

# The Fourier Transform (Part 1)

# Scope and Background Reading

This session continues our coverage of **Fourier Analysis** with an introduction to the **Fourier Transform**.

- ▶ **Fourier Series** is used when we are dealing with signals that are *periodic* in time. It is based on harmonics of the fundamental frequency  $\omega_0$  of the periodic signal where the period  $T = 2\pi/\omega_0$ .
- ▶ The line spectrum occur at integer multiples of the fundamental frequency  $k\omega_0$  and is a *discrete* (or sampled) function of frequency.
- ▶ As the period  $T$  is increased, the distance between harmonics decreases because  $\omega_0$  reduces.
- ▶ In the limit  $T \rightarrow \infty$ , the signal becomes **aperiodic** and  $k\omega_0 \rightarrow \omega$  which is a *continuous* function of frequency.

This is the basis of the **Fourier Transform** which is very important as the basis for data transmission, signal filtering, and the determination of system frequency response.

The material in this presentation and notes is based on Chapter 8 (Starting at Section 8.1) of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. from the **Required Reading List**. I also used Chapter 5 of Benoit Boulet, Fundamentals of Signals and Systems from the **Recommended Reading List**.

# Agenda

- ▶ Fourier Transform as the Limit of a Fourier Series
- ▶ Doing the Maths
- ▶ The Fourier Transform
- ▶ Properties of the Fourier Transform
- ▶ Some Examples
- ▶ Computing Fourier Transforms in Matlab

# Fourier Transform as the Limit of a Fourier Series

# Fourier Transform as the Limit of a Fourier Series

We start by considering the pulse train that we used in the last lecture and demonstrate that the discrete line spectra for the Fourier Series becomes a continuous spectrum as the signal becomes aperiodic.

This analysis is from Boulet pp 142—144 and 176—180.

Let  $\tilde{x}(t)$  be the Fourier series of the rectangular pulse train shown below:

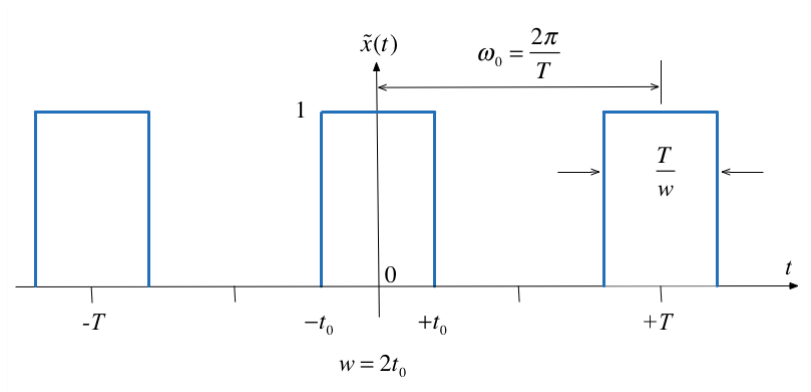


Figure 1: Rectangular pulse train



# Fourier Series

In the previous lecture we used

$$C_k = \frac{1}{2\pi} \int_{-\pi/w}^{\pi/w} A e^{-jk(\omega t)} d(\omega t) = \frac{A}{2\pi} \int_{-\pi/w}^{\pi/w} e^{-jk(\omega t)} d(\omega t)$$

to compute the line spectra.

## From the Time Point of View

If we instead take a time point-of-view and let  $A = 1$

$$C_k = \frac{1}{T} \int_{-t_0}^{t_0} e^{-jk\omega_0 t} dt.$$

We'll complete the analysis on the board.

# Solution

$$C_k = \frac{T}{2t_0} \text{sinc} \left( \pi k \frac{2t_0}{T} \right)$$

# The Sinc Function

The function,  $\sin(x)/x$  crops up again and again in Fourier analysis. The Fourier coefficients  $C_k$  are scaled *samples* of the real continuous *normalized sinc* function defined as follows:

$$\text{sinc } u := \frac{\sin \pi u}{\pi u}, \quad u \in \mathbb{R}.$$

The function is equal to 1 at  $u = 0^*$  and has zero crossings at  $u = \pm n$ ,  $n = 1, 2, 3, \dots$  as shown on the next slide.

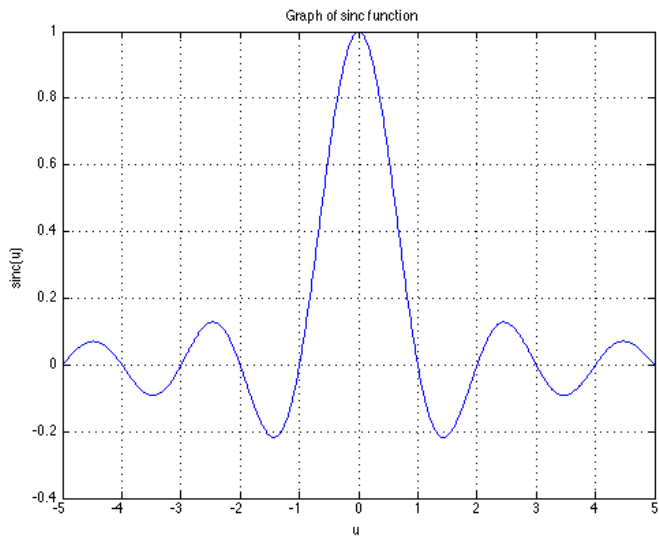


Figure 2: The sinc function

# Duty cycle

- ▶ We define the duty cycle  $\eta = 2t_0/T$  of the rectangular pulse train as the fraction of the time the signal is “on” (equal to 1) over one period.
- ▶ The duty cycle is often given as a percentage.

The spectral coefficients expressed using the normalized sinc function and the duty cycle can be written as

$$C_k = \frac{2t_0}{T} \frac{\sin\left(\frac{\pi k 2t_0}{T}\right)}{\frac{\pi k 2t_0}{T}} = \frac{2t_0}{T} \operatorname{sinc}\left(\frac{k 2t_0}{T}\right)$$

$$C_k = \eta \operatorname{sinc}(k\eta)$$

## Normalize the spectral coefficients

Let us normalize the spectral coefficients of  $\tilde{x}(t)$  by multiplying them by  $T$ , and assume  $t_0$  is fixed so that the duty cycle  $\eta = 2t_0/T$  will decrease as we increase  $T$ :

$$TC_k = T\eta \operatorname{sinc}(k\eta) = 2t_0 \operatorname{sinc}\left(k\frac{2t_0}{T}\right)$$

Then the normalized coefficients  $TC_k$  of the rectangular wave  $a$  is a sinc envelope with constant amplitude at the origin equal to  $2t_0$ , and a zero crossing at fixed frequency  $\pi/t_0$  rad/s, both independent of  $T$ .

Plots follow (use `duty_cycle.m` to reproduce).



## Duty cycle 50%

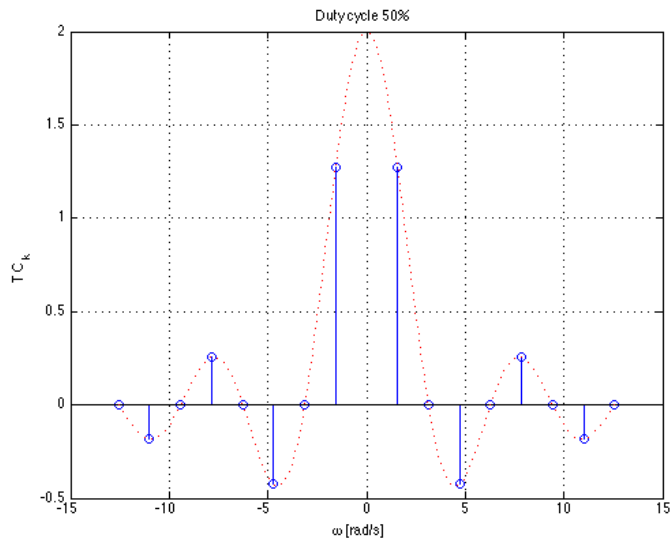


Figure 3:Duty cycle of 50%

## Duty cycle 12.5%

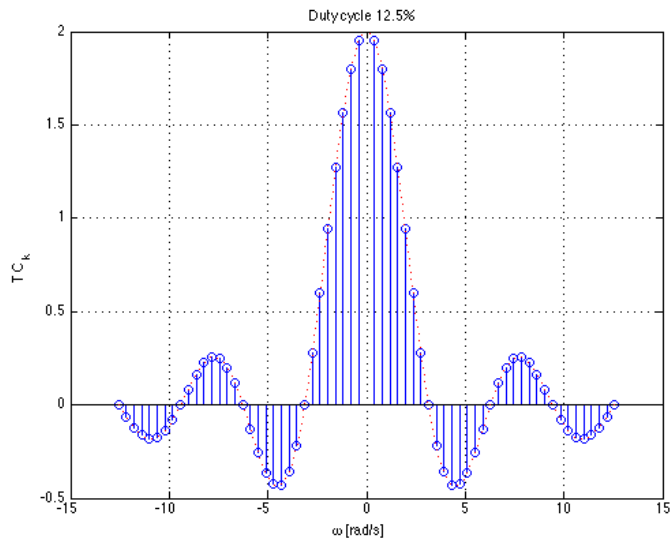


Figure 4:Duty cycle of 12.5%

# Intuition leading to the Fourier Transform

- ▶ An aperiodic signal that has been made periodic by “repeating” its graph every  $T$  seconds will have a line spectrum that becomes more and more dense as the fundamental period is made longer and longer.
- ▶ The line spectrum has the same continuous envelope.
- ▶ As  $T$  goes to infinity, the line spectrum will become a continuous function of  $\omega$ .
- ▶ The *envelope* is this function.

## Doing the Maths

# Doing the Maths

Fourier series coefficients for the rectangular wave are computed using:

$$C_k = \frac{1}{T} \int_{-t_0}^{t_0} \tilde{x}(t) e^{-jk(\omega_0 t)} dt$$

where  $\omega_0 = 2\pi/T$  as usual.

Now define a signal  $x(t)$  as a signal equal to  $\tilde{x}(t)$  over one period and zero elsewhere.

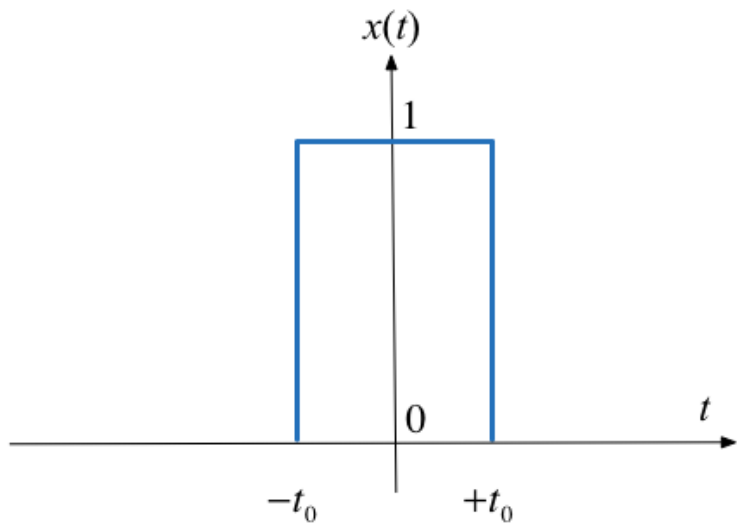


Figure 5: The rectangular pulse

# A rectangular pulse

The aperiodic signal, a single rectangular pulse, can be thought of as being periodic with an infinite fundamental period (we will let  $T \rightarrow +\infty$  later).

Since  $\tilde{x}(t) = x(t)$  over  $t \in [-T/2, T/2]$ , the spectral coefficients  $C_k$  of the periodic signal  $\tilde{x}(t)$  can be written in terms of the aperiodic signal  $x(t)$  as follows:

$$C_k = \frac{1}{T} \int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt.$$

Let us define the envelope  $X(j\omega)$  of  $TC_k$  (we already know this to be the sinc function):

$$X(j\omega) := \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt = \int_{-t_0}^{+t_0} e^{-j\omega t} dt = 2t_0 \operatorname{sinc}\left(\frac{t_0}{\pi}\omega\right).$$

The coefficients  $C_k$  are therefore samples of the continuous envelope  $X(j\omega)$ :

$$C_k = \frac{1}{T} X(jk\omega_0).$$



## Back to the periodic signal

$\tilde{x}(t)$  has the Fourier series representation

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}.$$

or equivalently, since  $\omega_0 = 2\pi/T$ :

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0.$$

## Now take the limits

At the limit, as  $T \rightarrow \infty$ , in

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0.$$

we get

- ▶  $\omega_0 \rightarrow d\omega$  – the fundamental frequency becomes infinitesimally small.
- ▶  $k\omega_0 \rightarrow \omega$  – harmonic frequencies get so close together they become a continuum.
- ▶ The summation becomes an integral.
- ▶  $\tilde{x}(t) \rightarrow x(t)$  – the periodic signal tends to the aperiodic signal.

# The Fourier Transform

# Define the Inverse Fourier and Fourier Transform

Putting these elements together we have an expression for the aperiodic signal in terms of its Fourier transform:

**Inverse Fourier Transform:**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega := \mathcal{F}^{-1} \{X(j\omega)\}$$

Similarly, given the expression we have already seen for an arbitrary  $x(t)$ :

**Fourier Transform:**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt := \mathcal{F} \{x(t)\}.$$

# Fourier Transform Pair

- ▶ The two equations on the previous slide are called the *Fourier transform pair*.
- ▶ They are analogous to the *Laplace transform pair* we have already seen and we can develop tables of properties and transform pairs in the same way.
- ▶ Equation  $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$ . gives the *Fourier transform* or the *frequency spectrum* of the signal  $x(t)$ .
- ▶ Equation  $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$ . is the *inverse Fourier transform*. It gives the function of time that has a given spectrum in the frequency domain.

# Properties of the Fourier Transform

# Properties of the Fourier Transform

Again, we will provide any properties that you might need in the examination.

For fun, you may like to read and even complete the proofs for these properties guided by the text books Karris (Section 8.3) or Boulet pp 180—185.

We are more interested in your ability to exploit the properties in problem solving.

A little more detail is given in the notes available on Blackboard.

# Table of Properties of the Fourier Transform

As was the case of the Laplace Transform, properties of Fourier transforms are usually summarized in Tables of Fourier Transform properties. For example this one: Properties of the Fourier Transform (Wikipedia) and Table 8.8 in Karris (page 8-17).

Some of the more important properties are summarized on this table of properties and there is some additional commentary in the notes.



## Some Examples

# Some Examples

1. Amplitude Modulation
2. Impulse response
3. Energy computation

## Example 1: Amplitude Modulation

Compute the result of multiplying a signal  $f(t)$  by a carrier waveform  $\cos \omega_c t$ .

*Hint* use Euler's identity and the frequency shift property.

## Example 2: Impulse response

A system has impulse response  $f(t) = e^{-t}u_0(t)$ . Compute the frequency spectrum of this system.

## Example 3: Energy computation

An aperiodic real signal  $f(t)$  has Fourier transform  $F(j\omega)$ .  
Compute the energy contained in the signal between 5kHz and 10kHz.

# Computing Fourier Transforms in Matlab

# Computing Fourier Transforms in Matlab

MATLAB has the built-in `fourier` and `ifourier` functions that can be used to compute the Fourier transform and its inverse. We will explore some of these in the lab.

For now, here's an example:

## Example

Use Matlab to confirm the Fourier transform pair:

$$e^{-\frac{1}{2}t^2} \Leftrightarrow \sqrt{2\pi}e^{-\frac{1}{2}\omega^2}$$

See script: ftex1.m.



# End of Part 1

## *Summary*

- ▶ Fourier Transform as the Limit of a Fourier Series
- ▶ Doing the Maths
- ▶ The Fourier Transform
- ▶ Properties of the Fourier Transform
- ▶ Some Examples
- ▶ Computing Fourier Transforms in Matlab

## *Next Time*

- ▶ Fourier transform of commonly occurring signals (Slides)