Chapter 2 homework

褚朱钇恒 **3200104144**

1 Theoretical questions

1.1 I

$x_0 = 1, f_0 = 1, x_1 = 2, f_2 = \frac{1}{2} \Rightarrow p_1(f;x) = -\frac{1}{x} + \frac{3}{2}$
So, $f(x) - p_1(f;x) = \frac{1}{x} + \frac{x}{2} - \frac{3}{2} = \frac{1}{\xi^3(x)}(x-1)(x-2)$
$\Rightarrow \xi(x) = \sqrt[3]{\frac{1}{2x}}$
So,$\max \xi(x) = \sqrt[3]{\frac{1}{2}}, \min \xi(x) = \sqrt[3]{\frac{1}{4}}, \max f^{''}(\xi(x)) = \max 4x = 8$

1.2 II

First, find an interpolation polynomial $p(x)$ of degree n that satisfies $p(x_i) = \sqrt{f_i}, i = 0, 1, \ldots, n$

Then, let $p_2(x) = p^2(x)$, then, we can verify than $p_2(x) \geq 0$, $p_2(x_i) = f_i$ and $p_2 \in \mathbb{P}_{2n}^+$

1.3 III

$\forall t,\ f[t] = f(t) = e^t = \frac{(e-1)^0}{0!}e^t$
Assume than when $n = 1, 2, \ldots, k, forall t \in \mathbb{R}, f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!}e^t$
Then $\forall t \in \mathbb{R}, f[t, t+1, \ldots, t+k+1] = \frac{f[t+1,t+2,\ldots,t+n+1]-f[t,t+1,\ldots,t+n]}{t+1-t} = \frac{(e-1)^{n+1}}{(n+1)!}e^t$
By induction, the original proposition is proved.
$f[0, 1, \ldots, n] = \frac{(e-1)^n}{n!}e^0 = \frac{(e-1)^n}{n!} = \frac{1}{n!}f^{(n)}(\xi)$
$\Rightarrow (e-1)^n = e^\xi$
So, $\xi = nlog(e-1)$ and it is located at the right side of $\frac{n}{2}$

1.4 IV

0	5			
1	3	-2		
3	5	1	1	
4	12	7	2	0.25

So, $p_3 = 5 - 2x + (x-1)x + 0.25x(x-1)(x-3)$
$p_3^{'}(x) = \frac{3x^2-9}{2} \Rightarrow x_{\min} = \sqrt{3}$

1.5 V

0	0					
1	1	1				
1	1	7	6			
1	1	7	21	15		
2	128	127	120	99	42	
2	128	448	321	201	102	30

$f[0, 1, 1, 1, 2, 2] = 30 = \frac{f^{(5)}(\xi)}{5!} = \frac{7*6*5*4*3x^2}{120}$
$\Rightarrow \xi = \sqrt{\frac{10}{7}}$

1.6 VI

0	1				
1	2	1			
1	2	-1	-2		
3	0	-1	0	$\frac{2}{3}$	
3	0	0	0.5	0.25	-$\frac{5}{36}$

$p(x) = 1 + x - 2(x-1)x + \frac{2}{3}(x-1)^2x - \frac{5}{36}(x-3)(x-1)^2x$

$f(2) \approx p(2) = \frac{11}{18}$

$|f(2) - p(2)| \leq \frac{|f^{(5)}(\xi|}{5!}(2-1)^2(2-3)^2(2) \leq \frac{M}{60}$

1.7 VII

$\forall x \in \mathbb{R}, \Delta^0 f(x) = 0!h^0 f[x_0]$

Assume that $\forall x \in \mathbb{R}, \Delta^k f(x) = k!h^k f[x_0, \ldots, x_k]$

Then $\Delta^{k+1} f(x) = \Delta^k f(x+h) - \Delta^k f(x) = k!h^k(f[x_1, \ldots, x_{k+1}] - f[x_0, \ldots, x_k]) = (k+1)!h^{k+1} f[x_0, \ldots, x_{k+1}]$

The second equation can prove similarly.

1.8 VIII

$$\frac{\partial}{x_0} f[x_0, \ldots, x_n] = \lim_{h\to 0} \frac{f[x_0+h, x_1, \ldots, x_n] - f[x_0, x_1, \ldots, x_n]}{h}$$

$$= \lim_{h\to 0} f[x_0, x_0+h, x_1, \ldots, x_n] = f[x_0, x_0, x_1, \ldots, x_n]$$

Similarly

$$\frac{\partial}{x_k} f[x_0, \ldots, x_n] = \lim_{h\to 0} f[x_0, x_k+h, x_1, \ldots, x_n] = f[x_k, x_0, x_1, \ldots, x_n]$$

1.9 IX

When $x \in [-1,1], a_0 = 1$, $\min\max_{x\in[-1,1]} |a_0 x^n + \cdots + a_n| = \frac{1}{2^{n-1}}$

Let $x = \frac{b-a}{2}(y+1), y \in [-1,1]$

then $p(x) = q(y) = a_0(\frac{b-a}{2}y + \frac{a+b}{2})^n + \cdots + a_n$

$\Rightarrow \min\max_{y\in[-1,1]} |\frac{2^n}{a_0(b-a)^n}q(y| = \frac{1}{2^{n-1}}$

$\Rightarrow \min\max_{x\in[a,b]} |p(x)| = \frac{a_0(b-a)^n}{2^{2n-1}}$

1.10 X

$T_n(x) = cos(n \arccos x)$

Assume $\exists p_0 \in \mathbb{P}_n^a, \|\hat{p}_n\|_\infty > \|p_0\|_\infty$, then $\max_{x\in[-1,1]} |p_0| < \frac{1}{T_n(a)}$

Let $Q(x) = \hat{p}(x) - p_0(x)$,then $Q(x_k^{'}) = \frac{(-1)^k}{T_n(a)} - p_0(x_k^{'})$, then $Q(x)$ has alternating signs at these $n+1$ points.

However, the degree of $Q(x)$ is no larger than n, so $Q(x) \equiv 0$, which is a contradiction to assumption. So the original proposition is true.

1.11 XI

$$\sum_{k=0}^{n} b_{n,k}(t) = ((1-t)+t)^n = 1$$

$$(p+q)^n = \sum_{k=0}^{n} \tbinom{n}{k} p^k q^{n-k}$$

$$\Rightarrow n(p+q)^{n-1} = \sum_{k=1}^{n} k \tbinom{n}{k} p^{k-1} q^{n-k}$$

$$\Rightarrow np(p+q)^{n-1} = \sum_{k=1}^{n} k \tbinom{n}{k} p^{k} q^{n-k}$$

Let $p = t, q = 1-t$, we have

$$\sum_{k=0}^{n} k b_{n,k}(t) = nt$$

$$(p+q)^n = \sum_{k=0}^{n} \tbinom{n}{k} p^k q^{n-k}$$

$$\Rightarrow np(p+q)^{n-1} = \sum_{k=1}^{n} k \tbinom{n}{k} p^{k} q^{n-k}$$

$$\Rightarrow n(n-1)p(p+q)^{n-2} + n(p+q)^{n-1} = \sum_{k=1}^{n} k^2 \tbinom{n}{k-1} p^{k} q^{n-k}$$

$$\Rightarrow n(n-1)p^2(p+q)^{n-2} + np(p+q)^{n-1} = \sum_{k=1}^{n} k^2 \tbinom{n}{k} p^{k} q^{n-k}$$

Let $p = t, q = 1-t$, we have

$$\sum_{k=0}^{n} k^2 b_{n,k}(t) = n(n-1)t^2 + nt$$

then

$$\sum_{k=0}^{n} (k-nt)^2 b_{n,k}(t) = \sum_{k=0}^{n} (k^2 - 2knt + n^2t^2) b_{n,k}(t)$$

$$= n(n-1)t^2 + nt - 2n^2t^2 + n^2t^2 = nt - nt^2 = nt(1-t)$$