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State-dependent Riccati equation-based robust dive plane control of AUV with control constraints

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Abstract

The paper treats the question of suboptimal dive plane control of autonomous underwater vehicles (AUVs) using the state-dependent Riccati equation (SDRE) technique. The SDRE method provides an effective mean of designing nonlinear control systems for minimum as well as nonminimum phase AUV models. It is assumed that the hydrodynamic parameters of the nonlinear vehicle model are imprecisely known, and in order to obtain a practical design, a hard constraint on control fin deflection is imposed. The problem of depth control is treated as a robust nonlinear output (depth) regulation problem with constant disturbance and reference exogenous signals. As such an internal model of first-order fed by the tracking error is constructed. A quadratic performance index is chosen for optimization and the algebraic Riccati equation is solved to obtain a suboptimal control law for the model with unconstrained input. For the design of model with fin angle constraints, a slack variable is introduced to transform the constrained control input problem into an unconstrained problem, and a suboptimal control law is designed for the augmented system using a modified performance index. Using the center manifold theorem, it is shown that in the closed-loop system, the system trajectories are regulated to a manifold (called output zeroing manifold) on which the depth tracking error is zero and the equilibrium state is asymptotically stable. Simulation results are presented which show that effective depth control is accomplished in spite of the uncertainties in the system parameters and control fin deflection constraints.

Keywords: AUV control; Robust output regulation; SDRE method; Nonlinear suboptimal control

1. Introduction

Autonomous underwater vehicles (AUVs) have great advantages for activities in deep sea; such as inspections, long range survey and oceanographic mapping, etc. Presently, efforts are also being made to develop biorobotic AUVs (BAUVs), equipped with biologically inspired control surfaces, which will play a greater role in oceanic research and exploration (Bandyopadhyay, 2005). However, the dynamics of AUVs and BAUVs are highly nonlinear and the hydrodynamic coefficient is not precisely known. As such the development of nonlinear control systems for AUVs with uncertain dynamics is of considerable importance.

Often, for simplicity, control laws for AUVs are designed using linearized models (Fossen, 1994; Prestero, 2001; Healey and Lienard, 1993; Jalving, 1994). Recent designs of dorsal and pectoral fin control systems for BAUVs have also ignored model nonlinearities (Bandyopadhyay et al., 1999; Singh et al., 2004; Narasimhan et al., 2006). For nonlinear models of AUVs with known dynamics, control laws have been designed using the Lyapunov stability theory, and the backstepping design technique (Fjellstad and Fossen, 1994; Fossen, 2002). For the control of AUV models in the presence of uncertainties, sliding mode control has been considered (Yoerger and Slotine, 1985; Demirci and Kerestecioglu, 2004; Healey and Lienard, 1993). Sliding mode control approach requires high-gain feedback for the compensation of uncertainties. Adaptive control laws have been also designed for the control of AUVs (Fossen, 2002; Li and Lee, 2005; Do et al., 2004; Narasimhan and Singh, 2006a,b). For adaptive

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control, dynamic feedback loop is used for generating the estimates of unknown controller parameters for compensation. A sliding mode fuzzy control law has been proposed by Guo et al. (2003). A digital control system has been also designed in which the unknown parameters are estimated using a discrete time parameter identifier (Yuh, 1990). A neural network based control system has been developed for AUVs by Ishii et al. (1995).

Adaptive and nonadaptive control systems designed using feedback linearization and sliding mode control technique are applicable only to minimum phase AUVs models. As indicated in Bandyopadhyay et al. (1999), often the dive plane dynamics exhibit nonminimum phase behavior. This difficulty in design for nonminimum phase systems is alleviated by appropriate output modification (Bandyopadhyay et al., 1999). However, such a modification cannot be done if the parameters of the system are not known. Yet another disadvantage of feedback linearization is that useful nonlinearities of the system get canceled in order to obtain linear error dynamics. Although, adaptive laws are effective in the control of AUVs in the presence of large parameter uncertainties, their synthesis is complicated because large number of control parameters must be adapted in the dynamic feedback loop. Moreover, adaptive systems are extremely sensitive to unmodeled dynamics.

Recently developed state-dependent Riccati equation (SDRE) techniques provide a systematic and effective means for the design of control systems for nonlinear dynamical systems (Mracek and Cloutier, 1997, 1998; Cloutier et al., 1998; Cloutier and Stansbery, 2001). This design method has been applied to variety of aerospace related problems. It is important to note that SDRE methods are applicable to minimum as well as nonminimum phase nonlinear systems. Furthermore, the control input magnitude constraints can be included in the design process using this method. For the design using the SDRE method, one selects an appropriate performance index for optimization. The choice of performance indices for optimization provides flexibility in satisfying the response characteristics in the closed-loop system. For the computation of control law, the SDRE method requires the solution of a state-dependent algebraic Riccati equation in a pointwise fashion as the system trajectory evolves in the state space. In a real situation, the allowable maximum control fin deflections are always limited. But the control system designs for AUVs indicated in the literature here do not consider hard limits on the control input. Also it appears from the literature that the effectiveness of SDRE methods for the control of AUVs has not been explored. As such it is interesting to develop SDRE-based control laws for maneuvering AUVs with control fin angle constraints, which may be minimum or nonminimum phase.

The contribution of this paper lies in the design of a robust suboptimal control system for the control of AUVs in the dive plane using the state-dependent Riccati equation method. The model of the AUV is nonlinear and, for a realistic design, a hard constraint on the control surface (control fin) deflection is imposed. Moreover, it is assumed that the hydrodynamic parameters are not known precisely. The problem of depth control is posed as a robust nonlinear output (depth) regulation problem in which the disturbance and reference output are constant exogenous signals. For this reason, a first-order internal model fed by the output tracking error is constructed. A quadratic performance index is chosen for optimization and first a suboptimal control law for the model without control fin constraint is derived using the solution of an algebraic Riccati equation. This is followed by the design for the AUV with fin angle constraints. The design is accomplished by transforming the constrained problem into an unconstrained design problem by the introduction of a slack variable. Then a suboptimal control law is derived for the augmented system by the optimization of a modified performance index. Using the center manifold theorem (Isidori, 1995), it is shown that in the closed-loop system, the control system designed using the SDRE method accomplishes robust regulation of the trajectories to a manifold (called output zeroing manifold) on which the depth tracking error vanishes and that the equilibrium state is asymptotically stable. Simulation results show that AUV can be effectively controlled in the dive phase in spite of the presence of parameter uncertainties and the constraints on the control fin deflection.

The organization of the paper is as follows. Section 2 presents the AUV model and the output regulation problem. Suboptimal control laws for the constrained and unconstrained cases are derived in Sections 3 and 4, respectively. Then simulation results are presented in Section 5.

2. Nonlinear AUV model and control problem

A schematic of the AUV model with its body-fixed coordinate system is shown in Fig. 1. The earth-fixed frame is treated as an inertial frame. The motion of the AUV lies in a vertical plane. Let (x_B, y_B, z_B) be the coordinates of the center of buoyancy. The origin of the body-fixed coordinate system is fixed at the center of buoyancy (i.e. $(x_B, y_B, z_B) = 0$). We denote the coordinates of the center of gravity of the vehicle with respect to the center of buoyancy by (x_G, y_G, z_G).

The heave and pitch equations of motion of the vehicle with respect to the body-fixed moving frame are described by a set of nonlinear differential equations. These equations of motion are given by (Prestero, 2001):

$$\begin{aligned} m[\dot{w} - uq - x_G\dot{q} - z_G q^2] &= Z_{\dot{q}}\dot{q} + Z_{\dot{w}}\dot{w} + Z_{uq}uq + Z_{uw}uw \\ &\quad + Z_{w|w|}w|w| + Z_{q|q|}q|q| + (W - B_o)\cos\theta + u^2 Z_{uu}\delta_s, \end{aligned}$$

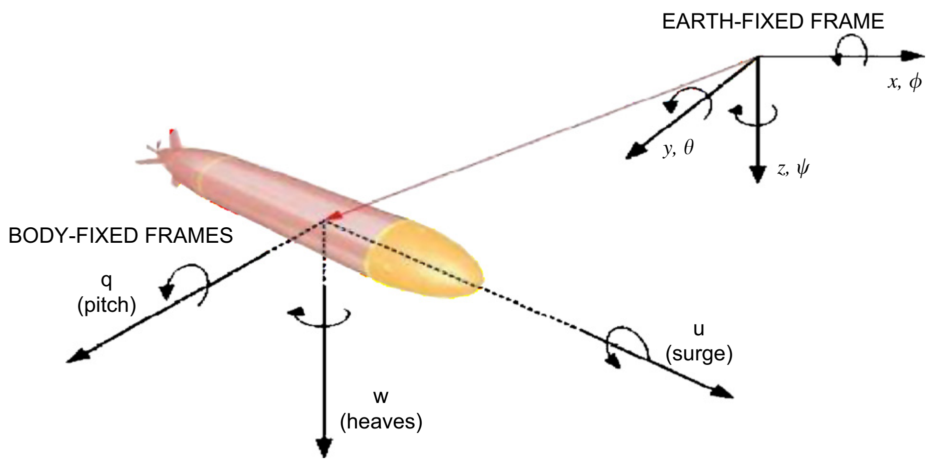

Fig. 1. AUV model.

$$
\begin{aligned}
&I_{yy}\dot{q} + m[x_G(uq - \dot{w}) + z_G wq] \\
&\quad = M_{\dot{q}}\dot{q} + M_{\dot{w}}\dot{w} + M_{uq}uq + M_{uw}uw + M_{w|w|}w|w| \\
&\qquad + M_{q|q|}q|q| - (x_G W - x_B B_o)\cos\theta \\
&\qquad - (z_G W - z_B B_o)\sin\theta + M_{uu}u^2\delta_s,
\end{aligned}
$$

$$
\begin{aligned}
&\dot{z} = w\cos\theta - u\sin\theta, \\
&\dot{\theta} = q, \qquad (1)
\end{aligned}
$$

where θ is the pitch angle, w is the heave velocity, δ_S is the control fin angle, I_{yy} is the moment of inertia of the vehicle about the pitch axis, u is the forward velocity, W denotes the vehicle's weight and B_o is the vehicle buoyancy. $f(x) = N(x)x$, where $N(x)$ is a state-dependent coefficient matrix. Now the representation of Eq. (1) in a linear-like form is considered.

The vehicle model Eq. (1) has $\sin\theta$ and $\cos\theta$ besides polynomial type nonlinearities. Since $\cos\theta = 1$ at $\theta = 0$, in order to express Eq. (1) in a linear form, we replace $\cos\theta$ and $\sin\theta$ by

$$
\begin{aligned}
\cos\theta &= \left(\frac{\cos\theta - 1}{\theta}\right)\theta + 1, \\
\sin\theta &= \left(\frac{\sin\theta}{\theta}\right)\theta. \qquad (2)
\end{aligned}
$$

Using Eq. (2) in (1), one can easily show that

$$
\begin{aligned}
\begin{pmatrix} \dot{w} \\ \dot{q} \end{pmatrix} &= M^{-1}\begin{bmatrix} Z_{uw}u + Z_{w|w|}|w| & Z_{uq} + Z_{q|q|}|q| + mz_G q + mu \\ M_{uw}u + M_{w|w|}|w| & M_{uq}u + M_{q|q|}|q| - m(x_G u + z_G w) \end{bmatrix}\begin{bmatrix} w \\ q \end{bmatrix} \\
&\quad + M^{-1}\begin{bmatrix} 0 & (W - B_o)(\cos\theta - 1)\theta^{-1} \\ 0 & (x_B B_o - x_G W)(\cos\theta - 1)\theta^{-1} - (z_G W - z_B B_o)\theta^{-1}\sin\theta \end{bmatrix}\begin{bmatrix} z \\ \theta \end{bmatrix} \\
&\quad + M^{-1}\begin{bmatrix} Z_{uu} \\ M_{uu} \end{bmatrix}u^2\delta_s + M^{-1}\begin{bmatrix} (W - B_o) \\ (x_B B_o - x_G W) \end{bmatrix} \\
&\triangleq A_1\begin{pmatrix} w \\ q \end{pmatrix} + A_2\begin{pmatrix} z \\ q \end{pmatrix} + B_1\delta_s + d_1, \qquad (3)
\end{aligned}
$$

Although, here $(x_B, y_B, z_B) = 0$, we have retained these parameters in Eq. (1) for generality. $Z_{\dot{q}}, Z_{\dot{w}}, Z_{uq}, M_{\dot{q}}$ and M_{uw} etc., are the hydrodynamics parameters. It is assumed that the forward velocity u is held constant by a control mechanism and the lateral velocity is zero.

Define the state vector $x = (w, q, z, \theta)^{\mathrm{T}} \in R^4$ (T denotes matrix transposition). For the application of the SDRE method, the nonlinear dynamics Eq. (1) must be represented by a linear structure having state-dependent coefficients matrices. For this purpose, any nonlinear vector function of the form $f(x)$ must be factored as

$$
\begin{aligned}
\begin{pmatrix} \dot{z} \\ \dot{\theta} \end{pmatrix} &= \begin{bmatrix} \cos\theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w \\ q \end{bmatrix} + \begin{bmatrix} 0 & -\theta^{-1}u\sin\theta \\ 0 & 0 \end{bmatrix}\begin{pmatrix} z \\ \theta \end{pmatrix} \\
&\triangleq A_3(x)\begin{pmatrix} w \\ q \end{pmatrix} + A_4(x)\begin{pmatrix} z \\ \theta \end{pmatrix}, \qquad (4)
\end{aligned}
$$

where A_i, $i = 1, \ldots, 4$ are defined in Eqs. (3) and (4), and

$$
M = \begin{bmatrix} m - Z_{\dot{w}} & -mx_G - Z_{\dot{q}} \\ -mx_G - M_{\dot{w}} & I_{yy} - M_{\dot{q}} \end{bmatrix},
$$

$$d_1 = M^{-1}[(W - B_o), (x_B B_o - x_G W)]^T,$$

$$B_1 = M^{-1}[Z_{uu}, M_{uu}]^T u^2. \tag{5}$$

It must be pointed out that the representation of the system Eq. (1) in linear-like form is not unique. Indeed one can obtain another representation by factoring the nonlinearity $mz_G wq$ as $[mz_G q]w$ instead of the factorization $[mz_G w]q$, which has been used in Eq. (3). Of course the control law will not be the same if one uses different form of the system.

It is assumed that the system parameters are not precisely known. Let $p_a \in R^p$ (p is the dimension of the unknown parameter vector) be the collection of all the unknown parameters in the matrices $A_i(x)$ and B_1, and p^* and $p \in \Omega_p$ be the nominal value of p_a and the unknown part of p_a, respectively, where $\Omega_p \subset R^p$ is a compact set. That is

$$p_a = p^* + p. \tag{6}$$

The nominal value of p_a is obtained if $p = 0$. Introducing the dependence of matrices A_i and B_1 on the perturbation vector p, one expresses these matrices as $A_i(x, p)$ and $B_1(p)$. In view of Eqs. (3) and (4), one obtains a new representation of Eq. (1) in the desired form given as

$$\dot{x} = \begin{bmatrix} A_1(x,p) & A_2(x,p) \\ A_3(x) & A_4(x) \end{bmatrix} x + \begin{bmatrix} B_1(p) \\ 0 \end{bmatrix} \delta_s + \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$$
$$\triangleq A(x,p)x + B(p)\delta_s + d, \tag{7}$$

where 0 denotes null matrices of appropriate dimensions, $B(p) = [B_1^T(p), 0^T]^T \in R^4$ and $d = [d_1^T \ \ 0]^T \in R^4$. We treat here d as a constant disturbance input since it is a function of the unknown parameters. The representation Eq. (7) has the desired linear-like structure with state-dependent coefficient matrices for the application of the SDRE method. Since we are interested in the dive plane control, consider a controlled output variable $y_c(t)$ as

$$y_c(t) = z(t) = Cx, \tag{8}$$

where $C = [0, 0, 1, 0]$.

Suppose that it is desired to control the AUV to a prescribed depth z_r, a given constant. Then the output tracking error is

$$e = z - z_r = Cx - z_r. \tag{9}$$

We are interested in deriving a control law such that in the closed-loop system, the tracking error tends to zero and the state vector x converges to an equilibrium state in spite of the uncertainties in the parameter vector p_a and the disturbance input d. Furthermore the control fin angle deflection is assumed to be limited.

3. Robust suboptimal control law: unconstrained fin angle

First, in this section, a control law is derived under the assumption that the control fin deflection is unconstrained. This is followed by the design of the control law with hard constraint on the control fin angle in the next section. The depth control problem for the system Eq. (7) is essentially a robust output regulation (servomechanism) problem. In the following, using the robust nonlinear output regulation (servomechanism) theory (Isidori, 1995; Huang, 2004) and the SDRE method a suboptimal nonlinear control law is derived.

We treat the signal v formed by the vector disturbance input d_1 and the command input z_r defined as

$$v = \begin{pmatrix} d_1 \\ z_r \end{pmatrix} \in \Omega_v \subset R^3 \tag{10}$$

as an exogenous signal, where Ω_v is an open neighborhood of $v = 0$. Of course, v can be generated by the exosystem

$$\dot{v} = 0, \quad v(0) = v_0. \tag{11}$$

The exosystem Eq. (11) is capable of generating any constant disturbance d_1 and command input z_r. For the design of an output regulator, according to the nonlinear output regulation (servomechanism) theory, it is sufficient to introduce a dynamic system (internal model of the exosystem) of the form

$$\dot{x}_{s1} = e = z - z_r = Cx - z_r. \tag{12}$$

The signal $x_{s1}(t)$ is the integral of the depth trajectory tracking error.

Define the augmented state vector as $x_{a1} = (x^T, x_{s1})^T \in \Omega_{a1} \subset R^5$, where Ω_{a1} is the open neighborhood of the origin. Then the composite system Eqs. (7) and (12) can be written as:

$$\dot{x}_{a1} = \begin{bmatrix} A(x,p) & 0_{4\times 1} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_{s1} \end{bmatrix} + \begin{bmatrix} B(p) \\ 0 \end{bmatrix} \delta_s + \begin{bmatrix} d \\ -z_r \end{bmatrix}$$
$$\triangleq A_{a1}(x,p)x_{a1} + B_{a1}(p)\delta_s + Ev, \tag{13}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In the sequel, the regions Ω_{ai}, Ω_v, and Ω_p will be allowed to be sufficiently small so that various arguments in the derivation of the control laws remain valid.

We are interested in deriving a control law $\delta_s = \delta_s(x_{a1})$ such that the closed-loop system has the following properties:

(i) For $v = 0$, the origin $x_{a1} = 0$ of the closed-loop system is exponentially stable.
(ii) For $v \neq 0$, the tracking error e converges to zero as $t \to 0$.

For the stabilization of the system Eq. (13) with $v = 0$, an optimal control problem is formulated. Consider the optimal control problem for minimizing the performance index

$$J_1 = \frac{1}{2}\int_0^{\infty} [x_{a1}^{\mathrm{T}} Q_1(x_{a1}) x_{a1} + R_1 \delta_s^2]\,\mathrm{d}t, \tag{14}$$

with respect to the state x_{a1} and input δ_s subject to the nominal nonlinear differential equation constraint:

$$\dot{x}_{a1} = A_{na1}(x)x_{a1} + B_{na1}\delta_s, \tag{15}$$

where $A_{na1}(x) = A_{a1}(x,0)$ and $B_{na1} = B_{a1}(0)$ are the matrices computed at the nominal value p^* (i.e. $p = 0$) of the parameter vector p_a, the matrix $Q_1(x_{a1})$ is a positive definite symmetric matrix (denoted as $Q_1(x_{a1})>0$) and R_1 is a positive real number. The weighting matrix $Q_1(x_{a1})$ and R_1 are properly selected to shape the response characteristics in the closed-loop system.

For deriving the optimal control law, one must solve the Hamilton–Jacobi–Bellman (HJB) equation which is a nonlinear partial differential equation. Since it is extremely difficult to solve this equation, instead, for simplicity, a suboptimal control law is designed using the SDRE method (Cloutier et al., 1998). This control law is obtained by solving a simplified state-dependent Riccati equation given by

$$A_{na1}^{\mathrm{T}}(x)P_1 + P_1 A_{na1}^{\mathrm{T}}(x) - P_1 B_{na1} R_1^{-1} B_{na1}^{\mathrm{T}} P_1 + Q_1(x_{a1}) = 0, \tag{16}$$

where $P_1(x_{a1})>0$. For the existence of solution for P_1 of Eq. (16), the pair $\{A_{na1}(x), B_{na1}\}$ has to be pointwise stabilizable for all $x_{a1} \in \Omega_{a1} \subset R^5$, the domain of interest. For the AUV model, the rank of the controllability matrix

$$C_0 = [B_{na1}, A_{na1}(0)B_{na1}, \ldots, A_{na1}^4(0)B_{na1}] \tag{17}$$

is 5, the dimension of x_{a1}. As such the system is pointwise controllable in a neighborhood of $x_{a1} = 0$ and the solution for $P_1(x_{a1})$ exists. The stabilizing control law is then given by

$$\delta_s = -R_1^{-1} B_{na1}^{\mathrm{T}} P(x_{a1}) x_{a1}. \tag{18}$$

Readers may refer to Cloutier et al. (1998) for the properties of the SDRE method. It is interesting to note that the suboptimal law satisfies

$$\frac{\mathrm{d}H(x_{a1},\lambda)}{\mathrm{d}\delta_s} = 0, \tag{19}$$

where the Hamiltonian of the nonlinear optimal control problem is

$$\begin{aligned} H(x_{a1},\lambda) = {} & \tfrac{1}{2}[x_{a1}^{\mathrm{T}} Q_1(x_{a1}) x_a + R_1\delta_s^2] \\ & + \lambda^{\mathrm{T}}[A_{na1}(x)x_{a1} + B_{na1}\delta_s] \end{aligned} \tag{20}$$

and $\lambda \in R^5$ is the costate or the Lagrange multiplier.

Substituting the control law Eq. (18) in Eq. (13) with $v = 0$ gives

$$\begin{aligned} \dot{x}_{a1} &= [A_{a1}(x,p) - B_{a1} R_1^{-1} B_{na1}^{\mathrm{T}} P_1(x_{a1})]x_{a1} \\ &\triangleq A_{c1}(x_{a1},p)x_{a1}. \end{aligned} \tag{21}$$

Let $A_{nc1}(x_{a1})$ be the nominal matrix $A_{c1}(x_{a1},0)$. Then the closed-loop matrix $A_{nc1}(x_{a1})$ is guaranteed to be Hurwitz in a neighborhood of the origin from the Riccati equation theory. Since the closed-loop matrix $A_{c1}(x_{a1},p)$ is a continuous function of the parameter p, it remains Hurwitz for (x_{a1},p) in a sufficiently small region $\Omega_{a1} \times \Omega_p$.

The composite closed-loop system (Eqs. (13) and (18)) and the exosystem Eq. (11) can be written as

$$\begin{aligned} \dot{x}_{a1} &= A_{c1}(x_{a1},p)x_{a1} + Ev, \\ \dot{v} &= 0. \end{aligned} \tag{22}$$

For the composite system, we state the following theorem.

Theorem 1. *Consider the closed-loop system including the AUV model Eq. (7), the internal model Eq. (12) and the control law Eq. (18). Then there exists a region $D_1 = \Omega_{a1} \times \Omega_v \subset R^5 \times R^3$ and a compact set Ω_p such that for $(x_{a1}(0), v(0)) \in D_1$, and for $p \in \Omega_p$, the trajectory $x_{a1}(t,p)$ converges to an equilibrium point, and the depth-tracking error $e(t)$ tends to zero as $t \to \infty$.*

Proof. In view of the Riccati equation theory, $A_{c1}(x_{a1},p)$ is Hurwitz in a domain $\Omega_{a1} \times \Omega_p$; and therefore, expanding A_{c1} about $x_{a1} = 0$ gives

$$\begin{aligned} \begin{bmatrix} \dot{x}_{a1} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} A_{c1}(0,p) & E \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{a1} \\ v \end{bmatrix} + \begin{bmatrix} A_r(x_{a1},p) \\ 0 \end{bmatrix} \\ &\triangleq A_{cc}(p) \begin{bmatrix} x_{a1} \\ v \end{bmatrix} + \begin{bmatrix} A_r(x_{a1},p) \\ 0 \end{bmatrix}, \end{aligned}$$

where $A_r(x_{a1},p)$ denotes the second and higher order terms in x_{a1}. In the triangular matrix A_{cc}, $A_{c1}(0,p)$ is Hurwitz for $p \in \Omega_p$ and its remaining eigenvalues are zero. Therefore, according to the center manifold theorem (Isidori, 1995; Huang, 2004), there exists a vector function $X_{a1}(v,p)$ defined for (v,p) belonging to a sufficiently small region $\Omega_v \times \Omega_p$, with $X_{a1}(0,p) = 0$, that satisfies

$$\frac{\partial X_{a1}(v,p)}{\partial v}\dot{v} = 0 = A_{c1}(X_{a1}(v,p),p)X_{a1}(v,p) + Ev. \tag{23}$$

Moreover,

$$\|x_{a1}(t,p) - X_{a1}(v(t),p)\| \leqslant \alpha \mathrm{e}^{-\beta t}\|x_{a1}(0) - X_{a1}(v(0),p)\|, \tag{24}$$

where $x_{a1}(t,p)$ and $v(t)$ are the solutions of Eq. (22) and α and β are positive real numbers. It follows from Eq. (24) that $x_{a1}(t,p) = (w,q,z,\theta,x_{s1})^{\mathrm{T}}$ converges to $X_{a1}(v(t),p)$ as $t \to \infty$. Note that one has $X_{a1}(v,p) = (X^{\mathrm{T}}(v,p), X_{s1}(v,p))^{\mathrm{T}}$, where on the center manifold $x = X(v,p)$ and $x_{s1} = X_{s1}(v,p)$. Therefore, the last equation of (23) gives

$$\frac{\partial X_{s1}(v,p)}{\partial v}\dot{v} = 0 = X_3(v,p) - v_3, \tag{25}$$

where $v_3 = z_r$, on the center manifold $z = X_3$, and X_k denotes the kth component of X. Thus on the manifold $x_{a1} = X_{a1}$, in view of Eq. (25), the tracking error vanishes; and indeed it is an output zeroing manifold. The tracking error is

$$e = z - z_r = z - X_{a13} + X_{a13} - z_r \\ \leqslant \|z - X_{a13}\| + \|X_{a13} - v_3\|. \quad (26)$$

(For simplicity, the arguments of X_{a1} are suppressed here.) In view of Eqs. (24)–(26), one has that $z(t) \to z_r$ as $t \to \infty$. Of course, the convergence of $x_{a1}(t,p)$ to an equilibrium point on the center manifold follows from Eq. (24) since the exogenous signal v is some constant. This establishes Theorem 1. □

The derivation of the control law Eq. (18) is based on the assumption that the control fin angle deflection is unlimited. However, this is not a valid assumption, and one must limit the control surface deflection to obtain a practical control law. The design of a constrained control law is considered in the following section.

4. Robust control law: control fin constrained

For the design of a constrained control law, now we introduce a hard constraint on the fin angle given by

$$|\delta_s| \leqslant \delta_{sm}, \quad (27)$$

where $\delta_{sm} > 0$ is the maximum permissible value of the fin angle. The SDRE methods provide means to include the control constraints Eq. (27) directly in the design process.

According to Cloutier et al. (1998), the design is accomplished by transforming the bounded control problem into an equivalent nonlinear regulator problem by introducing a slack variable x_{s2} that satisfies

$$\dot{x}_{s2} = u_{\text{n}}, \quad (28)$$

where u_{n} is the new control input. The fin angle takes the form of a saturation sin function, given by

$$\delta_s = \text{satsin}(\delta_{sm}, x_{s2}), \quad (29)$$

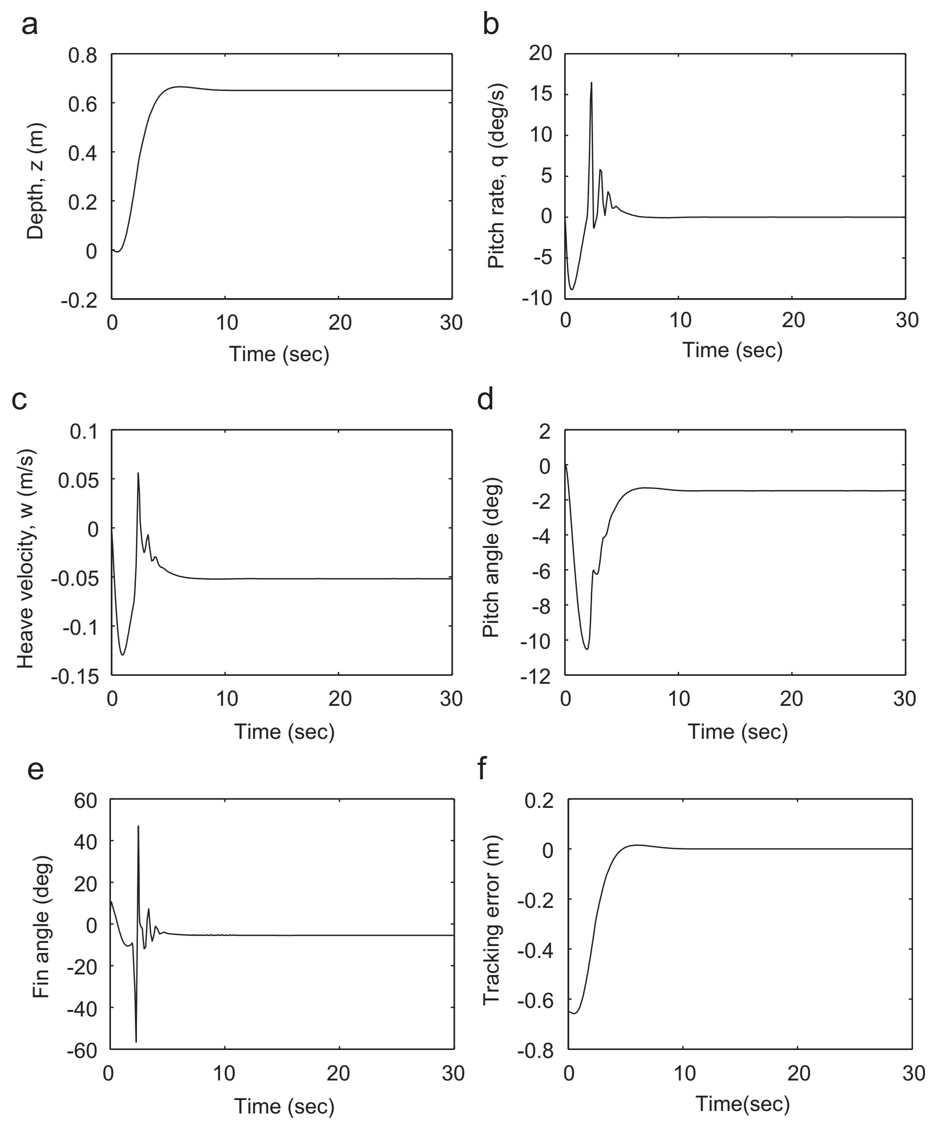

Fig. 2. Nominal REMUS control with unconstrained input: $u = 2\,\text{m/s}$, $z_r = 0.65\,\text{m}$. (a) Plunge displacement (m). (b) Pitch rate (deg/s). (c) Heave velocity (m/s). (d) Pitch angle (deg). (e) Fin angle (deg). (f) Tracking error (m).

where one defines

$$\operatorname{satsin}(\delta_{sm}, x_{s2}) = \begin{cases} \delta_{sm}\operatorname{sgn}(x_{s2}) & \text{for } |x_{s2}| > \dfrac{\pi}{2}, \\ \delta_{sm}\sin(x_{s2}) & \text{for } |x_{s2}| \leqslant \dfrac{\pi}{2}. \end{cases} \tag{30}$$

According to the definition Eq. (30) of the satsin function, fin angle is a function of the slack variable x_{s2} and does satisfy the control magnitude constraint for all $x_{s2} \in R$. But the new input u_n is unconstrained and now suboptimal regulator design is possible.

Define an augmented state vector $x_{a2} = (x_{a1}^T, x_{s2})^T \in \Omega_{a2} \subset R^6$ in an extended state space, where Ω_{a2} is an open set containing the origin. The composite system (13) and (28) can be written as

$$\dot{x}_{a2} = \begin{bmatrix} A_{a1}(x,p) & x_{s2}^{-1} B_{a1}(p)\operatorname{satsin}(\delta_{sm}, x_{s2}) \\ 0_{1\times 5} & 0 \end{bmatrix} x_{a2} + \begin{bmatrix} 0_{5\times 1} \\ 1 \end{bmatrix} u_n + \begin{bmatrix} E \\ 0 \end{bmatrix} v$$
$$\triangleq A_{a2}(x_{a2}, p)x_{a2} + B_{a2}u_n + E_a v, \tag{31}$$

where A_{a2}, B_{a2} and E_a are defined in Eq. (31). Consider an optimal control problem, in which for the system Eq. (31), the performance index of the form

$$J_2 = \frac{1}{2}\int_0^\infty [x_{a2}^T Q_2(x_{a2})x_{a2} + R_2 u_n^2]\,dt \tag{32}$$

is to be minimized, where $R_2 > 0$ is a design parameter, and the weighting matrix Q_2 is

$$Q_2 = \begin{bmatrix} Q_1(x_{a1}) & 0_{5\times 1} \\ 0_{1\times 5} & R_1 q_{s2}(x_{s2}) \end{bmatrix},$$

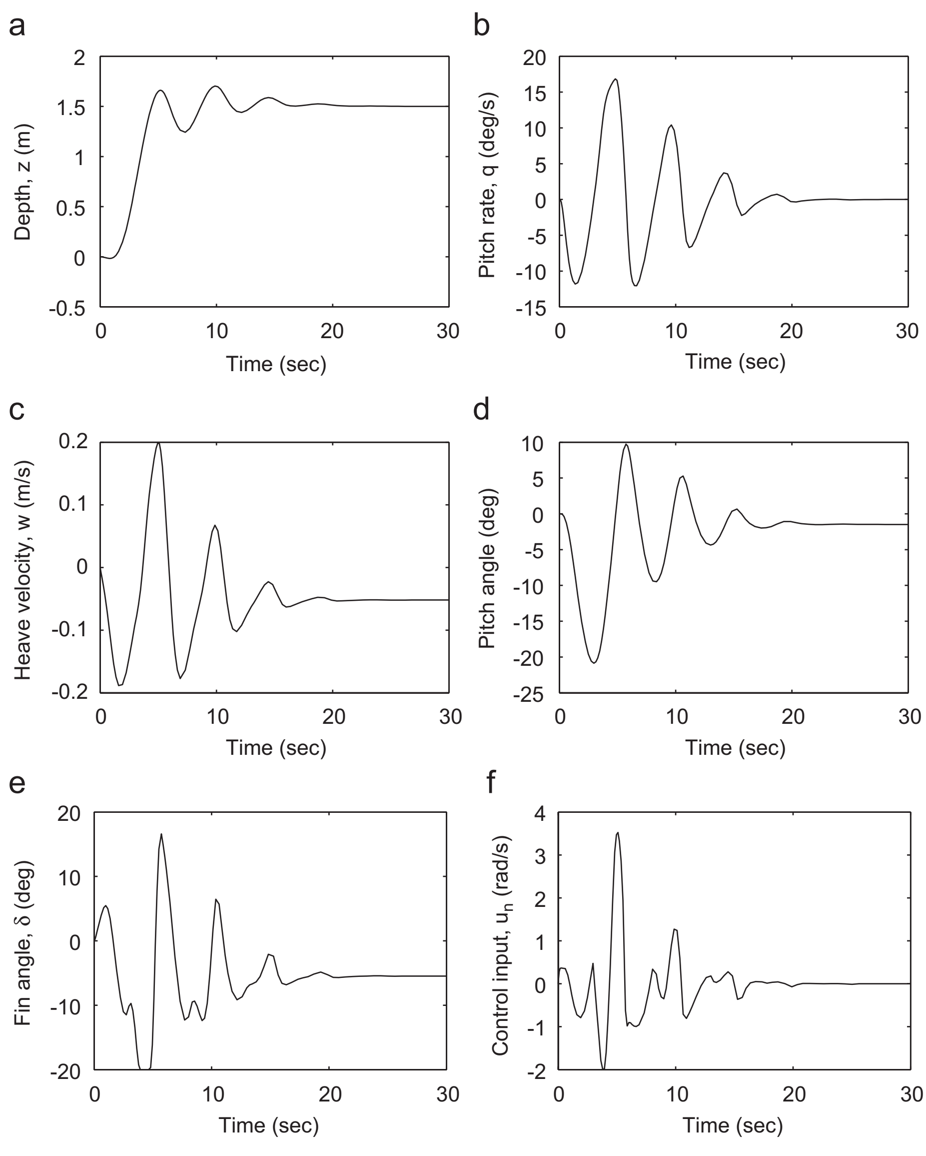

Fig. 3. Nominal REMUS control with saturating fin: $u = 2\,\mathrm{m/s}$, $z_r = 1.5\,\mathrm{m}$. (a) Plunge displacement (m). (b) Pitch rate (deg/s). (c) Heave velocity (m/s). (d) Pitch angle (deg). (e) Fin angle (deg). (f) Control input (rad/s).

$$q_{s2}(x_{s2}) = \begin{cases} [\text{satsin}(\delta_{sm}, x_{s2})/x_{s2}]^2, & |x_{s2}| \leqslant \dfrac{\pi}{2}, \\ (\delta_{sm}/x_s)^2, & |x_{s2}| > \dfrac{\pi}{2}. \end{cases} \tag{33}$$

Note that the performance index Eq. (32) is obtained by substituting Eq. (29) for δ_s in the performance index J_1 of the unconstrained control problem. The matrix Q_2 is a positive definite symmetric matrix for all x_{a2}.

For the AUV model, the pair $\{A_{a2}(x_{a2}), B_{a2}\}$ is controllable in a suitably chosen domain Ω_{a2}, and similar to the derivation in the previous section, one obtains a suboptimal control law by solving the state-dependent algebraic Riccati equation:

$$A_{na2}^{\mathrm{T}}(x_{a2})P_2 + P_2 A_{na2}(x_{a2}) - P_2 B_{a2} R_2^{-1} B_{a2}^{\mathrm{T}} P_2 + Q_2(x_{a2}) = 0, \tag{34}$$

where $A_{na2}(x_{a2}) = A_{a2}(x_{a2}, 0)$ is the nominal value evaluated at $p = 0$ and P_2 is the positive definite symmetric matrix. The new control law is given by

$$u_{\mathrm{n}} = -R_2^{-1} B_{a2}^{\mathrm{T}} P_2(x_{a2}) x_{a2}. \tag{35}$$

There exists a region $\Omega_{a2} \times \Omega_p$ such that the closed-loop matrix $A_{c2}(x_{a2}, p) = [A_{a2}(x_{a2}, p) - R_2^{-1} B_{a2} B_{a2}^{\mathrm{T}} P_2(x_{a2})]$ is pointwise Hurwitz. Since $A_{c2}(0, p)$ is Hurwitz, according to the center manifold theorem, there exists a function $X_{a2}(v, p)$ defined in a sufficiently small region $\Omega_{a2} \times \Omega_v \times \Omega_p$ that satisfies

$$\frac{\partial X_{a2}(v,p)}{\partial v} \dot{v} = 0 = A_{c2}(X_{a2}(v,p),p) X_{a2}(v,p) + E_a v. \tag{36}$$

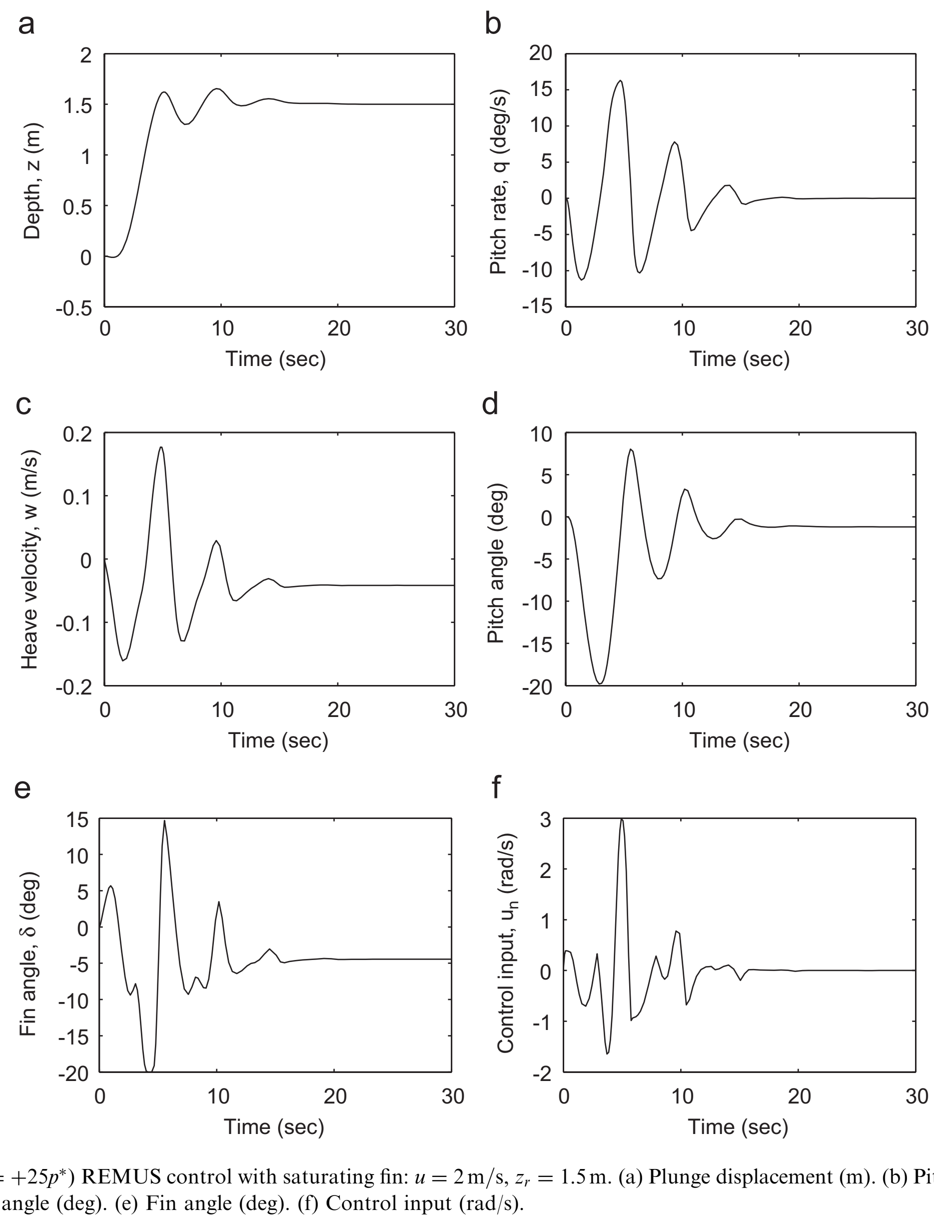

Fig. 4. Off-nominal ($p = +25p^*$) REMUS control with saturating fin: $u = 2\,\mathrm{m/s}$, $z_r = 1.5\,\mathrm{m}$. (a) Plunge displacement (m). (b) Pitch rate (deg/s). (c) Heave velocity (m/s). (d) Pitch angle (deg). (e) Fin angle (deg). (f) Control input (rad/s).

Moreover,

$$\|x_{a2}(t,p) - X_{a2}(v(t),p)\| \leqslant \alpha_2 e^{-\beta_2 t}\|x_{a2}(0) - X_{a2}(v(0),p)\|, \tag{37}$$

where α_2 and β_2 are positive numbers, and $x_{a2}(t,p)$ and $v(t)$ are the solutions of Eqs. (31) and (11).

Now in view of Eqs. (36) and (37), using an argument similar to the unconstrained case, the following theorem is obtained.

Theorem 2. *Consider the closed-loop system including the AUV model Eq.* (7), *the internal model Eq.* (12), *Eq.* (28) *for the slack variable*, *and the control law Eq.* (35). *Then there exists a region* $D_2 = \Omega_{a2} \times \Omega_v \subset R^6 \times R^3$ *and a compact set* Ω_p *such that for* $(x_{a2}(0), v(0)) \in D_2$, *and for* $p \in \Omega_p$, *the trajectory* $x_{a2}(t)$ *converges to an equilibrium point, and the depth tracking error* $e(t)$ *tends to zero as* $t \to \infty$. *Moreover the control fin constraint Eq.* (27) *is satisfied.*

The regulation property of the control systems designed using the SDRE method has been established in a sufficiently small region $(x_{a2}, v, p) \in \Omega_{a2} \times \Omega_v \times \Omega_p$ surrounding the origin. However, the results presented in the next section show that indeed the designed controller is capable of dive plane control for useful values of command inputs and large uncertainties in the system.

5. Simulation results

In this section, simulation results using *MATLAB* and *SIMULINK* for the depth control are presented. For the purpose of illustration, computer simulation is done for the

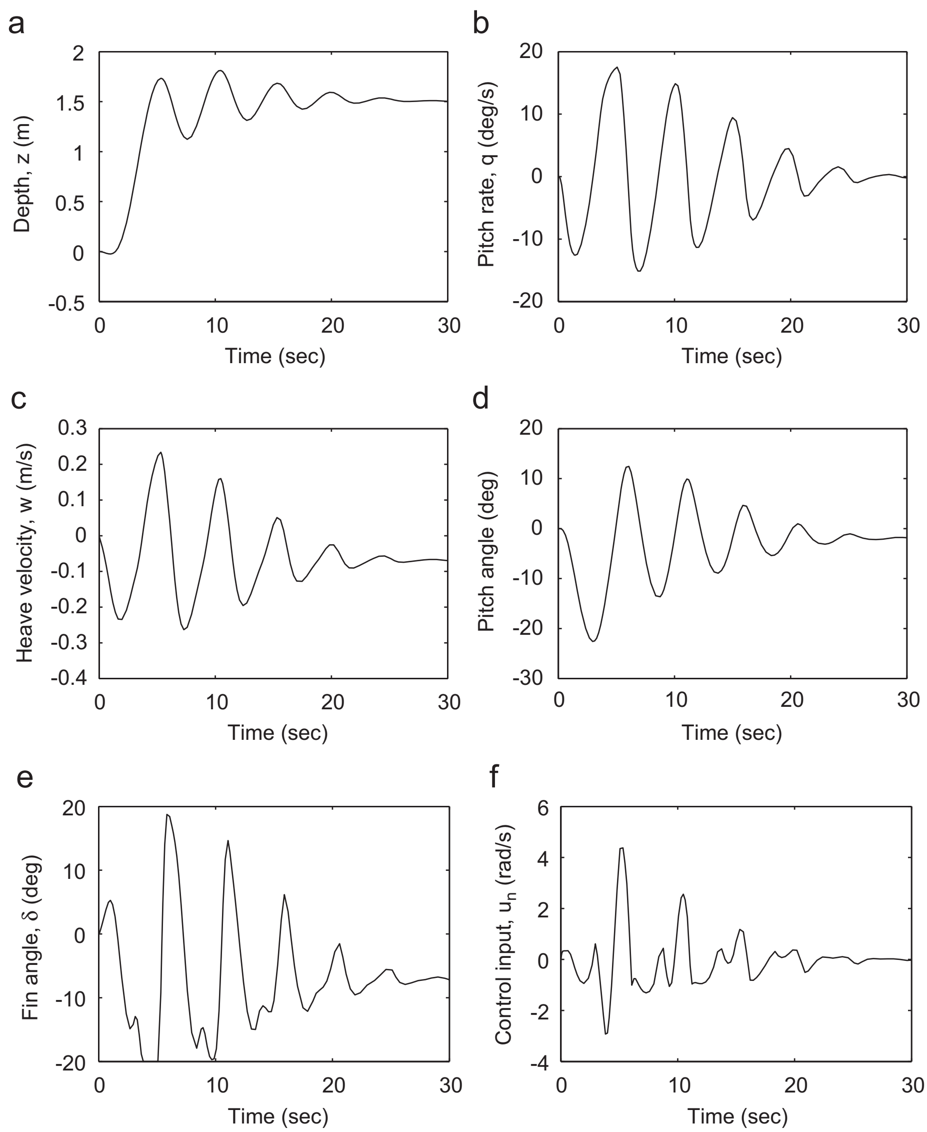

Fig. 5. Off-nominal ($p = -0.25p^*$) REMUS control with saturating fin: $u = 2\,\text{m/s}$, $z_r = 1.5\,\text{m}$. (a) Plunge displacement (m). (b) Pitch rate (deg/s). (c) Heave velocity (m/s). (d) Pitch angle (deg). (e) Fin angle (deg). (f) Control input (rad/s).

REMUS (Remote Environmental Unit) AUV (Prestero, 2001). REMUS is a low-cost, modular vehicle with applications in autonomous docking, long-range oceanographic survey, and shallow-water mine reconnaissance. The parameters of the dive plane model of the REMUS are collected in the Appendix. Here for the purpose of comparison, simulations are done using the controllers designed for the constrained as well as unconstrained fin angle. The performance of the optimal control systems depends on the choice of the weighting matrices in the performance index. Here the matrices Q_i and R_i have been selected by observing the simulated responses. The initial conditions chosen are $x(0) = 0$. Responses are obtained for different values of u and the reference input z_r.

Case A1: Nominal AUV control with unconstrained input: $u = 2\,\mathrm{m/s}$, $z_r = 0.65\,\mathrm{m}$.

First the closed-loop system Eq. (13) with the unconstrained control law Eq. (18) is simulated. The vehicle's parameters are assumed to be nominal. The performance index has $Q_1 = 100I_{5\times5}$ and $R_1 = 1$. The vehicle's velocity is $u = 2\,\mathrm{m/s}$ and it is desired to dive to a depth of $z_r = 0.65\,\mathrm{m}$. The responses are shown in Fig. 2. We observe that for the chosen performance index, the desired depth is attained, but the control fin angle required for maneuver is extremely large (more than 50°). Simulation results for larger command z_r show even larger fin deflections. This shows the limitation of unconstrained input design.

Case A2: Control of AUV with nominal parameters and saturating fin angle: $u = 2\,\mathrm{m/s}$, $z_r = 1.5\,\mathrm{m}$.

The AUV model Eq. (13) with the control law Eq. (28) is simulated. For an illustration, it is assumed that

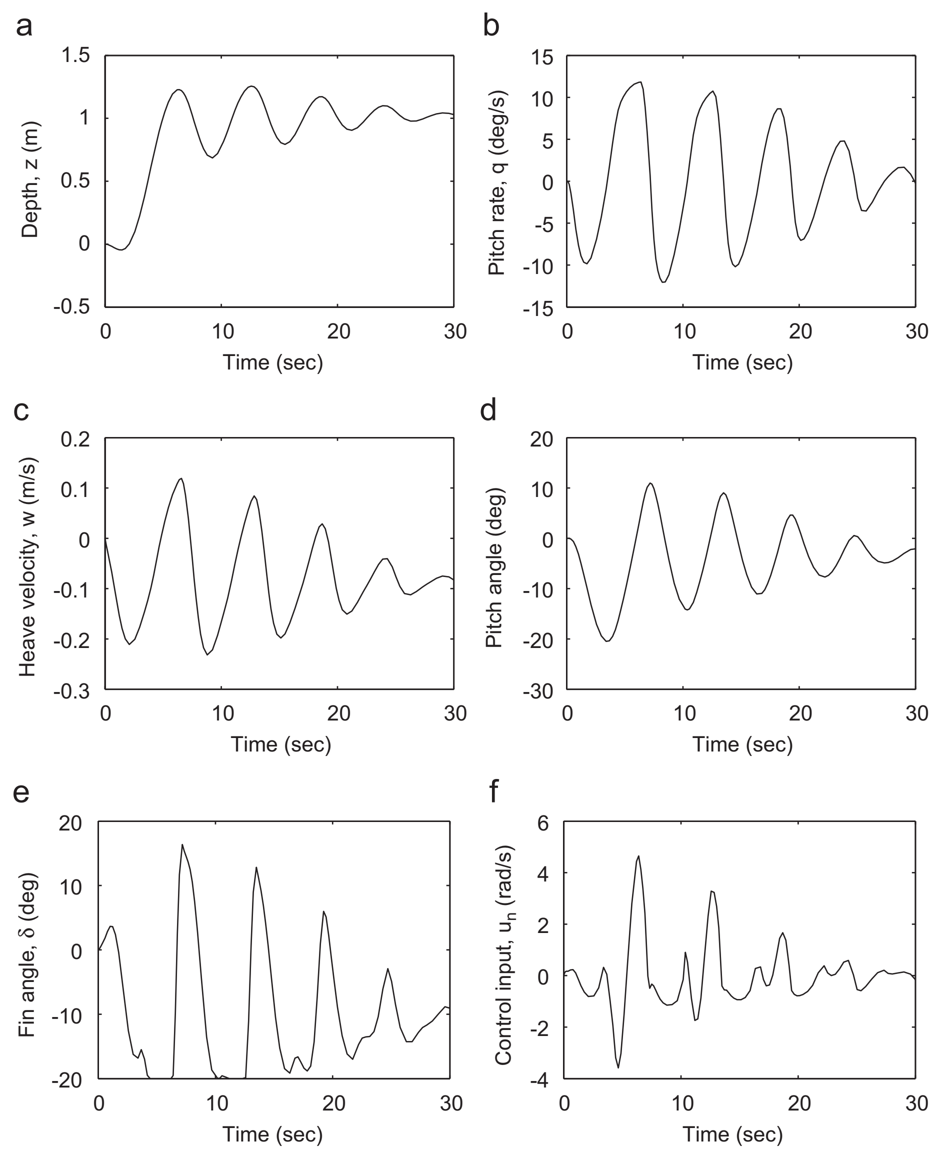

Fig. 6. Off-nominal ($p = -0.25p^*$) REMUS control with saturating: $u = 1.54\,\mathrm{m/s}$, $z_r = 1\,\mathrm{m}$. (a) Plunge displacement (m). (b) Pitch rate (deg/s). (c) Heave velocity (m/s). (d) Pitch angle (deg). (e) Fin angle (deg). (f) Control input (rad/s).

$|\delta_s(t)| \leqslant \delta_{sm} = 20°$. Noting that the performance index plays a key role in the design, for a meaningful comparison with the control system designed without any magnitude constraint on the fin angle, the weighting matrix Q_1 and the scalar parameter R_1 of the performance index J_1 of the unconstrained case A1 is retained in the performance index J_2. The initial conditions are $x_{a2} = 0$ and $R_2 = 5$ and the command input is $z_r = 1.5\,\text{m}$. We have given a larger command ($z_r = 1.5$ instead of 0.65 m of Case A1) to show the advantage of the saturating control law design. It is assumed that the vehicle parameters are known (i.e. $p = 0$). The responses are shown in Fig. 3. It is observed that the depth trajectory converges to the target value in about 15 s. The fin angle saturates over a brief period in the transient phase, but it causes no problem in performing the desired maneuver. As expected the state vector remains bounded and converges to an equilibrium state. We observe the pitch angle and w converge to nonzero values in steady-state. This is because the weight of the vehicle and vehicle buoyancy B_o are not equal, and as such the equilibrium state is not at the origin. (Later for comparison, responses for $W = B_o$ are presented.)

Case A3: Control of AUV with perturbed parameters and saturating fin angle: $u = 2\,\text{m/s}$, $z_r = 1.5\,\text{m}$.

In order to examine the robustness of the control system, simulation is done using +25% perturbation in the hydrodynamics parameters (i.e. $p_a = 1.25p^{\star}$, $p = 0.25p^{\star}$) of the AUV, but the controller designed using nominal parameter vector $p^{\star}$ of case A2 is retained. The responses are shown in Fig. 4. We observe that in spite of the uncertainty in the AUV model, the controller is effective in regulating the AUV to the desired depth and the state vector converges in about 15 s. Again, the control fin saturates in the transient phase.

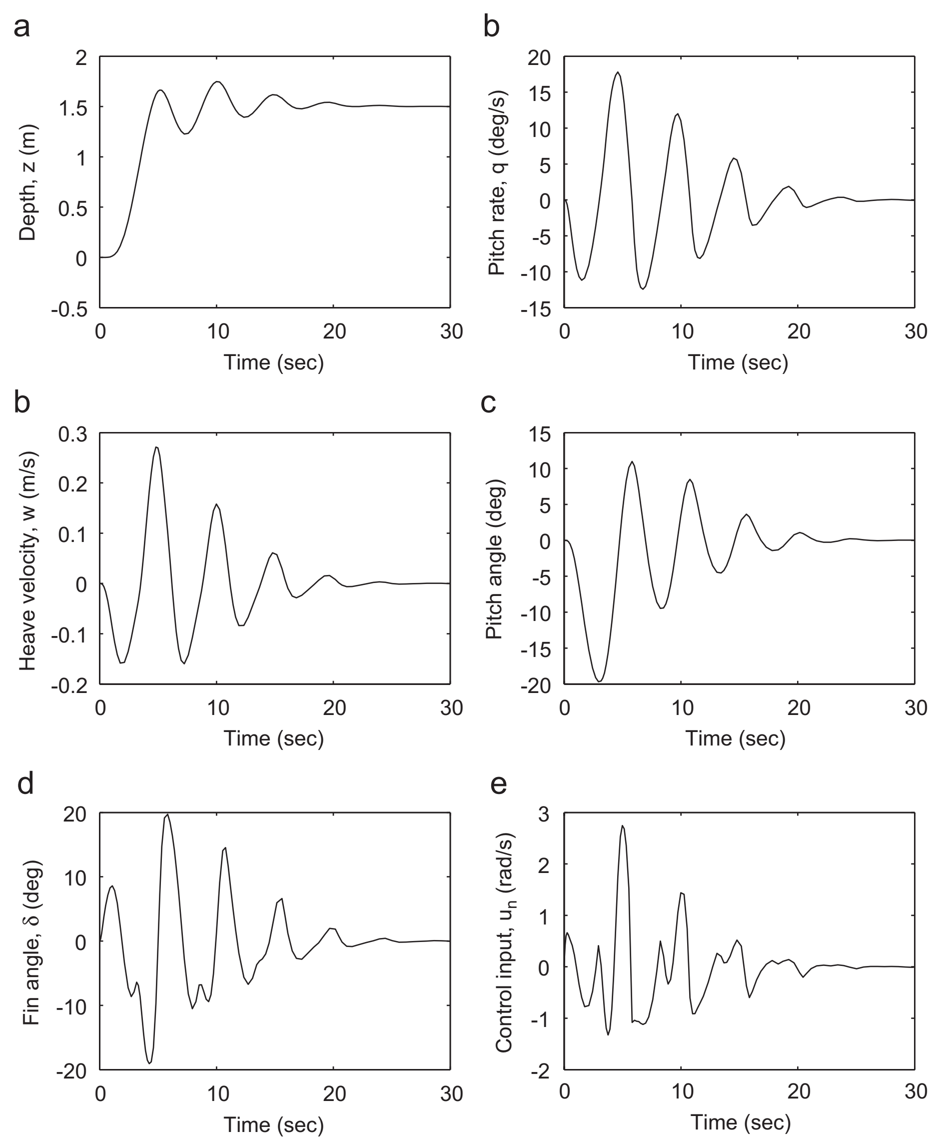

Fig. 7. Off-nominal ($p = -0.25p^{*}$) REMUS control with saturating control: $u = 2\,\text{m/s}$, $z_r = 1.5\,\text{m}$, $W = B_o$. (a) Plunge displacement (m). (b) Pitch rate (deg/s). (c) Heave velocity (m/s). (d) Pitch angle (deg). (e) Fin angle (deg). (f) Control input (rad/s).

Simulation is also performed with off-nominal lower values of the hydrodynamic parameters ($p = -25\%p^{\star}$, $p_a = 0.75p^{\star}$) of the AUV. Of course, the controller designed for the nominal AUV model is retained. The response is shown in Fig. 5. We observe that output regulation is accomplished in about 25 s. It seems that the controller designed using the underestimated values of the hydrodynamic parameters of the AUV model is more robust compared to controller designed using overestimated values. However, one must note that the weighting parameters in the performance index play an important role in shaping the closed-loop responses.

Case A4: Control of AUV with saturating control: $u = 1.54\,\mathrm{m/s}$, $z_r = 1\,\mathrm{m}$.

Simulation is performed for different velocity $u = 1.54\,\mathrm{m/s}$ and $z_r = 1\,\mathrm{m}$ using the nominal and off-nominal hydrodynamic parameters ($\pm 25\%$ uncertainties). We observed that in each case, the depth control is accomplished and the state vector converges to an equilibrium state. The control fin saturates only for a brief period in the transient phase. In order to save space, responses only for the worse off-nominal case ($p = -0.25p^{*}$) are shown in Fig. 6. As expected, we observe that controller performs better when the vehicle speed is larger. This is due to the increased control effectiveness of the fins at higher vehicle's speed (See Fig. 5 for comparison).

Case A5: Control of AUV with saturating control: $u = 2\,\mathrm{m/s}$, $z_r = 1.5\,\mathrm{m}$, $W = B_o$.

Simulation results of Cases A2–A4 have been obtained for the cases of unbalanced weight (W) and vehicle buoyancy (B_o) ($W \neq B_o$). Now simulation is done for the off-nominal case ($p = -0.25p^{*}$) for $u = 2\,\mathrm{m/s}$ and $z_r = 1.5\,\mathrm{m}$, but unlike the previous cases, one has $W = B_o$. The responses are shown in Fig. 7. It is observed that compared to Fig. 5, the responses are slightly better. The vehicle attains the desired depth, and in this case the state vector converges to the origin. The pitch angle and w tend to zero in the steady-state as expected.

6. Conclusion

In this paper, the dive plane control of AUVs using the state-dependent Riccati equation method was considered. For the design, nonlinearities in the AUV model were retained and it was assumed that the parameters of the vehicle were not known precisely. Furthermore, hard constraints on the control fin angle were imposed for a practical design. The dive plane control problem was posed as robust output (depth) regulation (servomechanism) problem. Using the SDRE method, control systems were designed with the constrained as well as unconstrained input (control fin angle). Using the center manifold theorem, it was shown that in the closed-loop system, the tracking error converges to zero and the state vector tends to an equilibrium state. Simulation results were presented which showed that the SDRE-based control system accomplishes depth control in spite of the control saturation and the presence of parameter uncertainties.

Appendix

The hydrodynamic parameters for the REMUS for simulation are (Prestero, 2001):

$M_{\dot q} = -4.88\,\mathrm{kg\,m^2/rad}$; $M_{\dot w} = -1.93\,\mathrm{kg\,m}$;
$M_{w|w|} = 3.18\,\mathrm{kg}$; $M_{q|q|} = -188\,\mathrm{kg\,m^2/rad^2}$;
$M_{uu\delta} = -6.15\,\mathrm{kg/rad}$;
$M_{uq} = -2\,\mathrm{kg\,m/rad}$; $M_{uw} = 24\,\mathrm{kg}$; $Z_{\dot w} = -35.5\,\mathrm{kg}$;
$Z_{\dot q} = -1.93\,\mathrm{kg\,m/rad}$; $Z_{ww} = -131\,\mathrm{kg/m}$;
$Z_{q|q|} = -0.632\,\mathrm{kg\,m/rad^2}$;
$Z_{uw} = -28.6\,\mathrm{kg/m}$; $Z_{uq} = -5.22\,\mathrm{kg/rad}$;
$Z_{uu\delta} = -6.15\,\mathrm{kg/(m\,rad)}$.

The vehicle physical parameters are:
$x_{cg} = 0$; $y_{cg} = 0$; $z_{cg} = 0.0196\,\mathrm{m}$;
$x_{\mathrm{B}} = 0$; $y_{\mathrm{B}} = 0$; $z_{\mathrm{B}} = 0$;
$W = 299\,\mathrm{N}$; $B_o = 306\,\mathrm{N}$; $m = 30.48\,\mathrm{kg}$;
$I_{yy} = 3.45\,\mathrm{kg\,m^2}$.

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