微分方程数值解（第九章理论作业）

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1 Exercise 9.5

$Ae = r \Rightarrow ||Ae||_2 = ||r||_2 \leq ||A||_2||e||_2$

$Ax = b \Rightarrow x = A^{-1}b \Rightarrow ||A^{-1}b||_2 = ||x||_2 \leq ||A^{-1}||_2||b||_2$ 且 $||Ax||_2 = ||b||_2 \leq ||A||_2||x||_2$

$Cond(A)\frac{||r||_2}{||b||_2} = \frac{||A||_2||A^{-1}||_2||r||_2}{||b||_2} \geq \frac{||A||_2||A^{-1}r||_2}{||A||_2||x||_2} = \frac{||e||_2}{||x||_2}$

$\frac{||r||_2}{Cond(A)||b||_2} = \frac{||r||_2}{||A||_2||A^{-1}||_2||b||_2} \leq \frac{||A||_2||x||_2}{||A||_2||e||_2} = \frac{||e||_2}{||x||_2}$

2 Exercise 9.8

由于 A 对称，则有 $||A||_2 = \sqrt{\lambda_{max}(A^TA)} = |\lambda_{max}(A)|$

同理 $||A^{-1}||_2 = \frac{1}{|\lambda_{min}(A)|}$

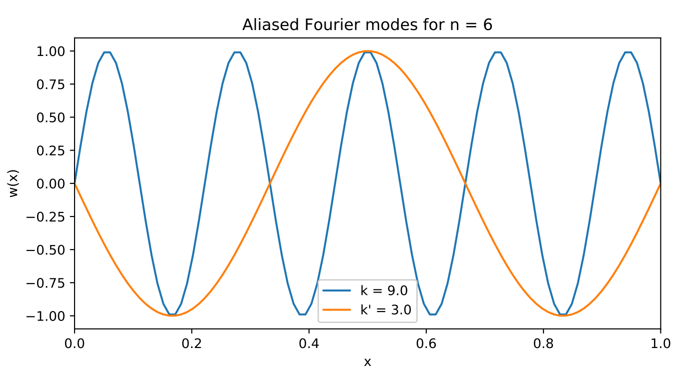
故有 $cond(A) = \frac{|\lambda_{max}(A)|}{|\lambda_{min}(A)|}$

$n = 8$ 时，$cond(A) = \frac{|\lambda_{max}(A)|}{|\lambda_{min}(A)|} = \frac{\frac{4}{h^2}sin^2\frac{k_{max}\pi}{2(m+1)}}{\frac{4}{h^2}sin^2\frac{k_{min}\pi}{2(m+1)}} = \frac{sin^2\frac{7\pi}{16}}{sin^2\frac{\pi}{16}}$

同理 $n = 1024$ 时，$cond(A) = \frac{|\lambda_{max}(A)|}{|\lambda_{min}(A)|} = \frac{\frac{4}{h^2}sin^2\frac{k_{max}\pi}{2(m+1)}}{\frac{4}{h^2}sin^2\frac{k_{min}\pi}{2(m+1)}} = \frac{sin^2\frac{1023\pi}{2048}}{sin^2\frac{\pi}{2048}}$

3 Exercise 9.14

可以发现，以下两条图像在 $\frac{i}{n}$ 处相交，所以在 $\Omega = (0, 1)$ 的 $h = \frac{1}{6}$ 的网格上无法区分两者。



4 Exercise 9.17

由 $\lambda(A)=\frac{4}{h^2}sin^2\frac{k\pi}{2(m+1)}$，设其特征向量为 x_k，则有
$T_\omega x_k = Ix_k - \frac{\omega h^2}{2}Ax_k = x_k - 2\omega sin^2\frac{k\pi}{2n}x_k$，故有
$\lambda(T_\omega) = 1 - 2\omega sin^2\frac{k\pi}{2n}$

5 Exercise 9.35

当 $v_1 = v_2 = 1$,FMG 的时间复杂度为 $\Sigma_{i=0}^{m}2^{-iDm}\Sigma_{i=0}^{m}2^{-iDm}2WD = \frac{2}{(1-2^{-D})^2}WU$
将 $D = 1, 2, 3$ 代入则可得到上界分别为 $8WU, \frac{32}{9}WU, \frac{128}{49}WU$

6 Exercise 9.45

$N(I_h^{2h}) = \{v^h \in \Omega^h \ and \ v_{2*i+1}^h = -(v_{2*i}^h + v_{2*i+2}^h)\}$
所以 $\dim N(I_h^{2h}) = \frac{n}{2}$
由于 I_{2h}^h 与 I_h^{2h} 互为逆变换，则 $\dim R(I_{2h}^h) = \dim R(I_h^{2h}) = \frac{n}{2} - 1$