Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Complex Analysis **MATH50001**

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Student Answers to Coursework 1

1. (a) To describe Ω we first recall $\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$. Since $|z| < 1$ we have $\ln|z| < 0$ and since $\text{Arg}(z) \in (-\pi/2, \pi/2)$. The logarithm $\ln|z|$ can take any value in $(-\infty, 0)$, therefore we're looking at numbers of the form $x + iy$ where $x < 0$ and $y \in (-\pi/2, \pi/2)$.

$$f(\Omega) = \{z \in \mathbb{C} : z = x + iy, x < 0, y \in (-\pi/2, \pi/2)\}$$

   (b) Similarly, we have $|z| > 1$ so $\ln|z| > 0$ and $\text{Arg}(z) \in [0, \pi]$. The logarithm $\ln|z|$ can take any value in $(0, \infty)$, therefore we're looking at numbers of the form $x + iy$ where $x > 0$ and $y \in [0, \pi]$.

$$f(\Omega) = \{z \in \mathbb{C} : z = x + iy, x > 0, y \in [0, \pi]\}$$

   (c) We're asked to find the branch cut of the function $f(z)$

2. (a) If $|\sin(z)| \leq 1$ and $z = x + iy$ and by definition of $|\sin(z)|$ we have

$$|\sin\ z| = \left|\frac{1}{2i}(e^{iz} - e^{-iz})\right| \leq 1$$

Now, we can write $e^{iz} - e^{-iz}$ as $e^{i(x+iy)} - e^{-i(x+iy)} = e^{ix-y} - e^{-ix+y} = e^{ix}e^{-y} - e^{-ix}e^{y}$ And using Euler's formula

$$e^{ix} = \cos(x) + i\sin(x) \quad \text{and} \quad e^{-ix} = \cos(x) - i\sin(x)$$

Therefore, we have

$$|\sin\ z| = \left|\frac{1}{2i}((\cos(x) + i\sin(x))e^{-y} - (\cos(x) - i\sin(x))e^{y})\right|$$

$$|\sin\ z| = \left|\frac{1}{2i}(\cos(x)e^{-y} + i\sin(x)e^{-y} - \cos(x)e^{y} + i\sin(x)e^{y})\right|$$

Since $1/i = -i$ this is the same as

$$|\sin\ z| = \left|\frac{1}{2}(i\cos(x)(e^{y} - e^{-y}) + \sin(x)(e^{-y} + e^{y}))\right|$$

And finally

$$|\sin\ z| = |(\sin(x)\cosh(y) + i\cos(x)\sinh(y))| = \sqrt{\sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y)} \leq 1$$

We can simplify this if we square both sides as the RHS is invariant. Now, we can use

$$\begin{aligned}(\sin(x)\cosh(y))^2 + (\cos(x)\sinh(y))^2 &= \sin^2(x)(1 + \sinh^2(y)) + \cos^2(x)\ \sinh^2(y) \\ &= \sin^2(x) + (\sin^2(x) + \cos^2(x))\sinh^2(y) \\ &= \sin^2(x) + \sinh^2(y)\end{aligned}$$

Therefore, we are looking at the complex numbers $z = x + iy$ such that

$$\sin^2(x) + \sinh^2(y) \leq 1 \tag{1}$$

The region described by this inequality can be visualized in Figure 1.

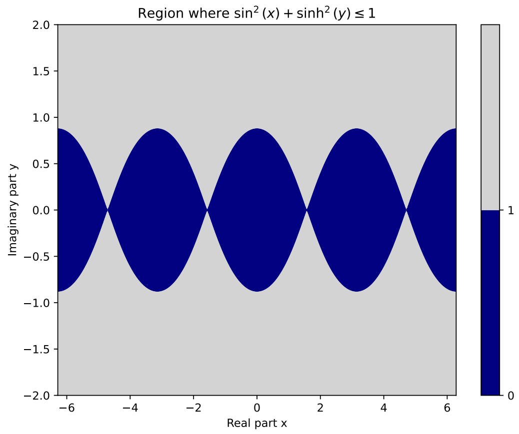

Figure 1: The region described by the inequality $\sin^2(x) + \sinh^2(y) \leq 1$

(b) Let's find Taylor series for $f = \frac{1}{(z+i)(z-2)}$. First, recall

$$f_{z_0}(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{2}$$

We're asked to set $z_0 = 1$. Notice that since $(z + i)(z - 2)$ is entire, 1 is entire and the denominator is not 0 at $z_0 = 1$ and thus f is holomorphic at $z_0 = 1$.
We can already see that the derivatives will be too complicated to compute. Let's try to simplify the function first. We can use partial fractions to write f as a sum of two simpler functions:

$$f(z) = \frac{1}{(z + i)(z - 2)} = \frac{A}{z + i} + \frac{B}{z - 2} \tag{3}$$

Multiplying both sides by $(z + i)(z - 2)$ we get $1 = A(z - 2) + B(z + i)$. We can solve for A and B by setting $z = -i$ and $z = 2$:
And so, $1 = (-i - 2)A$ and $1 = (2 + i)B$. Solving we get $A = \frac{1}{-i-2}$ and $B = \frac{1}{i+2}$. Now, the simplification leads to:

$$B = \frac{1}{i + 2}\frac{i - 2}{i - 2} = \frac{i - 2}{-5} = \frac{2}{5} - \frac{i}{5}$$
$$A = \frac{1}{-i - 2}\frac{-i + 2}{-i + 2} = \frac{-i + 2}{-5} = \frac{-2}{5} + \frac{i}{5}$$

Let $\kappa = -\frac{2}{5} + \frac{i}{5}$. Then we have:

$$f(z) = \frac{1}{(z + i)(z - 2)} = \kappa\left(\frac{1}{z + i} - \frac{1}{z - 2}\right) \tag{4}$$

And now, calculating derivatives becomes much easier:

$$
\begin{aligned}
f'(z) &= \kappa\left(-\frac{1}{(z+i)^2}+\frac{1}{(z-2)^2}\right)\\
f''(z) &= \kappa\left(\frac{2}{(z+i)^3}-\frac{2}{(z-2)^3}\right)\\
f'''(z) &= \kappa\left(-\frac{6}{(z+i)^4}+\frac{6}{(z-2)^4}\right)\\
&\dots\\
f^{(n)}(z) &= (-1)^n\kappa n!\left(\frac{1}{(z+i)^{n+1}}-\frac{1}{(z-2)^{n+1}}\right)
\end{aligned}
$$

And, applying this to $z_0 = 1$, we get:

$$f^{(n)}(1) = \kappa n!\left(\frac{(-1)^n}{(1+i)^{n+1}}+1\right) \tag{5}$$

Now, plugging (5) into (2) we get:

$$f_{z_0}(z) = \kappa\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(1+i)^{n+1}}+1\right)(z-1)^n \tag{6}$$

Where $\kappa = -\frac{2}{5}+\frac{i}{5}$.

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4. Since f is entire, we can consider its MacLaurin series:

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n = a_0 + a_1 z + a_2 z^2 + \dots$$

Where a_i is the complex coefficient of z^i in the MacLaurin expansion. Assume $a_i \neq 0$ for $i \geq 2$, then $|f(z)| = a_0 + a_1 z + a_2 z^2 + \dots$, so the limit becomes

$$\lim_{|z|\to\infty}\frac{|f(z)|}{|z|^2} = \lim_{|z|\to\infty}\frac{|a_0 + a_1 z + a_2 z^2 + \dots|}{|z|^2} = \lim_{|z|\to\infty}\left|\frac{a_0 + a_1 z + a_2 z^2 + \dots}{z^2}\right|$$

Applying the division, we get:

$$\lim_{|z|\to\infty}\left|\frac{a_0 + a_1 z + a_2 z^2 + \dots}{z^2}\right| = \lim_{|z|\to\infty}\left|\frac{a_0}{z^2}+\frac{a_1}{z}+a_2+\dots\right| = 0$$

This is true if and only if the complex number tends to 0, so we have:

$$\lim_{|z|\to\infty}\frac{a_0}{z^2}+\frac{a_1}{z}+a_2+\dots = 0$$

And since the limit of the sum is the sum of the limits:

$$\lim_{|z|\to\infty}\frac{a_0}{z^2}+\lim_{|z|\to\infty}\frac{a_1}{z}+\lim_{|z|\to\infty}a_2+\dots = 0$$

And we know that $\lim_{|z|\to\infty}\left|\frac{a_0}{z^2}\right| = 0$ and $\lim_{|z|\to\infty}\left|\frac{a_1}{z}\right| = 0$, because a_0 and a_1 are constants. Once again, since the modulus is 0, the number must be 0, so the limit becomes:

$$\lim_{|z|\to\infty} a_2 + \lim_{|z|\to\infty} a_3 z + \cdots = a_2 + \lim_{|z|\to\infty} \sum_{n=3}^{\infty} a_n z^{n-2} = 0$$

So the limit of the sum is $-a_2$, like so:

$$\lim_{|z|\to\infty} \sum_{n=3}^{\infty} a_n z^{n-2} = -a_2$$

Taking the modulus of both sides, we get:

$$\lim_{|z|\to\infty} \left| \sum_{n=3}^{\infty} a_n z^{n-2} \right| = |-a_2| = |a_2|$$

Since the LHS doesn't have singularities, and the limit is finite, we can bound the sum by a constant M:

$$\left| \sum_{n=3}^{\infty} a_n z^{n-2} \right| \leq M$$

But, by Liouville's theorem $(*)$, if $s(z) = \sum_{n=3}^{\infty} a_n z^{n-2}$ is bounded, then it is constant. Since the limit is $-a_2$, then

$$s(z) = \sum_{n=3}^{\infty} a_n z^{n-2} = -a_2$$

But since $s(0) = 0$, then $-a_2 = 0$, so $a_2 = 0$, and $s(z) = \sum_{n=3}^{\infty} a_n z^{n-2} = 0$

Going back to the Maclaurin expansion of f, we have:

$$f(z) = a_0 + a_1 z + a_2 z^2 + z^2 \sum_{n=3}^{\infty} a_n z^{n-2} = a_0 + a_1 z$$

If we rename $a_0 = a$ and $a_1 = b$, we get the final result:

$$\begin{aligned} f(z) &= a + bz \\ &a, b \in \mathbb{C} \end{aligned}$$

$(*)$ Here we need to be careful as $s(z)$ was obtained by $\frac{f(z) - a_2 z^2 - a_1 z - a_0}{z^2}$, so we would really be talking about a function which takes this fraction in $\mathbb{C} - \{0\}$ and 0 if $z = 0$, and this function is holomorphic at $\mathbb{C} - \{0\}$, but we need to check that it is also holomorphic at 0 to make sure that we can apply the theorem. Following the actual piecewise definition of $s(z)$

$$s(z) = \begin{cases} \frac{f(z) - a_2 z^2 - a_1 z - a_0}{z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

To check whether $s(z)$ is holomorphic at 0, we need to check if

$$\lim_{h\to 0} \frac{s(h) - s(0)}{h - 0} = \lim_{h\to 0} \frac{s(h)}{h}$$

converges. Note $f(h) - a_2h^2 - a_1h - a_0 = h^2 \sum_{n=3}^{\infty} a_n h^{n-2}$, so

$$\lim_{h\to 0} \frac{s(h)}{h} = \lim_{h\to 0} \frac{h^2 \sum_{n=3}^{\infty} a_n h^{n-2}}{h^2} = \lim_{h\to 0} \sum_{n=3}^{\infty} a_n h^{n-3} = a_3$$

So $s(z)$ is holomorphic at 0, and thus it is entire.