NMPC Design for AUV Dynamic Positioning Control with Incremental Input Constraints

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***Abstract*—This paper presents a novel nonlinear model predictive control (NMPC) method for the dynamic positioning (DP) problem of general autonomous underwater vehicle (AUV) systems. In addition to common bound constraints on the control input signal, the input increments are particularly considered for smooth actuation. With a Lyapunov-based auxiliary DP controller, we show that the input constraints and incremental input constraints can be fully respected. The widely used proportional-integral-derivative (PID) type control law is exploited as the auxiliary DP controller, and then conditions that guarantees the recursive feasibility and closed-loop stability are derived. The feasibility and stability do not rely on the optimality of the control solution, but the control performance improves as the solution approaches to the (local) optimum. Hence, the computational complexity and control performance can be well balanced by the user specified optimizer parameters. Simulation results reveal the effectiveness and advantages of the proposed NMPC method.**

***Index Terms*—Autonomous underwater vehicle, nonlinear model predictive control, dynamic positioning, input constraints**

I. Introduction

The study of dynamic positioning (DP) system design has a long history [1]. The first generation of DP system design could date back to the 1960s. In [2] the single-input single-output proportional-integral-derivative (PID) controllers were designed and applied to the surge, sway, and yaw subsystems respectively. From the late 1970s, the model-based control began to play an important role. The majority of DP systems were designed based on linear control theories [3], [4]. An H_∞ robust control design was proposed in [5], where the rejection of wave disturbance was particularly emphasized. Notably, the (Kalman filter based) observer-controller structure was widely adopted in DP systems [6], which formed the second generation of DP control design. Due to good robustness against external disturbances and model uncertainties, the Lyapunov-based backstepping control and the sliding mode control gradually became the mainstream methods for DP control design. Early work can be found in [7] and [8] which have inspired many other research work on marine control systems design (e.g., [9] [10]). More recent DP systems begin to consider more practical factors. One focused direction is to integrate the fault diagnostics and fault-tolerant control. One commonly used fault-tolerant control strategy is to use a supervisory controller. In [11], the fault detection was obtained by a Luenberger observer in combination with the parity space approach, and the fault-tolerant DP was achieved by a supervisory controller with a bank of reconfigurable discrete-time variable-structure controllers. On the other hand, the design constraints, such as the capacity of thrusters, vessel stability, and safety and operability, present another important practical factor and deserve to be handed in the DP design.

Model predictive control (MPC) is featured by the ability to address system constraints in the controller design [12]. In terms of AUV control applications, some early attempts were made in the new century. In [13], the linear MPC approach was proposed for the pipeline tracking problem. The formulated optimization problem was a quadratic program which was solved by a genetic algorithm. The NMPC formulation for AUV motion planning and tracking is reported in [14] where a goal-directed optimization algorithm was implemented to solve the NMPC problem. For the DP problem, an NMPC solution was proposed in [15] where the simulation results suggested that the NMPC could improve the DP performance to a large extent. However, due to the complexity of kinematics and dynamics of the AUV motion and the lack of closed-form solutions to the optimization, not until the most recent years was the rigorous analysis of closed-loop stability available for NMPC-based AUV control design. In [16], the combined path planning and trajectory tracking control problem was solved by a dynamic path planner cascaded to an NMPC controller. Conditions on stability were derived for the cascade planner-controller structure. In [17], the multiobjective model predictive control (MOMPC) concept was introduced for the AUV path following control. The path convergence requirement and speed assignment could be prioritized and guaranteed under the MOMPC formulation. In our previous paper [18], the AUV DP problem was investigated in the NMPC framework. A well-designed contraction constraint was imposed to guarantee the recursive feasibility and closed-loop stability.

In [18], however, we could occasionally observe big increments in the calculated control commands. Big increments should be avoided not only because they are forbidden by actuator physics, but also they significantly deteriorate the

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system performance. This does not occur for common state feedback controls since the system state is continuous in practice. However, for optimization-based control, such as MPC, the feedback control law could be discontinuous [19]. Therefore, we aim to extend the results in [18] and incorporate design constraints on the input increments. In this paper, novel optimization formulations are proposed, discussed and analyzed for the DP control. We show that the incremental input constraints can be fully incorporated, and stability and feasibility can be proved.

The main contributions of this paper are summarized:

- A novel NMPC solution is proposed for AUV DP system design where the input and incremental input constraints can be fully respected, and the thrust allocation sub-task is seamlessly integrated.
- Recursive feasibility of the proposed optimization formulation and asymptotic stability of the closed-loop system are rigorously proved.

The remaining part is organized as follows. Section II introduces the AUV system model and important properties. In Section III, the NMPC based DP control design is detailed and stability analysis is provided. In Section IV, we extend the developed NMPC controller to incorporate incremental input constraints. Simulation results are presented in Section V and conclusions are drawn in Section VI.

II. MODELING

The dynamic equations of AUV motion are as follows [20]:

$$\mathbf{M}\dot{\mathbf{v}} + \mathbf{C}(\mathbf{v})\mathbf{v} + \mathbf{D}(\mathbf{v})\mathbf{v} + \mathbf{g}(\boldsymbol{\eta}) = \boldsymbol{\tau} + \mathbf{w} \tag{1}$$

where $\mathbf{v}$ is the velocity vector represented in the body frame; $\boldsymbol{\eta}$ is the position vector in the inertial frame; $\boldsymbol{\tau}$ is the generalized control forces, and $\mathbf{w}$ is the external disturbance.

In this work, the DP control considers the AUV motion in the local level plane. The coefficient matrices in (1) are then simplified as $\mathbf{M}$ = diag$(M_{\dot{u}}, M_{\dot{v}}, M_{\dot{r}})$, $\mathbf{D}(\mathbf{v})$ = diag(X_u, Y_v, N_r) + diag$(D_u|u|, D_v|v|, D_r|r|)$, $\mathbf{g}(\boldsymbol{\eta}) = \mathbf{0}$ and

$$\mathbf{C}(\mathbf{v}) = \begin{bmatrix} 0 & 0 & -M_{\dot{v}}v \\ 0 & 0 & M_{\dot{u}}u \\ M_{\dot{v}}v & -M_{\dot{u}}u & 0 \end{bmatrix}$$

Here, $\mathbf{v} = [u, v, r]^{\mathrm{T}}$ denotes velocities in the surge, sway and yaw directions, and $\boldsymbol{\tau} = [F_u, F_v, F_r]^{\mathrm{T}}$ considers generalized control forces in these directions. $\boldsymbol{\eta} = [x, y, \psi]^{\mathrm{T}}$ denotes the position and orientation of the vehicle. $M_{\dot{u}}$, $M_{\dot{v}}$ and $M_{\dot{r}}$ are the inertia coefficients; X_u, Y_v and N_r are linear drag terms; and D_u, D_v and D_r are quadratic drag terms.

The kinematic equations of AUV motion are as follows:

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix} = \mathbf{R}(\psi)\mathbf{v} \tag{2}$$

For a specific AUV platform usually the generalized control forces are generated by multiple thrusters. For example, the Saab SeaEye Falcon model [21] has four thrusters $\mathbf{u} = [u_1, u_2, u_3, u_4]^{\mathrm{T}}$ responsible for the motion in the local level plane, and then the $\boldsymbol{\tau}$ and $\mathbf{u}$ are related by $\boldsymbol{\tau} = \mathbf{B}\mathbf{u}$.

Gathering the kinematics, dynamics and thrust distribution, we establish the AUV model used for the DP control design

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{R}(\psi)\mathbf{v} \\ \mathbf{M}^{-1}(\mathbf{B}\mathbf{u} - \mathbf{C}(\mathbf{v})\mathbf{v} - \mathbf{D}(\mathbf{v})\mathbf{v}) \end{bmatrix} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{3}$$

where $\mathbf{x} = [x, y, \psi, u, v, r]^{\mathrm{T}}$ is the state vector, and $\mathbf{u} = [u_1, u_2, u_3, u_4]^{\mathrm{T}}$ is the control vector. The following properties can be explored for the established model (3) and will be exploited in the DP control design:

P-1: The initial matrix is symmetric positive definite and bounded: $\infty > \bar{m}\mathbf{I} \geq \mathbf{M} = \mathbf{M}^{\mathrm{T}} \geq \underline{m}\mathbf{I} > 0$

P-2: The Coriolis and centripetal matrix $\mathbf{C}(\mathbf{v})$ is skew-symmetric, i.e., $\mathbf{C}(\mathbf{v}) = -\mathbf{C}^{\mathrm{T}}(\mathbf{v})$

P-3: The rotation matrix satisfies: $\mathbf{R}^{-1}(\psi) = \mathbf{R}^{\mathrm{T}}(\psi)$ and it preserves length: $\|\mathbf{R}(\psi)\mathbf{v}\|_2 = \|\mathbf{R}^{\mathrm{T}}(\psi)\mathbf{v}\|_2 = \|\mathbf{v}\|_2$.

P-4: The damping matrix is positive definite: $\mathbf{D}(\mathbf{v}) > 0$

P-5: The input matrix satisfies that $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ is non-singular.

III. NMPC FOR DYNAMIC POSITIONING

The following problem formulation is proposed for the NMPC DP control:

$$\min_{\hat{\mathbf{u}} \in S(\delta)} J = \int_0^T \|\hat{\mathbf{x}}(s)\|_Q^2 + \|\hat{\mathbf{u}}(s)\|_R^2 ds + \|\hat{\mathbf{x}}(T)\|_P^2 \tag{4a}$$

$$\text{s.t.} \quad \dot{\hat{\mathbf{x}}}(s) = \mathbf{f}(\hat{\mathbf{x}}(s), \hat{\mathbf{u}}(s)),\ \hat{\mathbf{x}}(0) = \mathbf{x}(t) \tag{4b}$$

$$|\hat{\mathbf{u}}(s)| \leq \mathbf{u}_{\max} \tag{4c}$$

$$\tfrac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\hat{\mathbf{x}}(0), \hat{\mathbf{u}}(0)) \leq \tfrac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\hat{\mathbf{x}}(0), h(\hat{\mathbf{x}}(0))) \tag{4d}$$

where $h(\cdot)$ is an auxiliary DP control law (that is stabilizing the error system) and $V(\cdot)$ is the corresponding Lyapunov function. The presence of constraint (4d) allows us to rigorously prove the closed-loop stability.

In principle, the design of auxiliary control law $h(\cdot)$ can be done by any method that provides the Lyapunov-based stability analysis. To facilitate the illustration of the proposed NMPC method, we investigate the multivariate nonlinear PD control as an example. Consider the following auxiliary control law:

$$\boldsymbol{\tau}(\mathbf{x}) = \boldsymbol{\tau}_a(\mathbf{x}) := -\mathbf{R}^{\mathrm{T}}(\psi)(\mathbf{K}_p\boldsymbol{\eta} + \mathbf{K}_d\dot{\boldsymbol{\eta}}) \tag{5}$$

where $\mathbf{K}_p$ and $\mathbf{K}_d$ are diagonal matrices, positive definite.

The corresponding Lyapunov function is constructed as

$$V = \frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{M}\mathbf{v} + \frac{1}{2}\boldsymbol{\eta}^{\mathrm{T}}\mathbf{K}_p\boldsymbol{\eta} \tag{6}$$

Taking time derivative of V along the closed-loop state trajectory yields

$$\dot{V} = \mathbf{v}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{v}} + \dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbf{K}_p\boldsymbol{\eta} \tag{7}$$

Substituting (3) and (5) into (7) we have

$$\dot{V} = -\mathbf{v}^{\mathrm{T}}[\mathbf{C}(\mathbf{v}) + \mathbf{D}(\mathbf{v}) + \mathbf{K}_d^*(\psi)]\mathbf{v} \tag{8}$$

where $\mathbf{K}_d^*(\psi) = \mathbf{R}^{\mathrm{T}}(\psi)\mathbf{K}_d\mathbf{R}(\psi) > 0$. By property P-2, we have $\mathbf{v}^{\mathrm{T}}\mathbf{C}(\mathbf{v})\mathbf{v} = 0$ and then

$$\dot{V} = -\mathbf{v}^{\mathrm{T}}[\mathbf{D}(\mathbf{v}) + \mathbf{K}_d^*(\psi)]\mathbf{v} \leq 0 \tag{9}$$

Since V is radially unbounded, by LaSalle's theorem [22], the closed-loop system with the multivariate PD control is globally asymptotically stable with respect to the desired configuration $[\boldsymbol{\eta}, \mathbf{v}] = [\mathbf{0}, \mathbf{0}]$. Then the constraint (4d) corresponding to the nonlinear PD control is

$$\hat{\mathbf{v}}(0)^{\mathrm{T}}(\mathbf{B}\hat{\mathbf{u}}(0) - \mathbf{C}(\hat{\mathbf{v}}(0))\hat{\mathbf{v}}(0) - \mathbf{D}(\hat{\mathbf{v}}(0))\hat{\mathbf{v}}(0) + \mathbf{R}^{\mathrm{T}}(\hat{\psi}(0)) \\ \mathbf{K}_p\hat{\boldsymbol{\eta}}(0)) \leq -\hat{\mathbf{v}}(0)^{\mathrm{T}}[\mathbf{D}(\hat{\mathbf{v}}(0)) + \mathbf{K}_d^*(\hat{\psi}(0))]\hat{\mathbf{v}}(0)$$

Now we analyze the recursive feasibility and stability. For simplicity, we assume that the thrusters have the same maximum capacity, i.e., $|u_i| \leq u_{\max}$. Then we have the following results characterizing the recursive feasibility.

Proposition 1 ([18]). *Consider the Moore-Penrose pseudoinverse solution for the thrust allocation (TA) problem:*

$$\mathbf{u} = (\mathbf{B}\mathbf{B}^{\mathrm{T}})^{-1}\mathbf{B}^{\mathrm{T}} = \mathbf{B}^{+}\boldsymbol{\tau} \tag{10}$$

and let the maximum allowable generalized control force be $\tau_{max} = \|\boldsymbol{\tau}_{max}\|_\infty$ with $\boldsymbol{\tau}_{max} = [F_{u,max}, F_{v,max}, F_{r,max}]^{\mathrm{T}}$. If the following relation can hold

$$\tau_{max} \leq \frac{u_{max}}{\bar{b}^{+}} \tag{11}$$

where $\bar{b}^{+} = \|\mathbf{B}^{+}\|_\infty$, then the TA is feasible for the AUV system, i.e., $\|\mathbf{u}\|_\infty \leq u_{max}$.

Lemma 1 ([18]). *Choose the control gain matrices $\mathbf{K}_p = k_p\mathbf{I}$ and $\mathbf{K}_d = k_d\mathbf{I}$ with k_p, k_d positive, and define $h(\mathbf{x}) = \mathbf{B}^{+}\boldsymbol{\tau}(\mathbf{x})$. If the following relation can hold*

$$(k_p + \sqrt{2}k_d)\|\mathbf{x}(0)\|_2 \leq \frac{u_{max}}{\sqrt{2}\bar{b}^{+}} \tag{12}$$

where $\mathbf{x}(0)$ is the initial error. Then for all $t \geq 0$ a feasible solution for the optimization (4) is

$$\hat{\mathbf{u}}_0(s) := h(\hat{\mathbf{x}}(s)) \tag{13}$$

where the state trajectory $\hat{\mathbf{x}}(s)$ follows

$$\dot{\hat{\mathbf{x}}}(s) = \mathbf{f}(\hat{\mathbf{x}}(s), \hat{\mathbf{u}}_0(s)),\ \hat{\mathbf{x}}(0) = \mathbf{x}(t) \tag{14}$$

Then the closed-loop stability can be inferred by the guaranteed recursive feasibility.

Theorem 1. *Suppose the recursive feasibility condition (12) can hold, then the NMPC-based DP control algorithm, based on solving (4) repeatedly at each sampling instant, makes the desired equilibrium $[\boldsymbol{\eta}, \mathbf{v}] = [\mathbf{0}, \mathbf{0}]$ asymptotically stable with an arbitrarily large region of attraction.*

Proof. Since the Lyapunov function (6) is continuously differentiable in $\mathbf{x}$ and radically unbounded, by converse Lyapunov theorems [22] there exist class $\mathcal{K}_\infty$ functions $\alpha_i(\cdot)$, $i = 1, 2, 3$ satisfying the following relation:

$$\alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|) \tag{15a}$$

$$\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, h(\mathbf{x})) \leq -\alpha_3(\|\mathbf{x}\|) \tag{15b}$$

With (4d) and the receding horizon implementation (i.e., the optimal solution $\mathbf{u}^*$ obtained for the current time is applied for only one sampling period), the following holds

$$\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u}^*(\mathbf{x}(t))) \leq \frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, h(\mathbf{x}(t))) \leq -\alpha_3(\|\mathbf{x}\|) \tag{16}$$

By standard Lyapunov arguments (Theorem 4.8 in [22]) we claim that the NMPC controlled closed-loop system is asymptotically stable with respect to the equilibrium $[\boldsymbol{\eta}, \mathbf{v}] = [\mathbf{0}, \mathbf{0}]$. Obviously, the stability is obtained unconditionally except for the feasibility condition (12). Therefore, the region of attraction (ROA) encompasses all points that render (12) true:

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^6 \mid (k_p + \sqrt{2}k_d)\|\mathbf{x}\|_2 \leq \frac{u_{\max}}{\sqrt{2}\bar{b}^{+}}\} \tag{17}$$

Since the control gains can be freely chosen, the ROA can be made arbitrarily large with small enough k_p and k_d. □

IV. NMPC DP Control with Incremental Input Constraints

Let $u_i^*(t_k)$ be the NMPC command for thruster i at time $t_k := k\delta$. Then the incremental input constraints can be expressed as

$$|u_i^*(t_k) - u_i^*(t_{k-1})| \leq \Delta u_{\max} \tag{18}$$

Here, $\Delta u_{\max}$ denotes the maximum increment allowed for the NMPC control command between neighboring sampling instants. We propose the problem formulation as follows:

$$\min_{\hat{\boldsymbol{\tau}} \in S(\delta)} J = \int_0^T \|\hat{\mathbf{x}}(s)\|_Q^2 + \|\hat{\boldsymbol{\tau}}(s)\|_R^2 ds + \|\hat{\mathbf{x}}(T)\|_P^2 \tag{19a}$$

$$\text{s.t.} \quad \dot{\hat{\mathbf{x}}}(s) = \mathbf{f}(\hat{\mathbf{x}}(s), \hat{\boldsymbol{\tau}}(s)),\ \hat{\mathbf{x}}(0) = \mathbf{x}(t) \tag{19b}$$

$$\|\hat{\boldsymbol{\tau}}(s)\|_\infty \leq \tau_{\max} \tag{19c}$$

$$\|\mathbf{f}(\hat{\mathbf{x}}(s), \hat{\boldsymbol{\tau}}(s))\|_\infty \leq H \tag{19d}$$

$$\|\hat{\boldsymbol{\tau}}(0) - \boldsymbol{\tau}_a(\hat{\mathbf{x}}(0))\|_\infty \leq \Delta\tau_{\max} \tag{19e}$$

$$\tfrac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\hat{\mathbf{x}}(0), \hat{\boldsymbol{\tau}}(0)) \leq \tfrac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\hat{\mathbf{x}}(0), \boldsymbol{\tau}_a(\hat{\mathbf{x}}(0))) \tag{19f}$$

where H and $\Delta\tau_{\max}$ are constants that are related to the AUV model parameters and the auxiliary control law. We will see shortly that the presence of (19d) and (19e) allows us to show the satisfaction of original incremental input constraints (18).

Since the auxiliary control (5) is differentiable in $\mathbf{x}$, there exists a (local) Lipschitz constant Λ satisfying

$$\|\boldsymbol{\tau}_a(\mathbf{x}) - \boldsymbol{\tau}_a(\mathbf{x}')\|_\infty \leq \Lambda\|\mathbf{x} - \mathbf{x}'\|_\infty \tag{20}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are two points on the closed-loop state trajectory $\mathbf{x}(t)$ which is the solution of $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\tau}_a(t))$. Then we have the following theorem characterizing the relationship between $\Delta u_{\max}$ and $\Delta\tau_{\max}$.

Theorem 2. *Suppose that the TA is implemented by the Moore-Penrose pseudoinverse method (10) and $\tau_{\max}$ follows (11), and the auxiliary control law $\boldsymbol{\tau}_a(\mathbf{x})$ is Lipschitz contin-*

uous in the ROA satisfying (20). For a given $\Delta u_{max} > 0$, if the following relation can hold

$$2\bar{b}^{+}\Delta\tau_{max} + \Lambda H\delta\bar{b}^{+} \leq \Delta u_{max} \tag{21}$$

then the original incremental input constraints (18) are satisfied during the DP control with the proposed NMPC (19).

Proof. Consider two successive states $x(t_k)$ and $x(t_{k-1})$ of the NMPC controlled closed-loop system. By definition,

$$|\mathbf{x}(t_k) - \mathbf{x}(t_{k-1})| = |\int_{t_{k-1}}^{t_k} \dot{\mathbf{x}}(t)dt| = \int_{t_{k-1}}^{t_k} |\mathbf{f}(\mathbf{x}(t), \boldsymbol{\tau}^*(t))|dt$$

Since (19d) is equivalent to $|\mathbf{f}(\hat{\mathbf{x}}(s), \hat{\boldsymbol{\tau}}(s))| \leq H$, we have

$$|\mathbf{x}(t_k) - \mathbf{x}(t_{k-1})| \leq \int_{t_{k-1}}^{t_k} H dt = H\delta \tag{22}$$

which follows

$$\|\mathbf{x}(t_k) - \mathbf{x}(t_{k-1})\|_\infty \leq H\delta \tag{23}$$

Eqn. (20) guarantees that

$$\|\boldsymbol{\tau}_a(\mathbf{x}(t_k)) - \boldsymbol{\tau}_a(\mathbf{x}(t_{k-1}))\|_\infty \leq \Lambda\|\mathbf{x}(t_k) - \mathbf{x}(t_{k-1})\|_\infty$$

Therefore

$$\|\boldsymbol{\tau}_a(\mathbf{x}(t_k)) - \boldsymbol{\tau}_a(\mathbf{x}(t_{k-1}))\|_\infty \leq \Lambda H\delta \tag{24}$$

Then the increment on optimal control $\boldsymbol{\tau}^*(t)$ satisfies

$$\begin{aligned} &\|\boldsymbol{\tau}^*(t_k) - \boldsymbol{\tau}^*(t_{k-1})\|_\infty \\ &= \|\boldsymbol{\tau}^*(t_k) - \boldsymbol{\tau}_a(\mathbf{x}(t_k)) - \boldsymbol{\tau}^*(t_{k-1}) + \boldsymbol{\tau}_a(\mathbf{x}(t_{k-1})) \\ &\quad + \boldsymbol{\tau}_a(\mathbf{x}(t_k)) - \boldsymbol{\tau}_a(\mathbf{x}(t_{k-1})\|_\infty \leq \|\boldsymbol{\tau}^*(t_k) \\ &\quad - \boldsymbol{\tau}_a(\mathbf{x}(t_k))\|_\infty + \|\boldsymbol{\tau}^*(t_{k-1}) - \boldsymbol{\tau}_a(\mathbf{x}(t_{k-1}))\|_\infty \\ &\quad + \|\boldsymbol{\tau}_a(\mathbf{x}(t_k)) - \boldsymbol{\tau}_a(\mathbf{x}(t_{k-1}))\|_\infty \end{aligned}$$

Combining (19e) and (24) we have

$$\|\boldsymbol{\tau}^*(t_k) - \boldsymbol{\tau}^*(t_{k-1})\|_\infty \leq 2\Delta\tau_{\max} + \Lambda H\delta \tag{25}$$

The TA is solved by (10), therefore, the following holds

$$\begin{aligned} \|\mathbf{u}^*(t_k) - \mathbf{u}^*(t_{k-1})\|_\infty &= \|\mathbf{B}^{+}\boldsymbol{\tau}^*(t_k) - \mathbf{B}^{+}\boldsymbol{\tau}^*(t_{k-1})\|_\infty \\ &\leq \bar{b}^{+}\|\boldsymbol{\tau}^*(t_k) - \boldsymbol{\tau}^*(t_{k-1})\|_\infty = 2\bar{b}^{+}\Delta\tau_{\max} + \Lambda H\delta\bar{b}^{+} \end{aligned}$$

If (21) can hold, then

$$\|\mathbf{u}^*(t_k) - \mathbf{u}^*(t_{k-1})\|_\infty \leq \Delta u_{\max} \tag{26}$$

which is the original incremental input constraints (18). □

Before proceeding to the derivation of feasibility and stability conditions, we would like to show that the key condition (21) can be satisfied by only adjusting the control gains of the auxiliary control law.

For simplicity, we set the control gain matrices by $\mathbf{K}_p = k_p\mathbf{I}$ and $\mathbf{K}_d = k_d\mathbf{I}$. Expand the auxiliary control law (5) into element-wise expression

$$\boldsymbol{\tau}_a(\mathbf{x}) = \begin{bmatrix} \tau_1(\mathbf{x}) \\ \tau_2(\mathbf{x}) \\ \tau_3(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} -k_p x \mathrm{c}\psi - k_p y \mathrm{s}\psi - k_d u \\ k_p x \mathrm{s}\psi - k_p y \mathrm{c}\psi - k_d v \\ -k_p\psi - k_d r \end{bmatrix} \tag{27}$$

where s· and c· are shorthand for sin and cos respectively. The Jacobian matrix $\nabla\boldsymbol{\tau}_a(\mathbf{x})$ at $\mathbf{x}$ is

$$\begin{bmatrix} -k_p\mathrm{c}\psi & -k_p\mathrm{s}\psi & k_p x\mathrm{s}\psi - k_p y\mathrm{c}\psi & -k_d & 0 & 0 \\ k_p\mathrm{s}\psi & k_p\mathrm{c}\psi & k_p x\mathrm{c}\psi - k_p y\mathrm{s}\psi & 0 & -k_d & 0 \\ 0 & 0 & -k_p & 0 & 0 & -k_d \end{bmatrix}$$

Then a Lipschitz constant could be

$$\Lambda = \sup\nolimits_{\mathbf{x}\in\mathcal{X}} \|\nabla\boldsymbol{\tau}_a(\mathbf{x})\|_\infty \tag{28}$$

By norm equivalence we have

$$\frac{\Lambda}{\sqrt{6}} \leq \Lambda_F = \sup_{\mathbf{x}\in\mathcal{X}} \|\nabla\boldsymbol{\tau}_a(\mathbf{x})\|_F \tag{29}$$

The calculation of $\|\nabla\boldsymbol{\tau}_a(\mathbf{x})\|_F$ is convenient

$$\begin{aligned} \|\nabla\boldsymbol{\tau}_a\|_F^2 &= k_p^2(\mathrm{c}\psi)^2 + k_p^2(\mathrm{s}\psi)^2 + k_p^2x^2(\mathrm{s}\psi)^2 + k_p^2y^2(\mathrm{c}\psi)^2 \\ &\quad + |2k_p^2xy\mathrm{s}\psi\mathrm{c}\psi| + 3k_d^2 + k_p^2(\mathrm{s}\psi)^2 + k_p^2(\mathrm{c}\psi)^2 \\ &\quad + k_p^2x^2(\mathrm{c}\psi)^2 + k_p^2y^2(\mathrm{s}\psi)^2 + |2k_p^2xy\mathrm{s}\psi\mathrm{c}\psi| + k_p^2 \\ &= 3k_p^2 + 3k_d^2 + k_p^2x^2 + k_p^2y^2 + 4k_p^2|xy\mathrm{s}\psi\mathrm{c}\psi| \\ &\leq 3k_p^2 + 3k_d^2 + k_p^2x^2 + k_p^2y^2 + 2k_p^2|xy| \\ &= 3k_p^2 + 3k_d^2 + k_p^2(|x| + |y|)^2 \end{aligned}$$

From above calculation we see that the Lipschitz constant $\Lambda \leq \sqrt{6}\Lambda_F$ is fully controlled by k_p and k_d. This is a desired property because in (21) the values of H, δ and $\bar{b}^{+}$ are usually difficult to change for a given AUV system.

Now we analyze the recursive feasibility. For the proposed problem (19), the value of H plays an important role. The following results reveal that appropriate H values can be found with the help of the auxiliary control law.

Lemma 2. *Suppose that the AUV is steered by the multivariate PD controller (5). Then the damping matrix $\mathbf{D}(\mathbf{v})$ is upper bounded satisfying*

$$\|\mathbf{D}(\mathbf{v})\|_2 \leq \bar{d} := \sqrt{3}\bar{d}_0\|\mathbf{x}(0)\|_2 \tag{30}$$

where $\bar{d}_0 := \max\{D_u + X_u/\|\mathbf{x}(0)\|_2, D_v + Y_v/\|\mathbf{x}(0)\|_2, D_r + N_r/\|\mathbf{x}(0)\|_2\}$.

Proof. Omitted. It can be obtained following the same lines in proof of *Lemma 1* in [23]. □

Theorem 3. *Suppose that the control gain matrices are $\mathbf{K}_p = k_p\mathbf{I}$ and $\mathbf{K}_d = k_d\mathbf{I}$ where k_p, k_d are positive satisfying*

$$(\sqrt{2}k_p + 2k_d)\|\mathbf{x}(0)\|_2 \leq \tau_{\max} \tag{31}$$

If the value of H satisfies the following relation

$$H \geq \frac{1}{\underline{m}}(\bar{d} + k_d + k_p + 1)\|\mathbf{x}(0)\|_2 \tag{32}$$

with $\underline{m}$ defined in P-1 and $\bar{d}$ in (30). Then for all $t \geq 0$ a feasible solution to the optimization problem (19) is

$$\hat{\boldsymbol{\tau}}_0(s) := \boldsymbol{\tau}_a(\hat{\mathbf{x}}(s)) \tag{33}$$

where the state trajectory $\hat{\mathbf{x}}(s)$ is the solution of

$$\dot{\hat{\mathbf{x}}}(s) = \mathbf{f}(\hat{\mathbf{x}}(s), \hat{\boldsymbol{\tau}}_0(s)),\ \hat{\mathbf{x}}(0) = \mathbf{x}(t) \tag{34}$$

Proof. As the auxiliary control law $\boldsymbol{\tau}_a(\hat{\mathbf{x}}(s))$ is taken as the feasible solution, the constraints (19b), (19e) and (19f) are already satisfied. The satisfaction of (19c) can be proved given (31) with the same line as in proof of *Lemma* 1 [18]. Therefore, in this proof, we only need to show that (19d) can be satisfied for $\boldsymbol{\tau}_a(\hat{\mathbf{x}}(s))$ if (32) holds.

Suppose that the auxiliary control law is in effect, then

$$\dot{V} = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \boldsymbol{\tau}_a(\mathbf{x})) = -\mathbf{v}^{\mathrm{T}}(\mathbf{D} + \mathbf{K}_d^*)\mathbf{v} \tag{35}$$

Since $\frac{\partial V}{\partial \mathbf{x}} = \mathbf{x}^{\mathrm{T}}\Pi$ with $\Pi = \mathrm{diag}(\mathbf{K_p}, \mathbf{M})$, we have

$$\mathbf{x}^{\mathrm{T}}\Pi\mathbf{f} = -\mathbf{v}^{\mathrm{T}}(\mathbf{D} + \mathbf{K}_d^*)\mathbf{v} \tag{36}$$

Considering $\mathbf{x} = [\boldsymbol{\eta}^{\mathrm{T}}, \mathbf{v}^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathbf{f} = [f_1^{\mathrm{T}}, f_2^{\mathrm{T}}]^{\mathrm{T}}$ with

$$f_1 = \mathbf{R}\mathbf{v}, \quad f_2 = \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{C}\mathbf{v} - \mathbf{D}\mathbf{v})$$

we have the following equation by expanding (36)

$$\boldsymbol{\eta}^{\mathrm{T}}\mathbf{K_p} f_1 + \mathbf{v}^{\mathrm{T}}\mathbf{M} f_2 = -\mathbf{v}^{\mathrm{T}}(\mathbf{D} + \mathbf{K}_d^*)\mathbf{v} \tag{37}$$

We have chosen $\mathbf{K_p} = k_p\mathbf{I}$, therefore,

$$\boldsymbol{\eta}^{\mathrm{T}}\mathbf{K_p} f_1 = k_p\boldsymbol{\eta}^{\mathrm{T}}\mathbf{R}\mathbf{v} = k_p\mathbf{v}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\boldsymbol{\eta} \tag{38}$$

Substitute (38) into (37)

$$\mathbf{v}^{\mathrm{T}}(k_p\mathbf{R}^{\mathrm{T}}\boldsymbol{\eta} + \mathbf{M} f_2) = -\mathbf{v}^{\mathrm{T}}(\mathbf{D} + \mathbf{K}_d^*)\mathbf{v}$$

Considering that (9) is obtained by substituting (3) and (5) into (7), we have the following

$$k_p\mathbf{R}^{\mathrm{T}}\boldsymbol{\eta} + \mathbf{M} f_2 = -(\mathbf{D} + \mathbf{K}_d^*)\mathbf{v}$$

Due to the system property P-1 and that $\mathbf{K_d} = k_d\mathbf{I}$, the following relation can be deduced

$$\underline{m} f_2 \leq \mathbf{M} f_2 = -(\mathbf{D} + k_d)\mathbf{v} - k_p\mathbf{R}^{\mathrm{T}}\boldsymbol{\eta} \tag{39}$$

Taking 2-norm on both sides of (39) and having the system property P-3 yields

$$\begin{aligned} \|f_2\|_2 &\leq \frac{1}{\underline{m}}(\|\mathbf{D}\|_2 + k_d)\|\mathbf{v}\|_2 + k_p\|\boldsymbol{\eta}\|_2 \\ &\leq \frac{1}{\underline{m}}(\|\mathbf{D}\|_2 + k_d + k_p)\|\mathbf{x}\|_2 \end{aligned}$$

By (30) and the fact that $\|\mathbf{x}(t)\|_2 \leq \|\mathbf{x}(0)\|_2$ due to stability (35), we have

$$\|f_2\|_2 \leq \frac{1}{\underline{m}}(\bar{d} + k_d + k_p)\|\mathbf{x}(0)\|_2 \tag{40}$$

Combining the knowledge that $\|\mathbf{f}\|_2 = \|f_1\|_2 + \|f_2\|_2$ and

$$\|f_1\|_2 = \|\mathbf{R}\mathbf{v}\|_2 = \|\mathbf{v}\|_2 \leq \|\mathbf{x}(0)\|_2$$

yields

$$\|\mathbf{f}\|_\infty \leq \|\mathbf{f}\|_2 \leq \frac{1}{\underline{m}}(\bar{d} + k_d + k_p + 1)\|\mathbf{x}(0)\|_2 \tag{41}$$

Therefore, it is guaranteed that the feasible solution $\boldsymbol{\tau}_a(\hat{\mathbf{x}}(s))$ satisfies (19d) if $\tau_{\max}$ is chosen such that (32) can hold. □

In addition, if $\tau_{\max}$ further satisfies (11), the Moore-Penrose pseudoinverse based TA (10) is always feasible for the real AUV system. The closed-loop stability is guaranteed because the critical contraction constraint (19f) remains unchanged in the optimization formulation.

Theorem 4. *Suppose that the controller parameters H, k_p, k_d and $\tau_{\max}$ are chosen such at (11), (31) and (32) can hold, then the NMPC-based DP control algorithm, based on solving (19) repeatedly at each sampling instant, makes the desired equilibrium $[\boldsymbol{\eta}, \mathbf{v}] = [\mathbf{0}, \mathbf{0}]$ asymptotically stable with an arbitrarily large region of attraction.*

Proof. Omitted (Similar to the proof of *Theorem 1*). □

V. Simulation Results

The controller parameters are set as follows: the sampling period is $\delta = 0.1$ second; the prediction horizon is $N = 5$; the weighting matrices are $Q = \mathrm{diag}(1e4, 1e4, 1e2, 1e1, 1e1, 1e1)$, $R = 5 * \mathrm{diag}(1e{-3}, 1e{-3}, 1e{-2})$ and $Q_f = \mathrm{diag}(1e3, 1e3, 1e1, 1e0, 1e0, 1e0)$; the constraints are $u_{\max} = 500$, $\Delta u_{\max} = 100$, $\tau_{\max} = 318$, $\Delta\tau_{\max} = 31.8$, and $H = 200$. The nonlinear PD control gain matrices are $\mathbf{K}_p = \mathbf{K}_d = 10 * \mathbf{I}_3$, and the initial state of the AUV is $\mathbf{x}(0) = [-5, 5, -\pi/2, 0, 0, 0]^{\mathrm{T}}$. The AUV model parameters can be found in [21].

In this simulation, the dynamic positioning task is performed by the multivariate nonlinear PD control (corresponding to black curves), the NMPC without incremental input constraints (corresponding to blue curves), and the NMPC with incremental input constraints (corresponding to red curves).

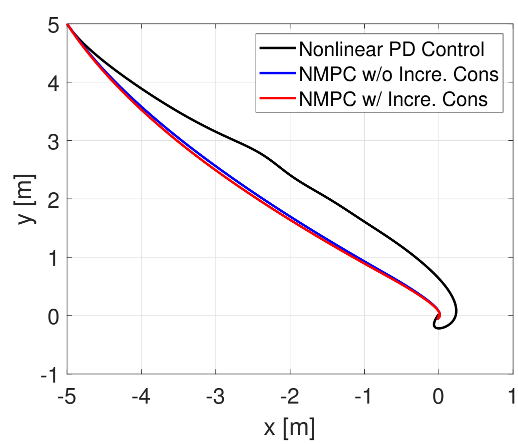

Fig. 1. The AUV footprints in the local level plane.

The AUV trajectories during the DP simulation are shown in Fig. 1. As we can see, all the three control methods are able to steer the AUV to reach the desired configuration. However, if we check the settling time, as can be observed in Fig. 2, the NMPC converges much faster than the PD control. Both NMPC controllers finish the DP task in about 30 seconds, while nonlinear PD controller takes more than 40 seconds to finish the job. Therefore, the efficiency is improved by 33.3%. The improvement is due to the online optimization associated

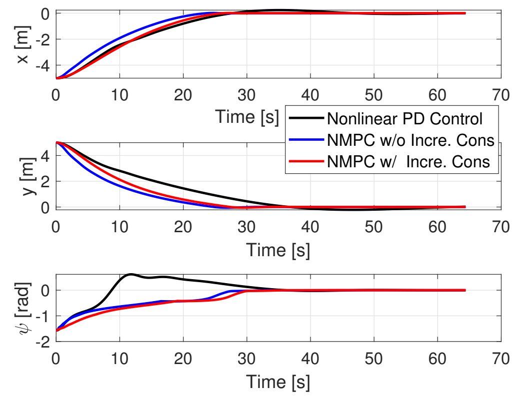

Fig. 2. The state trajectories with respect to time.

with the NMPC. Note that we have set relatively small control gains $\mathbf{K}_p$ and $\mathbf{K}_d$ to ensure a large region of attraction. The optimization makes full use of the onboard thrust capability to generate best possible performance while respecting the constraints all the time.

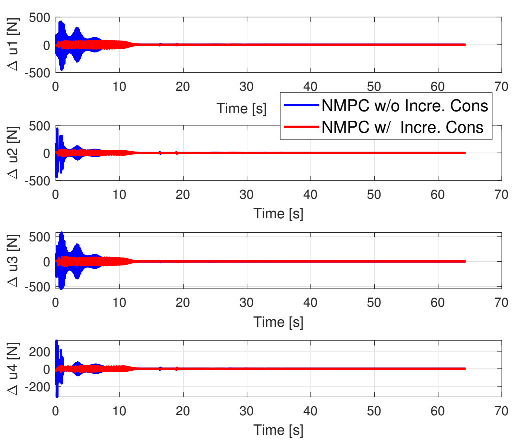

Fig. 3. The control signal increments on each thruster.

The signal increments are plotted in Fig. 3. As we can expect, if we do not impose incremental input constraints, the increments of control signal between neighboring sampling instants can be as large as 500 Newtons, which is obviously undesirable and is beyond the ability of the actuators. The results clearly demonstrate the necessity of considering the input increments in the DP control design.

VI. Conclusions

In this paper, we have developed a novel NMPC framework for solving the DP problem of AUVs. The incremental input constraints were particularly focused for better actuation. With an auxiliary DP control law, we showed that the input and incremental input constraints could be respected, and the recursive feasibility and closed-loop stability were guaranteed. The NMPC framework seamlessly integrated TA sub-task with the DP control and allowed users to balance the computational complexity and control performance for practical implementation. Simulation results showed significant improvement on the convergence rate and all constraints can be fully respected.

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