}(x, u(x))|u'_{m_j}(x)|\big)\, dx =: \varliminf J_\varepsilon(u'_{m_j}).$$

The functional

$$J_\varepsilon(v) = \int\limits_{-1}^{1} \chi_{L_\varepsilon}(x) F\big(u(x), \mathfrak{a}(x, u(x))|v(x)|\big)\, dx$$

is convex. Switching to a subsequence u_k again, we can assume that $\varliminf J_\varepsilon(u'_{m_j}) = \lim J_\varepsilon(u'_k)$. Since $u'_k \rightharpoonup u'$ in L_1, we can choose a sequence of convex combinations of u'_k, which converges to u' strongly (see [10, Theorem 3.13]). Namely, there are $\alpha_{k,l} \geqslant 0$ for $k \in \mathbb{N}$, $l \leqslant k$, such that $\sum_{l=1}^{k} \alpha_{k,l} = 1$ for every k and $w_k := \sum_{l=1}^{k} \alpha_{k,l} u'_l \to u'$ in L_1. Also, without loss of generality we can assume that the minimal index l of a nonzero coefficient $\alpha_{k,l}$ tends to infinity as k tends to infinity. Then

$$\lim J_\varepsilon(u'_k) = \lim \sum_{l=1}^{k} \alpha_{k,l} J_\varepsilon(u'_l).$$

By the convexity of J_ε, we have

$$\sum_{l=1}^{k} \alpha_{k,l} J_\varepsilon(u'_l) \geqslant J_\varepsilon(w_k).$$

Finally, since $w_k \to u'$ in $L_1(-1,1)$, we can assume, by switching to a subsequence, that $w_k(x) \to u'(x)$ almost everywhere. Moreover, since $\big|u'_j(x)\big| < \frac{R_0}{\varepsilon}$ holds for $x \in L_\varepsilon$, then $|w_k(x)| < \frac{R_0}{\varepsilon}$. Hence,

$$F\big(u(x), \mathfrak{a}(x, u(x))|w_k(x)|\big) \leqslant \max_{(x,M)} F\big(u(x), \mathfrak{a}(x, u(x))M\big) < \infty,$$

where the maximum is taken over a compact set $(x, M) \in [-1,1] \times [-\frac{R_0}{\varepsilon}, \frac{R_0}{\varepsilon}]$. Therefore, by the Lebesgue theorem, $\lim J_\varepsilon(w_k) = J_\varepsilon(u')$. Thus,

$$A \geqslant \lim J_\varepsilon(u'_k) = \lim \sum_{l=1}^{k} \alpha_{k,l} J_\varepsilon(u'_l) \geqslant \varliminf J_\varepsilon(w_k) = J_\varepsilon(u').$$

Since $\varepsilon > 0$ is arbitrary, $A \geqslant I(\mathfrak{a}, u)$ follows. □

Lemma 6. *Let $B \subset A \subset W_1^1(-1,1)$. Let inequality (2) hold for any $u \in B$. Suppose that for each $u \in A$ there is a sequence $u_k \in B$ such that relation (4) holds. Then inequality (2) holds for any $u \in A$.*

Proof. Let us pick some $u \in A$ and find an appropriating sequence $\{u_k\} \subset B$. By hypothesis, $I(\mathfrak{a}, u_k^*) \leqslant I(\mathfrak{a}, u_k) \to I(\mathfrak{a}, u)$. By [2, Theorem 1]

$$u_k \to u \text{ in } W_1^1(-1,1) \quad \Longrightarrow \quad \overline{u_k} \rightharpoondown \overline{u} \text{ in } W_1^1(-1,1).$$

Since $u_k^*(x) = \overline{u_k}(\frac{x-1}{2})$ and $u^*(x) = \overline{u}(\frac{x-1}{2})$, we have $u_k^* \rightharpoondown u^*$ in $W_1^1(-1,1)$. By Lemma 5, we obtain

$$I(\mathfrak{a}, u^*) \leqslant \underline{\lim} I(\mathfrak{a}, u_k^*) \leqslant \lim I(\mathfrak{a}, u_k) = I(\mathfrak{a}, u).$$

□

Corollary 1. *Let the weight $\mathfrak{a}$ be continuous, and let inequality* (2) *hold for nonnegative piecewise linear functions u. Then it holds for all nonnegative Lipschitz functions.*

Proof. By Theorem 1 in Section 6.6 [11], any Lipschitz function u can be approximated by $u_k \in C^1[-1,1]$ such that

$$u_k \rightrightarrows u, \qquad u_k' \to u' \text{ a.e.}, \qquad |u_k'| \leqslant const.$$

Relation (4) holds by the Lebesgue theorem. In turn, u_k can be approximated in the same way by piecewise linear functions. Using Theorem 2 and applying Lemma 6, we complete the proof. □

6 The inequality for $u \in W_1^1(-1,1)$ with an additional restriction on weight

In this section we prove inequality (2) under the additional condition: the weight is monotonic in x for $x \in [-1,0]$ and $x \in [0,1]$.

Lemma 7. *Let $\mathfrak{a}$ be a continuous function and let $\mathfrak{a}(\cdot, u)$ be increasing on $[-1,0]$ and decreasing on $[0,1]$ for all $u \geqslant 0$. Then for any function $u \in W_1^1(-1,1)$, $u \geqslant 0$, there exists a sequence $\{u_k\} \subset Lip[-1,1]$, such that relation* (4) *holds.*

Proof. We can assume that $I(\mathfrak{a}, u) < \infty$.

We prove the assertion for the functional

$$I_1(u) = \int_0^1 F\big(u(x), \mathfrak{a}(x, u(x))|u'(x)|\big)\, dx,$$

and the integral over $[-1,0]$ can be reduced to I_1 by changing the variable.

We modify the scheme from [1, Theorem 2.4]. A part of the proof overlaps with [1], but we present a complete proof here for the reader's convenience.

We need the following auxiliary assertion.

Proposition 1. [1, Lemma 2.7]. *Let $\varphi_h : [-1,1] \to \mathbb{R}_+$ be a sequence of Lipschitz functions satisfying the conditions: $\varphi_h' \geqslant 1$ for almost every x and all h, $\varphi_h(x) \to x$ for almost every x. Then for any $f \in L_1(\mathbb{R})$ we have $f(\varphi_h) \to f$ in $L_1(\mathbb{R})$.*

For $h \in \mathbb{N}$ we cover the set $\{x \in [0,1] : |u'(x)| > h\}$ with an open set A_h. Without loss of generality, we can assume that $A_{h+1} \subset A_h$, and $|A_h| \to 0$ for $h \to \infty$.

Denote by v_h the nonnegative continuous function on $[0,1]$, coinciding with u on $[0,1] \setminus A_h$ and linear on intervals forming A_h. Then $v_h \to u$ in W_1^1. Now we modify v_h to get Lipschitz functions.

Let $A_h = \cup_k \Omega_{h,k}$, where $\Omega_{h,k} = (b_{h,k}^-, b_{h,k}^+)$. Denote

$$\alpha_{h,k} := |\Omega_{h,k}|, \quad \beta_{h,k} := v_h(b_{h,k}^+) - v_h(b_{h,k}^-) = u(b_{h,k}^+) - u(b_{h,k}^-).$$

Then $v_h' = \frac{\beta_{h,k}}{\alpha_{h,k}}$ in $\Omega_{h,k}$. Note that

$$\sum_k |\beta_{h,k}| \leqslant \int\limits_{A_h} |u'|\, dx \leqslant \|u'\|_{L_1(-1,1)} < \infty,$$

and hence $\sum_k |\beta_{h,k}| \to 0$ as $h \to 0$ by the Lebesgue theorem.

We define the function $\varphi_h \in W_1^1(0,1)$ as follows:

$$\begin{aligned} \varphi_h(0) &= 0 \\ \varphi_h' &= 1 && \text{in } [0,1] \setminus A_h, \\ \varphi_h' &= \max\Big(\frac{|\beta_{h,k}|}{\alpha_{h,k}}, 1\Big) && \text{in } \Omega_{h,k}. \end{aligned}$$

Note that $\int_0^1 |\varphi_h'|\, dx \leqslant 1 + \sum_k |\beta_{h,k}| < \infty$.

Next, $\varphi_h' \to 1$ in $L_1(0,1)$:

$$\int |\varphi_h' - 1|\, dx = \sum_k \Big(\max\Big(\frac{|\beta_{h,k}|}{\alpha_{h,k}}, 1\Big) - 1\Big)\alpha_{h,k} \leqslant \sum_k |\beta_{h,k}| \to 0.$$

Thus φ_h satisfies the conditions of Proposition 1.

Consider now $\varphi_h^{-1} : [0,1] \to [0,1]$, the restriction to $[0,1]$ of the inverse to φ_h. Then

$$\begin{aligned} \varphi_h^{-1}(0) &= 0 \\ (\varphi_h^{-1})' &= 1 && \text{in } [0,1] \setminus \varphi_h(A_h), \\ (\varphi_h^{-1})' &= \min\Big(\frac{\alpha_{h,k}}{|\beta_{h,k}|}, 1\Big) && \text{in } [0,1] \cap \varphi_h(\Omega_{h,k}). \end{aligned}$$

Let $u_h = v_h(\varphi_h^{-1})$. Note that $u_h(0) = u(0)$, and

$$\begin{aligned} u_h' &= v_h'(\varphi_h^{-1}) \cdot (\varphi_h^{-1})' = u'(\varphi_h^{-1}) && \text{in } [0,1] \setminus \varphi_h(A_h), \\ u_h' &= v_h'(\varphi_h^{-1}) \cdot (\varphi_h^{-1})' = \operatorname{sign} \beta_{h,k} \cdot \min\Big(1, \frac{|\beta_{h,k}|}{\alpha_{h,k}}\Big) && \text{in } [0,1] \cap \varphi_h(\Omega_{h,k}). \end{aligned}$$

Thus, u_h is Lipschitz since u' is bounded in $[0,1] \setminus A_h$.

We claim that $u_h \to u$ in $W_1^1(0,1)$. Indeed, it is sufficient to estimate

$$\|u_h' - u'\|_{L_1} \leqslant \int\limits_{[0,1]\setminus\varphi_h(A_h)} |u_h' - u'| + \int\limits_{[0,1]\cap\varphi_h(A_h)} |u_h'| + \int\limits_{[0,1]\cap\varphi_h(A_h)} |u'| =: P_h^1 + P_h^2 + P_h^3.$$

$$P_h^1 = \int\limits_{[0,1]\setminus\varphi_h(A_h)} \left|u'(\varphi_h^{-1}) - u'\right| dx = \int\limits_{\varphi_h^{-1}([0,1])\setminus A_h} \left|u' - u'(\varphi_h)\right| dz \leqslant \int\limits_{[0,1]} \left|u' - u'(\varphi_h)\right| dz.$$

By Proposition 1, $P_h^1 \to 0$. Further,

$$P_h^2 \leqslant |\varphi_h(A_h)| = \sum_k |\varphi_h(\Omega_{h,k})| = \sum_k \max(|\beta_{h,k}|, \alpha_{h,k}) \leqslant \sum_k \alpha_{h,k} + \sum_k |\beta_{h,k}| \to 0.$$

Finally, $P_h^3 \to 0$ by the absolute continuity of the integral, and the assertion is proved.

It remains to show that $I_1(u_h) \to I_1(u)$.

$$I_1(u_h) = \int\limits_{[0,1]\setminus\varphi_h(A_h)} F\big(u_h(x), \mathfrak{a}(x, u_h(x))|u_h'(x)|\big)\, dx +$$
$$\int\limits_{[0,1]\cap\varphi_h(A_h)} F\big(u_h(x), \mathfrak{a}(x, u_h(x))|u_h'(x)|\big)\, dx =: \hat{P_h^1} + \hat{P_h^2}.$$

Since $u \in W_1^1(0,1)$ then $u \in L_\infty([0,1])$. Denote $\|u\|_\infty = r$. Then $\|u_h\|_\infty < 2r$ for sufficiently large h. Also, $|u_h'| \leqslant 1$ almost everywhere in $\varphi_h(A_h)$. Then $\hat{P_h^2} \leqslant M_F\,|\varphi_h(A_h)| \to 0$, where

$$M_F = \max_{[-2r,2r]\times[-M_\mathfrak{a}, M_\mathfrak{a}]} F; \quad M_\mathfrak{a} = \max_{[0,1]\times[-2r,2r]} \mathfrak{a}.$$

Further,

$$\hat{P_h^1} = \int\limits_{[0,1]\setminus\varphi_h(A_h)} F\big(u(\varphi_h^{-1}(x)), \mathfrak{a}(x, u(\varphi_h^{-1}(x))|u'(\varphi_h^{-1}(x))(\varphi_h^{-1})'|)\big)\, dx$$
$$= \int\limits_{\varphi_h^{-1}([0,1])\setminus A_h} F\big(u(z), \mathfrak{a}(\varphi_h(z), u(z))|u'(z)|\big)\, dz$$
$$= \int\limits_{[0,1]} F\big(u(z), \mathfrak{a}(\varphi_h(z), u(z))|u'(z)|\big)\chi_{\varphi_h^{-1}([0,1])\setminus A_h}\, dz.$$

The last equality, generally speaking, does not make sense, since $\varphi_h(z)$ can take values outside $[0,1]$. Let us define $\mathfrak{a}(z,u) = \mathfrak{a}(1,u)$ for $z > 1$. Now the expression is correct. Note that $\chi_{\varphi_h^{-1}([0,1])\setminus A_h}$ increases, since sets $\varphi_h^{-1}([0,1])$ increase and sets A_h decay, i.e. $\varphi_{h_1}^{-1}([0,1]) \subset \varphi_{h_2}^{-1}([0,1])$ and $A_{h_1} \supset A_{h_2}$ for $h_1 \leqslant h_2$. Since $\mathfrak{a}$ is decreasing on $[0,1]$ (in fact, on $\varphi_h([0,1])$) and $\varphi_h(z)$ is decreasing in h, then $\mathfrak{a}(\varphi_h(z))$ is increasing in h. We apply the monotone convergence theorem and get

$$\hat{P_h^1} \to \int\limits_{[0,1]} F\big(u(z), \mathfrak{a}(z, u(z))|u'(z)|\big)\, dz.$$

□

Remark 4. *Obviously, the proof works for any interval* $[x_0, x_1]$ *with function* u *pinned at* x_0*, provided the weight* $\mathfrak{a}$ *is decreasing in* x *on* $[x_0, x_1]$*. This means that there exists* $\{u_h\}$*, such that*

$$u_h(x_0) = u(x_0); \qquad u_h \to u \ \textit{in}\ W_1^1(x_0, x_1);$$
$$\int\limits_{x_0}^{x_1} F\big(u_h(x), \mathfrak{a}(x, u_h(x))\,|u_h'(x)|\,\big) \to \int\limits_{x_0}^{x_1} F\big(u(x), \mathfrak{a}(x, u(x))\,|u'(x)|\,\big).$$

Similarly, if $\mathfrak{a}$ *is increasing in* x*, the same works for functions* u *pinned at the right end of the segment.*

Corollary 2. *Suppose that the function* $\mathfrak{a}$ *is continuous, even in* x*, decreasing on* $[0, 1]$ *and satisfies* (5)*. Then for every* $u \in W_1^1(-1, 1)$ *inequality (2) holds.*

Proof. The statement follows from Lemmata 6 and 7 immediately. □

7 The result in the general case

Now we want to get rid of the monotonicity restriction on the weight. We do this in several steps.

To begin, we note that all properties of the function $\mathfrak{a}$ are of interest only in the neighborhood of the graphs of functions u and u^*.

We introduce the following conditions, each of which, when added to the previous ones, defines a smaller class of weight functions:

$(H1)$ $\mathfrak{a}(x, v)$ satisfies (5), is even in x and $I(\mathfrak{a}, u) < \infty$.

$(H2)$ the number of zeros of $\mathfrak{a}(\cdot, v)$ is bounded by a constant independent of v for all $v \in [\min u(x), \max u(x)]$ such that $\mathfrak{a}(\cdot, v) \not\equiv 0$.

$(H3)$ If $\mathfrak{a}(x_0, u(x_0)) = 0$ for some x_0, then $\mathfrak{a}(\cdot, u(x_0)) \equiv 0$. Moreover, $\lim\limits_{k\to\infty} D_k(\mathfrak{a}, U(\mathfrak{a})) = 0$, where

$$U(\mathfrak{a}) := \{v \in [\min u(x), \max u(x)] : \mathfrak{a}(\cdot, v) \not\equiv 0\},$$
$$D_k(\mathfrak{a}, U) := \sup_{v\in U} \frac{\max\limits_{|x_1 - x_2| \leqslant \frac{2}{k}} |\mathfrak{a}(x_1, v) - \mathfrak{a}(x_2, v)|}{\min\limits_{\operatorname{dist}(x, u^{-1}(v)) \leqslant \frac{2}{k}} \mathfrak{a}(x, v)}. \tag{10}$$

$(H4)$ There exists an even k, such that $\mathfrak{a}(\cdot, v)$ are linear for each v on each of the segments $[-1 + \frac{2i}{k}, -1 + \frac{2(i+1)}{k}]$.

$(H5)$ The difference between the set of $v \in \mathbb{R}_+$, for which $\mathfrak{a}(\cdot, v)$ has segments of constant values, and the set of $v \in \mathbb{R}_+$ such that $\mathfrak{a}(\cdot, v) \equiv 0$ has zero measure.

$(H6)$ The segment $[-1,1]$ can be represented as a unity of touching segments on each of which $\mathfrak{a}$ does not change the monotonicity with respect to x in a v-neighborhood of the graph of the function u.

$(H7)$ Let $x_1 < x_2 < x_3$, let $\mathfrak{a}(\cdot, v)$ decrease for $x \in [x_1, x_2]$ in a v-neighborhood of the graph of the function u, and let $\mathfrak{a}(\cdot, v)$ increase for $x \in [x_1, x_2]$ in a v-neighborhood of the graph of the function u. Then we have $\mathfrak{a}(\cdot, v) \equiv 0$ in a v-neighborhood of $u(x_2)$.

The weights satisfying $(H1)$ will be called *admissible for a given u.*

Now we can formulate the main assertion of our work.

Theorem 3. *Suppose $F \in \mathfrak{F}$, the function $u \in W_1^1(-1,1)$ is nonnegative, and the weight function $\mathfrak{a} : [-1,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and admissible for u. Then inequality* (2) *holds.*

We prove inequality (2) under conditions $(H1) - (H7)$, and then get rid of extra conditions one by one.

To prove it, we need the following facts.

Proposition 2. [9, Theorem 6.19] *For every $u \in W_1^1(-1,1)$ and for an arbitrary set $A \subset \mathbb{R}_+$ of zero measure, $u'(x) = 0$ almost everywhere in $u^{-1}(A)$.*

Lemma 8. *Suppose that $u \in W_1^1(-1,1)$ is nonnegative. Let a closed set $W \subset \mathbb{R}_+$ be such that the set of $v \in W$, for which $\mathfrak{a}(\cdot, v) \not\equiv 0$, has a zero measure. Then there exists an increasing sequence of weights $\mathfrak{b}_\ell$, which satisfy*

1) $\mathfrak{b}_\ell(\cdot, v) \rightrightarrows \mathfrak{a}(\cdot, v)$ for almost all v;

2) $\mathfrak{b}_\ell(\cdot, v) \equiv 0$ for every v in some neighborhood of W (the neighborhood depends on ℓ);

3) $I(\mathfrak{b}_\ell, u) \to I(\mathfrak{a}, u)$ and $I(\mathfrak{b}_\ell, u^*) \to I(\mathfrak{a}, u^*)$.

Remark 5. *If a is admissible for u, then b_ℓ are also admissible.*

Proof. Take $\rho(d) := \min(1, \max(0, d))$,

$$\mathfrak{b}_\ell(x, v) := \mathfrak{a}(x, v) \cdot \rho(\ell \operatorname{dist}(v, W) - 1) \leqslant \mathfrak{a}(x, v).$$

This weight is equal to zero in the $\left(\frac{1}{\ell}\right)$-neighborhood of W. In addition, $\mathfrak{b}_\ell \equiv \mathfrak{a}$ outside the $\left(\frac{2}{\ell}\right)$-neighborhood of W and $\mathfrak{b}_\ell(x, v)$ increases in ℓ. Thus, $\mathfrak{b}_\ell(\cdot, v) \rightrightarrows \mathfrak{a}(\cdot, v)$ for almost all v. By the monotone convergence theorem $I(u^{-1}(\mathbb{R}_+ \setminus W), \mathfrak{b}_\ell, u) \nearrow I(u^{-1}(\mathbb{R}_+ \setminus W), \mathfrak{a}, u)$.

Divide the set W into $W_1 := \{v \in W : \mathfrak{a}(\cdot, v) \equiv 0\}$ and $W_2 = W \setminus W_1$. Then

$$I(u^{-1}(W_1), \mathfrak{b}_\ell, u) = I(u^{-1}(W_1), \mathfrak{a}, u),$$

$$I(u^{-1}(W_2), \mathfrak{b}_\ell, u) = \int\limits_{x \in u^{-1}(W_2)} F\big(u(x), \mathfrak{b}_\ell(x, u(x))|u'(x)|\big)\, dx.$$

By Proposition 2, $u'(x) = 0$ almost everywhere on $u^{-1}(W_2)$. Thus

$$I(u^{-1}(W_2), \mathfrak{b}_\ell, u) = \int\limits_{x \in u^{-1}(W_2)} F\big(u(x), 0\big)\, dx = 0.$$

Similarly, $I(u^{-1}(W_2), \mathfrak{a}, u) = 0$. Hence $I(\mathfrak{b}_\ell, u) \to I(\mathfrak{a}, u)$. The second relation in *3)* is proved by the same arguments. □

We proceed to the proof of the theorem.

Step 1. *Let $u \in W_1^1(-1,1)$ and let the weight $\mathfrak{a}$ satisfy the conditions $(H1)-(H7)$. Then inequality (2) holds.*

Divide the segment $[-1,1]$ into touching subsegments Δ_j, each consisting of two parts. On the left part of each Δ_j the weight $\mathfrak{a}$ increases in x in a neighborhood of the graph of $u(x)$. On the right part it decreases. On each Δ_j, we can apply the construction from the previous section for approximating u with Lipschitz functions u_n. This gives us $I(\Delta_j, \mathfrak{a}, u_n) \to I(\Delta_j, \mathfrak{a}, u)$.

However, the approximating functions u_n have discontinuities at the borders of the segments Δ_j (denote them by $\hat{x}_j$).

Note that according to the condition $(H7)$ one can choose points $\hat{x}_j$ so that $\mathfrak{a} \equiv 0$ in (x,v)-neighborhoods of the points $(\hat{x}_j, u(\hat{x}_j))$.

Next, substitute functions u_n in these neighborhoods of $\hat{x}_j$ with linear pieces making u_n continuous on $[-1,1]$. In view of the above, this does not change the integrals $I(\Delta_j, \mathfrak{a}, u_n)$, and we get $I(\mathfrak{a}, u_n) \to I(\mathfrak{a}, u)$.

By Lemma 6 we obtain (2).

Step 2. *Let the weight $\mathfrak{a}$ satisfy the conditions $(H1)-(H6)$. Then inequality (2) holds.*

We apply Lemma 8 using the following set W: the set of all v, at which the graph of $u(x)$ traverses from a rectangle, in which the weight decreases in x, to a rectangle in which the weight increases. Obviously, the resulting functions $\mathfrak{b}_\ell$ satisfy $(H1)-(H7)$. By Step 1, $I(\mathfrak{b}_\ell, u^*) \leqslant I(\mathfrak{b}_\ell, u)$. Passing to the limit, we obtain (2).

Step 3. *Let the weight $\mathfrak{a}$ satisfy the conditions $(H1)-(H5)$. Then inequality (2) holds.*

Consider abscissas of nodes of $\mathfrak{a}$ and ordinates, for which $\mathfrak{a}$ has constant pieces. They define a division of the rectangle $[-1,1] \times [\min u(x), \max u(x)]$ into smaller rectangles, in each of which the weight $\mathfrak{a}$ is monotone in x. However, the number of rectangles can be infinite. Also, if the graph of u crosses a horizontal boundary of some rectangle, monotonicity in the v-neighborhood of the point of intersection may change.

Consider a set W containing all v, for which the weight $\mathfrak{a}$ has constant pieces. Due to $(H5)$ the set of all $v \in W$ such that $a(\cdot, v) \not\equiv 0$ has a zero measure.

We apply Lemma 8 and obtain a sequence of weights $\mathfrak{b}_\ell$. We claim that each of them has only a finite number of monotonicity rectangles. Indeed, any two vertically adjacent rectangles with different monotonicities are separated by a stripe of $\frac{2}{\ell}$ width with zero values.

The weight b_ℓ can change monotonicity along the graph of u either at the points $x = -1 + \frac{2i}{k}$ or where the graph crosses a stripe of zero values. Note that only a finite number of such crossings can arise since $\int |u'|$ gains at least $\frac{2}{\ell}$ at any crossing and $u' \in L_1(-1,1)$.

Thereby, each $\mathfrak{b}_\ell$ satisfies $(H1)-(H6)$. By Step 2, $I(\mathfrak{b}_\ell, u^*) \leqslant I(\mathfrak{b}_\ell, u)$. Passing to the limit, we obtain (2).

Step 4. *Let the weight $\mathfrak{a}$ satisfy the conditions $(H1)-(H3)$. Then inequality (2) holds.*

Suppose that the function $\mathfrak{a}$ satisfies $(H1)-(H3)$, in particular $I(\mathfrak{a},u)<\infty$.

We fix an arbitrary even k. For each v we interpolate $\mathfrak{a}$ with piecewise linear functions with nodes $(-1+\frac{2i}{k},\mathfrak{a}(-1+\frac{2i}{k},v))$. The resulting function $\mathfrak{a}_k(x,v)$ is continuous, even in x and satisfies (5) by Lemma 4. In addition, $\mathfrak{a}_k\to\mathfrak{a}$ when $k\to\infty$, moreover the convergence is uniform on compact sets. However, the inequality $\mathfrak{a}_k(x,u(x))\leqslant\mathfrak{a}(x,u(x))$ can be violated, and thus $\mathfrak{a}_k$ may be non-admissible for u.

Set $\mathfrak{c}_k:=(1-D_k(\mathfrak{a}_k,U(\mathfrak{a}_k)))\mathfrak{a}_k$, where D_k is defined in (10). $D_k(\mathfrak{a}_k,U(\mathfrak{a}_k))$ are positive and tend to zero, thus $\mathfrak{c}_k\to\mathfrak{a}$ while $k\to\infty$. We claim that $\mathfrak{c}_k(x,u(x))\leqslant\mathfrak{a}(x,u(x))$.

Indeed, consider some $x\in[-1+\frac{2i}{k},-1+\frac{2(i+1)}{k}]=:[x_i,x_{i+1}]$. Then $\mathfrak{c}_k(x,u(x))\leqslant\max(\mathfrak{c}_k(x_i,u(x)),\mathfrak{c}_k(x_{i+1},u(x)))$, because $\mathfrak{c}_k$ is piecewise linear in x. Moreover,

$$\begin{aligned}\mathfrak{c}_k(x_i,u(x))&=(1-D_k(\mathfrak{a}_k,U(\mathfrak{a}_k)))\cdot\mathfrak{a}(x_i,u(x))\\&\leqslant\mathfrak{a}(x_i,u(x))-\frac{\mathfrak{a}(x_i,u(x))-\mathfrak{a}(x,u(x))}{\mathfrak{a}(x_i,u(x))}\cdot\mathfrak{a}(x_i,u(x))=\mathfrak{a}(x,u(x)).\end{aligned}$$

Similarly, $\mathfrak{c}_k(x_{i+1},u(x))\leqslant\mathfrak{a}(x,u(x))$. Thus, $\mathfrak{c}_k(x,u(x))\leqslant\mathfrak{a}(x,u(x))$ for any x, and $\mathfrak{c}_k$ are admissible for u. Thereby the functions $\mathfrak{c}_k$ satisfy $(H1)-(H4)$.

For a given $k\in\mathbb{N}$, we approximate the function $\mathfrak{c}_k=:\mathfrak{c}$ with weights satisfying $(H1)-(H5)$. Consider the auxiliary function $\Lambda(x)=1-|x|$, satisfying (5).

Take

$$t(v):=D_k(\mathfrak{c},U(\mathfrak{c}))\cdot\max\{\tau\geqslant 0:\forall x\in u^{-1}(v)\quad\tau\Lambda(x)\leqslant\mathfrak{c}(x,u(x))\}.$$

The function t depends on k, but we omit this fact in presentation.

It is clear that the maximum τ is zero only if $\mathfrak{c}(\cdot,v)\equiv 0$, since otherwise the condition $(H3)$ is violated.

Function t may be discontinuous. However, it is easy to see that it is lower semicontinuous. Next, we take

$$\tilde{t}(v):=\inf_{w\in u([-1,1])}\{t(w)+|v-w|\}.$$

It is obvious that $\tilde{t}\leqslant t$, and the set of zeros of t and $\tilde{t}$ coincide.

We claim that $\tilde{t}$ is continuous (and even Lipschitz). Indeed, take some v_1. Then there is an arbitrarily small $\varepsilon>0$ and $w_1\in u([-1,1])$ satisfying $\tilde{t}(v_1)=t(w_1)+|v_1-w_1|-\varepsilon$. For every v_2, we have $\tilde{t}(v_2)\leqslant t(w_1)+|v_2-w_1|$. And thus $\tilde{t}(v_2)-\tilde{t}(v_1)\leqslant|v_1-v_2|+\varepsilon$. By the arbitrariness of v_1, v_2 and ε, the claim follows therefrom.

For $\alpha\in[0,1]$ the function $\mathfrak{d}_\alpha(x,v):=\mathfrak{c}(x,v)+\alpha\Lambda(x)\tilde{t}(v)$ is even in x, satisfies (5) in concordance with Lemma 3, and does not exceed $\mathfrak{a}(x,v)$ due to the construction of the function $\tilde{t}$. Thus, $\mathfrak{d}_\alpha$ is an admissible weight. Also, it is obvious that $\mathfrak{d}_\alpha$ satisfies $(H1)-(H4)$.

Let us show that there exists a sequence $\alpha_j\searrow 0$ such that $\mathfrak{d}_{\alpha_j}(\cdot,v)$ has no segments of constant values, unless $\mathfrak{d}_{\alpha_j}(\cdot,v)\equiv 0$ or v belongs to a zero measure set. We introduce the set of α which are "bad" on $[x_i,x_{i+1}]$:

$$\begin{aligned}A_i:=\Big\{\alpha\in[0,1]:&\\&meas\{v\in[\min u,\max u]:\frac{\mathfrak{c}(x_{i+1},v)-\mathfrak{c}(x_i,v))}{\frac{2}{k}}+\alpha\chi_i\tilde{t}(v)=0\}>0\Big\},\end{aligned}$$

where $\chi_i = 1$ if $[x_i, x_{i+1}] \subset [0,1]$, and $\chi_i = -1$ if $[x_i, x_{i+1}] \subset [-1,0]$.

Consider the following function

$$\begin{aligned} h_i(v) =& \frac{\mathfrak{c}(x_{i+1}, v) - \mathfrak{c}(x_i, v)}{\tilde{t}(v)} \qquad \text{if } \tilde{t}(v) \neq 0 \\ h_i(v) =& 0 \qquad\qquad\qquad\qquad\qquad \text{if } \tilde{t}(v) = 0. \end{aligned}$$

We have $\operatorname{card}(A_i) = \operatorname{card}(\{\alpha \in [0,1] : meas\{v \in [\min u, \max u] : h_i(v) \pm \frac{2}{k}\alpha = 0\} > 0\})$. Then $\operatorname{card}(A_i) \leqslant \aleph_0$, and $\operatorname{card}(\cup_i A_i) \leqslant \aleph_0$. Thus, there exists a sequence of weights $\mathfrak{d}_{\alpha_j} \searrow \mathfrak{c}$, satisfying $(H1)-(H5)$. By Step 3, $I(\mathfrak{d}_{\alpha_j}, u^*) \leqslant I(\mathfrak{d}_{\alpha_j}, u)$. Passing to the limit, we get $I(\mathfrak{c}, u^*) \leqslant I(\mathfrak{c}, u)$.

Further, for $x \in [-1,1]$ we have

$$F\big(u(x), \mathfrak{c}_k(x, u(x))|u'(x)|\big) \to F\big(u(x), \mathfrak{a}(x, u(x))|u'(x)|\big) \tag{11}$$

as $k \to \infty$. Moreover, $F\big(u(x), \mathfrak{a}(x, u(x))|u'(x)|\big)$ is an integrable majorant for the left-hand side in (11). By the Lebesgue theorem, we have $I(\mathfrak{c}_k, u) \to I(\mathfrak{a}, u)$. Since $I(\mathfrak{c}_k, u^*) \leqslant I(\mathfrak{c}_k, u)$, Lemma 6 proves inequality (2).

Step 5. *Let the weight $\mathfrak{a}$ satisfy only the condition $(H1)$. Then inequality (2) holds.*

We approximate $\mathfrak{a}$ by weights satisfying $(H1)-(H2)$. To do this we apply Lemma 8 with $W = \{v \in \mathbb{R}_+ : \mathfrak{a}(\cdot, v) \equiv 0\}$. Let us introduce the notation

$$Z_{\mathfrak{a}}(v) := \{x \in [-1,1] : \mathfrak{a}(x, v) = 0\}.$$

Note that the sets $Z_{\mathfrak{b}_\ell}(v)$ are either $Z_{\mathfrak{a}}(v)$ or $[-1,1]$.

Let us show that $\mathfrak{b}_\ell$ satisfies $(H2)$. Indeed, otherwise there is a sequence v_m, for which $m < \operatorname{card}(Z_{\mathfrak{b}_\ell})(v_m) < \infty$. After passing to a subsequence, we have $v_m \to v_0$. Part 2 of Lemma 2 implies that the set $Z_{\mathfrak{b}_\ell}(v_m) = Z_{\mathfrak{a}}(v_m)$ is periodic with a period less or equal to $\frac{2}{m-1}$. Take some $x \in [-1,1]$. For each m there exists x_m such that $|x - x_m| \leqslant \frac{1}{m-1}$ and $\mathfrak{a}(x_m, v_m) = 0$. But $\mathfrak{a}(x_m, v_m) \to \mathfrak{a}(x, v_0)$. Therefore, $\mathfrak{a}(x, v_0) = 0$.

Thus $Z_{\mathfrak{a}}(v_0) = [-1,1]$. But this means that for every v such that $|v - v_0| \leqslant \frac{1}{\ell}$, we have $\mathfrak{b}_\ell(\cdot, v) \equiv 0$, which contradicts $\operatorname{card}(Z_{\mathfrak{b}_\ell})(v_m) < \infty$.

Now we fix $\ell \in \mathbb{N}$ and denote $\mathfrak{b}_\ell =: \mathfrak{b}$. Let us approximate the function $\mathfrak{b}$ with weights satisfying $(H1)-(H3)$. It follows from $(H2)$ that there exists a set $T \subset [-1,1]$ consisting of a finite number of elements, such that if $x \notin T$ and $\mathfrak{b}(x, v) = 0$ for some v, then $\mathfrak{b}(\cdot, v) \equiv 0$.

We use Lemma 8 with $W = u(T) \cup u^*(T)$. The weights $\mathfrak{c}_j$, given by the Lemma, satisfy $(H1)-(H2)$, since they are just $\mathfrak{b}$ multiplied by a factor less than one, which depends only on v.

For any k sufficiently large, there exists $j = j(k)$ such that

$$u\Big(\Big\{x \in [-1,1] : dist(x, T) \leqslant \frac{4}{k}\Big\}\Big) \subset \Big\{v \in \mathbb{R}_+ : dist(v, u(T)) \leqslant \frac{1}{2j}\Big\},$$

and $j(k) \to \infty$ as $k \to \infty$ by continuity of u. This implies that $\min\limits_{dist(x, u^{-1}(v)) \leqslant \frac{2}{k}} c_j(x, v) > 0$ for all $v \in U(c_j)$. Moreover, for $v \in U(c_j)$ we have

$$\frac{\max\limits_{|x_i - x_{i+1}| \leqslant \frac{2}{k}} |\mathfrak{c}_j(x_i, v) - \mathfrak{c}_j(x_{i+1}, v)|}{\min\limits_{\operatorname{dist}(x, u^{-1}(v)) \leqslant \frac{2}{k}} \mathfrak{c}_j(x, v)} = \frac{\max\limits_{|x_i - x_{i+1}| \leqslant \frac{2}{k}} |\mathfrak{b}(x_i, v) - \mathfrak{b}(x_{i+1}, v)|}{\min\limits_{\operatorname{dist}(x, u^{-1}(v)) \leqslant \frac{2}{k}} \mathfrak{b}(x, v)}.$$

Note, that the denominator of the right-hand side is separated from zero for $v \in U(\mathfrak{c}_j)$. Thus, $D_k(\mathfrak{c}_j, U(\mathfrak{c}_j))$ is bounded.

Since D_k does not change if we multiply the first argument by a positive factor independent of x, and $U(\mathfrak{c}_j) \nearrow U(\mathfrak{b})$, we have

$$D_k(\mathfrak{c}_j, U(\mathfrak{c}_j)) = D_k(\mathfrak{b}, U(\mathfrak{c}_j)) \leqslant D_k(\mathfrak{b}, U(\mathfrak{b})) \to 0$$

as $k \to \infty$.

Thus, the weights $\mathfrak{c}_{j(k)}$ satisfy $(H1)-(H3)$. By Step 4, $I(\mathfrak{c}_{j(k)}, u^*) \leqslant I(\mathfrak{c}_{j(k)}, u)$. Passing to the limit, we get $I(\mathfrak{b}_\ell, u^*) \leqslant I(\mathfrak{b}_\ell, u)$, and consequently inequality (2).

Thus, Theorem 3 is proved. □

Now we consider the case where the function u satisfies the additional condition $u(-1) = 0$.

Theorem 4. *Suppose that $F \in \mathfrak{F}$, the function $u \in W_1^1(-1,1)$ is nonnegative, $u(-1) = 0$, and the weight function $\mathfrak{a} : [-1,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies* (5)*. Then inequality* (2) *holds.*

Proof. We follow the proof of Theorem 3, but we change $(H1)$ and $(H7)$ to the following conditions:

$(H1')$ $\mathfrak{a}(x,v)$ satisfies (5), and $I(\mathfrak{a}, u) < \infty$.

$(H7')$ Assumption $(H7)$ is satisfied and $\mathfrak{a}(\cdot, v) \equiv 0$ in some v-neighborhood of zero.

Step 1. *Let $u \in W_1^1(-1,1)$, $u(-1) = 0$ and let the weight $\mathfrak{a}$ satisfy the conditions $(H1'), (H2)-(H6), (H7')$. Then inequality (2) holds.*

To prove this, we approximate the function u in the same way as in the first step in the proof of Theorem 3, changing u in a neighborhood of $x = -1$ to a linear function with $u_n(-1) = 0$ preserved.

Step 2. *Let the weight $\mathfrak{a}$ satisfy conditions $(H1'), (H2)-(H6)$. Then inequality (2) holds.*

To prove this, we add zero to the set W from the second step in the proof of Theorem 3, and repeat the rest of the proof.

Further steps are unchanged. □

8 Appendix. The case of symmetric rearrangement

8.1 Necessary conditions for the weight

Lemma 9. *If inequality* (3) *holds for all $F \in \mathfrak{F}$ and all piecewise linear u, then the weight $\mathfrak{a}$ satisfies*

$$\forall s,t \in [-1,1], \forall v \in \mathbb{R}_+ \quad \mathfrak{a}(s,v) + \mathfrak{a}(t,v) \geqslant \mathfrak{a}\Big(\frac{s-t}{2}, v\Big) + \mathfrak{a}\Big(\frac{t-s}{2}, v\Big). \tag{12}$$

Proof. Assume that inequality (12) is not satisfied. Then there are $-1 \leqslant s < t \leqslant 1$, $\varepsilon, \delta > 0$ $(2\varepsilon < t - s)$ and $\bar{v} \in \mathbb{R}_+$, such that for any $0 \leqslant z \leqslant \varepsilon$ and any $\bar{v} \leqslant v \leqslant \bar{v} + \varepsilon$ the following holds:

$$\mathfrak{a}(s+z, v+z) + \mathfrak{a}(t-z, v+z) + 2\delta < \mathfrak{a}\Big(\frac{s-t}{2} + z, v+z\Big) + \mathfrak{a}\Big(\frac{t-s}{2} - z, v+z\Big). \quad (13)$$

Consider the function u defined in (6). We have

$$\begin{cases} \bar{u}(x) = \bar{v}, & x \in [-1, \frac{s-t}{2}] \cup [\frac{t-s}{2}, 1] \\ \bar{u}(x) = \bar{v} + x - \frac{s-t}{2}, & x \in [\frac{s-t}{2}, \frac{s-t}{2} + \varepsilon] \\ \bar{u}(x) = \bar{v} + \varepsilon, & x \in [\frac{s-t}{2} + \varepsilon, \frac{t-s}{2} - \varepsilon] \\ \bar{u}(x) = \bar{v} + \frac{t-s}{2} - x, & x \in [\frac{t-s}{2} - \varepsilon, \frac{t-s}{2}]. \end{cases}$$

Hence we obtain

$$\begin{aligned} 0 &\leqslant I(\mathfrak{a}, u) - I(\mathfrak{a}, \overline{u}) \\ &= \int_0^\varepsilon F\big(u(s+z), \frac{\mathfrak{a}(s+z, u(s+z))}{\varepsilon}\big)dz + \int_0^\varepsilon F\big(u(t-z), \frac{\mathfrak{a}(t-z, u(t-z))}{\varepsilon}\big)dz \\ &\quad - \int_0^\varepsilon F\big(\bar{u}(\frac{s-t}{2} + z), \frac{\mathfrak{a}(\frac{s-t}{2} + z, \bar{u}(\frac{s-t}{2} + z))}{\varepsilon}\big)dz \\ &\quad - \int_0^\varepsilon F\big(\bar{u}(\frac{t-s}{2} - z), \frac{\mathfrak{a}(\frac{t-s}{2} - z, \bar{u}(\frac{t-s}{2} - z))}{\varepsilon}\big)dz =: J. \end{aligned}$$

Take $F(v,p) := f(p) := p + \gamma p^2$, where $\gamma > 0$. Then

$$\begin{aligned} J = \int_0^\varepsilon \big(f(\frac{\mathfrak{a}(s+z, \bar{v}+z)}{\varepsilon}) + f(\frac{\mathfrak{a}(t-z, \bar{v}+z)}{\varepsilon}) \\ - f(\frac{\mathfrak{a}(\frac{s-t}{2} + z, \bar{v}+z)}{\varepsilon}) - f(\frac{\mathfrak{a}(\frac{t-s}{2} - z, \bar{v}+z)}{\varepsilon})\big)dz. \end{aligned}$$

We define A by relation (8). If we take $\gamma := \frac{\delta/\varepsilon}{(A/\varepsilon)^2} > 0$, then for $p \leqslant \frac{A}{\varepsilon}$ we have $p \leqslant f(p) \leqslant p + \frac{\delta}{\varepsilon}$, and

$$J \leqslant \frac{1}{\varepsilon} \int_0^\varepsilon \big(\mathfrak{a}(s+z, \bar{v}+z) + \mathfrak{a}(t-z, \bar{v}+z) + 2\delta - \mathfrak{a}(\frac{s-t}{2} + z, \bar{v}+z) - \mathfrak{a}(\frac{t-s}{2} - z, \bar{v}+z)\big)dz < 0$$

(the last inequality follows from (13)).

Thus, we get a contradiction, hence (12) holds. $\square$

Lemma 10. *Let relation* (12) *hold for a function* $\mathfrak{a} \in C([-1,1] \times \mathbb{R}_+)$. *Then* $\mathfrak{a}$ *is even and convex with respect to the first argument.*

Proof. Assume first that $\mathfrak{a}(\cdot, v) \in C^1([-1,1])$ for each v. We fix arbitrary $s \in [-1,1]$ and $v \in \mathbb{R}_+$ and consider the function

$$b(x) := \mathfrak{a}(s,v) + \mathfrak{a}(x,v) - \mathfrak{a}(\frac{s-x}{2}, v) - \mathfrak{a}(\frac{x-s}{2}, v) \geqslant 0.$$

$x = -s$ is the minimum point of b, since $b(-s) = 0$. Hence,

$$b'(-s) = \mathfrak{a}'_x(-s,v) + \frac{1}{2}\mathfrak{a}'_x(s,v) - \frac{1}{2}\mathfrak{a}'_x(-s,v) = 0,$$

that is $\mathfrak{a}'_x(s,v) = -\mathfrak{a}'_x(-s,v)$. Thus, the function $\mathfrak{a}(\cdot, v)$ is even.

Now consider the case of a continuous $\mathfrak{a}$.

Define $\mathfrak{a}(x,v) := \mathfrak{a}(-1,v)$ for $x < -1$ and $\mathfrak{a}(x,v) := \mathfrak{a}(1,v)$ for $x > 1$. Consider the mollification of the function:

$$\mathfrak{a}_\rho(x,v) = \int_{\mathbb{R}} \omega_\rho(z)\mathfrak{a}(x-z,v)dz = \int_{\mathbb{R}} \omega_\rho(z)\mathfrak{a}(x+z,v)dz,$$

where $\omega_\rho(z)$ is a smoothing kernel with radius ρ. Then we have for $-1+\rho \leqslant s,t \leqslant 1-\rho$

$$\mathfrak{a}_\rho(s,v) + \mathfrak{a}_\rho(t,v) - \mathfrak{a}_\rho(\frac{s-t}{2}, v) - \mathfrak{a}_\rho(\frac{t-s}{2}, v) =$$
$$\int_{\mathbb{R}} \omega_\rho(z)\big(\mathfrak{a}(s-z,v) + \mathfrak{a}(t+z,v) - \mathfrak{a}(\frac{s-t}{2} - z, v) - \mathfrak{a}(\frac{t-s}{2} + z, v)\big)dz \geqslant 0.$$

So $\mathfrak{a}_\rho(\cdot, v)$ is even on $[-1+\rho, 1-\rho]$. Pushing $\rho \to 0$, we obtain that $\mathfrak{a}(\cdot, v)$ is even.

Finally, for any s, t and v, we have

$$\mathfrak{a}(s,v) + \mathfrak{a}(t,v) = \mathfrak{a}(s,v) + \mathfrak{a}(-t,v) \geqslant 2\mathfrak{a}\big(\frac{s+t}{2}, v\big).$$

□

8.2 The proof of inequality (3)

Theorem 5. *Suppose that $F \in \mathfrak{F}$, the function $u \in W_1^1(-1,1)$ is nonnegative, and the continuous weight function $\mathfrak{a} : [-1,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is even and convex with respect to the first argument. Then inequality* (3) *holds.*

Proof. As we mentioned in the introduction, the statement is proved for Lipschitz functions u in paper [2]. Thus, we need only to extend it to W_1^1-functions.

The case of convex weight is much simpler than the case considered in Section 7. Namely, the function $\mathfrak{a}$ decreases for $x < 0$ and increases for $x > 0$ regardless of v. Thus, assumption $(H6)$ of Theorem 3 is satisfied. To fulfil assumption $(H7)$, we apply Lemma 8 with $W = \{u(0)\}$. Then we can use immediately Step 1 of the proof of Theorem 3. This gives us (3). Since Step 1 uses assumptions $(H1)$, $(H6)$, $(H7)$ only, we do not need to check $(H2) - (H5)$. □

We are grateful to Professor V.G. Osmolovskii for valuable comments, which helped to improve the text of the paper.

Authors were supported by RFBR grant 12-01-00439. The second author was also supported by St. Petersburg University grant 6.38.670.2013.

References

[1] G. Alberti, F. Serra Cassano: Non-occurrence of gap for one-dimentional autonomous functionals, Proceedings of “Calc. Var., Homogen. and Cont. Mech.”, G. Bouchitté, G. Buttazzo, P. Suquet, ed.: World Sci., Singapore, p. 1–17, 1994

[2] F. Brock: Weighted Dirichlet-type inequalities for Steiner symmetrization, Calc. Var. and PDEs **8**, p. 15–25, 1999

[3] B. Kawohl: Rearrangements and convexity of level sets in PDE, Lecture notes in mathematics **1150**. Berlin; Springer Verlag, 1985. 134 pp.

[4] R. Landes: Some remarks on rearrangements and functionals with non-constant density, Math. Nachr. **280**, N5–6, p. 560–570, 2007

[5] S. Bankevich, A. Nazarov: A generalization of the Pólya–Szegö inequality for one-dimensional functionals, Doklady RAN **438**, N1, p. 11–13, 2011 (in Russian). English translation in: Doklady Mathematics **83**, N3, p. 287–289, 2011

[6] G. Buttazzo, M. Giaquinta, S. Hildebrandt: One-Dimentional Variational Problems. An Introduction, Oxford Lecture Series in Mathematics and Its Applications **15**, Oxford University Press, New York, 1998. viii+262 pp.

[7] V. Zhikov: Weighted Sobolev spaces, Mat. Sb. **189**, N8, p. 27–58, 1998 (in Russian). English translation in: Sb. Math. **189**, N8, p. 1139–1170, 1998

[8] V. Zhikov: To the Problem of Passage to the Limit in Divergent Nonuniformly Elliptic Equations, Funkts. Anal. Prilozh. **35**, N1, p. 23–39, 2001 (in Russian). English translation in: Funct. Anal. Appl. **35**, N1, p. 19–33, 2001

[9] E. Lieb, M. Loss: Analysis, second edition, American Mathematical Soc., 2001. 346 pp.

[10] W. Rudin: Functional analysis, McGraw-Hill, 1991. 424 pp.

[11] L.C. Evans, R.F. Gariepy: Measure Theory and Fine Properties of Functions, CRC Press, 1991. 288 pp.