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Brief paper

# Offset-free reference tracking with model predictive control[☆]

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ABSTRACT

The standard way to achieve offset-free tracking in MPC is to add the disturbance dynamics to the prediction model and then use an observer to estimate the real disturbance. Existing algorithms only consider piecewise constant signals, while in practice it is often desirable to have a wider choice of reference and disturbance dynamics, such as sinusoids and ramps. This work provides a generalization of the disturbance estimation approach to arbitrary unstable dynamics. Zero offset is achieved under the assumption that the disturbance and reference dynamics are appropriately included in the prediction model and feasibility of the commanded reference is given.

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## 1. Introduction

Model predictive control (MPC) employs an explicit prediction model of the plant to optimize future plant behaviour. At each time step, an open loop optimal control sequence is obtained by means of solving an optimization problem. The first element of this sequence is applied to the plant, the rest is discarded. This optimization procedure is repeated at every time step. For linear systems, the optimization problem can be posed as linear or quadratic program, which can be readily solved by many commercial software products. Constraints on input and state variables can be easily incorporated into the optimization problem, which renders MPC a particularly attractive control scheme in practice.

MPC problems are often formulated using state-space models. This is the natural formulation for regulation problems, which address either initial state problems or equivalently the rejection of stochastic or impulsive disturbances. In practice however, it is often required to track a changing set point. This type of problem is usually referred to as reference tracking or servo problem (Davison, 1976).

A further natural extension to reference tracking is that zero offset is achieved in presence of persistent disturbances and plant-model mismatch (robust servo problem). In the unconstrained case, this problem has been extensively studied (Davison, 1972; Francis, Sebakhy, & Wonham, 1974; Francis & Wonham, 1976; Wonham, 1973; Wonham & Pearson, 1974), leading to the fundamental result of the Internal Model Principle. However, the synthesis procedures motivated by these works (Davison, 1976; Davison & Smith, 1971; Johnson, 1970; Pearson, Shields, & Staats, 1974) are not straightforward to apply to MPC. In most approaches, the tracking error is fed into a specific block (called servo compensator) of the controller which contains explicit models of the disturbance and reference dynamics. This strategy is comparable to the integration of the error in PID. Since the error integration is independent of the controller, this method may lead to windup in constrained systems, even when MPC is used. Thus, it requires the addition of anti-windup mechanisms.

The methods in Maeder, Borrelli, and Morari (2009), Maeder and Morari (2007), Muske and Badgwell (2002), Pannocchia and Bemporad (2007), Pannocchia and Kerrigan (2003), Pannocchia and Rawlings (2003), and Qin and Badgwell (2003) aim to avoid this problem by employing a disturbance estimator approach. Thereby, the state update equations used for the prediction are augmented by the reference and disturbance dynamics. An observer is used to estimate the disturbance states, and the MPC is designed to reject the estimated disturbance and track the reference. Such controllers do not suffer from windup (Maeder & Morari, 2007).

The existing methods essentially consider constant disturbances and references and hence remove offset at steady-state. For more general signals, such as ramps and sines, these methods will fail to remove offset. This work generalizes the previous methods and provides a synthesis procedure for the case when disturbances and references are generated by arbitrary, unstable dynamics.

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### 1.1. Notation

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Let $A \in \mathbb{R}^{n\times m}$ be a matrix. Then $\mathcal{N}(A) = \{x \in \mathbb{R}^m | Ax = 0\}$ is the null space of $A$ and $\mathcal{R}(A) = \{Ax | x \in \mathbb{R}^n\}$ denotes the range space of $A$. If $m = n$, then $\lambda_i \in \sigma(A)$ denotes the $i$-th eigenvalue of $A$. The Jordan chain of length $p_i$ corresponding to this eigenvalue is given by nontrivial solutions to $(A - \lambda_i I)v_j = v_{j-1}, j = 1, \ldots, p_i$ where $v_0 := 0$. The space $\mathcal{N}((A - \lambda_i I)^{p_i}) = \mathcal{R}[v_1, v_2, \ldots, v_{p_i}]$ is called generalized (right) eigenspace corresponding to $\lambda_i$. The sequence $\{s(t)\}_{t=0}^{\infty}$ denotes a time signal. Where it is clear, we will use the notation $s(\cdot)$ for brevity. Given signals $s_1(\cdot), s_2(\cdot)$, define $s_1(\cdot) + s_2(\cdot) \triangleq \{s_1(t) + s_2(t)\}_{t=0}^{\infty}$. The term $s(t)$ denotes a signal value at time $t$, while $s_t(k)$ denotes the $k$-step prediction into the future at time $t$.

## 2. Preliminaries

We consider discrete-time systems of the form

$$\begin{aligned} x_\phi(t+1) &= f(x_\phi(t), u(t)) \\ y_\phi(t) &= g(x_\phi(t)) \end{aligned} \tag{1}$$

where $x_\phi(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$ and $y_\phi(t) \in \mathbb{R}^{n_y}$.

For the controller design, we employ a linear model of (1)

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2}$$

with $y(t) \in \mathbb{R}^{n_y}$, $x(t) \in \mathbb{R}^{n_x}$, where $n_x$ need not be equal to $n$. We assume $(A, B)$ controllable, $(C, A)$ observable and $C$ to have full row rank. Let the reference signal $r(t) \in \mathbb{R}^{n_y}$ be generated by the autonomous dynamic system

$$\begin{aligned} x_r(t+1) &= A_r x_r(t) \\ r(t) &= C_r x_r(t) \end{aligned} \tag{3}$$

with $A_r \in \mathbb{R}^{n_r\times n_r}$ and $C_r \in \mathbb{R}^{n_y\times n_r}$ and $(C_r, A_r)$ observable. Without loss of generality, we assume that $A_r$ is unstable, i.e. $|\lambda| \geq 1$, $\forall \lambda \in \sigma(A_r)$. The internal state $x_r(t)$ of the generating system can either be known or estimated from $r(t)$. The goal is to design an MPC which achieves offset-free tracking of the reference signal

$$y(t) \to r(t) \tag{4}$$

as $t \to \infty$ under the input and state constraints

$$u(t) \in \mathcal{U}, \qquad x(t) \in \mathcal{X}, \tag{5}$$

where $\mathcal{X}$ and $\mathcal{U}$ are convex polytopic sets given by

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^{n_x} | H_x x \leq K_x\}, \\ \mathcal{U} &= \{u \in \mathbb{R}^{n_u} | H_u u \leq K_u\}. \end{aligned} \tag{6}$$

For the analysis of offset, the dynamic modes of the reference signal and the disturbance are of paramount importance. In fact, we will see that the analysis can be restricted to these modes. For instance, if the reference is assumed to be constant ($A_r = I$), all signals can be assumed to be in steady-state which greatly simplifies analysis (Muske & Badgwell, 2002; Pannocchia, 2003; Pannocchia & Bemporad, 2007).

For more general signals, we introduce the following definition.

**Definition 1.** Let $\{s(t)\}_{t=0}^{\infty}$ be a signal with $s(t) \in \mathbb{C}$. We say the signal is generated by mode $\lambda$ with order $p$, if there exists a generating linear system with $x_s(0) \in \mathbb{C}^p$ and $C_s \in \mathbb{R}^{1\times p}$ such that

$$s(t) = C_s x_s(t), \qquad x_s(t+1) = J_{\lambda,p} x_s(t) \quad t = 0, 1, \ldots \tag{7}$$

where the $J_{\lambda,p}$ is a Jordan block matrix for $\lambda$ with order $p$, i.e. $J_{\lambda,p} \in \mathbb{C}^{p\times p}$ and

$$J_{\lambda,p} = \begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \ldots & & \lambda \end{bmatrix}. \tag{8}$$

Since the generating system (7), (8) is not unique, it is not straightforward to check if a given signal is generated by a specific mode. Moreover, for every component in a vector signal, the parameters $C_s$ and $x_s(0)$ have to be determined. In the following proposition, we present a simple check which can be used in the analysis.

**Proposition 1.** *Consider the signal $s(\cdot) \in \mathbb{C}$. Define the sequence*

$$\begin{aligned} s_p^\lambda(t) &= s(t), \\ s_{j-1}^\lambda(t) &= s_j^\lambda(t+1) - \lambda s_j^\lambda(t), \quad j = 1, \ldots, p,\ t = 0, 1, \ldots. \end{aligned} \tag{9}$$

*The signal is generated by mode $\lambda$ with order $p$ if and only if*

$$s_0^\lambda(t) = 0, \quad t = 0, 1, \ldots. \tag{10}$$

**Proof.** ($\Rightarrow$) Given $\{s(t)\}_{t=0}^{\infty}$ and assuming (9) holds, choose $C_s = [1\ 0\ \ldots\ 0]$ and $x_s(t) = [s_p(t)^T \ldots s_1(t)^T]^T$. Using (7) yields $x_{s,j-1}^\lambda(t) = x_{s,j}^\lambda(t+1) - \lambda x_{s,j}^\lambda(t) = [s_{j-1}(t)^T \ldots s_1(t)^T\ 0^T \ldots 0^T]^T$. It follows that $C_s x_{s,j}^\lambda(t) = s_j^\lambda(t)$ and $C_s x_{s,0}^\lambda(t) = 0$. Hence the parameters $x_s(t)$ and $C_s$ in fact generate the sequence $\{s(t)\}_{t=0}^{\infty}$ at time $t$. By induction, we again determine an initial condition for $t+1$: $x_s(t+1) = [s_p(t+1)^T \ldots s_1(t+1)^T]^T$ which generates a valid sequence. Using (7) and (9) yields $J_{\lambda,p}x(t) = [\lambda s_p(t) + s_{p-1}(t) \ldots \lambda s_2(t) + s_1(t), \lambda s_1(t)] = x(t+1)$. Hence the initial condition $x_s(t)$ generates a valid signal also for $t+1$, which concludes the induction step. Clearly we can choose $t = 0$ in the first step, thus (9) holds for all $t \geq 0$.

($\Leftarrow$) Assume $s(t)$ is generated by the mode $\lambda$ with order $p$ and $x_s(0)$, $C_s$ are known such that (7) holds. By substituting into (9) we get $s_p(t) = C(J_{\lambda,p} - \lambda I)^p J_{\lambda,p}^t x(0)$. Since $J_{\lambda,p}^t - \lambda I$ is nilpotent of degree $p$, $(J_{\lambda,p}^t - \lambda I)^p = 0$. Eq. (10) follows. □

Of further interest is the discrete composition of modal signals:

$$s(\cdot) = \sum_i s_{p_i}^{\lambda_i}(\cdot) \tag{11}$$

where each $s_{p_i}^{\lambda_i}(\cdot)$ is generated by the mode $\lambda_i$ with order $p_i$. Henceforth, the notation $s_{p_i}^{\lambda_i}(\cdot)$ will be used to denote a signal generated by the given mode.

Eqs. (9)–(11) can be used to determine whether a given signal $s(\cdot)$ admits a modal decomposition. As the equations are linear, they can be easily integrated into the optimization problem as constraints, as will be seen later.

## 3. Disturbance model

To account for plant-model mismatch and disturbances entering the plant, system (2) is augmented with a disturbance model

$$\begin{aligned} \begin{bmatrix} x(t+1) \\ d(t+1) \end{bmatrix} &= \begin{bmatrix} A & B_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C\ C_d \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, \end{aligned} \tag{12}$$

with $d(k) \in \mathbb{R}^{n_d}$, $A_d \in \mathbb{R}^{n_d\times n_d}$, $B_d \in \mathbb{R}^{n_x\times n_d}$ and $C_d \in \mathbb{R}^{n_y\times n_d}$.

In order to capture plant-model mismatch, the disturbance model must contain a model of the reference dynamics. Additional modes can be added to reject specific disturbance dynamics which are not part of the reference signal. Both types of modes need to be added for every output channel. We have the following definition.

**Definition 2.** We say $A_d$ incorporates an internal model of $A_r$ if the following holds:

(1) $A_d$ contains all unique eigenvalues of $A_r$.
(2) The geometric multiplicity of every eigenvalue of $A_d$ is $n_y$, i.e. $dim(\mathcal{N}(\lambda_i I - A_d)) = n_y$ for $\lambda_i \in \sigma(A_d)$.
(3) Every Jordan chain of $A_d$ corresponding to the eigenvalue $\lambda_i \in \sigma(A_d)$ is of the same length as the longest Jordan chain in $A_r$ corresponding to this eigenvalue.

In order to counteract disturbances and to follow reference signals, the model needs to be output controllable with respect to the modes given by $A_d$ and $A_r$. A mathematical condition for this is given in the following definition.

**Definition 3.** The reference tracking problem for model (3), (12) is said to be well-posed, if the following holds

$$rank\begin{bmatrix} A-\lambda_i I & B \\ C & 0 \end{bmatrix} = n_x + n_y, \quad \forall \lambda_i \in \sigma(A_d). \tag{13}$$

Eq. (13) implies that $n_u \geq n_y$. Furthermore, none of the unstable poles of $A_d$ (and thus $A_r$) must coincide with the transmission zeros of (2). Specifically, if (13) does not hold, one can always find a linear combination of output channels which cannot be controlled at the given mode, hence rendering reference tracking or disturbance rejection impossible.

It was shown in Davison (1976), that both the internal model condition and well-posedness have to hold for a tracking controller to exist.

In the following, we will present a method to create a canonical disturbance model such that $A_d$ incorporates an internal model of $A_r$ as given by Definition 2.

**Algorithm 1.** (1) Let $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m\}$ be the set of unique eigenvalues of $A_r$, i.e. $\tilde{\lambda}_i \in \sigma(A_r)$, $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ if $i \neq j$ and $m \leq n_r$. Furthermore, let $p_i$ be the maximum length of any Jordan chain in $A_r$ associated with the eigenvalue $\tilde{\lambda}_i$.
(2) Let $\tilde{A}_d = diag([J_{\tilde{\lambda}_1,p_1}, \ldots, J_{\tilde{\lambda}_m,p_m}])$.
(3) Select

$$A_d = \underbrace{\begin{bmatrix} \tilde{A}_d & & 0 \\ & \ddots & \\ 0 & & \tilde{A}_d \end{bmatrix}}_{n_y \times n_y}.$$

It is straightforward to check that the internal model condition holds. Next, the observability of the augmented system (12) hast to be ensured.

**Theorem 1.** *Suppose $(C, A)$ is observable. Then, there exist $A_d, B_d, C_d$ such that $A_d$ incorporates an internal model of $A_r$ and the augmented system (12) is observable.*

**Proof.** By employing the Hautus observability condition, a necessary and sufficient condition for (12) to be observable is

$$rank\begin{bmatrix} A-\lambda I & B_d \\ 0 & A_d-\lambda I \\ C & C_d \end{bmatrix} = n_x + n_d, \quad \forall \lambda \in \mathbb{C}.$$

By assumption of observability of $(C, A)$, $rank([(A-\lambda I)^T\ C^T]^T) = n_x\ \forall \lambda \in \mathbb{C}$, hence we need to check only the eigenvalues of $A_d$ and assure the right part of the matrix contributes $n_d$ linearly independent column vectors. Assume henceforth that $A_d$ is constructed by Algorithm 1. Then, $rank(A_d - \lambda_i I) = n_d - n_y$ for $\lambda_i \in \sigma(A_d)$, and there are $n_y$ zero columns in $(A_d - \lambda_i I)$. Since the $n_d - n_y$ non-zero columns are clearly linearly independent to the left part of the matrix irrespective of the choice of $B_d$, $C_d$, they can safely be removed from the Hautus condition, yielding

$$rank\begin{bmatrix} A-\lambda_i I & \bar{B}_d^{\lambda_i} \\ C & \bar{C}_d^{\lambda_i} \end{bmatrix} = n_x + n_y, \quad \forall \lambda_i \in \sigma A_d,$$

with $\bar{B}_d^{\lambda_i} \in \mathbb{R}^{n_x \times n_y}$, $\bar{D}_d^{\lambda_i} \in \mathbb{R}^{n_y \times n_y}$. It is clear that $\bar{B}_d^{\lambda_i}$ and $\bar{D}_d^{\lambda_i}$ can be chosen freely such that the condition holds. Since every $\lambda_i$ selects different columns of $A_d$, this argument can be repeated for all $\lambda_i$. □

Assuming observability of the augmented system, a standard linear observer is employed to obtain estimates of the state and disturbance vectors:

$$\begin{bmatrix} \hat{x}(t+1) \\ \hat{d}(t+1) \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L_x \\ L_d \end{bmatrix} (y_\phi(t) - C\hat{x}(t) - C_d\hat{d}(t)), \tag{14}$$

where $L_x$ and $L_d$ are chosen such that the estimator is stable. Let the estimation error be given by

$$\epsilon(t) = C\hat{x}(t) + C_d\hat{d}(t) - y_\phi(t). \tag{15}$$

The specific method used for choosing $L_x$ and $L_d$ is not relevant in this context, any method can be employed. In the following propositions, we will analyze the properties of the estimator at the modes given by $A_r$ and $A_d$. Since $A_d$ is assumed to contain all modes of $A_r$, we may in fact restrict analysis to the modes of $A_d$. These results will later be used in the analysis of tracking offset.

**Proposition 2.** *Assume the observer (14) is stable and $A_d$ incorporates an internal model of $A_r$. Then, $rank(L_d) = n_y$ and for all $\lambda_i \in \sigma(A_d)$, $\mathcal{R}(L_d) \cap \mathcal{R}(A_d - \lambda_d I) = \{0\}$.*

**Proof.** By stability of (14), the matrix

$$M(\lambda) = \begin{bmatrix} A - L_xC - \lambda I & B_d - L_xC_d \\ -L_dC & A_d - L_dC_d - \lambda I \end{bmatrix}$$

must have full row rank for $|\lambda| \geq 1$. Consider the particular cases where $\lambda_i \in \sigma(A_d)$, which are unstable eigenvalues by assumption. For $M(\lambda_i)$ to have full rank, a necessary condition is $rank[L_dC\ (A_d - L_dC_d - \lambda_i I)] = n_d$. By the internal model condition, $rank(A_d - \lambda_i I) = n_d - n_y$. Hence, $N = L_d[C\ C_d]$ must contribute $n_y$ dimensions to the column space. Since $N \in \mathbb{R}^{n_d \times n_y}$, $N$ contributes at most $n_y$ dimensions. Furthermore, $rank(N) \leq rank(L_d)$. Hence, for $N$ to contribute $n_y$ dimensions, $rank(L_d) = n_y$ must hold and every column vector of $N$ must be disjoint from the column space of $(A_d - \lambda_i I)$. Because of the full row rank of $C$, a necessary condition is that the range spaces of $L_d$ and $(A_d - \lambda_i I)$ are disjoint. □

**Proposition 3.** *Consider observer (14). Assume the observer is stable and the following modal decompositions exist*

$$y_\phi(\cdot) = \sum_{i=1}^{m} y_{p_i}^{\lambda_i}(\cdot), \qquad u(\cdot) = \sum_{i=1}^{m} u_{p_i}^{\lambda_i}(\cdot), \tag{16}$$

*where $\lambda_i$ is the ith eigenvalue of $A_d$ with longest Jordan chain of length $p_i$. Then the following holds*

$$(A - \lambda_i I)\hat{x}_j^{\lambda_i}(t) + Bu_j^{\lambda_i}(t) + B_d\hat{d}_j^{\lambda_i}(t) = \hat{x}_{j-1}^{\lambda_i}(t), \tag{17a}$$

$$(A_d - \lambda_i I)\hat{d}_j^{\lambda_i}(t) = \hat{d}_{j-1}^{\lambda_i}(t), \tag{17b}$$

$$\epsilon_j^{\lambda_i}(t) = 0 \tag{17c}$$

$$j = 1, \ldots, p,\ i = 1, \ldots, m,\ t = 0, 1, \ldots$$

*with*

$$\hat{x}(\cdot) = \sum_{i=1}^{m} \hat{x}_{p_i}^{\lambda_i}(\cdot), \qquad \hat{d}(\cdot) = \sum_{i=1}^{m} \hat{d}_{p_i}^{\lambda_i}(\cdot). \tag{18}$$

**Proof.** The proof is by induction. Inserting (16) and (9) into (14) and introducing $\epsilon_j^{\lambda_i}(t) = C\hat{x}_j^{\lambda_i}(t) + C_d\hat{d}_j^{\lambda_i}(t) - y_p^{\lambda_i}(t)$ yields

$$\begin{bmatrix} \hat{x}_{j-1}^{\lambda_i}(t) \\ \hat{d}_{j-1}^{\lambda_i}(t) \end{bmatrix} = \begin{bmatrix} A-\lambda_i I & B_d \\ 0 & A_d-\lambda_i I \end{bmatrix} \begin{bmatrix} \hat{x}_j^{\lambda_i}(t) \\ \hat{d}_j^{\lambda_i}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_j^{\lambda_i}(t) + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \epsilon_j^{\lambda_i}(t)$$

for $j = 1, \ldots, p_i$. Let $E_j = \mathcal{N}(A_d - \lambda_i I)^j$ and fix $\hat{d}_0^{\lambda_i}(t) = 0$. For $j = 1$, it follows $0 = (A_d - \lambda_i I)\hat{d}_1^{\lambda_i}(t) - L_d\epsilon_1^{\lambda_i}(t) \Rightarrow L_d\epsilon_1^{\lambda_i}(t) = 0 \Rightarrow \epsilon_1^{\lambda_i}(t) = 0$ by Proposition 2; obviously, $\hat{d}_1^{\lambda_i}(t) \in E_1$. For any $\hat{d}_{j-1}^{\lambda_i}(t) \in E_{j-1}$, $\exists\hat{d}_j^{\lambda_i}(t)$ such that $(A_d - \lambda_i I)\hat{d}_j^{\lambda_i}(t) = \hat{d}_{j-1}^{\lambda_i}(t)$. Hence, $\hat{d}_{j-1}^{\lambda_i}(t) \in E_{j-1} \Rightarrow \hat{d}_{j-1}^{\lambda_i}(t) \in \mathcal{R}(A_d - \lambda_i I)$ and thus $L_d\epsilon_j^{\lambda_i}(t) = 0 \Rightarrow \epsilon_j^{\lambda_i}(t) = 0$ for $j = 1, \ldots, p_i$. □

**Remark 1.** From (17b) and (17c), we see that a necessary and sufficient condition for $\hat{d}(t)$ to be a signal generated by mode $\lambda_i$ with order $p_i$ is that it lies in the generalized eigenspace of $A_d$ associated with $\lambda_i$. That is, $\hat{d}(t) \in \mathcal{N}(A_d - \lambda_i I)^{p_i}$.

## 4. Controller

In the following, we will present a method for the construction of an MPC which achieves target tracking.

### *4.1. Target trajectory*

In regulation problems, the goal is to bring the system state to the origin, where the system is in equilibrium. When tracking unstable reference signals such as sinusoids, we want to allow some modes to be non-zero. We capture this by introducing the notion of target trajectories, which describe the set of 'good' states, analogous to the origin in the regulator problem.

A target trajectory is defined as a sequence of states and inputs yielding the desired output for a given reference and disturbance signal. It is defined for all future time instants $t' \geq t$. For simplicity, we will use $k = t' - t$ for signal predictions at time $t$. The future predicted disturbance is given based on the current estimate by

$$\bar{d}_t(k) = A_d^k\hat{d}(t). \tag{19}$$

Similarly, as the reference signal is generated by (3), it is given by

$$\bar{r}_t(k) = C_r A_r^k x_r(t). \tag{20}$$

Define the target trajectory at time $t$ to be $\{(\bar{x}_t(k), \bar{u}_t(k))\}_{k=0}^{\infty}$. It must satisfy

$$\begin{bmatrix} \bar{x}_t(k+1) \\ \bar{r}_t(k) \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_t(k) \\ \bar{u}_t(k) \end{bmatrix} + \begin{bmatrix} B_d\bar{d}_t(k) \\ C_d\bar{d}_t(k) \end{bmatrix} \quad k = 0, 1, \ldots, \tag{21}$$

Eq. (21) could potentially be used directly to find a target trajectory when only a finite horizon is considered. However, if $\bar{x}_t(k)$, $\bar{u}_t(k)$ solve (21) for a given disturbance and reference for $k \in \{0, \ldots, T\}$ with $T < \infty$, there is no guarantee that the problem remains feasible for a larger horizon $T' = T + 1$.

In the following, we propose a method to establish the invariant property of a target solution at a given time instant, thus automatically preserving feasibility for the next time instant. We tackle the problem by employing a modal decomposition of (21).

Let $A_d$ have $m$ distinct eigenvalues with $\lambda_i$ being the $i$th eigenvalue. Denote the length of the longest Jordan chain associated with this eigenvalue by $p_i$. Assume $A_d$ incorporates an internal model of $A_r$ as given by Definition 2. Then, the following modal decomposition exists

$$\bar{r}_t(\cdot) = \sum_{i=1}^{m} \bar{r}_{t,p_i}^{\lambda_i}(\cdot), \qquad \bar{d}_t(\cdot) = \sum_{i=1}^{m} \bar{d}_{t,p_i}^{\lambda_i}(\cdot). \tag{22}$$

Introducing the signals $\bar{x}_{t,j}^{\lambda_i}(\cdot)$ and $\bar{u}_{t,j}^{\lambda_i}(\cdot)$, (21) is expressed in modal form. We first state the modal sub-problem:

$$\begin{bmatrix} A-\lambda_i I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{t,j}^{\lambda_i}(k) \\ \bar{u}_{t,j}^{\lambda_i}(k) \end{bmatrix} - \begin{bmatrix} \bar{x}_{t,j-1}^{\lambda_i}(k) \\ 0 \end{bmatrix} = \begin{bmatrix} -B_d\bar{d}_{t,j}^{\lambda_i}(k) \\ \bar{r}_{t,j}^{\lambda_i}(k) - C_d\bar{d}_{t,j}^{\lambda_i}(k) \end{bmatrix}, \quad j = 1, \ldots, p_i, \tag{23}$$

where we fix $\bar{x}_{t,0}^{\lambda_i}(k) := 0$. From the modal decomposition, we also have $\bar{r}_{t,0}^{\lambda_i}(k) = 0$ and $\bar{d}_{t,0}^{\lambda_i}(k) = 0$. To retain equivalence to (21), the modal sub-problem is required to hold for all modes and all time

$$i = 1, \ldots, m, \quad k = 0, 1, \ldots. \tag{24}$$

The solution of the target trajectory problem is recovered by superposition of the solutions to the modal sub-problems

$$\bar{x}_t(\cdot) = \sum_{i=1}^{m} \bar{x}_{t,p_i}^{\lambda_i}(\cdot), \qquad \bar{u}_t(\cdot) = \sum_{i=1}^{m} \bar{u}_{t,p_i}^{\lambda_i}(\cdot). \tag{25}$$

**Proposition 4.** *Consider the modal target trajectory sub-problem (23) for mode $\lambda_i$ with corresponding chain length $p_i$ at time $k$. Assume the tracking problem is well-posed, i.e. (13) holds. Then, a solution $\{\bar{x}_{t,p_i}^{\lambda_i}(k), \bar{y}_{t,p_i}^{\lambda_i}(k)\}_{k=0}^{\infty}$ exists for any given sequences $\{\bar{d}_t^{\lambda_i}(k)\}_{k=0}^{\infty}$ and $\{\bar{r}_t^{\lambda_i}(k)\}_{k=0}^{\infty}$ generated by mode $\lambda_i$ with order $p_i$.*

**Proof.** Rewriting (23) yields

$$\begin{bmatrix} A-\lambda_i I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{t,j}^{\lambda_i}(k) \\ \bar{u}_{t,j}^{\lambda_i}(k) \end{bmatrix} = \begin{bmatrix} \bar{x}_{t,j-1}^{\lambda_i}(k) \\ 0 \end{bmatrix} + \begin{bmatrix} -B_d\bar{d}_{t,j}^{\lambda_i}(k) \\ \bar{r}_{t,j}^{\lambda_i}(k) - C_d\bar{d}_{t,j}^{\lambda_i}(k) \end{bmatrix}, \quad j = 1, \ldots, p_i.$$

By the well-posedness, the matrix on the left hand side has full row rank, hence a solution exists for $j = 1$, since $\bar{x}_{t,0}^{\lambda_i}(k) = 0$, $\bar{d}_{t,0}^{\lambda_j}(k) = 0$. The argument can then be repeated for $j \leftarrow j + 1$, until $\bar{x}_{t,p_i}^{\lambda_i}(k)$ and $\bar{u}_{t,p_i}^{\lambda_j}(k)$ are determined. □

Define

$$\bar{v}_{t,j}^{\lambda_i}(k) = \begin{bmatrix} \bar{x}_{t,j}^{\lambda_i}(k) \\ \bar{u}_{t,j}^{\lambda_i}(k) \end{bmatrix}, \qquad \bar{w}_{t,j}^{\lambda_i}(k) = \begin{bmatrix} \bar{d}_{t,j}^{\lambda_i}(k) \\ \bar{r}_{t,j}^{\lambda_i}(k) \end{bmatrix} \tag{26}$$

and write (23) as

$$M_1^{\lambda_i}\bar{v}_{t,j}^{\lambda_i}(k) + M_2\bar{v}_{t,j-1}^{\lambda_i}(k) = M_3\bar{w}_{t,j}^{\lambda_i}(k), \quad j = 1, \ldots, p_i. \tag{27}$$

**Proposition 5.** *Consider the modal target trajectory sub-problem (27) for mode $\lambda_i$ with order $p_i$. Let $\{\bar{v}_j^{\star}\}_{j=0}^{p_i}$ with $\bar{v}_0^{\star} = 0$ denote a solution of (27) at time $k$. Then, $\{\lambda_i\bar{v}_j^{*} + \bar{v}_{j-1}^{*}\}_{j=0}^{p_i}$ with $\bar{v}_{-1}^{\star} = 0$ is a solution at time $k + 1$.*

**Proof.** Assume the proposition holds. From (9) we get $\bar{w}_{t,j}^{\lambda_i}(k+1) = \lambda \bar{w}_{t,j}^{\lambda_i}(k) + \bar{w}_{t,j-1}^{\lambda_i}(k)$. Inserting the new solution into (27) yields

$$\begin{aligned} &M_1^{\lambda_i}(\lambda_i \bar{v}_j^\star + \bar{v}_{j-1}^*) + M_2(\lambda_i \bar{v}_{j-1}^* + \bar{v}_{j-2}^*) \\ &\quad = M_3(\lambda \bar{w}_{t,j}^{\lambda_i}(k) + \bar{w}_{t,j-1}^{\lambda_i}(k)), \quad j = 1, \dots, p_i \end{aligned}$$

which clearly holds since $\{\bar{v}_j^*\}_{j=0}^{p_i}$ is a solution of (27). □

**Remark 2.** Contrary to the time-domain target problem (21), trajectories resulting from the modal problem (22)–(25) are invariant. Given a solution at time $k$, a solution exists at $k+1$. Moreover, future solutions can be found by linear combination of the current solution. Therefore, this method allows for generating infinite-time target trajectories.

### *4.2. The MPC algorithm*

Define the set of predicted input variables by

$$U_t = \{u_t(k)\}_{k=0}^{N-1}. \tag{28}$$

The decision variables introduced by the target trajectory problem are

$$T_t = \bigcup_{i=1}^{m} \bigcup_{j=0}^{p_j} \{(\bar{x}_{t,j}^{\lambda_i}(k), \bar{u}_{t,j}^{\lambda_i}(k))\}_{k=0}^{N}. \tag{29}$$

The MPC optimization problem is then posed as follows.

$$\begin{aligned} \min_{U_t, T_t} \quad & \sum_{k=0}^{N-1} \|x_t(k) - \bar{x}_t(k)\|_Q^2 + \|u_t(k) - \bar{u}_t(k)\|_R^2 \\ & + \|x_t(N) - \bar{x}_t(N)\|_P^2 \\ \text{s.t.} \quad & x_t(k) \in \mathcal{X}, \quad k = 1, \dots, N, \\ & u_t(k) \in \mathcal{U}, \quad k = 0, \dots, N-1, \\ & x_t(k+1) = Ax_t(k) + Bu_t(k) + B_d \hat{d}_t(k), \\ & k = 0, \dots, N \\ & x_t(0) = \hat{x}(t), \\ & (19), (20), (22)\text{–}(25) \end{aligned} \tag{30}$$

with $\|x\|_M^2 \triangleq x^T M x$. We assume $Q$, $P$, $R$ and $N$ are selected for the nominal closed loop system to be stable, and that the optimization problem is feasible for all time instants.

Let $U_t^\star = \{u_t^\star(0), \dots, u_t^\star(N-1)\}$ be the optimal input obtained by solving (30) at time $t$. Then, the first sample of $U_t^\star$ is applied to the system (1)

$$u(t) = u_t^\star(0). \tag{31}$$

**Remark 3.** The target trajectory problem introduces numerous decision variables to the optimization problem. One could expect that this might adversely affect the time needed by the solver for solving one problem instance. In practice however, the structure added by the (22)–(25) is very sparse and can usually be exploited well by standard solvers, such that the impact on the computation time is small.

## 5. Analysis of tracking offset

In Maeder and Morari (2007), Muske and Badgwell (2002), Pannocchia (2003) and Pannocchia and Bemporad (2007), the signals of the closed loop were analyzed at steady-state to derive conditions for offset-free control. In a similar way, we will analyze the behaviour of the closed loop when all modes except those contained in $A_d$ are zero. We will denote such signals by the superscript "$\infty$" in the following.

Consider system (1) under closed loop control using controller (14), (30). Since (1) is formulated in a completely general form, we need the following assumptions.

**Assumption 1.** The MPC (30) is feasible for all $t \geq 0$.

**Assumption 2.** The measurement signal $y_\phi(t)$ converges such that

$$y_\phi(t) \to y^\infty(t) \tag{32}$$

as $t \to \infty$ and the modal decomposition

$$y^\infty(\cdot) = \sum_{i=1}^{m} y_{p_i}^{\lambda_i}(\cdot) \tag{33}$$

holds where $\lambda_i$ is the $i$th unique eigenvalue of $A_d$, $p_i$ the length of the longest Jordan chain associated with this eigenvalue, and the number of unique eigenvalues of $A_d$ is $m$ with $m \leq n_d$.

**Remark 4.** Assumption 2 requires the controller to stabilize all modes not contained in $A_d$. The remaining, non-zero modes may originate either from disturbances acting on the plant, the reference signal or the controller compensating for disturbances or reference.

**Assumption 3.** There exists $t^\star \geq 0$ such that for $t \geq t^\star$ the MPC control law is strictly feasible and given by the linear feedback policy

$$u(t) - \bar{u}_t^\star(0) = K_x(\hat{x}(t) - \bar{x}_t^\star(0)) \tag{34}$$

where $\bar{u}_t^\star(0)$ and $\bar{x}_t^\star(0)$ are the optimal values determined by the MPC. Furthermore, let $A + BK_x$ be stable.

By (34) and Proposition 3, it is clear that the remaining signals can be defined accordingly. Let

$$u(t) \to u^\infty(t), \qquad \hat{x}(t) \to \hat{x}^\infty(t), \qquad \hat{d}(t) \to \hat{d}^\infty(t) \tag{35}$$

for $t \to \infty$, where

$$\begin{bmatrix} u^\infty(\cdot) \\ \hat{x}^\infty(\cdot) \\ \hat{d}^\infty(\cdot) \end{bmatrix} = \sum_{i=1}^{m} \begin{bmatrix} u_{p_i}^{\lambda_i}(\cdot) \\ \hat{x}_{p_i}^{\lambda_i}(\cdot) \\ \hat{d}_{p_i}^{\lambda_i}(\cdot) \end{bmatrix} \tag{36}$$

holds.

**Remark 5.** In general, Assumptions 2 and 3 can only hold if (1) is a linear system. Trivial exceptions for nonlinear plants exist for step disturbance models, or if some modes $\lambda_i$ are not excited by a disturbance or reference signal. In most cases however, if (1) contains nonlinearities, Assumptions 2 and 3 will not hold. In these cases, the proposed method may help to reduce – but not eliminate – the offset, depending on the nonlinearities.

Denote the tracking error of the attractive trajectory by

$$e^\infty(t) = y^\infty(t) - r^\infty(t). \tag{37}$$

In the following, we will show that under the given assumptions, $e^\infty(t) = 0$ and hence, due to the convergence properties (32), (35), offset is removed from the output.

First, consider the MPC problem (30) for the attractive trajectory at time $t$. Introducing the variables

$$\delta x_t(k) = x_t(k) - \bar{x}_t(k), \qquad \delta u_t(k) = u_t(k) - \bar{u}_t(k), \tag{38}$$

the target dynamics induced by disturbance estimate and reference signal can be decoupled from the system dynamics. Eq. (30) is then formulated as follows

$$\begin{aligned} \min_{\delta U_t, T_t} \quad & \sum_{k=0}^{N-1} \|\delta x_t(k)\|_Q^2 + \|\delta u_t(k)\|_R^2 + \|\delta x_t(N)\|_P^2 \\ \text{s.t.} \quad & \delta x_t(k) + \bar{x}_t(k) \in \mathcal{X}, \quad k = 1, \dots, N, \\ & \delta u_t(k) + \bar{u}_t(k) \in \mathcal{U}, \quad k = 0, \dots, N-1, \\ & \delta x_t(k+1) = A\delta x_t(k) + B\delta u_t(k), \quad k = 0, \dots, N \\ & \delta x_t(0) = \hat{x}^\infty(t) - \bar{x}_t(0), \\ & (19), (20), (22)\text{–}(25). \end{aligned} \tag{39}$$

Denote by $\delta U_t^\star$, $T_t^\star$ the optimal solution to (39). The input value applied to the plant is

$$u(t) = \delta u_t^\star(0) + \bar{u}_t^\star(0). \tag{40}$$

Assuming $t \geq t^{\star}$ and using (34) yields

$$\delta u(t) = K_x \delta x(t). \tag{41}$$

From (22)–(25), we see that the target trajectory also admits the modal decomposition

$$\begin{bmatrix} \bar{x}_t^{\star}(\cdot) \\ \bar{u}_t^{\star}(\cdot) \end{bmatrix} = \sum_{i=1}^{m} \begin{bmatrix} \bar{x}_{t,p_i}^{\lambda_i}(\cdot) \\ \bar{u}_{t,p_i}^{\lambda_i}(\cdot) \end{bmatrix}. \tag{42}$$

Combining (17), (23) and (41) yields

$$\begin{bmatrix} A-\lambda_i I & B_d & B & 0 & 0 & L_x \\ 0 & 0 & 0 & 0 & 0 & L_d \\ 0 & B_d & 0 & A-\lambda_i I & B & 0 \\ -C & 0 & 0 & C & 0 & I \\ K_x & 0 & -I & -K_x & I & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_j^{\lambda_i}(k) \\ \hat{d}_j^{\lambda_i}(k) \\ u_j^{\lambda_i}(k) \\ \bar{x}_{t,j}^{\lambda_i}(k) \\ \bar{u}_{t,j}^{\lambda_i}(k) \\ \epsilon_j^{\lambda_i}(k) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{x}_{j-1}^{\lambda_i}(k) \\ 0 \\ \bar{x}_{t,j-1}^{\lambda_i}(k) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e_j^{\lambda_i}(k) \\ 0 \end{bmatrix},$$

$$j = 1, \dots, p_i \; i = 1, \dots, m, \; k = 0, 1, \dots \tag{43}$$

for $j = 1, \dots, p_i$, $i = 1, \dots, m$ and $k = 0, 1, \dots$. Using $\delta x_j^{\lambda_i}(k) = \hat{x}_j^{\lambda_i}(k) - \bar{x}_{t,j}^{\lambda_i}(k)$ we obtain

$$\begin{bmatrix} A+BK_x-\lambda_i I & L_x \\ -C & I \\ 0 & L_d \end{bmatrix} \begin{bmatrix} \delta x_j^{\lambda_i}(k) \\ \epsilon_j^{\lambda_i}(k) \end{bmatrix} = \begin{bmatrix} \delta x_{j-1}^{\lambda_i}(k) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_j^{\lambda_i}(k) \\ 0 \end{bmatrix},$$

$$j = 1, \dots, p_i, \; i = 1, \dots, m, \; k = 0, 1, \dots \tag{44}$$

**Theorem 2.** *Consider the closed loop system with estimator (14) and controller (30), (31). Assume $A_d$ incorporates an internal model of $A_r$, the problem is well-posed, the estimator is stable and Assumptions 1–3 are satisfied. Then, $e(t) \to 0$ for $t \to \infty$.*

**Proof.** Since $e(t) \to e^{\infty}(t)$ as $t \to \infty$ by assumption, it suffices to show that $e^{\infty}(t) = 0$ for all $t$. Consider mode $\lambda_i$ with order $p_i$. By Proposition 2, $L_d$ has full column rank. Hence, $\epsilon_j^{\lambda_i} = 0$ in (44). Rewriting yields

$$\begin{bmatrix} A+BK_x-\lambda_i I \\ -C \end{bmatrix} \delta x_j^{\lambda_i}(t) = \begin{bmatrix} \delta x_{j-1}^{\lambda_i}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_j^{\lambda_i}(t) \end{bmatrix}$$

By stability of $A + BK_x$, $\lambda_i \in \sigma(A_d)$ is not an eigenvalue of $A + BK_x$. It follows from the first row that $\delta x_j^{\lambda_i}(t) = 0$ for $j = 1, \dots, p_i$. From $e_j^{\lambda_i}(t) = -C\delta x_j^{\lambda_i}(t)$ we have that $e_j^{\lambda_i}(t) = 0$. By $e^{\infty}(t) = \sum_{i=1}^{m} e_{p_i}^{\lambda_i}(t)$ it follows $e^{\infty}(t) = 0$. □

### 5.1. Method summary

We briefly summarize the main steps of the procedure proposed in this paper. These are:

(1) Choose a reference model $A_r$, $C_r$.
(2) Choose a disturbance model $A_d$, $B_d$, $C_d$. The dynamics matrix $A_d$ must incorporate an internal model of $A_r$. It may contain additional dynamics of expected disturbances. The parameters $B_d$ and $C_d$ are chosen such that the augmented system (12) is observable.
(3) Compute the estimator gain $L$.

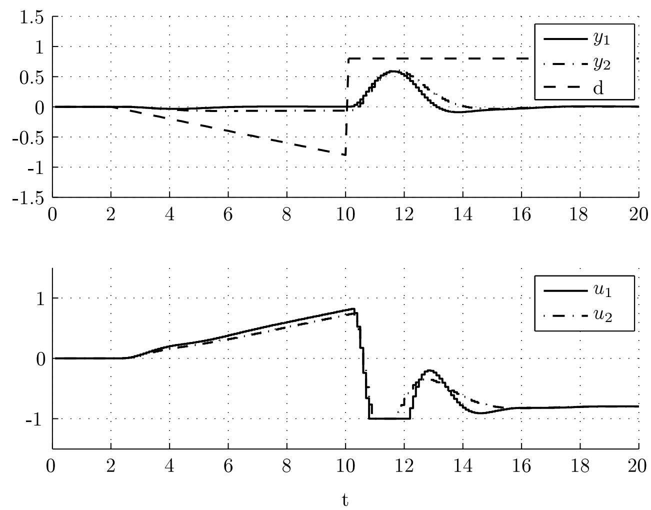


**Fig. 1.** Comparison of closed loop responses to a disturbance signal. $(y_1, u_1)$ shows the response for the proposed control scheme, $(y_2, u_2)$ for the controller in Maeder and Morari (2007), Muske and Badgwell (2002) and Pannocchia and Rawlings (2003).

(4) Compute the set of constraints for the target trajectory problem (22)–(25). Integrate them into MPC problem (30).

The crucial point of the procedure is clearly the choice of a disturbance model for which the internal model condition holds and which preserves observability of the system.

## 6. Example

As an example, we study a simple damped spring–mass system:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -\rho \end{bmatrix} x(t) + k/m \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{45}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{46}$$

with $k = 1$, $m = 1$ and $\rho = 0.1$. The real plant however shall be slightly perturbed with $k = 1.2$ and $\rho_m = 0.09$. The goal is to track ramp references and reject ramp disturbances with zero offset. A further requirement is that the input variable is to be constrained

$$|u(t)| < 1. \tag{47}$$

First, the discrete-time matrices are obtained for a sampling time of $T_s = 0.1$ s. To be able to track the ramp, we chose the following reference signal generator

$$A_r = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C_r = [1\ 0]. \tag{48}$$

Since the internal generator state $x_r(t)$ is not accessible, a standard linear estimator is employed to produce the estimate $\hat{x}_r(t)$. Eq. (20) is thus changed to

$$\bar{r}_t(k) = C_r A_r^k \hat{x}_r(t). \tag{49}$$

We choose a disturbance model which contains the reference dynamics and which enters the plant at the input

$$A_d = A_r, \qquad B_d = BC_r, \qquad C_d = 0. \tag{50}$$

The size of the largest Jordan block of $A_d$ and $A_r$ is 2. Since there is only one measured output, we need to add only one disturbance block to satisfy the internal model condition. The observer gain $L$ is computed by solving the discrete-time algebraic Riccati equation with unit weights. The target trajectory problem is posed as follows. First note that the modal decomposition (22) of reference and disturbance is trivial, as there is only the mode $\lambda_1 = 1$. The mode subscript $\lambda_i$ will thus be omitted in the following.

Stating the chain conditions yields

$$\begin{aligned}
&\bar{d}_{t,2}(k) = A_d^k \hat{d}(t),\\
&\bar{d}_{t,1}(k) = \bar{d}_{t,2}(k+1) - \bar{d}_{t,2}(k),\\
&0 = \bar{d}_{t,1}(k+1) - \bar{d}_{t,1}(k),\\
&\bar{r}_{t,2}(k) = C_r A_r^k \hat{x}_r(t),\\
&\bar{r}_{t,1}(k) = \bar{r}_{t,2}(k+1) - \bar{r}_{t,2}(k),\\
&0 = \bar{r}_{t,1}(k+1) - \bar{r}_{t,1}(k).
\end{aligned} \tag{51}$$

The target trajectory conditions are

$$\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{t,2}(k) \\ \bar{u}_{t,2}(k) \end{bmatrix} - \begin{bmatrix} \bar{x}_{t,1}(k) \\ 0 \end{bmatrix} = \begin{bmatrix} -B_d \bar{d}_{t,2}(k) \\ \bar{r}_{t,2}(k) - C_d \bar{d}_{t,2}(k) \end{bmatrix}, \tag{52}$$

$$\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{t,1}(k) \\ \bar{u}_{t,1}(k) \end{bmatrix} = \begin{bmatrix} -B_d \bar{d}_{t,1}(k) \\ \bar{r}_{t,1}(k) - C_d \bar{d}_{t,1}(k) \end{bmatrix}.$$

Recovering the target trajectory is straightforward. We have

$$\bar{x}_t(k) = \bar{x}_{t,2}(k), \qquad \bar{u}_t(k) = \bar{u}_{t,2}(k). \tag{53}$$

The MPC problem is given by

$$\begin{aligned}
\min_{U_t, T_t} \quad & \sum_{k=0}^{N-1} \|x_t(k) - \bar{x}_t(k)\|_Q^2 + \|u_t(k) - \bar{u}_t(k)\|_R^2 \\
& + \|x_t(N) - \bar{x}_t(N)\|_P^2 \\
\text{s.t.} \quad & x_t(k) \in \mathcal{X}, \quad k = 1, \ldots, N, \\
& u_t(k) \in \mathcal{U}, \quad k = 0, \ldots, N-1, \\
& x_t(k+1) = Ax_t(k) + Bu_t(k) + B_d \bar{d}_{t,2}(k), \quad k = 0, \ldots, N \\
& x_t(0) = \hat{x}(t), \\
& (51)\text{–}(53).
\end{aligned} \tag{54}$$

In the following plots, $y_1$ designates the output of the system under closed loop control of the proposed controller. The method proposed in Maeder and Morari (2007), Muske and Badgwell (2002) and Pannocchia and Rawlings (2003) has also been implemented for illustrative purposes, designated $y_2$ in the plots. Essentially, the controller in Maeder and Morari (2007), Muske and Badgwell (2002) and Pannocchia and Rawlings (2003) contains a step disturbance model.

Fig. 1 shows the closed loop response to a ramp disturbance at $t = 2$ and a step disturbance at $t = 10$. Similarly, Fig. 2 shows the closed loop response to reference changes. It can be observed that the proposed controller ($y_1$) both achieves tracking of the reference and rejection of the disturbance, while the controller from literature ($y_2$) achieves this only for the case when disturbance and reference are constant. The transient performance of both controllers is very similar.

To demonstrate the ability of the proposed control scheme to handle arbitrary unstable disturbances, we finally consider the disturbance generated by the following unstable and oscillating dynamics

$$\begin{aligned}
\dot{x}_d(t) &= \begin{bmatrix} .1 & \pi \\ -\pi & .1 \end{bmatrix} x_d(t), \\
d(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(t).
\end{aligned}$$

For brevity, we do not repeat the (straightforward) construction of the controller here. Closed loop responses are depicted in Fig. 3.

## 7. Conclusion

In this article, we presented a method for offset-free reference tracking and disturbance rejection of constrained systems by means of an MPC controller. We have generalized existing results by extending the class of considered disturbance and reference signals to signals generated by arbitrary unstable linear models. The crucial points of the method are the choice of a disturbance model satisfying the internal model condition, and the addition of target trajectory conditions to the MPC problem. The method was shown to remove offset under the assumptions of stability and feasibility of the closed loop.

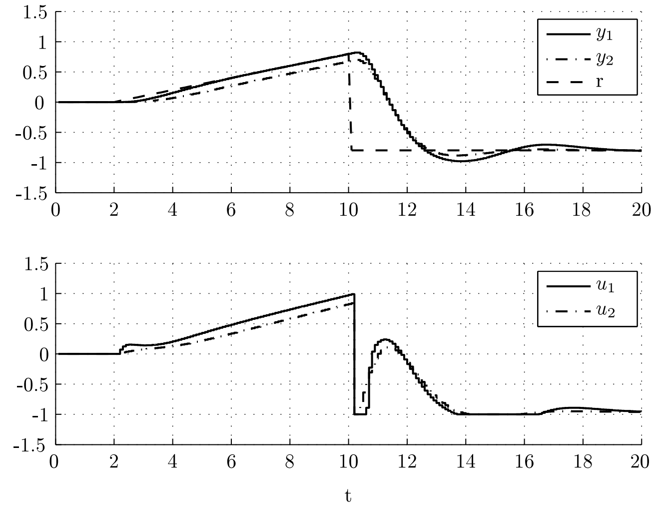


**Fig. 2.** Comparison of closed loop response to a reference signal. ($y_1, u_1$) shows the response for the proposed control scheme, ($y_2, u_2$) for the controller in Maeder and Morari (2007), Muske and Badgwell (2002) and Pannocchia and Rawlings (2003).

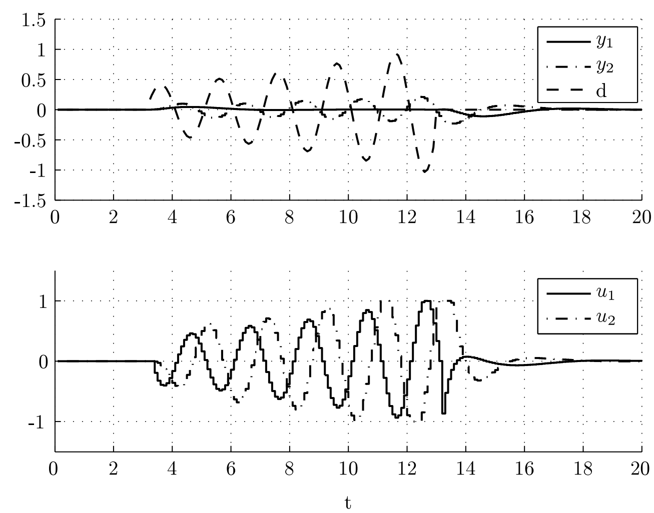


**Fig. 3.** Comparison of closed loop response to an unstable disturbance signal. ($y_1, u_1$) shows the response for the proposed control scheme, ($y_2, u_2$) for the controller in Maeder and Morari (2007), Muske and Badgwell (2002) and Pannocchia and Rawlings (2003).

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