# The Inverse Z-Transform

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You can view the notes for this presentation in HTML and PDF.

The source code of this presentation is available in Markdown format from GitHub: i-z-transform.md.

The GitHub repository EG-247 Resources also contains the source code for all the Matlab/Simulink examples and the Laboratory Exercises.

## Scope and Background Reading

This session we will talk about the Inverse Z-Transform and illustrate its use through an examples class.

The material in this presentation and notes is based on Chapter 9 (Starting at Section 9.6) of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. from the **Required Reading List**.

## Agenda

- Inverse Z-Transform
- Examples using PFE
- Examples using Long Division
- Analysis in Matlab

## The Inverse Z-Transform

The inverse Z-Transform enables us to extract a sequence $f[n]$ from $F(z)$. It can be found by any of the following methods:

- Partial fraction expansion
- The inversion integral
- Long division of polynomials

# Partial fraction expansion

## Partial fraction expansion

We expand $F(z)$ into a summation of terms whose inverse is known. These terms have the form:

$$k, \ \frac{r_1 z}{z-p_1}, \ \frac{r_1 z}{(z-p_1)^2}, \ \frac{r_3 z}{z-p_2}, \ldots$$

where $k$ is a constant, and $r_i$ and $p_i$ represent the residues and poles respectively, and can be real or complex [1]

**Notes**

$$\frac{r_i z}{z-p_i} + \frac{r_i^* z}{z-p_i^*}$$

## Step 1: Make Fractions Proper

- Before we expand $F(z)$ into partial fraction expansions, we must first express it as a *proper* rational function.
- This is done by expanding $F(z)/z$ instead of $F(z)$
- That is we expand

$$\frac{F(z)}{z} = \frac{k}{z} + \frac{r_1}{z-p_1} + \frac{r_2}{z-p_2} + \cdots$$

## Step 2: Find residues

- Find residues from

$$r_k = \lim_{z \to p_k} (z-p_k) \frac{F(z)}{z} = (z-p_k) \left. \frac{F(z)}{z} \right|_{z=p_k}$$

## Step 3: Map back to transform tables form

- Rewrite $F(z)/z$:

$$z\frac{F(z)}{z} = F(z) = k + \frac{r_1 z}{s-p_1} + \frac{r_2 z}{s-p_2} + \cdots$$

[1] If complex, the poles and residues will be in complex conjugate pairs

## Example 1

Karris Example 9.4: use the partial fraction expansion to compute the inverse z-transform of

$$F(z) = \frac{1}{(1-0.5z^{-1})(1-0.75z^{-1})(1-z^{-1})}$$

## Answer to Example 1

$$f[n] = 2\left(\frac{1}{2}\right)^n - 9\left(\frac{3}{4}\right)^n + 8$$

## Matlab solution

See example1.m

Uses Matlab functions:

- `collect` – expands a polynomial
- `sym2poly` – converts a polynomial into a numeric polymial (vector of coefficients in descending order of exponents)
- `residue` – calculates poles and zeros of a polynomial
- `ztrans` – symbolic z-transform
- `iztrans` – symbolic inverse ze-transform
- `stem` – plots sequence as a "lollipop" diagram

## Stem ("Lollipop") Plot

## Example 2

Karris example 9.5: use the partial fraction expansion method to to compute the inverse z-transform of

$$F(z) = \frac{12z}{(z+1)(z-1)^2}$$

## Answer to Example 2

$$f[n] = 3(-1)^n + 6n - 3$$

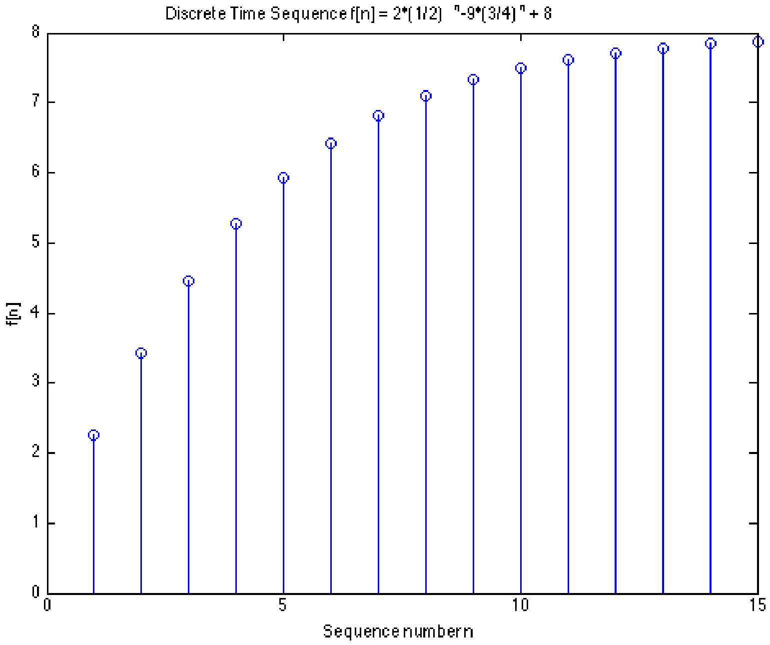


Figure 1: Solution to Example 1

## Matlab solution

See example2.m

Uses additional Matlab functions:

- `dimpulse` – computes and plots a sequence $f[n]$ for any range of values of $n$

## Lollipop Plot

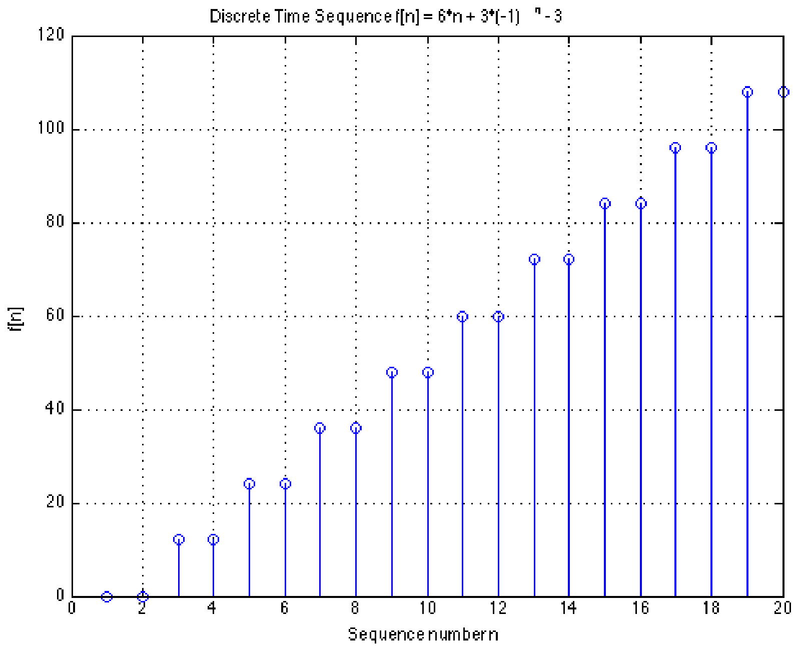


Figure 2: Solution to Example 2

## Staircase Plot

Simulates output of Zero-Order-Hold (ZOH) or Digital Analogue Converter (DAC)

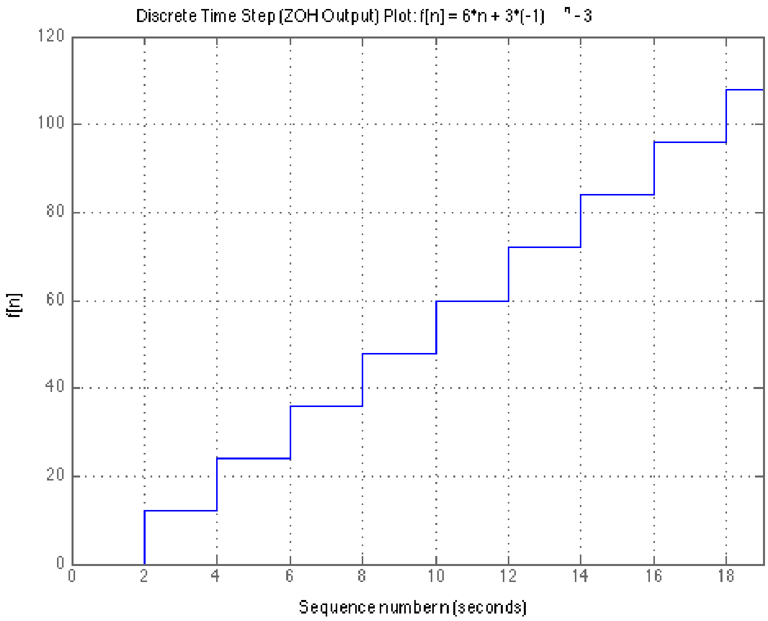


Figure 3: Solution to Example 2 as DAC Output

## Example 3

Karris example 9.6: use the partial fraction expansion method to to compute the inverse z-transform of

$$F(z) = \frac{z+1}{(z-1)(z^2+2z+2)}$$

## Answer to Example 3

$$f[n] = -0.5\delta[n] + 0.4 + \frac{(\sqrt{2})^n}{10}\cos\frac{3n\pi}{4} - \frac{3(\sqrt{2})^n}{10}\sin\frac{3n\pi}{4}$$

## Matlab solution

See example3.m

## Lollipop Plot

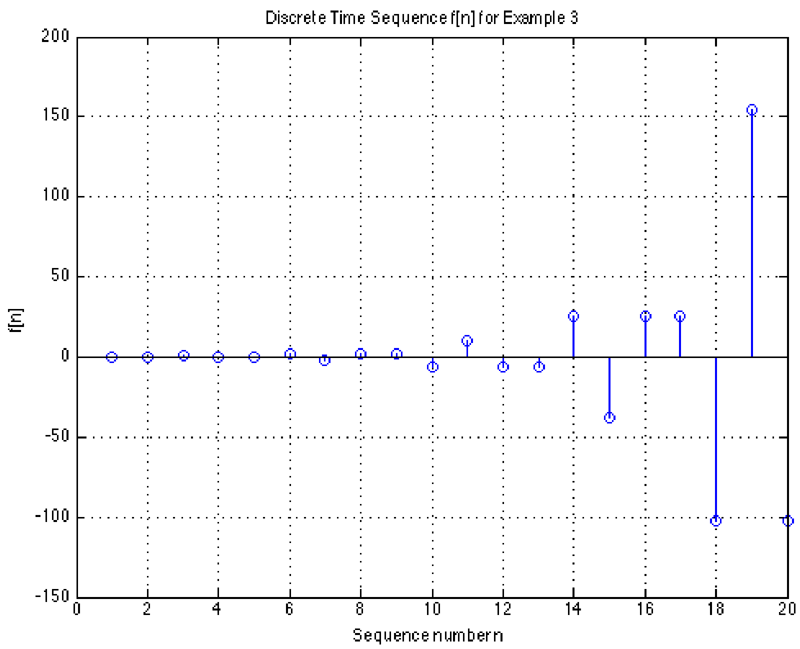


Figure 4: Solution to Example 3

**Staircase Plot**

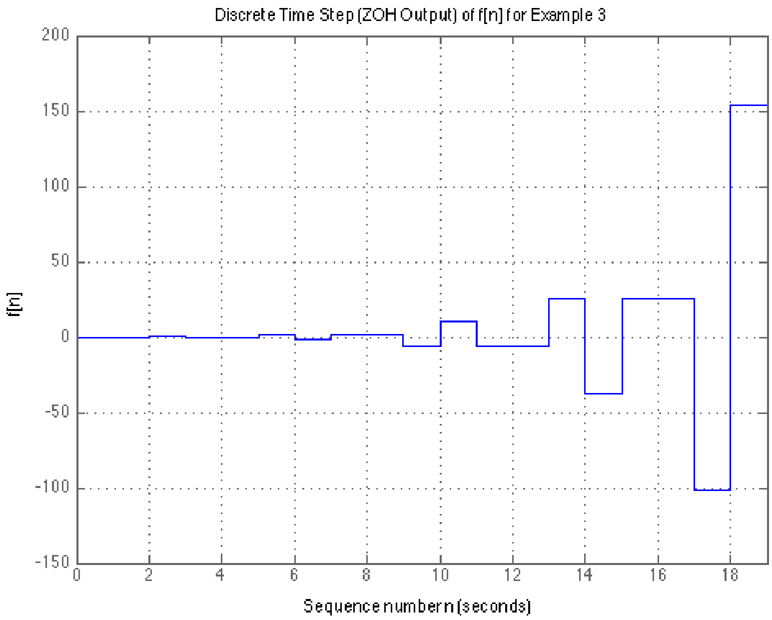


Figure 5: Solution to Example 3 as DAC Output

# Inverse Z-Transform by the Inversion Integral

## Inverse Z-Transform by the Inversion Integral

The inversion integral states that:

$$f[n] = \frac{1}{j2\pi} \oint_C F(z) z^{n-1}\, dz$$

where $C$ is a closed curve that encloses all poles of the integrant.

This can (*apparently*) be solved by Cauchy's residue theorem!!

Fortunately (:-), this is beyond the scope of this module!

See Karris Section 9.6.2 (pp 9-29—9-33) if you want to find out more.

# Inverse Z-Transform by the Long Division

## Inverse Z-Transform by the Long Division

To apply this method, $F(z)$ must be a rational polynomial functuon, and the numerator and denominator must be polynomials arranged in descending powers of $z$.

## Example 4

Karris example 9.9: use the long division method to determine $f[n]$ for $n = 0,\ 1,\ \text{and}\ 2$, given that

$$F(z) = \frac{1 + z^{-1} + 2z^{-2} + 3z^{-3}}{(1 - 0.25z^{-1})(1 - 0.5z^{-1})(1 - 0.75z^{-1})}$$

## Answer 4

$f[0] = 1,\ f[1] = 5/2,\ f[2] = 81/16.$

## Matlab

See example4.m

```
sym_den =

z^3 - (3*z^2)/2 + (11*z)/16 - 3/32


fn =

    1.0000
    2.5000
    5.0625
    ....
```

## Combined Staircase/Lollipop Plot

## Methods of Evaluation of the Inverse Z-Transform

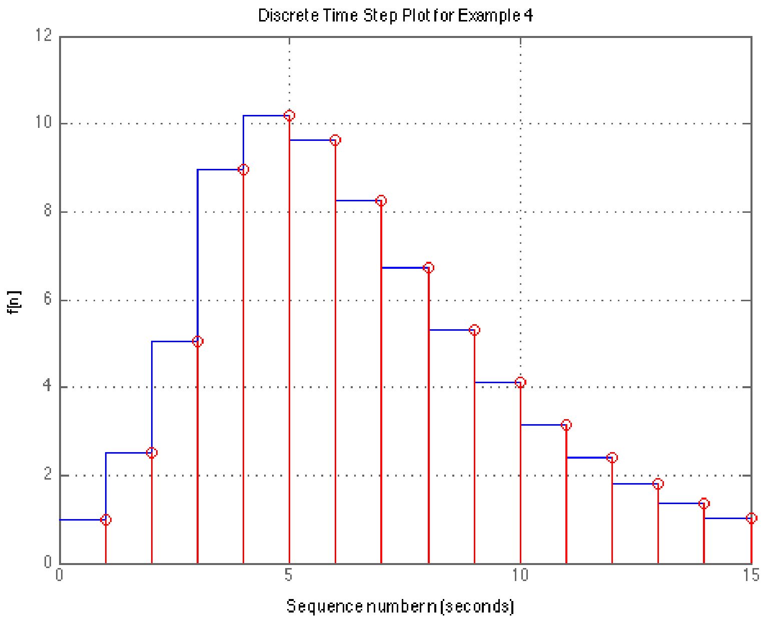


Figure 6: Combined Staircase/Lollipop Plot

| Method | Advantages | Disadvantages |
|---|---|---|
| Partial Fraction Expansion | • Most familiar.<br>• Can use Matlab `residue` function. | • Requires that $F(z)$ is a proper rational function. |
| Inversion Integral | • Can be used whether $F(z)$ is rational or not | • Requires familiarity with the *Residues theorem* \| \| \| \| of complex variable analysis. |
| Long Division | • Practical when only a small sequence of numbers is desired.<br>• Useful when z-transform has no closed-form solution.<br>• Can use Matlab `dimpulse` function to compute a large sequence of numbers. | • Requires that $F(z)$ is a proper rational<br>• Division may be endless. |

## Summary

- Inverse Z-Transform
- Examples using PFE
- Examples using Long Division
- Analysis in Matlab

*Next time*

- DT transfer functions, continuous system equivalents, and modelling DT systems in Matlab and Simulink.

## Homework

Attempt the end of the chapter exercises 4-7 (Section 9.10) from Karris. Don't look at the answers until you have attempted the problems.