

1 Exercise 11.9

验证 real positivity, 由 $\|T\| \geq \inf\{M \geq 0\} = 0 \Rightarrow \|T\| \geq 0$

验证 point separation, $\|T\| = 0 \Rightarrow \forall x, \|Tx\| \leq M\|x\| = 0 \Rightarrow T = 0$

验证 absolute homogeneity, $\forall a \in \mathbb{F}, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

有 $\|av\| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |avx| = |a||vx| \leq M|x|\}$

若 $|a| = 0, \|av\| = 0 = |a|\|v\|$, 否则, 有 $\|av\| = \inf M \geq 0 : \forall x \in \mathbb{F}^n, |vx| \leq \frac{M}{|a|}|x|$, 也有 $\|av\| = |a|\|v\|$

验证 triangle inequality, $\forall u, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

$\|u+v\| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |(u+v)x| = |ux+vx| \leq M|x|\}$

设 $\|u\| = M_1, \|v\| = M_2$, 则有 $\forall x \in \mathbb{F}^n, |ux| \leq M_1|x|, |vx| \leq M_2|x| \Rightarrow |ux| + |vx| \leq (M_1 + M_2)|x|$

有 $\|u+v\| \leq M_1 + M_2 = \|u\| + \|v\|$

所以有 $\|\cdot\|$ 是一个范数

2 Exercise 11.13

验证 non-negativity, 由范数的非负性, $d(S, T) = \|S - T\| \geq 0$

验证 identity of indiscernibles, $d(S, T) = 0 \Leftrightarrow \|S - T\| = 0 \Leftrightarrow$

$\forall x|(S - T)x| = 0 = |Sx - Tx| \Leftrightarrow \forall x Sx = Tx \Leftrightarrow S = T$

验证 symmetry, $d(S, T) = \|S - T\| = \|T - S\| = d(T, S)$

验证 triangle inequality, $d(A, B) = \|A - B\| \leq \|A - C\| + \|C - B\| = d(A, C) + d(C, B)$

3 Exercise 11.16

real positivity, point separation, absolute homogeneity 显然, 下证 triangle inequality:

$$\|u+v\|^2 = \sum_{j=1}^n \|ue_j + ve_j\|^2 \quad (1)$$

$$= \sum_{j=1}^n (\|ue_j\|^2 + \|ve_j\|^2 + 2\langle ue_j, ve_j \rangle) \quad (2)$$

$$= \|u\|^2 + \|v\|^2 + 2 \sum_{j=1}^n \langle ue_j, ve_j \rangle \quad (\text{由内积的性质}) \quad (3)$$

$$\leq \|u\|^2 + \|v\|^2 + 2 \sqrt{\sum_{j=1}^n \|ue_j\|^2 \sum_{j=1}^n \|ve_j\|^2} \quad (\text{由柯西不等式}) \quad (4)$$

$$\leq \|u\|^2 + \|v\|^2 + 2 \sqrt{\sum_{j=1}^n \|ue_j\|^2 \sum_{j=1}^n \|ve_j\|^2} \quad (5)$$

$$\leq \|u\|^2 + \|v\|^2 + 2\sqrt{\|u\|^2 \|v\|^2} \quad (6)$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \quad (7)$$

$$\leq (\|u\| + \|v\|)^2 \quad (8)$$

$$(9)$$

两边同时开平方即可得 $\|u + v\| \leq \|u\| + \|v\|$

4 Exercise 11.20

由 11.18 有 $|TSx| \leq |T||Sx|$

则 $|TS|^2 = \sum_{j=1}^k |TSe_j|^2 \leq \sum_{j=1}^k (|T||Se_j|)^2 = |T|^2 \sum_{j=1}^k (|Se_j|)^2 = |T|^2 |S|^2$

两边同时开平方即可得 $|TS| \leq |T||S|$

5 Exercise 11.36

设 $X = PJP^{-1}$, J 为约当标准型, P 为可逆矩阵, 则有 $\det(X) = \det(PJP^{-1}) = \det(P)\det(J)\det(P^{-1}) = \det(J)$ 且 $\text{trace}(X) = \text{trace}(PJP^{-1}) = \text{trace}(P)\text{trace}(J)\text{trace}(P^{-1}) = \text{trace}(J)$

设 X 的特征值为 $\lambda_1, \dots, \lambda_n$, 则有 $\text{trace}(X) = \sum_{i=1}^n \lambda_i$

所以有 $\det(e^X) = e^{\text{trace}(X)} = e^{\sum_{i=1}^n \lambda_i} = e^{\text{trace}(X)}$