Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

MPC for tracking of constrained nonlinear systems

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo and E.F. Camacho

***Abstract*— This paper deals with the tracking problem for constrained nonlinear systems using a model predictive control (MPC) law. MPC provides a control law suitable for regulating constrained linear and nonlinear systems to a given target steady state. However, when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference. In this paper, a novel MPC for tracking changing constant references is presented. The main characteristics of this controller are: (i) considering an artificial steady state as a decision variable, (ii) minimizing a cost that penalizes the error with the artificial steady state, (iii) adding to the cost function an additional term that penalizes the deviation between the artificial steady state and the target steady state (the so-called *offset cost function*) and (iv) considering an invariant set for tracking as extended terminal constraint. The calculation of the stabilizing parameters of the proposed controller is studied and some methods are proposed. The properties of this controller has been tested on a constrained CSTR simulation model.**

I. INTRODUCTION

Model predictive control (MPC) is one of the most successful techniques of advanced control in the process industry. This is due to its control problem formulation, the natural usage of the model to predict the expected evolution of the plant, the optimal character of the solution and the explicit consideration of hard constraints in the optimization problem. Thanks to the recent developments of the underlying theoretical framework, MPC has become a mature control technique capable to provide controllers ensuring stability, robustness, constraint satisfaction and tractable computation for linear and for nonlinear systems [1].

The control law is calculated by predicting the evolution of the system and computing the admissible sequence of control inputs which makes the system evolve satisfying the constraints and minimizing the predicted cost. This problem can be posed as an optimization problem. To obtain a feedback policy, the obtained sequence of control inputs is applied in a receding horizon manner, solving the optimization problem at each sample time. Considering a suitable penalization of the terminal state and an additional terminal constraint, asymptotic stability and constraints satisfaction of the closed loop system can be proved [2].

Most of the results on MPC consider the regulation problem, that is steering the system to a fixed steady-state (typically the origin), but when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference [3], [4], [5], [6]. Tracking control of constrained nonlinear systems is an interesting problem due to the nonlinear nature of many processes in industry mainly when large transitions are required, as in the case of changing operating point.

In [7] a nonlinear predictive control for set point families is presented, which considers a pseudolinearization of the system and a parametrization of the set points. The stability is ensured thanks to a quasi-infinite nonlinear MPC strategy, but the solution of the tracking problem is not considered.

In [8] an output feedback receding horizon control algorithm for nonlinear discrete-time systems is presented, which solves the problem of tracking exogenous signals and asymptotically rejecting disturbances generated by a properly defined exosystem. In [9] an MPC algorithm for nonlinear systems is proposed, which guarantees local stability and asymptotic tracking of constant references. This algorithm needs the presence of an integrator preliminarily plugged in front of the system to guarantee the solution of the asymptotic tracking problem.

Another approach to the tracking of nonlinear systems problem are the so-called reference governors [10], [4], [11]. A reference governor is a nonlinear device which manipulates on-line a command input to a suitable pre-compensated system so as to satisfy constraints. This can be seen as adding an artificial reference, computed at each sampling time to ensure the admissible evolution of the system, converging to the desired reference.

In [12] the tracking problem for constrained linear systems is solved by means of an approach called dual mode: the dual mode controller operates as a regulator in a neighborhood of the desired equilibrium wherein constraints are feasible, while it switches to a feasibility recovery mode, whenever this is lost due to a set point change, which steers the system to the feasibility region of the MPC as quickly as possible. In [13] this approach is extended to nonlinear systems, considering constraint-admissible invariant sets as terminal regions, obtained by means of a LPV model representation of the nonlinear plant.

In [14], [15] an MPC for tracking of constrained linear systems is proposed, which is able to lead the system to any admissible set point in an admissible way. The main characteristics of this controller are: an artificial steady state is considered as a decision variable, a cost that penalizes the error with the artificial steady state is minimized, an additional term that penalizes the deviation between the artificial steady state and the target steady state is added to

The authors acknowledge MEC-Spain for funding this work (contracts DPI2007-66718-C04-01 and DPI2005-04568).

A. Ferramosca, D. Limon, I. Alvarado, T. Alamo and E.F. Camacho are with the Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n. 41092 Sevilla, SPAIN.
E-mail: {ferramosca,limon,alvarado,alamo,eduardo}@cartuja.us.es

978-1-4244-3872-3/09/$25.00 ©2009 IEEE

the cost function (the so-called *offset cost function*) and an invariant set for tracking is considered as extended terminal constraint. This controller ensures that under any change of the target steady state, the closed loop system maintains the feasibility of the controller and ensures the convergence to the target if admissible. Furthermore, the domain of attraction of this controller is potentially larger than those of reference governors and standard predictive controllers.

This controller has been extended to the nonlinear case in [16], where a preliminary formulation of the controller is presented. In this paper, an enhanced formulation of this controller is presented. Moreover, the properties of the controller are shown and a novel method to design the stabilizing controller is proposed.

The paper is organized as follows. In section II the constrained tracking problem is stated. In section III the new MPC for tracking is presented and in section IV some procedures for the stabilizing design of the proposed controller are shown. In section V an illustrative example is shown. Finally, in section VI some conclusions are drawn.

II. PROBLEM STATEMENT

Consider a system described by a nonlinear invariant discrete time model

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x,u) \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. It is assumed that this model is perfect and that the function model $f(x,u)$ is continuous. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j,x,\mathbf{u})$ where $x = \phi(0,x,\mathbf{u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively.

The system is subject to hard constraints on state and control:

$$x(k) \in X, \quad u(k) \in U \quad (2)$$

for all $k \geq 0$. $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compact convex polyhedra containing the origin in its interior.

The steady state, input and output of the plant (x_s,u_s,y_s) are such that (1) is fulfilled, i.e.

$$x_s = f(x_s,u_s) \quad (3)$$

$$y_s = h(x_s,u_s) \quad (4)$$

Due to the relation derived from these equalities, it is possible to find a parameter vector $\theta \in \mathbb{R}^q$ which univocally defines each triplet (x_s,u_s,y_s), i.e., these can be posed as

$$x_s = g_x(\theta), \; u_s = g_u(\theta), \; y_s = g_y(\theta) \quad (5)$$

This parameter is typically the controlled output y_s although another parameter could be chosen for convenience.

The problem we consider is the design of an MPC controller $\kappa(x,y_t)$ to track a (possible time-varying) target steady output y_t, such that the system is steered as close as possible to the target while fulfilling the constraints.

III. MPC FOR TRACKING

In this section, the proposed MPC for tracking is presented. The aim of this novel formulation is to guarantee recursive feasibility for any (possibly changing) output target to be tracked and, if possible, the convergence of the output of the plant to the target. The way this controller handles the tracking problem is characterized by (i) considering an artificial steady state and input as decision variables, (ii) using a cost function that penalizes the deviation of the predicted trajectory from the artificial steady state and input, (iii) adding an offset-cost function to penalize the deviation between the artificial steady output and the target output, and (iv) considering an extended terminal constraint.

The proposed controller, as standard in MPC, relies on the calculation of a suitable terminal control law $u = \kappa(x,\theta)$, where θ defines the target to be reached. The knowledge of this control law allows the use of an horizon for prediction N_p larger than the one for control N_c, in such a way that the control action are extended using the terminal control law [17]. The proposed cost function of the MPC is given by:

$$\begin{aligned} V_{N_c,N_p}(x,y_t;\mathbf{u},\theta) = & \sum_{i=0}^{N_c-1} \ell((x(i)-x_s),(u(i)-u_s)) \\ & + \sum_{i=N_c}^{N_p-1} \ell((x(i)-x_s),(\kappa(x(i),\theta)-u_s)) \\ & + V_f(x(N_p)-x_s,y_t) + V_O(y_s-y_t) \end{aligned}$$

where $x(j) = \phi(j,x,\mathbf{u})$, $x_s = g_x(\theta)$, $u_s = g_u(\theta)$ and $y_s = g_y(\theta)$; y_t is the target of the controlled variables.

The controller is derived from the solution of the optimization problem $P_{N_c,N_p}(x,y_t)$ given by:

$$\begin{aligned} & \min_{\mathbf{u},\theta} \; V_{N_c,N_p}(x,y_t;\mathbf{u},\theta) \\ & s.t. \\ & x(0) = x, \\ & x(j+1) = f(x(j),u(j)), && j = 0,\cdots,N_c-1 \\ & x(j+1) = f(x(j),\kappa(x(j),\theta)), && j = N_c,\cdots,N_p-1 \\ & x(j) \in X, && j = 0,\cdots,N_p-1 \\ & u(j) \in U, && j = 0,\cdots,N_c-1 \\ & \kappa(x(j),\theta) \in U, && j = N_c,\cdots,N_p-1 \\ & x_s = g_x(\theta), u_s = g_u(\theta), y_s = g_y(\theta) \\ & (x(N_p),\theta) \in \Gamma \end{aligned}$$

The optimal cost and the optimal decision variables will be denoted as $V^*_{N_c,N_p}(x,y_t)$ and $(\mathbf{u}^*,\theta^*)$ respectively. Considering the receding horizon policy, the control law is given by

$$\kappa^{MPC}_{N_c,N_p}(x,y_t) = u^*(0;x,y_t)$$

Since the set of constraints of $P_{N_c,N_p}(x,y_t)$ does not depend on y_t, its feasibility region does not depend on the target operating point y_t. Then there exists a region $\mathcal{X}_{N_c,N_p} \subseteq X$

such that for all $x \in \mathcal{X}_{N_c,N_p}$ and for all $y_t \in \mathbb{R}^p$, $P_{N_c,N_p}(x, y_t)$ is feasible.

Consider the following assumption on the controller parameters:

Assumption 1:

1) Let the function $g_x(\theta)$ be Lipschitz continuous in $\Theta \triangleq \{\theta : g_x(\theta) \in X, g_u(\theta) \in U\}$.
2) Let $k(x,\theta)$ be a piecewise continuous control law such that for all $\theta \in \Theta$, the system $x^+ = f(x, k(x,\theta))$ has $x_s = g_x(\theta)$ and $u_s = g_u(\theta)$ as steady state and input, and it is asymptotically stable.
3) Let $\Gamma \subset \mathbb{R}^{n+q}$ be a set such that for all $(x,\theta) \in \Gamma$, $x \in X$, $k(x,\theta) \in U$ and $(f(x,k(x,\theta)),\theta) \in \Gamma$. Besides, there exists a $\epsilon > 0$ such that for all $\theta \in Proj_\theta(\Gamma)$ and $z \in \{z : \|z\| \le \epsilon\}$, $(g_x(\theta) + z, \theta) \in \Gamma$.
4) Let $V_f(x - g_x(\theta), \theta)$ be a Lyapunov function for system $x^+ = f(x, k(x,\theta))$:
$$\begin{aligned} V_f(f(x,k(x,\theta)) - g_x(\theta),\theta) - V_f(x - g_x(\theta),\theta) \\ \le -l(x - g_x(\theta), k(x,\theta) - g_u(\theta)) \end{aligned}$$
for all $(x,\theta) \in \Gamma$. Moreover, there exist $b > 0$ and $\sigma > 1$ which verify $V_f(x_1 - x_2, y_t) \le b\|x_1 - x_2\|^\sigma$ for all (x_1,θ) and (x_2,θ) contained in Γ.
5) Let $l(x,u)$ be a positive definite function and let the offset cost function $V_O : \mathbb{R}^p \to \mathbb{R}$ be a convex, positive definite and subdifferentiable function.

Notice that the assumptions on the terminal ingredients are similar to the standard ones but extended to a set of equilibrium points. This fact makes them more difficult to obtain. In the following section, practical methods to calculate these ingredients are shown.

The set of admissible steady outputs consistent with the invariant set for tracking Γ is given by:

$$\mathcal{Y}_s = \{y_s = g_y(\theta) : x_s = g_x(\theta), \text{ and } (x_s,\theta) \in \Gamma\}$$

The following theorem proves asymptotic stability and constraints satisfaction of the controlled system.

Theorem 1 (Stability): Consider that assumption 1 holds and consider a given target operation point y_t. Then for any feasible initial state $x_0 \in \mathcal{X}_{N_c,N_p}$, the system controlled by the proposed MPC controller $\kappa^{MPC}_{N_c,N_p}(x, y_t)$ is stable, converges to an equilibrium point, fulfils the constraints along time and besides

(i) If $y_t \in \mathcal{Y}_s$ then $\lim_{k\to\infty} \|y(k) - y_t\| = 0$.
(ii) If $y_t \notin \mathcal{Y}_s$, then $\lim_{k\to\infty} \|y(k) - \tilde{y}_s\| = 0$, where

$$\tilde{y}_s = \arg\min_{y_s \in \mathcal{Y}_s} V_O(y_s - y_t)$$

Proof:

Feasibility. The first part of the proof is devoted to prove the feasibility of the controlled system, that is $x(k+1) \in \mathcal{X}_{N_c,N_p}$, for all $x(k) \in \mathcal{X}_{N_c,N_p}$ and y_t. Assume that $x(k)$ is feasible and consider the optimal solution of $P_{N_c,N_p}(x(k), y_t)$, $\mathbf{u}^*(x(k), y_t)$, $\theta^*(x(k), y_t)$. Define the following sequences:

$$\begin{aligned} \mathbf{u}(x(k+1), y_t) &\triangleq [u^*(1; x(k), y_t), \cdots, u^*(N_c - 1; x(k), y_t), \\ &\quad \kappa(x^*(N_c; x(k), y_t), \theta^*(x(k), y_t)), \cdots, \\ &\quad \kappa(x^*(N_p; x(k), y_t), \theta^*(x(k), y_t))] \\ \bar{\theta}(x(k+1), y_t) &\triangleq \theta^*(x(k), y_t) \end{aligned}$$

Then, due to the fact that $x(k+1) = f(x(k), u^*(0; x(k), y_t))$ and to condition 3 in assumption 1, it is easy to see that $\mathbf{u}(x(k+1), y_t)$ and $\bar{\theta}(x(k+1), y_t)$ are feasible solutions of $P_{N_c,N_p}(x(k+1), y_t)$. Consequently, $x(k+1) \in \mathcal{X}_{N_c,N_p}$.

Convergence. Consider the feasible solution at time $k+1$ previously presented. Following standard steps in the stability proofs of MPC [2], we get that

$$\begin{aligned} &V^*_{N_c,N_p}(x(k+1), y_t) - V^*_{N_c,N_p}(x(k), y_t) \\ &\le -l(x(k) - g_x(\theta^*(x(k), y_t)), u(k) - g_u(\theta^*(x(k), y_t))) \end{aligned}$$

Due to the definite positiveness of the optimal cost and its non-increasing evolution, we infer that $\lim_{k\to\infty} \|x(k) - g_x(\theta^*(x(k), y_t)\| = 0$ and $\lim_{k\to\infty} \|u(k) - g_u(\theta^*(x(k), y_t)\| = 0$.

Optimality. Define $x^*_s(x(k),y_t) = g_x(\theta^*(x(k),y_t)$ and $u^*_s(x(k),y_t) = g_u(\theta^*(x(k),y_t)$. Define also the set $\tilde{\mathcal{Y}}_s \triangleq \{\tilde{y} : \tilde{y} = \arg\min_{y \in \mathcal{Y}_s} V_O(y - y_t)\}$

We proceed by contradiction. Consider that $y^* \notin \tilde{\mathcal{Y}}_s$ and pick a $\tilde{y} \in \tilde{\mathcal{Y}}_s$, then $V_O(y^* - y_t) > V_O(\tilde{y} - y_t)$.

In virtue of the continuity of the model and the control law, there exists a $\hat{\lambda} \in [0,1)$ such that, for every $\lambda \in [\hat{\lambda}, 1)$, the parameter $\bar{\theta} = \lambda\theta^* + (1-\lambda)\tilde{\theta}$ fulfils $(x^*_s, \bar{\theta}) \in \Gamma$. Defining as $\mathbf{u}$ the sequence of control actions derived from the control law $k(x, \bar{\theta})$, it is inferred that $(\mathbf{u}, x^*_s, \bar{\theta})$ is a feasible solution for $P_{N_c,N_p}(x^*_s, y_t)$. Then from assumption 1 and using standard procedures in MPC, we have that

$$\begin{aligned} V^*_{N_c,N_p}(x^*_s, y_t) &= V_O(y^* - y_t) \\ &\le V_{N_c,N_p}(x^*_s, y_t; \mathbf{u}, \bar{\theta}) \\ &= \sum_{i=0}^{N_p-1} \ell((x(i) - \bar{x}), (k(x(i), \bar{\theta}) - \bar{u})) \\ &\quad + V_f(x(N_p) - \bar{x}, \bar{\theta}) + V_O(\bar{y} - y_t) \\ &\le V_f(x^*_s - \bar{x}, \bar{\theta}) + V_O(\bar{y} - y_t) \\ &\le L_{V_f}\|\theta^* - \bar{\theta}\|^\sigma + V_O(\bar{y} - y_t) \\ &= L_{V_f}(1-\lambda)^\sigma\|\theta^* - \tilde{\theta}\|^\sigma + V_O(\bar{y} - y_t) \end{aligned}$$

where $\bar{y} = g_y(\bar{\theta})$, $L_{V_f} = L_g^\sigma b$ and L_g is the Lipshitz constant of $g_x(\cdot)$.

Define $W(x^*_s, y_t, \lambda) \triangleq L_{V_f}(1-\lambda)^\sigma\|\theta^* - \tilde{\theta}\|^\sigma + V_O(\bar{y} - y_t)$ and notice that $W(x^*_s, y_t, \lambda) = V^*_N(x^*_s, y_t)$ for $\lambda = 1$. Taking the partial of W about λ we have that

$$\frac{\partial W}{\partial \lambda} = -L_{V_f}\sigma(1-\lambda)^{\sigma-1}\|\theta^* - \tilde{\theta}\|^\sigma + g^T(y^* - \tilde{y})$$

where $g^T \in \partial V_O(\bar{y} - y_t)$, defining $\partial V_O(\bar{y} - y_t)$ as the subdifferential of $V_O(\bar{y} - y_t)$. Evaluating this partial for $\lambda = 1$ we obtain that:

$$\left.\frac{\partial W}{\partial \lambda}\right|_{\lambda=1} = g^{*T}(y^* - \tilde{y})$$

where $g^{*T} \in \partial V_O(y^* - y_t)$, defining $\partial V_O(y^* - y_t)$ as the subdifferential of $V_O(y^* - y_t)$. Taking into account that V_O is a subdifferentiable function, from convexity [18] we can state for every y^* and $\tilde{y}$ that

$$g^{*T}(y^* - \tilde{y}) \geq V_O(y^* - y_t) - V_O(\tilde{y} - y_t)$$

Taking into account that $y_s^* \notin \tilde{\mathcal{Y}}_s$, $V_O(y^* - y_t) - V_O(\tilde{y} - y_t) > 0$, it can be derived that

$$\left.\frac{\partial W}{\partial \lambda}\right|_{\lambda=1} \geq V_O(y^* - y_t) - V_O(\tilde{y} - y_t) > 0$$

This means that there exists a $\lambda \in [\hat{\lambda}, 1)$ such that $W(x_s^*, y_t, \lambda)$ is smaller than the value of $W(x_s^*, y_t, \lambda)$ for $\lambda = 1$, which equals to $V_N^*(x_s^*, y_t)$.

This contradicts the optimality of the solution and hence the result is proved, finishing the proof. ■

Remark 1 (Changing operation points): Considering that problem $P_{N_c,N_p}(x, y_t)$ is feasible for any y_t, then the proposed controller is able to track changing operation points maintaining the recursive feasibility and admissibility.

Remark 2 (Stability for any admissible steady state): Since the property of Remark 1 holds for any value of the horizons N_c and N_p, it can be derived that the proposed controller is able to track any admissible set point $y_t \in \mathcal{Y}_s$, even for $N_c = N_p = 1$, if the system starts from a feasible initial state.
Typically, the starting point of the controller is an equilibrium point. If this point is reachable, i.e. $x_s = g_x(\theta)$, $\theta \in Proj_\theta(\Gamma)$, then the system can be steered to any reachable equilibrium point, for any N_c and N_p.

Remark 3 (Enlargement of the domain of attraction): The domain of attraction of the MPC is the set of states that can be admissible steered to $\Omega \triangleq Proj_x\Gamma$. The fact that this set is an invariant set for any equilibrium points makes this set (potentially) larger than the domain of a MPC calculated for regulation to a fixed equilibrium point or the domain of a reference governor with a fixed inner controller. This property is particularly interesting for small values of the control horizon.

Remark 4 (Steady state optimization): It is not unusual that the output target y_t is not contained in $\mathcal{Y}_s$. This may happen when there not exists an admissible operating point which steady output equals to the target or when the target is not a possible steady output of the system. To deal with this situation in predictive controllers, the standard solution is to add an upper level steady state optimizer to decide the best reachable target of the controller [19].

From the latter theorem it can be clearly seen that in this case, the proposed controller steers the system to the optimal operating point according to the offset cost function $V_O(\cdot)$. Then it can be considered that the proposed controller has a steady state optimizer built in and $V_O(\cdot)$ defines the function to optimize. See that the only mild assumptions on this function are to be convex, positive definite, subdifferentiable and zero when the entry is null (to ensure offset-free control if $y_t \in \mathcal{Y}_s$).

IV. Calculation of the terminal ingredients

The conditions for the stabilizing design of the controllers require the calculation of a control law capable to locally asymptotically stabilize the system to any steady states contained in set. This problem is also present in the design of another controllers for tracking, such as the command governors, [10], [20], [13], [12].

A remarkable property of the proposed MPC is that the controller must only stabilize the system locally, and hence a number of existing techniques could be used. The local nature of the obtained controller can be enhanced by using a prediction horizon larger than the control horizon. In what follows, some practical techniques to cope with this problem are briefly presented.

1) *Terminal equality constraint*:
   This is the simplest choice of the ingredients. The terminal region is chosen as the set of steady states and its corresponding value of θ, i.e.

   $$\Gamma = \{(g_x(\theta), \theta) : g_x(\theta) \in X, g_u(\theta) \in U\}$$

   This terminal constraint can be posed as the following equality constraint $x(N_c) = g_x(\theta)$ together with the inequalities $g_x(\theta) \in X$ and $g_u(\theta) \in U$.
   The considered terminal control law is $\kappa(x, \theta) = g_u(\theta)$ and the cost function is chosen as $V_f(x - g_x(\theta), \theta) \triangleq 0$. This choice of terminal ingredients ensures that the system can be steered to any admissible target $y_t \in \mathcal{Y}_s$, starting from any admissible steady state. In this case, it is sensible to choose $N_p = N_c$.
   These terminal ingredients do not fulfil some of the conditions in assumption 1, but it can be demonstrated that the stabilizing properties of the controller hold in this case under the assumption that the system is locally controllable at each admissible equilibrium point.
2) *Feedback linearization*:
   There exist some classes for model functions that allow to find a suitable feedback aimed to linearize [21] or pseudolinearize [7] the plant. Once the plant is represented by a linear model, the ingredients can be calculated as proposed in [14] and [15] for linear systems.
3) *LTV modeling of the plant*:
   Linear Time-Varying systems have been widely used to represent the local dynamics of nonlinear systems. The main advantage is that the region of the validity

of the model is larger than in the standard linearization model, but this is achieved at expense of a more complex design. Successful design methods based on the solution of Linear Matrix Inequalities have been proposed. This technique has been used for the design of command governors [22] or predictive tracking strategies [13].

4) *LTV modeling of the plant in partitions*:
This novel method exploits the LTV modelling technique and the partition method proposed in [23]. In what follows, this method is briefly presented.
Choose the sets $B_x \in X$ and $B_u \in U$ (typically small) and define

$$\tilde{\Theta}=\{\theta : g_x(\theta)+z \in X,\ g_u(\theta)+v \in U,\ \forall z \in B_x, v \in B_u\}.$$

Choose a set of regions Θ_i such that it is a partition of $\tilde{\Theta}$, i.e. $\bigcup_i \tilde{\Theta}_i = \tilde{\Theta}$. Then a suitable LTV representation of the model function must be found for each region

$$X_i = \{x : \exists z \in B_x, \theta \in \tilde{\Theta}_i | x = g_x(\theta) + z\}$$

$$U_i = \{u : \exists v \in B_u, \theta \in \tilde{\Theta}_i | u = g_u(\theta) + v\}$$

Based on the LTV model, calculate a stabilizing control gain K_i and a suitable Lyapunov matrix P_i for the LTV. Then compute an invariant set for the LTV Ω_i contained in $\{z : z \in B_x,\ K_i z \in B_u\}$. Finally define

$$\Gamma_i = \{(x, \theta) : \theta \in \tilde{\Theta}_i, \text{ and } \exists z \in \Omega_i | x = g_x(\theta) + z,\}$$

Then it can be proved that Γ_i is an admissible invariant set for tracking in the partition, and for all $(x, \theta) \in \Gamma_i$, the control law given by

$$\kappa_i(x, \theta) = K_i(x - g_x(\theta)) + g_u(\theta)$$

is admissible and

$$V_{f_i}(x - g_x(\theta), \theta) = (x - g_x(\theta))^T P_i (x - g_x(\theta))$$

is a suitable terminal cost function.
The union of all the ingredients for each partition of $\tilde{\Theta}$ constitutes the terminal ingredients for the optimization problem.
Continuity of the model and local controllability ensures that there exists a suitable partition of $\tilde{\Theta}$ and sets B_x and B_u which allows to calculate the stabilizing ingredients.

V. Example

This section presents the application of the proposed controller to the highly nonlinear model of a continuous stirred tank reactor (CSTR), [13], [17], [22], [23]. Assuming constant liquid volume, the CSTR for an exothermic, irreversible reaction, $A \to B$, is described by the following model:

$$\begin{aligned} \dot{C}_A &= \frac{q}{V}(C_{Af} - C_A) - k_o e^{(\frac{-E}{RT})} C_A \qquad (6) \\ \dot{T} &= \frac{q}{V}(T_f - T) - \frac{\Delta H}{\rho C_p} k_o e^{(\frac{-E}{RT})} C_A + \frac{UA}{V \rho C_p}(T_c - T) \end{aligned}$$

where C_A is the concentration of A in the reactor, T is the reactor temperature and T_c is the temperature of the coolant stream. The nominal operating conditions are: $q = 100$ l/min, $T_f = 350$ K, $V = 100$ l, $\rho = 1000$ g/l, $C_p = 0.239$ J/g K, $\Delta H = -5 \times 10^4$ J/mol, $E/R = 8750$ K, $k_0 = 7.2 \times 10^{10} \mathrm{min}^{-1}$, $UA = 5 \times 10^4$ J/ min K and $C_{Af} = 1$ mol/l.

The objective is to regulate $y = x_2 = T$ and $x_1 = C_A$ by manipulating $u = T_c$. The constraints are $0 \leq C_A \leq 1$ mol/l, $280\mathrm{K} \leq T \leq 370\mathrm{K}$ and $280\mathrm{K} \leq T_c \leq 370$ K. The nonlinear discrete time model of system (6) is obtained by discretizing equation (6) using a 5-th order Runge-Kutta method and taking as sampling time 0.03 min. The set of reachable output is given by $304.17\mathrm{K} \leq T \leq 370\mathrm{K}$. We considered an MPC with $N_c = N_p = 5$ and with $Q = diag(1, 1/100)$ and $R = 1/100$ as weighting matrices. The terminal ingredients has been calculated choosing the first technique of Section IV, that is the terminal equality constraint. The function $V_O = 50\|y - y_t\|^2$ as been chosen as offset cost function. The controller has been implemented in MATLAB 7.3 and the function `fmincon` to solve the optimization problem.

The output $y = x_2$ has been chosen as the parameter θ. To illustrate the proposed controller, two references has been considered, $y_{t,1} = 310$ K, $y_{t,2} = 370$ K. The initial conditions are $x_0 = (0.2057, 370)$ and $u_0 = 300.1261$. In Figure 1 the evolution of the states (solid lines), the artificial references (dashed lines) and the real one (dashed-dotted line) are shown. See how the controller leads the system to track the artificial reference when the real one is infeasible. The evolution of C_A and T in the state-space (dashed-dotted line) is drawn in Figure 2. The solid thick line represents the curve of all admissible equilibrium points for system (6). The domain of attraction of the controller has been estimated by gridding and it is also depicted in Figure 2.

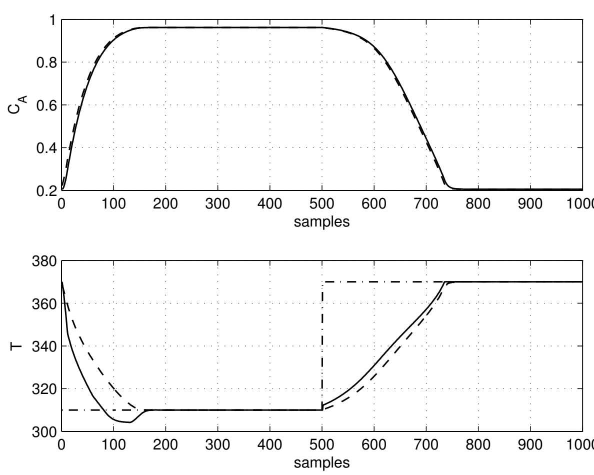

Fig. 1. Evolutions of C_A and T.

To the aim of illustrating the properties of the offset cost function, a simulation for tracking the operating point $y_t = 290$ K has been made. This point is not admissible, because its corresponding equilibrium input, $T_c = 260.7561$ K, does

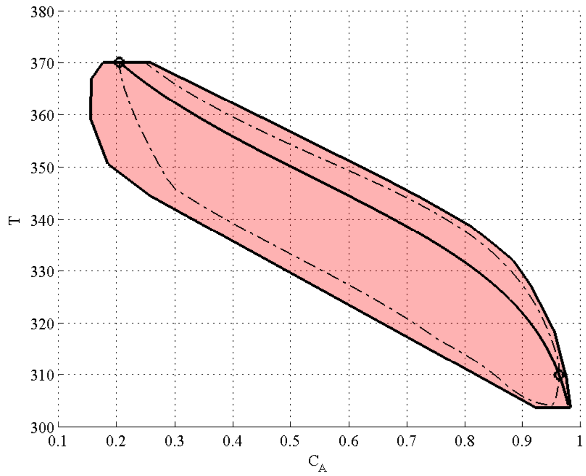

Fig. 2. Feasibility region $\mathcal{X}_5$ and evolutions of the states.

not satisfy the constraints. The point $x_0 = (0.8725, 325)$ has been considered as initial state. In Figure 3, the evolution of T (solid line), the artificial reference (dashed lines) and the real one (dashed-dotted line) are depicted. See how the controller steers the system to the closest admissible point $T = 304.17$ K when the target operating point is not admissible. This point is the one that minimizes the offset cost function, $V_O(y - y_t)$.

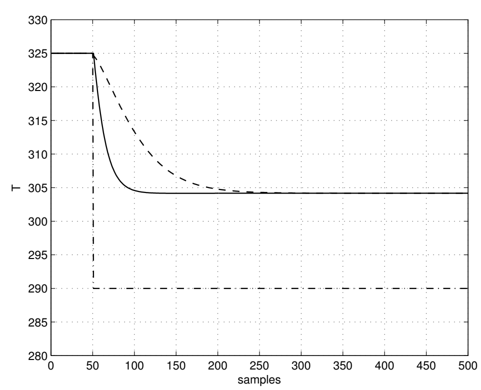

Fig. 3. Evolutions of T in case of not admissible reference.

VI. Conclusion

In this paper a novel MPC controller for tracking changing references for constrained nonlinear systems has been presented. This controller ensures feasibility by means of adding an artificial steady state and input as decision variable of the optimization problem. Convergence to an admissible target steady state is ensured by using a modified cost function and a stabilizing extended terminal constraint. Optimality is ensured by means of an offset cost function which penalizes the difference between the artificial reference and the real one. Several methods to calculate the stabilizing ingredients are provided. In order to enlarge the domain of attraction of the controller and to enhance the closed-loop performance, the proposed controller has been formulated with a prediction horizon larger than the control horizon. The properties of the controller have been illustrated in an example.

References

[1] E. F. Camacho and C. Bordons, *Model Predictive Control*, 2nd ed. Springer-Verlag, 2004.

[2] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[3] J. Rossiter, B. Kouvaritakis, and J. Gossner, "Guaranteeing feasibility in constrained stable generalized predictive control." *IEEE Proc. Control theory Appl.*, vol. 143, pp. 463–469, 1996.

[4] A. Bemporad, A. Casavola, and E. Mosca, "Nonlinear control of constrained linear systems via predictive reference management." *IEEE Transactions on Automatic Control*, vol. 42, pp. 340–349, 1997.

[5] G. Pannocchia and E. Kerrigan, "Offset-free reciding horizon control of constrained linear systems," *AIChE Journal*, vol. 51, pp. 3134–3146, 2005.

[6] I. Alvarado, "Model predictive control for tracking constrained linear systems," Ph.D. dissertation, Univ. de Sevilla., 2007.

[7] R. Findeisen, H. Chen, and F. Allgöwer, "Nonlinear predictive control for setpoint families." in *Proc. Of the American Control Conference*, 2000.

[8] L. Magni, G. D. Nicolao, and R. Scattolini, "Output feedback and tracking of nonlinear systems with model predictive control." *Automatica*, vol. 37, pp. 1601–1607, 2001.

[9] L. Magni and R. Scattolini, "On the solution of the tracking problem forn non-linear systems with mpc." *Int. J. of System Science*, vol. 36:8, pp. 477–484, 2005.

[10] D. Angeli and E. Mosca, "Command governors for constrained nonlinear systems." *IEEE Transactions on Automatic Control*, vol. 44, p. 816820, 1999.

[11] E. Gilbert and I. Kolmanovsky, "Nonlinear tracking control in the presence of state and control constraints: A generalized reference governor." *Automatica*, vol. 38, p. 20632073, 2002.

[12] L. Chisci and G. Zappa, "Dual mode predictive tracking of piecewise constant references for constrained linear systems," *Int. J. Control*, vol. 76, pp. 61–72, 2003.

[13] L. Chisci, P. Falugi, and G. Zappa, "Predictive tracking control of constrained nonlinear systems," *IEE Proceedings - Control Theory and Applications*, vol. 152:3, pp. 309–316, 2005.

[14] D. Limon, I. Alvarado, T. Alamo, and E. Camacho, "MPC for tracking of piece-wise constant references for constrained linear systems," *Automatica*, vol. 44, pp. 2382–2387, 2008.

[15] A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho, "MPC for tracking with optimal closed-loop performance," *Automatica*, vol. 45, pp. 1975–1978, 2009.

[16] D. Limon, A. Ferramosca, I. Alvarado, T. Alamo, and E. Camacho, "MPC for tracking of constrained nonlinear systems," in *Proceedings of 3rd International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control, NMPC 2008*, Pavia, Italia, September, 5-9 2008.

[17] L. Magni, G. De Nicolao, L. Magnani, and R. Scattolini, "A stabilizing model-based predictive control algorithm for nonlinear systems," *Automatica*, vol. 37, pp. 1351–1362, 2001.

[18] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2006.

[19] C. Rao and J. Rawlings, "Steady states and constraints in model predictive control," *AIChE Journal*, vol. 45, pp. 1266–1278, 1999.

[20] A. Bemporad, "Reference governor for constrained nonlinear systems." *IEEE Transactions on Automatic Control*, vol. 43, p. 415419, 1998.

[21] S. Rakovic, A. Teel, D. Mayne, and A. Astolfi, "Simple robust control invariant tubes for some classes of nonlinear discrete time systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2006.

[22] D. Angeli, A. Casavola, and E. Mosca, "Command governors for constrained nonlinear systems: Direct nonlinear vs. linearization-based strategies," *Int. J. of Robust and Nonlinear Control*, vol. 9, pp. 677–699, 1999.

[23] Z. Wan and M. Kothare, "An efficient off-line formulation of robust model predictive control using linear matrix inequalities," *Automatica*, vol. 39, pp. 837–846, 2003.