Fourier Series (Part 1)

Scope and Background Reading

This session launches our Introduction to Fourier Transforms with a look at Wave Analysis and Trigonometric Fourier Series.

As we shall see, any periodic waveform can be approximated by a DC component (which may be 0) and the sum of a fundamental and harmomic sinusoidal waveforms. This has important applications in many applications of electronics but is particularly important for signal processing and communications.

The material in this presentation and notes is based on Chapter 7 of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition..

Revision?

I believe that this subject may have been covered in EG-150 Signals and Systems.

Is that true?

Agenda

- Motivating examples
- Wave analysis and the Trig. Fourier Series
- Symmetry in Trigonometric Fourier Series
- Computing coefficients of Trig. Fourier Series in Matlab
- Gibbs Phenomenon

Motivating Examples

Motivating Examples

This Fourier Series demo, developed by Members of the Center for Signal and Image Processing (CSIP) at the School of Electrical and Computer Engineering at the Georgia Institute of Technology, shows how periodic signals can be synthesised by a sum of sinusoidal signals.

It is here used as a motivational example in our introduction to Fourier Series. (See also Fourier Series from Wolfram MathWorld referenced in the **Quick Reference** on Blackboard.)

To install this example, download the zip file and unpack it somewhere on your MATLAB path.

Wave analysis and the Trig. Fourier Series

Wave Analysis

- Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) discovered that any *periodic signal* could be represented as a series of *harmonically related* sinusoids.
- A *harmonic* is a frequency whose value is an integer multiple of some *fundamental frequency*
- For example, the frequencies 2 MHz, 3 Mhz, 4 MHz are the second, third and fourth harmonics of a sinusoid with fundamental frequency 1 Mhz.

The Trigonometric Fourier Series

Any periodic waveform $f(t)$ can be represented as

$$f(t) = \frac{1}{2}a_0 + a_l \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \cdots + a_n \cos n\omega t + \cdots$$
$$+ b_l \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \cdots + b_n \sin n\omega t + \cdots$$

or equivalently (if more confusingly)

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

Notation

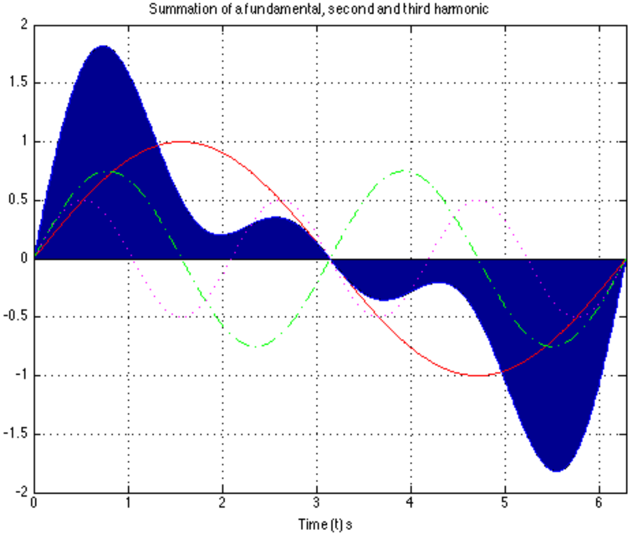
- The first term $a_o/2$ is a constant and represents the DC (average) component of the signal $f(t)$
- The terms with coefficients a_1 and b_1 together represent the fundamental frequency component of $f(t)$ at frequency ω.
- The terms with coefficients a_2 and b_2 together represent the second harmonic frequency component of $f(t)$ at frequency 2ω.

And so on.

Since any periodic function $f(t)$ can be expressed as a Fourier series, it follows that the sum of the DC, fundamental, second harmonic and so on must produce the waveform $f(t)$.

Sums of sinusoids

In general, the sum of two or more sinusoids does not produce a sinusoid as shown below.



Evaluation of the Fourier series coefficients

The coefficients are obtained from the following expressions (valid for any periodic waveform with fundamental frequency ω_0 so long as we integrate over one period $0 \to T_0$ where $T_0 = 2\pi/\omega_0$):

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt$$

Symmetry in Trigonometric Fourier Series

Symmetry in Trigonometric Fourier Series

There are simplifications we can make if the original periodic properties has certain properties:

- If $f(t)$ is odd, $a_0 = 0$ and there will be no cosine terms so $a_n = 0 \; \forall n > 0$
- If $f(t)$ is even, there will be no sine terms and $b_n = 0 \; \forall n > 0$. The DC may or may not be zero.
- If $f(t)$ has *half-wave symmetry* only the odd harmonics will be present. That is a_n and b_n is zero for all even values of n (0, 2, 4, ...)

Odd, Even and Half-wave Symmetry

Recall

- An *odd* function is one for which $f(t) = -f(-t)$. The function $\sin t$ is an *odd* function.
- An *even* function is one for which $f(t) = f(-t)$. The function $\cos t$ is an *even* function.

Half-wave symmetry

- A periodic function with period T is a function for which $f(t) = f(t + T)$
- A periodic function with period T, has *half-wave symmetry* if $f(t) = -f(t + T/2)$

Symmetry in Common Waveforms

To reproduce the following waveforms (without annotation) publish the script waves.m.

Squarewave

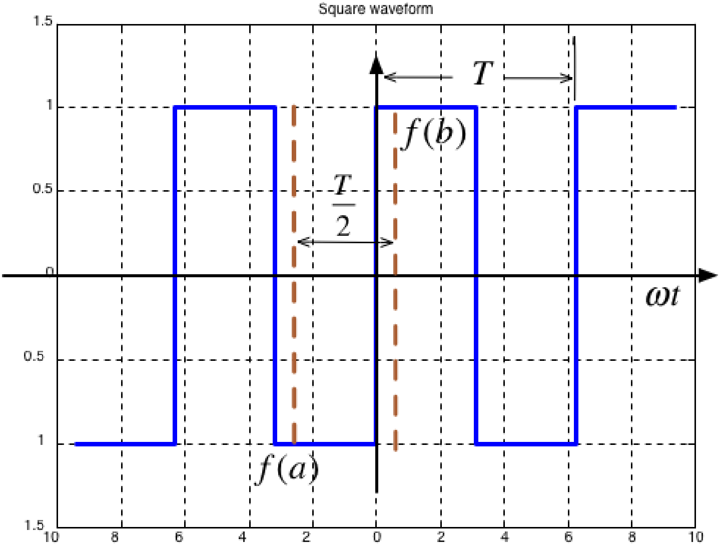


Figure 2:Squarewave

- Average value over period T is ...?
- It is an **odd**/**even** function?
- It **has/doesn't have** half-wave symmetry $f(t) = -f(t + T/2)$?

- Average value over period T is zero.
- It is an *odd* function $f(t) = -f(-t)$
- It *has* half-wave symmetry $f(t) = -f(t + T/2)$

Shifted Squarewave

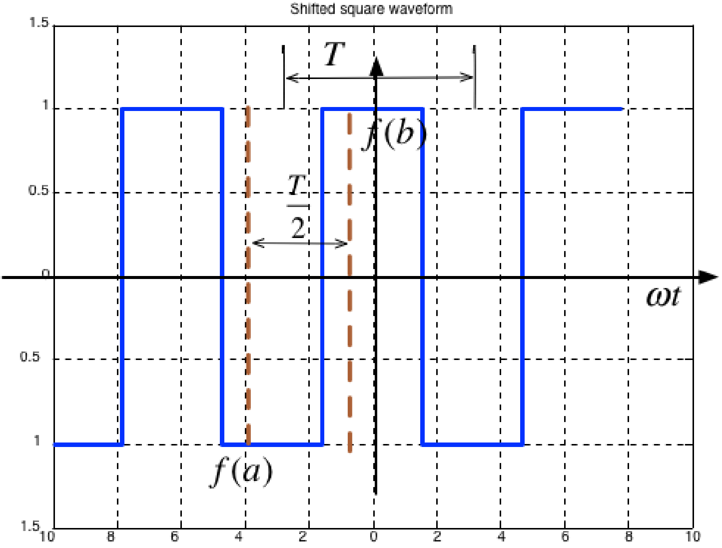


Figure 3:Shifted squarewave

- Average value over period T is ...?
- It is an **odd**/**even** function?
- It **has/doesn't have** half-wave symmetry $f(t) = -f(t + T/2)$?

- Average value over period T is zero.
- It is an *even* function $f(t) = f(-t)$
- It has *half-wave symmetry* $f(t) = -f(t + T/2)$

Sawtooth

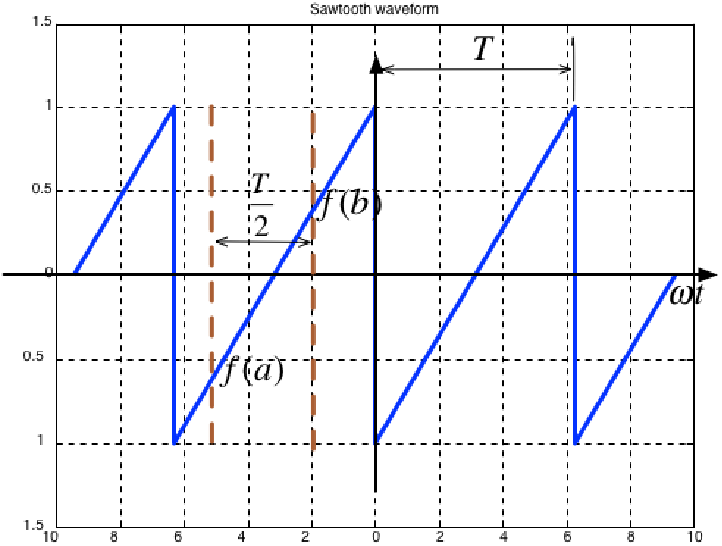


Figure 4:Sawtooth

- Average value over period T is ...?
- It is an **odd**/**even** function?
- It **has/doesn't have** half-wave symmetry $f(t) = -f(t + T/2)$?

- Average value over period T is zero.
- It is an *odd* function $f(t) = -f(-t)$
- It *doesn't have* half-wave symmetry $f(t) \neq -f(t + T/2)$

Triangle

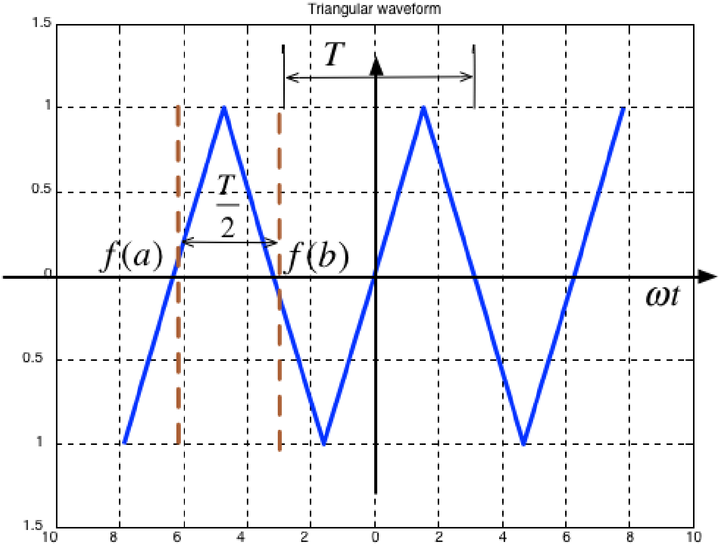


Figure 5:Triangle

- Average value over period T is ...?
- It is an **odd**/**even** function?
- It **has/hdoesn't have** half-wave symmetry $f(t) = -f(t + T/2)$?

- Average value over period T is zero.
- It is an *odd* function $f(t) = -f(-t)$
- It *has* half-wave symmetry $f(t) = -f(t + T/2)$

Symmetry in fundamental, second and third Harmonics

In the following, $T/2$ is taken to be the half-period of the fundamental sine-wave.

Fundamental

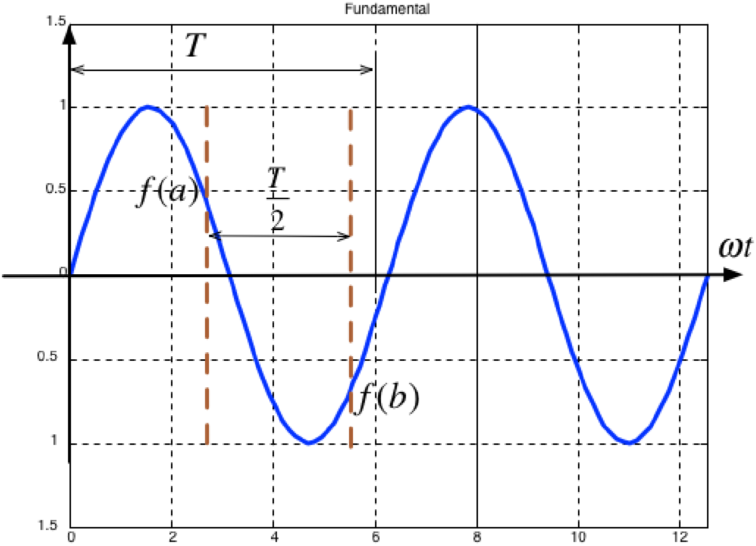


Figure 6:Fundamental

- Average value over period T is ...?
- It is an **odd**/**even** function?
- It **has/doesn't have** half-wave symmetry $f(t) = -f(t + T/2)$?

- Average value over period T is zero.
- It is an *odd* function $f(t) = -f(-t)$
- It *has* half-wave symmetry $f(t) = -f(t + T/2)$

Second Harmonic

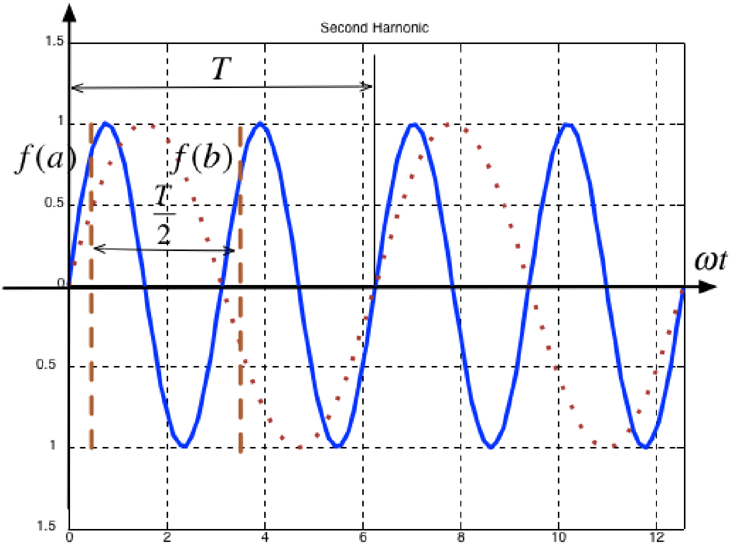


Figure 7:Second harmonic

- Average value over period T is ...?
- It is an **odd**/**even** function?
- It **has/doesn't have** half-wave symmetry $f(t) = -f(t + T/2)$?

- Average value over period T is zero.
- It is an *odd* function $f(t) = -f(-t)$
- It *doesn't have* half-wave symmetry $f(t) \neq -f(t + T/2)$

Third Harmonic

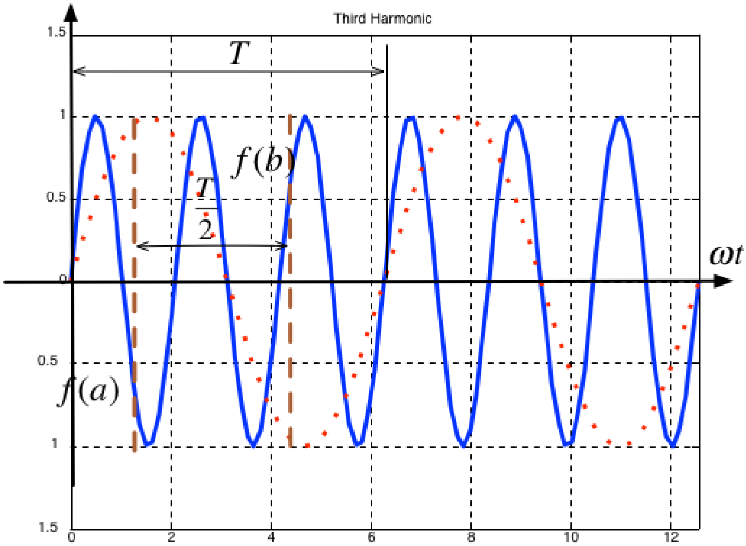


Figure 8:Third harmonic

- Average value over period T is ...?
- It is an **odd**/**even** function?
- It **has/doesn't have** half-wave symmetry $f(t) = -f(t + T/2)$?

- Average value over period T is zero.
- It is an *odd* function $f(t) = -f(-t)$
- It *doesn't have* half-wave symmetry $f(t) = -f(t + T/2)$

Some simplifications that result from symmetry

- The limits of the integrals used to compute the coefficients a_n and b_n of the Fourier series are given as $0 \to 2\pi$ which is one period T
- We could also choose to integrate from $-\pi \to \pi$
- If the function is *odd*, or *even* or has *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \to \pi$ and multiplying by 2.
- If we have *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \to \pi/2$ and multiplying by 4.

(For more details see page 7-10 of the textbook)

Computing coefficients of Trig. Fourier Series in Matlab

Computing coefficients of Trig. Fourier Series in Matlab

As an example let's take a square wave with amplitude $\pm A$ and period T.

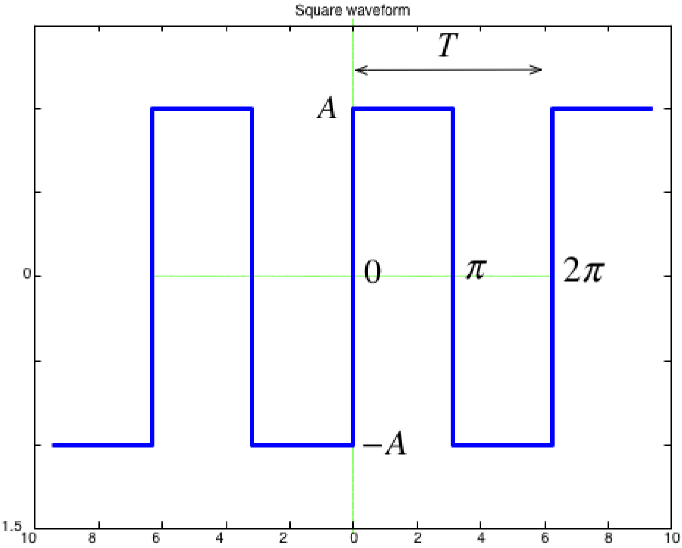


Figure 9:Matlab example

Solution

Solution: See square_ftrig.m. Script confirms that:

- $a_0 = 0$
- $a_i = 0$: function is odd
- $b_i = 0$: for i even - half-wave symmetry

```
ft =

(4*A*sin(t))/pi + (4*A*sin(3*t))/(3*pi) + ..
(4*A*sin(5*t))/(5*pi) + (4*A*sin(7*t))/(7*pi) + ..
(4*A*sin(9*t))/(9*pi) + (4*A*sin(11*t))/(11*pi)
```

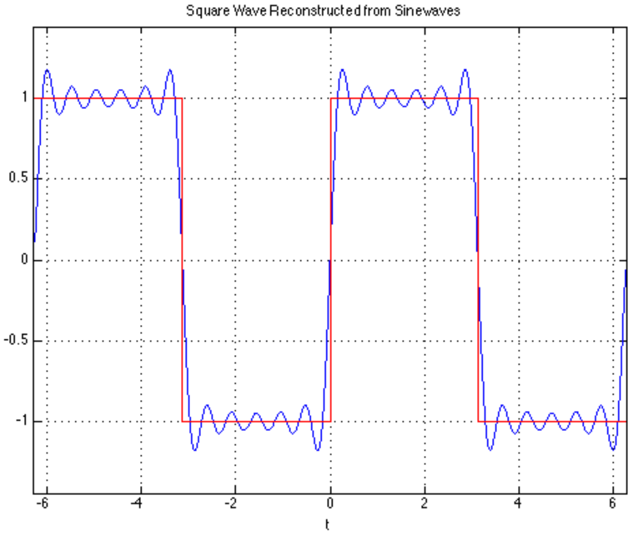
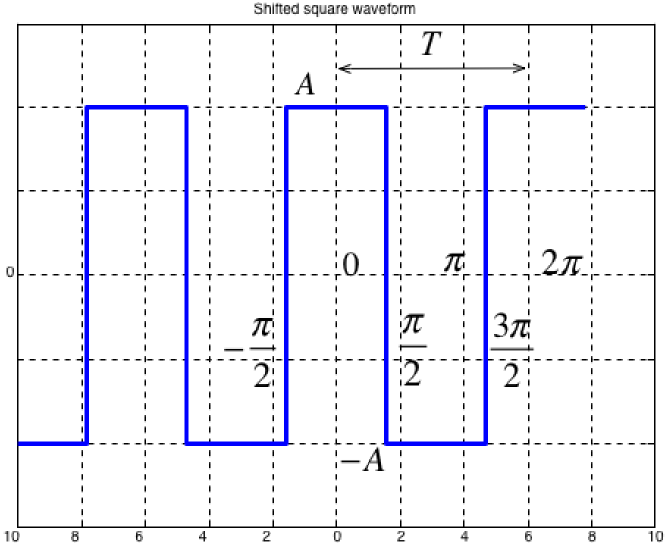


Figure 10:Trig. fourier series for square wave

Using symmetry - computing the Fourier series coefficients of the shifted square wave



- As before $a_0 = 0$
- We observe that this function is even, so all b_k coefficients will be zero
- The waveform has half-wave symmetry, so only odd indexed coefficients will be present.
- Further more, because it has half-wave symmetry we can just integrate from $0 \to \pi/2$ and multiply the result by 4.

See shifted_sq_ftrig.m.

```
ft =

(4*A*cos(t))/pi - (4*A*cos(3*t))/(3*pi) + ..
(4*A*cos(5*t))/(5*pi) - (4*A*cos(7*t))/(7*pi) + ..
(4*A*cos(9*t))/(9*pi) - (4*A*cos(11*t))/(11*pi)
```

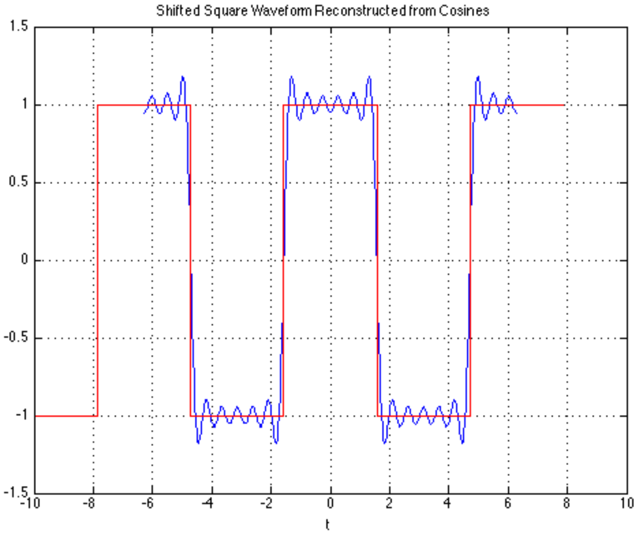


Figure 12:Trig. Fourier series for shifted squarewave

More Examples

We will compute other examples from the text-book in the Lab.

Gibbs Phenomenon

Gibbs Phenomenon

In an earlier slide we found that the trigonometric for of the Fourier series of the square waveform is

$$f(t) = \frac{4A}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right) = \frac{4A}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\sin n\omega t$$

The next slide shows the approximation for the first 11 harmonics:

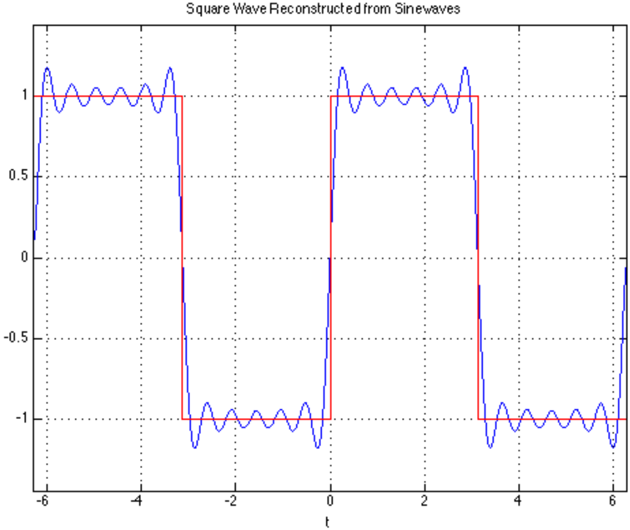


Figure 13:Approximation for the first 11 harmonics

As we add more harmonics, the sum looks more and more like a square wave. However the crests do not become flattened; this is known as *Gibbs Phenomenon* and it occurs because of the discontinuity of the perfect sqare waveform as it changes from $+A$ to $-A$ and *vice versa*.

End of Part 1

This concludes our introduction to Fourier series.

Next Time

- The exponential Fourier series
- Line spectra
- Power in periodic signals

Lab Work

In the lab in week 7 we will compute some Trigonometric Fourier series