

# The Inverse Z-Transform

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Digital Technium 123

Office Hours: 12:00-13:00 Mondays

You can view the notes for this presentation in [HTML](#) and [PDF](#).

The source code of this presentation is available in Markdown format from GitHub: [i-z-transform.md](#).

The GitHub repository [EG-247 Resources](#) also contains the source code for all the Matlab/Simulink examples and the Laboratory Exercises.

## Scope and Background Reading

This session we will talk about the Inverse Z-Transform and illustrate its use through an examples class.

The material in this presentation and notes is based on Chapter 9 (Starting at Section 9.6) of [Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition](#). from the **Required Reading List**.

## Agenda

- Inverse Z-Transform
- Examples using PFE
- Examples using Long Division
- Analysis in Matlab

## The Inverse Z-Transform

The inverse Z-Transform enables us to extract a sequence  $f[n]$  from  $F(z)$ . It can be found by any of the following methods:

- Partial fraction expansion
- The inversion integral
- Long division of polynomials

## Partial fraction expansion

### Partial fraction expansion

We expand  $F(z)$  into a summation of terms whose inverse is known. These terms have the form:

$$k, \frac{r_1 z}{z - p_1}, \frac{r_1 z}{(z - p_1)^2}, \frac{r_3 z}{z - p_2}, \dots$$

where  $k$  is a constant, and  $r_i$  and  $p_i$  represent the residues and poles respectively, and can be real or complex <sup>1</sup>

Notes

$$\frac{r_i z}{z - p_i} + \frac{r_i^* z}{z - p_i^*}$$

### Step 1: Make Fractions Proper

- Before we expand  $F(z)$  into partial fraction expansions, we must first express it as a *proper* rational function.
- This is done by expanding  $F(z)/z$  instead of  $F(z)$
- That is we expand

$$\frac{F(z)}{z} = \frac{k}{z} + \frac{r_1}{z - p_1} + \frac{r_2}{z - p_2} + \dots$$

### Step 2: Find residues

- Find residues from

$$r_k = \lim_{z \rightarrow p_k} (z - p_k) \frac{F(z)}{z} = (z - p_k) \left. \frac{F(z)}{z} \right|_{z=p_k}$$

### Step 3: Map back to transform tables form

- Rewrite  $F(z)/z$ :

$$\frac{F(z)}{z} = F(z) = k + \frac{r_1 z}{s - p_1} + \frac{r_2 z}{s - p_2} + \dots$$

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<sup>1</sup>If complex, the poles and residues will be in complex conjugate pairs

### Example 1

Karris Example 9.4: use the partial fraction expansion to compute the inverse z-transform of

$$F(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.75z^{-1})(1 - z^{-1})}$$

### Answer to Example 1

$$f[n] = 2 \left(\frac{1}{2}\right)^n - 9 \left(\frac{3}{4}\right)^n + 8$$

### Matlab solution

See [example1.m](#)

Uses Matlab functions:

- `collect` – expands a polynomial
- `sym2poly` – converts a polynomial into a numeric polynomial (vector of coefficients in descending order of exponents)
- `residue` – calculates poles and zeros of a polynomial
- `ztrans` – symbolic z-transform
- `iztrans` – symbolic inverse ze-transform
- `stem` – plots sequence as a “lollipop” diagram

### Stem (“Lollipop”) Plot

### Example 2

Karris example 9.5: use the partial fraction expansion method to to compute the inverse z-transform of

$$F(z) = \frac{12z}{(z+1)(z-1)^2}$$

### Answer to Example 2

$$f[n] = 3(-1)^n + 6n - 3$$

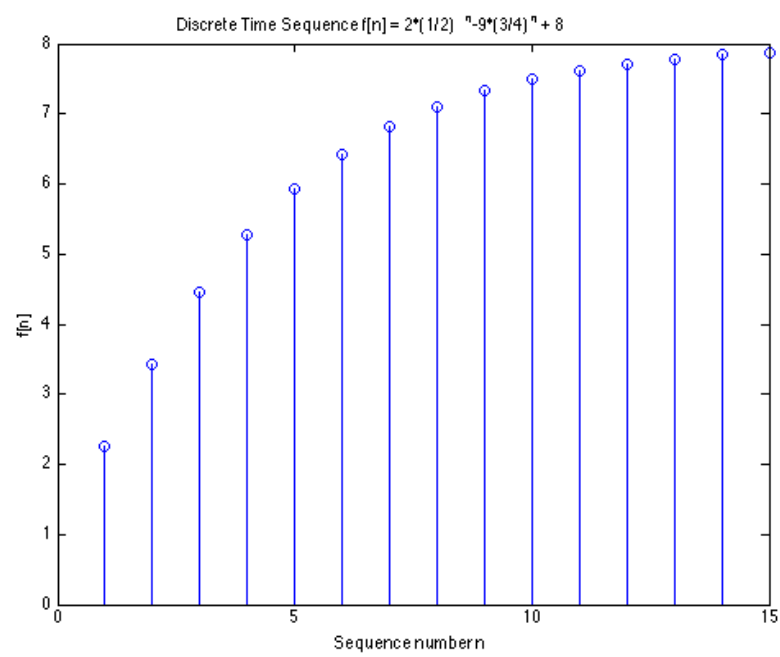


Figure 1: Solution to Example 1

## Matlab solution

See [example2.m](#)

Uses additional Matlab functions:

- `dimpulse` – computes and plots a sequence  $f[n]$  for any range of values of  $n$

## Lollipop Plot

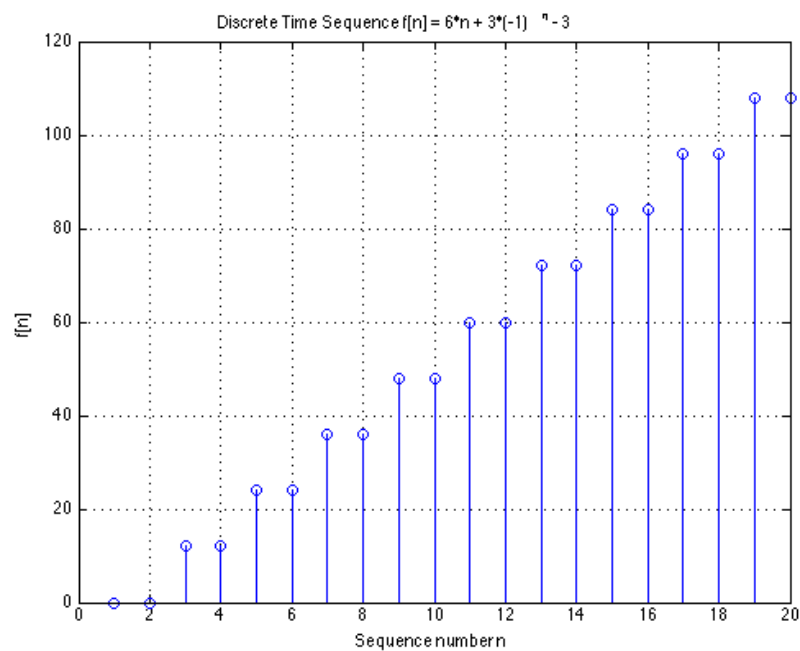


Figure 2: Solution to Example 2

## Staircase Plot

Simulates output of Zero-Order-Hold (ZOH) or Digital Analogue Converter (DAC)

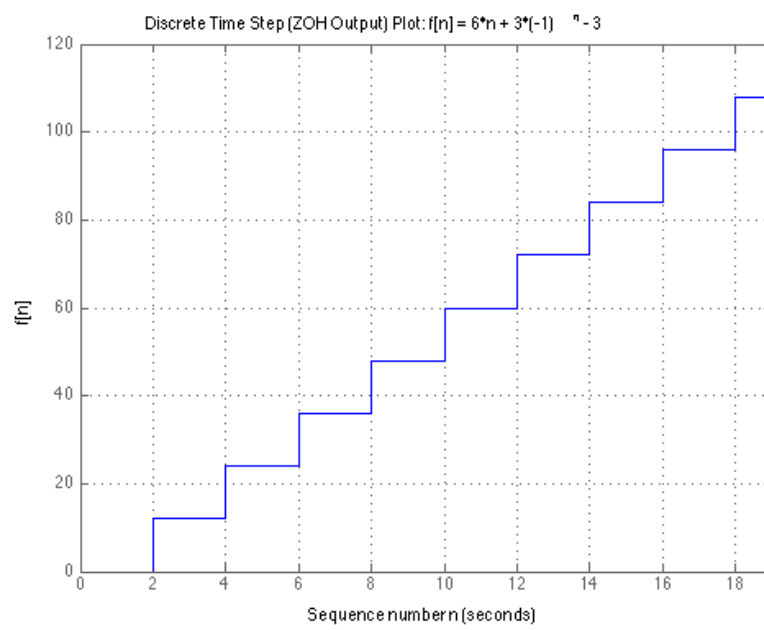


Figure 3: Solution to Example 2 as DAC Output

### Example 3

Karris example 9.6: use the partial fraction expansion method to compute the inverse z-transform of

$$F(z) = \frac{z+1}{(z-1)(z^2+2z+2)}$$

### Answer to Example 3

$$f[n] = -0.5\delta[n] + 0.4 + \frac{(\sqrt{2})^n}{10} \cos \frac{3n\pi}{4} - \frac{3(\sqrt{2})^n}{10} \sin \frac{3n\pi}{4}$$

### Matlab solution

See [example3.m](#)

### Lollipop Plot

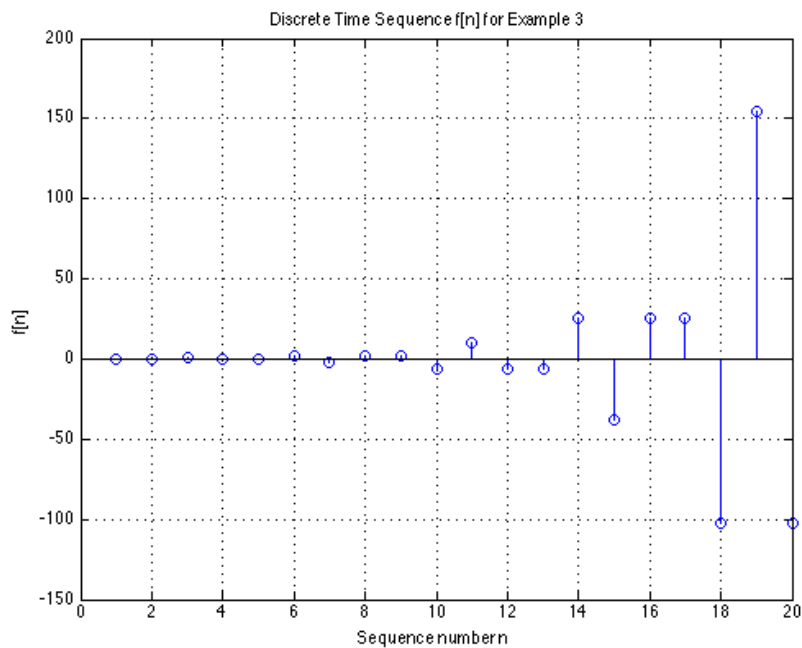


Figure 4: Solution to Example 3

## Staircase Plot

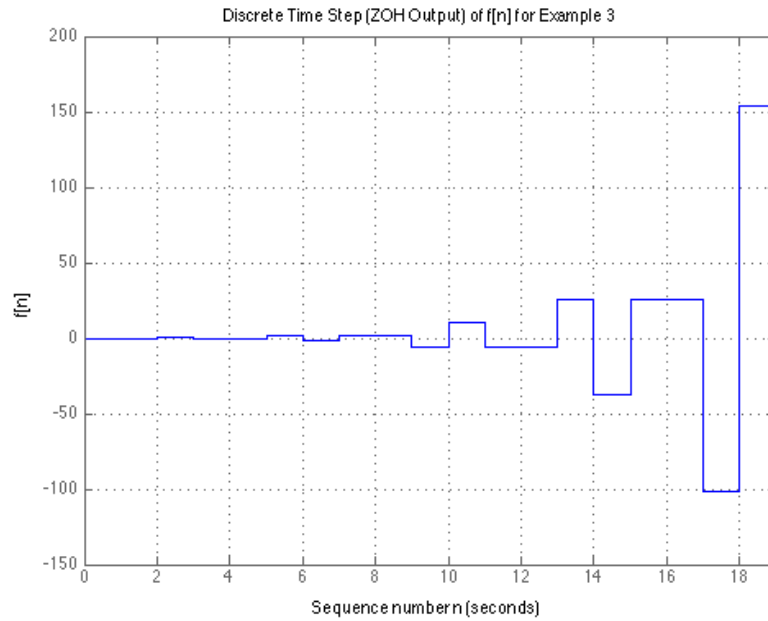


Figure 5: Solution to Example 3 as DAC Output

## Inverse Z-Transform by the Inversion Integral

### Inverse Z-Transform by the Inversion Integral

The inversion integral states that:

$$f[n] = \frac{1}{j2\pi} \oint_C F(z) z^{n-1} dz$$

where  $C$  is a closed curve that encloses all poles of the integrand.

This can (*apparently*) be solved by Cauchy's residue theorem!!

Fortunately (:-), this is beyond the scope of this module!

See Karris Section 9.6.2 (pp 9-29—9-33) if you want to find out more.



## Inverse Z-Transform by the Long Division

### Inverse Z-Transform by the Long Division

To apply this method,  $F(z)$  must be a rational polynomial function, and the numerator and denominator must be polynomials arranged in descending powers of  $z$ .

#### Example 4

Karris example 9.9: use the long division method to determine  $f[n]$  for  $n = 0, 1$ , and  $2$ , given that

$$F(z) = \frac{1 + z^{-1} + 2z^{-2} + 3z^{-3}}{(1 - 0.25z^{-1})(1 - 0.5z^{-1})(1 - 0.75z^{-1})}$$

#### Answer 4

$$f[0] = 1, f[1] = 5/2, f[2] = 81/16.$$

#### Matlab

See [example4.m](#)

```
sym_den =
```

```
z^3 - (3*z^2)/2 + (11*z)/16 - 3/32
```

```
fn =
```

```
1.0000  
2.5000  
5.0625  
....
```

#### Combined Staircase/Lollipop Plot

#### Methods of Evaluation of the Inverse Z-Transform

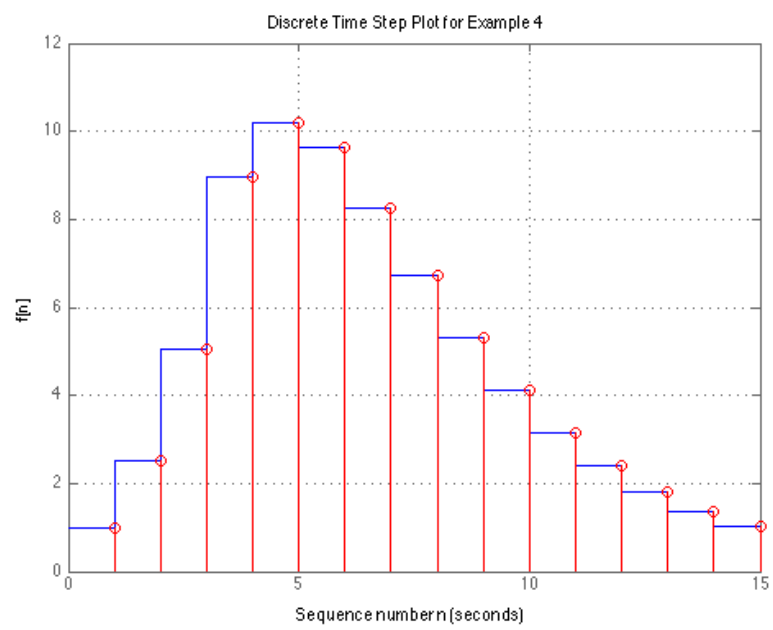


Figure 6: Combined Staircase/Lollipop Plot

Method	Advantages	Disadvantages
Partial Fraction Expansion	<ul style="list-style-type: none"> <li>• Most familiar.</li> <li>• Can use Matlab <b>residue</b> function.</li> </ul>	<ul style="list-style-type: none"> <li>• Requires that <math>F(z)</math> is a proper rational function.</li> </ul>
Inversion Integral	<ul style="list-style-type: none"> <li>• Can be used whether <math>F(z)</math> is rational or not</li> </ul>	<ul style="list-style-type: none"> <li>• Requires familiarity with the <i>Residues theorem</i>       of complex variable analysis.</li> </ul>
Long Division	<ul style="list-style-type: none"> <li>• Practical when only a small sequence of numbers is desired.</li> <li>• Useful when z-transform has no closed-form solution.</li> <li>• Can use Matlab <b>dimpulse</b> function to compute a large sequence of numbers.</li> </ul>	<ul style="list-style-type: none"> <li>• Requires that <math>F(z)</math> is a proper rational</li> <li>• Division may be endless.</li> </ul>

## Summary

- Inverse Z-Transform
- Examples using PFE
- Examples using Long Division
- Analysis in Matlab

## Next time

- DT transfer functions, continuous system equivalents, and modelling DT systems in Matlab and Simulink.

## Homework

Attempt the end of the chapter exercises 4-7 (Section 9.10) from Karris. Don't look at the answers until you have attempted the problems.