Fourier Series (Part 2)

Agenda

Last time

- Trigonometric Fourier Series

This Time

- Exponents and Euler's Equation
- The Exponential Fourier series
- Symmetry in Exponential Fourier Series
- Examples

Next Time

- Line spectra
- Power in periodic signals
- Steady-State Response of an LTI System to a Periodic Signals

Scope and Background Reading

This session continues our introduction to Fourier Series.

Trigonometric Fourier series uses integration of a periodic signal multiplied by sines and cosines at the fundamental and harmonic frequencies. If performed by hand, this can a painstaking process. Even with the simplifications made possible by exploiting waveform symmetries, there is still a need to integrate cosine and sine terms, be aware of and able to exploit the tigonometrc identities, and the properties of *orthogonal functions* before we can arrive at the simplified solutions. This is why I concentrated on the properties and left the computation to a computer.

However, by exploiting the exponential function e^{at}, we can derive a method for calculating the coefficients of the harmonics that is much easier to calculate by hand and convert into an algorithm that can be executed by computer.

The result is called the *Exponential Fourier Series* and we will develop it in this session.

The material in this presentation and notes is based on Chapter 7 (Starting at Section 7.8) of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. from the Required Reading List. Some clarificattion was needed and I used Chapter 4 of Benoit Boulet, Fundamentals of Signals and Systems from the Recommended Reading List for this.

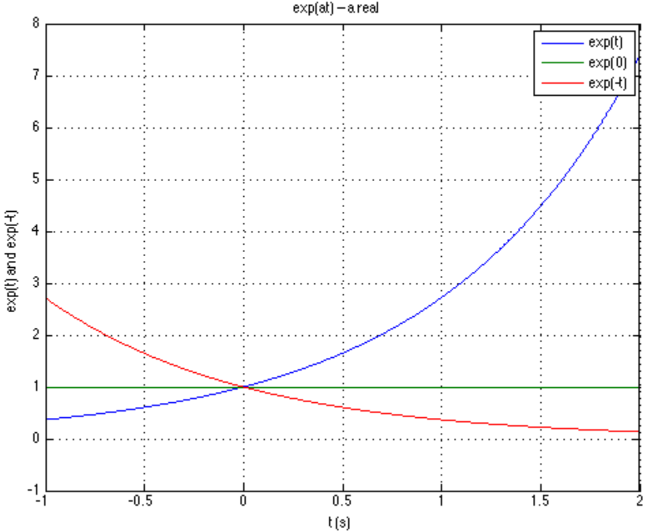
Exponents and Euler's Equation

The Exponential Function e^{at}

- You should already be familiar with e^{at} because it appears in the solution of differential equations; the transition matrix $\phi(t)$ used in the solution of state-space models; and the definition of the impulse response of a system.
- It's also a function that appears in the definition of the Laplace and Inverse Laplace Transform
- It's pops up again and again in tables and properties of the Laplace Transform.

Case when a is real.

When a is real the function e^{at} will take one of the two forms illustrated below:



Case when a is imaginary

This is the case that helps us simplify the computation of *sinusoidal Fourier series*.

It was Leonhard Euler who discovered the formula

$$exp(j\omega t) = cos(\omega t) + jsin(\omega t)$$

visualized on the next slide.

Case when a is imaginary

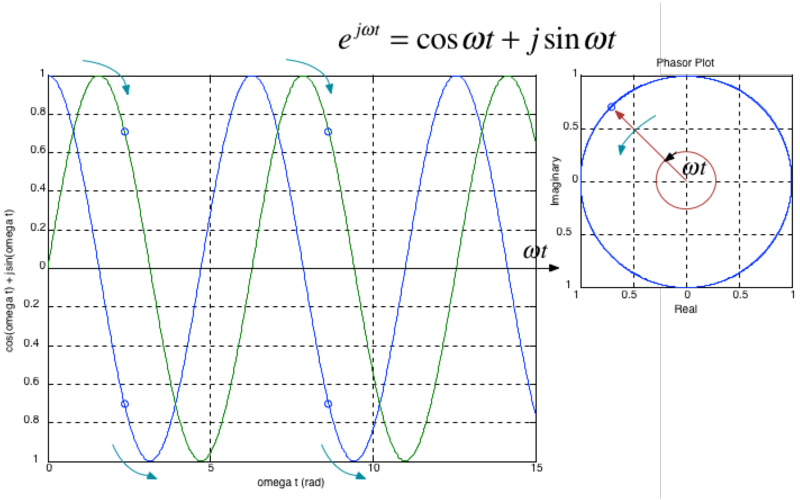


Figure 2: Case when a is imaginary

Some important values of ωt

These are useful when simplifying expressions that result from integrating functions that involve the imaginary exponential

Give the following:

- $e^{j\omega t}$ when $\omega t = 0$
- $e^{j\omega t}$ when $\omega t = \pi/2$
- $e^{j\omega t}$ when $\omega t = \pi$
- $e^{j\omega t}$ when $\omega t = 3\pi/2$
- $e^{j\omega t}$ when $\omega t = 2\pi$

Solution

- When $\omega t = 0$: $e^{j\omega t} = e^{j0} = 1$
- When $\omega t = \pi/2$: $e^{j\omega t} = e^{j\pi/2} = j$
- When $\omega t = \pi$: $e^{j\omega t} = e^{j\pi} = -1$
- When $\omega t = 3\pi/2$: $e^{j\omega t} = e^{j3\pi/2} = -j$
- When $\omega t = 2\pi$: $e^{j\omega t} = e^{j2\pi}e^{j0} = 1$

Case where a is complex

We shall not say much about this case except to note that the Laplace transform equation includes such a number. The variable s in the Laplace Transform

$$\int_0^\infty f(t)e^{-st}dt$$

is a *complex exponential*.

The consequences of a complex s have particular significance in the development of system stabilty theories and in control systems analysis and design. Look out for them in EG-243.

Two Other Important Properties

By use of trig. identities, it is relatively straight forward to show that:

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

and

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{j2}$$

We can use this result to convert the *Trigonometric Fourier Series* into an *Exponential Fourier Series* which has only one integral term to solve per harmonic.

The Exponential Fourier Series

The Exponential Fourier Series

As we saw last time any periodic waveform $f(t)$ can be represented as

$$f(t) = \frac{1}{2}a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots$$

If we replace the cos and sin terms with their imaginary exponential equivalents:

$$f(t) = \frac{1}{2}a_0 + a_1 \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) + a_2 \left(\frac{e^{j2\omega t} + e^{-j2\omega t}}{2} \right) + \cdots$$
$$+ b_1 \left(\frac{e^{j\omega t} - e^{-j\omega t}}{j2} \right) + b_2 \left(\frac{e^{j2\omega t} - e^{-j2\omega t}}{j2} \right) + \cdots$$

Gouping terms with same exponents

$$f(t) = \cdots + \left(\frac{a_2}{2} - \frac{b_2}{j2}\right) e^{-j2\omega t} + \left(\frac{a_1}{2} - \frac{b_1}{j2}\right) e^{-j\omega t} + \frac{1}{2}a_0 + \left(\frac{a_1}{2} + \frac{b_1}{j2}\right) e^{j\omega t} + \left(\frac{a_2}{2} + \frac{b_2}{j2}\right) e^{j2\omega t} + \cdots$$

New coefficents

The terms in parentheses are usually denoted as

$$C_{-k} = \frac{1}{2}\left(a_k - \frac{b_k}{j}\right) = \frac{1}{2}\left(a_k + jb_k\right)$$

$$C_k = \frac{1}{2}\left(a_k + \frac{b_k}{j}\right) = \frac{1}{2}\left(a_k - jb_k\right)$$

$$C_0 = \frac{1}{2}a_0$$

The Exponential Fourier Series

$$f(t) = \cdots + C_{-2}e^{-j2\omega t} + C_{-1}e^{-j\omega t} + C_0 + C_1 e^{j\omega t} + C_2 e^{j2\omega t} + \cdots$$

or more compactly

$$f(t) = \sum_{k=-n}^{n} C_k e^{-jk\omega t}$$

Important

The C_k coefficents, except for C_0 are *complex* and appear in conjugate pairs so

$$C_{-k} = C_k^*$$

Evaluation of the complex coefficients

The coefficients are obtained from the following expressions*:

$$C_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jk(\omega t)}\,d(\omega t)$$

or

$$C_k = \frac{1}{T}\int_0^{T} f(t)e^{-jk\omega t}\,dt$$

These are much easier to derive and compute than the equivalent Trigonometric Fourier Series coefficients.

Trigonometric Fourier Series from Exponential Fourier Series

By substituting C_{-k} and C_k back into the original expansion

$$C_k + C_{-k} = \frac{1}{2}\left(a_k - jb_k + a_k + jb_k\right)$$

so

$$a_k = C_k + C_{-k}$$

Similarly

$$C_k - C_{-k} = \frac{1}{2}\left(a_k - jb_k - a_k - jb_k\right)$$

so

$$b_k = j\left(C_k - C_{-k}\right)$$

Thus we can easily go back to the Trig. Fourier series if we want to.

Symmetry in Exponential Fourier Series

Symmetry in Exponential Fourier Series

Since the coefficients of the Exponential Fourier Series are complex numbers, we can use symmetry to determine the form of the coefficients and thereby simplify the computation series for wave forms that have symmetry.

Even Functions

For even functions, all coefficients C_k are real.

Odd Functions

For odd functions, all coefficients C_k are imaginary.

By a similar argument, all odd functions have no cosine terms so the a_k coefficients are 0. Therefore both C_{-k} and C_k are imaginary.

Half-wave symmetry

If there is *half-wave symmetry*, $C_k = 0$ for n even.

No symmetry

If there is no symmetry the Exponential Fourier Series of $f(t)$ is complex.

Relation of C_{-k} to C_k

C_{-k} is **always** the complex conjugate of C_k

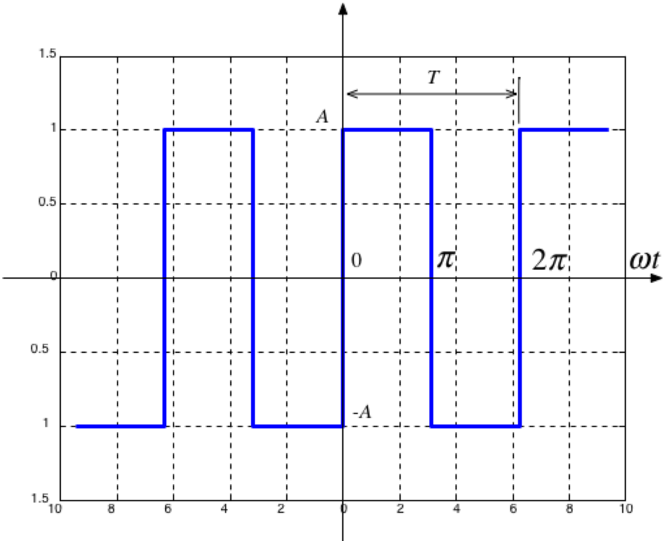
So

$$C_{-k} = C_k^*$$

Examples

Example 1

Compute the Exponential Fourier Series for the square wave shown below assuming that $\omega = 1$



Some questions for you

- Square wave is an [**odd/even/neither**] function?
- DC component is [**zero/non-zero**]?
- Square wave [**has/does not have**] half-wave symmetry?

Hence

- $C_0 = $ [**?**]
- Coefficients C_k are [**real/imaginary/complex**]?
- Subscripts k are [**odd only/even only/both odd and even**]?
- What is the integral that needs to be solved for C_k?

Some answers for you

- Square wave is an **odd** function!
- DC component is **zero**!
- Square wave **has** half-wave symmetry!

Hence

- $C_0 = 0$
- Coefficients C_k are **imaginary**!
- Subscripts k are **odd only**!
- What is the integral that needs to be solved for C_k?

$$C_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-jk(\omega t)}\, d(\omega t)$$

$$C_k = \frac{1}{2\pi} \left[\int_0^{\pi} A e^{-jk(\omega t)}\, d(\omega t) + \int_{\pi}^{2\pi} (-A) e^{-jk(\omega t)}\, d(\omega t) \right]$$

Solution

See notes for detailed evaluation of the integral that leads to this result

$$\begin{aligned} \underset{k=\text{odd}}{C_k} &= \frac{A}{2j\pi k}\left(e^{-jk\pi}-1\right)^2 \\ &= \frac{A}{2j\pi k}(-1-1)^2 \\ &= \frac{A}{2j\pi k}(-2)^2 \\ &= \frac{2A}{j\pi k} \end{aligned}$$

exponential Fourier series for the square wave with odd symmetry

From the definition of the exponential Fourier series

$$f(t) = \cdots + C_{-2}e^{-j2\omega t} + C_{-1}e^{-j\omega t} + C_0 + C_1 e^{j\omega t} + C_2 e^{j2\omega t} + \cdots$$

the *exponential Fourier series for the square wave with odd symmetry* is

$$f(t) = \frac{2A}{j\pi}\left(\cdots - \frac{1}{3}e^{-j3\omega t} - e^{-j\omega t} + e^{j\omega t} + \frac{1}{3}e^{j3\omega t} + \cdots\right)$$

$$f(t) = \frac{2A}{j\pi}\sum_{k=\text{odd}} \frac{1}{k}e^{jk\omega t}$$

Trig. Fourier Series from Exponential Fourier Series

Since

$$f(t) = \frac{2A}{j\pi}\left(\cdots - \frac{1}{3}e^{-j3\omega t} - e^{-j\omega t} + e^{j\omega t} + \frac{1}{3}e^{j3\omega t} + \cdots\right)$$

gathering terms at each harmonic frequency gives

$$f(t) = \frac{4A}{\pi}\left(\cdots + \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) + \frac{1}{3}\left(\frac{e^{j3\omega t} - e^{-j3\omega t}}{2j}\right) + \cdots\right)$$

$$f(t) = \frac{4A}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \cdots\right) = \frac{4A}{\pi}\sum_{k=\text{odd}} \frac{1}{k}\sin k\omega t$$

Example 2

Verify the result of Example 1 using Matlab.

Solution

Solution: See efs_sqw.m.

Script confirms that:

- $C_0 = 0$
- C_k is imaginary: function is odd
- $C_k = 0$: for i even – half-wave symmetry

```
X =

[ (A*2*i)/(5*pi), 0, (A*2*i)/(3*pi), 0, (A*2*i)/pi, 0, -(A*2*
-(A*2*i)/(3*pi), 0, -(A*2*i)/(5*pi)]


w =

    -5    -4    -3    -2    -1     0     1     2     3     4     5
```

Plot of Result

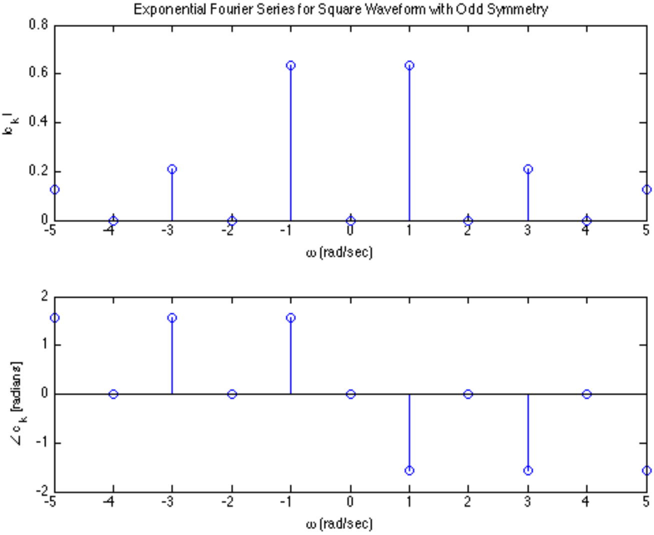


Figure 4: Result of example 2

End of Part 2

Summary

- Exponents and Euler's Equation
- The exponential Fourier series
- Symmetry in Exponential Fourier Series
- Examples

Next Time

- Line spectra
- Power in periodic signals
- Steady-State Response of an LTI System to a Periodic Signals

Home work

Compute the exponential Fourier series for the wave forms shown in the last lecture.

Lab Work

In the lab, in week 7, we will explore Fourier Series and their applications