(a) Denote the sentences in Φ as (1), (2), (3):

- $\forall x(\neg Above(x,x))$ (1)
- $\forall x \forall y \forall z((Above(x,y) \wedge Above(x,z) \wedge \neg(y=z)) \rightarrow (Above(z,y) \vee Above(y,z)))$ (2)
- $\forall x \forall y \forall z((Above(x,y) \wedge Above(y,z)) \rightarrow Above(x,z))$ (3)

(1): nothing is above itself

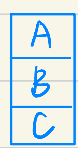
(2): if x is directly or indirectly above y and x is directly or indirectly above z and y is not the same as z, $y \neq z$, then z is directly or indirectly above y or y is directly or indirectly above z.

(3): if x is directly or indirectly above y and y is directly or indirectly above z, then x is directly or indirectly above z.

let model be:

$M = \{A, B, C\}$ (domain)

$Above^M = \{(A,B), (B,C), (A,C)\}$ $\Rightarrow$



$Under^M = \{(A, C)\}$

$Clear^M = \{A\}$

$OnTable^M = \{C\}$

M satisfies all sentences in Φ:

(1) $\because$ $\neg(Above(A,A))$ is true

$\neg(Above(B,B))$ is true

$\neg(Above(C,C))$ is true

$\therefore$ A, B, C are all not above itself, sentence 1 is satisfied.

(2) we only need to check if it's possible to have $Above(x,y) \wedge Above(x,z) \wedge \neg(y=z)$ be true while $Above(z,y) \vee Above(y,z)$ to be false.

$\because$ to make predicate true, x has to be A, y and z can either be B or C, in any case, $Above(B,C) \vee Above(C,B)$ is true.

$\therefore$ It's not possible to have a true predicate, false consequence in this case.

$\therefore$ sentence 2 is satisfied.

(3) we only need to check if it's possible to have: $Above(x,y) \wedge Above(y,z)$ is true, while $Above(x,z)$ is false.

$\because$ to make predicate true, x has to be A, y has to be B, z has to be C. In this case, $Above(A,B) \wedge Above(B,C)$ is true, and $Above(A,C)$ is also true

$\therefore$ It's not possible to have a true predicate, false consequence in this case

$\therefore$ sentence 2 is satisfied.

Based on $Above^M$, $Under^M(A,B)$ does not satisfy the English description of Under.

(b) Φ should be:

① $\forall x(\neg Above(x,x))$

② $\forall x\forall y\forall z(Above(x,y)\wedge Above(x,z)\wedge\neg(y=z))\rightarrow Above(z,y)$
$\vee\ Above(y,z)))$

③ $\forall x\forall y\forall z((Above(x,y)\wedge Above(y,z))\rightarrow Above(x,z))$

④ $\forall x\forall y[x\neq y\rightarrow(Under(y,x)\leftrightarrow(Above(x,y)\wedge\forall z(Above(z,y))\wedge x=z))]$

⑤ $\forall x(Under(x,x)\longleftrightarrow OnTable(x)\longleftrightarrow\neg\exists y(Above(x,y))$

⑥ $\forall x(Clear(x)\longleftrightarrow\neg\exists y(Above(y,x)))$

2. Φ is not satisfiable.

Justification:

let $M = \{A\}$, $between^M = \{(A, A, A)\}$

Assume M satisfies (1) – (4)

Show M doesn't satisfy (5)

For (1), $between(x, y, z)$ and $between(z, y, x)$ are both $between(A, A, A)$.

$between(A, A, A) \to between(A, A, A)$ is true $\to$ true, so (1) is true.

For (2), $between(x, y, z)$ and $between(y, x, z)$ are both $between(A, A, A)$. $between(x, y, z) \wedge between(y, x, z)$ is true, and $(x = y)$ is always $(A = A)$ which is also true.

$between(A, A, A) \wedge between(A, A, A) \to (A = A)$ is true.

For (3), between(y, x, z), between(y, x, w) and between(z, x, w) are between(A, A, A).
Therefore, between$(A, A, A) \rightarrow$ (between(A, A, A) $\vee$ between(A, A, A)) is true.

For (4), between(y, x, z), between(z, y, x), between(x, z, y) are all between(A, A, A)
Therefore, between(A, A, A) $\vee$ between(A, A, A) $\vee$ between(A, A, A) is true.

For (5), As between(x, y, z) and between(y, x, z) are both between(A, A, A), and between(A, A, A) is true. Therefore, between$(A, A, A) \rightarrow \neg$between$(A, A, A)$ is false.

$\therefore$ This structure satisfies (1)-(4), but doesn't satisfy (5).

Modification 1: Remove (5) from original Φ

New Φ:

$\forall x \forall y \forall z(between(x,y,z) \rightarrow between(z,y,x))$. (1)

$\forall x \forall y \forall z((between(x,y,z) \wedge between(y,x,z)) \rightarrow (x=y))$. (2)

$\forall x \forall y \forall z \forall w(between(y,x,z) \rightarrow (between(y,x,w) \vee between(z,x,w)))$. (3)

$\forall x \forall y \forall z(between(y,x,z) \vee between(z,y,x) \vee between(x,z,y))$. (4)

let $M = \{A\}$, $between^M = \{(A,A,A)\}$

This structure is the same as the previous example,

as shown before, this structure satisfies (1) - (4).

Hence, this set is satisfiable.

Modification 2: Remove (3) from original Φ, and modify (4).

New Φ:

$\forall x \forall y \forall z(between(x,y,z) \rightarrow between(z,y,x))$. (1)

$\forall x \forall y \forall z((between(x,y,z) \wedge between(y,x,z)) \rightarrow (x=y))$. (2)

$\forall x \forall y \forall z(((x \neq y) \vee (y \neq z) \vee (x \neq z)) \rightarrow between(y,x,z) \vee between(z,y,x) \vee between(x,z,y))$. (4)

$\forall x \forall y \forall z(between(x,y,z) \rightarrow \neg between(y,x,z))$. (5)

let $M = \{A, B\}$, $between^M = \{(A,B,A), (B,A,B)\}$

for (1): if $between(x,y,z)$ is $between(A,B,A)$, then between (z, y, x) is between (A, B, A), so (1) is true in this case;

if between(x, y, z) is between(B, A, B), then between (z, y, x) is between (B, A, B), so (1) is true in this case;

All other cases are trivially true as between(x, y, z) is false.

For (2): we only need to check if between$(x, y, z) \wedge$ between(y, x, z) is true while $x = z$ is false.
However, only between(A, B, A) and between(B, A, B) are true, either case, between (y, x, z) is false as between(B, A, A) and between(A, B, B) are false respectively.
Therefore, (2) is trivially true.

For (4): if there cannot be all three elements are the same, then it's guaranteed that between(A, B, A) or between(B, A, B) will be present, making between(y, x, z) $\vee$ between(z, y, x) $\vee$ between(x, z, y) be true.

For (5): if between(x, y, z) is between(A, B, A), then between(y, x, z) is between(B, A, A), so between(A, B, A) $\longrightarrow \neg$ between(B, A, A) is true $\rightarrow$ true which evaluates to true.

if between(x, y, z) is between(B, A, B), then between(y, x, z) is between(A, B, B), so between(B, A, B) $\longrightarrow \neg$ between(A, B, B) is true $\rightarrow$ true which evaluates to true.

All other cases are trivially true as between(x, y, z) is false.

Hence, this set is satisfiable.

a) $P(C_i=1, H=1) = P(C_i=1) \times P(H=1 \mid C_i=1) = \frac{1}{5} \times 0 = 0$

$P(C_i=2, H=1) = P(C_i=2) \times P(H=1 \mid C_i=2) = \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$

$P(C_i=3, H=1) = P(C_i=3) \times P(H=1 \mid C_i=3) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$

$P(C_i=4, H=1) = P(C_i=4) \times P(H=1 \mid C_i=4) = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20}$

$P(C_i=5, H=1) = P(C_i=5) \times P(H=1 \mid C_i=5) = \frac{1}{5} \times 1 = \frac{1}{5}$

$P(C_i=1, H=0) = P(C_i=1) \times P(H=0 \mid C_i=1) = \frac{1}{5} \times (1-0) = \frac{1}{5}$

$P(C_i=2, H=0) = P(C_i=2) \times P(H=0 \mid C_i=2) = \frac{1}{5} \times (1-\frac{1}{4}) = \frac{3}{20}$

$P(C_i=3, H=0) = P(C_i=3) \times P(H=0 \mid C_i=3) = \frac{1}{5} \times (1-\frac{1}{2}) = \frac{1}{10}$

$P(C_i=4, H=0) = P(C_i=4) \times P(H=0 \mid C_i=4) = \frac{1}{5} \times (1-\frac{3}{4}) = \frac{1}{20}$

$P(C_i=5, H=0) = P(C_i=5) \times P(H=0 \mid C_i=5) = \frac{1}{5} \times (1-1) = 0$

b) $P(H=1) = \sum_{i=1} P(H=1 \mid C_i) P(C_i)$

$= (0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1) \cdot \frac{1}{5}$

$= \frac{1}{2}$

$P(C_i=1 \mid H=1) = \dfrac{\frac{1}{5} \times 0}{\frac{1}{2}} = 0$

$P(C_i=2 \mid H=1) = \dfrac{\frac{1}{5} \times \frac{1}{4}}{\frac{1}{2}} = \frac{1}{10}$

$P(C_i=3 \mid H=1) = \dfrac{\frac{1}{5} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{5}$

$P(C_i=4 \mid H=1) = \dfrac{\frac{1}{5} \times \frac{3}{4}}{\frac{1}{2}} = \frac{3}{10}$

$P(C_i=5 \mid H=1) = \dfrac{\frac{1}{5} \times 1}{\frac{1}{2}} = \frac{2}{5}$

c) $P(H_2=1 \mid H_1=1) = \dfrac{P(H_2=1 \cap H_1=1)}{P(H_1=1)}$

$$= \frac{\sum_{i=1}^{5} P(H_2=1 \cap H_1=1 \mid C_i) \cdot P(C_i)}{\sum_{i=1}^{5} P(H_1=1 \mid C_i)\, P(C_i)}$$

$$= \frac{0^2 \cdot \frac{1}{5} + (\frac{1}{4})^2 \cdot \frac{1}{5} + (\frac{1}{2})^2 \cdot \frac{1}{5} + (\frac{3}{4})^2 \cdot \frac{1}{5} + 1^2 \cdot \frac{1}{5}}{0 \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{5} + \frac{3}{4} \cdot \frac{1}{5} + 1 \cdot \frac{1}{5}}$$

$$= \frac{3}{4}$$

d) $P(C_i \mid B_4) = \dfrac{P(B_4 \mid C_i)\, P(C_i)}{P(B_4)} = \dfrac{P(B_4 \mid C_i)\, P(C_i)}{\sum_{i=1}^{5} P(B_4 \mid C_i)\, P(C_i)}$

$$P(B_4) = [(1-0)^3 \times 0 + (1-\tfrac{1}{4})^3 \times \tfrac{1}{4} + (1-\tfrac{1}{2})^3 \times \tfrac{1}{2} + (1-\tfrac{3}{4})^3 \times \tfrac{3}{4} + (1-1)^3 \times 1] \times \tfrac{1}{5}$$

$$= \frac{23}{640}$$

$$P(C_{i=1} \mid B_4) = \frac{0 \times \frac{1}{5}}{\frac{23}{640}} = 0$$

$$P(C_{i=2} \mid B_4) = \frac{(1-\frac{1}{4})^3 \times \frac{1}{4} \times \frac{1}{5}}{\frac{23}{640}} = \frac{27}{46}$$

$$P(C_{i=3} \mid B_4) = \frac{(1-\frac{1}{2})^3 \times \frac{1}{2} \times \frac{1}{5}}{\frac{23}{640}} = \frac{8}{23}$$

$$P(C_i = 4 \mid B_4) = \frac{(1-\frac{3}{4})^3 \times \frac{3}{4} \times \frac{1}{5}}{\frac{23}{640}} = \frac{3}{46}$$

$$P(C_i = 5 \mid B_4) = \frac{0 \times \frac{1}{5}}{\frac{23}{640}} = 0$$

4\. a) $4 \times 3 \times 2 = 24$ rows

b)

i) $P_{S_1}(G, D, C) = P_{S_1}(D) \times P_{S_1}(G \mid D) \times P(C \mid D)$

ii) For $P_{S_1}(D)$: 3 rows

For $P_{S_1}(G \mid D)$: $4 \times 3 = 12$ rows

For $P_{S_1}(C \mid D)$: $2 \times 3 = 6$ rows

Total rows: $3 + 12 + 6 = 21$ rows

c)

i) $P_{S_2}(G, D, C) = P_{S_2}(G, D) \times P_{S_2}(C)$

ii) For $P_{S_2}(G, D) = 4 \times 3 = 12$ rows

For $P_{S_2}(C) = 2$ rows

Total rows: $12 + 2 = 14$ rows