The Fourier Transform (Part 2)

Scope and Background Reading

This session continues our introduction to the Fourier Transform with several examples extracted from tables of transform tables.

The material in this presentation and notes is based on Chapter 8 (Starting at Section 8.4) of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. from the Required Reading List. I also used Chapter 5 of Benoit Boulet, Fundamentals of Signals and Systems from the Recommended Reading List.

Agenda

Last time

- Fourier Transform as the Limit of a Fourier Series
- Doing the Maths
- The Fourier Transform
- Properties of the Fourier Transform
- Some Examples
- Computing Fourier Transforms in Matlab

This Time

- Tables of Transform Pairs
- Examples of Selected Transforms
- Relationship between Laplace and Fourier
- Fourier Transforms of Common Signals

Reminder of the Definitions

The Fourier Transform

In the signals and systems context, the Fourier Transform is used to convert a function of time $f(t)$ to a function of radian frequency $F(\omega)$:

$$\mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(e)e^{-j\omega t}\,dt = F(\omega).$$

The Inverse Fourier Transform

In the signals and systems context, the *Inverse Fourier Transform* is used to convert a function of frequency $F(\omega)$ to a function of time $f(t)$:

$$\mathcal{F}^{-1}\left\{F(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega = f(t).$$

Note, the factor 2π is introduced because we are changing units from radians/second to seconds.

Duality of the transform

Note the similarity of the Fourier and its Inverse. This has important consequences in filter design and later when we consider sampled data systems.

Table of Common Fourier Transform Pairs

This table is adapted from Table 8.9 of Karris. See also: Wikibooks: Engineering Tables/Fourier Transform Table and Fourier Transfom—WolframMathworld for more complete references.

Examples of Selected Transforms

The Dirac Delta

$$\delta(t) \Leftrightarrow 1$$

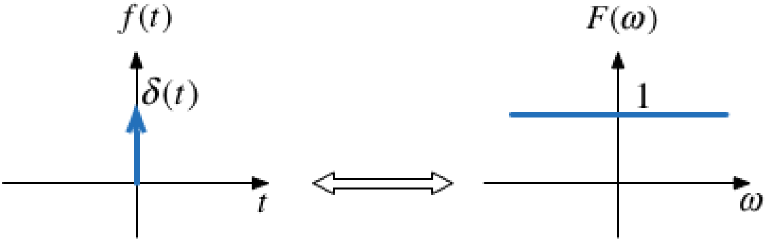

Figure 1:Fourier transform of $\delta(t)$)

DC

$$1 \Leftrightarrow 2\pi\delta(\omega)$$

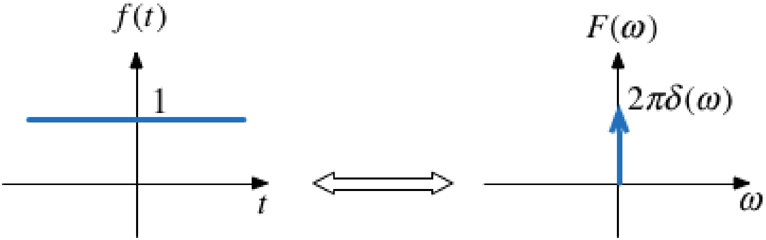

Figure 2:Fourier transform of DC

Cosine wave (Sinewave with even symmetry)

$$\cos(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \Leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

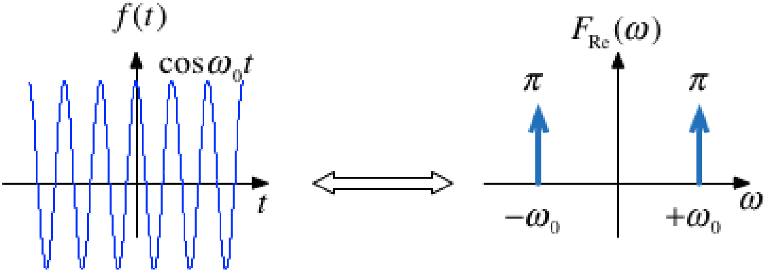

Figure 3:Fourier transform of cosine wave

Sine wave

$$\sin(t) = \frac{1}{j2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \Leftrightarrow j\pi\delta(\omega - \omega_0) - j\pi\delta(\omega + \omega_0)$$

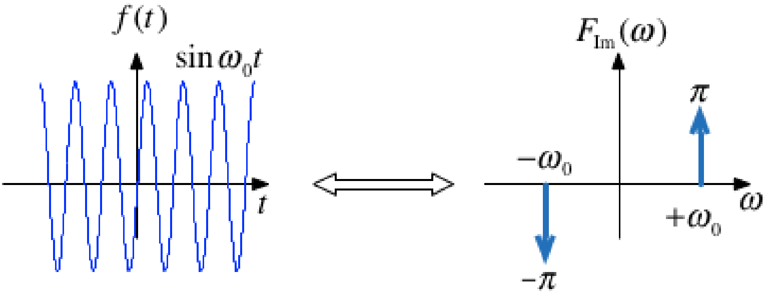

Figure 4:Fourier transform of sine wave

Signum (Sign)

The signum function is a function whose value is equal to

$$\operatorname{sgn} x = \begin{cases} -1 \ x < 1 \\ 0 \ x = 0 \\ +1 \ x > 0 \end{cases}$$

This function is often used to model a *voltage comparitor* in circuits.

The transform is:

$$\operatorname{sgn} x = u_0(t) - u_0(-t) = \frac{2}{j\omega}$$

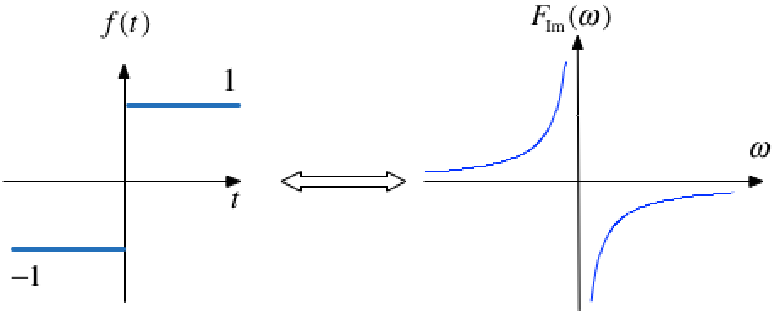

Figure 5:Signum function

Example 1: Unit Step

Use the signum function to show that

$$\mathcal{F}\left\{u_0(t)\right)\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

Clue

Define

$$u_0(t) = 2\,\mathrm{sgn}\, x - 1$$

Does that help?

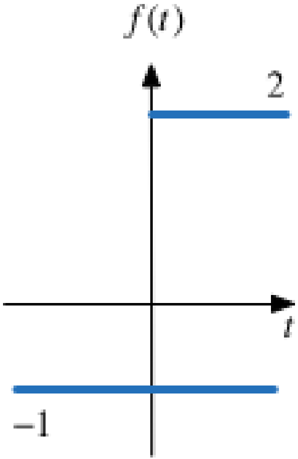

Figure 6:Unit step defined using signum

Graph of unit step

$$u_0(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

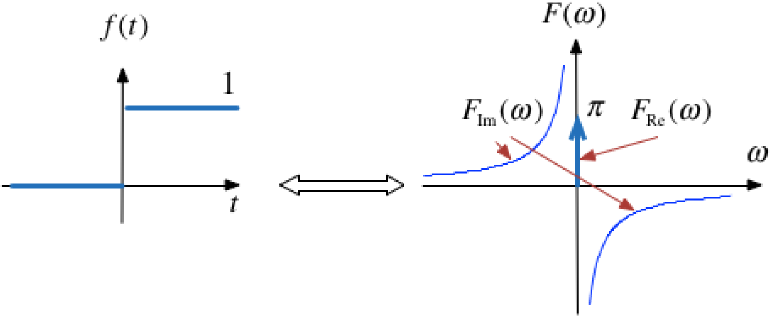

Figure 7:FT of unit step

Example 2

Use the results derived so far to show that

$$e^{j\omega_0 t}u_0(t) \Leftrightarrow \pi\delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)}$$

Hint: linearity plus frequency shift property.

Example 3

Use the results derived so far to show that

$$\sin \omega_0 t \; u_0(t) \Leftrightarrow \frac{\pi}{j2} \left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right] + \frac{\omega_0}{\omega_0^2 - \omega^2}$$

Hint: Euler's formula plus solution to example 2.

Example 4

Use the result of Example 3 to determine the Fourier transform of $\cos \omega_0 t \; u_0(t)$.

Derivation of the Fourier Transform from the Laplace Transform

If a signal is a function of time $f(t)$ which is zero for $t \leq 0$, we can obtain the Fourier transform from the Laplace transform by substituting s by $j\omega$.

Example 5: Single Pole Filter

Given that

$$\mathcal{L}\left\{e^{-at}u_0(t)\right\} = \frac{1}{s+a}$$

Compute

$$\mathcal{F}\left\{e^{-at}u_0(t)\right\}$$

Boulet gives the graph of this function.

Example 6: Complex Pole Pair (cos term)

Given that

$$\mathcal{L}\left\{e^{-at}\cos\omega_0 t\; u_0(t)\right\} = \frac{s+a}{(s+a)^2+\omega_0^2}$$

Compute

$$\mathcal{F}\left\{e^{-at}\cos\omega_0 t\; u_0(t)\right\}$$

Boulet gives the graph of this function.

Fourier Transforms of Common Signals

We shall conclude this session by computing as many of the the Fourier transform of some common signals as we have time for.

- rectangular pulse
- triangular pulse
- periodic time function
- unit impulse train (model of regular sampling)

I will not provide notes for these, but you will find more details in Chapter 8 of Karris and Chapter 5 of Boulet and I have created some worked examples to help with revision which you'll find on Blackboard.

Suggestions for Further Reading

Boulet has several interesting amplifications of the material presented by Karris. You would be well advised to read these. Particular highlights which we will not have time to cover:

- Time multiplication and its relation to amplitude modulation (pp 182—183).
- Fourier transform of the complex exponential signal $e^{(\alpha+j\beta)t}$ with graphs (pp 184—187).
- Use of inverse Fourier series to determine $f(t)$ from a given $F(j\omega)$ and the "ideal" low-pass filter (pp 188—191).
- The Duality of the Fourier transform (pp 191—192).

End of Part 2

Summary

- Tables of Transform Pairs
- Examples of Selected Transforms
- Relationship between Laplace and Fourier
- Fourier Transforms of Common Signals

Next Time

- The Fourier Transform for Systems and Circuit Analysis

Homework

Attempt Questions 1—6 of the End of Chapter Problems (Section 8.10) in Karris.