Appendix F

Fourier Analysis

Definition F.1. The L^2-*norm* of a Lebesgue-measurable function $u : \mathbb{R} \to \mathbb{C}$ is a nonnegative or infinite real number

$$\|u\| = \left[\int_{-\infty}^{\infty} |u(x)|^2 \mathrm{d}x\right]^{\frac{1}{2}}. \tag{F.1}$$

Notation 33. Denote by L^2 the set of all functions whose L^2-norms are finite, i.e.,

$$L^2 = \{u : \|u\| < \infty\}. \tag{F.2}$$

Similarly, L^1 and L^∞ respectively denote the sets of functions with finite L^1- and L^∞- norms,

$$\|u\|_1 = \int_{-\infty}^{\infty} |u(x)|\mathrm{d}x, \quad \|u\|_\infty = \sup_{-\infty<x<\infty} |u(x)|. \tag{F.3}$$

Since the L^2 norm is the norm used in most applications, we have reserved the symbol $\|\cdot\|$ without a subscript for it.

F.1 L^p spaces

Definition F.2. Let Ω be a *domain* in $\mathbb{R}^n$, i.e. a Lebesgue-measurable subset of $\mathbb{R}^n$ with nonempty interior. The L^p-*norm of a Lebesgue-measurable function* $f : \Omega \to \mathbb{R}$ is

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \mathrm{d}x\right)^{1/p} & \text{if } p \in [1, +\infty); \\ \operatorname{ess\,sup}\{|f(x)| : x \in \Omega\} & \text{if } p = \infty, \end{cases} \tag{F.4}$$

where "$\mathrm{d}x$" denotes the Lebesgue measure, "$\int_\Omega$" the Lebesgue integral, and "ess sup" the essential supremum:

$$\operatorname{ess\,sup}\{|f(x)| : x \in \Omega\} := \inf_{\mathrm{meas}(\Omega')=0} \sup_{x\in\Omega\setminus\Omega'} |f(x)|. \tag{F.5}$$

The L^p-*space* or the *Lebesgue space* is the set of all Lebesgue-measurable functions on Ω with bounded L^p-norms, written

$$L^p(\Omega) := \left\{f : \|f\|_{L^p(\Omega)} < \infty\right\}. \tag{F.6}$$

Theorem F.3 (Minkowski's Inequality)**.** For $p \in [1, \infty]$ and $f, g \in L^p(\Omega)$, we have

$$\|f + g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \tag{F.7}$$

Theorem F.4 (Hölder's Inequality)**.** For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ imply $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}. \tag{F.8}$$

Theorem F.5 (Schwarz's Inequality)**.** $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$ imply $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg|\mathrm{d}x \le \|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}. \tag{F.9}$$

Theorem F.6. $L^p(\Omega)$ is a Banach space for any $p \in [1, \infty]$.

Definition F.7. A *Hilbert space* is a Banach space equipped with an inner product.

Theorem F.8 (Lebesgue's Dominated Convergence)**.** Let $\{f_n\}_{n=0}^\infty$ be sequence of functions in $L^1(\Omega)$ such that

(i) $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in Ω;

(ii) there is a function $g \in L^1(\Omega)$ such that, for all n, $|f_n(x)| \le g(x)$ a.e. in Ω.

Then $f \in L^1(\Omega)$ and $\lim_{n\to\infty} f_n = f$ in $L^1(\Omega)$.

F.2 Fourier transforms

Definition F.9. The *Fourier transform of a function* $u \in L^2(\mathbb{R})$ is the function $\hat{u} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathbf{i}\xi x} u(x)\mathrm{d}x. \tag{F.10}$$

Example F.10. The Fourier transform of $u \in L^1(\mathbb{R})$ is the function $\hat{u} : \mathbb{R} \to \mathbb{C}$ given by (F.10). Then $\hat{u} \in \mathcal{C}_b(\mathbb{R})$, i.e., $\hat{u}$ is a bounded continuous function, because

$$\begin{aligned} |\hat{u}(\xi)| &= \frac{1}{\sqrt{2\pi}} \left|\int_{-\infty}^{\infty} e^{-\mathbf{i}\xi x} u(x)\mathrm{d}x\right| \\ &< \int_{-\infty}^{\infty} \left|e^{-\mathbf{i}\xi x}\right| |u(x)|\mathrm{d}x = \int_{-\infty}^{\infty} |u(x)|\mathrm{d}x. \end{aligned}$$

Therefore we have $\hat{\cdot} \in \mathcal{CL}\left(L^1(\mathbb{R}), \mathcal{C}_b(\mathbb{R})\right)$, i.e. the Fourier transform $\hat{\cdot} : \left(L^1(\mathbb{R}), \|\cdot\|_1\right) \to (\mathcal{C}_b(\mathbb{R}), \|\cdot\|_\infty)$ is a bounded linear transformation.

Theorem F.11. If $u \in L^2(\mathbb{R})$, then its Fourier transform (F.10) also belongs to L^2, and u can be recovered from $\hat{u}$ by the *inverse Fourier transform*

$$u(x) = (\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathbf{i}\xi x}\hat{u}(\xi)\mathrm{d}\xi. \tag{F.11}$$

Definition F.12. The *Fourier transform of a grid function* $U \in L^2(\mathbb{Z})$ is the function $\hat{U} : \mathbb{Z} \to \mathbb{C}$ given by

$$\hat{U}(\xi) = (\mathcal{F}U)(\xi) := \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} e^{-\mathbf{i}m\xi} U_m \tag{F.12}$$

where $\xi \in [-\pi, \pi]$. Its *inverse Fourier transform* is given by

$$U_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} e^{\mathbf{i}m\xi}\hat{U}(\xi)\mathrm{d}\xi. \tag{F.13}$$

Theorem F.13. Let $u, v \in L^2$ have Fourier transforms $\hat{u} = \mathcal{F}u, \hat{v} = \mathcal{F}v$. Then

(a) Linearity. If $c \in \mathbb{R}$, then

$$\mathcal{F}\{u + v\}(\xi) = \hat{u}(\xi) + \hat{v}(\xi), \tag{F.14}$$
$$\mathcal{F}\{cu\}(\xi) = c\hat{u}(\xi). \tag{F.15}$$

(b) Translation. If $x_0 \in \mathbb{R}$, then

$$\mathcal{F}\{u(x + x_0)\}(\xi) = e^{\mathbf{i}\xi x_0}\hat{u}(\xi). \tag{F.16}$$

(c) Modulation. If $\xi_0 \in \mathbb{R}$, then

$$\mathcal{F}\{e^{\mathbf{i}\xi_0 x}u(x)\}(\xi) = \hat{u}(\xi - \xi_0). \tag{F.17}$$

(d) Dilation. If $c \in \mathbb{R}$ with $c \neq 0$, then

$$\mathcal{F}\{u(cx)\}(\xi) = \frac{1}{|c|}\hat{u}\Big(\frac{\xi}{c}\Big). \tag{F.18}$$

(e) Conjugation.

$$\mathcal{F}\{\bar{u}\}(\xi) = \overline{\hat{u}(-\xi)}. \tag{F.19}$$

(f) Differentiation. If $u_x \in L^2$, then

$$\mathcal{F}\{u_x\}(\xi) = \mathbf{i}\xi\hat{u}(\xi). \tag{F.20}$$

(g) Inversion.

$$\mathcal{F}^{-1}\{u\}(\xi) = \hat{u}(-\xi). \tag{F.21}$$

Proof. Most of the conclusions follow directly from the definition (F.10). We only show (F.20) by

$$\begin{aligned}\widehat{u_x}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathbf{i}\xi x}u_x(x)\mathrm{d}x \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-\mathbf{i}\xi)e^{-\mathbf{i}\xi x}u(x)\mathrm{d}x = \mathbf{i}\xi\hat{u}(\xi),\end{aligned}$$

where we have applied integration by parts with assumptions that $u(x)$ is smooth and decays at ∞. □

Definition F.14. A function $u(x)$ is *even, odd, real, or imaginary* if $u(x) = u(-x)$, $u(x) = -u(-x)$, $u(x) = \overline{u(x)}$, or $u(x) = -\overline{u(x)}$, respectively; $u(x)$ is *Hermitian or skew-Hermitian* if $u(x) = \overline{u(-x)}$ or $u(x) = -\overline{u(-x)}$, respectively.

Theorem F.15. Denote by $\hat{u} = \mathcal{F}u$ the Fourier transform of $u \in L^2$. Then

(a) $u(x)$ is even (odd) $\Leftrightarrow$ $\hat{u}(\xi)$ is even (odd).
(b) $u(x)$ is real (imaginary) $\Leftrightarrow$ $\hat{u}(\xi)$ is Hermitian (skew-Hermitian) and therefore
(c) $u(x)$ is real and even $\Leftrightarrow$ $\hat{u}(\xi)$ is real and even.
(d) $u(x)$ is real and odd $\Leftrightarrow$ $\hat{u}(\xi)$ is imaginary and odd.
(e) $u(x)$ is imaginary and even $\Leftrightarrow$ $\hat{u}(\xi)$ is imaginary and even.
(f) $u(x)$ is imaginary and odd $\Leftrightarrow$ $\hat{u}(\xi)$ is real and odd.

Definition F.16. The *convolution* of two functions u, v is the function $u * v$ defined by

$$\begin{aligned}&(u * v)(x) = (v * u)(x) \\ &= \int_{-\infty}^{\infty} u(x - y)v(y)\mathrm{d}y = \int_{-\infty}^{\infty} u(y)v(x - y)\mathrm{d}y,\end{aligned}$$

assuming these integrals exist.

Theorem F.17. If $u \in L^2$ and $v \in L^1$ (or vice versa), then $u * v \in L^2$, and $\widehat{u * v}$ satisfies

$$\widehat{u * v}(\xi) = \sqrt{2\pi}\hat{u}(\xi)\hat{v}(\xi). \tag{F.22}$$

Proof. By Definition F.16 and F.9, we calculate

$$\begin{aligned}\widehat{u * v}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} u(y)v(x - y)\mathrm{d}y\Big)e^{-\mathbf{i}\xi x}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(y)v(x - y)e^{-\mathbf{i}\xi x}\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(y)v(z)e^{-\mathbf{i}\xi(y+z)}\mathrm{d}z\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(y)e^{-\mathbf{i}\xi y}\mathrm{d}y \int_{-\infty}^{\infty} v(z)e^{-\mathbf{i}\xi z}\mathrm{d}z \\ &= \sqrt{2\pi}\hat{u}(\xi)\hat{v}(\xi),\end{aligned}$$

where the third step follows from the variable substitution $z = x - y$. □

Theorem F.18. The L^2-norms of u and $\hat{u}$ are related by *Parseval's equality*, a.k.a. the *Plancherel theorem*,

$$\forall u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \qquad \|\hat{u}\| = \|u\|. \tag{F.23}$$

Proof. We calculate

$$\begin{aligned}\int |u(x)|^2\mathrm{d}x &= \int \overline{u(x)}u(x)\mathrm{d}x = \left(\overline{u(x)} * u(-x)\right)(0) \\ &= \frac{1}{\sqrt{2\pi}} \int \widehat{\overline{u(x)} * u(-x)}(\xi)\mathrm{d}\xi \\ &= \int \widehat{\overline{u(x)}}(\xi)\widehat{u(-x)}(\xi)\mathrm{d}\xi \\ &= \int \overline{\hat{u}(-\xi)}\,\hat{u}(-\xi)\mathrm{d}\xi = \int |\hat{u}(\xi)|^2\mathrm{d}\xi,\end{aligned}$$

where in the first step $\overline{u(x)}$ denotes the conjugate of $u(x)$, the second step follows from Definition F.16, the third from

(F.11), the fourth from Theorem F.17, the fifth from (F.18) and (F.19), and the last from the symmetry of the integral limits. This is almost a proof except that we need to verify that $\overline{u} * u$ is in L^2 and that $\widehat{\overline{u} * u} = |\hat{u}(\xi)|^2$ is in L^1. These are out of the scope of this course and we refer the reader to a standard text on Fourier analysis. □

Example F.19 (B-splines)**.** For the function

$$u(x) = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } x \in [-1, 1]; \\ 0 & \text{otherwise}, \end{cases} \tag{F.24}$$

(F.1) yields $\|u\| = \sqrt{\pi}$, and (F.10) gives

$$\hat{u}(\xi) = \frac{1}{2}\int_{-1}^{1} e^{-\mathbf{i}\xi x}\mathrm{d}x = \left.\frac{e^{-\mathbf{i}\xi x}}{-2\mathbf{i}\xi}\right|_{-1}^{1} = \frac{\sin\xi}{\xi}, \tag{F.25}$$

where the function $\xi \mapsto \frac{\sin\xi}{\xi}$ is known as the *sinc function*. From (F.1) and the indispensable identity

$$\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}\mathrm{d}s = \pi, \tag{F.26}$$

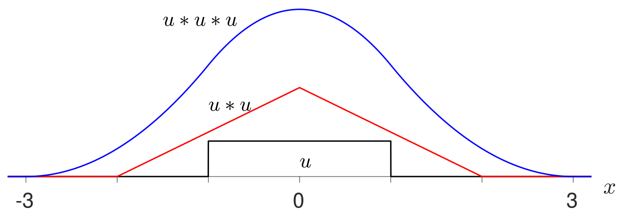
which can be derived by complex contour integration, we calculate $\|\hat{u}\| = \sqrt{\pi}$, which confirms (F.23).

By Definition F.16, it is readily verified that

$$(u * u)(x) = \begin{cases} \frac{\pi}{2}(2 - |x|) & \text{if } x \in [-2, 2]; \\ 0 & \text{otherwise}, \end{cases} \tag{F.27}$$

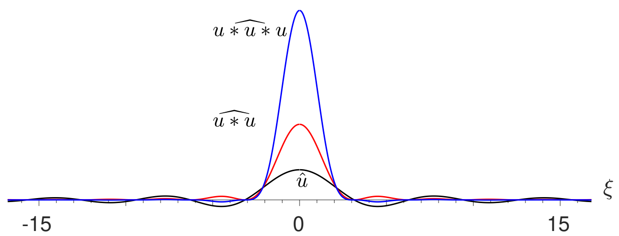
and

$$(u * u * u)(x) = \begin{cases} \frac{3(\sqrt{2\pi})^3}{8} - \frac{(\sqrt{2\pi})^3}{8}x^2 & \text{if } |x| \le 1; \\ \frac{(\sqrt{2\pi})^3}{16}(9 - 6|x| + x^2) & \text{if } 1 \le |x| \le 3; \\ 0 & \text{otherwise}. \end{cases}$$



By (F.22) and (F.25), their Fourier transforms are

$$\widehat{u * u}(\xi) = \sqrt{2\pi}\frac{\sin^2\xi}{\xi^2}, \quad \widehat{u * u * u}(\xi) = 2\pi\frac{\sin^3\xi}{\xi^3}. \tag{F.28}$$



In general, a convolution $u_{(p)}$ of p copies of u has the Fourier transform

$$\widehat{u_{(p)}}(\xi) := \mathcal{F}\{u * u * \cdots * u\}(\xi) = (2\pi)^{\frac{p-1}{2}}\left(\frac{\sin\xi}{\xi}\right)^p.$$

Example F.20. The function

$$u(x) = \begin{cases} \frac{\pi}{2}, & \text{for } -2 \le x < 0, \\ -\frac{\pi}{2}, & \text{for } 0 < x \le 2, \\ 0, & \text{otherwise}, \end{cases} \tag{F.29}$$

has its Fourier transform as

$$\begin{aligned} \hat{u}(\xi) &= \frac{\sqrt{2\pi}}{4}\int_{-2}^{0} e^{-\mathbf{i}\xi x}\mathrm{d}x - \frac{\sqrt{2\pi}}{4}\int_{0}^{2} e^{-\mathbf{i}\xi x}\mathrm{d}x \\ &= \frac{\sqrt{2\pi}}{-4\mathbf{i}\xi}(1 - e^{2\mathbf{i}\xi} - e^{-2\mathbf{i}\xi} + 1) \\ &= \frac{\sqrt{2\pi}}{4\mathbf{i}\xi}(e^{\mathbf{i}\xi} - e^{-\mathbf{i}\xi})^2 = \sqrt{2\pi}\frac{i\sin^2\xi}{\xi}. \end{aligned}$$

Hence $\hat{u}(\xi)$ is $\mathbf{i}\xi$ times the Fourier transform (F.28) of the triangular hat function (F.27), which confirms (F.20).

Definition F.21. A function u defined on $\mathbb{R}$ is said to have *bounded variation* if there is a constant M such that for any finite m and any points $x_0 < x_1 < \cdots < x_m$,

$$\sum_{j=1}^{m} |u(x_j) - u(x_{j-1})| \le M. \tag{F.30}$$

Theorem F.22. Let u be a function in L^2.

(a) If u has $p-1$ continuous derivatives in L^2 for some $p \ge 0$, and a pth derivative in L^2 that has bounded variation, then

$$\hat{u}(\xi) = O(|\xi|^{-p-1}) \quad \text{as } |\xi| \to \infty. \tag{F.31}$$

(b) If u has infinitely many continuous derivatives in L^2, then we have

$$\hat{u}(\xi) = O(|\xi|^{-M}) \quad \text{as } |\xi| \to \infty \text{ for all } M. \tag{F.32}$$

The converse also holds.

(c) If u can be extended to an entire function such that

$$\exists a > 0 \text{ s.t. } |u(z)| = O\left(e^{a|z|}\right) \quad \text{as } |z| \to \infty,$$

then $\hat{u}$ has compact support contained in $[-a, a]$, i.e.,

$$\hat{u}(\xi) = 0 \quad \text{for all } |\xi| > a. \tag{F.33}$$

The converse also holds.

Example F.23. The square wave u of Example F.19 satisfies condition (a) of Theorem F.22 with $p = 0$, so its Fourier transform should satisfy

$$|\hat{u}(\xi)| = O(|\xi|^{-1}),$$

as is verified by (F.25). On the other hand, suppose we interchange the role of u and $\hat{u}$ and apply the theorem again. The function $u(\xi) = \frac{\sin\xi}{\xi}$ is entire, and since $\sin(\xi) = \frac{1}{2\mathrm{i}}(e^{\mathbf{i}\xi} - e^{-\mathbf{i}\xi})$, it satisfies

$$u(\xi) = O(e^{|\xi|}) \quad \text{as } |\xi| \to \infty$$

(with ξ now taking complex values). By part (c) of Theorem F.22, it follows that $u(x)$ must have compact support contained in $[-1,1]$, as indeed it does.

Repeating the example for $u * u$, condition (a) now applies with $p=1$, and the Fourier transform (F.28) is indeed of magnitude $O(|\xi|^{-2})$, as required. Interchanging u and $\hat{u}$, we note that $\frac{\sin^2\xi}{\xi^2}$ is an entire function of magnitude $O(e^{2|\xi|})$ as $|\xi|\to\infty$, and $u*u$ has support contained in $[-2,2]$.

F.3 Green's function of the heat equation in $(-\infty,+\infty)$

Lemma F.24. The Fourier transform of a Gaussian centered at the origin is another such Gaussian.

Proof. First we consider the case $f(x)=e^{-x^2}$, then

$$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-x^2}e^{-\mathbf{i}\xi x}\,\mathrm{d}x,$$

Differentiating with respect to ξ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{f}(\xi)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-x^2}(-\mathbf{i}x)e^{-\mathbf{i}\xi x}\,\mathrm{d}x\\&=\frac{i}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}e^{-x^2}}{\mathrm{d}x}e^{-\mathbf{i}\xi x}\,\mathrm{d}x\\&=-\frac{\xi}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-x^2}e^{-\mathbf{i}\xi x}\,\mathrm{d}x\\&=-\frac{\xi}{2}\hat{f}(\xi),\end{aligned}$$

where the third line follows from the integration by parts formula. The unique solution to this ODE is given by

$$\hat{f}(\xi)=c\cdot e^{-\frac{\xi^2}{4}},$$

where $c=\hat{f}(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-x^2}\,\mathrm{d}x=\frac{\sqrt{2}}{2}$. The proof is completed by the dilation property (F.18) of Fourier transform. In particular, the Fourier transform of a Gaussian with $a=1$, $b=0$, and c is another Gaussian with $a'=c$, $b'=0$, and $c'=\frac{1}{c}$. ☐

Lemma F.25. For any $u\in L^2$ satisfying

$$\forall n\in\mathbb{N},\qquad \lim_{x\to\pm\infty}u^{(n)}(x)=0, \tag{F.34}$$

we have

$$\widehat{\frac{\mathrm{d}^2u}{\mathrm{d}x^2}}=-\xi^2\hat{u}. \tag{F.35}$$

Proof. Repeated application of (F.34) yields

$$\begin{aligned}\sqrt{2\pi}\cdot\widehat{\frac{\mathrm{d}^2u}{\mathrm{d}x^2}}&=\int_{-\infty}^{+\infty}e^{-\mathbf{i}\xi x}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{d}x\\&=e^{-\mathbf{i}\xi x}\frac{\mathrm{d}u}{\mathrm{d}x}\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{\mathrm{d}x}(-\mathbf{i}\xi)e^{-\mathbf{i}\xi x}\mathrm{d}x\\&=\mathbf{i}\xi\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{\mathrm{d}x}e^{-\mathbf{i}\xi x}\mathrm{d}x\\&=\mathbf{i}\xi(e^{-\mathbf{i}\xi x}u)\Big|_{-\infty}^{+\infty}+(\mathbf{i}\xi)^2\int_{-\infty}^{+\infty}ue^{-\mathbf{i}\xi x}\mathrm{d}x\\&=-\xi^2\int_{-\infty}^{+\infty}ue^{-\mathbf{i}\xi x}\mathrm{d}x=-\xi^2\sqrt{2\pi}\hat{u},\end{aligned}$$

where the first and last lines follow from Definition F.9, the second and fourth lines from the integration by parts formula, and the third line from (F.34). ☐

Theorem F.26. The solution to the heat equation

$$u_t=\nu u_{xx}\text{ on }(-\infty,+\infty) \tag{F.36}$$

with the initial condition $\eta(x)=e^{-\beta x^2}$ is

$$u(x,t)=\frac{1}{\sqrt{4\beta\nu t+1}}e^{-\frac{x^2}{4\nu t+1/\beta}}. \tag{F.37}$$

Proof. By Lemma F.25, the Fourier transform of (F.36) leads to the ODE

$$\hat{u}_t(\xi,t)=-\nu\xi^2\hat{u}(\xi,t),$$

the solution of which with the initial data $\hat{u}(\xi,0)=\hat{\eta}(\xi)$ yields

$$\hat{u}(\xi,t)=e^{-\nu\xi^2t}\hat{\eta}(\xi).$$

Then

$$\begin{aligned}u(x,t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{u}(\xi,t)e^{\mathbf{i}\xi x}\mathrm{d}\xi\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\nu\xi^2t}\hat{\eta}(\xi)e^{\mathbf{i}\xi x}\mathrm{d}\xi\\&=\frac{1}{2\sqrt{\pi\beta}}\int_{-\infty}^{+\infty}e^{-\xi^2(\nu t+\frac{1}{4\beta})}e^{\mathbf{i}\xi x}\mathrm{d}\xi.\end{aligned}$$

Define $C=\frac{1}{4\nu t+1/\beta}$, then

$$\begin{aligned}u(x,t)&=\frac{1}{2\sqrt{\pi\beta}}\int_{-\infty}^{+\infty}e^{-\frac{\xi^2}{4C}}e^{\mathbf{i}\xi x}\mathrm{d}\xi\\&=\frac{1}{2\sqrt{\pi\beta}}\sqrt{4\pi C}\cdot e^{-x^2C}\\&=\frac{1}{\sqrt{4\beta\nu t+1}}e^{-\frac{x^2}{4\nu t+1/\beta}}.\end{aligned}$$

As t increases this Gaussian becomes more spread out and the magnitude decreases. ☐

Corollary F.27. A translation of the initial condition

$$\eta(x)=e^{-\beta(x-\bar{x})^2} \tag{F.38}$$

of the heat equation (F.36) leads to a translation of the solution, i.e.,

$$u(x,t)=\frac{1}{\sqrt{4\beta\nu t+1}}e^{-\frac{(x-\bar{x})^2}{4\nu t+1/\beta}}. \tag{F.39}$$

Corollary F.28. For the heat equation (F.36) with the initial condition as

$$\omega_\beta(x,0;\bar{x}) = \sqrt{\frac{\beta}{\pi}} e^{-\beta(x-\bar{x})^2}, \tag{F.40}$$

its solution is

$$\omega_\beta(x,t;\bar{x}) = \frac{1}{\sqrt{4\pi\nu t + \pi/\beta}} e^{-\frac{(x-\bar{x})^2}{4\nu t + 1/\beta}}. \tag{F.41}$$

Definition F.29. The *Green's function*

$$G(x,t;\bar{x}) := \lim_{\beta\to+\infty} \omega_\beta(x,t;\bar{x}) = \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(x-\bar{x})^2}{4\nu t}} \tag{F.42}$$

is the solution of the heat equation (F.36) with its initial condition as the Dirac delta function in Definition 5.41.

F.4 Fourier solutions of linear PDEs

Lemma F.30. For a linear PDE of the form

$$\frac{\partial u}{\partial t} + \sum_{n=1}^{N} a_n \frac{\partial^n u}{\partial x^n} = 0, \tag{F.43}$$

the evolution of a single Fourier mode of wavenumber ξ satisfies the ODE

$$\frac{\partial \hat{u}}{\partial t}(\xi,t) + \sum_{n=1}^{N} a_n (\mathbf{i}\xi)^n \hat{u}(\xi,t) = 0. \tag{F.44}$$

Proof. Differentiations of the inverse Fourier transform (F.11) with respect to t and x yield, respectively,

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_t(\xi,t) e^{\mathbf{i}\xi x} \mathrm{d}\xi, \\ \frac{\partial^n u(x,t)}{\partial x^n} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi,t)(\mathbf{i}\xi)^n e^{\mathbf{i}\xi x} \mathrm{d}\xi. \end{aligned}$$

Plug these equations into (F.43) and we have (F.44). □

Lemma F.31. The solution to the linear PDE (F.43) with initial condition $u(x,0) = \eta(x)$ is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{\mathbf{i}(\xi x - \omega t)} \mathrm{d}\xi, \tag{F.45}$$

where the frequency is

$$\omega := \sum_{n=1}^{N} a_n \xi^n \mathbf{i}^{n-1}. \tag{F.46}$$

Proof. Rewrite (F.44) as

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} = -\mathbf{i}\omega \hat{u}(\xi,t),$$

and we have from Duhamel's principle (Theorem 11.51)

$$\hat{u}(\xi,t) = e^{-\mathbf{i}\omega t}\hat{\eta}(\xi). \tag{F.47}$$

Then the inverse Fourier transform yields

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi,t) e^{\mathbf{i}\xi x} \mathrm{d}\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{\mathbf{i}(\xi x - \omega t)} \mathrm{d}\xi. \end{aligned}$$ □

Theorem F.32. If the initial condition $\eta(x) := u(x,0)$ of the linear PDE (F.43) belongs to L^2, the L_2 norm of $u(x,t)$ remains a constant for any $t \geq 0$.

Proof. By Lemma F.31, we have

$$\begin{aligned} \|u(x,t)\|^2 &= \int_{-\infty}^{+\infty} |u(x,t)|^2 \mathrm{d}x = \int_{-\infty}^{+\infty} |\hat{u}(\xi,t)|^2 \mathrm{d}\xi \\ &= \int_{-\infty}^{+\infty} |\hat{\eta}(\xi,t)|^2 \mathrm{d}\xi = \int_{-\infty}^{+\infty} |\eta(x,t)|^2 \mathrm{d}x, \end{aligned}$$

where the first step follows from $u \in L^2$, the third from (F.47), and the second and the fourth from Parseval's equality (Theorem F.18). □

F.5 Dissipation and dispersion

Definition F.33. The *dispersion relation* of a PDE is the relation between the frequency ω and the wavenumber ξ,

$$\omega = \omega(\xi). \tag{F.48}$$

Example F.34. The dispersion relation of (F.43) is (F.46). In particular, the linear Korteweg-deVries (KdV) equation

$$\varphi_t + c_0\varphi_x + \nu\varphi_{xxx} = 0 \tag{F.49}$$

is characterized by its dispersion relation $\omega = c_0\xi - \nu\xi^3$.

Example F.35. The beam equation

$$\varphi_{tt} + \gamma^2\varphi_{xxxx} = 0 \tag{F.50}$$

is characterized by its dispersion relation $\omega = \pm\gamma\xi^2$ since it is equivalent to $\varphi_t = \pm\gamma\varphi_{xx}$.

F.6 Phase velocity and group velocity

Definition F.36. For a wave function of the form

$$\varphi(x,t) = \varphi_0 e^{\mathbf{i}\theta(x,t)} = \varphi_0 e^{\mathbf{i}\omega t} e^{\mathbf{i}\xi x} e^{-\mathbf{i}\theta_0}, \tag{F.51}$$

φ_0 is called the *amplitude*, $e^{\mathbf{i}\omega t}$ the *oscillation*, $e^{\mathbf{i}\xi x}$ the *propogation*, θ the *phase*, and the constant θ_0 the *phase offset*.

Definition F.37. The *phase velocity* of a monochromatic wave with wavenumber ξ is the speed that a point of constant phase propogates:

$$C_p(\xi) := \frac{\omega(\xi)}{\xi}. \tag{F.52}$$

Example F.38. The speed of light is a phase velocity,

$$C_p = \frac{\lambda}{T} = \frac{\omega}{\xi},$$

where λ and T are the wavelength and the period, respectively.

Definition F.39. The system (F.43) is said to be *hyperbolic* if the PDE is hyperbolic; it is *dissipative* if ω is purely imaginary; it is *dispersive* if $\omega(\xi)$ is real and $\omega(\xi)/\xi$ is not a constant.

Definition F.40. The *group velocity* of a monochromatic wave with wavenumber ξ is

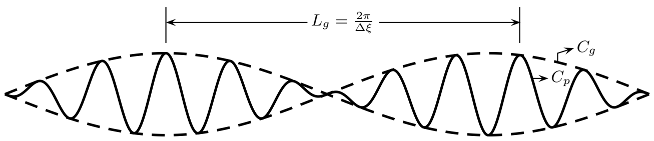
$$C_g(\xi) := \frac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi}. \tag{F.53}$$

Example F.41. For the linear PDE (F.43), the phase velocity of a single Fourier mode is

$$C_p(\xi) = \sum_{n=1}^{N} a_n \xi^{n-1}\mathbf{i}^{n-1}$$

while the group velocity is

$$C_g(\xi) = \sum_{n=1}^{N} n a_n \xi^{n-1}\mathbf{i}^{n-1}.$$



Lemma F.42. The group velocity and the phase velocity are related via

$$C_g(\xi) = \frac{C_p(\xi)}{1 - \xi\frac{\mathrm{d}C_p}{\mathrm{d}\omega}}. \tag{F.54}$$

Proof. Definition F.40 yields

$$C_g(\xi) = \frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \frac{\mathrm{d}\omega}{\mathrm{d}\frac{\omega}{C_p}} = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\omega}{C_p}\right)},$$

which implies (F.54). □

Example F.43. For the advection equation, we have $a_1 = a$ and $a_j = 0$ for all $j > 1$. Consequently, we have

$$\omega = a\xi, \qquad C_p = a = C_g,$$

hence Lemma F.31 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\eta}(\xi)e^{\mathbf{i}\xi(x-at)}\mathrm{d}\xi = \eta(x-at).$$

Thus all wave modes that constitute the initial data $\eta(x)$ move at the same phase speed, which is also the moving speed of energy.

Example F.44. For the heat equation

$$u_t = \nu u_{xx},$$

we have $a_2 = -\nu < 0$, $a_1 = a_3 = a_4 = \ldots = 0$, and thus

$$\omega = a_2\xi^2\mathbf{i}, \qquad C_p = a_2\xi\mathbf{i}, \quad C_g = 2a_2\xi\mathbf{i}.$$

Lemma F.31 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\eta}(\xi)e^{\mathbf{i}\xi x}e^{a_2\xi^2 t}\mathrm{d}\xi,$$

where the term "$e^{\mathbf{i}\xi x}$" denotes the initial mode ξ that does not move while "$e^{-\nu\xi^2 t}$" represents the exponential decay with respect to time.

Example F.45. For the dispersion equation

$$u_t = u_{xxx},$$

we have $a_3 = -1$, $a_1 = a_2 = a_4 = a_5 = \ldots = 0$,

$$\omega = \xi^3, \qquad C_p = \xi^2, \quad C_g = 2\xi^2.$$

Lemma F.31 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\eta}e^{\mathbf{i}\xi(x-\xi^2 t)}\mathrm{d}\xi,$$

thus there is no damping, but different phases move with different speed ξ^2.

Example F.46. For the equation

$$u_t + au_x + bu_{xxx} = 0,$$

we have $a_1 = a$, $a_3 = b$, $a_2 = a_4 = a_5 = \ldots = 0$, and

$$\omega = a\xi - b\xi^3, \qquad C_p = a - b\xi^2, \quad C_g = a - 3b\xi^2.$$

Lemma F.31 yields

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\eta}(\xi)e^{\mathbf{i}\xi(x-(a-b\xi^2)t)}\mathrm{d}\xi.$$