Journal of Process Control 20 (2010) 248–260

Contents lists available at ScienceDirect

Journal of Process Control

journal homepage: www.elsevier.com/locate/jprocont

Robust tube-based MPC for tracking of constrained linear systems with additive disturbances

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ARTICLE INFO

Article history:
Received 10 June 2008
Received in revised form 11 November 2009
Accepted 14 November 2009

Keywords:
Robust control
Model predictive control
Tracking
Constrained linear systems
Offset-free

ABSTRACT

This paper is devoted to solve the problem that the predictive controllers may present when the target operation point changes. Model predictive controllers (MPC) are capable to steer an uncertain system to a given target operation point fulfilling the constraints. But if the target changes significantly the controller may not success due to the loss of feasibility of the optimization problem and the inadequacy of the terminal conditions.

This paper presents a novel formulation of a robust model predictive controller (MPC) for tracking changing targets based on a single optimization problem. The plant is assumed to be modelled as a linear system with additive uncertainties confined to a bounded known polyhedral set. Under mild assumptions, the proposed MPC is feasible under any change of the target and steers the uncertain system to (a neighborhood of) the target if this is admissible. If the target is not admissible, and hence unreachable, the system is steered to the closest admissible operating point.

The controller formulation has some parameters which provide extra degrees of freedom. These new parameters allow control objectives such as disturbance rejection, output offset prioritization or enlargement of the domain of attraction to be dealt with. The paper shows how these parameters can be calculated off-line.

In order to demonstrate the benefits of the proposed controller, it has been tested on a real plant: the four tanks plant which is a multivariable nonlinear system configured to exhibit non-minimum phase transmission zeros. Experimental results show the robust stability and offset-free tracking of the controlled plant.

1. Introduction

Process control techniques have been applied with a hierarchical approach for years. The main layers of the hierarchy are the lower layers of feedback (regulatory) control and the upper layers of optimization. Fig. 1 shows a typical functional multilayer structure [23].

The direct control layer (known also as basic control layer) is responsible for basic safety of dynamic processes in the plant. Algorithms of direct control should be robust and relatively easy to tune and supervise, that is why classic PID algorithms are still dominant. However, capabilities of modern Distributed Control Systems (DCS) enable more computationally advanced solutions and more advanced control algorithms can be employed, in particular modifications of the PID algorithm and, recently, implementations of MPC algorithms.

The constraint control layer is linked to advanced control algorithms, in fact almost exclusively the MPC algorithms, devoted to the satisfaction of the constraints. This control layer provides the references to the local controllers.

Optimization of set-points for feedback controllers is performed by the local optimization layer also called Steady State Target Optimizers [16]. The goal is to calculate the process optimal operating point or optimal operating trajectory to be used by feedback controllers of directly subordinate layers. These values result generally from the plant-wide optimization problem which normally uses economic criteria [20].

The aim of this paper is design a controller able to cope with the local optimization, the constraint control and the direct control for a given (possibly changing) target, verifying the constraints while guaranteeing the robust stability of the system.

For a given target, the local optimization layer provides the set-point compatible with the constraints to the MPC, suitably synthesized for the provided set-point. Model predictive controllers are capable to regulate the controlled variable ensuring constraint satisfaction. However, when the target is changed by the plant-wide optimization layer, and hence the set-points, the stabilizing design

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doi:10.1016/j.jprocont.2009.11.007

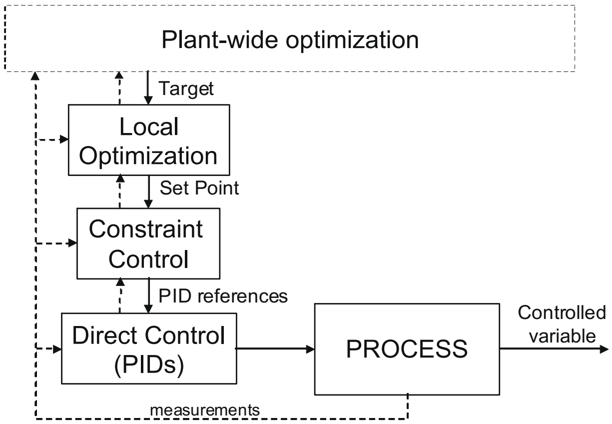

Fig. 1. Hierarchical control structure.

of the predictive controller may be not valid anymore and/or a the feasibility of the controller may be lost [21].

In order to overcome this problem several solutions have been proposed: in [22,7] an auxiliary controller that is able to recover the feasibility in finite time is used leading to a switching strategy. The controllers proposed in [18,17] consider the change of the set point as a disturbance to be rejected; thus, this technique is able to steer the system to the desired set point for small enough variations of the set point leading to a conservative solution.

A different approach has been proposed in the context of reference governors [9,3]. This control technique assumes that the system is robustly stabilized by a local controller, and a nonlinear filtering of the reference is designed to ensure robust satisfaction of the constraints. These controllers ensure robust tracking without considering the performance of the obtained controller nor the domain of attraction.

In this paper, a novel formulation of robust MPC for tracking is proposed. This is capable of leading the system to any robustly admissible target in an admissible way. The proposed controller follows the novel MPC formulation presented in [12] aimed at controlling constrained linear systems to track piece-wise constant references in absence of uncertainties.

This controller has been extended to control uncertain linear systems by incorporating the notion of tube-based robust control presented in [15]. The obtained robust controller is based on the solution of a single Quadratic Programming problem. Under mild conditions, the proposed controller ensures robust and admissible convergence to (a neighborhood of) the desired steady state, and maintains these properties under any change of reference. Moreover, offset-free control can be achieved by means of a simple procedure. The paper also presents a method for the synthesis of the controller which allows us to fulfil specifications for tracking as well as for disturbance rejection.

In order to demonstrate the applicability of the proposed controller it has been applied to an experimental tank system developed at the University of Seville. This plant is based on the well-known quadruple-tank process [10] and it is a multivariable laboratory plant of interconnected tanks that can be easily configured to exhibit the effect of multivariable zeros (minimum and non-minimum phase) on the system behavior, as well as the effect of non linear dynamics, saturation, constraints, etc.

The paper is organized as follows: firstly, the problem to be solved is described and then some preliminary and existing results are recalled. In Section 4 the proposed controller is presented, and stability conditions together with some interesting properties are shown in Section 5. Section 6 presents a simple method to cancel the possible offset on the outputs. The synthesis of the controller is shown in Section 7. In Section 8, the experimental results of the application of the proposed controller to the four tanks plant are given and finally, some conclusions are provided.

Notation: A positive definite symmetric matrix T is denoted as $T > 0$ and $T > P$ denotes that $T - P > 0$. For a given symmetric matrix $P > 0$, $\|x\|_P^2 \triangleq x^\top Px$. Consider $a \in \mathbf{R}^{n_a}$ and $b \in \mathbf{R}^{n_b}$, then $(a,b) \triangleq [a^\top, b^\top]^\top \in \mathbf{R}^{n_a+n_b}$ for a set $\Gamma \subset \mathbf{R}^{n_a+n_b}$, the projection of Γ onto a is defined as $Proj_a(\Gamma) = \{a \in \mathbf{R}^{n_a} : \exists b \in \mathbf{R}^{n_b},\ (a,b) \in \Gamma\}$. A vector $\mathbf{t}$ denotes a finite sequence of vectors, that is, a vector defined as $(t(0), t(1), \ldots, t(N))$, where N is deduced from the context. A matrix $\mathbf{0}_{n,m} \in \mathbf{R}^{n\times m}$ denotes a matrix of zeros and $I_n \in \mathbf{R}^{n\times n}$ denotes the identity matrix. Given two sets $\mathscr{U}$ and $\mathscr{V}$, such that $\mathscr{U} \subset \mathbf{R}^n$ and $\mathscr{V} \subset \mathbf{R}^n$, the Minkowski sum is defined by $\mathscr{U} \oplus \mathscr{V} \triangleq \{u + v : u \in \mathscr{U},\ v \in \mathscr{V}\}$, the Pontryagin set difference is: $\mathscr{U} \ominus \mathscr{V} \triangleq \{u : u \oplus \mathscr{V} \subseteq \mathscr{U}\}$; given a matrix $M \in \mathbf{R}^{p\times n}$, the set $M\mathscr{U} \subset \mathbf{R}^p$ is defined as $M\mathscr{U} \triangleq \{Mu : u \in \mathscr{U}\}$; for a given $\lambda, \lambda\mathscr{U} \triangleq (\lambda I_n)\mathscr{U}$. The set of integer numbers $\{0, 1, \ldots, N-1\}$ is denoted as $\mathbb{Z}_{[0,N-1]}$.

2. Problem statement

Consider that the plant to be controlled can be described by the following uncertain discrete-time LTI system

$$x^+ = Ax + Bu + w \qquad (1)$$
$$y = Cx + Du$$

where $x \in \mathbf{R}^n$ is the state of the system at the current time instant, x^+ denotes the successor state, that is, the state of the system at next sampling time, $u \in \mathbf{R}^m$ is the manipulated control input, $y \in \mathbf{R}^p$ is the controlled variables and $w \in \mathbf{R}^n$ is an unknown but bounded state disturbance. In what follows, $x(k), u(k), y(k)$ and $w(k)$ denote the state, the manipulable variable, controlled variable and the disturbance respectively, at sampling time k.

The state and input trajectories must satisfy the following constraints for any possible uncertain trajectory:

$$(x(k), u(k)) \in \mathscr{Z} \qquad (2)$$

where $\mathscr{Z}$ is a polyhedral set given by

$$\mathscr{Z} \triangleq \{z \in \mathbf{R}^{n+m} : A_z z \leqslant b_z\}.$$

The plant is assumed to fulfil the following hypotheses:

Assumption 1.

(i) The pair (A, B) is controllable.
(ii) The uncertainty vector w is bounded and lies in the following compact convex polyhedron.

$$\mathscr{W} = \{w \in \mathbf{R}^n : A_w w \leqslant b_w\} \qquad (3)$$

that is, $w(k) = (x(k+1) - Ax(k) - Bu(k)) \in \mathscr{W}$ for all $(x(k), u(k)) \in \mathscr{Z}$.
(iii) The state of the system is measured, and hence $x(k)$ is known at each sample time.

It is remarkable that no assumption is considered in the number of inputs m and outputs p, allowing thin plants ($p > m$), square plants ($p = m$) and flat plants ($p < m$). Moreover, it is not assumed that (A, B, C, D) is a minimal realization of the state–space model. This allows us to use state-space models derived from input–output models, that is, using as state a collection of past inputs and outputs of the plant [6]. The necessity of an observer is also avoided while the global uncertainty and the noise can be posed as additive uncertainties in the state-space model (1).

Control Objective: The main aim of this paper is to obtain a control law $u(k) = \kappa_N(x(k), y_t)$ such that, for a given target controlled variable y_t, the controlled plant

$x^+ = Ax + B\kappa_N(x, y_t) + w$

fulfils the plant constraints $(x(k), u(k)) \in \mathcal{Z}$ despite the uncertainties. Furthermore, if the target y_t is reachable, then controlled variable $y(k)$ should be steered to (a neighborhood of) the target y_t; if y_t is not reachable, then the controlled system should be steered to (a neighborhood of) the closest possible steady controlled variable.

In the following section, some known relevantly important results of the proposed predictive control law are summarized in order to make the paper clearer and more self-contained.

3. Preliminary results

3.1. Tube of trajectories and robust MPC for regulation

The proposed controller for tracking is based on the robust MPC for regulation proposed in [15]. This calculates the control input using nominal predictions and the notion of tube of trajectories. In this section these concepts are briefly introduced.

For a plant modelled by (1), its nominal model is the one obtained by ignoring the disturbances. This is given by

$$\begin{aligned} \bar{x}^+ &= A\bar{x} + B\bar{u} \\ \bar{y} &= C\bar{x} + D\bar{u} \end{aligned} \tag{4}$$

where $\bar{x}, \bar{u}$ and $\bar{y}$ denote the nominal state, input and controlled variable, respectively.

Assume that a given sequence of control actions $\bar{\mathbf{u}}$ has been calculated for the nominal plant of the model, then the predicted nominal trajectory starting from x is given by the recursion $\bar{x}(i+1) = A\bar{x}(i) + B\bar{u}(i)$ with $\bar{x}(0) = x$. Since the real system may be disturbed, the future trajectory of the disturbed plant will probably differ from the nominal prediction. To counteract the effect of the disturbances, it is desirable to force the trajectory to lie as close to the nominal trajectory as possible; this can be done by choosing the control action $u(i)$ as in [15]:

$$u(i) = \bar{u}(i) + K(x(i) - \bar{x}(i)) \tag{5}$$

From (1), (4) and (5) we infer that the dynamics of the error signal between the nominal and the disturbed plant $e \triangleq (x - \bar{x})$ is given by

$$e^+ = A_K e + w, \quad \text{with} \quad A_K = (A + BK) \tag{6}$$

If the feedback control gain K is such that A_K is Hurwitz, then the evolution of $e(i)$ is bounded and hence the real trajectory $x(i)$ lies in a neighborhood of the predicted one $\bar{x}(i)$, which can be seen as a tube of trajectories [13]. In order to limit the difference $e(i)$, the notion of robust positively invariant (RPI) set [11,19] is used.

Definition 1. A set ϕ_K is called a robust positively invariant (RPI) set for the uncertain system (6) if $A_K\phi_K \oplus \mathcal{W} \subseteq \phi_K$.

Based on this, the following result can be stated [15]: consider that ϕ_K is an RPI for the system (6) and that $x(0)$ and $\bar{x}(0)$ are such that $e(0) = x(0) - \bar{x}(0) \in \phi_K$; then, the trajectory of the uncertain system controlled by (5) is such that $x(i) \in \bar{x}(i) \oplus \phi_K$, for any possible realization of the disturbances. This is the so called tube of trajectories.

Moreover, if $x(0) \in \bar{x}(0) \oplus \phi_K$ and the nominal control sequence $\bar{\mathbf{u}}$ is such that

$$(\bar{x}(i), \bar{u}(i)) \in \overline{\mathcal{Z}} \tag{7}$$

where $\overline{\mathcal{Z}} \triangleq \mathcal{Z} \ominus (\phi_{\mathcal{K}} \times \mathcal{K}\phi_{\mathcal{K}})$. Then, the trajectory and control actions derived from the disturbed plant model (1) and (5) satisfy $(x(i), K(x(i) - \bar{x}(i)) + \bar{u}(i)) \in \mathcal{Z}$ for any possible realization of the disturbances. This implies that forcing a suitable tighter set of constraints for the nominal system, the evolution of the uncertain system controlled by (6) is robustly admissible [13].

Note that a primary condition to be satisfied is that the tighter set of constraints $\overline{\mathcal{Z}}$ is a non empty set. This implicitly defines a constraint on the selection of the control gain K which is not easy to fulfil. In Section 7.2 an LMI based method for calculating K is proposed.

The property of the tubes is exploited in [15] to derive a robust MPC for regulation. Thus, for a given target y_t the Steady State Target Optimizer must solve an optimization problem providing a setpoint $(\bar{\mathrm{x}}_t, \bar{\mathrm{u}}_t)$ for the nominal prediction model. Based on this, the following optimization problem must be solved for the current state of the plant:

$$\begin{aligned} \min_{\bar{\mathbf{u}}, \bar{x}(0)} \quad & \sum_{i=0}^{N-1} \|\bar{x}(i) - \bar{\mathrm{x}}_t\|_Q^2 + \|\bar{u}(i) - \bar{\mathrm{u}}_t\|_R^2 + \|\bar{x}(N) - \bar{\mathrm{x}}_t\|_P^2 \\ s.t. \quad & \bar{x}(0) \in x \oplus (-\Phi_K) \\ & \bar{x}(i+1) = A\bar{x}(i) + B\bar{u}(i), i \in \mathbb{Z}_{[0,N-1]} \\ & (\bar{x}(i), \bar{u}(i)) \in \overline{\mathcal{Z}}, i \in \mathbb{Z}_{[0,N-1]} \\ & (\bar{x}(N) - \bar{\mathrm{x}}_t) \in \Omega \end{aligned}$$

Then the control law for regulation is given by $K(x - \bar{x}^*) + \bar{u}^*(0)$. The domain of attraction of this controller is the feasibility region of the optimization problem and it is denoted as $X_N(y_t)$.

The terminal weighting matrix P and the terminal region Ω are chosen to ensure robust stability to a neighborhood of $\bar{\mathrm{x}}_t$. If the target y_t changes to a different value, then the stabilizing design may be not suitable for the new target and then the optimization problem can become not feasible or the feasibility (as well as the admissibility) can be lost along the evolution of the system. This can be a consequence of one or both of the two following causes: (i) the terminal set shifted to the new operating point may not be an admissible invariant set, which means that the recursive feasibility property may be lost, and (ii) the terminal region at the new set-point could be unreachable in N steps, which means that the optimization problem is unfeasible, making a re-calculation of an appropriate value of the prediction horizon necessary to ensure feasibility. Therefore, this would require an on-line re-design of the controller for each set point, which can be computationally unaffordable. In this paper, a novel robust MPC for tracking based on the notion of tubes is proposed. This controller is capable to maintain the feasibility for any changing target.

The feasibility loss when the target changes is demonstrated in the following illustrative example.

3.2. Illustrative example: double integrator

Consider a constrained sampled double integrator

$$\begin{aligned} x^+ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix} u + w \\ y &= [1 \quad 0] x \end{aligned} \tag{8}$$

where the disturbances are such that $\|w(k)\|_\infty \leqslant 0.1$ and the system must fulfil the following constraints: $\|x(k)\|_\infty \leqslant 5$, $\|u(k)\|_\infty \leqslant 0.3$.

Matrix K has been chosen as the LQR for $Q = I_2$ and $R = 10 \times I_2$ and it is the following:

$$K = \begin{bmatrix} -0.1183 & -0.5234 \\ -0.1739 & -0.4356 \end{bmatrix}$$

The minimal robust invariant set ϕ_K and the set $K\phi_K$ have been calculated using the methods proposed in [19,1] and are admissible. The loss of feasibility under a setpoint change is illustrated in Fig. 2. Consider that the current state is x_0 and the MPC has been designed to steer the system to the target $y_t = r_1$. The terminal region is chosen as the set $O_\infty(r_1)$ which is the maximal invariant set for the system controlled by $u = K(x - x_1) + u_1$ (the pair (x_1, u_1) is the steady state and input for the steady output r_1). The prediction

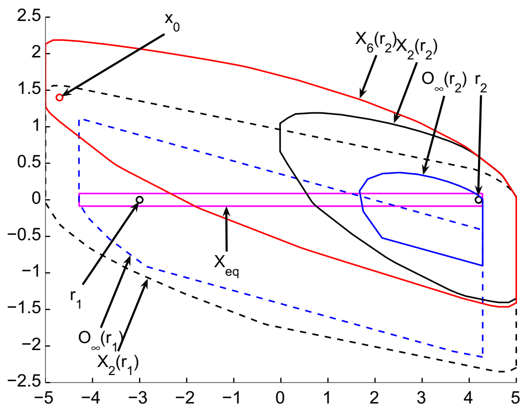

Fig. 2. Loss of feasibility due to a change of the target.

horizon has been chosen as $N = 2$. The region of attraction of the MPC is $X_2(r_1)$. Consider that the target is changed to $y_t = r_2$ at the current sampling instant. The first consequence is that set $O_\infty(r_1)$ translated to the steady state corresponding to r_2 is not an admissible invariant set and then this should be recalculated to avoid a possible loss of feasibility. Another issue is that, due to the fact that x_0 is out of $X_2(r_2)$, the MPC with $N = 2$ would not be feasible; in order to keep the feasibility, the prediction horizon should be enlarged to $N = 6$ (or even higher, because r_1 is out of $\mathscr{X}_6(r_2)$).

3.3. Characterization of the nominal steady states and inputs

Consider the nominal model of the plant (4) subject to the constraints on the nominal state and input given by (7). Every nominal steady state and input $\bar{z}_s = (\bar{x}_s, \bar{u}_s)$ is a solution of the equation

$$[A - I_n \quad B]\begin{bmatrix} \bar{x}_s \\ \bar{u}_s \end{bmatrix} = \mathbf{0}_{n,1} \tag{9}$$

and hence it is an element of the null space of the linear transformation given by matrix $[A - I_n B]$. Since it is assumed that (A, B) is controllable, the dimension of this null space is equal to m. Therefore, there exists a matrix $M_\theta \in \mathbf{R}^{(n+m)\times m}$ such that every nominal steady state and input can be posed as

$$\bar{z}_s = M_\theta \bar{\theta} \tag{10}$$

for certain $\bar{\theta} \in \mathbf{R}^m$. The subspace of nominal steady outputs is then given by

$$\bar{y}_s = N_\theta \bar{\theta} \tag{11}$$

where $N_\theta \triangleq [C\ D]M_\theta$. This parametrization is adopted to simplify the derivation of the proposed controller.

The existence of constraints (7) limits the set of admissible nominal steady states and inputs and the set of admissible nominal controlled variables, which are given by

$$\overline{\mathscr{Z}}_s \triangleq \{(\bar{x}_s, \bar{u}_s) \in \overline{\mathscr{Z}} : (A - I_n)\bar{x}_s + B\bar{u}_s = \mathbf{0}_{n,1}\}$$

$$\overline{\mathscr{Y}}_s \triangleq \{C\bar{x}_s + D\bar{u}_s : (\bar{x}_s, \bar{u}_s) \in \overline{\mathscr{Z}}_s\}$$

3.4. Calculation of an invariant set for tracking

Consider that the nominal system (4) is controlled by the following control law:

$$\bar{u} = \overline{K}(\bar{x} - \bar{x}_s) + \bar{u}_s = \overline{K}\bar{x} + L\bar{\theta} \tag{12}$$

where $L = [-\overline{K}\ I_m]M_\theta$. If $\overline{K}$ is such that matrix $A + B\overline{K}$ is Hurwitz then this control law steers the system to the steady state and input $(\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}$. The existence of constraints limits the set of initial states and steady states and inputs that can admissibly be stabilized. This leads to the following definition.

Definition 2 (*Invariant set for tracking*). An invariant set for tracking is the set of initial states and steady states and inputs (characterized by $\bar{\theta}$) that can be stabilized by the control law (12) fulfilling the constraints (7) throughout its evolution.

This set can be computed as an admissible invariant set for the augmented system $x^a \triangleq (\bar{x}, \bar{\theta}) \in \mathbf{R}^{n+m}$. Then the closed-loop system can be posed as:

$$\underbrace{\begin{bmatrix} \bar{x} \\ \bar{\theta} \end{bmatrix}^+}_{x_a^+} = \underbrace{\begin{bmatrix} A + B\overline{K} & BL \\ 0 & I_m \end{bmatrix}}_{A_a} \underbrace{\begin{bmatrix} \bar{x} \\ \bar{\theta} \end{bmatrix}}_{x_a} \tag{13}$$

subject to the set of constraints (7), that can be posed as

$$\mathscr{X}^a = \{x_a = (\bar{x}, \bar{\theta}) : (\bar{x}, \overline{K}\bar{x} + L\bar{\theta}) \in \overline{\mathscr{Z}}, M_\theta\bar{\theta} \in \overline{\mathscr{Z}}\}$$

Set $\Omega^a_{t,\overline{K}} \subset \mathscr{X}^a$ is an admissible invariant set for tracking, for system (13) constrained to $\mathscr{X}^a$, if $A_a\Omega^a_{t,\overline{K}} \subseteq \Omega^a_{t,\overline{K}}$ and $\Omega^a_{t,\overline{K}} \subseteq \mathscr{X}^a$. See that for any $(x(0), \bar{\theta}) \in \Omega^a_{t,\overline{K}}$, the trajectory of the system $\bar{x}(i+1) = A\bar{x}(i) + B\bar{u}(i)$ controlled by $\bar{u}(i) = \overline{K}\bar{x}(i) + L\bar{\theta}$ is confined in $\Omega_{t,\overline{K}} = Proj_x(\Omega^a_{t,\overline{K}})$[1] and tends to $(\bar{x}_s, \bar{u}_s) = M_\theta\bar{\theta}$.

Although the maximal invariant set is not needed, it is convenient in order to provide the largest possible region of attraction of the proposed robust MPC controller. The maximal admissible invariant set for system (13) may not be finitely determined due to the unitary eigenvalues of the plant. Fortunately, in this case, taking as constraints $\mathscr{X}^a_\lambda = \{x^a = (\bar{x}, \bar{\theta}) : (\bar{x}, \overline{K}\bar{x} + L\bar{\theta}) \in \overline{\mathscr{Z}},\ M_\theta\bar{\theta} \in \lambda\overline{\mathscr{Z}}\}$, the associated maximal admissible invariant set is finitely determined for any $\lambda \in (0, 1)$, resulting in a polyhedral region [8,1]. Thus taking a λ arbitrarily close to 1, the resulting invariant set is arbitrarily close (in the Hausdorff sense) to the maximal one.

It is interesting to characterize what will be the set of the nominal steady states, inputs and controlled variables that could be reached from an initial state contained $\Omega_{t,\overline{K}}$. This can be done by defining the following set of parameters $\overline{\Theta}$

$$\overline{\Theta} \triangleq \{\bar{\theta} : (\bar{x}_s, \bar{u}_s) = M_\theta\bar{\theta} \in \overline{\mathscr{Z}}, \bar{x}_s \in \Omega_{t,\overline{K}}\} \tag{14}$$

This set is equal to the projection of $\Omega^a_{t,\overline{K}}$ onto $\bar{\theta}$. Then the set of reachable nominal steady controlled variables is given by

$$\overline{\mathscr{Y}}_t = N_\theta\overline{\Theta} \tag{15}$$

Notice that if the calculation method proposed in [1] is used to compute $\Omega^a_{t,\overline{K}}$, then this set $\overline{\mathscr{Y}}_t$ is potentially equal to the maximal one $\overline{\mathscr{Y}}_s$ since $\overline{\mathscr{Y}}_t \subseteq \lambda\overline{\mathscr{Y}}_s$ and λ can be chosen arbitrarily close to 1.

4. Proposed robust MPC for tracking

This section is devoted to presenting the main contribution of the paper: a robust predictive controller $\kappa_N(x, y_t)$ which copes with the proposed control problem and it is derived from the solution of a single quadratic programming (QP) problem at each sampling time.

In order to ensure the feasibility of the problem for any target y_t, an artificial steady state $\bar{x}_s$, input $\bar{u}_s$ and controlled variable $\bar{y}_s$ are introduced as decision variables in the minimization of the performance index. In order to reduce the number of decision variables of

[1] In what follows, superscript a denotes that set $\Omega^a_{t,\overline{K}}$ is defined in the augmented state, while no superscript denotes that set $\Omega_{t,\overline{K}}$ is defined in the state vector space x, i.e. $\Omega_{t,\overline{K}} = Proj_x(\Omega^a_{t,\overline{K}})$.

the resulting optimization problem, these artificial steady variables are characterized by $\bar{\theta}$ since $(\bar{x}_s, \bar{u}_s) = M_\theta\bar{\theta}$, and $\bar{y}_s = N_\theta\bar{\theta}$.

The proposed performance index penalizes the deviation between the predicted nominal evolution of the plant and the artificial steady conditions throughout the prediction horizon N. Moreover, robust convergence to (a neighborhood of) the target y_t is ensured by adding to the cost function a term called (*offset cost*) of the form $\|\bar{y}_s - y_t\|_T^2$ which penalizes the deviation between the target and the artificial target. Then, the proposed cost function for a given state x and the target y_t is as follows

$$V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}(0), \bar{\theta}) \triangleq \sum_{i=0}^{N-1} \|\bar{x}(i) - \bar{x}_s\|_Q^2 + \|\bar{u}(i) - \bar{u}_s\|_R^2 + \|\bar{x}(N) - \bar{x}_s\|_P^2 + \|\bar{y}_s - y_t\|_T^2$$

where $\bar{x}(i+1) = A\bar{x}(i) + B\bar{u}(i), (\bar{x}_s, \bar{u}_s) = M_\theta\bar{\theta}$, and $\bar{y}_s = N_\theta\bar{\theta}$. The current state x and the controlled variable target y_t are parameters of the optimization problem, while the decision variables are: (i) the sequence of the future actions of the nominal system $\bar{\mathbf{u}}$, (ii) the initial state of the nominal trajectory $\bar{x}(0)$ and (iii) the parameter vector $\bar{\theta}$ that determines the artificial target steady state, input and output $(\bar{x}_s, \bar{u}_s, \bar{y}_s)$.

The optimization problem $\mathscr{P}_N(x, y_t)$ to be solved is:

$$\min_{\bar{\mathbf{u}}, \bar{x}(0), \bar{\theta}} \quad V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}(0), \bar{\theta})$$

$$\begin{aligned} s.t. \quad & \bar{x}(0) \in x \oplus (-\Phi_K) && (16)\\ & \bar{x}(i+1) = A\bar{x}(i) + B\bar{u}(i), i \in \mathbb{Z}_{[0,N-1]} \\ & (\bar{x}_s, \bar{u}_s) = M_\theta\bar{\theta} \\ & \bar{y}_s = N_\theta\bar{\theta} \\ & (\bar{x}(i), \bar{u}(i)) \in \overline{\mathscr{Z}}, i \in \mathbb{Z}_{[0,N-1]} && (17)\\ & (\bar{x}(N), \bar{\theta}) \in \Omega^a_{t,\overline{K}} && (18) \end{aligned}$$

where Φ_K is a robust positive invariant set for the system (1) controlled by (5) and $\overline{\mathscr{Z}}$ is given by (7). The constraint (16) forces the initial error $(\bar{x} - x)$ to be inside the tube and (17) ensures the robust satisfaction of the constraints [15]. The terminal constraint (18) is added for stability reasons.

Notice that this controller has two auxiliary control gains: control gain $\overline{K}$ used for the computation of the terminal ingredients and control gain K used for the tube (i.e., the calculation of Φ_K). This adds extra degrees of freedom to be exploited in the synthesis of the controller as it will be shown in Section 7.

Notice also that $\mathscr{P}_N(x, y_t)$ is a Quadratic Programming (QP) problem, that can be efficiently solved using specialized algorithms. Moreover, since the set of constraints of $\mathscr{P}_N(x, y_t)$ does not depend on the target output y_t, the optimization problem $\mathscr{P}_N(x, y_t)$ has a feasible solution for all x contained in a polyhedral set denoted as $\mathscr{X}_N \subset \mathbf{R}^n$, regardless of the choice of the target y_t. Defining $\overline{\mathscr{X}}_N$ as the set of states of the nominal model that can be steered to $\Omega_{t,\overline{K}}$ in N steps fulfilling the constraint (17), the feasibility set $\mathscr{X}_N$ is given by

$$\mathscr{X}_N = \overline{\mathscr{X}}_N \oplus \Phi_K$$

In the following, $V_N^*(x, y_t)$ will denote the optimal cost, $\bar{\mathbf{u}}^*(x, y_t)$, $\bar{x}^*(x, y_t)$ and $\bar{\theta}^*(x, y_t)$ the optimal value of the decision variables, $\bar{\mathbf{x}}^*(x, y_t)$ the nominal optimal state trajectory and $(\bar{x}_s^*(x, y_t), \bar{u}_s^*(x, y_t), \bar{y}_s^*(x, y_t))$ will denote the optimal artificial reference. The MPC control law is derived from the optimal solution as follows

$$\kappa_N(x, y_t) = K(x - \bar{x}^*(x, y_t)) + \bar{u}^*(0; x, y_t) \tag{19}$$

In the following section it will be shown that, under mild and standard assumptions, the proposed controller guarantees robust stability of the closed-loop system.

5. Stability and convergence of the Robust MPC for tracking

The proposed controller (19) is based on the solution of the optimization problem $\mathscr{P}_N(x, y_t)$. This problem has a number of ingredients such as: the definition of the stage cost Q, R matrices, the terminal state weighting matrix P, the offset weighting matrix T, the auxiliary robust control gain K, the terminal control gain $\overline{K}$, the section of the tube Φ_K, and the set of extended terminal constraint $\Omega^a_{t,\overline{K}}$. These ingredients will be chosen to satisfy the following assumption in order to give suitable properties the controller.

Assumption 2

(i) Q and R are symmetric matrices such that $Q \geqslant 0$, $R > 0$ and the pair $(Q^{\frac{1}{2}}, A)$ is observable.
(ii) T is a symmetric matrix such that $T > 0$.
(iii) The eigenvalues $A + BK$ are in the interior of the unitary circle and Φ_K is an admissible robust positively invariant (RPI) set of (1) subject to constraints (2) controlled by $u = Kx$.
(iv) The eigenvalues $A + B\overline{K}$ are in the interior of the unit circle and P is the definite positive matrix solution of

$$P - (A + B\overline{K})^\top P(A + B\overline{K}) = Q + \overline{K}^\top R\overline{K} \tag{20}$$

(v) The set $\Omega^a_{t,\overline{K}}$ is an invariant set for tracking for the system (4) subject to the constraints (7) for the gain matrix $\overline{K}$.

This conditions are an extension of the standard stabilizing conditions in MPC for linear systems [15] for the proposed robust MPC for tracking. Notice that the terminal region is assumed to be an invariant set for tracking and two auxiliary control gains K and $\overline{K}$ are considered. The determination of these ingredients will be studied in Section 7. Under these conditions, the following theorem can be formulated.

Theorem 1. *Consider that Assumptions 1 and 2 hold. Let $\overline{\mathscr{Y}}_t$ be the set of reachable nominal steady controlled variables defined in (15) . Consider the control law (19) resulting from the optimal solution of problem $\mathscr{P}_N(\cdot,\cdot)$. Then system (1) controlled by this law guarantees that:*

(i) *For all initial condition $x(0) \in \mathscr{X}_N$ and for every y_t, the evolution of the system is robustly feasible and admissible, that is, $x(i) \in \mathscr{X}_N$ and $(x(i), \kappa_N(x(i), y_t)) \in \mathscr{Z}, \forall w(k) \in \mathscr{W}, k = 0, 1, \ldots, i-1$.*
(ii) *If $y_t \in \overline{\mathscr{Y}}_t$ then the controlled variable $y(i)$ converges asymptotically at set $y_t \oplus (C + DK)\phi_K$.*
(iii) *If $y_t \notin \overline{\mathscr{Y}}_t$ then the controlled variable $y(i)$ converges asymptotically at set $\tilde{y}_s \oplus (C + DK)\phi_K$, where $\tilde{y}_s$ is the reachable nominal steady controlled output which minimizes the offset cost function, that is*

$$\tilde{y}_s = arg \min_{\bar{y}_s \in \overline{\mathscr{Y}}_t} \|\bar{y}_s - y_t\|_T^2$$

The proof can be found in the appendix section

From this result, further properties can be derived that result very interesting for the real application of the controller:

Property 1 (Stability under target changes). *Since $\mathscr{P}_N(x, y_t)$ is feasible $\forall x \in \mathscr{X}_N$, for an initial state such that $x(0) \in \mathscr{X}_N$, the evolution of the controlled uncertain system will be feasible, that is $x(k) \in \mathscr{X}_N$. This property holds for any value of y_t, even if this one is time varying. Therefore, if $y_t(k)$ is a piece-wise constant sequence or if this converges to a constant value, the controller steers the system to (a neighborhood of) the steady target. Because of this, the proposed controller is suitable to control systems with changing points of operation.*

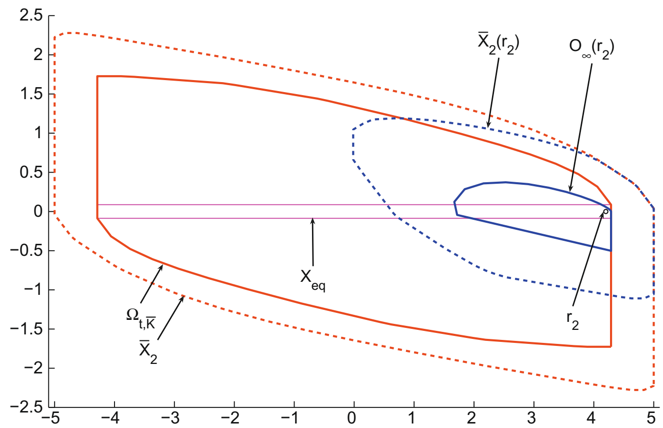

Fig. 3. Comparison of the domain of attraction of MPC for regulation and for tracking.

Property 2 (Enlarged domain of attraction). *The invariant set for tracking is (potentially) larger than the invariant set for a given steady state typically used for the regulation problem [15]. Therefore, the domain of attraction of the proposed robust MPC for tracking is (potentially) larger than the one of the regulation problem, and hence the presented controller can be of interest even when the set point is not changing.*

In order to demonstrate this property, the domains of attraction for the regulation robust MPC and the proposed robust MPC for tracking have been calculated for the double integrator of Section 3.2 for a certain target. As it can be seen in Fig. 3, the domain of attraction of the proposed controller is substantially larger than the one for regulation.

Property 3 (Controller implementation). *The proposed optimization problem $\mathscr{P}_N(x, y_t)$ is a Quadratic Programming problem affine on its parameters (x, y_t). Then the control law is a piece-wise affine function of (x, y_t) defined in a state partition of $\{\mathscr{X}_N \times \mathbf{R}^p\}$ that can be calculated off-line by means of well-known algorithms [4]. This property allows to implement the controller algorithm using simpler and faster software suitable for low order models.*

6. Output offset cancellation

As it was proven in the previous section, if the target y_t is reachable, i.e. $y_t \in \overline{\mathscr{Y}}_t$, the proposed controller steers the output of the uncertain system to a neighborhood of the target given by $y_t \oplus (C + DK)\Phi_K$. If the uncertainty signal $w(k)$ tends to a steady value $w(\infty)$, then, the closed-loop system state tends to a steady output $y(\infty)$ such that

$$y(\infty) = y_t + Hw(\infty)$$

where $H \triangleq (C + DK)(I_n - (A + BK))^{-1}$. Then the system may exhibit a steady output offset $y(\infty) - y_t$.

From these calculations, it is easy to see that if, for a given desired set point y_t, the system is controlled by $u = \kappa_N(x, \hat{y}_t)$ with $\hat{y}_t = y_t - Hw(\infty)$ and this is reachable, then the output of the controlled system tends to $y(\infty) = \hat{y}_t + Hw(\infty) = y_t$, and hence the offset is removed.

In order to use this idea to remove the offset on-line, an estimation of the disturbance must be calculated by means of an appropriate observer [18]. Thus, for a given estimation of the disturbance $\hat{w}(k)$, the controller with offset cancellation is given by

$$\begin{aligned} u(k) &= \kappa_N(x(k), \hat{y}_t(k)) \\ \hat{y}_t(k) &= y_t - H\hat{w}(k) \end{aligned} \tag{21}$$

The only assumption on the disturbance estimator is asymptotic stability and convergence of $\hat{w}(k)$ to the real steady disturbance $w(\infty)$.

The modified target together with the estimator dynamics (21) adds an outer feedback loop to the controller. Given that the proposed controller is feasible for all target, the variation of $\hat{y}_t(k)$ does not affect to the feasibility of the MPC. Furthermore, since the estimator dynamics does not depend on the MPC controller, the estimator converges at the steady value of the disturbance and then the whole system is stable.

Furthermore, the steady offset will be removed if the modified target $\hat{y}_t(\infty)$ is reachable, that is, $\hat{y}_t(\infty) \in \overline{\mathscr{Y}}_t$. If the target y_t and the steady disturbance $w(\infty)$ are such that $y_t - Hw(\infty)$ is contained in $\overline{\mathscr{Y}}_t$, then $\hat{y}_t(k)$ converges to $\hat{y}_t(\infty) \in \overline{\mathscr{Y}}_t$ and hence the real output converges to the desired reference $y(\infty) = y_t$.

6.1. Offset minimization

In the case that the steady modified target $\hat{y}_t(\infty)$ is not contained in $\overline{\mathscr{Y}}_t$, then the steady controlled variable $y(\infty)$ will present offset despite de proposed cancellation loop. As it will shown next, in this case the whole controller ensures that the offset is minimized.

It can be proved that, in this case, the real offset is given by

$$\begin{aligned} y(\infty) - y_t &= [\bar{y}_s^*(x(\infty), \hat{y}_t(\infty)) + Hw(\infty)] - y_t \\ &= \bar{y}_s^*(x(\infty), \hat{y}_t(\infty)) - \hat{y}_t(\infty) \end{aligned}$$

i.e. the deviation between the artificial steady output and the target of the control law. In virtue of Theorem 1, this offset is minimized by the controller according to the offset cost function. Consequently, the proposed controller with the offset cancellation loop steers the system to an steady output such that

$$\|y(\infty) - y_t\|_T^2 = \min_{\bar{y}_s \in \overline{\mathscr{Y}}_t} \|(\bar{y}_s + Hw(\infty) - y_t\|_T^2$$

Notice that matrix T can not make the offset null, but determines the actual offset of the system.

6.2. Illustrative example: the double integrator

This example shows how the offset can be removed by means of the proposed method, i.e. providing a target corrected by the estimated disturbance. Fig. 4 shows the response of the controlled uncertain system under extreme changes of the target. The evolution of the disturbances is depicted in plot (a) while plot (b) shows the trajectories of the controlled output and auxiliary variables for the robust MPC for tracking without offset cancellation loop. It can be seen how when there are non zero disturbances there exist offset.

Plot (c) shows the response of the controlled system with the offset cancellation loop. This is based on the estimation of the disturbance using a first order filter

$$\hat{w}(k) = b(x(k) - (Ax(k-1) + Bu(k-1))) + (1 - b)\hat{w}(k-1)$$

with a constant of $b = 0.95$. Notice that, due to the disturbances, the corrected reference $\hat{y}_t(k)$ becomes unfeasible in a period of time. Then, in this period, the offset can not be removed but minimized, keeping the feasibility.

7. Synthesis of the proposed controller

The proposed controller has some tuning parameters that, satisfying the sufficient conditions for stability, exhibit some degrees of freedom. These allow us to achieve an enhanced design by means of an optimized synthesis procedure.

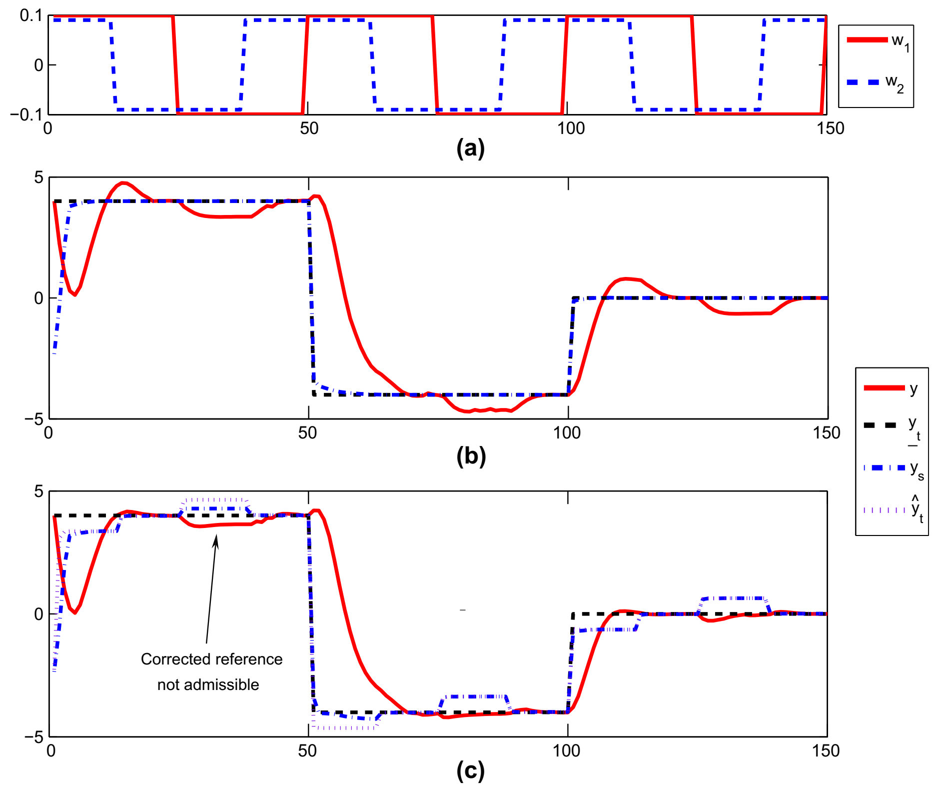

Fig. 4. Comparison of the robust MPC response without offset cancellation and with offset cancellation.

Matrices Q and R and the prediction horizon N are chosen as usual in MPC. The terminal ingredients $\overline{K}$ and P are typically chosen from the solution of the Riccati equation for optimality reasons. Once $\overline{K}$ is fixed, the invariant set for tracking $\Omega^a_{t,\overline{K}}$ can be calculated as proposed in [12,1]. Efficient methods to calculate the minimal robust invariant set ϕ_K and the invariant set for tracking $\Omega^a_{t,\overline{K}}$ are omitted due to lack of space and they can be found in [1] and references there in. Matrices T and K can be chosen as indicated in the following sections.

7.1. The offset cost weighting matrix T

This matrix defines the nominal offset cost term in the cost function $\|\bar{y}_s - y_t\|^2_T$. According to the properties of the proposed controller, the effects of this parameter on the closed-loop system are the following :

(i) *Set-point filtering*: if matrix T is chosen to penalize more heavily the offset cost, then the convergence of optimal artificial nominal controlled variable $\bar{y}^*_s(k)$ to the target y_t (assuming that $y_t \in \overline{\mathscr{Y}}_t$) is faster. Moreover, the optimality loss (due to the addition of the artificial steady state and input) can be arbitrarily reduced by scaling matrix T [1] producing an enhanced closed-loop performance.

(ii) *Offset minimization*: The controlled output $y(k)$ of the closed-loop system may exhibit steady offset, $y(\infty) - y_t$, due to: (i) the chosen target y_t is not nominally reachable, i.e. it is not compatible with the model or with the constraints, and (ii) the effect of disturbances, i.e. $w(\infty) \neq 0$.

Assume that the output offset cancellation loop is added, then the offset $(y(\infty) - y_t)$ is such that

$$\|y(\infty) - y_t\|^2_T = \min_{\bar{y}_s \in \overline{\mathscr{Y}}_t} \|(\bar{y}_s + Hw(\infty) - y_t\|^2_T$$

and hence it is minimized according to the chosen matrix T. Then, if the outer loop does not cancel the offset, this is minimized by the controller and this minimum is determined by matrix T. This allows us for instance to prioritize some of the outputs (by weighting its corresponding term in matrix T more heavily) to achieve a minimum nominal offset on these outputs. This is specially interesting for the case of thin plants (i.e. $p > m$), where the dimension of the subspace of reachable outputs is lower than p. It is also remarkable that since the closed-loop stability does not rely on the choice of matrix T, this can be adapted online according to a given criterium, as for instance, depending on regions in the output targets subspace.

7.2. The control gain matrix K

This parameter has an important role in the proposed robust controller. This control gain is used to compensate the deviation from the nominal predictions in case of disturbances by means of the control law (5). Therefore, this gain characterizes the dynamics of the closed-loop system in the presence of disturbances and should be designed according to a robustness or disturbance rejection criterium.

The control gain K should be chosen: (i) to ensure the existence of an admissible robust positively invariant set ϕ_K such that the set $\overline{\mathscr{Z}} \triangleq \mathscr{Z} \ominus (\phi_K \times K\phi_K)$ is not empty, and sufficiently large to provide to the MPC control law degrees of freedom to optimize the performance; and (ii) to reduce the effect of the disturbances on the closed-loop system by minimizing the size of ϕ_K.

In this paper, an LMI based design method is proposed to cope with both conditions. For this case it is assumed that $\mathscr{Z} = \mathscr{X} \times \mathscr{U}$ and, w.l.o.g., it is considered that $\mathscr{X} = \{x : |h_i^\top x| \leqslant 1,\ i = 1, \ldots, n_{rx}\}$ and $\mathscr{U} = \{u : |\ell_j^\top u| \leqslant 1,\ j = 1, \ldots, n_{ru}\}$. Then, the synthesis problem to solve is to calculate the control law $u = Kx$ such that the size of the ellipsoid $\mathcal{E}(P,1) = \{x \in \mathbf{R}^n : x^\top Px \leqslant 1\}$ is minimized while it is ensured that:

(i) $\mathcal{E}(P,1)$ is a robust invariant set for system (6).
(ii) For all $x \in \mathcal{E}(P,1), |\ell_j^\top Kx| \leqslant \rho_j$ for all $j = 1, \ldots, n_{ru}$ and $\rho_j \in (0,1]$. The role of the parameter ρ_j is to restrict the set of admissible control inputs to guarantee a given control range of the MPC controller i.e. the set $\overline{\mathcal{U}} = \mathcal{U} \ominus K\phi_K$ is not empty.
(iii) In order to minimize the size of the ellipsoid $\mathcal{E}(P,1)$, a suitable measure of this set must be chosen. In this paper, we propose a parameter $\gamma > 0$ as the measure such that $\mathcal{E}(P,1) \subseteq \sqrt{\gamma}\mathcal{X}$. Therefore, minimizing the size of $\mathcal{E}(P,1)$ is posed as minimizing the parameter γ. Obviously, admissibility of the solution requires that $\gamma \leqslant 1$.

Applying standard operations of LMIs (see [5]), the proposed synthesis procedure can be formulated as the solution of the following convex optimization problem:

$$\begin{aligned}
&\min_{Y,W,\gamma} \quad \gamma \\
&s.t. \\
&\qquad \begin{bmatrix} \lambda W & * & * \\ 0 & 1-\lambda & * \\ AW+BY & w & W \end{bmatrix} > 0, \quad \forall w \in vert(\mathcal{W}) \\
&\qquad \begin{bmatrix} \rho_i^2 & * \\ Y^\top \ell_i & W \end{bmatrix} > 0, \quad i = 1, \ldots, n_{ru} \\
&\qquad \begin{bmatrix} \gamma & * \\ Wh_i & W \end{bmatrix} > 0, \quad i = 1, \ldots, n_{rx}
\end{aligned}$$

for a given $\lambda \geqslant 0$. If feasible, the ellipsoid is given by $P = W^{-1}$ and the control gain is $K = YW^{-1}$. It is worth remarking that any robust criterium that can be posed as LMIs can be added to this synthesis problem.

Once the control gain K is designed, an admissible robust positively invariant (RPI) set (as small as possible) must be calculated. A procedure to achieve this can be found in [1].

8. Application of the robust MPC for tracking to the four tanks plant

In order to demonstrate the benefits of the proposed controller, this has been applied to a real experimental plant: the four tanks plant developed at the University of Seville [2].

The four tanks plant is a multivariable laboratory plant of interconnected tanks with nonlinear dynamics and subject to state and input constraints. One important property of this plant is that the dynamics present multivariable transmission zeros which can be located in the right hand side of the s plane for some operating conditions. This plant is based on the well-known quadruple-tank process [10], and its scheme can be seen in Fig. 5a. In the original plant, the inputs are the voltages of the two pumps and the outputs are the water levels in the lower two tanks. Fig. 5b shows the scheme of the real plant. The main difference is that a control valve regulates the inlet flow of each tank. The three-way valve ratio is imposed by a suitable choice of the references of the flows.

A state-space continuous time model of the quadruple-tank process system [10] can be derived from first principles as follows

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_a}{A_1}q_a \\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_b}{A_2}q_b \\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{A_3}q_b \\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{A_4}q_a
\end{aligned} \tag{22}$$

The estimated parameters of the real plant and the considered intervals of admissible variation of the levels and flows are shown in the following table:

	Value	Unit	Description
H_{1max}	1.36	m	Maximum level of the tank 1
H_{2max}	1.36	m	Maximum level of the tank 2
H_{3max}	1.30	m	Maximum level of the tank 3
H_{4max}	1.30	m	Maximum level of the tank 4
H_{min}	0.3	m	Minimum level in all cases
Q_{1max}	2.8	m^3/h	Maximal inflow of tank 1
Q_{2max}	2.45	m^3/h	Maximal inflow of tank 2
Q_{3max}	2.3	m^3/h	Maximal inflow of tank 3
Q_{4max}	2.4	m^3/h	Maximal inflow of tank 4
Q_{min}	0	m^3/h	Minimal inflow in all cases
Q_a^0	1.6429	m^3/h	Equilibrium flow $(Q_1 + Q_4)$
Q_b^0	2.0000	m^3/h	Equilibrium flow $(Q_2 + Q_3)$
a_1	1.341e-4	m^2	Discharge constant of tank 1
a_2	1.533e-4	m^2	Discharge constant of tank 2
a_3	9.322e-5	m^2	Discharge constant of tank 3
a_4	9.061e-5	m^2	Discharge constant of tank 4
A	0.06	m^2	Cross-section of all tanks
γ_a	0.3		Parameter of the 3-ways valve
γ_b	0.4		Parameter of the 3-ways valve
h_1^0	0.627	m	Equilibrium level of tank 1
h_2^0	0.636	m	Equilibrium level of tank 2
h_3^0	0.652	m	Equilibrium level of tank 3
h_4^0	0.633	m	Equilibrium level of tank 4

The minimum level of the tanks has been taken greater than zero to prevent eddy effects in the discharge of the tank. The values of γ_a and γ_b have been chosen in order to obtain a system with non-minimum phase multivariable zeros.

Linearizing the model at an operating point given by h_i^0 and defining the deviation variables $x_i = h_i - h_i^o$ and $u_j = q_j - q_j^o$ where $j = a, b$ and $i = 1, \cdots, 4$ we have that:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{A_3}{A_1\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{A_4}{A_2\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_a}{A_1} & 0 \\ 0 & \frac{\gamma_b}{A_2} \\ 0 & \frac{(1-\gamma_b)}{A_3} \\ \frac{(1-\gamma_a)}{A_4} & 0 \end{bmatrix} u$$

where $\tau_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}} \geqslant 0$, $i = 1, \ldots, 4$, are the time constants of each tank. This model has been discretized using the zero-order hold method with a sampling time of 5 s.

The main sources of deviation between the nonlinear model and the real plant are: (i) the linearization error; (ii) the hypothesis that parameters a_i do not depend on the levels of the tank; and (iii) the actuator dynamics since the modelled input to the plant is the reference of the PID that controls the flow of each pipe.

For the identification of $\mathcal{W}$, a worst case scenario input signal made of a series of step changes was applied to the plant. The level trajectories of this experiment allowed us to obtain a worst case bound on the disturbances. The set $\mathcal{W}$ identified from the experimental data is defined by the following inequalities:

$$\mathcal{W} \triangleq \{w \in \mathbf{R}^4 : \|w\|_\infty \leqslant 5 \times 10^{-3}\} \tag{23}$$

The proposed controller has been designed as described above. The defining matrices of the stage cost of the performance criterion have been chosen as

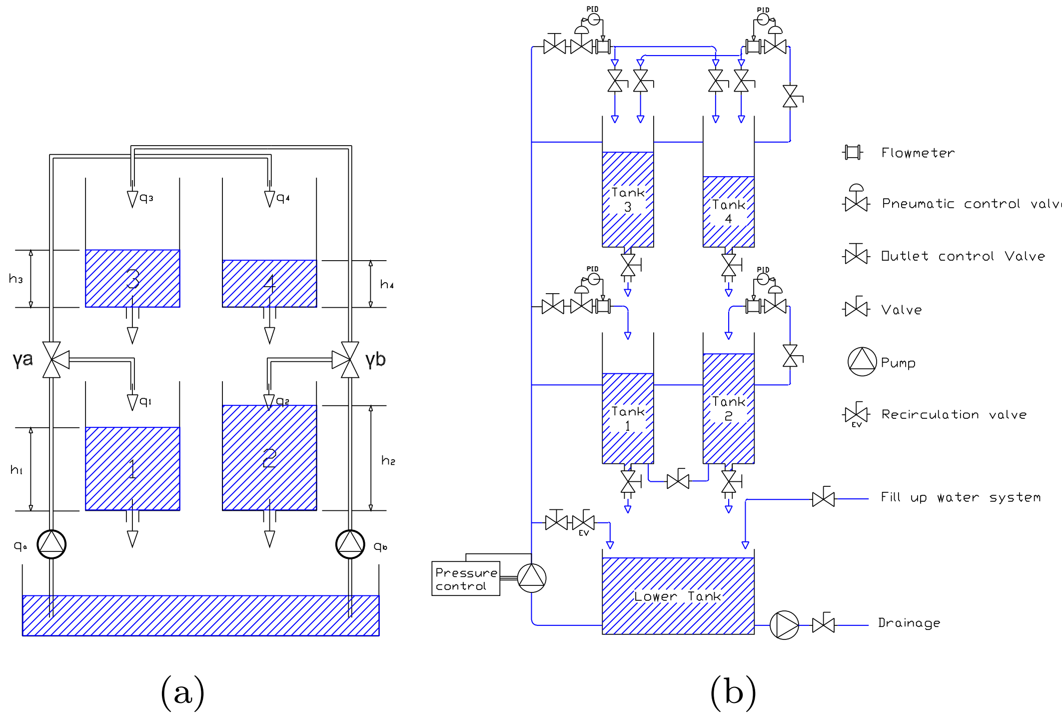

Fig. 5. Schemes of the quadruple-tank process (a) and of the real plant (b).

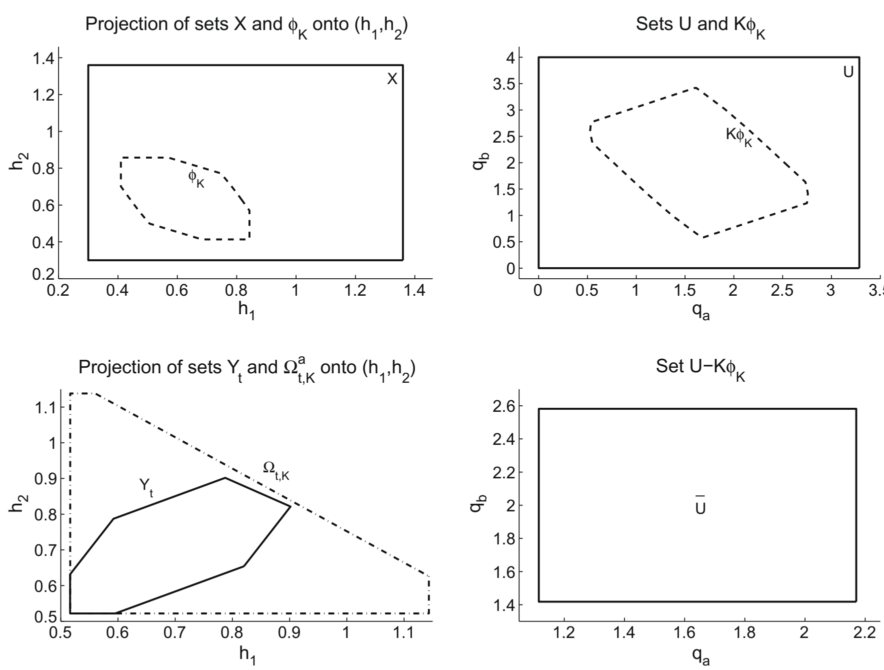

Fig. 6. Different sets of the MPC for tracking applied to the quadruple-tank process.

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = 10^{-4} I_4 \tag{24}$$

For the calculation of K, the LMI based method proposed in Section 7.2 has been used with a parameter $\rho_1 = 1$ and $\rho_2 = 1$. The obtained value of K is:

$$K = \begin{bmatrix} -5.9997 & -18.7429 & 6.2544 & -37.0666 \\ -20.6413 & -12.8487 & -29.7042 & -3.0337 \end{bmatrix}$$

The terminal control gain $\overline{K}$ has been chosen as the LQR gain for the matrices (24) and matrix P is the solution of the Ricatti equation. The offset cost weighting matrix T has been chosen as $T = 100P$. Finally the control horizon has been chosen as $N = 3$. The resulting regions are shown in Fig. 6.

The derived controller has been tested on the nominal model, and then applied on the real plant. This has been implemented and executed in MATLAB connected with the SCADA by means of OPC (OLE for Process Control) protocol. Fig. 7 shows the real evolution of levels h_1 and h_2, the set of the reachable targets and the targets.

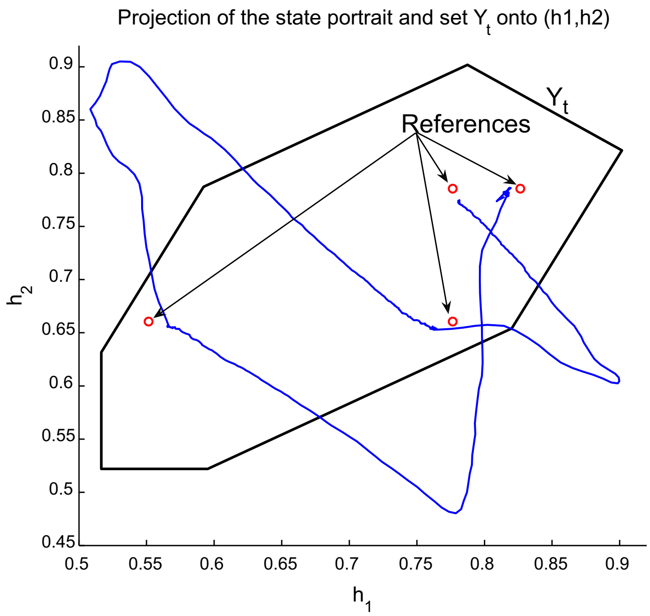

Fig. 7. Evolution of the levels h_1 and h_2.

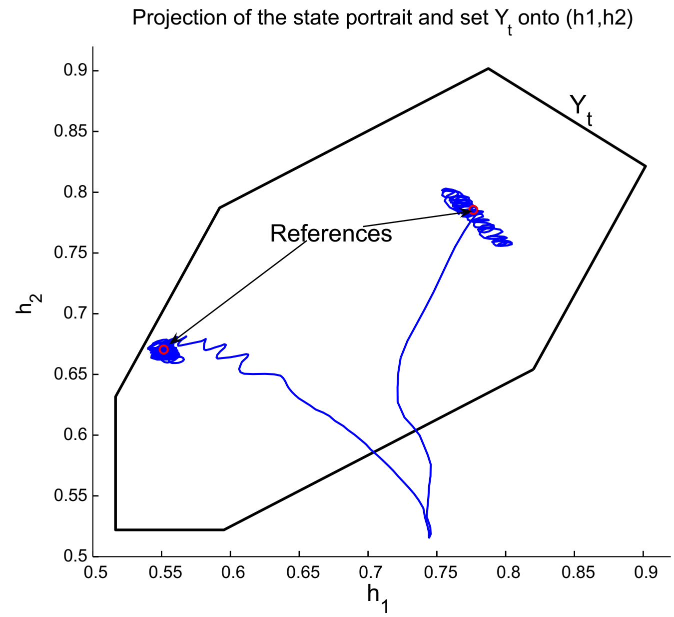

Fig. 9. Evolution of the outputs.

Fig. 8 shows the level trajectories (solid lines) and the references (dashed line and semi dashed line, in the case of h_1 and h_2). The plot in the middle shows the control actions while the plot in the bottom shows the levels h_3 and h_4. See that the system exhibits the classic non-minimum phase system response. It can be also seen how the setpoint changes are performed satisfying the constraints, but there is an offset as consequence of the disturbances and/or model discrepancies.

In order to cancel the existing offset, the method proposed in Section 6 has been tested. To this aim, the following simple estimator that guarantees that $\hat{w}(k)$ converges at the steady value $w(\infty)$ with a rate of convergence λ_f has been used:

$$\hat{w}(k) = \lambda_f \hat{w}(k-1) + (1-\lambda_f)(x(k) - Ax(k-1) - Bu(k-1))$$

The value of λ_f has been chosen as $\lambda_f = 0.98$.

Figs. 9 and 10 show the time evolution of the plant with the disturbance rejection. It can be seen how the offset is removed thanks to the additional disturbance estimator. However, the system also exhibits an oscillatory behavior. This is derived from the dynamics of the level due to the falling water. Thus, the measured level signal

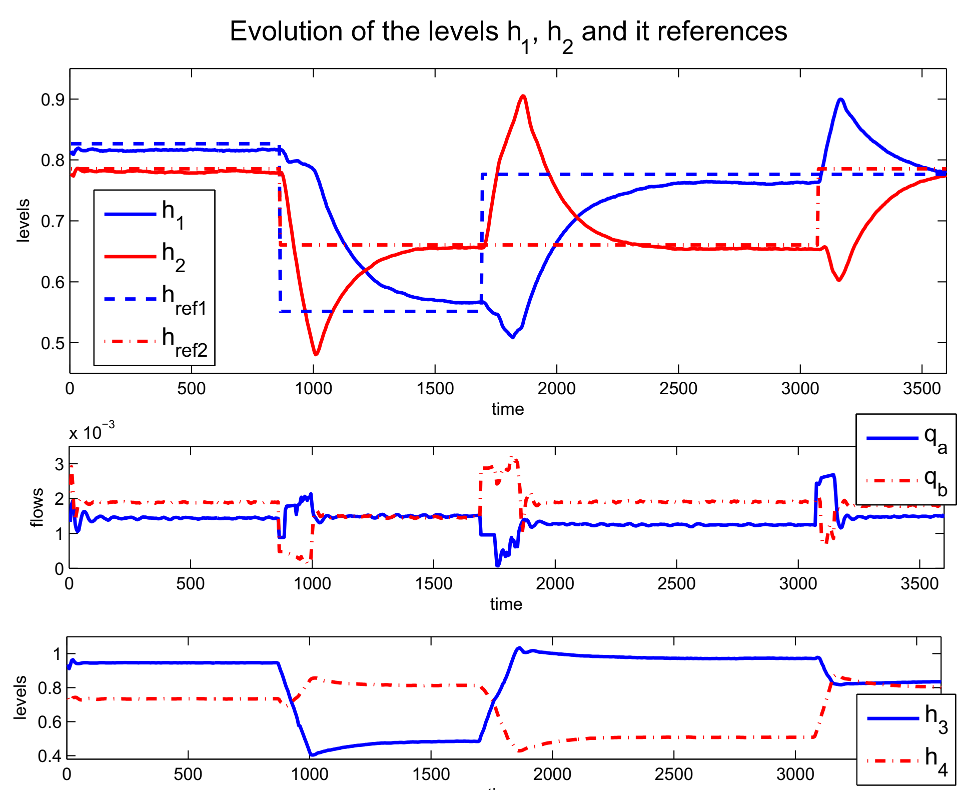

Fig. 8. Time evolution of the plant.

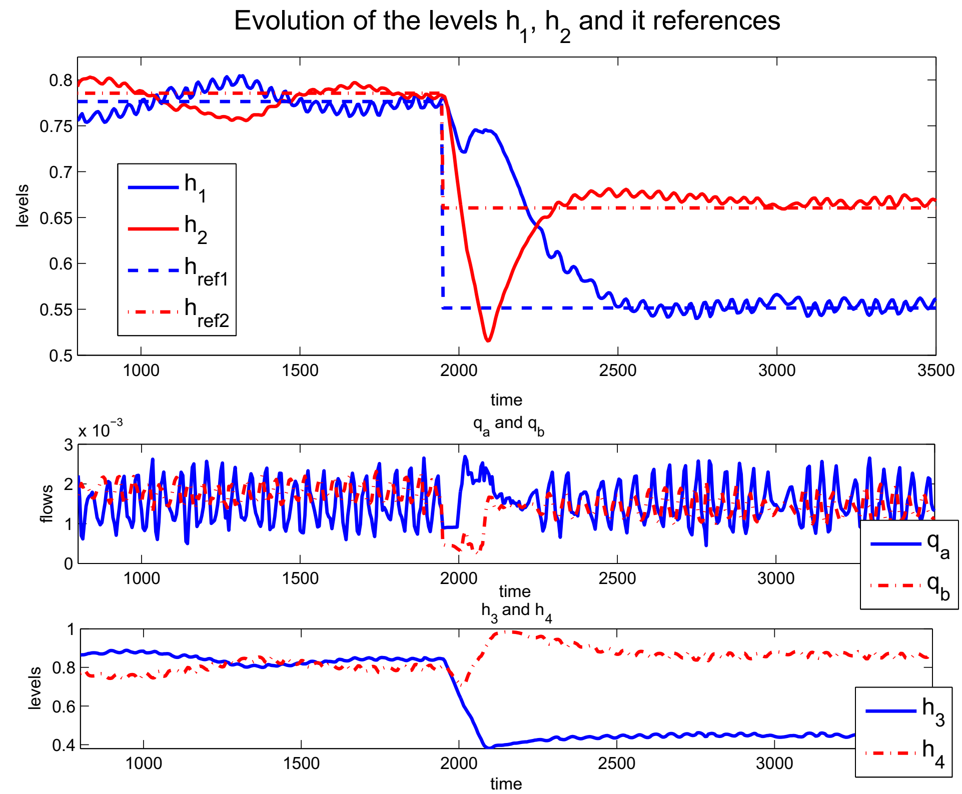

Fig. 10. Time evolution of the plant.

produces an oscillatory evolution of the modified set-point through the disturbance estimator feedback loop. The parameter λ_f can be adjusted to reduce this effect at expense of a slower evolution of the estimator and hence a slower convergence to the target. Then, a trade off has been achieved in the presented results. On the other hand, a more involved estimator could be chosen to obtain a response with less oscillations. However, this choice has not been considered in this paper because the objective is to show the proposed controller and the offset removal, and this is clearly demonstrated in the obtained results.

9. Conclusions

In this paper a novel robust MPC to track piece-wise references has been presented. The robustness of controller is achieved by using the notion of tubes. Feasibility of the problem for any desired admissible steady state is guaranteed by adding an artificial target and considering an extended terminal constraint. Robust convergence is ensured by minimizing a performance index which penalizes the error with the desired steady state and the deviation between the desired steady state and the artificial one. The control law is derived from the solution of a QP problem.

The proposed controller has been successfully applied to the quadruple-tank process, which is a nonlinear uncertain multivariable process. The operation point is characterized by non-minimum phase multivariable behaviour. The proposed robust MPC for tracking successfully solves this control problem.

Acknowledgements

The authors acknowledge MCYT-Spain (DPI2007-66718-C04-01) and the European FP7 STREP project HD-MPC (FP7-ICT-223854) for funding this work.

Appendix A. Proof of Theorem 1

The proof is based on the following lemma:

Lemma 1. *Consider that system (1) subject to constraints (2) fulfils Assumption 1 . Suppose that the ingredients of* $\mathscr{P}_N(\cdot,\cdot)$ $(Q, R, T, P, K$ *and* $\overline{K}$, *and sets* Φ_K *and* $\Omega^a_{t,\overline{K}})$ *satisfy Assumption 2 . Consider a given controlled variables target* y_t *and assume that for a given state* x *the optimal solution of* $\mathscr{P}_N(x, y_t)$ *is such that* $\bar{x}^*(x, y_t) = \bar{x}^*_s(x, y_t)$. *Let* $\tilde{\theta}$ *be given by*

$$\tilde{\theta} = \arg\min_{\bar{\theta}\in\overline{\Theta}} \|N_\theta\bar{\theta} - y_t\|^2_T$$

where $\overline{\Theta}$ *is defined in (14) . Let* $\tilde{x}_s, \tilde{u}_s$ *and* $\tilde{y}_s$ *be the steady state, input and controlled output associated to* $\tilde{\theta}$ *(i.e.* $(\tilde{x}_s, \tilde{u}_s) = M_\theta\tilde{\theta}$ *and* $\tilde{y}_s = N_\theta\tilde{\theta} = C\tilde{x}_s + D\tilde{u}_s$*). Then*

$$\bar{x}^*_s(x, y_t) = \tilde{x}_s,\ \bar{u}^*_s(0; x, y_t) = \tilde{u}_s,\ \bar{y}^*_s(x, y_t) = \tilde{y}_s$$

Proof. Consider that the optimal solution of $\mathscr{P}_N(x, y_t)$ is $(\bar{x}^*_s, \bar{u}^*_s) = M_\theta\bar{\theta}^*$ [2] and hence the optimal cost function is $V^*_N(x, y_t) = \|\bar{y}^*_s - y_t\|^2_T$.

In order to prove the lemma by contradiction, it will be assumed that $\bar{\theta}^* \neq \tilde{\theta}$.

Let us define a parameter $\hat{\theta}$ given by

$$\hat{\theta} = \beta\bar{\theta}^* + (1-\beta)\tilde{\theta} \quad \beta \in [0, 1]$$

From continuity arguments it can be derived that there exists a $\hat{\beta} \in [0, 1)$ such that for every $\beta \in [\hat{\beta}, 1)$, the state $\bar{x}^*_s$ is contained in the maximal admissible invariant set for the nominal system controlled by $u = \bar{K}(x - \bar{x}_s) + \bar{u}_s$, where $(\bar{x}_s, \bar{u}_s) = M_\theta\hat{\theta}$. (see [1] for further details).

[2] In this proof, the dependence of the optimal solution to (x, y_t) will be omitted for the sake of clarity.

Therefore, defining $\bar{\mathbf{u}}$ as the sequence of control actions derived from this control law, it is easily inferred that $(\bar{\mathbf{u}}, \bar{x}_s^*, \hat{\theta})$ is a feasible solution for $\mathscr{P}_N(x, y_t)$. The cost function of this feasible solution satisfies the following

$$\begin{aligned} V_N^*(x, y_t) &\leqslant V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}, \hat{\theta}) = \|\bar{\theta}^* - \hat{\theta}\|^2_{M_x^\top P M_x} + \|N_\theta \hat{\theta} - y_t\|_T^2 \\ &= (1-\beta)^2 \|\bar{\theta}^* - \tilde{\theta}\|^2_{M_x^\top P M_x} + \|N_\theta \hat{\theta} - y_t\|_T^2 \end{aligned}$$

where M_x is the matrix formed by the first n rows of M_θ.

It will be proved that a value of $\beta \in [\hat{\beta}, 1)$ exists such that this feasible solution is lower than the optimal cost, yielding a contradiction. To this aim, it is observed that the partial derivative of the cost function with respect to β is:

$$\frac{\partial V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}, \hat{\theta})}{\partial \beta} = -2(1-\beta)\|\theta - \theta\|^2_{M_x^\top P M_x} + 2(N_\theta \hat{\theta} - y_t)^\top T (N_\theta(\bar{\theta}^* - \tilde{\theta}))$$

For $\beta = 1$:

$$\left.\frac{\partial V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}, \hat{\theta})}{\partial \beta}\right|_{\beta=1} = 2(N_\theta \bar{\theta}^* - y_t)^\top T (N_\theta(\bar{\theta}^* - \tilde{\theta}))$$

Using the first order condition for convex functions [5] and considering that $\tilde{\theta} \neq \bar{\theta}^*$, we have thats

$$2(N_\theta \bar{\theta}^* - y_t)^\top T (N_\theta(\bar{\theta}^* - \tilde{\theta})) < \|N_\theta \tilde{\theta} - y_t\|_T^2 - \|N_\theta \bar{\theta}^* - y_t\|_T^2$$

Since $\tilde{\theta}$ is the minimizer of the offset cost, the assumption that $\tilde{\theta} \neq \bar{\theta}^*$ yields to $\|N_\theta \bar{\theta} - y_t\|_T^2 - \|N_\theta \tilde{\theta} - y_t\|_T^2 > 0$ and then $\left.\frac{\partial V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}, \hat{\theta})}{\partial \beta}\right|_{\beta=1} < 0$. This means that a value of $\beta \in [\hat{\beta}, 1)$ exists arbitrarily close to 1 such that $V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}, \hat{\theta})$ for β is lower than $V_N(x, y_t; \bar{\mathbf{u}}, \bar{x}, \hat{\theta})$ for $\beta = 1$, which is equal to $V_N^*(x, y_t)$. This fact leads to a contradiction, proving the lemma. □

A.1. Proof of Theorem 1

The first part of the proof is devoted to proving the recursive feasibility of the controlled system, that is, $x(k+1) \in \mathscr{X}_N$, for all $x(k) \in \mathscr{X}_N, w(k) \in \mathscr{W}$ and y_t. This property implies the robust constraint satisfaction of the controlled system.

Consider the optimal solution of $\mathscr{P}_N(x(k), y_t)$, then the successor state is

$$x(k+1) = Ax(k) + B\kappa_N(x(k), y_t) + w(k),$$

where $w(k) \in \mathscr{W}$. Define the following sequences:

$$\begin{aligned} &\bar{\mathbf{u}}(x(k+1), y_t) \triangleq [\bar{u}^*(1; x(k), y_t), \ldots, \bar{u}^*(N-1; x(k), y_t), \\ &\overline{K}(\bar{x}^*(N-1; x(k), y_t) - \bar{x}_s^*(x(k), y_t)) + \bar{u}_s^*(x(k), y_t)] \\ &\bar{x}(x(k+1), y_t) \triangleq \bar{x}^*(1; x(k), y_t) \quad \bar{\theta}(x(k+1), y_t) \triangleq \bar{\theta}^*(x(k), y_t) \end{aligned} \tag{25}$$

The proposed solution $(\bar{\mathbf{u}}, \bar{x}, \bar{\theta})$ is feasible for the optimization problem $\mathscr{P}_N(x(k+1), y_t)$ due to the following arguments:

- From the notion of the tube of trajectories it is possible to deduce that $x(k+1) \in \bar{x}(x(k+1), y_t) \oplus \Phi_K$.
- Since $\bar{x}(x(k+1), y_t) = \bar{x}^*(1; x(k), y_t)$, it can be derived that

  $$\bar{x}(i; x(k+1), y_t) = \bar{x}^*(i+1; x(k), y_t), \quad i = 0, 1, \ldots, N-1$$

  and then, the first $N-1$ terms of the nominal trajectory are admissible. Admissibility of $\bar{x}(N; x(k+1), y_t)$ stems from the fact that

  $$(\bar{x}(N-1; x(k+1), y_t), \bar{\theta}(x(k+1), y_t)) \in \Omega^a_{t,\overline{K}}$$

  and hence the control action $\bar{u}(N-1; x(k+1), y_t)$ ensures that

  $$(\bar{x}(N; x(k+1), y_t), \bar{\theta}(x(k+1), y_t)) \in \Omega^a_{t,\overline{K}}$$

- Feasibility of $\bar{\mathbf{u}}^*(x(k), y_t)$ and admissibility of set $\Omega^a_{t,\overline{K}}$ ensures the feasibility of $\bar{\mathbf{u}}(x(k+1), y_t)$.
- The terminal constraint satisfaction stems from the invariance for tracking of $\Omega^a_{t,\overline{K}}$.

Convergence is demonstrated by showing that the statement (iii) holds for any target controlled variable y_t. That is, for any y_t, the controlled system evolves to the set $\tilde{y}_s \oplus (C + DK)\phi_K$, where

$$\tilde{y}_s = arg \min_{\bar{y}_s \in \overline{\mathscr{Y}}_t} \|\bar{y}_s - y_t\|_T^2$$

Notice that if $y_t \in \overline{\mathscr{Y}}_t$, then $\tilde{y}_s = y_t$ and hence statement (ii) is derived.

As it is standard in MPC, this is proved by showing that the optimal cost is a Lyapunov function. Taking into account the properties of the feasible nominal trajectories for $x(k+1)$, the condition (iv) of Assumption 2 and using standard procedures in MPC [14], it can be obtained that the proposed feasible solution (25) can be made to fulfil the following condition:

$$\begin{aligned} &V_N(x(k+1), y_t; \bar{\mathbf{u}}, \bar{x}, \bar{\theta}) - V_N^*(x(k), y_t) \\ &\quad \leqslant -\|\bar{x}^*(x(k), y_t) - \bar{x}_s^*(x(k), y_t)\|_Q^2 - \|\bar{u}^*(0; x(k), y_t) - \bar{u}_s^*(x(k), y_t)\|_R^2 \\ &\quad \leqslant -\|\bar{x}^*(x(k), y_t) - \bar{x}_s^*(x(k), y_t)\|_Q^2 \end{aligned}$$

By optimality, we have that

$$V_N^*(x(k+1), y_t) \leqslant V_N(x(k+1), y_t; \bar{\mathbf{u}}, \bar{x}, \bar{\theta})$$

and then:

$$V_N^*(x(k+1), y_t) - V_N^*(x(k), y_t) \leqslant -\|\bar{x}^*(x(k), y_t) - \bar{x}_s^*(x(k), y_t)\|_Q^2$$

Taking into account that $\left(Q^{\frac{1}{2}}, A\right)$ is observable, we have that

$$\lim_{k \to \infty} \|\bar{x}^*(x(k), y_t) - \bar{x}_s^*(x(k), y_t)\| = 0$$

which implies that

$$\lim_{k \to \infty} \|\bar{y}^*(x(k), y_t) - \bar{y}_s^*(x(k), y_t)\| = 0$$

where $\bar{y}^*(x(k), y_t) = C\bar{x}^*(x(k), y_t) + D\bar{u}^*(0; x(k), y_t)$.

In virtue of Lemma 1, we can deduce that if

$$\|\bar{x}^*(x(k), y_t) - \bar{x}_s^*(x(k), y_t)\| \ \to \ 0$$

then $\|\bar{x}_s^*(x(k), y_t) - \tilde{x}_s\| \ \to \ 0$ and $\|\bar{u}^*(0; x(k), y_t) - \tilde{u}_s\| \ \to \ 0$.

Thus, from the notion of tubes, it is derived that the real state of the plant $x(k)$ tends to $\tilde{x}_s \oplus \Phi_K$ and the applied control input $u(k)$ tends to $\tilde{u}_s \oplus K\Phi_K$. Therefore, the real controlled variable $y(k) = Cx(k) + Du(k)$ tends to $C(\tilde{x}_s \oplus \Phi_K) \oplus D(\tilde{u}_s \oplus K\Phi_K) = \tilde{y}_s \oplus [C + DK]\Phi_K$. □

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