1 Exercise 11.9

验证 real positivity，由 $||T|| \geq \inf\{M \geq 0\} = 0 \Rightarrow ||T|| \geq 0$

验证 point separation，$||T|| = 0 \Rightarrow \forall x, ||Tx|| \leq M||x|| = 0 \Rightarrow T = 0$

验证 absolute homogeneity，$\forall a \in \mathbb{F}, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

有 $||av|| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |avx| = |a||vx| \leq M|x|\}$

若 $|a| = 0, ||av|| = 0 = |a|||v||$, 否则，有 $||av|| = \inf M \geq 0 : \forall x \in \mathbb{F}^n, |vx| \leq \frac{M}{|a|}|x|$, 也有 $||av|| = |a|||v||$

验证 triangle inequality，$\forall u, v \in \mathcal{V}(\mathbb{F}^n, \mathbb{F}^m)$

$||u + v|| = \inf\{M \geq 0 : \forall x \in \mathbb{F}^n, |(u + v)x| = |ux + vx| \leq M|x|\}$

设 $||u|| = M_1, ||v|| = M_2$，则有 $\forall x \in \mathbb{F}^n, |ux| \leq M_1|x|, |vx| \leq M_2|x| \Rightarrow |ux| + |vx| \leq (M_1 + M_2)|x|$

有 $||u + v|| \leq M_1 + M_2 = ||u|| + ||v||$

所以有 $||\cdot||$ 是一个范数

2 Exercise 11.13

验证 non-negativity，由范数的非负性，$d(S, T) = ||S - T|| \geq 0$

验证 identity of indiscernibles，$d(S, T) = 0 \Leftrightarrow ||S - T|| = 0 \Leftrightarrow$

$\forall x|(S - T)x| = 0 = |Sx - Tx| \Leftrightarrow \forall x Sx = Tx \Leftrightarrow S = T$

验证 symmetry，$d(S, T) = ||S - T|| = ||T - S|| = d(T, S)$

验证 triangle inequality，$d(A, B) = ||A - B|| \leq ||A - C|| + ||C - B|| = d(A, C) + d(C, B)$

3 Exercise 11.16

real positivity,point separation,absolute homogeneity 显然，下证 triangle inequality：

$$
\begin{aligned}
||u + v||^2 &= \sum_{j=1}^{n} ||ue_j + ve_j||^2 && (1)\\
&= \sum_{j=1}^{n} (||ue_j||^2 + ||ve_j||^2 + 2\langle ue_j, ve_j\rangle) && (2)\\
&= ||u||^2 + ||v||^2 + 2\sum_{j=1}^{n} \langle ue_j, ve_j\rangle (\text{由内积的性质}) && (3)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{\sum_{j=1}^{n} ||ue_j||^2||ve_j||^2} (\text{由柯西不等式}) && (4)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{\sum_{j=1}^{n} ||ue_j||^2 \sum_{j=1}^{n} ||ve_j||^2} && (5)\\
&\leq ||u||^2 + ||v||^2 + 2\sqrt{||u||^2||v||^2} && (6)\\
&\leq ||u||^2 + ||v||^2 + 2||u||||v|| && (7)\\
&\leq (||u|| + ||v||)^2 && (8)\\
& && (9)
\end{aligned}
$$

两边同时开平方即可得 $||u+v|| \leq ||u||+||v||$

4 Exercise 11.20

由 11.18 有 $|TSx| \leq |T||Sx|$

则 $|TS|^2 = \sum_{j=1}^{k} ||TSe_j||^2 \leq \sum_{j=1}^{k} (|T||Se_j||)^2 = |T|^2 \sum_{j=1}^{k} (|Se_j||)^2 = |T|^2|S|^2$

两边同时开平方即可得 $|TS| \leq |T||S|$

5 Exercise 11.36

设 $X = PJP^{-1}$,J 为约当标准型，P 为可逆矩阵，则有 $det(X) = det(PJP^{-1}) = det(P)det(J)det(P^{-1}) = det(J)$ 且 $trace(X) = trace(PJP^{-1}) = trace(P)trace(J)trace(P^{-1}) = trace(J)$

设 X 的特征值为 $\lambda_1, \ldots, \lambda_n$, 则有 $trace(X) = \Sigma_{i=1}^{N} \lambda_i$

所以有 $det(e^X) = e^J = e^{\Sigma_{i=1}^{N} \lambda_i} = e^{trace(X)}$

6 Exercise 11.50

由于 v 和 w 都满足相同的初值问题，因此我们有：

$$
\begin{aligned}
|v(t)-w(t)| &= |v(a)-w(a)+\int_a^t f(v(s),s)-f(w(s),s)ds| \\
&\leq |v(a)-w(a)| + \left|\int_a^t f(v(s),s)-f(w(s),s)ds\right| \\
&\leq |v(a)-w(a)| + L\int_a^t |v(s)-w(s)|ds,
\end{aligned}
$$

其中 L 是 f 关于其第一个变量的 Lipschitz 常数。

于是有

$$
|v(t)-w(t)| - L\int_a^t |v(s)-w(s)|ds \leq |v(a)-w(a)|
$$

$$
(e^{-tL}\int_a^t |v(s)-w(s)|ds)' \leq e^{-tL}|v(a)-w(a)|
$$

两边求积分得

$$
(e^{-tL}\int_a^t |v(s)-w(s)|ds) \leq e^{-tL}|v(a)-w(a)|\frac{e^{-aL}-e^{-tL}}{L}
$$

两边除以 e^{-tL} 再求导即可得到

$$
|v(t)-w(t)| \leq |v(a)-w(a)|\exp(L(t-a))
$$

证毕

7 Exercise 11.100

对于 trapezoidal rule，$s=1,\alpha_1=1,\alpha_0=-1,\beta_0=\beta_1=\frac{1}{2}$

$$C_0=\sum_{j=0}^{s}\alpha_j=0$$

$$C_1=\sum_{j=0}^{s}(j\alpha_j-\beta_j)=0$$

$$C_2=\sum_{j=0}^{s}(\frac{1}{2!}j^2\alpha_j-j\beta_j)=0$$

$$C_3=\sum_{j=0}^{s}(\frac{1}{3!}j^3\alpha_j-\frac{1}{2!}j^2\beta_j)=-\frac{1}{12}$$

$$C_4=\sum_{j=0}^{s}(\frac{1}{4!}j^4\alpha_j-\frac{1}{3!}j^3\beta_j)=-\frac{1}{24}$$

对于 midpoint method，$s=2,\alpha_2=1,\alpha_1=0,\alpha_0=-1,\beta_0=0,\beta_1=2$

$$C_0=\sum_{j=0}^{s}\alpha_j=0$$

$$C_1=\sum_{j=0}^{s}(j\alpha_j-\beta_j)=0$$

$$C_2=\sum_{j=0}^{s}(\frac{1}{2!}j^2\alpha_j-j\beta_j)=0$$

$$C_3=\sum_{j=0}^{s}(\frac{1}{3!}j^3\alpha_j-\frac{1}{2!}j^2\beta_j)=\frac{1}{3}$$

$$C_4=\sum_{j=0}^{s}(\frac{1}{4!}j^4\alpha_j-\frac{1}{3!}j^3\beta_j)=\frac{1}{3}$$

8 Exercise 11.102

为了达到三阶精度，有 $C_0=C_1=C_2=0$, 即
$\sum_{j=0}^{s}\alpha_j=0,\sum_{j=0}^{s}(j\alpha_j-\beta_j)=0,\sum_{j=0}^{s}(\frac{1}{2!}j^2\alpha_j-j\beta_j)=0$
等价于 $\rho(1)=0,\sigma(1)=\rho^{'}(1),\frac{1}{2}\sigma^{'}(1)=\rho^{'}(1)+\rho^{''}(1)$

9 Exercise 11.103

使用以下 matlab 代码分别计算 Adams-Bashforth formulas，Adams-Moulton formulas ，Backward differentiation formulas 的系数

```
clc; clear;
format rat;
for s=1:5
    a=zeros(s+1,1);
    co=zeros(s,s);
    rhs=zeros(s,1);
    a(s+1)=1;
```

```
    a(s)=-1;
    for q=1:s
        for j=0:s-1
            co(q,j+1)=j^(q-1)/factorial(q-1);
            rhs(q)=rhs(q)+j^(q)/factorial(q)*a(j+1);
        end
        rhs(q)=rhs(q)+s^(q)/factorial(q)*a(s+1);
    end
    b=(co\rhs).'
end

for s=1:5
    a=zeros(s+1,1);
    co=zeros(s+1,s+1);
    rhs=zeros(s+1,1);
    a(s+1)=1;
    a(s)=-1;
    for q=1:s+1
        for j=0:s
            co(q,j+1)=j^(q-1)/factorial(q-1);
            rhs(q)=rhs(q)+j^(q)/factorial(q)*a(j+1);
        end
    end
    b=(co\rhs).'
end

for s=1:5
    co=zeros(s+2,s+2);
    rhs=zeros(s+2,1);
    for q=0:s
        for j=0:s
            co(q+1,j+2)=j^(q)/factorial(q);
        end
        if(q>=1)co(q+1)=-s^(q-1)/factorial(q-1);end
    end
    co(s+2,s+2)=1;
    rhs(s+2)=1;
    ab=(co\rhs).'
end
```

结果如下:

```
  AdamsBashforth_co =

        1            0            0            0            0
```

−1/2	3/2	0	0	0
5/12	−4/3	23/12	0	0
−3/8	37/24	−59/24	55/24	0
251/720	−637/360	109/30	−1387/360	1901/720

AdamsMoulton_co =

1/2	1/2	0	0	0
−1/12	2/3	5/12	0	0
1/24	−5/24	19/24	3/8	0
−19/720	53/360	−11/30	323/360	251/720

Backwarddifferentiation_co =

1	−1	1	0	0	0
2/3	1/3	−4/3	1	0	0
6/11	−2/11	9/11	−18/11	1	0
12/25	3/25	−16/25	36/25	−48/25	1

10 Exercise 10.108

$\rho(z) = -\frac{2}{11} + \frac{9}{11}z - \frac{18}{11}z^2 + z^3$

$\sigma(z) = \frac{6}{11}z^3$

根据洛必达法则，$\lim_{z\to 1} \frac{\frac{\rho(z)}{\sigma(z)} - \log z}{(z-1)^4} = \frac{-120z^{-7}+180z^{-6}-72z^{-5}+6z^{-4}}{24} = -\frac{1}{4} = \frac{C_{p+1}}{\sigma(1)}$

11 Exercise 10.109

12 Exercise 10.113

证明 $zI - M$ 的行列式为 $z^s + \sum_{i=0}^{s-1} a_i z^i$ 即可

则有

$$\begin{aligned}
&\det(zI-M)\\
&=\det\begin{pmatrix} z & 0 & & \\ & z & & \\ \vdots & \vdots & \ddots & -1 \\ a_0 & a_1+a_0z^{-1} & \cdots & z+a_{s-1}\end{pmatrix}\\
&=det\begin{pmatrix} z & 0 & 0 & 0 & \cdots & 0 \\ 0 & z & 0 & 0 & \cdots & 0 \\ 0 & 0 & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1+a_0z^{-1} & a_2+a_1z^{-1}+a_0z^{-2} & a_3 & \cdots & z+a_{s-1}\end{pmatrix}
\end{aligned}$$

根据同理可去掉副对角线上的其他-1 项，则有

$$\begin{aligned}
&\det(zI-M)\\
&det\begin{pmatrix} z & 0 & 0 & 0 & \cdots & 0 \\ 0 & z & 0 & 0 & \cdots & 0 \\ 0 & 0 & z & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1+a_0z^{-1} & a_2+a_1z^{-1}+a_0z^{-2} & a_3+a_2z^{-1}+a_1z^{-2}+a_0z^{-3} & \cdots & z+\sum_{i=0}^{s-1}a_iz^{i-s+1}\end{pmatrix}\\
&=z^s+\sum_{i=0}^{s-1}a_iz^i
\end{aligned}$$

13 Exercise 10.119

根据 11.76，有

$$\begin{bmatrix} y_{s-1} \\ y_{s-2} \\ y_{s-3} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_{s-2} & \theta_{s-1} \\ 0 & 1 & \theta_1 & \cdots & \theta_{s-3} & \theta_{s-2} \\ 0 & 0 & 1 & \cdots & \theta_{s-4} & \theta_{s-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \theta_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_{s-1} \\ \tilde{y}_{s-2} \\ \tilde{y}_{s-3} \\ \vdots \\ \tilde{y}_1 \\ \tilde{y}_0 \end{bmatrix}.$$

假设对于 $m>0$，原式在 $n=0,...,m+s-1$ 上成立，下证 $n=m+s$ 时也成立

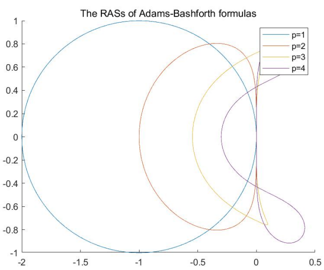
由题意有，$y_{n+s} = \psi_{n+s} - \sum_{i=0}^{s-1} \alpha_i y_{n+i}$

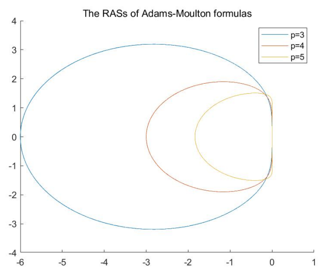
$$
\begin{aligned}
y_{m+s} &= \psi_{m+s} - \sum_{i=0}^{s-1} \alpha_i y_{m+i} \\
&= \psi_{m+s} - \sum_{i=0}^{s-1} \alpha_i (\sum_{j=0}^{s-1} \theta_{m+i-j} \tilde{y}_j + \sum_{j=s}^{m+i} \theta_{m+i-j} \Phi_j) \\
&= \psi_{m+s} - \sum_{j=0}^{s-1} \tilde{y}_j \sum_{i=0}^{s-1} \alpha_i \theta_{m+i-j} - \sum_{j=s}^{m+s-1} \psi_j \sum_{i=0}^{s-1} \alpha_i \theta_{m+i-j} \\
&= \psi_{m+s} + \sum_{j=0}^{s-1} \tilde{y}_j \theta_{m+s-i} + \sum_{j=s}^{m+s-1} \psi_j \theta_{m+s-j} \\
&= \sum_{1}^{s-1} \tilde{y}_j \theta_{m+s-j} + \sum_{j=s}^{m+s} \psi_j \theta_{m+s-j}
\end{aligned}
$$

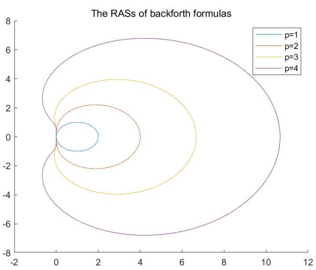
故 11.118 成立

14 Exercise 10.124

15 Exercise 10.141







使用的代码如下:

```
theta = linspace(0, 2*pi, 1000)*1i;
for p=1:4
    hold on;
    z=Adams_Bahsforth_r(p,theta);
    plot(real(z), imag(z));
end
legend('p=1','p=2','p=3','p=4')
title('The RASs of Adams-Bashforth formulas')
saveas(gcf, 'Adams-Bashforth.jpg');

cla
for p=3:5
    hold on;
    z=Adams_Moulton_r(p,theta);
    plot(real(z), imag(z));
end
```

```
legend('p=3','p=4','p=5')
title('The RASs of Adams-Moulton formulas')
saveas(gcf, 'Adams-Moulton.jpg');

cla
for p=1:4
    hold on;
    z=backforth_r(p,theta);
    plot(real(z), imag(z));
end
legend('p=1','p=2','p=3','p=4')
title('The RASs of backforth formulas')
saveas(gcf, 'backforth.jpg');
function [re]=Adams_Bahsforth_r(p,theta)
    if(p==1)
        re=exp(theta)-1;
    else if(p==2)
            re=2*(exp(2*theta)-exp(theta))./(3*exp(theta)-1);
        else
            if(p==3)
                re=12*(exp(3*theta)-exp(2*theta))./(23*exp(2*theta)-16*exp(theta)+5);
            else
                if(p==4)
                    re=24*(exp(4*theta)-exp(3*theta))./(55*exp(3*theta)-59*exp(2*theta)+
                end
            end
        end
    end
end
function [re]=Adams_Moulton_r(p,theta)
    if(p==3)
        re=12*(exp(2*theta)-exp(theta))./(5*exp(2*theta)+8*exp(theta) - 1);
    else if(p==4)
            re=24*(exp(3*theta)-exp(2*theta))./(9*exp(3*theta)+19*exp(2*theta)-5*exp(the
        else
            if(p==5)
                re=720*(exp(4*theta)-exp(3*theta))./(251*exp(4*theta)+646*exp(3*theta)-2
            end
        end
    end
end

function [re]=backforth_r(p,theta)
    if(p==1)
```

```
        re=(exp(theta)-1)./exp(theta);
    else if(p==2)
            re=(3*exp(2*theta)-4*exp(theta)+1)./(2*exp(2*theta));
        else
            if(p==3)
                re=(11*exp(3*theta)-18*exp(2*theta)+9*exp(theta)-2)./(6*exp(3*theta));
            else
                if(p==4)
                    re=(25*exp(4*theta)-48*exp(3*theta)+36*exp(2*theta)-16*exp(theta)+3
                end
            end
        end
    end
end
```