Chapter 1 homework

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1 Theoretical questions

1.1 I

The width of the interval at the nth step is 2^{-n+1}, the max possible distance between r and midpoint is 1.

1.2 II

At the nth step, the width of interval is $\frac{b_0-a_0}{2^n}$, the max absolute error is $\frac{b_0-a_0}{2^{n+1}}$, so the max relative error is $\frac{\frac{b_0-a_0}{2^{n+1}}}{a_0}$

As the relative error is no greater than ϵ, so $\frac{\frac{b_0-a_0}{2^{n+1}}}{a_0} \leq \epsilon$.

Then we have $log_2(b_0 - a_0) - n - 1 - log(a_0) \leq log(\epsilon)$.

So if $n \geq \frac{log(b_0-a_0)-log(\epsilon)-log(a_0)}{log(2)} - 1$, the relative error is no greater than ϵ.

1.3 III

$p^{'}(x) = 12x^2 - 4x$

The result is in the following table.

n	x_n	$p(x_n)$	$p^{'}(x_n)$	$x_n - \frac{p(x_n)}{p^{'}(x_n)}$
0	-1.0000	-3.0000	16.0000	-0.8125
1	-0.8125	-0.4658	11.1719	-0.7708
2	-0.7708	-0.0201	10.2129	-0.7688
3	-0.7688	-3.9801	10.1686	-0.7688
4	-0.7688			

1.4 IV

Assume α is the true root, then from Taylor's expansion we have

$$f(x_n) = f(\alpha) + f^{'}(\xi)(x_n - \alpha) = f^{'}(\xi)(x_n - \alpha), \xi \in [\alpha, x_n]$$

And combine with $x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_0)}$, there is

$$x_{n+1} = x_n - \frac{f^{'}(\xi)(x_n - \alpha)}{f^{'}(x_0)}$$

Subtract α on both side, we get

$$|x_{n+1} - \alpha| = |x_n - \alpha - \frac{f^{'}(\xi)(x_n - \alpha)}{f^{'}(x_0)}|$$

This is equivalent to $e_{n+1} = |1 - \frac{f^{'}(\xi)}{f^{'}(x_0)}|e_n$.

As

$$\lim_{n\to\infty} |1 - \frac{f^{'}(\xi)}{f^{'}(x_0)}| = |1 - \frac{f^{'}(\alpha)}{f^{'}(x_0)}|$$

then we have $C = |1 - \frac{f^{'}(\alpha)}{f^{'}(x_0)}|$ and $s = 1$ meet the question's requirement.

1.5 V

Because $0 \leq tan^{-1}(x) \leq x(x > 0)$, so $0 \leq x_{n+1} = tan^{-1}(x_n) \leq x_n(x_n > 0)$.
So if $x_0 \geq 0$, the interval is converge.
If $x_0 < 0$, then $\{-x_n\}$is converge, so $\{x_n\}$ is converge too.

1.6 VI

Let $f(x) = \frac{1}{x+p}$, then $x_0 = 1, x_1 = f(x_0), x_2 = f(x_1) \ldots$
As $p > 1$, then $f(x)$ is a continuous contraction on $[0, 1]$, and it has a fixed point $\alpha = \frac{\sqrt[2]{p^2+4}-p}{2}$.
$\lambda = \max|f^{'}(x) = -\frac{1}{(x+p)^2}| < 1$, from **Theorem 1.38**, we know $|x_n - \alpha| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|$.
So $x = \alpha = \frac{\sqrt[2]{p^2+4}-p}{2}$

1.7 VII

If we need absolute error no larger than δ, then we have $\frac{b_0-a_0}{2^{n+1}} \leq \delta$, which means $n \geq \frac{log(b_0-a_0)-logn}{log2} - 1$.
As the true root might be 0, the relative error isn't an appropriate measure.