1. **[13 marks]**

   (a) **Solution:**
   Any structure with size 3 that satisfies Φ but doesn't satisfy the definition of *under* in the solution of part (b) is acceptable.
   Here's an example:

   $$M = \{a,b,c\} \quad Above^{\mathscr{M}} = \{\langle a,b\rangle\} \quad Under^{\mathscr{M}} = \{\langle a,b\rangle\} \quad Clear^{\mathscr{M}} = \{b,c\} \quad OnTable^{\mathscr{M}} = \{a,c\}$$

   (b) **Solution:**

   $$\forall x \forall y (Under(y,x) \leftrightarrow ((ontable(x) \wedge (x = y)) \vee ((Above(x,y) \wedge \neg \exists z (Above(z,y) \wedge Above(x,z))))$$

   $$\forall x (OnTable(x) \leftrightarrow (\neg \exists y (Above(x,y))))$$

   $$\forall x (Clear(x) \leftrightarrow (\neg \exists y (Above(y,x))))$$

   Note: There are other sentences about *Above* that are needed in order to capture the English description of the predicates. But students are not required to provide those sentences.
   Anyone who does provide them could perhaps receive a bonus mark.

2. **[10 marks]** Φ is not satisfiable.
   Consider an arbitrary structure $\mathscr{M}$ that satisfies sentences (1) to (4).
   Then, according to sentence (4), for every element a in the domain of $\mathscr{M}$, we have

   $$\langle a,a,a\rangle \in between^{\mathscr{M}}.$$

   But then sentence (5) is not satisfied (let $x = y = z = a$).

   If we remove either sentence (4) or sentence (5) from Φ, the resulting sets will be satisfiable.

3. **[15 marks]**

   (a) **[4] Solution:**
   For the joint probability we have $P(C_i,H) = P(H|C_i)P(C_i)$, where $P(C+i) = \frac{1}{5}$ because we sample the coin uniformly at random. Thus $P(C_i,H) = \frac{p_i}{5}$, yielding the following joint probabilities:

$P(Ci = 1, H = 1)$	0
$P(Ci = 1, H = 0)$	$\frac{1}{5}$
$P(Ci = 2, H = 1)$	$\frac{1}{20}$
$P(Ci = 2, H = 0)$	$\frac{3}{20}$
$P(Ci = 3, H = 1)$	$\frac{1}{10}$
$P(Ci = 3, H = 0)$	$\frac{1}{10}$
$P(Ci = 4, H = 1)$	$\frac{3}{20}$
$P(Ci = 4, H = 0)$	$\frac{1}{20}$
$P(Ci = 5, H = 1)$	$\frac{1}{5}$
$P(Ci = 5, H = 0)$	0

NOTE: I suspect most students will provide the probability table, but there are several alternatives for expressing the joint distribution mathematically, and we will accept most of them as long as they respect the factorization $P(H|C_i)P(C_i)$.

(b) **[5] Solution:** We have

$$P(C_i) = \frac{P(C_i,H)}{P(H)} = \frac{P(C_i,H)}{\sum_j P(C_j,H)} = \frac{P(C_i)P(H|C_i)}{\sum_j P(C_j)P(H|C_j)}$$

Recall that coin selection is uniform so $P(C_i) = \frac{1}{5} \forall i \in \{1,\dots,5\}$. Meanwhile, the coin flip probabilities were given by $P(H|C_i) = p_i$. So the above expression simplifies as

$$P(C_i|H) = \frac{P(C_i)P(H|C_i)}{\sum_j P(C_j)P(H|C_j)} = \frac{\frac{1}{5}\cdot p_i}{\sum_j \frac{1}{5} * p_j} = \frac{\frac{1}{5}\cdot p_i}{\frac{1}{5}\sum_j p_j} = \frac{p_i}{\sum_j p_j} = \frac{p_i}{\frac{5}{2}} = \frac{2\cdot p_i}{5}.$$

Therefore,

$$P(C_1|H) = 0,\ P(C_2|H) = \frac{1}{10},\ P(C_3|H) = \frac{1}{5},\ P(C_3|H) = \frac{3}{10},\ P(C_5|H) = \frac{2}{5}.$$

(c) **[3] Solution:**

$$P(H_2|H_1) = \frac{P(H_2,H_1)}{H_1} = \frac{\sum_j P(C_j,H_1,H_2)}{\sum_{j'} P(C_{j'},H_1)} = \frac{\sum_j \frac{1}{5}\cdot p_j^2}{\sum_{j'} \frac{1}{5}\cdot p_{j'}} = \frac{\sum_j \cdot p_j^2}{\sum_{j'} \cdot p_{j'}}$$

$$= \frac{0^2 + (\frac{1}{4})^2 + (\frac{1}{2})^2 + (\frac{3}{4})^2 + 1^2}{0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1} = \frac{0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1}{\frac{5}{2}} = \frac{\frac{30}{16}}{\frac{5}{2}} = \frac{3}{4}$$

(d) **[3] Solution:** We have

$$P(C_i|B_4) = \frac{P(C_i,B_4)}{P(B_4)} = \frac{P(C_i,B_4)}{\sum_j P(C_j,B_4)},$$

and $P(C_i,B_4) = \frac{1}{5}(1-p_i)^3 p_i$ (this is the event of picking C_i uniformly at random, followed by three (conditionally independent) heads flip and one tails flip. Skipping over the details (TODO(): make sure the math works out), we have

$$P(C_1|B_4) = 0\ \ P(C_2|B_4) = \frac{27}{46}\ \ P(C_3|B_4) = \frac{8}{23}\ \ P(C_4|B_4) = \frac{3}{46}\ \ P(C_5|B_4) = 0$$

4. **[10 marks]**

(a) **[2] Solution:** For $P_{Bookstore}(C,D,G)$, which does not have any "compact" factorization, the total number of rows in the probability table is equal to the number of unique (C,D,G) in the sample space. Let $Rows(P)$ denote the number of rows in the probability table for P. Then we have

$$Rows(P_{Bookstore}(C,D,G)) = |Dom(G)||Dom(D)||Dom(C)| = 4 \cdot 3 \cdot 2 = \mathbf{24}.$$

(b) **[4] Solution:** The joint distribution for *Shopper 1* factorizes as follows:

$$P_{S1}(C,D,G) = P_{S1}(D)P_{S1}(C|D)P_{S1}(G|D).$$

To determine the overall number of rows, we first compute the number of rows in the (conditional) probability table of each factor:

$$Rows(P_{S1}(D)) = |Dom(D)| = 3;$$

$$Rows(P_{S1}(C|D))) = |Dom(C)||Dom(D)| = 2 \cdot 3 = 6;$$

$$Rows(P_{S1}(G|D))) = |Dom(G)||Dom(D)| = 4 \cdot 3 = 12.$$

So overall we have

$$Rows(P_{S1}(D))) + Rows(P_{S1}(C|D))) + Rows(P_{S1}(G|D))) = 3 + 6 + 12 = \mathbf{21}$$

(c) **[4] Solution:** The joint distribution for *Shopper 1* factorizes as follows:

$$P_{S2}(C,D,G) = P_{S2}(G,D)P_{S2}(C).$$

To determine the overall number of rows, we first compute the number of rows in the (conditional) probability table of each factor:

$$Rows(P_{S2}(G,D))) = |Dom(G)||Dom(D)| = 4 \cdot 3 = 12;$$

$$Rows(P_{S2}(C)) = |Dom(C)| = 2;$$

So overall we have

$$Rows(P_{S2}(G,D))) + Rows(P_{S2}(C))) = 12 + 2 = \mathbf{14}$$

GOOD LUCK!