Introduction to Filters

Scope and Background Reading

This session is Based on the section **Filtering** from Chapter 5 of Benoit Boulet, Fundamentals of Signals and Systems from the **Recommended Reading List**.

This material is an introduction to analogue filters. You will find much more in-depth coverage on Pages 11-1—1-48 of Karris.

Agenda

- Frequency Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

Introduction

- Filter design is an important application of Fourier transform
- Filtering is a rich topic often taught in graduate courses so we give only an introduction.
- Our introduction *will* illustrate the usefulness of the frequency domain viewpoint.
- We will explore how filters can shape the spectrum of a signal.

Other applications of the Fourier transform are sampling theory (introduced next week) and modulation.

Frequency Selective Filters

Frequency Selective Filters

An ideal frequency-selective filter is a system that let's the frequency components of a signal through undistorted while components at other components are completely cut off.

- The range of frequencies which are let through belong to the **pass Band**
- The range of frequencies which are cut-off by the filter are called the **stopband**
- A typical scenario where filtering is needed is when noise $n(t)$ is added to a signal $x(t)$ but that signal has most of its energy outside the bandwidth of a signal.

Typical filtering problem

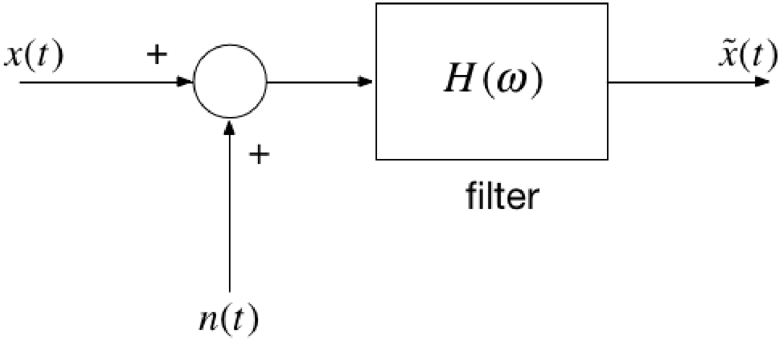


Figure 1:Typical filtering problem

Signal

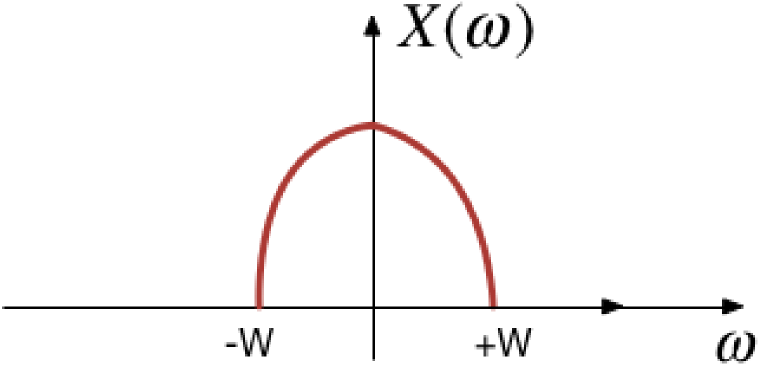


Figure 2:Signal

Out-of Bandwidth Noise

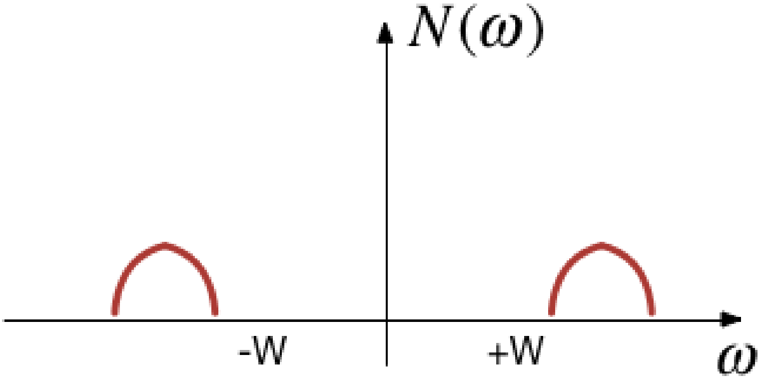


Figure 3:Noise

Signal plus Noise

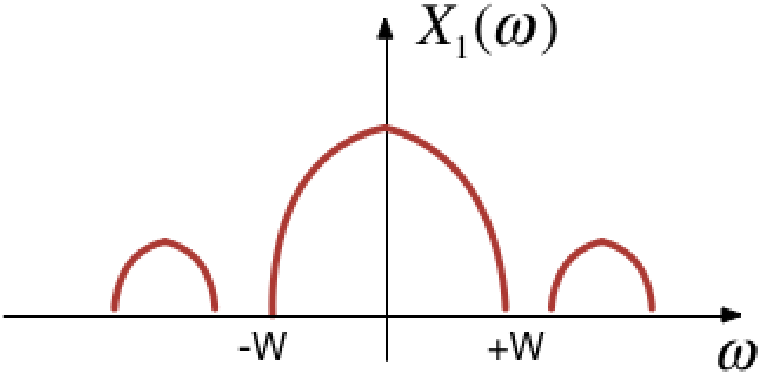


Figure 4:Signal plus noise

Filtering

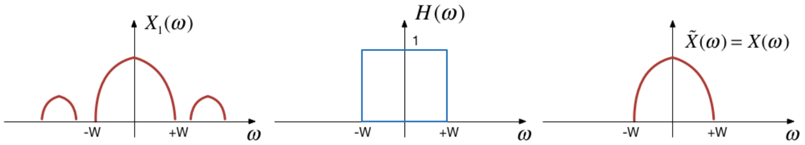


Figure 5:filtering

Motivating example

Filtering in Matlab using 'built-in' filter design techniques by David Dorran

YouTube.

For script see: Filter Design Using Matlab Demo

Ideal Low-Pass Filter

Ideal Low-Pass Filter

An ideal low pass filter cuts-off frequencies higher than its *cutoff frequency*, ω_c.

$$H_{\mathrm{lp}}(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases}$$

Frequency response

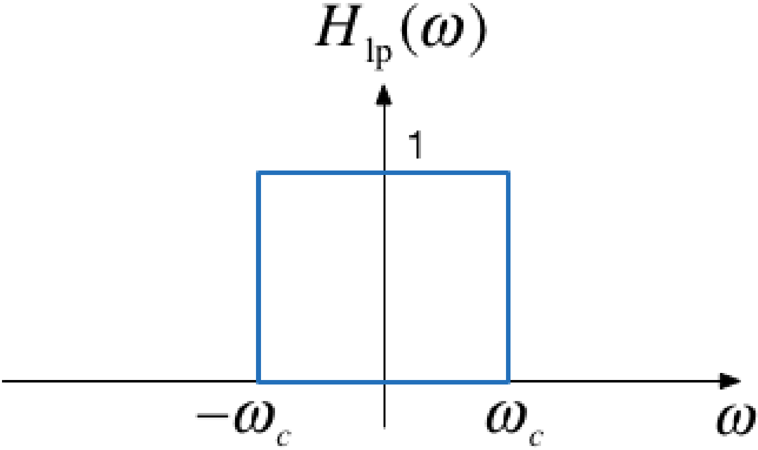


Figure 6:Ideal low-pass filter

Impulse response

$$h_{\mathrm{lp}}(t) = \frac{\omega_c}{\pi} \mathrm{sinc}\left(\frac{\omega_c}{\pi} t\right)$$

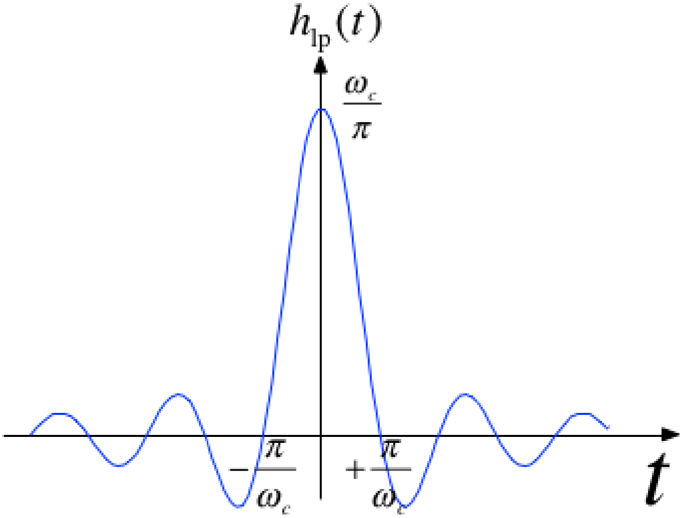


Figure 7:Impulse response of ideal low-pass filter

Filtering is Convolution

The output of an LTI system with impulse response

$$h(t) \Leftrightarrow H(\omega)$$

subject to an input signal

$$x(t) \Leftrightarrow X(\omega)$$

is given by

$$y(t) = h(t) * x(t) \Leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Issues with the "ideal" filter

This is the step response:

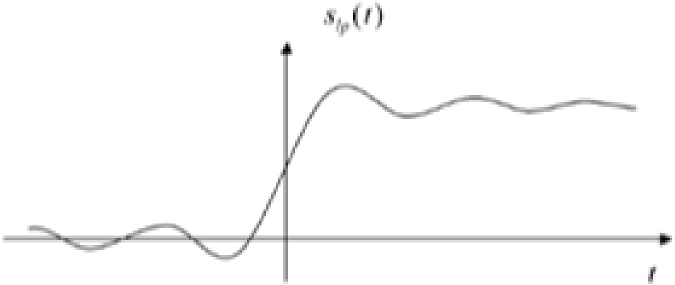


Figure 8:Step response of ideal filter

(reproduced from Boulet Fig. 5.23 p. 205)

Butterworth low-pass filter

Butterworth low-pass filter

N-th Order Butterworth Filter

$$|H_B(\omega)| = \frac{1}{\left(1 + \left(\frac{\omega}{\omega_c}\right)^{2N}\right)^{\frac{1}{2}}}$$

Remarks

- DC gain is $|H_B(j0)| = 1$
- Attenuation at the cut-off frequency is $|H_B(j\omega_c)| = 1/\sqrt{2}$ for any N

More about the Butterworth filter: Wikipedia Article

Example 1: Second-order BW Filter

The second-order Butterworth Filter is defined by its *characteristic equation* (CE):

$$p(s) = s^2 + \omega_c\sqrt{2}s + \omega_c^2 = 0^*$$

Calculate the roots of $p(s)$ (the poles of the filter transfer function) in both Cartesian and polar form.

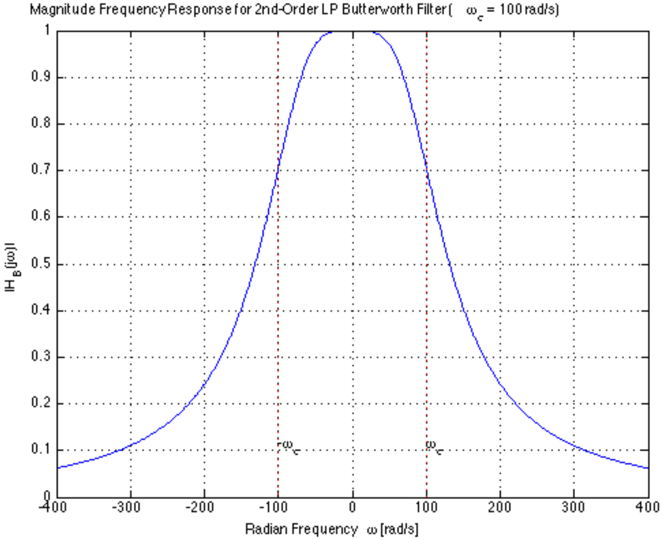
Example 2

Derive the differential equation relating the input $x(t)$ to output $y(t)$ of the 2nd-Order Butterworth low-pass filter with cutoff frequency ω_c.

Example 3

Determine the frequency response $H_B(\omega) = Y(\omega)/X(\omega)$

Magnitude of frequency response of a 2nd-order Butterworth Filter



Bode-plot of a 2nd-order Butterworth Filter

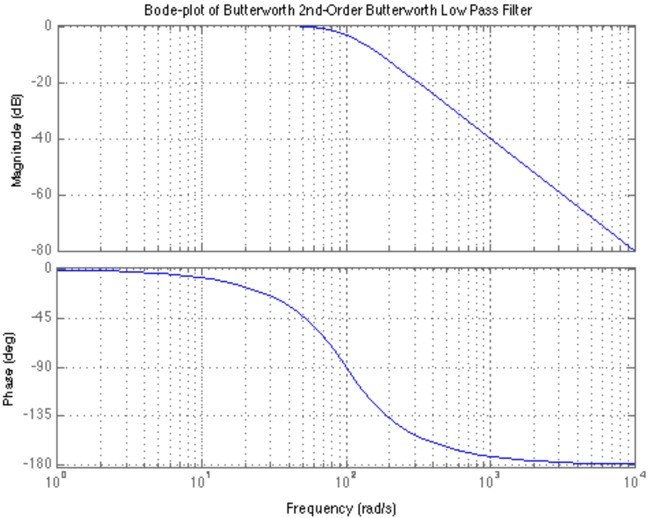


Figure 10:Bode plot of Butterworth filter

Example 4

Determine the impulse response of the Butterworth filter.

You will find this Fourier transform pair useful:

$$e^{-at} \sin \omega_0 t \; u_0(t) \Leftrightarrow \frac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$$

Impulse response

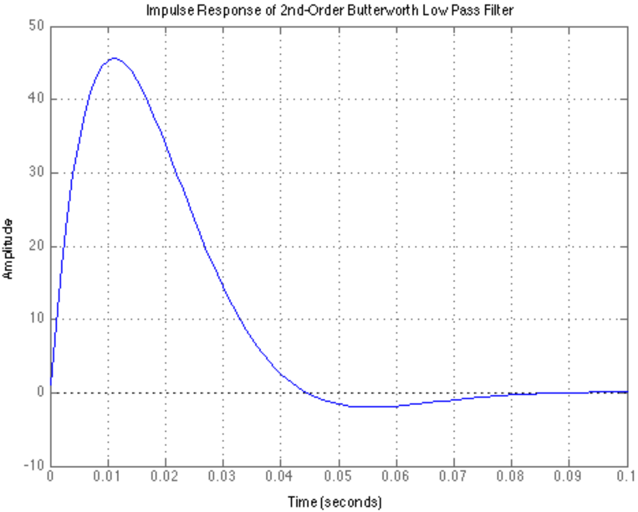


Figure 11:Impulse response of Butterworth filter

Step response of of a 2nd-order Butterworth Filter

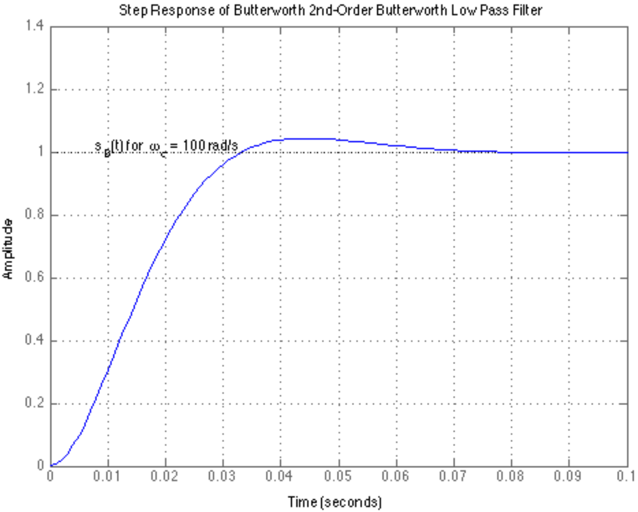


Figure 12:Step response of Butterworth filter

High-pass filter

High-pass filter

An ideal high-pass filter cuts-off frequencies lower than its *cutoff frequency*, ω_c.

$$H_{\mathrm{hp}}(\omega) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

Frequency response

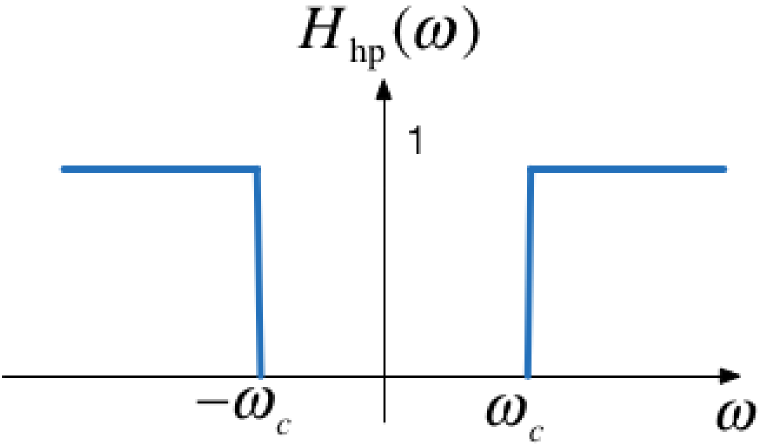


Figure 13:Frequency respons of a high-pass filter

Responses

Frequency response

$$H_{\mathrm{hp}}(\omega) = 1 - H_{\mathrm{lp}}(\omega)$$

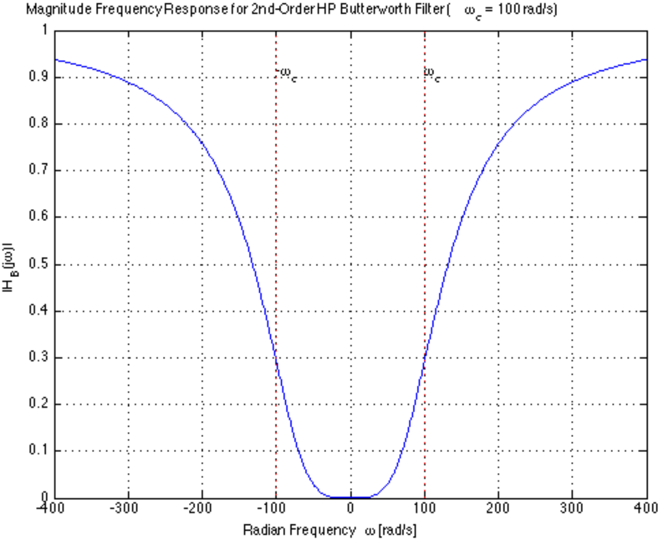
Impulse response

$$h_{\mathrm{hp}}(t) = \delta(t) - h_{\mathrm{lp}}(t)$$

Example 5

Determine the frequency response and impulse response of a 2nd-order butterworth highpass filter

Magnitude of frequency response of a 2nd-order Butterworth High-Pass Filter



Bandpass filter

Bandpass filter

An ideal bandpass filter cuts-off frequencies lower than its first *cutoff frequency* ω_{c1}, and higher than its second *cutoff frequency* ω_{c2}.

$$H_{\mathrm{bp}}(\omega) = \begin{cases} 1, & \omega_{c1} < |\omega| < \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

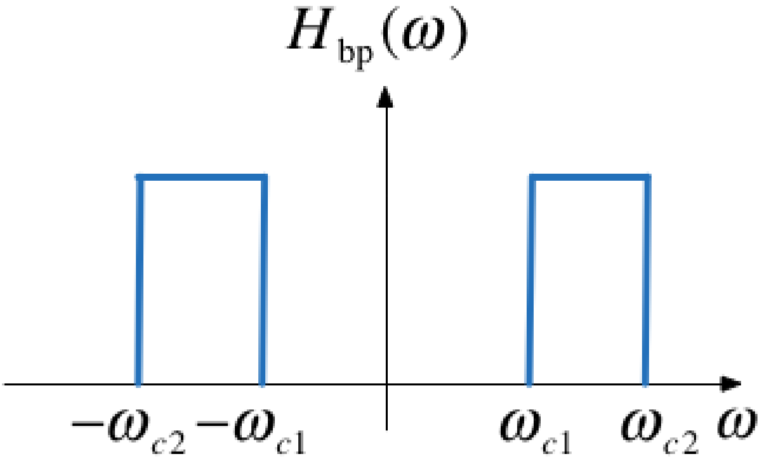


Figure 15:Bandpass filter

Bandpass filter design

A bandpass filter can be obtained by multiplying the frequency responses of a low-pass filter by a high-pass filter.

$$H_{\mathrm{bp}}(\omega) = H_{\mathrm{hp}}(\omega) H_{\mathrm{lp}}(\omega)$$

- The high-pass filter should have cut-off frequency of ω_{c1}
- The low-pass filter should have cut-off frequency of ω_{c2}

Summary

- Frequency Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

Next Session – sampling theory

Lab Work

In the lab we will look at frequency response analysis and filtering.