1 Exercise 11.9

验证 real positivity，由 $||T|| \geq \inf\{M \geq 0\} = 0 \Rightarrow ||T|| \geq 0$

验证 point separation，$||T|| = 0 \Rightarrow \forall x, ||Tx|| \leq M||x|| = 0 \Rightarrow T = 0$

验证 absolute homogeneity，$\forall a \in \mathbb{F}, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

有 $||av|| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |avx| = |a||vx| \leq M|x|\}$

若 $|a| = 0, ||av|| = 0 = |a|||v||$, 否则，有 $||av|| = \inf M \geq 0 : \forall x \in \mathbb{F}^n, |vx| \leq \frac{M}{|a|}|x|$, 也有 $||av|| = |a|||v||$

验证 triangle inequality，$\forall u, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

$||u+v|| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |(u+v)x| = |ux + vx| \leq M|x|\}$

设 $||u|| = M_1, ||v|| = M_2$，则有 $\forall x \in \mathbb{F}^n, |ux| \leq M_1|x|, |vx| \leq M_2|x| \Rightarrow |ux| + |vx| \leq (M_1 + M_2)|x|$

有 $||u+v|| \leq M_1 + M_2 = ||u|| + ||v||$

所以有 $||\cdot||$ 是一个范数

2 Exercise 11.13

验证 non-negativity，由范数的非负性，$d(S,T) = ||S-T|| \geq 0$

验证 identity of indiscernibles，$d(S,T) = 0 \Leftrightarrow ||S-T|| = 0 \Leftrightarrow$

$\forall x |(S-T)x| = 0 = |Sx - Tx| \Leftrightarrow \forall x Sx = Tx \Leftrightarrow S = T$

验证 symmetry，$d(S,T) = ||S-T|| = ||T-S|| = d(T,S)$

验证 triangle inequality，$d(A,B) = ||A-B|| \leq ||A-C|| + ||C-B|| = d(A,C) + d(C,B)$

3 Exercise 11.16

real positivity,point separation,absolute homogeneity 显然，下证 triangle inequality：

$$
\begin{aligned}
||u+v||^2 &= \sum_{j=1}^{n} ||ue_j + ve_j||^2 && (1)\\
&= \sum_{j=1}^{n} (||ue_j||^2 + ||ve_j||^2 + 2\langle ue_j, ve_j\rangle) && (2)\\
&= ||u||^2 + ||v||^2 + 2\sum_{j=1}^{n} \langle ue_j, ve_j\rangle (\text{由内积的性质}) && (3)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{\sum_{j=1}^{n} ||ue_j||^2 ||ve_j||^2} (\text{由柯西不等式}) && (4)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{\sum_{j=1}^{n} ||ue_j||^2 \sum_{j=1}^{n} ||ve_j||^2} && (5)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{||u||^2 ||v||^2} && (6)\\
&\leq ||u||^2 + ||v||^2 + 2||u||||v|| && (7)\\
&\leq (||u|| + ||v||)^2 && (8)\\
& && (9)
\end{aligned}
$$

两边同时开平方即可得 $||u+v|| \leq ||u|| + ||v||$

4 Exercise 11.20

由 11.18 有 $|TSx| \leq |T||Sx|$
则 $|TS|^2 = \sum_{j=1}^{k} ||TSe_j||^2 \leq \sum_{j=1}^{k}(|T||Se_j||)^2 = |T|^2 \sum_{j=1}^{k}(|Se_j||)^2 = |T|^2|S|^2$
两边同时开平方即可得 $|TS| \leq |T||S|$