1 Exercise 11.9

验证 real positivity，由 $||T|| \geq \inf\{M \geq 0\} = 0 \Rightarrow ||T|| \geq 0$

验证 point separation，$||T|| = 0 \Rightarrow \forall x, ||Tx|| \leq M||x|| = 0 \Rightarrow T = 0$

验证 absolute homogeneity，$\forall a \in \mathbb{F}, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

有 $||av|| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |avx| = |a||vx| \leq M|x|\}$

若 $|a| = 0, ||av|| = 0 = |a|||v||$, 否则，有 $||av|| = \inf M \geq 0 : \forall x \in \mathbb{F}^n, |vx| \leq \frac{M}{|a|}|x|$, 也有 $||av|| = |a|||v||$

验证 triangle inequality，$\forall u, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

$||u + v|| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |(u + v)x| = |ux + vx| \leq M|x|\}$

设 $||u|| = M_1, ||v|| = M_2$，则有 $\forall x \in \mathbb{F}^n, |ux| \leq M_1|x|, |vx| \leq M_2|x| \Rightarrow |ux| + |vx| \leq (M_1 + M_2)|x|$

有 $||u + v|| \leq M_1 + M_2 = ||u|| + ||v||$

所以有 $||\cdot||$ 是一个范数

2 Exercise 11.13

验证 non-negativity，由范数的非负性，$d(S, T) = ||S - T|| \geq 0$

验证 identity of indiscernibles，$d(S, T) = 0 \Leftrightarrow ||S - T|| = 0 \Leftrightarrow$

$\forall x|(S - T)x| = 0 = |Sx - Tx| \Leftrightarrow \forall xSx = Tx \Leftrightarrow S = T$

验证 symmetry，$d(S, T) = ||S - T|| = ||T - S|| = d(T, S)$

验证 triangle inequality，$d(A, B) = ||A - B|| \leq ||A - C|| + ||C - B|| = d(A, C) + d(C, B)$

3 Exercise 11.16

real positivity,point separation,absolute homogeneity 显然，下证 triangle inequality：

$$
\begin{aligned}
||u + v||^2 &= \sum_{j=1}^{n} ||ue_j + ve_j||^2 && (1)\\
&= \sum_{j=1}^{n} (||ue_j||^2 + ||ve_j||^2 + 2\langle ue_j, ve_j\rangle) && (2)\\
&= ||u||^2 + ||v||^2 + 2\sum_{j=1}^{n} \langle ue_j, ve_j\rangle(\text{由内积的性质}) && (3)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{\sum_{j=1}^{n} ||ue_j||^2||ve_j||^2}(\text{由柯西不等式}) && (4)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{\sum_{j=1}^{n} ||ue_j||^2 \sum_{j=1}^{n} ||ve_j||^2} && (5)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{||u||^2||v||^2} && (6)\\
&\leq ||u||^2 + ||v||^2 + 2||u||||v|| && (7)\\
&\leq (||u|| + ||v||)^2 && (8)\\
& && (9)
\end{aligned}
$$

两边同时开平方即可得 $||u+v|| \leq ||u|| + ||v||$

4 Exercise 11.20

由 11.18 有 $|TSx| \leq |T||Sx|$

则 $|TS|^2 = \sum_{j=1}^{k} ||TSe_j||^2 \leq \sum_{j=1}^{k} (|T||Se_j||)^2 = |T|^2 \sum_{j=1}^{k} (|Se_j||)^2 = |T|^2|S|^2$

两边同时开平方即可得 $|TS| \leq |T||S|$

5 Exercise 11.36

设 $X = PJP^{-1}$,J 为约当标准型，P 为可逆矩阵，则有 $det(X) = det(PJP^{-1}) = det(P)det(J)det(P^{-1}) = det(J)$ 且 $trace(X) = trace(PJP^{-1}) = trace(P)trace(J)trace(P^{-1}) = trace(J)$

设 X 的特征值为 $\lambda_1, \ldots, \lambda_n$, 则有 $trace(X) = \Sigma_{i=1}^{N} \lambda_i$

所以有 $det(e^X) = e^J = e^{\Sigma_{i=1}^{N} \lambda_i} = e^{trace(X)}$

6 Exercise 11.50

由于 v 和 w 都满足相同的初值问题，因此我们有：

$$\begin{aligned} |v(t) - w(t)| &= |v(a) - w(a) + \int_a^t f(v(s), s) - f(w(s), s) ds| \\ &\leq |v(a) - w(a)| + \left| \int_a^t f(v(s), s) - f(w(s), s) ds \right| \\ &\leq |v(a) - w(a)| + L \int_a^t |v(s) - w(s)| ds, \end{aligned}$$

其中 L 是 f 关于其第一个变量的 Lipschitz 常数。

于是有

$$|v(t) - w(t)| - L \int_a^t |v(s) - w(s)| ds \leq |v(a) - w(a)|$$

$$(e^{-tL} \int_a^t |v(s) - w(s)| ds)' \leq e^{-tL} |v(a) - w(a)|$$

两边求积分得

$$(e^{-tL} \int_a^t |v(s) - w(s)| ds) \leq e^{-tL} |v(a) - w(a)| \frac{e^{-aL} - e^{-tL}}{L}$$

两边除以 e^{-tL} 再求导即可得到

$$|v(t) - w(t)| \leq |v(a) - w(a)| \exp(L(t-a))$$

证毕

7 Exercise 11.100

对于 trapezoidal rule，$s=1, \alpha_1=1, \alpha_0=-1, \beta_0=\beta_1=\frac{1}{2}$

$$C_0=\sum_{j=0}^{s}\alpha_j=0$$
$$C_1=\sum_{j=0}^{s}(j\alpha_j-\beta_j)=0$$
$$C_2=\sum_{j=0}^{s}(\frac{1}{2!}j^2\alpha_j-j\beta_j)=0$$
$$C_3=\sum_{j=0}^{s}(\frac{1}{3!}j^3\alpha_j-\frac{1}{2!}j^2\beta_j)=-\frac{1}{12}$$
$$C_4=\sum_{j=0}^{s}(\frac{1}{4!}j^4\alpha_j-\frac{1}{3!}j^3\beta_j)=-\frac{1}{24}$$

对于 midpoint method，$s=2, \alpha_2=1, \alpha_1=0, \alpha_0=-1, \beta_0=0, \beta_1=2$

$$C_0=\sum_{j=0}^{s}\alpha_j=0$$
$$C_1=\sum_{j=0}^{s}(j\alpha_j-\beta_j)=0$$
$$C_2=\sum_{j=0}^{s}(\frac{1}{2!}j^2\alpha_j-j\beta_j)=0$$
$$C_3=\sum_{j=0}^{s}(\frac{1}{3!}j^3\alpha_j-\frac{1}{2!}j^2\beta_j)=\frac{1}{3}$$
$$C_4=\sum_{j=0}^{s}(\frac{1}{4!}j^4\alpha_j-\frac{1}{3!}j^3\beta_j)=\frac{1}{3}$$

8 Exercise 11.102

为了达到三阶精度，有 $C_0=C_1=C_2=0$, 即
$\sum_{j=0}^{s}\alpha_j=0, \sum_{j=0}^{s}(j\alpha_j-\beta_j)=0, \sum_{j=0}^{s}(\frac{1}{2!}j^2\alpha_j-j\beta_j)=0$
等价于 $\rho(1)=0, \sigma(1)=\rho^{'}(1), \frac{1}{2}\sigma^{'}(1)=\rho^{'}(1)+\rho^{''}(1)$

9 Exercise 11.103