

## The Impulse Response and Convolution (Part 2)

# Scope and Background Reading

This session continues our introduction to time convolution.

As we shall see, in the determination of a system's response to a signal input, time convolution involves integration by parts and is a tricky operation. But time convolution becomes *multiplication* in the Laplace Transform domain, and is much easier to apply.

The material in this presentation and notes is based on Chapter 6 of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. and builds on the time response of a state-space model that was developed in the previous session.

# Agenda

The material to be presented will need two sessions.

## *Last Session*

- ▶ The Impulse Response of a System in Time Domain
- ▶ Even and Odd Functions of Time

## *This Session*

- ▶ Time Convolution
- ▶ Graphical Evaluation of the Convolution Integral
- ▶ System Response by Convolution
- ▶ System Response by Laplace

# Time Convolution

# Time Convolution

Consider a system whose input is the Dirac delta ( $\delta(t)$ ), and its output is the impulse response  $h(t)$ . We can represent the inpt-output relationship as a block diagram

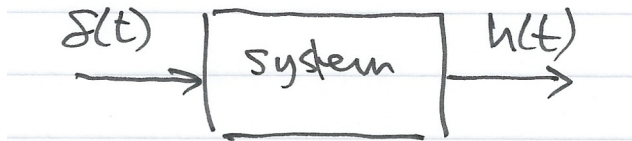


Figure 1: Impulse response

In general

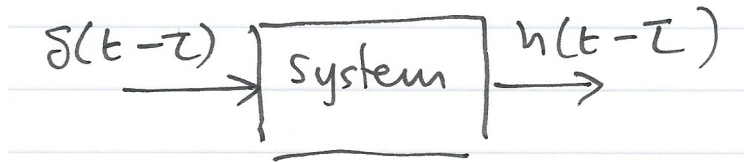


Figure 2: General impulse response

## Add an arbitrary input

Let  $u(t)$  be any input whose value at  $t = \tau$  is  $u(\tau)$ , Then because of the sampling property of the delta function

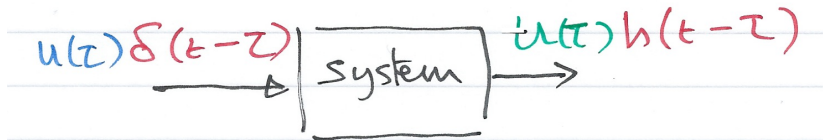


Figure 3: Response to an arbitrary input (1)

(output is  $u(\tau)h(t - \tau)$ )

## Integrate both sides

Integrating both sides over all values of  $\tau$  ( $-\infty < \tau < \infty$ ) and making use of the fact that the delta function is even, i.e.

$$\delta(t - \tau) = \delta(\tau - t)$$



We have:

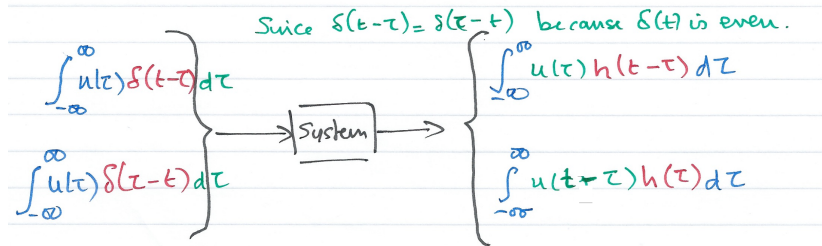


Figure 4: Response to an arbitrary input (2)

## Use the sifting property of delta

The second integral on the left side reduces to  $u(t)$

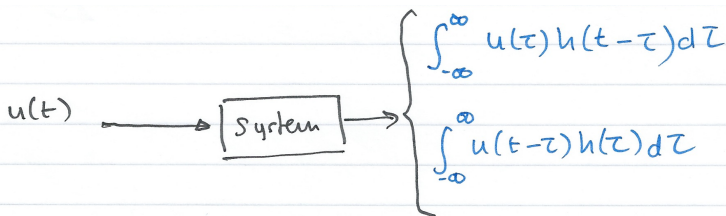


Figure 5: Response to an arbitrary input (3)

# The Convolution Integral

The integral

$$\int_{-\infty}^{\infty} u(\tau)h(t - \tau)d\tau$$

or

$$\int_{-\infty}^{\infty} u(t - \tau)h(\tau)d\tau$$

is known as the *convolution integral*; it states that if we know the impulse response of a system, we can compute its time response to any input by using either of the integrals.

The convolution integral is usually written  $u(t) * h(t)$  or  $h(t) * u(t)$  where the asterisk ( $*$ ) denotes convolution.

# Convolution and State-Space Models

In the previous session, we found that the impulse response of a SISO system (with  $d = 0$ ) was

$$h(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$$

Therefore, if we know  $h(t)$ , we can use the convolution integral to compute the response  $y(t)$  to any input  $u(t)$  using the relation

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau \\ h(t) &= \mathbf{C}e^{\mathbf{A}t} \int_{-\infty}^{\infty} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau \end{aligned}$$

# Graphical Evaluation of the Convolution Integral

# Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. The text book gives three examples (6.4-6.6) which we will demonstrate using a graphical visualization tool developed by Teja Muppirala of the Mathworks.

The tool: `convolutiondemo.m` (see `license.txt`).

# Convolution by Graphical Method - Summary of Steps

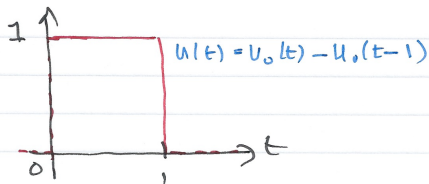
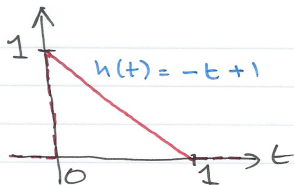
For simplicity, we give the rules for  $u(t)$ , but the procedure is the same if we reflect and slide  $h(t)$

1. Substitute  $u(t)$  with  $u(\tau)$  – this is a simple change of variable. It doesn't change the definition of  $u(t)$ .
2. Reflect  $u(\tau)$  about the vertical axis to form  $u(-\tau)$
3. Slide  $u(-\tau)$  to the right a distance  $t$  to obtain  $u(t - \tau)$
4. Multiply the two signals to obtain the product  $u(t - \tau)h(\tau)$
5. Integrate the product over all  $t$  from  $-\infty$  to  $\infty$ .

## Example 1

(This is example 6.4 in the textbook)

The signals  $h(t)$  and  $u(t)$  are shown below. Compute  $h(t) * u(t)$  using the graphical technique.





## Prepare for convolutiondemo

To prepare this problem for evaluation in the convolutiondemo tool, we need to determine the Laplace Transforms of  $h(t)$  and  $u(t)$ .

$h(t)$

The signal  $h(t)$  is the straight line  $f(t) = -t + 1$  but this is defined only between  $t = 0$  and  $t = 1$ . We thus need to gate the function by multiplying it by  $u_0(t) - u_0(t - 1)$  as illustrated below:

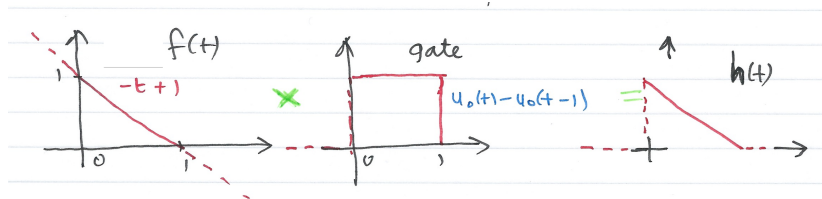


Figure 6: Gating  $f(t)$  with  $u_0(t) - u_0(t - 1)$  to get  $h(t)$

Thus

$$\begin{aligned}h(t) &= (-t + 1)(u_0(t) - u_0(t - 1)) \\&= (-t + 1)u_0(t) - (-(t - 1)u_0(t - 1)) \\&= -tu_0(t) + u_0(t) + (t - 1)u_0(t - 1) \Leftrightarrow H(s) = -\frac{1}{s^2} + \frac{1}{s} + \frac{e^{-s}}{s^2}\end{aligned}$$

$$H(s) = \frac{s^2 + e^{-s} - 1}{s^2}$$

$u(t)$

The input  $u(t)$  is the gating function:

$$u(t) = u_0(t) - u_0(t - 1)$$

so

$$U(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}$$

## convolutiondemo settings

- ▶ Let  $g = (1 - \exp(-s))/s$
- ▶ Let  $h = (s + \exp(-s) - 1)/s^2$
- ▶ Set range  $-2 < \tau < 2$

## Summary of result

1. For  $t < 0$ :  $u(t - \tau)h(\tau) = 0$
2. For  $t = 0$ :  $u(t - \tau) = u(-\tau)$  and  $u(-\tau)h(\tau) = 0$
3. For  $0 < t \leq 1$ :  
$$h * u = \int_0^t (1)(-\tau + 1)d\tau = \tau - \tau^2/2 \Big|_0^t = t - t^2/2$$
4. For  $1 < t \leq 2$ :  
$$h * u = \int_{t-1}^1 (-\tau + 1)d\tau = \tau - \tau^2/2 \Big|_{t-1}^1 = t^2/2 - 2t + 2$$
5. For  $2 \leq t$ :  $u(t - \tau)h(\tau) = 0$

## Example 2

This is example 6.5 from the text book.

$$h(t) = e^{-t}$$

$$u(t) = u_0(t) - u_0(t - 1)$$

$$y(t) = \begin{cases} 0 : t \leq 0 \\ 1 - e^{-t} : 0 < t \leq 1 \\ e^{-t}(e - 1) : 1 < t \leq 2 \\ 0 : 2 \leq t \end{cases}$$

## Example 3

This is example 6.6 from the text book.

$$h(t) = 2(u_0(t) - u_0(t - 1))$$

$$u(t) = u_0(t) - u_0(t - 2)$$

$$y(t) = \begin{cases} 0 : t \leq 0 \\ 2t : 0 < t \leq 1 \\ 2 : 1 < t \leq 2 \\ -2t + 6 : 2 < t \leq 3 \\ 0 : 3 \leq t \end{cases}$$



## System Response by Convolution

## Example 4

This is example 6.7 from the textbook.

For the circuit shown below, use the convolution integral to find the capacitor voltage when the input is the unit step function  $u_0(t)$  and  $v_c(0^-) = 0$

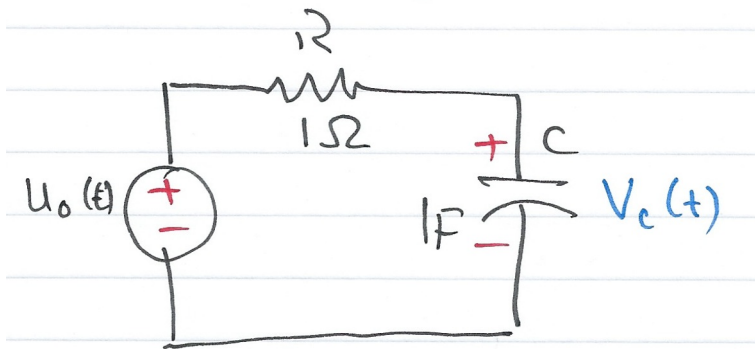


Figure 7: Example 4

## Solution

$$h(t) = \frac{1}{RC} e^{-t/RC} u_0(t)$$

which when  $C = 1 \text{ F}$  and  $R = 1 \text{ } \Omega$  reduces to

$$h(t) = e^{-t} u_0(t)$$

It is relatively straight forward to show that

$$y(t) = (1 - e^{-t}) u_0(t)$$

# System Response by Laplace

# System Response by Laplace

In the discussion of Laplace, we stated that

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

We can use this property to make the solution of convolution problems even simpler.

## Example 5

Solve Example 4 using Laplace.

## Solution

$$h(t) = e^t u_0(t) \Leftrightarrow H(s) = \frac{1}{s+1}$$

$$u(t) = u_0(t) \Leftrightarrow U(s) = \frac{1}{s}$$

$$y(t) = h(t) * u(t) \Leftrightarrow Y(s) = H(s)U(s) = \left(\frac{1}{s}\right) \times \left(\frac{1}{s+1}\right)$$

By PFE

$$Y(s) = \frac{r_1}{s} + \frac{r_2}{s+1}$$

The residues are  $r_1 = 1$ ,  $r_2 = -1$ , so

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} \Leftrightarrow y(t) = (1 - e^{-t}) u_0(t)$$



# Impulse Response and Transfer Functions

A consequence of Laplace is that the transform of the impulse response of a transfer function  $G(s)$  is given by the transfer function itself.

$$y(t) = g(t) * \delta(t) \Leftrightarrow Y(s) = G(s).1 = G(s)$$

Thus the Laplace transform of any system subject to an input  $u(t)$  is simply

$$Y(s) = G(s)U(s)$$

and

$$y(t) = \mathcal{L}^{-1} \{G(s)U(s)\}$$

Using partial fraction expansion (See lecture on the Inverse-Laplace transform) and transform tables, solution of a convolution problem by Laplace is usually simpler than using the convolution integral directly.

And if the system is particularly complex we can always fall back on the State-Space solution:

$$y(t) = \mathbf{C}e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}\tau} \mathbf{B}u(\tau) d\tau$$

# Homework

You should be able to do any of the questions from Section 6.7 of the textbook.

# Lab Work

In the lab we will get you to play with `convolutiondemo`. We will also demonstrate that the solution of the examples in this presentation can readily be solved using Laplace.