12.24. $(sh2x)/x-2.$

$x \to 2x,$

$$sh(2x) = 2x + \frac{2^3 \cdot x^3}{3!} + \frac{2^5 \cdot x^5}{5!} + \dots + \frac{2^{2n-1} x^{2n-1}}{(2n-1)!} + \dots$$

$$\frac{sh(2x)}{x} - 2 = \cancel{-2} + \cancel{2} + \frac{2^3 x^2}{3!} + \frac{2^5 x^4}{5!} + \dots + \frac{2^{2n-1} x^{2n-2}}{(2n-1)!} + \dots$$

$$\frac{sh(2x)}{x} - 2 = \sum_{2}^{\infty} \frac{2^{2n-1} x^{2n-2}}{(2n-1)!}$$

13.24. $\int\limits_0^{0,5} ln\left(1+x^2\right)dx.$

$$\int_0^{\frac{1}{2}} \ln(1+x^2)\,dx$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\int_0^{\frac{1}{2}} \ln(1+x^2)\,dx = \int_0^{\frac{1}{2}} \left[x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \dots\right] dx = \frac{x^3}{2} - \frac{x^4}{2\cdot 3} + \frac{x^5}{4\cdot 3} - \frac{x^6}{5\cdot 4} + \dots \Big|_0^{1/2} \approx$$

$$\approx \frac{\frac{1}{8}}{2} - \frac{\frac{1}{16}}{6} + \frac{\frac{1}{32}}{12} - \frac{\frac{1}{64}}{20} = \frac{1}{16} - \frac{1}{106} + \frac{1}{384} - \frac{1}{1280} \approx 0,0549$$

14.24.

$$f(x)=\begin{cases}2x-1, & -\pi\le x\le 0,\\ 0, & 0<x\le\pi.\end{cases}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(2x-1)dx+\frac{1}{\pi}\int_{0}^{\pi}0dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}2xdx-\int_{-\pi}^{0}dx\right)=$$

$$=\frac{1}{\pi}\left(x^2\Big|_{-\pi}^{0}-x\Big|_{-\pi}^{0}\right)=\frac{\pi^2}{\pi}-\frac{-\pi}{\pi}=\pi+1$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}(2x-1)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}0\cos nx\,dx=$$

$$=\frac{2}{\pi}\int_{-\pi}^{0}x\cos nx\,dx-\frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx=$$

Используем интегрирование по частям для:

$$\int_{-\pi}^{0}x\cos nx\,dx;\quad \int_{-\pi}^{0}u\,dv=uv-\int_{-\pi}^{0}v\,du;\quad \left[\begin{array}{l|l}u=x & dV=\cos nx\,dx\\ du=dx & V=\int\cos nx\,dx=\frac{1}{n}\sin nx\end{array}\right]$$

$$=\frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_{-\pi}^{0}-\frac{1}{n}\int_{-\pi}^{0}\sin(nx)dx\right)-\frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx=$$

$$=\frac{2}{\pi}\left(0+\frac{\pi}{n}\underbrace{\sin(-n\pi)}_{0}+\frac{1}{n^2}\cos(nx)dx\Big|_{-\pi}^{0}\right)-\frac{1}{\pi n}\left(\sin nx\right)\Big|_{-\pi}^{0}=$$

$$=\frac{2}{\pi n^2}\left(\cos 0-\cos(-\pi n)\right)-\frac{1}{\pi n}\left(\underbrace{\sin 0}_{0}-\underbrace{\sin(-\pi n)}_{0}\right)=\frac{2}{\pi n^2}\left(1-(-1)^n\right)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}(2x-1)\sin nx\,dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}2x\sin nx\,dx-\int_{-\pi}^{0}\sin nx\,dx\right)=$$

Используем интегрирование по частям для:

$$\int_{-\pi}^{0}2x\sin nx\,dx;\quad \int_{-\pi}^{0}u\,dv=uv-\int_{-\pi}^{0}v\,du;\quad \left[\begin{array}{l|l}u=2x & dV=\sin nx\,dx\\ du=2dx & V=\int\sin nx\,dx=-\frac{1}{n}\cos nx\end{array}\right]$$

$$=\frac{1}{\pi}\left(-x\frac{2}{n}\cos nx\Big|_{-\pi}^{0}+\frac{2}{n}\int_{-\pi}^{0}\cos nx\,dx-\int_{-\pi}^{0}\sin nx\,dx\right)=$$

$$=\frac{1}{\pi}\left(0+\frac{2\pi}{n}\cos n\pi+\frac{2}{n^2}\sin nx\,dx\Big|_{-\pi}^{0}+\frac{1}{n}\cos nx\,dx\Big|_{-\pi}^{0}\right)=$$

$$=\frac{1}{\pi}\left(\frac{2\pi(-1)^n}{n}+\frac{2}{n^2}\cdot 0-\frac{2}{n^2}\cdot 0+\frac{1-(-1)^n}{n}\right)=\frac{2(-1)^n}{n}+\frac{1-(-1)^n}{\pi n}=\frac{2\pi(-1)^n+1-(-1)^n}{\pi n}=$$

$$=\frac{(-1)^n(2\pi-1)+1}{\pi n}$$

$$a_0=\pi+1$$

$$a_n=\frac{2}{\pi n^2}\left(1-(-1)^n\right)$$

$$b_n=\frac{(-1)^n(2\pi-1)+1}{\pi n}$$

Ответ $f(x)=\frac{\pi+1}{2}+\sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}\left(1-(-1)^n\right)\cos nx+\frac{(-1)^n(2\pi-1)+1}{\pi n}\cdot\sin nx\right)$

15.24. $f(x)=2^x.$

Предположим нечетным образом:

Если функция нечетна, то a0 = 0, an = 0, а bn:

$$b_n=\frac{2}{\pi}\int_0^{\pi}2^x\sin nx\,dx=$$

Воспользуемся методом интегрирования по частям:

$$\left[\begin{array}{l|l} u=2^x & dV=\sin nx\,dx \\ du=2^x\ln 2\,dx & V=-\frac{1}{n}\cos nx \end{array}\right]$$

$$=\frac{2}{\pi}\left(-\frac{2^x}{n}\cos nx\Big|_0^{\pi}+\frac{\ln 2}{n}\int_0^{\pi}2^x\cos nx\,dx\right)=$$

Повторно воспользуемся методом интегрирования по частям:

$$\left[\begin{array}{l|l} u=2^x & dV=\cos nx\,dx \\ du=2^x\ln 2\,dx & V=\frac{1}{n}\sin nx \end{array}\right]$$

$$=-\frac{2^x}{n}\cos nx\Big|_0^{\pi}+\frac{\ln 2}{n}\left(\frac{2^x}{n}\sin nx\Big|_0^{\pi}-\frac{\ln 2}{n}\int_0^{\pi}2^x\sin nx\right)=$$

$$=-\frac{2^x}{n}\cos nx\Big|_0^{\pi}+\frac{2^x\ln 2}{n^2}\sin nx\Big|_0^{\pi}-\frac{(\ln 2)^2}{n^2}\int_0^{\pi}2^x\sin nx$$

Отсюда выразим исходную функцию:

$$1\int_{-\pi}^{0}2^x\sin nx=-\frac{2^x}{n}\cos nx\Big|_0^{\pi}+\frac{2^x\ln 2}{n^2}\sin nx\Big|_0^{\pi}-\frac{(\ln 2)^2}{n^2}\int_{-\pi}^{0}2^x\sin nx$$

$$\frac{(\ln 2)^2}{n^2}\int_{-\pi}^{0}2^x\sin nx+1\int_{-\pi}^{0}2^x\sin nx=\frac{2^x\ln 2}{n^2}\sin nx\Big|_0^{\pi}-\frac{2^x}{n}\cos nx\Big|_0^{\pi}$$

$$\left(\frac{(\ln 2)^2}{n^2}+1\right)\int_{-\pi}^{0}2^x\sin nx=\frac{2^x\ln 2}{n^2}\sin nx\Big|_0^{\pi}-\frac{2^x}{n}\cos nx\Big|_0^{\pi}$$

$$\int_{-\pi}^{0}2^x\sin nx=\frac{\cancel{\frac{2^{\pi}\ln 2}{n^2}\sin n\pi}^{\,0}-\cancel{\frac{2^0\ln 2}{n^2}\sin n0}^{\,0}-\frac{2^{\pi}}{n}\cos n\pi+\frac{2^0}{n}\cos n0}{\frac{(\ln 2)^2}{n^2}+1}$$

$$\int_{-\pi}^{0}2^x\sin nx=\frac{\frac{1}{n}-\frac{2^{\pi}}{n}(-1)^n}{\frac{(\ln 2)^2}{n^2}+1}=\frac{\frac{n^2}{n}-\frac{n^2 2^{\pi}(-1)^n}{n}}{(\ln 2)^2+n^2}=\frac{n-n\cdot 2^{\pi}(-1)^n}{(\ln 2)^2+n^2}=\frac{n(1-2^{\pi}(-1)^n)}{(\ln 2)^2+n^2}$$

$$b_n=\frac{n(1-2^{\pi}(-1)^n)}{(\ln 2)^2+n^2}$$

Тогда ряд Фурье:

$$f(x)=\sum_{n=1}^{\infty}\frac{n(1-2^{\pi}(-1)^n)}{(\ln 2)^2+n^2}\cdot\sin nx$$

Предположим четным образом:

Если функция четная, то bn = 0, а a0 и an:

$$a_0=\frac{2}{\pi}\int_0^{\pi}2^x\,dx=\frac{2}{\pi}\frac{2^x}{\ln 2}\Big|_0^{\pi}=\frac{2^{\pi}-1}{\ln 2}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}2^x\cos nx\,dx=$$

Воспользуемся методом интегрирования по частям:

$$\left[\begin{array}{l|l} u=2^x & dV=\cos nx\,dx \\ du=2^x\ln 2\,dx & V=\frac{1}{n}\sin nx \end{array}\right]$$

$$=\frac{2^x}{n}\sin nx\Big|_0^{\pi}-\frac{\ln 2}{n}\int_0^{\pi}2^x\sin nx=$$

Воспользуемся методом интегрирования по частям:

$$\left[\begin{array}{l|l} u=2^x & dV=\sin nx\,dx \\ du=2^x\ln 2\,dx & V=-\frac{1}{n}\cos nx \end{array}\right]$$

$$=\frac{2^x}{n}\sin nx\Big|_0^{\pi}-\frac{\ln 2}{n}\left(-\frac{2^x}{n}\cos nx\Big|_0^{\pi}+\frac{\ln 2}{n}\int_0^{\pi}2^x\cos nx\,dx\right)=$$

$$=\frac{2^x}{n}\sin nx\Big|_0^{\pi}+\frac{\ln 2\cdot 2^x}{n^2}\cos nx\Big|_0^{\pi}-\frac{(\ln 2)^2}{n^2}\int_0^{\pi}2^x\cos nx\,dx$$

Отсюда выразим исходную функцию:

$$\int_0^{\pi}2^x\cos nx\,dx=\frac{2^x}{n}\sin nx\Big|_0^{\pi}+\frac{\ln 2\cdot 2^x}{n^2}\cos nx\Big|_0^{\pi}-\frac{(\ln 2)^2}{n^2}\int_0^{\pi}2^x\cos nx\,dx$$

$$\left(1+\frac{(\ln 2)^2}{n^2}\right)\int_0^{\pi}2^x\cos nx\,dx=\cancel{\frac{2^{\pi}}{n}\sin\pi n}^{\,0}-\cancel{\frac{2^0}{n}\sin 0}^{\,0}+\frac{\ln(2)2^{\pi}}{n^2}\cos\pi n-\frac{\ln(2)2^0}{n^2}\cos 0$$

$$\int_0^{\pi}2^x\cos nx\,dx=\frac{\frac{\cancel{\ln(2)}(2^{\pi}\cdot(-1)^n-\ln 2)}{\cancel{n^2}}}{\frac{n^2+(\ln 2)^2}{\cancel{n^2}}}=\frac{2^{\pi}\cdot(-1)^n-\ln 2}{n^2+\ln 2}$$

$$a_0=\frac{2^{\pi}-1}{\ln 2}\qquad a_n=\frac{2^{\pi}\cdot(-1)^n-\ln 2}{n^2+\ln 2}$$

$$f(x)=\frac{2(2^{\pi}-1)}{\ln 2}+\sum_{n=1}^{\infty}\frac{2^{\pi}\cdot(-1)^n-\ln 2}{n^2+\ln 2}\cos nx$$

Ответ:

Если нечетным образом:

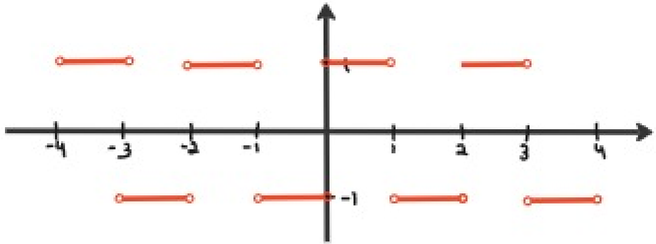
$$f(x)=\sum_{n=1}^{\infty}\frac{n(1-2^{\pi}(-1)^n)}{(\ln 2)^2+n^2}\cdot\sin nx$$

Если четным образом:

$$f(x)=\frac{2(2^{\pi}-1)}{\ln 2}+\sum_{n=1}^{\infty}\frac{2^{\pi}\cdot(-1)^n-\ln 2}{n^2+\ln 2}\cos nx$$

16.24.

$$f(x)=\begin{cases}1, & 0<x<1,\\ -1, & 1<x<2,\ l=1.\end{cases}$$



$$f(x)=\begin{cases}-1, & 1<x<2 \\ 1, & 0<x<1\end{cases} \qquad x\in(0;2)$$

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{2}\left(\int_0^1 dx+\int_1^2 -dx\right)=x\Big|_0^1-x\Big|_1^2=1-(2-1)=0$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{\pi n x}{l}\,dx=\frac{2}{1}\left(\int_0^1 \cos\frac{\pi n x}{2}-\int_1^2\cos\frac{\pi n x}{2}\right)=$$

$$=\frac{4}{\pi n}\sin\frac{\pi n x}{2}\Big|_0^1-\frac{4}{\pi n}\sin\frac{\pi n x}{2}\Big|_1^2=\frac{4}{\pi n}\left(0-\cancel{\sin\frac{\pi n}{2}}-\underbrace{\sin\frac{2\pi n}{2}}_{0}+\cancel{\sin\frac{\pi n}{2}}\right)=0$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{\pi n x}{l}\,dx=\frac{2}{1}\left(\int_0^1 \sin\pi n x\,dx+\int_1^2\sin\pi n x\,dx\right)=$$

$$=-2\left(\cos\pi n x\Big|_0^1+\cos\pi n x\Big|_1^2\right)=-2\left(\cancel{\cos\pi n}-\cancel{\cos 0}+\cancel{\cos 2\pi n}-\cancel{\cos\pi n}\right)=0$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)=0$$