Appendix E

Functional Analysis

Example E.1. A copper mining company mines in a mountain that has an estimated total amount of Q tonnes of copper. Let $x(t)$ denote the amount of copper removed during the period $[0, t]$, with $x(0) = 0$ and $x(T) = Q$. Assume x is a continuous function $[0, T] \to \mathbb{R}$ and the cost of extracting copper per unit tonne at time t is

$$c(t) = ax(t) + bx'(t), \tag{E.1}$$

where $a, b \in \mathbb{R}^+$. What is the optimal mining operation $x(t)$ that minimizes the cost function

$$f(x) = \int_0^T (ax(t) + bx'(t))x'(t)\mathrm{d}t?$$

In math terms, we would like to minimize $f : \mathcal{C}_Q^1[0, T] \to \mathbb{R}^+$ where $\mathcal{C}_Q^1[0, T]$ is the set of continuously differentiable functions $x : [0, T] \to \mathbb{R}$ satisfying $x(0) = 0$ and $x(T) = Q$.

In calculus, the minimizer x_* of a function $f \in C^2$ is usually found by the condition $f'(x_*) = 0$ and $f''(x_*) > 0$. However, the above problem does not fit into the usual framework of calculus, since x is not a number but a function that belongs to an infinite-dimensional function space. Solving this problem requires a number of techniques in functional analysis.

E.1 Normed and Banach spaces

E.1.1 Metric spaces

Definition E.2. The ℓ^∞ *sequence space* is a metric space (ℓ^∞, d), where ℓ^∞ is the set of all bounded sequences of complex numbers,

$$\ell^\infty := \left\{(\xi_1, \xi_2, \ldots) : \exists c_x \in \mathbb{R}, \text{ s.t. } \sup_{i \in \mathbb{N}^+} |\xi_i| \le c_x.\right\} \tag{E.2}$$

and the metric is given by

$$d(x, y) = \sup_{i \in \mathbb{N}^+} |\xi_i - \eta_i|$$

where $y = (\eta_1, \eta_2, \ldots) \in \mathcal{X}$.

Exercise E.3. Let $\mathcal{X}$ be the set of all bounded and unbounded sequences of complex numbers. Show that the following is a metric on $\mathcal{X}$,

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}, \tag{E.3}$$

where $x = (\xi_j)$ and $y = (\eta_j)$.

Definition E.4. For a real number $p \ge 1$, the ℓ^p *sequence space* is the metric space (ℓ^p, d) with

$$\ell^p := \left\{(\xi_j)_{j=1}^{\infty} : \xi_j \in \mathbb{C}; \sum_{j=1}^{\infty} |\xi_j|^p < \infty\right\}; \tag{E.4}$$

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{1/p}, \tag{E.5}$$

where $x = (\xi_j)$ and $y = (\eta_j)$ are both in $\mathcal{X}$. In particular, the *Hilbert sequence space* ℓ^2 is the ℓ^p space with $p = 2$.

Definition E.5. A pair of *conjugate exponents* are two real numbers $p, q \in [1, \infty]$ satisfying

$$p + q = pq, \text{ i.e., } \frac{1}{p} + \frac{1}{q} = 1. \tag{E.6}$$

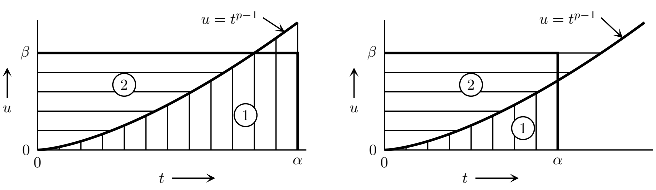
Lemma E.6. Any two positive real numbers α, β satisfy

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \tag{E.7}$$

where p and q are conjugate exponents and the equality holds if $\beta = \alpha^{p-1}$.

Proof. By (E.6), we have

$$u = t^{p-1} \Rightarrow t = u^{q-1}.$$



It follows that

$$\alpha\beta \le \int_0^\alpha t^{p-1}\mathrm{d}t + \int_0^\beta u^{q-1}\mathrm{d}u = \frac{\alpha^p}{p} + \frac{\beta^q}{q},$$

where the equality holds if $\beta = \alpha^{p-1}$ since $p = q(p-1)$. $\square$

Corollary E.7. A pair of conjugate exponents p, q satisfy

$$\forall a, b \in [0, +\infty), \quad a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{a}{p} + \frac{b}{q}. \tag{E.8}$$

Proof. This follows directly from Lemma E.6. $\square$

Theorem E.8 (Hölder's inequality)**.** For $n \in \mathbb{N}^+ \cup \{+\infty\}$, $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and conjugate exponents $p, q \in [1, \infty]$, we have

$$\sum_{j=1}^n |x_j y_j| \le \|\mathbf{x}\|_p \|\mathbf{y}\|_q, \tag{E.9}$$

where $\|\cdot\|_p$ is the Euclidean norm. For $p, q \in (1, \infty)$, the equality in (E.9) holds if

$$\exists c \in \mathbb{R} \text{ s.t. } \forall j = 1, \ldots, n,\ |x_j|^p = c|y_j|^q. \tag{E.10}$$

Proof. If $\sum_{j=1}^n |x_j|^p = 0$ or $\sum_{j=1}^n |y_j|^p = 0$ or $p = \infty$ or $q = \infty$, then (E.9) holds trivially. Otherwise we define

$$a_i := \frac{|x_i|^p}{\sum_{j=1}^n |x_j|^p}, \quad b_i := \frac{|y_i|^q}{\sum_{j=1}^n |y_j|^q}.$$

It follows from (E.8) that

$$\frac{|x_i y_i|}{\left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q\right)^{\frac{1}{q}}} \le \frac{|x_i|^p}{p\sum_{j=1}^n |x_j|^p} + \frac{|y_i|^q}{q\sum_{j=1}^n |y_j|^q}.$$

Sum up all equations for $i = 1, \ldots, n$ and we have

$$\frac{\sum_{j=1}^n |x_i y_i|}{\|\mathbf{x}\|_p \|\mathbf{y}\|_q} \le \frac{1}{p} + \frac{1}{q} = 1,$$

which yields (E.9). Substitute (E.10) into (E.9) and we have the equality. $\square$

Example E.9. Cauchy-Schwarz inequality in Theorem B.171 is a special case of the Hölder inequality (E.9) for $p = q = 2$.

Exercise E.10. Prove that (E.5) satisfies the triangular inequality and is indeed a metric.

(a) The Hölder inequality implies the *Minkowski inequality*, i.e. for any $p \ge 1$, $(\xi_j) \in \ell^p$, and $(\eta_j) \in \ell^p$,

$$\left(\sum_{j=1}^\infty |\xi_j + \eta_j|^p\right)^{1/p} \le \left(\sum_{k=1}^\infty |\xi_k|^p\right)^{1/p} + \left(\sum_{m=1}^\infty |\eta_m|^p\right)^{1/p}. \tag{E.11}$$

(b) The Minkowski inequality implies that the triangular inequality holds for (E.5).

E.1.2 Normed spaces

Example E.11. $(\mathbb{R}^n, \|\cdot\|_p)$ is a normed space, where $\|\cdot\|_p$ is the Euclidean norm in Definition B.154 with $p \in [1, \infty)$:

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}}.$$

Exercise E.12. Prove the backward triangle inequality of a norm $\|\cdot\|$, i.e.,

$$\forall u, v \in V, \quad |\|u\| - \|v\|| \le \|u - v\|. \tag{E.12}$$

Exercise E.13. Use Holder's inequality to verify the triangle inequality for the Euclidean norm in Example E.11.

Definition E.14. In a normed space $(\mathcal{X}, \|\cdot\|)$, an *open ball* $B_r(x)$ centered at $x \in \mathcal{X}$ with radius $r > 0$ is the subset

$$B_r(x) := \{y \in \mathcal{X} : \|x - y\| < r\}. \tag{E.13}$$

Lemma E.15. Any open ball in a normed space is a convex set as in Definition 1.18.

Proof. For $\alpha \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in B_r(\mathbf{0})$, we have

$$\begin{aligned}\|\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\| &\le \|\alpha\mathbf{x}\| + \|(1-\alpha)\mathbf{y}\| \\ &\le \alpha\|\mathbf{x}\| + (1-\alpha)\|\mathbf{y}\| < \alpha r + (1-\alpha)r = r,\end{aligned}$$

where we have applied properties of norms. Hence $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in B_r(\mathbf{0})$. $\square$

Exercise E.16. Show that the Euclidean norm $\|\cdot\|_p$ in Example E.11 satisfies a monotonicity property:

$$1 \le p \le q \le \infty \ \Rightarrow\ \forall \mathbf{x} \in \mathbb{R}^n\ \|\mathbf{x}\|_q \le \|\mathbf{x}\|_p.$$

Example E.17. $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a normed space, where $\|\cdot\|_\infty$ is the Euclidean norm in Definition B.154:

$$\|\mathbf{x}\|_\infty = \max_j |x_j|.$$

Definition E.18. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The *p-norm of a continuous scalar function* in the linear space $\mathcal{C}(\overline{\Omega})$ is

$$\forall v \in \mathcal{C}(\overline{\Omega}), \quad \|v\|_p := \left[\int_\Omega |v(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right]^{\frac{1}{p}} \tag{E.14}$$

and the ∞*-norm* or *maximum norm* is given by

$$\forall v \in \mathcal{C}(\overline{\Omega}), \quad \|v\|_\infty := \max_{\mathbf{x}\in\overline{\Omega}} |v(\mathbf{x})|. \tag{E.15}$$

Example E.19. $(\mathcal{C}(\overline{\Omega}), \|\cdot\|_\infty)$ in Definition E.18 is a normed space, so is $(\mathcal{C}(\overline{\Omega}), \|\cdot\|_p)$ for any $p \in [1, \infty)$.

Example E.20. For the ℓ^∞ sequence space in (E.2),

$$\ell^\infty := \left\{(a_n)_{n\in\mathbb{N}} : \sup_{n\in\mathbb{N}} |a_n| < \infty.\right\},$$

define $\|\cdot\|_\infty : \ell^\infty \to \mathbb{R}^+ \cup \{0\}$ as

$$\|(a_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |a_n|. \tag{E.16}$$

Then $(\ell^\infty, \|\cdot\|_\infty)$ is a normed space.

Example E.21. For the ℓ^p space in (E.4) with $p \in [1, \infty)$,

$$\ell^p := \left\{ (a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{C}; \sum_{n\in\mathbb{N}} |a_n|^p < \infty \right\},$$

we have

$$\begin{cases} (a_n)_{n\in\mathbb{N}} \in \ell^p, \ (b_n)_{n\in\mathbb{N}} \in \ell^p \\ |a+b|^p \le (|a|+|b|)^p \le 2^p(\max(|a|,|b|))^p \le 2^p(|a|^p+|b|^p) \end{cases}$$
$$\Rightarrow (a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} \in \ell^p,$$

where the comparison test is applied.

Then $(\ell^p, \|\cdot\|_p)$ is a normed space where

$$\|(a_n)_{n\in\mathbb{N}^+}\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}. \tag{E.17}$$

Notation 30. Let c_{00} denote the space of all sequences that are eventually 0, c_0 the space of all sequences that converge to 0, and c the space of all sequences that converge.

Definition E.22 (Convergence of sequences)**.** A sequence $\{u_n\}$ in a normed space $(V, \|\cdot\|)$ is *convergent* to $u \in V$ iff

$$\lim_{n\to\infty} \|u_n - u\| = 0. \tag{E.18}$$

E.1.3 The topology of normed spaces

Example E.23. The topology of a normed space is the metric topology in Definition D.189 because a normed space is always a metric space.

Definition E.24. Let $(\mathcal{X}, d)$ be a metric space. A point $x_0 \in \mathcal{X}$ is an *adherent point* or a *closure point* of $E \subset \mathcal{X}$ or a *point of closure* or a *contact point* iff

$$\forall r > 0, \ E \cap B_r(x_0) \neq \emptyset. \tag{E.19}$$

Example E.25. Any point in the set K in (D.28) is a closure point of K, so is 0.

Definition E.26. A point x is an *accumulation point* (or a *limit point*) of A iff

$$\forall r > 0, \ (B_r(x) \setminus \{x\}) \cap A \neq \emptyset. \tag{E.20}$$

Example E.27. The only accumulation point of the set K in (D.28) is 0.

Example E.28. Each number in $\mathbb{R}$ is an accumulation point of $\mathbb{Q}$.

Definition E.29. Let $V_1 \subset V_2$ be two subsets in a normed space V. The set V_1 is *dense* in V_2 iff

$$\forall u \in V_2, \forall \epsilon > 0, \exists v \in V_1 \text{ s.t. } \|v - u\| < \epsilon. \tag{E.21}$$

Theorem E.30. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise E.31. Show that c_{00} is dense in ℓ^2.

Example E.32. The set of polynomials is dense in $(\mathcal{C}[a,b], \|\cdot\|_\infty)$, c.f. Theorem 2.53.

Definition E.33. A normed space is *separable* if it has a countable dense set.

Example E.34. By Definitions E.29 and E.33, $L^p(\Omega)$ is separable since the set of all polynomials with rational coefficients is countable and is dense in $L^p(\Omega)$.

Lemma E.35. ℓ^∞ is not separable.

Proof. Suppose ℓ^∞ is separable. Then there exists in ℓ^∞ a dense subset $D = \{x_1, x_2, x_3, \ldots, \}$. For the set A of sequences with each term being either 0 or 1, we have

$$\forall a, b \in A, \quad a \neq b \ \Leftrightarrow \ \|a - b\|_\infty = 1.$$

It follows from D being dense in ℓ^∞ that

$$\forall a \in A, \exists x_{n(a)} \in D \text{ s.t. } x_{n(a)} \in B_{\frac{1}{2}}(a).$$

Because the open balls are pairwise distinct, the map $f : A \to \mathbb{N}$ given by $f(a) = n(a)$ is injective. However, the construction of A implies that A is uncountable because A has a one-to-one correspondence with all real numbers in $[0, 1)$ via binary expansion. Thus $f : A \to \mathbb{N}$ cannot be injective and this completes the proof. □

Exercise E.36. Prove that ℓ^p is separable for all $p \in [1, \infty)$.

E.1.4 Bases of infinite-dimensional spaces

Definition E.37. An infinite dimensional normed space V has a *countably-infinite basis* iff

$$\exists \{v_i\}_{i\ge1} \subset V \text{ s.t. } \forall v \in V, \exists \{\alpha_{n,i}\}_{i=1}^n \text{ where } n \in \mathbb{N}^+, \alpha_{n,i} \in \mathbb{R}$$
$$\text{s.t. } \lim_{n\to\infty} \left\| v - \sum_{i=1}^n \alpha_{n,i} v_i \right\| = 0. \tag{E.22}$$

The sequence $\{v_i\}_{i\ge1}$ is a *basis* if any finite subset of it is linearly independent.

Definition E.38. A *Schauder basis* of an infinite-dimensional normed linear space V is a sequence $\{v_n\}_{n\ge1}$ of elements in V such that

$$\forall v \in V, \exists! \{\alpha_n\}_{n\ge1} \text{ where } \alpha_n \in \mathbb{R} \text{ s.t. } v = \sum_{n=1}^{\infty} \alpha_n v_n. \tag{E.23}$$

Example E.39. The sequence space ℓ^2 in Definition E.4 has a Schauder basis

$$\left\{ e_j = (0, \ldots, 0, 1, 0, 0, \ldots) \right\}_{j=1}^{\infty}$$

since any $\xi = (\xi_1, \xi_2, \ldots) \in \ell^2$ can be uniquely written as $\xi = \sum_{j=1}^\infty \xi_j e_j$.

Example E.40. It can be proved that the set $\{1, \cos nx, \sin nx\}_{n=1}^\infty$ is a Schauder basis of $L^p(-\pi, \pi)$ for $p \in (1, \infty)$.

E.1.5 Sequential compactness

Definition E.41. A subset K of a normed space $(X, \|\cdot\|)$ is *sequentially compact* if every sequence in K has a convergent subsequence that converges in K,

$$\forall (x_n)_{n\in\mathbb{N}} \subset K,\ \exists n_k : \mathbb{N} \to \mathbb{N},\ \exists L \in K \text{ s.t. } \lim_{n\to+\infty} x_{n_k} = L. \tag{E.24}$$

Example E.42. Any interval $[a, b]$ is sequentially compact in $\mathbb{R}$. Indeed, any sequence in $[a, b]$ is bounded, and by the Bolzano-Weierstrass theorem (Theorem C.13) it has a convergent subsequence, of which the limit must be in $[a, b]$, thanks to the completeness of $\mathbb{R}$ (Theorem C.15),

Example E.43. (a, b) is not sequentially compact since the sequence $\left(a + \frac{b-a}{2n}\right)_{n\in\mathbb{N}^+}$ is contained in (a, b), but its limit a is not contained in (a, b).

Example E.44. $\mathbb{R}$ is not sequentially compact because the sequence $(n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ cannot have a convergent subsequence: the distance bewteen any two terms on any subsequence is at least 1.

Lemma E.45. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof. We prove this statement by an induction on n. The induction basis is the Bolzano-Weierstrass theorem (Theorem C.13). Suppose the statement holds for $n \geq 1$. For a bounded sequence $(\mathbf{x}_m)_{m\in\mathbb{N}} \subset \mathbb{R}^{n+1}$, we split each $\mathbf{x}_m$ as $\mathbf{x}_m = (\alpha_n, \beta_n)$, where $\alpha_n \in \mathbb{R}^n$ and $\beta_n \in \mathbb{R}$. Since $\mathbf{x}_m$ is bounded and $\|\alpha_n\|_2 \leq \|\mathbf{x}_m\|_2$, α_m is also bounded. By the induction hypothesis, $(\alpha_m)_{m\in\mathbb{N}}$ has a convergent subsequence, say $(\alpha_{m_k})_{k\in\mathbb{N}}$, that converges to $\alpha \in \mathbb{R}^k$. Then $(\beta_{m_k})_{k\in\mathbb{N}}$ is bounded and by Theorem C.13 it has a convergent subsequence $(\beta_{m_{k_p}})_{p\in\mathbb{N}}$ that converges to $\beta \in \mathbb{R}$. Therefore we have

$$\lim_{p\to\infty} \mathbf{x}_{m_{k_p}} = \lim_{p\to\infty} \left(\alpha_{m_{k_p}}, \beta_{m_{k_p}}\right) = (\alpha, \beta) \in \mathbb{R}^{n+1},$$

which completes the proof. □

Theorem E.46. In a metric space, sequential compactness is equivalent to compactness.

Lemma E.47. A sequentially compact subset K of a normed space X must be closed and bounded.

Proof. Supppose K is compact but not bounded. Then

$$\forall n \in \mathbb{N}, \exists \mathbf{x}_n \in K \text{ s.t. } \|\mathbf{x}_n\| \geq n.$$

Hence no subsequence of $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset K$ converges and this contradicts the compactness of K.

For any convergent sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset K$, Definition E.41 implies that it has a convergent subsequence that converges in K. The uniqueness of limit (Lemma C.6) dictates that the two sequences converge to the same limit in K. Now that any convergent sequence converges to some limit point in K, Corollary D.98 implies that K is closed. □

Theorem E.48. A subset K of $\mathbb{R}^n$ is sequentially compact if and only if K is closed and bounded.

Proof. The necessity follows from Lemma E.47, we only prove the sufficiency. Any sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset K$ is bounded because K is bounded. Then Lemma E.45 dictates that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Because each term $\mathbf{x}_n \in K$ and K is closed, Corollary D.98 implies that the limit point of $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is also in K. The proof is then completed by Definition E.41. □

Example E.49. The intervals $(a, b]$, $[a, b)$, $(-\infty, b]$, and $[a, +\infty)$ are not sequentially compact in $\mathbb{R}$.

Definition E.50. The *Cantor set* is a subset of $\mathbb{R}$ give by $C := \cap_{n=1}^{+\infty} F_n$ where $F_1 = [0, 1]$ and each F_{n+1} is obtained by deleting from F_n the open middle third of each closed interval.

Example E.51. The Cantor set is an intersection of closed set and thus it is closed. It is also bounded and thus it is sequentially compact.

Corollary E.52. A subset K of a finite-dimensional normed space X is sequentially compact if and only if K is closed and bounded.

Example E.53. The closed unit ball in $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$

$$K := \{f \in \mathcal{C}[0, 1] : \|f\|_\infty \leq 1\} \tag{E.25}$$

is closed and bounded, but K is not sequentially compact. Consider the hat function

$$B_n(x) = \begin{cases} \frac{x-a_n}{b_n-a_n} & x \in [a_n, b_n], \\ \frac{x-c_n}{b_n-c_n} & x \in [b_n, c_n], \\ 0 & \text{otherwise}, \end{cases} \tag{E.26}$$

where $a_n = 1 - \frac{1}{2^n}$, $c_n = a_{n+1}$, and $b_n = \frac{a_n+c_n}{2}$. Then the sequence $(B_n)_{n\in\mathbb{N}}$ has no convergent subsequence.

Example E.54. The closed unit ball in ℓ^2,

$$K := \left\{\mathbf{x} \in \ell^2 : \|\mathbf{x}\|_2 \leq 1\right\}, \tag{E.27}$$

is closed and bounded, but is not sequentially compact. For

$$\mathbf{e}_n = (0, \cdots, 0, 1, 0, \cdots, 0) \in K \subset \ell^2$$

where all terms are zero except than the nth term is 1, the sequence $(\mathbf{e}_n)_{n\in\mathbb{N}^+}$ has no convergent subsequence.

Example E.55. The *Hilbert cube* in the normed space ℓ^2,

$$C := \left\{(x_n)_{n\in\mathbb{N}^+} :\ x_n \in \left[0, \frac{1}{n}\right]\right\}, \tag{E.28}$$

can be shown to be a sequentially compact subset.

Definition E.56. An *open cover of a topological space* X is collection of open subsets of X such that any element of X belongs to some open subset in the collection.

Definition E.57. A subset K in a topological space is *compact* if and only if every open cover of K has a finite subcover.

E.1.6 Continuous maps of normed spaces

Definition E.58. Let X and Y be normed spaces. A function $f : X \to Y$ is *continuous at* $x_0 \in X$ iff

$$\begin{aligned}&\forall \epsilon > 0, \exists \delta > 0 \text{ s.t.}\\ &\forall x \in X, \|x - x_0\|_X < \delta \ \Rightarrow\ \|f(x) - f(x_0)\|_Y < \epsilon.\end{aligned} \tag{E.29}$$

The function $f : X \to Y$ is *continuous* iff it is continuous at every $x_0 \in X$.

Lemma E.59. Let X and Y be normed spaces. A function $f : X \to Y$ is *continuous at* $x \in X$ iff, for any sequence with $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Exercise E.60. Prove Lemma E.59.

Lemma E.61. The norm function $\|\cdot\|$ is continuous.

Proof. By Definition E.58, we have $\lim_{n\to\infty} \|u_n - u\| = 0$ from $\lim_{n\to\infty} u_n = u$. The rest of the proof follows from the backward triangle inequality (E.12). □

Exercise E.62. For $V = \mathcal{C}[0, 1]$ and $x_0 \in [0, 1]$, define a function $\ell_{x_0} : V \to \mathbb{R}$ as

$$\ell_{x_0}(v) = v(x_0).$$

Show that ℓ_{x_0} is continuous on $\mathcal{C}[0, 1]$.

Example E.63. The function $S : (\mathcal{C}[0, 1], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$,

$$S(f) = \int_0^1 f^2(x)\mathrm{d}x, \tag{E.30}$$

is continuous. Indeed, for any $g \in \mathcal{C}[0, 1]$, we have

$$\begin{aligned}|S(f) - S(g)| &= \left|\int_0^1 f^2(x)\mathrm{d}x - \int_0^1 g^2(x)\mathrm{d}x\right|\\ &\le \int_0^1 |f(x) - g(x)|\,|f(x) - g(x) + 2g(x)|\,\mathrm{d}x\\ &\le \int_0^1 \|f - g\|_\infty \left(\|f - g\|_\infty + 2\|g\|_\infty\right)\mathrm{d}x,\end{aligned}$$

which implies

$$\begin{aligned}&\forall \epsilon > 0, \exists \delta = \min\left(1, \frac{\epsilon}{1 + 2\|g\|_\infty}\right) \text{ s.t. } \|f - g\|_\infty < \delta \ \Rightarrow\\ &\|f - g\|_\infty \left(\|f - g\|_\infty + 2\|g\|_\infty\right) < \|f - g\|_\infty \left(1 + 2\|g\|_\infty\right) < \epsilon\\ &\Rightarrow\ |S(f) - S(g)| < \epsilon.\end{aligned}$$

Example E.64. The differentiation map

$$\frac{\mathrm{d}}{\mathrm{d}t} : (\mathcal{C}^1[a, b], \|\cdot\|_\infty) \to (\mathcal{C}[a, b], \|\cdot\|_\infty)$$

is not continuous, but can be made continuous if we change the norm on $\mathcal{C}^1[a, b]$ to

$$\|f\|_{1,\infty} := \|f\|_\infty + \|f'\|_\infty. \tag{E.31}$$

Indeed, for $f_n(t) = \frac{1}{\sqrt{n}}\cos(2\pi n t)$, we have

$$\forall n \in \mathbb{N}^+, \quad \|f_n' - \mathbf{0}'\|_\infty = 2\pi\sqrt{n} > 1,$$

yet $\|f_n - \mathbf{0}\|$ can be made arbitrarily small as $n \to \infty$. In contrast, $D : (\mathcal{C}^1[a, b], \|\cdot\|_{1,\infty}) \to (\mathcal{C}[a, b], \|\cdot\|_\infty)$ is continuous because

$$\begin{aligned}&\forall \epsilon > 0,\ \exists \delta = \epsilon,\ \text{s.t.}\ \ \forall f, g \in \mathcal{C}^1[0, 1],\ \|f - g\|_{1,\infty} < \delta \ \Rightarrow\\ &\|Df - Dg\|_\infty = \|f' - g'\|_\infty \le \|f - g\|_{1,\infty} < \delta = \epsilon.\end{aligned}$$

Exercise E.65. Show that the arc length function $L : \mathcal{C}^1[0, 1] \to \mathbb{R}$,

$$L(f) := \int_0^1 \sqrt{1 + (f'(t))^2}\mathrm{d}t, \tag{E.32}$$

is not continuous if the norm of $\mathcal{C}^1[0, 1]$ is $\|\cdot\|_\infty$, whereas it is continuous if we equip $\mathcal{C}^1[0, 1]$ with (E.31).

Exercise E.66. Is the function $S : (c_{00}, \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$,

$$S\left((a_n)_{n\in\mathbb{N}}\right) = \sum_{n=1}^{\infty} a_n^2 \tag{E.33}$$

continuous?

Theorem E.67. A map $f : X \to Y$ between normed spaces is continuous if and only if the preimage $f^{-1}(V)$ of each open set V in Y is open in X.

Corollary E.68. A map $f : X \to Y$ between normed spaces is continuous if and only if the preimage $f^{-1}(V)$ of each closed set V in Y is closed in X.

Lemma E.69. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions between normed spaces, then the composition map $g \circ f : X \to Z$ is continuous.

Lemma E.70. Let X, Y be normed spaces and let K be a compact subset of X. If $f : X \to Y$ is continuous at each $x \in K$, then $f(K)$ is a compact subset of Y.

Proof. For a sequence $(y_n)_{n\in\mathbb{N}} \subset f(K)$, there exists for each $n \in \mathbb{N}$ an $x_n \in K$ such that $f(x_n) = y_n$. This defines a sequence $(x_n)_{n\in\mathbb{N}} \subset K$. Because K is compact, Definition E.41 implies the existence of a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to $L \in K$. Since f is continuous, Lemma E.59 implies that $(y_n)_{n\in\mathbb{N}}$ converges to $f(L) \in f(K)$. □

Theorem E.71 (Weierstrass)**.** Suppose K is a nonempty compact subset of a normed space X and the function $f : X \to \mathbb{R}$ is continuous at each $x \in K$. Then

$$\exists a, b \in K \text{ s.t. } \begin{cases} f(a) = \max\{f(x) : x \in K\},\\ f(b) = \min\{f(x) : x \in K\}.\end{cases}$$

Proof. It suffices to only prove the first clause. By Lemma E.70, $f(K)$ is compact, and thus by Lemma E.47 $f(K)$ is bounded. $f(K)$ is also nonempty because K is nonempty. Then Theorem A.28 implies that $f(K) \subset \mathbb{R}$ must have a unique supremum

$$M := \sup\{f(x) : x \in K\} \in \mathbb{R},$$

and hence there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset K$ satisfying $\lim_{n\to\infty} f(x_n) = M$. By Definition E.41, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some $c \in K$. The continuity of f, Lemma E.59 and Lemma C.9 yield

$$\lim_{k\to\infty} f(x_{n_k}) = f(c) = M.$$ □

Example E.72. Since the set $K = \{\mathbf{x} \in \mathbf{R}^3 : \|\mathbf{x}\|_2 = 1\}$ is compact in $\mathbb{R}^3$ and the function $\mathbf{x} \mapsto \sum_{j=1}^{3} x_j$ is continuous, the optimization problem

$$\begin{cases} \text{minimize } \sum_{j=1}^{3} x_j, \\ \text{subject to } \|\mathbf{x}\|_2 = 1, \end{cases}$$

has a minimizer.

E.1.7 Norm equivalence

Example E.73. The optimal mining problem in Example E.1 concerns $\mathcal{C}^1[a,b]$. Since $\mathcal{C}^1[a,b]$ is a subspace of $\mathcal{C}[a,b]$, we could use the norm $\|\cdot\|_\infty$ for $\mathcal{C}[a,b]$ as a norm for $\mathcal{C}^1[a,b]$. But by Example E.64, the differentiation map would not be continuous; instead, if we equip $\mathcal{C}^1[a,b]$ with (E.31), then the differentiation map is continuous. Also, it might be more appropriate to regard two functions in $\mathcal{C}^1[a,b]$ as being close to each other if both their function values and their function derivatives are close.

Definition E.74. Two norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on V are *equivalent*, written $\|\cdot\|_A \sim \|\cdot\|_B$, iff

$$\exists c_1, c_2 \in \mathbb{R}^+ \text{ s.t. } \forall v \in V, \quad c_1\|v\|_A \le \|v\|_B \le c_2\|v\|_A. \tag{E.34}$$

Example E.75. The Euclidean norms in Definition B.154 satisfy

$$\forall \mathbf{x} \in \mathbb{R}^n, \quad \|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_p \le \sqrt[p]{n}\|\mathbf{x}\|_\infty. \tag{E.35}$$

Therefore all the Euclidean ℓ_p norms are equivalent.

Exercise E.76. Show that $\sim$ in Definition E.74 defines an equivalence relation on the set of all norms on V.

Exercise E.77. Show that two norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on a linear space V are equivalent if and only if each sequence converging with respect to $\|\cdot\|_A$ also converges with respect to $\|\cdot\|_B$.

Theorem E.78. All norms are equivalent on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Proof. Since $\sim$ is an equivalence relation on the set of all norms, it suffices to prove that any norm is equivalent to the 2-norm. Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ be a basis of $\mathbb{R}^n$. Then any vector $\mathbf{x} \in \mathbb{R}^n$ can be expressed as $\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{e}_i$. Thus

$$\|\mathbf{x}\| = \left\|\sum_{i=1}^{n} x_i\mathbf{e}_i\right\| \le \sum_{i=1}^{n} |x_i|\|\mathbf{e}_i\| \le M\|\mathbf{x}\|_2,$$

where $M = \sqrt{\sum_{i=1}^{n} \|\mathbf{e}_i\|}$ and the last inequality follows from the Cauchy-Schwarz inequality (Theorem B.171). Set

$$K := \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y}\|_2 = 1\}.$$

Since K is a compact set and the norm $\|\cdot\| : K \to \mathbb{R}$ is a continuous function (Lemma E.61), $\|\cdot\|$ must attain its minimum value m on K. Furthermore, $m > 0$ since $\mathbf{0} \notin K$. For $\mathbf{x} \ne 0$, we have $\mathbf{y} := \frac{\mathbf{x}}{\|\mathbf{x}\|_2} \in K$ since $\|\mathbf{y}\|_2 = 1$. Then $\|y\| \ge m$ implies $m\|\mathbf{x}\|_2 \le \|\mathbf{x}\|$. □

Corollary E.79. Over a finite dimensional space, any two norms are equivalent.

Proof. This follows from Theorem E.78 and the isomorphism of linear spaces. □

Example E.80. In the normed space $V := \mathcal{C}[0,1]$, consider a sequence of functions $\{u_n\}$ given by

$$u_n(x) := \begin{cases} 1 - nx, & x \in [0, \frac{1}{n}]; \\ 0, & x \in (\frac{1}{n}, 1]. \end{cases}$$

For the p-norm in (E.14), we have

$$\|u_n\|_p = [n(p+1)]^{-\frac{1}{p}}$$

and thus the sequence $\{u_n\}$ converges to $u = 0$. However, for the ∞-norm in (E.15), we have

$$\|u_n\|_\infty = 1$$

and thus the sequence $\{u_n\}$ does not converge to $u = 0$.

E.1.8 Banach spaces

Definition E.81. A *Cauchy sequence in a normed space V* is a sequence $\{u_n\} \subset V$ satisfying

$$\lim_{m,n\to+\infty} \|u_m - u_n\| = 0. \tag{E.36}$$

Definition E.82 (Banach spaces)**.** A *Banach space* (or a *complete* normed space) is a normed space V such that every Cauchy sequence in V converges to an element in V.

Example E.83 ($\mathbb{Q}$ is not complete)**.** The sequence

$$x_1 = \frac{3}{2}; \qquad \forall n > 1,\ x_n = \frac{4 + 3x_{n-1}}{3 + 2x_{n-1}} \tag{E.37}$$

is bounded below by $\sqrt{2}$ and is monotonically decreasing. By Theorem C.12, (x_n) is convergent in $\mathbb{R}$. However, although (x_n) is Cauchy in $\mathbb{Q}$, it is not convergent in $\mathbb{Q}$ because

$$L = \frac{4 + 3L}{3 + 2L} \ \Rightarrow\ L = \sqrt{2}.$$

Example E.84. The sequence $(\sum_{k=1}^{n} \frac{1}{k^k})_{n\in\mathbb{N}}$ is Cauchy, but we do not know yet whether the limit is rational or irrational.

Theorem E.85. $(\mathcal{C}[a,b], \|\cdot\|_\infty)$ is a Banach space.

Proof. It is straightforward to show that $\|\cdot\|_\infty$ is a norm. We only show the completeness in three steps.

First, at any fixed $t \in [a,b]$, we can reduce a Cauchy sequence $\{f_n\}_{n\ge1} \subset \mathcal{C}[a,b]$ to a sequence $\{f_n(t)\} \subset \mathbb{R}$. The completeness of $\mathbb{R}$ (Theorem C.15) yields $\lim_{n\to\infty} f_n(t) \in \mathbb{R}$. For any Cauchy sequence $\{f_n\} \subset \mathcal{C}[a,b]$, this process furnishes a function $f : [a,b] \to \mathbb{R}$ given by

$$f(t) = \lim_{n\to\infty} f_n(t).$$

Second, we show $f \in \mathcal{C}[a,b]$, i.e., f is continuous. The sequence $\{f_n\} \subset \mathcal{C}[a,b]$ being Cauchy implies

$$\forall \epsilon > 0, \exists N-1 \in \mathbb{N} \text{ s.t. } \forall m, n > N-1, \qquad \|f_m - f_n\|_\infty < \frac{\epsilon}{3}.$$

In particular, set $m = N$, let $n \to \infty$, and we have

$$\forall t \in [a,b], \quad |f_N(t) - f(t)| \le \|f_N - f_n\|_\infty < \frac{\epsilon}{3}.$$

The condition of $f_N \in \mathcal{C}[a,b]$ implies

$$\begin{aligned}&\forall t \in [a,b], \forall \epsilon > 0, \exists \delta > 0 \text{ s.t.}\\ &\qquad |t-\tau| < \delta \Rightarrow |f_N(t) - f_N(\tau)| < \frac{\epsilon}{3}.\end{aligned}$$

The above two equations yield

$$\begin{aligned}&\forall t \in [a,b], \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |t-\tau| < \delta \Rightarrow\\ &|f(t) - f(\tau)| \le |f(t) - f_N(t)| + |f_N(t) - f_N(\tau)|\\ &\qquad + |f_N(\tau) - f(\tau)|\\ &\qquad \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,\end{aligned}$$

which shows that f is continuous at every $t \in [a,b]$.

Finally, we show that $\{f_n\}_{n\ge1}$ indeed converges to f. The sequence $\{f_n\} \subset \mathcal{C}[a,b]$ being Cauchy implies

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m,n > N, \qquad \|f_m - f_n\|_\infty < \epsilon.$$

For a fixed $n > N$, we have

$$\forall m > N, \forall t \in [a,b], \quad |f_n(t) - f_m(t)| \le \|f_n - f_m\|_\infty < \epsilon,$$

which implies

$$\forall t \in [a,b], \quad |f_n(t) - f(t)| = \left| f(t) - \lim_{m\to\infty} f_m(t) \right| < \epsilon.$$

It follows that

$$\|f_n - f\|_\infty = \max_{t\in[a,b]} |f_n(t) - f(t)| < \epsilon.$$

In the above process, we could have fixed any $n > N$ at the outset to obtain the same result. Therefore we have

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, \ \|f_n - f\|_\infty < \epsilon,$$

which implies $\lim_{n\to\infty} f_n = f$. □

Exercise E.86. Define $\mathcal{C}_b[0,\infty)$ as the set of all functions f that are continuous on $[0,\infty)$ and satisfy

$$\|f\|_\infty := \sup_{x\ge0} |f(x)| < \infty.$$

Show $\mathcal{C}_b[0,\infty)$ with this norm is complete.

Exercise E.87. Define $\mathcal{C}^\alpha[a,b]$ as the set of all functions $f \in \mathcal{C}[a,b]$ satisfying

$$M_\alpha(f) := \sup_{x,y\in[a,b];x\ne y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

Define $\|f\|_\alpha = \|f\|_\infty + M_\alpha(f)$. Show that $(\mathcal{C}^\alpha[a,b], \|\cdot\|_\alpha)$ is a Banach space.

Example E.88. For $p \in [1,\infty)$, $\left(\mathcal{C}(\overline{\Omega}), \|\cdot\|_p\right)$ is not a Banach space. Consider $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}[0,1]$ given by

$$u_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{2} - \frac{1}{2n}];\\ nx - \frac{n-1}{2}, & x \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}];\\ 1 & x \in [\frac{1}{2} + \frac{1}{2n}, 1].\end{cases} \tag{E.38}$$

$(u_n)_{n\in\mathbb{N}}$ is clearly Cauchy and we have

$$\lim_{n\to\infty} u_n = u(x) = \begin{cases} 0, & x \in [0, \frac{1}{2});\\ 1, & x \in (\frac{1}{2}, 1].\end{cases}$$

But $u(x)$ cannot be in $\mathcal{C}(\overline{\Omega})$ no matter how we define $u(\frac{1}{2})$.

Exercise E.89. Show that the sequence space $(\ell^p, \|\cdot\|_p)$ is complete for $p \in [1,+\infty]$.

Theorem E.90. In a Banach space, absolutely convergent series converge. More precisely, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in a Banach space $(X, \|\cdot\|)$ such that $\sum_{n=1}^\infty \|x_n\|$ converges, then $\sum_{n=1}^\infty x_n$ converges in X. Furthermore,

$$\left\| \sum_{n=1}^\infty x_n \right\| \le \sum_{n=1}^\infty \|x_n\|. \tag{E.39}$$

Proof. Since X is Banach, it suffices to prove that the sequence $(s_n = \sum_{i=1}^n x_i)_{n\in\mathbb{N}}$ is Cauchy. Since the real sequence $(\sigma_n = \sum_{i=1}^n \|x_i\|)_{n\in\mathbb{N}}$ is Cauchy, we have

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N}^+ \text{ s.t. } \forall n > m > N, \ \sum_{i=m+1}^n \|x_i\| < \epsilon,$$

which implies that $(s_n = \sum_{i=1}^n x_i)_{n\in\mathbb{N}}$ is Cauchy:

$$\left\| \sum_{i=m+1}^n x_i \right\| \le \sum_{i=m+1}^n \|x_i\| < \epsilon.$$

Set $L := \sum_{i=m+1}^n x_i$ and we have

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, \ \|s_n - L\| < \epsilon,$$

which implies

$$\|L\| \le \|s_n - L\| + \|s_n\| < \epsilon + \sigma_n < \epsilon + \sum_{n=1}^\infty \|x_n\|,$$

where the second inequality follows from the triangle inequality and the third from n being a finite number. Then (E.39) holds because ϵ can be made arbitrarily small. □

Example E.91. The series $\sum_{n=1}^\infty \frac{1}{n^2}\sin(nx)$ converges in $(\mathcal{C}[0,2\pi], \|\cdot\|_\infty)$ since $\sum_{n=1}^\infty \frac{1}{n^2}$ converges in $\mathbb{R}$. Hence $x \mapsto \sum_{n=1}^\infty \frac{1}{n^2}\sin(nx)$ defines a continuous function.

Exercise E.92. Prove the converse of Theorem E.90, i.e., a normed space X is complete if every absolutely convergent series converges in X.

Theorem E.93. For each normed space V, there exists another normed space W and a dense subspace $\widehat{V} \subset W$ such that one can find an *isometric isomorphism* between V and $\widehat{V}$, i.e., a bijective linear function $\mathcal{I} : V \to \widehat{V}$ satisfying

$$\forall v \in V, \quad \|\mathcal{I}v\|_W = \|v\|_V. \tag{E.40}$$

Furthermore, the complete normed space W is unique up to the isometric isomorphism.

Definition E.94. The normed space W in Theorem E.93 is called the *completion of the normed space* V.

Example E.95. If V is the normed space $\mathbb{Q}$ of rational numbers, then $W = \mathbb{R}$ is a completion of $\mathbb{Q}$, where each element is an equivalence class of Cauchy sequences of rational numbers.

E.2 Continuous linear maps

E.2.1 The space $\mathcal{CL}(X, Y)$

Notation 31. $\mathcal{CL}(X, Y)$ denotes the *set of all continuous linear transformations or bounded linear transformations* from the normed space X to the normed space Y,

$$\mathcal{CL}(X, Y) := \mathcal{C}(X, Y) \cap \mathcal{L}(X, Y). \tag{E.41}$$

For $Y = X$, we write $\mathcal{CL}(X)$.

Theorem E.96. For any map $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:

(1) T is continuous,

(2) T is continuous at $\mathbf{0}$,

(3) $\exists M \in \mathbb{R}^+$ s.t. $\forall x \in X$, $\|Tx\|_Y \le M\|x\|_X$.

Proof. (1)⇒(2) follows from Definition E.58. For (2)⇒(3), the continuity of T at $\mathbf{0}$ implies

$$\text{for } \epsilon = 1, \exists \delta > 0 \text{ s.t. } \|x\| < \delta \;\Rightarrow\; \|Tx\| < 1.$$

Replacing x with $y = \frac{\delta}{2}\frac{x}{\|x\|}$ in the above inequalities yields $\|Tx\| \le M\|x\|$ with $M = \frac{2}{\delta}$. Finally, (3)⇒(1) follows from

$$\forall \epsilon > 0, \exists \delta = \frac{\epsilon}{M} \text{ s.t. } \|x - y\| < \delta \;\Rightarrow$$
$$\|Tx - Ty\| = \|T(x - y)\| \le M\|x - y\| = \frac{\epsilon}{\delta}\|x - y\| < \epsilon. \qquad \square$$

Example E.97. The left shift operator $L : \ell^2 \to \ell^2$ and right shift operator $R : \ell^2 \to \ell^2$,

$$L(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots), \tag{E.42}$$
$$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots), \tag{E.43}$$

are linear operators. Furthermore, $L, R \in \mathcal{CL}(\ell^2)$ because they are bounded:

$$\|L(a_n)_{n\in\mathbb{N}}\| \le \|(a_n)_{n\in\mathbb{N}}\|, \quad \|R(a_n)_{n\in\mathbb{N}}\| = \|(a_n)_{n\in\mathbb{N}}\|.$$

Example E.98. The linear map $T : (\mathcal{C}[a, b], \|\cdot\|_\infty) \to \mathbb{R}$ given by $T(f) = \int_a^b f(t)\mathrm{d}t$ is continuous because

$$|T(f)| = \left|\int_a^b f(t)\mathrm{d}t\right| \le \int_a^b \|f\|_\infty \mathrm{d}t = (b - a)\|f\|_\infty.$$

By Lemma E.59, T preserves convergent sequences:

$$\lim_{n\to\infty} f_n = f \;\Rightarrow\; \lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

In other words, the continuity of T under $\|\cdot\|_\infty$ guarantees that T and $\lim_{n\to\infty}$ are commutative; see Section C.7.

Theorem E.99 (Existence and uniqueness of ODEs)**.** The IVP

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t), t) \tag{E.44}$$

with initial condition $x(0) = x_0 \in \mathbb{R}$ has a unique solution $x \in \mathcal{C}^1[0, T]$ for some $T > 0$, if $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is Lipschitz continuous in space and continuous in time.

Proof. For existence, we define $y_0(t) = x_0$ and

$$(*): \quad y_{n+1}(t) = x_0 + \int_0^t f(y_n(\tau), \tau)\mathrm{d}\tau.$$

For any $t \in [0, \frac{1}{2L}]$ where L is the Lipschitz constant,

$$\begin{aligned}
|y_{n+1}(t) - y_n(t)| &= \left|\int_0^t f(y_n(\tau), \tau) - f(y_{n-1}(\tau), \tau)\mathrm{d}\tau\right| \\
&\le \int_0^t |f(y_n(\tau), \tau) - f(y_{n-1}(\tau), \tau)|\,\mathrm{d}\tau \\
&\le \int_0^t L|y_n(\tau) - y_{n-1}(\tau)|\mathrm{d}\tau \\
&\le \int_0^t L\|y_n - y_{n-1}\|_\infty \mathrm{d}\tau \\
&\le \tfrac{1}{2}\|y_n - y_{n-1}\|_\infty.
\end{aligned}$$

Hence we have

$$\|y_{n+1} - y_n\|_\infty \le \frac{1}{2}\|y_n - y_{n-1}\|_\infty \le \frac{1}{2^n}\|y_1 - y_0\|_\infty.$$

It follows that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists $y \in \mathcal{C}^1[0, T]$ such that $\lim_{n\to\infty} y_n = y$. Similarly, $(f(y_n, t))_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists $f(y, t)$ such that $\lim_{n\to\infty} f(y_n, t) = f(y, t)$. Take $\lim_{n\to\infty}(*)$, apply Example E.98, and we have

$$(**): \quad y(t) = x_0 + \int_0^t f(y(\tau), \tau)\mathrm{d}\tau.$$

It is trivial to check that the above $y(t)$ solves (E.44).

For uniqueness, suppose for two solutions x and y of (E.44) there exists $t^* \in (0, T)$ satisfying

$$t^* := \max\{t \in [0, T] : \forall \tau \le t,\ y(\tau) = x(\tau)\}.$$

We choose

$$\begin{aligned}
N &:= \max\left\{2,\ \tfrac{1}{L(T - t^*)}\right\}, \\
M &:= \max_{t\in[t_*, t_* + \frac{1}{LN}]} |x(t) - y(t)|
\end{aligned}$$

to obtain $t_* + \frac{1}{LN} \le T$. Then (**) implies

$$\begin{aligned}
&\forall t \in [t^*, t_* + \tfrac{1}{LN}],\\
&|x(t) - y(t)| = \left|\int_{t^*}^{t} [f(x(\tau),\tau) - f(y(\tau),\tau)]\,\mathrm{d}\tau\right|\\
&\le \int_{t^*}^{t} |f(x(\tau),\tau) - f(y(\tau),\tau)|\,\mathrm{d}\tau \le \int_{t^*}^{t} L\,|x(\tau) - y(\tau)|\,\mathrm{d}\tau\\
&\le LM(t - t^*) \le \tfrac{M}{N},
\end{aligned}$$

which yields $M \le \frac{M}{N}$ and contradicts $N \ge 2$. Hence the uniqueness is proved by the non-existence of such a t^*. ☐

Example E.100. The continuity of differentiation maps in Example E.64 can be determined by Theorem E.96. For $\|\cdot\|_{1,\infty}$, we have $\|D\mathbf{x}\|_\infty \le \|\mathbf{x}\|_{1,\infty}$, and thus the operator $D : (\mathcal{C}[0,1], \|\cdot\|_{1,\infty}) \to (\mathcal{C}[0,1], \|\cdot\|_\infty)$ is continuous.

In comparison, $D : (\mathcal{C}[0,1], \|\cdot\|_\infty) \to (\mathcal{C}[0,1], \|\cdot\|_\infty)$ is not continuous: for $\mathbf{x}_n = t^n$, we have $\|x_n\|_\infty = 1$ yet $\lim_{n\to\infty} \|\mathbf{x}'_n\| = \infty$.

Corollary E.101. For finite-dimensional normed spaces X and Y, we have $\mathcal{L}(X,Y) = \mathcal{CL}(X,Y)$.

Proof. Each linear transformation $T_A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ has a matrix $A \in \mathbb{R}^{m\times n}$ such that

$$\begin{aligned}
\|T_A\mathbf{x}\|_2^2 = \|A\mathbf{x}\|_2^2 &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2\\
&\le \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right)^2\left(\sum_{j=1}^{n} x_j\right)^2 = \|\mathbf{x}\|_2^2 \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2,
\end{aligned}$$

where the inequality follows from the Cauchy-Schwarz inequality. The proof is completed by Theorem E.96, Theorem E.78, and the isomorphism of linear spaces. ☐

Exercise E.102. For an infinite-dimensional matrix A satisfying $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^2 < \infty$, define $T_A : \ell^2 \to \ell^2$ by

$$\forall \mathbf{x} = (x_j)_{j\in\mathbb{N}} \in \ell^2, \quad T_A\mathbf{x} = A\mathbf{x} = \left(\sum_{j=1}^{\infty} a_{ij}x_j\right)_{i\in\mathbb{N}^+}.$$

Prove $T_A \in \mathcal{CL}(\ell^2)$.

Exercise E.103. For $A, B \in \mathcal{C}[a,b]$ and

$$S := \{f \in \mathcal{C}^1[a,b] : f(a) = f(b) = 0\}, \tag{E.45}$$

show that the map $L : (S, \|\cdot\|_{1,\infty}) \to \mathbb{R}$ given by

$$L(f) = \int_a^b \left[A(t)f(t) + B(t)f'(t)\right]\mathrm{d}t$$

is a bounded linear transformation.

Exercise E.104. For $A \in \mathbb{R}^{m\times n}$, show that the subspace $\ker A$ is closed in $\mathbb{R}^n$.

Theorem E.105. Every subspace of $\mathbb{R}^n$ is closed.

Exercise E.106. Prove Theorem E.105.

Lemma E.107. $\mathcal{CL}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$.

Exercise E.108. Prove Lemma E.107.

Lemma E.109. The *operator norm* $\|\cdot\| : \mathcal{CL}(X,Y) \to \mathbb{R}$,

$$\forall T \in \mathcal{CL}(X,Y),\ \|T\| := \sup\big\{\|Tx\| : x \in X, \|x\| \le 1\big\}, \tag{E.46}$$

is well defined, i.e., $\|T\|$ is a unique bounded real number.

Proof. By Theorem A.28, it suffices to show that

$$S := \big\{\|Tx\| : x \in X, \|x\| \le 1\big\} \tag{E.47}$$

is a nonempty bounded subset of $\mathbb{R}$. S is nonempty because $\mathbf{0} \in X$ and $T\mathbf{0} = \mathbf{0}_Y$, imply $0 \in S$. The boundedness of S follows from Theorem E.96(3) and $\|x\|_X \le 1$. ☐

Lemma E.110. For any $T \in \mathcal{CL}(X,Y)$, we have

$$(\forall x \in X, \|Tx\| \le M\|x\|) \ \Rightarrow\ \|T\| \le M. \tag{E.48}$$

Proof. M is an upper bound of the set S in (E.47) while $\|T\|$ is the least upper bound of S. ☐

Lemma E.111. $\forall T \in \mathcal{CL}(X,Y)$, $\forall x \in X$, $\|Tx\| \le \|T\|\|x\|$.

Proof. The statement holds trivially for $x = \mathbf{0}$. Otherwise for $y = \frac{x}{\|x\|}$ we have $\|Ty\| \in S$ where S is in (E.47). Hence

$$\|Ty\| \le \|T\| \ \Rightarrow\ \|Tx\| \le \|T\|\|x\|.$$ ☐

Lemma E.112. $\forall S \in \mathcal{CL}(X,Y)$, $\forall T \in \mathcal{CL}(Y,Z)$, we have $\|ST\| \le \|S\|\|T\|$.

Proof. This follows from Lemmas E.110 and E.111. ☐

Theorem E.113. $\big(\mathcal{CL}(X,Y), \|\cdot\|\big)$ is a normed space.

Exercise E.114. Prove Theorem E.113.

Lemma E.115. For a normed space X, $\big(\mathcal{CL}(X,Y), \|\cdot\|\big)$ is a Banach space if Y is a Banach space.

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{CL}(X,Y)$. For any $x \in X$, $(T_nx)_{n\in\mathbb{N}} \subset Y$ is Cauchy as Lemma E.111 yields

$$\|T_nx - T_mx\| \le \|T_n - T_m\|\|x\|.$$

Since Y is complete, $(T_nx)_{n\in\mathbb{N}}$ converges to some $L(x) \in Y$. This defines a map $T(x) = L(x)$.

The second step is to show $T \in \mathcal{CL}(X,Y)$.

The third step is to show $\lim_{n\to\infty} T_n = T$. ☐

Exercise E.116. Supplement the proof of Lemma E.115 with all details.

Corollary E.117. If X is a normed space over $\mathbb{R}$, then the dual space of X, $X' = \mathcal{CL}(X,\mathbb{R})$, is a Banach space with the operator norm.

Proof. This follows directly from Lemma E.115. ☐

Corollary E.118. If X is a Banach space, then $\mathcal{CL}(X)$ is a Banach space with the operator norm.

Proof. This follows directly from Lemma E.115. ☐

Definition E.119. An *algebra* is a vector space V with an associative and distributive multiplication $V \times V \to V$,

$$\forall u, v, w \in V,\ \forall \alpha \in \mathbb{F},$$
$$\begin{cases} u(vw) = (uv)w, \\ (u+v)w = uw + vw, \quad u(v+w) = uv + uw, \\ \alpha(uv) = u(\alpha v) = (\alpha u)v. \end{cases} \tag{E.49}$$

The *multiplicative identity* is the element $e \in V$ such that $\forall v \in V$, $ev = v = ve$.

Definition E.120. A *normed algebra* is an algebra V with a norm $\|\cdot\|$ satisfying

$$\forall u, v \in V, \quad \|uv\| \le \|u\|\|v\|. \tag{E.50}$$

A *Banach algebra* is a normed algebra that is complete.

E.2.2 The topology of $\mathcal{CL}(X, Y)$

Notation 32. For a vector space X and its subsets A, A_1, A_2, we write

$$\begin{aligned} \forall \alpha \in \mathbb{R}, \quad \alpha A &:= \{\alpha a, a \in A\}; \\ \forall w \in X, \quad A + w &:= \{a + w, a \in A\}. \\ \forall A_1, A_2 \subset X, \quad A_1 + A_2 &:= \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}. \end{aligned} \tag{E.51}$$

Definition E.121. A linear map $T : X \to Y$ between normed spaces X and Y is *open* if its image of any open set is open.

Lemma E.122. Let X and Y be normed spaces. A bounded linear map $T \in \mathcal{CL}(X, Y)$ is open if and only if the image of the unit open ball in X under T contains some open ball centered at $\mathbf{0}_Y$ in Y, i.e.,

$$\exists \delta > 0 \text{ s.t. } B(\mathbf{0}_Y, \delta) \subset T\left(B(\mathbf{0}_X, 1)\right). \tag{E.52}$$

Proof. For necessity, T being an open map implies that the image $T(B(\mathbf{0}_X, 1))$ is open. The linearity of T implies $\mathbf{0}_Y \in T(B(\mathbf{0}_X, 1))$. Then Lemma D.190 yields (E.52).

For sufficiency, let $U \subset X$ be open, we need to show

$$(*): \quad \forall y_0 \in T(U),\ \exists r_Y > 0 \text{ s.t. } B(y_0, r_Y) \subset T(U).$$

$y_0 \in T(U)$ implies there exists $x_0 \in U$ such that $Tx_0 = y_0$. Since U is open, we have

$$(**): \quad \exists r_X > 0 \text{ s.t. } B(x_0, r_X) \subset U.$$

Choose $r_Y = \delta r_X$ and we have

$$\begin{aligned} B(Tx_0, r_Y) &= Tx_0 + B(\mathbf{0}_Y, \delta r_X) \subset Tx_0 + TB(\mathbf{0}_X, r_X) \\ &= TB(x_0, \delta r_X) \subset T(U), \end{aligned}$$

where the second step follows from (E.52). □

Lemma E.123. If a closed set F in a normed space X does not contain any open set, then $X \setminus F$ is dense in X.

Proof. We need to show

$$\forall x \in X, \exists r > 0 \text{ s.t. } B(x, r) \cap (X \setminus F) \neq \emptyset.$$

If $x \in (X \setminus F)$, then we are done. Otherwise $x \in F$ implies that $B(x, r)$ is not contained in F for any $r > 0$. Therefore,

$$\forall r > 0, \forall x \in X,\ \exists y \in B(x, r) \subset (X \setminus F) \text{ s.t. } \|y - x\| < r.$$

Then the proof is completed by Lemma D.190. □

Theorem E.124 (Baire)**.** Suppose $(F_n)_{n\in\mathbb{N}}$ is a sequence of closed sets in a Banach space X such that $X = \cup_{n\in\mathbb{N}} F_n$. Then there exists an $n \in \mathbb{N}$ and a nonempty open set U such that $U \subset F_n$.

Proof. Suppose that no F_n contains any nonempty open set. Then Lemma E.123 implies that $X \setminus F_n$ is dense in X for each $n \in \mathbb{N}$. Therefore we have

$$\exists x_1 \in (X \setminus F_1), \exists r_1 > 0 \text{ s.t. } \overline{B(x_1, r_1)} \subset (X \setminus F_1).$$

Both $B(x_1, r_1)$ and $(X \setminus F_1)$ are open and thus their intersection $D_2 := B(x_1, r_1) \cap (X \setminus F_1)$ is also open. Hence,

$$\exists x_2 \in D_2, \exists r_2 \in \left(0, \frac{r_1}{2}\right) \text{ s.t. } \overline{B(x_2, r_2)} \subset D_2;$$

Proceed inductively and we have

$$\exists x_n \in D_n, \exists r_n \in \left(0, \frac{r_{n-1}}{2}\right) \text{ s.t. } \overline{B(x_n, r_n)} \subset D_n,$$

where $D_n := B(x_{n-1}, r_{n-1}) \cap (X \setminus F_{n-1})$. By construction, $n > m$ implies $B(x_n, r_n) \subset B(x_m, r_m)$ and

$$\|x_n - x_m\| < r_m < \frac{r_1}{2^{m-1}}.$$

Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and converges to x in the Banach space X. For any $m \in \mathbb{N}$, we have

$$x \in \overline{B(x_m, r_m)} \subset \left(X \setminus \cup_{i=1}^m F_m\right),$$

which contradicts $X = \cup_{n\in\mathbb{N}} F_n$ as $m \to \infty$. □

Lemma E.125 (Unit open ball)**.** Suppose X and Y are Banach spaces and $T \in \mathcal{CL}(X, Y)$ is surjective. Then the image $T(B_0)$ of the open ball $B_0 := B(\mathbf{0}_X, 1)$ contains an open ball about $\mathbf{0}_Y$.

Proof. Define

$$B_n := B\left(\mathbf{0}_X, \frac{1}{2^n}\right)$$

and we show $\overline{T(B_1)}$ contains an open ball. Indeed

$$\forall x \in X,\ \exists k > 2\|x\| \text{ s.t. } x \in kB_1$$

and thus $X = \cup_{k\in\mathbb{N}} kB_1$. T being surjective implies

$$Y = T(X) = T\left(\bigcup_{k\in\mathbb{N}^+} kB_1\right) = \bigcup_{k\in\mathbb{N}^+} kT(B_1) = \bigcup_{k\in\mathbb{N}^+} \overline{kT(B_1)},$$

where the last step follows from the condition of Y being a Banach space. By Theorem E.124, there exists some $\overline{kT(B_1)}$ that contains a nonempty open ball, which implies that $\overline{T(B_1)}$ also contains an open ball, say,

$$B(y_0, \epsilon) \subset \overline{T(B_1)},$$

which implies

$$B(\mathbf{0}_Y, \epsilon) = B(y_0, \epsilon) - y_0 \subset \overline{T(B_1)} + \overline{T(B_1)} \\ \subset \overline{T(B_1) + T(B_1)} = \overline{T(B_0)}.$$

To sum up the above arguments, we have

$$(*): \quad B(\mathbf{0}_Y, \epsilon) \subset \overline{T(B_0)}.$$

Define $V_n := B(\mathbf{0}_Y, \frac{\epsilon}{2^n})$. To complete the proof, we show

$$V_1 = B\left(\mathbf{0}_Y, \frac{\epsilon}{2}\right) \subset T(B_0).$$

The linearity of T and $(*)$ imply

$$(\triangle): \quad \forall n \in \mathbb{N},\ V_n \subset \overline{T(B_n)}.$$

For $y \in V_1$, we have $y \in \overline{T(B_1)}$. Since both T and $\|\cdot\|$ are continuous, the map $x \mapsto \|y - Tx\|$ is also continuous, and therefore

$$\exists x_1 \in B_1 \text{ s.t. } \|y - Tx_1\| < \frac{\epsilon}{4}.$$

By definition of V_n and $(\triangle)$, $y - Tx_1 \in V_2 \subset \overline{T(B_2)}$. Thus

$$\exists x_2 \in B_2 \text{ s.t. } \|(y - Tx_1) - Tx_2\| < \frac{\epsilon}{8}.$$

Proceed inductively and we have

$$\forall k = 1, 2, \ldots, n,\ \exists x_k \in B_k \text{ s.t. } \left\| y - T\sum_{k=1}^{n} x_k \right\| < \frac{\epsilon}{2^{n+1}}.$$

Take limit of the above and we have

$$(\square): \quad y = T \lim_{n\to\infty} \sum_{k=1}^{n} x_k.$$

For each k, $x_k \in B_k$ implies $\|x_k\| < \frac{1}{2^k}$. Hence

$$\sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

The completeness of X and Theorem E.90 yield

$$\exists x \in X \text{ s.t. } \ x = \sum_{k=1}^{\infty} x_k, \quad \|x\| < 1.$$

Then $(\square)$ yields $y = Tx \in T(B_0)$. □

Theorem E.126 (Open mapping)**.** For Banach spaces X and Y, any surjective map $T \in \mathcal{CL}(X, Y)$ is open.

Proof. This follows from Lemmas E.122 and E.125. □

Example E.127. The following function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x+1 & \text{if } x \in (-\infty, -1]; \\ 0 & \text{if } x \in (-1, +1); \\ x-1 & \text{if } x \in [+1, +\infty), \end{cases}$$

is surjective and continuous; but since $f(-1, 1) = \{0\}$ is closed, f is not open. By the open mapping theorem, if a map between two Banach spaces is not open but surjective and continuous, then it must be nonlinear.

E.2.3 Invertible operators

Lemma E.128. In a finite-dimensional vector space X, if two operators $T, S \in \mathcal{L}(X)$ satisfy $TS = I$, then $ST = I$.

Proof. $TS = I$ implies $\ker S = \{\mathbf{0}\}$ because

$$Sx = \mathbf{0} \ \Rightarrow\ TSx = \mathbf{0} \ \Rightarrow\ x = \mathbf{0}.$$

Thus for any basis $(v_i)_{i=1}^n$ of X, $(Sv_i)_{i=1}^n$ is also a basis.

$$\forall x \in X,\ \exists (\beta_i)_{i=1}^n \text{ s.t. } x = \sum_{i=1}^{n} \beta_i S v_i = S \sum_{i=1}^{n} \beta_i v_i.$$

It follows that

$$\forall x \in X,\ STx = STS\sum_{i=1}^{n} \beta_i v_i = S\sum_{i=1}^{n} \beta_i v_i = x,$$

which implies $ST = I$. □

Example E.129. For the shift operators on ℓ^2 in Example E.97, we have $LR = I$ but $RL \neq I$,

$$RL(1, 0, 0, \ldots) = (0, 0, 0, \ldots).$$

Definition E.130. For vector spaces X and Y, a map $A \in \mathcal{L}(X, Y)$ is *invertible* if there exists $B \in \mathcal{L}(Y, X)$ such that $AB = I \in \mathcal{L}(Y)$ and $BA = I \in \mathcal{L}(X)$. Then B is called the *inverse* of A.

Exercise E.131. Prove that the inverse of $A \in \mathcal{L}(X, Y)$ is unique if A is invertible.

Lemma E.132. For any vector spaces X and Y, if a linear map $A \in \mathcal{L}(X, Y)$ is invertible, then A is bijective.

Proof. A is injective because

$$Ax = Ay \ \Rightarrow\ A^{-1}Ax = A^{-1}Ay \ \Rightarrow\ x = y.$$

A is surjective because $\forall y \in Y$, $A^{-1}y \in X$ implies $y = Ax$ for some $x \in X$. □

Lemma E.133. For any vectors space X and Y, if a map $A \in \mathcal{L}(X, Y)$ is invertible, then its inverse A^{-1} is linear.

Proof. For any $x, y \in X$, set $z = A^{-1}(x + y)$ and we have

$$x + y = Az \ \Rightarrow\ A^{-1}x + A^{-1}y = z = A^{-1}(x + y).$$

Similarly, for any $\alpha \in \mathbb{F}$, set $z = A^{-1}(\alpha x)$ and we have

$$Az = \alpha x \ \Rightarrow\ A\frac{z}{\alpha} = x \ \Rightarrow\ \frac{z}{\alpha} = A^{-1}x \\ \Rightarrow \alpha A^{-1}x = z = A^{-1}(\alpha x).$$ □

Lemma E.134. For finite-dimensional vector space X and Y, if a map $A \in \mathcal{L}(X, Y)$ is bijective, then A is invertible.

Proof. For the bijective map A, define a map $B : Y \to X$,

$$\forall v \in X, \quad A(Bv) = v.$$

The existence and uniqueness of Bv are guaranteed by the surjectivity and injectivity of A. Therefore, $AB = I$. Furthermore, $BA = I$ follows from the injectivity of A and

$$\forall v \in X, \quad A(BAv) = (AB)Av = Av.$$

Finally, Lemma E.133 implies that B is a linear map. □

Theorem E.135. Suppose X and Y are finite-dimensional normed spaces. Then a map $A \in \mathcal{CL}(X, Y)$ is invertible with $A^{-1} \in \mathcal{CL}(Y, X)$ if and only if A is bijective.

Proof. This follows from Lemmas E.132, E.133, E.134, and Corollary E.101. □

Example E.136. The map $A : c_{00} \to c_{00}$ given by

$$\forall (x_n)_{n\in\mathbb{N}} \in c_{00}, \quad A(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$$

is linear, bijective, and continuous (since $\|A\mathbf{x}\|_\infty \le \|\mathbf{x}\|_\infty$). However, it is not invertible in $\mathcal{CL}(c_{00})$. Suppose it is and $B \in \mathcal{CL}(c_{00})$ is the inverse of A. Then for the sequences $\mathbf{e}_m := (0, \ldots, 0, 1, 0, \ldots)$ where all terms are 0 except that the mth term is 1, we have

$$1 = \|\mathbf{e}_m\|_\infty = \|BA\mathbf{e}_m\|_\infty \le \|B\|\|A\mathbf{e}_m\|_\infty = \frac{\|B\|}{m}.$$

Hence $\forall m \in \mathbb{N}$, $\|B\| \ge m$ and this contradicts Lemma E.109.

Theorem E.137 (Banach)**.** For Banach spaces X and Y, a map $T \in \mathcal{CL}(X, Y)$ is invertible with $T^{-1} \in \mathcal{CL}(Y, X)$ if and only if T is bijective.

Proof. The necessity follows from Lemma E.132. For sufficiency, the bijective map T induces a map $T^{-1} : Y \to X$,

$$\forall y = Tx \in Y, \quad T^{-1}(y) = x.$$

Since the bijectiveness of T guarantees that T^{-1} is well defined, T^{-1} is indeed an inverse of T. By Lemma E.133, T^{-1} is linear. It remains to show that T^{-1} is continuous. By the surjectivity of T and Theorem E.126, T is open. Hence $T(U)$ is open whenever U is open. Meanwhile we have

$$\begin{aligned}\left(T^{-1}\right)^{-1}(U) &= \left\{y \in Y : T^{-1}y \in U\right\} \\ &= \{y \in Y : y \in T(U)\} = T(U).\end{aligned}$$

Thus T^{-1} is continuous by Theorem E.67. □

Definition E.138. A pair of *isomorphic normed spaces* are Banach spaces X and Y for which there exists a bijective map $T \in \mathcal{CL}(X, Y)$. Then T is called an *isomorphism of normed spaces* and we write $X \simeq Y$.

Theorem E.139 (Closed graph)**.** For two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, a map $T \in \mathcal{L}(X, Y)$ is continuous if and only if its graph $\mathcal{G}(T) := \{(x, Tx) : x \in X\}$ is closed in $(X \times Y, \|\cdot\|_\infty)$ where

$$\forall (x, y) \in X \times Y, \quad \|(x, y)\|_\infty := \max(\|x\|_X, \|y\|_Y). \quad \text{(E.53)}$$

Exercise E.140. Prove Theorem E.139.

E.2.4 Series of operators

Theorem E.141 (Neumann series)**.** Suppose X is a Banach space and $A \in \mathcal{CL}(X)$ has $\|A\| < 1$. Then we have

(NST-1) $I - A$ is invertible in $\mathcal{CL}(X)$,

(NST-2) $(I - A)^{-1} = I + A + \cdots + A^n + \cdots = \sum_{n=0}^\infty A^n$,

(NST-3) $\left\|(I - A)^{-1}\right\| \le \frac{1}{1-\|A\|}$.

Proof. Since X is a Banach space, Corollary E.118 states that $\mathcal{CL}(X)$ is also a Banach space. By Theorem E.90, the convergence of $\sum_{n=0}^\infty \|A\|^n$ implies that the sequence $(S_n)_{n\in\mathbb{N}}$ with

$$S_n = \sum_{k=0}^n A^k$$

converges to some $S \in \mathcal{CL}(X)$. It follows that

$$\begin{aligned} & S_nA = AS_n = \textstyle\sum_{k=1}^{n+1} A^k = S_{n+1} - I \\ \Rightarrow\ & \begin{cases} \|AS_n - AS\| \le \|A\|\|S_n - S\|, \\ \|S_nA - SA\| \le \|A\|\|s - S_n\|. \end{cases} \\ \Rightarrow\ & SA = AS = S - I \\ \Rightarrow\ & (I - A)S = I = S(I - A) \\ \Rightarrow\ & (I - A)^{-1} = S = \textstyle\sum_{n=0}^\infty A^n, \end{aligned}$$

where the last step follows from Definition E.130. Finally, (NST-3) follows from

$$\left\|(I - A)^{-1}\right\| = \left\|\sum_{n=0}^\infty A^n\right\| \le \sum_{n=0}^\infty \|A^n\| \le \sum_{n=0}^\infty \|A\|^n = \frac{1}{1 - \|A\|}$$

where the first inequality follows from Theorem E.90 and the second inequality from Lemma E.112. □

Theorem E.142. Suppose X is a Banach space. Then the exponential of $A \in \mathcal{CL}(X)$, defined as

$$e^A := \sum_{n=0}^\infty \frac{1}{n!} A^n, \quad \text{(E.54)}$$

converges in $\mathcal{CL}(X)$.

Proof. By Lemma E.112, we have

$$\sum_{n=0}^\infty \left\|\frac{1}{n!} A^n\right\| \le \sum_{n=0}^\infty \frac{1}{n!} \|A\|^n = e^{\|A\|}.$$

By the comparison test, $\sum_{n=0}^\infty \frac{1}{n!} A^n$ converges absolutely. The rest of the proof follows from Theorem E.90. □

Lemma E.143. For a Banach space X, $A \in \mathcal{CL}(X)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tA} := Ae^{tA} = e^{tA}A. \quad \text{(E.55)}$$

Lemma E.144. For a Banach space X, if $A, B \in \mathcal{CL}(X)$ commute, i.e. $AB = BA$, then

$$e^{A+B} := e^A e^B. \quad \text{(E.56)}$$

Corollary E.145. For a Banach space X and $A \in \mathcal{CL}(X)$, e^A is always invertible with its inverse as e^{-A}.

Theorem E.146 (Existence and uniqueness of ODEs)**.** For a Banach space X and $A \in \mathcal{CL}(X)$, the IVP

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t) \tag{E.57}$$

with initial condition $x(0) = x_0 \in X$ has a unique solution $x(t) = e^{tA}x_0$ for $t \in \mathbb{R}$.

Proof. If $x(t)$ solves (E.57), then

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-tA}x(t)) = e^{-tA}(-A)x(t) + e^{-tA}\frac{\mathrm{d}}{\mathrm{d}t}(x(t)) = 0,$$

which implies $e^{-tA}x(t) = x_0$ and thus $x(t) = e^{tA}x_0$. □

E.2.5 Uniform boundedness

Lemma E.147. Suppose X is a normed space and a subset $A \subset X$ satisfies

- A is symmetric, i.e., $-A = A$;
- A is mid-point convex, i.e., $\forall x, y \in A$, $\frac{x+y}{2} \in A$,
- there exists a nonempty open set $U \subset A$.

Then there exists $\delta > 0$ such that $B(\mathbf{0}_X, \delta) \subset A$.

Proof. For $\alpha \neq 0$ and $a \in X$, the maps $x \mapsto x + a$ and $x \mapsto \alpha a$ are both continuous with continuous inverses. By Theorem E.67, U being open in X implies that its preimage $U + \{-a\}$ under $x \mapsto x + a$ is also open in X. Adopting notations in (E.51), we find that the set

$$U + (-A) := \cup_{a\in A}(U + \{-a\})$$

is open since it is a union of open sets. For $a \in U$, we have

$$\mathbf{0}_X = \frac{a-a}{2} \in \frac{U+(-A)}{2} \subset \frac{A+(-A)}{2} = \frac{A+A}{2} = A,$$

where the last two equalities follows from A being symmetric and mid-point convex, respectively. The proof is completed by Lemma D.190 and $\frac{U+(-A)}{2}$ being open. □

Theorem E.148 (Uniform boundedness principle)**.** Suppose X is a Banach space and Y is a normed linear space. For a family of maps $T_i \in \mathcal{CL}(X, Y)$, $i \in I$, "pointwise boundedness" implies "uniform boundedness,"

$$\forall x \in X, \quad \sup_{i\in I} \|T_i x\| < +\infty \ \Rightarrow\ \sup_{i\in I} \|T_i\| < +\infty.$$

Proof. For any given $n \in \mathbb{N}$, we define

$$F_n := \cap_{i\in I}\{x \in X : \|T_i x\| \le n\} = \{x \in X : \sup_{i\in I} \|T_i x\| \le n\}.$$

As intersection of closed sets, each F_n is closed. By pointwise boundedness, we have $X = \cup_{n\in\mathbb{N}} F_n$. The Baire theorem E.124 implies that there exists some F_n that contains a nonempty open subset. Since F_n is also symmetric and mid-point convex, Lemma E.147 implies that F_n contains an open ball $B(\mathbf{0}_X, \delta)$. Consequently, $x \in B(\mathbf{0}_X, \delta) \subset F_n$ implies

$$\|x\| < \delta \ \Rightarrow\ \forall i \in I, \|T_i x\| \le n.$$

Thus for any $x \in X$, there exists $y = \frac{\delta}{2}\frac{x}{\|x\|}$ such that

$$\forall i \in I,\ \|T_i y\| \le n \ \Rightarrow\ \|T_i x\| \le \frac{2n}{\delta}\|x\|,$$

and the proof is completed by Lemma E.110. □

Example E.149. Many PDEs can be written in the form

$$Tx = y,$$

where y is a known vector incorporating initial and boundary conditions, x is the unknown, and T is a continuous linear operator. If the PDE is well-posed, we can often assume that T is a bijection, hence by Theorem E.137 the inverse of T is a bounded linear operator and we write $x = T^{-1}y$. In numerically solving the PDE, we usually approximate y by a grid function y_n and approximate T^{-1} by a discrete operator T_n^{-1}. The convergence usually means

$$\forall y \in \mathcal{C}^r(\overline{\Omega}), \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} T_n^{-1} y_n = x,$$

i.e., $\sup_{n\to\infty} \|T_n^{-1} y_n\| < \infty$. Theorem E.148 then implies $\sup_{n\in\mathbb{N}} \|T_n^{-1}\| < \infty$, which usually implies some form of numerical stability.

Theorem E.150 (Banach-Steinhaus)**.** Suppose X and Y are Banach spaces. If a sequence $(T_n)_{n\in\mathbb{N}} \in \mathcal{CL}(X, Y)$ has

$$\forall x \in X,\ \lim_{n\to\infty} \|T_n x\| < \infty,$$

then the map $x \mapsto \lim_{n\to\infty} T_n x$ belongs to $\mathcal{CL}(X, Y)$.

Proof. Clearly the map $T(x) = \lim_{n\to\infty} T_n x$ is linear, it remains to show that it is continuous. Because the limit $\lim_{n\to\infty} T_n x$ exists, we have $\sup_{n\in\mathbb{N}} \|T_n x\| < \infty$ for all $x \in X$. Then Theorem E.148 implies $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. Hence

$$\exists M \in \mathbb{R},\ \forall x \in X, \forall n \in \mathbb{N},\ \|T_n x\| \le M\|x\|;$$

the limit of the above yields $\forall x \in X,\ \|Tx\| \le M\|x\|$. The proof is completed by Theorem E.96. □