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Brief paper

MPC for tracking piecewise constant references for constrained linear systems☆

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ABSTRACT

In this paper, a novel model predictive control (MPC) for constrained (non-square) linear systems to track piecewise constant references is presented. This controller ensures constraint satisfaction and asymptotic evolution of the system to any target which is an admissible steady-state. Therefore, any sequence of piecewise admissible setpoints can be tracked without error. If the target steady state is not admissible, the controller steers the system to the closest admissible steady state.

These objectives are achieved by: (i) adding an artificial steady state and input as decision variables, (ii) using a modified cost function to penalize the distance from the artificial to the target steady state (iii) considering an extended terminal constraint based on the notion of invariant set for tracking. The control law is derived from the solution of a single quadratic programming problem which is feasible for any target. Furthermore, the proposed controller provides a larger domain of attraction (for a given control horizon) than the standard MPC and can be explicitly computed by means of multiparametric programming tools. On the other hand, the extra degrees of freedom added to the MPC may cause a loss of optimality that can be arbitrarily reduced by an appropriate weighting of the offset cost term.

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1. Introduction

Model predictive control (MPC) is one of the few control techniques able to consider constraints, on both state and inputs of the system, in the design of the control law for linear, nonlinear or uncertain systems (Mayne, Rawlings, Rao, & Scokaert, 2000; Mhaskar, El-Farra, & Christofides, 2005). This is achieved by predicting the evolution of the system and computing the admissible sequence of control inputs which makes the system evolves satisfying the constraints. This calculation can be posed as an optimization problem which is solved at each sampling time, deriving the control law by means of a receding horizon policy. The theoretical foundations of MPC are well-known and under some assumptions, asymptotic stability of the origin is guaranteed. This is usually achieved by means of a suitable penalization of the terminal state and an additional terminal constraint (Mayne et al., 2000).

For practical application of model predictive controllers, these must be able to handle non-zero target steady states which are typically provided by a steady state target optimizer (Muske, 1997). The standard technique to deal with this problem is shifting the system state to the desired steady state (Muske & Rawlings, 1993). The finite control horizon and the constraint imposed on the terminal state may cause that the target steady state is not reachable and the feasibility of the optimization problem is not always ensured. This can be solved by re-calculating the control horizon and the terminal set for the new target steady state. Given the complexity of the necessary algorithms, the on-line re-calculation can be unaffordable. An off-line design of the controller for a finite set of target steady states might be a solution, but the set of targets to be tracked is limited. This tracking problem (the loss of feasibility) has motivated several solutions proposed in the literature.

Among the existing results dealing with the tracking problem in presence of constraints, a remarkable approach is the so-called command governors (Gilbert, Kolmanovsky, & Tan, 1994); this technique is based on the addition of a nonlinear low-pass filter of the reference to guarantee the admissible evolution of the system to the reference. This can be seen as adding an artificial reference (the output of the filter) which is computed at each sampling time to ensure the admissible evolution of the system, converging on the desired reference. In Bemporad, Casavola, and Mosca (1997) a command governor is designed to minimize a performance index of the predicted evolution of the system. In Blanchini and Miani (2000) it is proved that any control invariant set for the constrained system is a tracking domain of attraction and an interpolation-based control law is proposed.

In Rossiter, Kouvaritakis, and Gossner (1996), a Constrained Stable Generalized Predictive Controllers (CSGPC) for SISO plants is presented; this ensures feasibility by adding an artificial reference as decision variable and convergence is ensured by means of a

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contractive constraint based on the closest reachable reference. In Chisci and Zappa (2003) a dual-mode strategy for tracking based on MPC is presented: if the MPC is not feasible, the controller switches to a feasibility recovery mode which steers the system to the feasibility region of the MPC. In Pannocchia and Kerrigan (2005), the offset-free tracking problem is analyzed for uncertain systems with unmeasured disturbances; the proposed MPC considers the time-varying set point as a disturbance to be rejected. This means that the set of set points to be tracked is potentially small and the domain of attraction reduced.

In this paper, a novel formulation of the MPC is proposed for general non-square linear systems to track any admissible target steady state in an admissible evolution. The main ingredients are: (i) artificial steady state and input are considered as decision variables, (ii) the cost function penalizes the deviation between the artificial steady state and the desired one, and (iii) extended stabilizing terminal conditions are considered consisting of adding a tracking error penalty term in the cost function and adding a terminal constraint in both the terminal state and the artificial steady state and input.

This controller drives the system to any admissible target steady state, and if this is not admissible, the system is steered to the closest admissible steady state. Since the control law is derived from the solution of a single QP (for a given target steady state), the control law can be explicitly calculated by means of multiparametric programming tools (Bemporad, Morari, Dua, & Pistikopoulos, 2002), and therefore, could be applied on fast systems. Compared to the standard MPC, the proposed controller provides a larger domain of attraction (for the same control horizon), but the local optimality property cannot be ensured due to the extra degree of freedom added. Fortunately, the loss of optimality can be arbitrarily reduced by weighting the tracking error penalty term. Further results related to this controller can be found in Alvarado (2007).

This paper is organized as follows. In the next section, the constrained tracking problem is stated, and then the set of admissible steady states is analyzed and characterized and the notion of an invariant set for tracking is presented. The proposed novel MPC strategy is presented in Section 3. In Section 4, an illustrative example and some conclusions are shown in Section 5. The paper finishes with an appendix containing the proof of the stability theorem.

Notation. vector (a, b) denotes $[a^{\mathrm{T}}, b^{\mathrm{T}}]^{\mathrm{T}}$; for a given λ, $\lambda X = \{\lambda x : x \in X\}$; $int(X)$ denotes the interior of set X; a definite positive matrix T is denoted as $T > 0$ and $T > P$ denotes that $T - P > 0$. For a given symmetric matrix $P > 0$, $\|x\|_P$ denotes the weighted euclidean norm of x, i.e. $\|x\|_P = \sqrt{x^{\mathrm{T}}Px}$. Matrix $\mathbf{0}_{n,m} \in \mathbb{R}^{n\times m}$ denotes a matrix of zeros. Consider $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$, and set $\Gamma \subset \mathbb{R}^{n_a+n_b}$, then projection operation is defined as $\mathrm{Proj}_a(\Gamma) = \{a \in \mathbb{R}^{n_a} : \exists b \in \mathbb{R}^{n_b},\ (a, b) \in \Gamma\}$.

1.1. Problem description

Let a discrete-time linear system be described by:

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^m$ is the current input, $y \in \mathbb{R}^p$ is the current output and x^+ is the successor state. The state of the system and the control input applied at sampling time k are denoted as x_k and u_k respectively.

The system (not necessarily square) is assumed to fulfill the following condition:

Assumption 1. The pair (A, B) is stabilizable.

The system is subject to hard constraints on state and input:

$$(x_k, u_z) \in Z = \{z \in \mathbb{R}^{n+m} : A_z z \le b_z\}, \quad \forall k \ge 0, \tag{2}$$

where the set Z is assumed to be a non-empty compact convex polyhedron containing the origin in its interior.

The objective of the paper is the calculation of a control law $u_k = K_N(x_k, \hat{\mathrm{x}}_s)$ such that for a given target steady state $\hat{\mathrm{x}}_s$, the state of the system is steered as close as possible to the target while fulfilling the constraints.

2. Preliminary results

2.1. Characterization of the steady states and inputs

For a given target output variable y_t, any steady state of the system $\mathrm{z}_s = (\mathrm{x}_s, \mathrm{u}_s)$ associated to this target output must satisfy the following equation

$$\begin{bmatrix} A - I_n & B & \mathbf{0}_{n,1} \\ C & D & -I_p \end{bmatrix} \begin{bmatrix} \mathrm{x}_s \\ \mathrm{u}_s \\ y_t \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n,1} \\ \mathbf{0}_{p,1} \end{bmatrix}. \tag{3}$$

Assumption 1 is necessary and sufficient to ensure that Eq. (3) has a non-trivial solution (Muske & Rawlings, 1993). This solution can be parametrized as

$$\begin{aligned} \mathrm{z}_s &= M_\theta \theta, \\ y_t &= N_\theta \theta, \end{aligned} \tag{4}$$

where $\theta \in \mathbb{R}^{n_\theta}$ is a parameter vector which characterizes any solution, and M_θ and N_θ are suitable matrices. This parametrization allows us to characterize the subspace of steady states and inputs by a minimal number of variables (θ), which simplifies further calculations necessary for the derivation of the proposed controller.

The existence of constraints limits the set of admissible steady states and inputs

$$Z_s = \{(\mathrm{x}_s, \mathrm{u}_s) = M_\theta \theta : M_\theta \theta \in Z\}.$$

Thus, the sets of admissible steady states and inputs are $X_s = \mathrm{Proj}_x(Z_s)$ and $U_s = \mathrm{Proj}_u(Z_s)$ respectively.

2.2. Calculation of an invariant set for tracking

Consider that system (1) is controlled by the following control law

$$u = K(x - \mathrm{x}_s) + \mathrm{u}_s = Kx + L\theta, \tag{5}$$

where $L = [-K\ I_m]M_\theta$. If matrix $A+BK$ is Hurwitz, then the closed loop system evolves to the steady state and input $(\mathrm{x}_s, \mathrm{u}_s) = M_\theta\theta$. An admissible invariant set for tracking is the set of initial states and steady states and inputs (characterized by θ) that can be admissibly stabilized by this control law.

Considering the extended state $w = (x, \theta)$, the closed loop system can be expressed as $w^+ = A_w w$, where

$$A_w = \begin{bmatrix} A + BK & BL \\ 0 & I_{n_\theta} \end{bmatrix}.$$

Let W_λ be a convex polyhedron defined as

$$W_\lambda = \{w = (x, \theta) : (x, Kx + L\theta) \in Z,\ M_\theta\theta \in \lambda Z\}.$$

Then, we say that a set Ω^w is an admissible invariant set for tracking if for all $w \in \Omega^w$, then $A_w w \in \Omega^w$ and $\Omega^w \subseteq W_1$.

The maximal admissible invariant set for tracking is given by $\mathcal{O}^w_\infty = \{w : A^i_w w \in W_1, \forall i \ge 0\}$. Due to the unitary eigenvalues of A_w, this set might not be finitely determined, i.e. described by a

finite set of constraints (Gilbert & Tan, 1991). Fortunately, we can state that the following set

$$\mathcal{O}^w_{\infty,\lambda} = \{w : A^i_w w \in W_\lambda, \forall i \geq 0\}$$

is finitely determined for any $\lambda \in (0, 1)$ and hence is a convex polyhedron (Gilbert & Tan, 1991). Given that $\lambda\mathcal{O}^w_\infty \subset \mathcal{O}^w_{\infty,\lambda} \subset \mathcal{O}^w_\infty$ and since λ can be chosen arbitrarily close to 1, the obtained invariant set can be used as a reliable polyhedral approximation to the maximal invariant set $\mathcal{O}^w_\infty$. Moreover, it is easy to see that $\lambda X_s \subset \mathcal{O}_{\infty,\lambda}$, where $\mathcal{O}_{\infty,\lambda} = \mathrm{Proj}_x(\mathcal{O}^w_{\infty,\lambda})$.

3. MPC for tracking

In this section the MPC proposed for tracking is presented. As has been previously stated, this predictive controller is based on the addition of the steady state and input as decision variables, the usage of a modified cost function and an extended terminal constraint. To this end, the following assumption is considered.

Assumption 2. (1) Let $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $T \in \mathbb{R}^{n\times n}$ be positive definite matrices.
(2) Let $K \in \mathbb{R}^{m\times n}$ be a stabilizing control gain such that $(A + BK)$ is Hurwitz.
(3) Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix such that

$$(A + BK)^{\mathrm{T}}P(A + BK) - P = -(Q + K^{\mathrm{T}}RK)$$

(4) Let $\mathcal{X}^w_f \subseteq \mathbb{R}^{n+n_\theta}$ be an admissible polyhedral invariant set for tracking for system (1) subject to (2) and a gain controller K.

For a given state x and a given target steady state $\hat{\mathrm{x}}_s$, the following cost function is proposed

$$V_N(x, \hat{\mathrm{x}}_s, \mathbf{u}, \theta) = \sum_{i=0}^{N-1} \|x(i) - \mathrm{x}_s\|^2_Q + \|u(i) - \mathrm{u}_s\|^2_R + \|x(N) - \mathrm{x}_s\|^2_P + \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|^2_T,$$

where $\mathbf{u}$ is a sequence of N future control inputs, i.e. $\mathbf{u} = \{u(0), \ldots, u(N-1)\}$, $(\mathrm{x}_s, \mathrm{u}_s) = M_\theta\theta$, $x(i)$ is the predicted state of the system at time i given by $x(i+1) = Ax(i)+Bu(i)$, with $x(0) = x$. Note that, in contrast to the standard MPC, this cost function penalizes the deviation from the artificial steady state and input; moreover, a term penalizing the deviation between the artificial steady state and the target steady state (tracking error cost) has been added.

Based on this cost function, the proposed MPC optimization problem $P_N(x, \hat{\mathrm{x}}_s)$ is given by

$$\begin{aligned} V^*_N(x, \hat{\mathrm{x}}_s) = \quad & \min_{\mathbf{u},\theta} V_N(x, \hat{\mathrm{x}}_s, \mathbf{u}, \theta) \\ s.t.\ & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & (x(j), u(j)) \in Z, \quad j = 0, \ldots, N-1 \\ & (\mathrm{x}_s, \mathrm{u}_s) = M_\theta\theta, \\ & (x(N), \theta) \in \mathcal{X}^w_f. \end{aligned}$$

Note that $\mathbf{u}$ and θ are the decision variables and x and $\hat{\mathrm{x}}_s$ are parameters of the proposed optimization problem $P_N(x, \hat{\mathrm{x}}_s)$. Moreover, this optimization problem is a standard (parametric) Quadratic Programming problem that can be efficiently solved by specialized algorithms. Applying the receding horizon strategy, the control law is given by $K_N(x, \hat{\mathrm{x}}_s) = u^*(0)$, where $u^*(0)$ is a function of x and $\hat{\mathrm{x}}_s$.

Given that the constraints of $P_N(x, \hat{\mathrm{x}}_s)$ do not depend on the target $\hat{\mathrm{x}}_s$, there exists a (polyhedral) region $X_N \subset \mathbb{R}^n$ such that for all $x \in X_N$, $P_N(x, \hat{\mathrm{x}}_s)$ is feasible (for all $\hat{\mathrm{x}}_s \in \mathbb{R}^n$) and moreover, $\mathcal{X}_f = \mathrm{Proj}_x(\mathcal{X}^w_f) \subseteq X_N$.

The main result of the paper is presented in the following theorem:

Theorem 1 (*Stability*). *Consider that Assumptions 1 and 2 hold. Suppose that $\mathcal{X}^w_f = \mathcal{O}^w_{\infty,\lambda}$ for a given $\lambda \in (0, 1)$. Consider that the target steady state $\hat{\mathrm{x}}_s$ is admissible. Then, for any feasible initial state $x_0 \in X_N$, the proposed MPC controller $u_k = K_N(x_k, \hat{\mathrm{x}}_s)$ asymptotically steers the system to $\hat{\mathrm{x}}_s$ in an admissible way.*

The proof can be found in the Appendix. The following remarks point out some properties of the proposed controller.

Remark 1. The set of admissible steady states that can be tracked without steady error is λX_s. Since $\lambda X_s \subseteq \mathcal{X}_f \subseteq X_N$ and since the evolution of the system state remains in X_N, any sequence of piecewise admissible targets can be tracked without steady error.

If the desired steady state $\hat{\mathrm{x}}_s$ is not admissible, then the controller steers the system to the closest admissible steady state (Alvarado, 2007).

Remark 2. Consider a given admissible target steady state $\hat{\mathrm{x}}_s \in \lambda X_s$ and design a standard MPC with the maximal admissible invariant set $\mathcal{O}_\infty(\hat{\mathrm{x}}_s)$ as terminal set. Consider the proposed MPC with the same ingredients (Q, R, P, K, and N) and the set $\mathcal{O}^w_{\infty,\lambda}$ is used as terminal constraint. Then:

(1) Since $\mathcal{O}_\infty(\hat{\mathrm{x}}_s) \subseteq \mathrm{Proj}_x(\mathcal{O}^w_{\infty,\lambda})$, the domain of attraction of the proposed MPC is larger than the one of the standard MPC.
(2) Contrary to the standard MPC, the proposed MPC controller may not guarantee the local optimality property. Fortunately, this optimality loss can be arbitrarily reduced by more heavily penalizing the tracking error term by means of matrix T (Alvarado, 2007).

Thus, taking an arbitrarily large matrix T, the MPC for tracking provides a larger domain of attraction and a control law which is locally nearly optimal.

Remark 3. The optimization problem to be solved belongs to the class of parametric quadratic programming problems that can be analyzed by the multiparametric quadratic programming tools. This allows the explicit calculation of the control law, making possible its application to fast systems (Fiacchini, Alvarado, Limon, Alamo, & Camacho, 2006).

4. Example

Consider a non-square LTI system given by:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.0 & 0.5 \\ 1.0 & 0.5 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

which is constrained to $\|x\|_\infty \leq 5$ and $\|u\|_\infty \leq 0.3$. The steady state and input are characterized by $\theta \in \mathbb{R}^2$ through the matrices

$$M_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}^{\mathrm{T}}, \qquad N_\theta = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The weighting matrices have been chosen as $Q = I_2$ and $R = I_2$. The considered local controller gain K and the Lyapunov matrix P are the LQR and the associated Riccati equation solution respectively. Matrix T has been chosen as $T = 100P$.

The invariant set for tracking $\mathcal{X}^w_f$ has been computed taking $\lambda = 0.99$ and the interval of admissible references which can be tracked without steady error is $[-4.95, 4.95]$. The system is controlled by the proposed MPC with a control horizon $N = 3$. The domain of attraction of this controller X_3 and set $\mathcal{X}_f$ are depicted in Fig. 1.

Fig. 2 shows the evolution of the output signal for three changes on the target: first $y_t = 4.95$ (admissible), then $y_t = -5.5$ (not admissible) and finally $y_t = 2$ (admissible). The target steady state has been chosen as $\hat{\mathrm{x}}_s = (y_t, 0)$. The states evolution of this

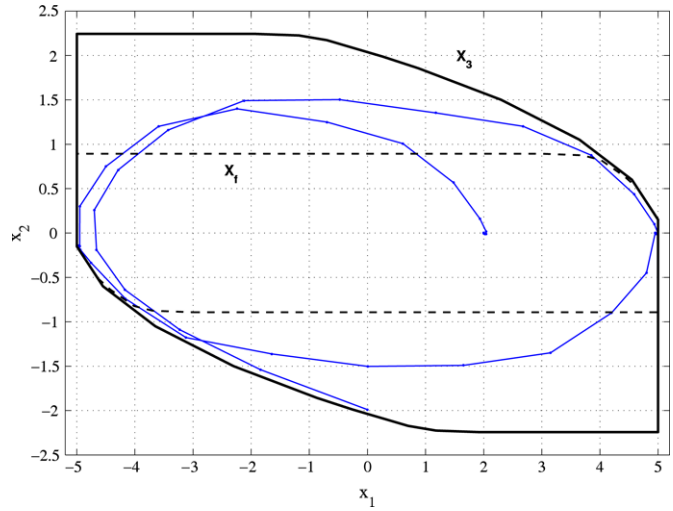

Fig. 1. Domain of attraction and state portrait of the closed loop system.

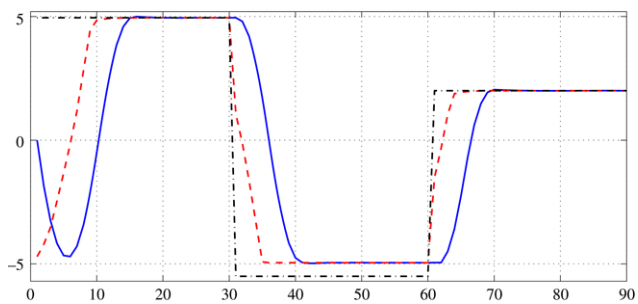

Fig. 2. Evolution of the output (solid line), artificial reference (dashed line) and the output target (dash-dotted line).

simulation is also depicted in Fig. 1. As can be seen, the evolution is admissible and the system evolves to the target steady state when this is admissible. In the case of non-admissible target, the system evolves to the closest admissible steady state.

In order to compare the MPC proposed for tracking with the standard MPC to regulate the system to the origin, the maximal admissible invariant set $\mathcal{O}_\infty(0)$ for the local control law $u = Kx$ and the domain of attraction of the MPC for regulation using $\mathcal{O}_\infty(0)$ as terminal set have been computed. This set is shown in Fig. 3, together with those sets computed for the MPC for tracking. As can be seen, the MPC for tracking provides a significant enlargement of the domain of attraction.

5. Conclusions

In this paper, a novel model predictive controller to track changing steady states for constrained linear systems is proposed. This controller ensures feasibility by means of adding an artificial steady state and input as decision variable of the optimization problem. Convergence to an admissible target steady state is ensured by using a modified cost function and a stabilizing extended terminal constraint based on the notion of invariance for tracking. The optimization problem to be solved is a QP which allows the multi-parametric programming tools to be used. This controller ensures feasibility for any change of steady state and for any prediction horizon. The properties of the controller have been illustrated in an example.

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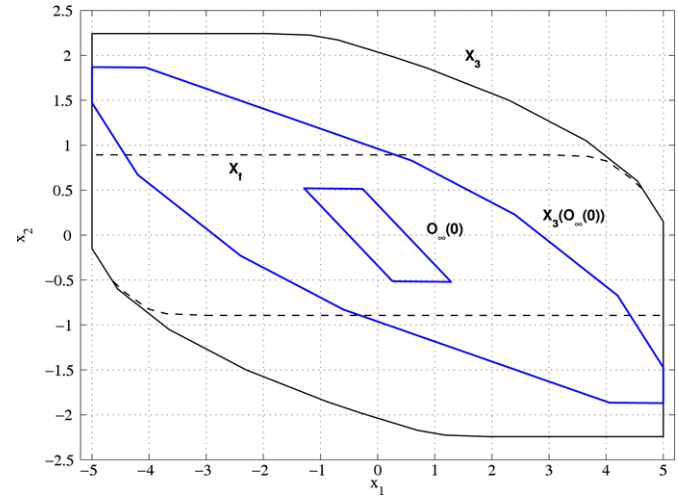

Fig. 3. Comparison of the domain of attraction between MPC for tracking $X_3(\mathcal{X}_f)$ and regulation to the origin $X_3(\mathcal{O}_\infty(0))$.

Appendix. Proof of Theorem 1

In what follows, we denote the optimal solution to the optimization problem by the superscript $*$.

Lemma 1. *Let $\hat{x}_s$ be an admissible steady state and let $(x_s, u_s) \in int(Z_s)$ be a steady state and input for system (1). Let K be a controller gain and let P be a Lyapunov matrix such that Assumption 2 holds. Then there exists $\lambda \in [0, 1)$ such that the steady state $\bar{x}_s = \lambda x_s + (1-\lambda)\hat{x}_s$ fulfils the following properties:*

(1) $x_s \in \mathcal{O}_\infty(\bar{x}_s)$.
(2) $\|x_s - \bar{x}_s\|_P^2 + \|\bar{x}_s - \hat{x}_s\|_T^2 < \|x_s - \hat{x}_s\|_T^2$, *for all matrix $T > 0$ and* $x_s \neq \hat{x}_s$.

Proof. The first statement is proved as follows. Let P be a Lyapunov matrix of the closed loop system $x^+ = (A + BK) \cdot x$. Let denote the ellipsoid $E(x_0, \tau) = \{x \in \mathbb{R}^n : \|x - x_0\|_P^2 \leq \tau\}$.

Let θ, $\hat{\theta}$ and $\bar{\theta}$ be such that $z_s = (x_s, u_s) = M_\theta\theta$, $\hat{z}_s = (\hat{x}_s, \hat{u}_s) = M_\theta\hat{\theta}$ and $\bar{z}_s = (\bar{x}_s, \bar{u}_s) = M_\theta\bar{\theta}$, where $\bar{\theta} = \lambda\theta + (1-\lambda)\hat{\theta}$. Given that $z_s \in int(Z)$ there exists constants $\epsilon > 0$ and $\gamma \in (0, 1)$ such that for all $x \in E(x_s, \epsilon)$, $(x, Kx + L\theta) \in \gamma Z$. Take a value of $\lambda \in (0, 1)$ sufficiently large such that $(0, L(\bar{\theta} - \theta)) = (0, (1-\lambda)L(\hat{\theta} - \theta)) \in (1-\gamma)Z$. Find a $\beta > 0$ such that $x_s \in E(\bar{x}_s, \beta) \subset E(P, \epsilon)$.

Then, for all $x \in E(\bar{x}_s, \beta)$ we have that $(x, Kx + L\bar{\theta}) = (x, Kx + L\theta) + (0, L(\bar{\theta} - \theta))$. Given that for all $x \in E(\bar{x}_s, \beta)$, $(x, Kx + L\theta) \in \gamma Z$ and $(0, L(\bar{\theta} - \theta)) \in (1-\gamma)Z$, we have that $(x, Kx + L\bar{\theta}) \in Z$. Therefore $x_s \in E(\bar{x}_s, \beta) \subset \mathcal{O}_\infty(\bar{x}_s)$.

The second fact is proved in virtue of the previous statement.

Considering the expression of $\bar{x}_s$, it is easy to see that $x_s - \bar{x}_s = (1-\lambda)(x_s - \hat{x}_s)$ and $\bar{x}_s - \hat{x}_s = \lambda(x_s - \hat{x}_s)$. Then we have that $\|x_s - \bar{x}_s\|_P = (1-\lambda)\|x_s - \hat{x}_s\|_P$ and $\|\bar{x}_s - \hat{x}_s\|_T = \lambda\|x_s - \hat{x}_s\|_T$. Thus, we can state that

$$\begin{aligned}\|x_s - \bar{x}_s\|_P^2 + \|\bar{x}_s - \hat{x}_s\|_T^2 &= (1-\lambda)^2\|x_s - \hat{x}_s\|_P^2 + \lambda^2\|x_s - \hat{x}_s\|_T^2 \\ &= \|x_s - \hat{x}_s\|_H^2,\end{aligned}$$

where $H = \lambda^2 T + (1-\lambda)^2 P$. Thus, it suffices to prove that a sufficiently large λ exists such that $H < T$, which implies that $\|x_s - \hat{x}_s\|_H^2 < \|x_s - \hat{x}_s\|_T^2$ for all $x_s \neq \hat{x}_s$.

Given that $P, T > 0$, there exists a constant $\gamma > 0$ such that $P < \gamma T$. Considering this, it follows that

$$\begin{aligned} T - H &= (1-\lambda^2)T - (1-\lambda)^2 P \\ &> \left((1-\lambda^2) - (1-\lambda)^2\gamma\right) T \\ &= (1-\lambda)\left((1+\lambda) - (1-\lambda)\gamma\right) T. \end{aligned}$$

Therefore, for every $\lambda > \frac{\gamma-1}{\gamma+1}$, we have that $T - H > 0$. ■

Lemma 2. *Consider system (1) subject to constraint (2). Let K be a controller gain and let P be a Lyapunov matrix such that Assumption 2 holds for some given definite positive matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$. Consider any $x \in \mathcal{X}_f = \mathrm{Proj}_x(\mathcal{X}_f^w)$. Consider a given target steady state x_s such that $x \in \mathcal{O}_\infty(\mathrm{x}_s)$. Then we have that*

$$V_N^*(x, \hat{\mathrm{x}}_s) \le \|\mathrm{x} - \mathrm{x}_s\|_P^2 + \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T^2.$$

Proof. Firstly, note that the sequence of control inputs obtained from the control law $u = K(x - \mathrm{x}_s) + \mathrm{u}_s$, denoted as $\mathbf{u}_s$, is a feasible solution for the MPC optimization problem since $x \in \mathcal{O}_\infty(\mathrm{x}_s)$.

Defining $A_K = A + BK$ and $Q^* = Q + K^{\mathrm{T}}RK$, from Lyapunov equation $P - A_K^{\mathrm{T}}PA_K = Q^*$ we derive that $\|x(i) - \mathrm{x}_s\|_P^2 - \|x(i+1) - \mathrm{x}_s\|_P^2 = \|x(i) - \mathrm{x}_s\|_{Q^*}^2$ for all $x(i)$. Therefore, summing up these terms, we obtain that

$$\begin{aligned} V_N^*(x, \hat{\mathrm{x}}_s) &\le \sum_{i=0}^{N-1} \|x(i) - \mathrm{x}_s\|_Q^2 + \|K(x(i) - \mathrm{x}_s)\|_R^2 \\ &\quad + \|x(N) - \mathrm{x}_s\|_P^2 + \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T^2 \\ &= \|\mathrm{x} - \mathrm{x}_s\|_P^2 + \|\mathrm{x}_s - \hat{\mathrm{x}}_s\|_T^2. \quad ■ \end{aligned}$$

Based on the previous lemmas, the following one can be proved:

Lemma 3. *Consider system (1) subject to contraint (2). Let K be a controller gain and let P be a Lyapunov matrix such that Assumption 2 holds for some given definite positive matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$. Consider a target steady state $\hat{\mathrm{x}}_s \in \lambda X_s$, where $\lambda \in (0, 1]$ is a given parameter. If, for a given state x, the optimal solution of $P_N(x, \hat{\mathrm{x}}_s)$ is such that $\|x - \mathrm{x}_s^*\|_Q = 0$ (i.e. $x = \mathrm{x}_s^*$), then $\|x - \hat{\mathrm{x}}_s\|_T = 0$.*

Proof. It is proved by contradiction. Assume that $x = \mathrm{x}_s^*$ and $x \ne \hat{\mathrm{x}}_s$. Since $x = \mathrm{x}_s^*$ is a steady state of the system, the control sequence given by the steady input is the optimal solution of $P_N(\mathrm{x}_s^*, \hat{\mathrm{x}}_s)$ and hence $V_N^*(\mathrm{x}_s^*, \hat{\mathrm{x}}_s) = \|\mathrm{x}_s^* - \hat{\mathrm{x}}_s\|_T^2$.

Since $\mathrm{x}_s^* \ne \hat{\mathrm{x}}_s$, in virtue of Lemma 1 it is inferred that there exists a steady state $\bar{\mathrm{x}}_s$ (and an input $\bar{\mathrm{u}}_s$), described by $\bar{\theta}$, such that $\mathrm{x}_s^* \in \mathcal{O}_\infty(\bar{\mathrm{x}}_s)$. Then, the sequence $\bar{\mathbf{u}}$ derived from the control law $u = K(x - \bar{\mathrm{x}}_s) + \bar{\mathrm{u}}_s$ is admissible and hence, from Lemma 2, we have that

$$V_N(\mathrm{x}_s^*, \hat{\mathrm{x}}_s, \bar{\mathbf{u}}, \bar{\theta}) \le \|\mathrm{x}_s^* - \bar{\mathrm{x}}_s\|_P^2 + \|\bar{\mathrm{x}}_s - \hat{\mathrm{x}}_s\|_T^2.$$

In virtue of Lemma 1 we have that

$$V_N(\mathrm{x}_s^*, \hat{\mathrm{x}}_s, \bar{\mathbf{u}}, \bar{\theta}) < \|\mathrm{x}_s^* - \hat{\mathrm{x}}_s\|_T^2 = V_N^*(\mathrm{x}_s^*, \hat{\mathrm{x}}_s)$$

which contradicts the optimality of $V_N^*(\mathrm{x}_s^*, \hat{\mathrm{x}}_s)$, and then $x = \mathrm{x}_s^* = \hat{\mathrm{x}}_s$. ■

Proof of Theorem 1. In what follows $(\mathbf{u}^*(k), \theta^*(k))$ denotes the optimal solution obtained in the optimization problem solved at sampling time k. Furthermore, $\mathrm{x}_s^*(k)$ and $\mathrm{u}_s^*(k)$ denote the optimal steady state and input associated to $\theta^*(k)$ respectively. $x^*(i; k)$ denotes the optimal predicted evolution of the system, i.e. $x^*(i; k) = Ax^*(i-1; k) + Bu^*(i-1; k)$ where $x^*(0; k) = x_k$.

Feasibility: Assume that the state at the current sample time k, x_k, is such that $x_k \in X_N$. Also assume that the optimal solution is $(\mathbf{u}^*(k), \theta^*(k))$ with an optimal cost $V_N^*(x_k, \hat{\mathrm{x}}_s)$. Let x_{k+1} be the state at the next sampling time. Consider $\theta(k+1) = \theta^*(k)$ and a control sequence

$$\begin{aligned} \mathbf{u}(k+1) = \{&u^*(1; k), \ldots, u^*(N-1; k), \\ &K(x^*(N; k) - \mathrm{x}_s^*(k)) + \mathrm{u}_s^*(k)\}. \end{aligned}$$

Then, it is easy to see that $(\mathbf{u}(k+1), \theta(k+1))$ is feasible due to the feasibility of the optimal solution at k and the positive invariance of $\mathcal{X}_f^w$. Consequently, $x_{k+1} \in X_N$.

Convergence: Consider the feasible solution at time $k+1$ previously presented. Following standard steps in the stability proofs of MPC (Mayne et al., 2000), we get that

$$\begin{aligned} V_N^*(x_{k+1}, \hat{\mathrm{x}}_s) &\le V_N(x_{k+1}, \hat{\mathrm{x}}_s, \mathbf{u}(k+1), \theta(k+1)) \\ &\le V_N^*(x_k, \hat{\mathrm{x}}_s) - \|x_k - \mathrm{x}_s^*(k)\|_Q^2. \end{aligned}$$

Due to the definite positiveness of the optimal cost and its non-increasing evolution, we infer that $\lim_{k\to\infty} \|x_k - \mathrm{x}_s^*(k)\|_Q = 0$ and from Lemma 3 we have that $\lim_{k\to\infty} \|x_k - \hat{\mathrm{x}}_s\|_Q = 0$. Consequently, the system is steered to $\hat{\mathrm{x}}_s$. ■

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