

Chapter 3 homework

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1 Theoretical questions

1.1 I

$$s(1) = 1, s'(1) = 3, s''(1) = 6$$

插值可得 $p(x) = 7x^3 - 18x^2 + 12x$, 故 $s''(0) = -36 \neq 0$

故不是自然样条

1.2 II

1.2.1 a

在每个区间上, f 有三个待定系数, 故共有 $3(n-1)$ 个待定系数。

在每个中间节点上, 有 $f_{i-1} = f_i, f'_{i-1} = f'_i$, 引入两个条件。

在每个形值点上, 有 $f_i = f(x_i)$, 引入 n 个条件。

故还需要确定 $3(n-1) - 2(n-2) - n = 1$ 个条件。

1.2.2 b

在 x_i 处做泰勒展开得 $p_i(x) = f_i + m_i(x - x_i) + a_i(x - x_i)^2$, 将 $p_i(x_{i+1}) = f_{i+1}$, 得
$$p_i(x) = f_i + m_i(x - x_i) + \frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}(x - x_i)^2$$

1.2.3 c

根据 (b) 得, $m_{i+1} = -m_i + 2\frac{f_{i+1} - f_i}{x_{i+1} - x_i}$, 故可以递推求得 m_2, \dots, m_{n-1}

1.3 III

$$s(0) = 1 + c, s'(0) = 3c, s''(0) = 6c, \text{ 故 } s_2(x) = 1 + c + 3cx + 3cx^2 + ax^3$$

由 s 为自然样条, $s''(1) = 6c + 6a = 0$, 故 $a = -c$, 即 $s_2(x) = 1 + c + 3cx + 3cx^2 - cx^3$

$$s(1) = -1 \Rightarrow c = -\frac{1}{3}$$

1.4 IV

1.4.1 a

$$\text{设 } s_1(x) = a_1x^3 + bx^2 + cx + 1, s_2(x) = a_2x^3 + bx^2 + cx + 1,$$

由 $f(-1) = f(1) = 0, s''(-1) = s''(1) = 0$, 解得

$$s_1(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1, s_2(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$$

1.4.2 b

$$\int_{-1}^1 [s''(x)]^2 dx = 6$$

(i)

$$g(x) = -x^2 + 1$$

$$\int_{-1}^1 [g''(x)]^2 dx = 8 > \int_{-1}^1 [s''(x)]^2 dx$$

(ii)

$$\int_{-1}^1 [f''(x)]^2 dx = \frac{\pi^4}{16} \approx 6.08 > \int_{-1}^1 [s''(x)]^2 dx$$

1.5 V

1.5.1 a

$$\begin{aligned} \text{当 } x \in [t_{i-1}, t_i] \text{ 时, } B_i^2(x) &= \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} \\ \text{当 } x \in [t_i, t_{i+1}] \text{ 时, } B_i^2(x) &= \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} \\ \text{当 } x \in [t_{i+1}, t_{i+2}] \text{ 时, } B_i^2(x) &= \frac{(x-t_{i+2})^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} \end{aligned}$$

1.5.2 b

$$\begin{aligned} \frac{d}{dx} B_i^2(t_i^-) &= \frac{2}{t_{i+1}-t_{i-1}} = \frac{d}{dx} B_i^2(t_i^+) \\ \frac{d}{dx} B_i^2(t_{i+1}^-) &= -\frac{2}{t_{i+2}-t_{i-1}} = \frac{d}{dx} B_i^2(t_{i+1}^+) \\ \text{故 } \frac{d}{dx} B_i^2 &\text{ 在 } t_i \text{ 和 } t_{i+1} \text{ 上连续。} \end{aligned}$$

1.5.3 c

$$\begin{aligned} \text{当 } x \in (t_{i-1}, t_i) \text{ 时, } \frac{d}{dx} B_i^2 &= \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} \neq 0 \\ \text{当 } x \in (t_i, t_{i+1}) \text{ 时, } \frac{d}{dx} B_i^2 &\text{ 为线性函数, 故只有一处 } x^* \text{ 为 } 0. \\ x^* &= \frac{(t_{i+1}+t_{i-1})(t_{i+2}-t_i)+(t_{i+2}+t_i)(t_{i+1}-t_{i-1})}{2(t_{i+1}+t_{i+2}-t_{i-1}-t_i)} \end{aligned}$$

1.5.4 d

只需考虑边界点和极值点, $B_i^2(t_i) = 0, B_i^2(x^*) < 1$
故 $B_i^2(x) \in [0, 1)$

1.5.5 e

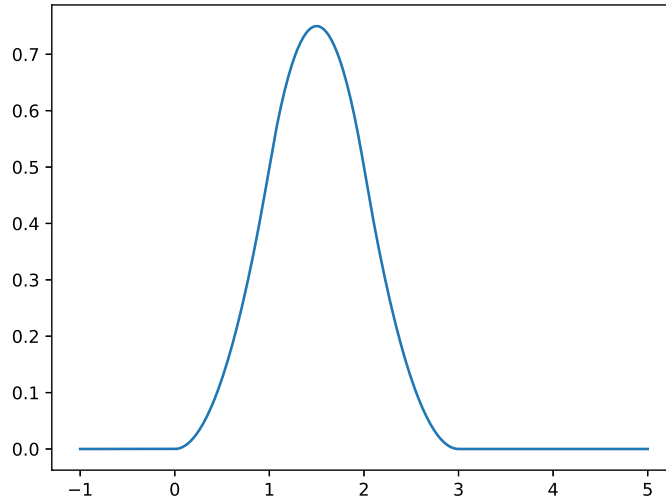


图 1: $B_1^2(x)$ 的图像

1.6 VI

$$LHS = [t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_{i-1}, t_i, t_{i+1}](t-x)_+^2$$

当 $x \leq t_{i-1}$ 时, $LHS = 0 = RHS$

当 $t_{i-1} < x \leq t_i$ 时, $LHS = \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} = RHS$

当 $t_i < x \leq t_{i+1}$ 时, $LHS = \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = RHS$

当 $t_{i+1} < x \leq t_{i+2}$ 时, $LHS = \frac{(x-t_{i+2})^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} = RHS$

当 $x > t_{i+2}$ 时, $LHS = 0 = RHS$

综上, 原等式成立。

1.7 VII

由 B 样条的微分性质, $\frac{d}{dx}B_i^n(x) = \frac{nB_i^{n-1}(x)}{t_{i+n-1}-t_{i-1}} - \frac{nB_{i+1}^{n-1}(x)}{t_{i+n}-t_i}$

那么有 $\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}} dx - \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1}-t_i} dx =$

$\int_{t_{i-1}}^{t_{i+n+1}} \left[\frac{B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{B_{i+1}^n(x)}{t_{i+n+1}-t_i} \right] dx =$

$\frac{1}{n} B_i^{n+1}(x) \Big|_{t_{i-1}}^{t_{i+n+1}} = 0$

故 $\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}} dx = \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1}-t_i} dx$, 原命题成立

1.8 VIII

该命题可表示为 $\forall m \in N, \forall n = 0, 1, \dots, m$

$\tau_{m-n}(x_1, \dots, x_n) = [(x_1, \dots, x_n)x_m]$

(a) 当 $m = 4, n = 2$ 时