Model Predictive Control
Algorithm, Feasibility and Stability

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Infinite Time Constrained Optimal Control (what we would like to solve)

$$
\begin{aligned}
J_0^*(x(0)) = \min \sum_{k=0}^{\infty} q(x_k, u_k)& \\
\text{s.t. } & x_{k+1} = Ax_k + Bu_k, k = 0, \ldots, N-1 \\
& x_k \in \mathcal{X}, u_k \in \mathcal{U}, k = 0, \ldots, N-1 \\
& x_0 = x(0)
\end{aligned}
$$

- **Stage cost** $q(x, u)$ describes "cost" of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
  ... but we can't compute it: there are an **infinite number of variables**

Receding Horizon Control (what we can sometimes solve)

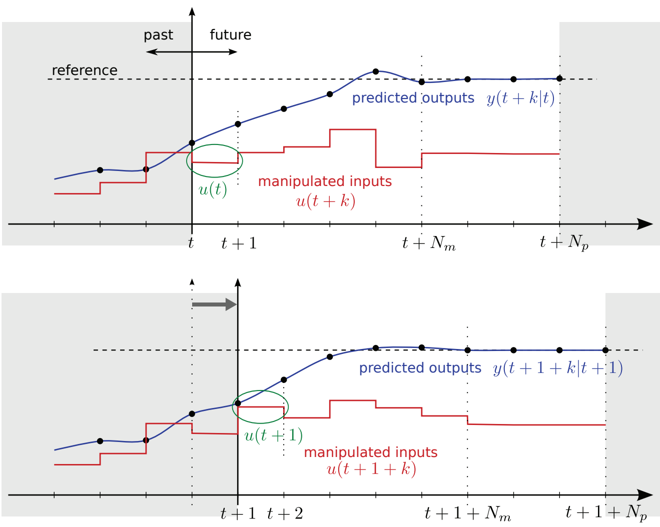
$$
\begin{array}{lll}
J_t^*(x(t)) = & \min\limits_{U_t} & p(x_{t+N}) + \sum\limits_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) \\
& \text{subj. to} & x_{t+k+1} = Ax_{t+k} + Bu_{t+k},\ k = 0, \ldots, N-1 \\
& & x_{t+k} \in \mathcal{X},\ u_{t+k} \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& & x_{t+N} \in \mathcal{X}_f \\
& & x_t = x(t)
\end{array} \quad (1)
$$

where $\mathcal{U}_t = \{u_t, \ldots, u_{t+N-1}\}$.

Truncate after a finite horizon:

- $p(x_{t+N})$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

On-line Receding Horizon Control



1. At each sampling time, solve a **CFTOC**.
2. Apply the optimal input **only during** $[t, t+1]$
3. At $t+1$ solve a CFTOC over a **shifted horizon** based on new state measurements
4. The resultant controller is referred to as **Receding Horizon Controller (RHC)** or **Model Predictive Controller (MPC)**.

On-line Receding Horizon Control

1) MEASURE the state $x(t)$ at time instance t
2) OBTAIN $U_t^*(x(t))$ by solving the optimization problem in (1)
3) IF $U_t^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element u_t^* of U_t^* to the system
5) WAIT for the new sampling time $t+1$, GOTO 1)

Note that, we need a constrained optimization solver for step 2).

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History of MPC

- **A. I. Propoi, 1963**, "Use of linear programming methods for synthesizing sampled-data automatic systems", *Automation and Remote Control*.
- **J. Richalet et al., 1978** "Model predictive heuristic control- application to industrial processes". *Automatica*, 14:413-428.
  - known as **IDCOM (Identification and Command)**
  - impulse response model for the plant, linear in inputs or internal variables (**only stable plants**)
  - quadratic performance objective over a finite prediction horizon
  - future plant output behavior specified by a reference trajectory
  - **ad hoc** input and output constraints
  - optimal inputs computed using a heuristic iterative algorithm, interpreted as the dual of identification
  - controller was not a transfer function, hence called **heuristic**

History of MPC

- 1970s: Cutler suggested MPC in his PhD proposal at the University of Houston in 1969 and introduced it later at Shell under the name Dynamic Matrix Control. **C. R. Cutler, B. L. Ramaker, 1979** "Dynamic matrix control – a computer control algorithm". *AICHE National Meeting*, Houston, TX.
  - successful in the petro-chemical industry
  - linear step response model for the plant
  - quadratic performance objective over a finite prediction horizon
  - future plant output behavior specified by trying to follow the set-point as closely as possible
  - input and output constraints included in the formulation
  - optimal inputs computed as the solution to a least–squares problem
  - **ad hoc** input and output constraints. Additional equation added online to account for constraints. Hence a **dynamic matrix** in the least squares problem.
- **C. Cutler, A. Morshedi, J. Haydel, 1983**. "An industrial perspective on advanced control". *AICHE Annual Meeting*, Washington, DC.
  - Standard QP problem formulated in order to systematically account for constraints.

History of MPC

- Mid 1990s: extensive theoretical effort devoted to provide conditions for guaranteeing feasibility and closed-loop stability
- 2000s: development of tractable robust MPC approaches; nonlinear and hybrid MPC; MPC for very fast systems
- 2010s: stochastic MPC; distributed large-scale MPC; economic MPC

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RHC Notation

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ \forall t \geq 0$$

The CFTOC Problem

$$
\begin{aligned}
J_t^*(x(t)) = \quad & \min_{U_{t\rightarrow t+N|t}} && p(x_{t+N|t}) + \sum_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t}) \\
& \text{subj. to} && x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t},\ k = 0, \ldots, N-1 \\
& && x_{t+k|t} \in \mathcal{X},\ u_{t+k|t} \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& && x_{t+N|t} \in \mathcal{X}_f \\
& && x_{t|t} = x(t)
\end{aligned}
$$

with $U_{t\rightarrow t+N|t} = \{u_{t|t}, \ldots, u_{t+N-1|t}\}$.

RHC Notation

- $x(t)$ is the state of the system at time t.
- $x_{t+k|t}$ is the state of the model at time $t+k$, predicted at time t obtained by starting from the current state $x_{t|t} = x(t)$ and applying to the system model

  $$x_{t+1|t} = Ax_{t|t} + Bu_{t|t}$$

  the input sequence $u_{t|t}, \ldots, u_{t+k-1|t}$.
- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $t = 1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $t = 2$ starting from the current state $x(2)$.
- Similarly $u_{t+k|t}$ is read as "the input u at time $t+k$ computed at time t".

RHC Notation

- Let $U^*_{t\to t+N|t} = \{u^*_{t|t}, \dots, u^*_{t+N-1|t}\}$ be the optimal solution. The first element of $U^*_{t\to t+N|t}$ is applied to system

$$u(t) = u^*_{t|t}(x(t)).$$

- The CFTOC problem is reformulated and solved at time $t+1$, based on the new state $x_{t+1|t+1} = x(t+1)$.

Receding horizon control law

$$f_t(x(t)) = u^*_{t|t}(x(t))$$

Closed loop system

$$x(t+1) = Ax(t) + Bf_t(x(t)) \triangleq f_{cl}(x(t)),\ \ t \geq 0$$

RHC Notation: Time-invariant Systems

As the system, the constraints and the cost function are time-invariant, the solution $f_t(x(t))$ becomes a time-invariant function of the initial state $x(t)$. Thus, we can simplify the notation as

$$\begin{aligned} J_0^*(x(t)) = \quad \min_{U_0} \quad & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ \text{subj. to} \quad & \\ & x_{k+1} = Ax_k + Bu_k,\ \ k = 0, \dots, N-1 \\ & x_k \in \mathcal{X},\ \ u_k \in \mathcal{U},\ \ k = 0, \dots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(t) \end{aligned}$$

where $U_0 = \{u_0, \dots, u_{N-1}\}$.

The control law and closed loop system are **time-invariant** as well, and we write $f_0(x_0)$ for $f_t(x(t))$.

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MPC Features

Pros

- Any model
  - linear
  - nonlinear
  - single/multivariable
  - time delays
  - constraints
- Any objective:
  - sum of squared errors
  - sum of absolute errors (i.e., integral)
  - worst error over time
  - economic objective

Cons

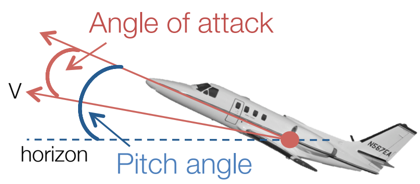
- Computationally demanding in the general case
- May or may not be stable
- May or may not be feasible

Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: 0, 0, $-1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$J_\infty(x(t)) = \min \sum_{k=0}^{\infty} x_t^T Q x_t + u_k^T R u_k$$
$$\text{s.t. } x_{k+1} = A x_k + B u_k$$
$$x_0 = x(t)$$

Assume: $Q = Q^T \succeq 0$, $R = R^T \succ 0$

MPC

$$J_0^*(x(t)) = \min_{U_0} \sum_{k=0}^{N-1} {x_k}^T Q x_k + {u_k}^T R u_k$$
$$\text{s.t. } x_{k+1} = A x_k + B u_k$$
$$x_k \in \mathcal{X}, \ u_k \in \mathcal{U}$$
$$x_0 = x(t)$$

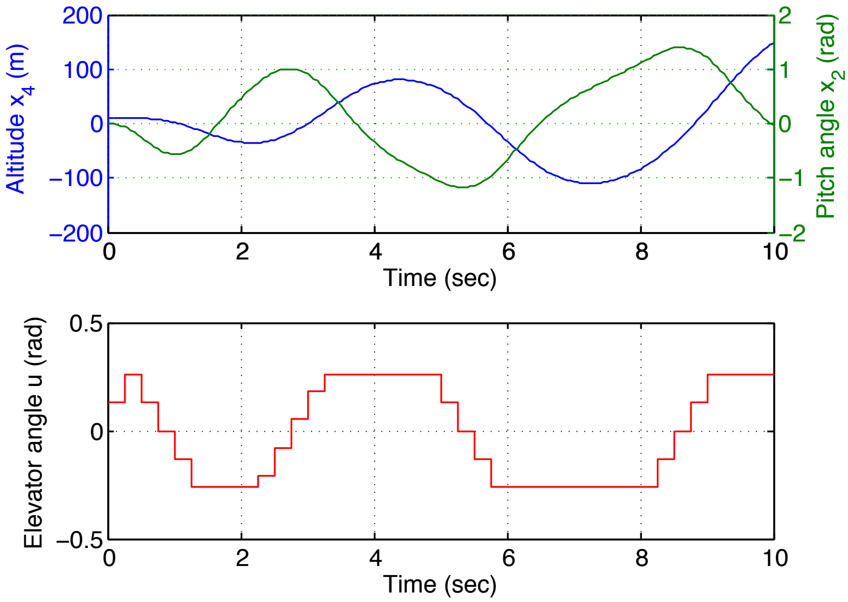
Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$

Problem parameters:

Sampling time 0.25sec,
$Q = I$, $R = 10$



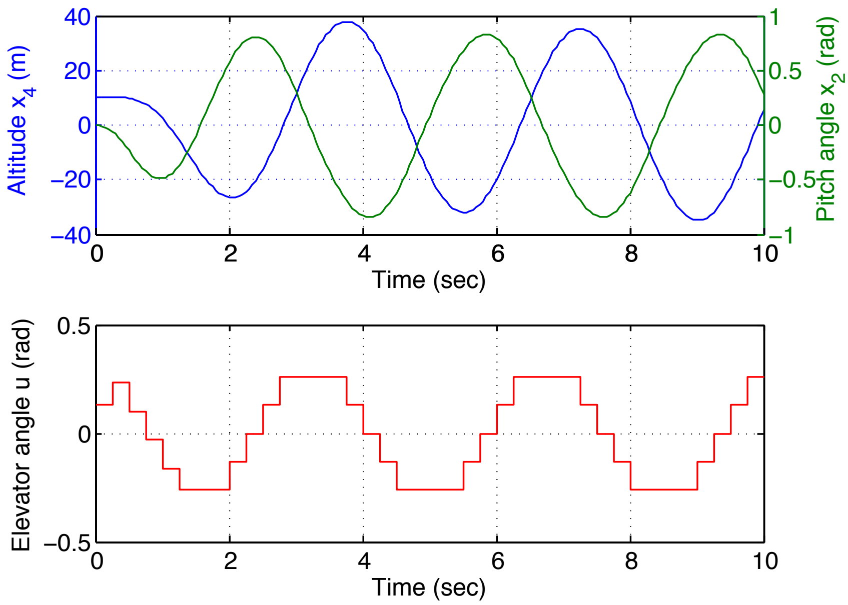
- Closed-loop system is unstable
- Applying LQR control and saturating the controller can lead to instability!

Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_i| \leq 0.262$

Problem parameters:

Sampling time 0.25sec,
$Q = I$, $R = 10$, $N = 10$



The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

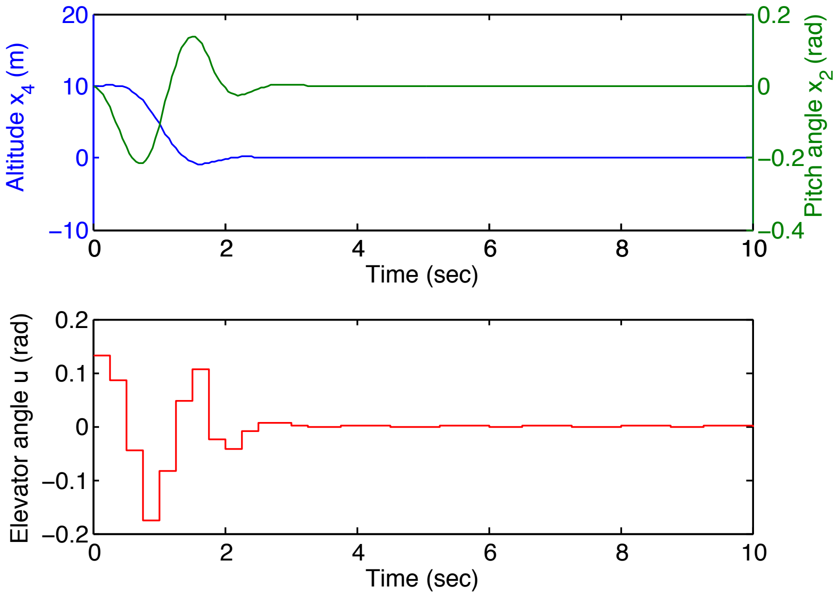
$\Rightarrow$ System does not converge to desired steady-state but to a limit cycle

Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$



The MPC controller considers all constraints on the actuator

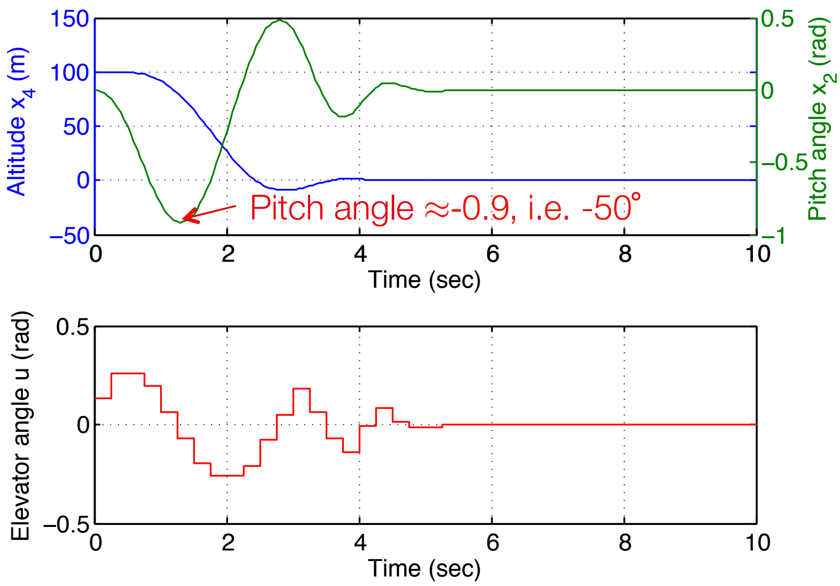
- Closed-loop system is stable
- Efficient use of the control authority

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$



Increase step:
At time $t = 0$ the plane is flying with a deviation of 100m of the desired altitude, i.e. $x_0 = [0; 0; 0; 100]$

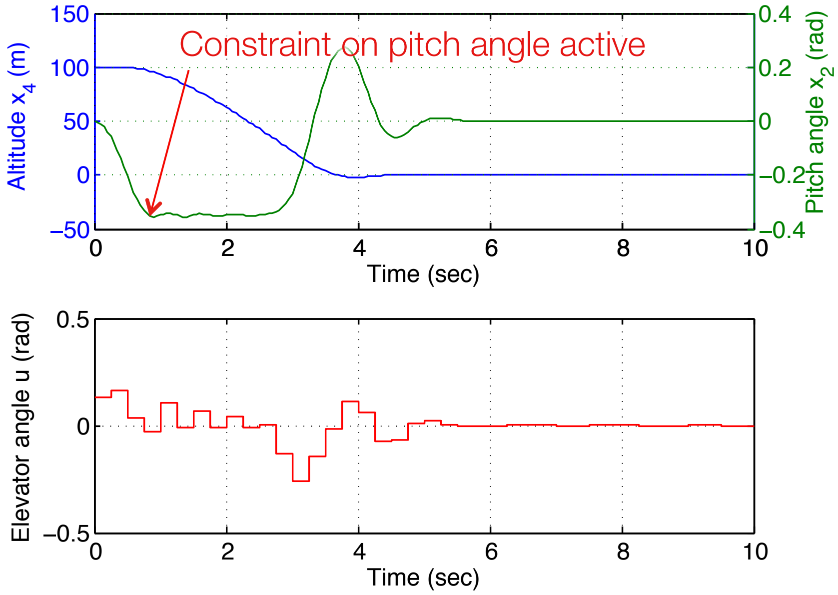
- Pitch angle too large during transient

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\text{Sampling time } 0.25\text{sec}, \; Q = I, \; R = 10, \; N = 10}$



Add state constraints for passenger comfort:

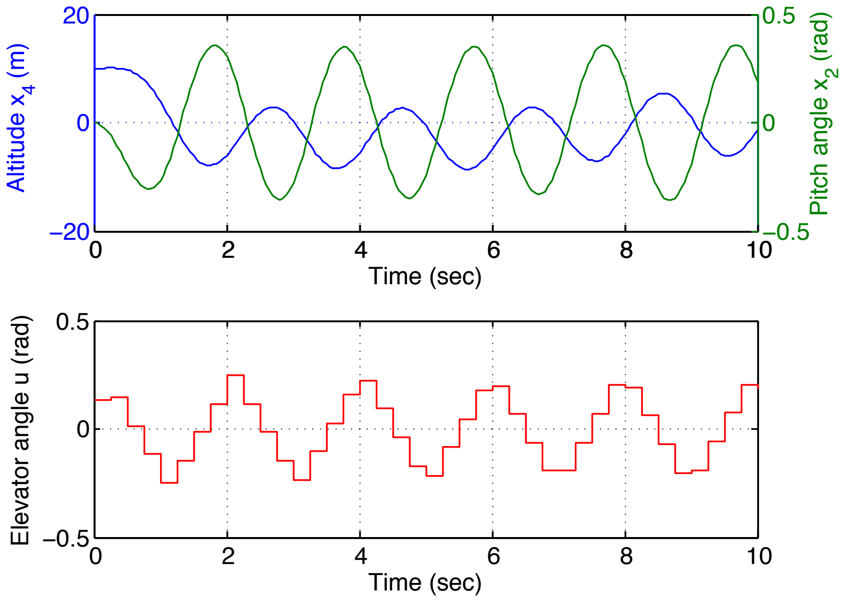
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\text{Sampling time } 0.25\text{sec}, \; Q = I, \; R = 10, \; N = 4}$



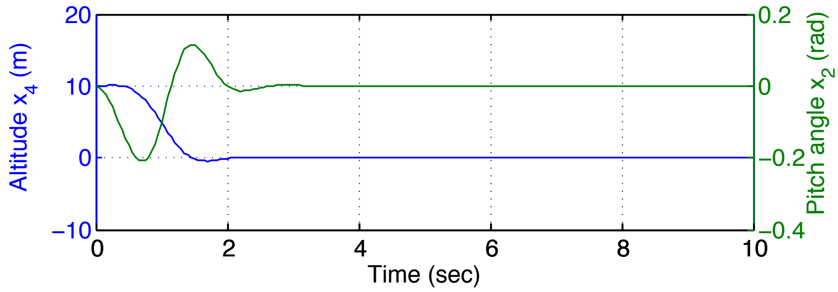
Decrease in the prediction horizon causes loss of the stability properties

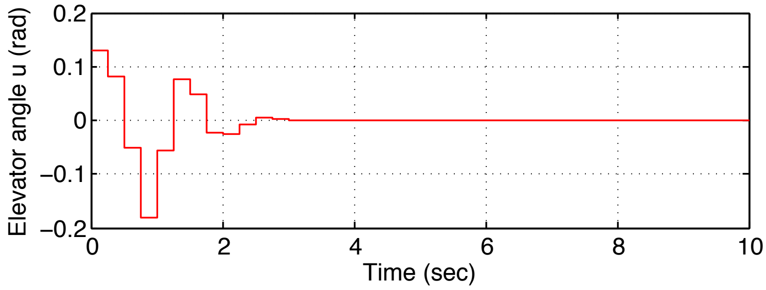
Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 4$





Inclusion of terminal cost and constraint provides stability

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Loss of Feasibility and Stability

What can go wrong with "standard" MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution
- No stability guarantee, i.e., trajectories may not converge to the origin

Example: Loss of feasibility - Double Integrator

Consider the double integrator

$$\begin{cases} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to the input constraints

$$-0.5 \leq u(t) \leq 0.5$$

and the state constraints

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Compute a receding horizon controller with quadratic objective with

$$N = 3,\ P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = 10.$$

Example: Loss of feasibility - Double Integrator

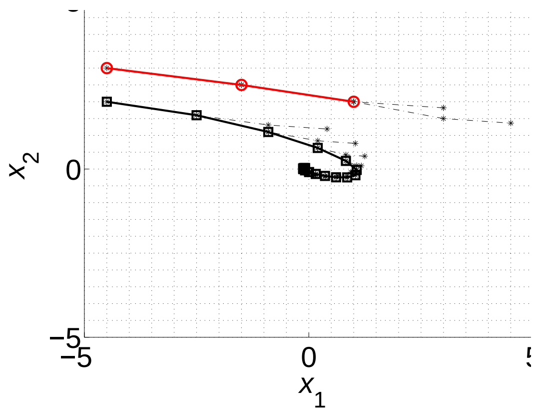
The QP problem associated with the RHC is

$$H = \begin{bmatrix} 13.50 & -10.00 & -0.50 \\ -10.00 & 22.00 & -10.00 \\ -0.50 & -10.00 & 31.50 \end{bmatrix}, \quad F = \begin{bmatrix} -10.50 & 10.00 & -0.50 \\ -20.50 & 10.00 & 9.50 \end{bmatrix}, \quad Y = \begin{bmatrix} 14.50 & 23.50 \\ 23.50 & 54.50 \end{bmatrix}$$

$$G_0 = \begin{bmatrix} 0.50 & -1.00 & 0.50 \\ -0.50 & 1.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ -0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & 0.50 \\ -1.00 & 0.00 & 0.00 \\ 0.00 & -1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 \\ -0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0.50 & 0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ -0.50 & -0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 1.00 \\ -0.50 & -0.50 \\ -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.50 & -1.50 \\ 0.50 & 1.50 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.00 & -1.00 \end{bmatrix}, \quad w_0 = \begin{bmatrix} 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \end{bmatrix}$$

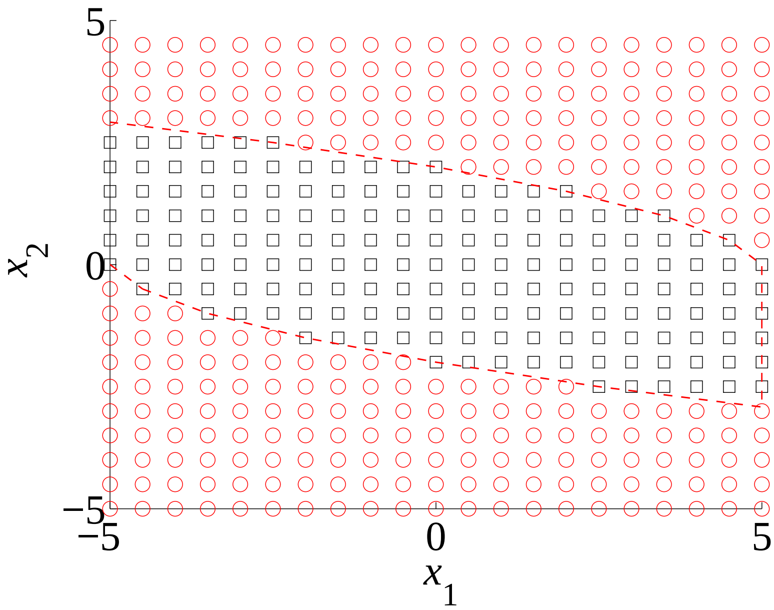
Example: Loss of feasibility - Double Integrator

1) MEASURE the state $x(t)$ at time instance t
2) OBTAIN $U_0^*(x(t))$ by solving the optimization problem in (1)
3) IF $U_0^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element u_0^* of U_0^* to the system
5) WAIT for the new sampling time $t+1$, GOTO 1)



Depending on initial condition, closed loop trajectory may lead to states for which optimization problem is infeasible.

Example: Loss of feasibility - Double Integrator



Boxes (*Circles*) are initial points leading (not leading) to feasible closed-loop trajectories

Example: Feasibility and stability are function of tuning

Unstable system $x(t+1) = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

Input constraints $-1 \leq u(t) \leq 1$

State constraints $\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

Parameters: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

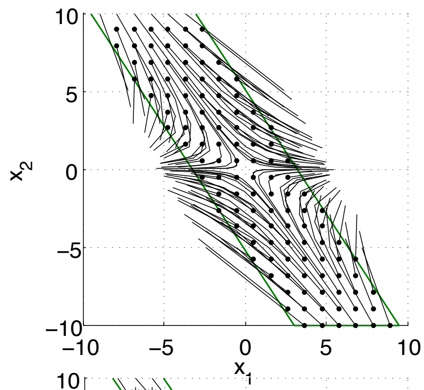
Investigate the stability properties for different horizons N and weights R by solving the finite-horizon MPC problem in a receding horizon fashion...

Example: Feasibility and stability are function of tuning

1. $R = 10,\ N = 2$: all trajectories unstable.
2. $R = 2,\ N = 3$: some trajectories stable.
3. $R = 1,\ N = 4$: more stable trajectories.

* Initial points with convergent trajectories
∘ Initial points that diverge

①



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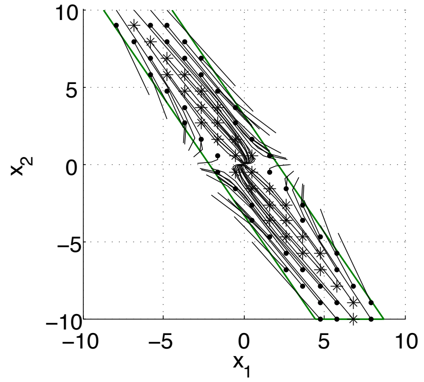

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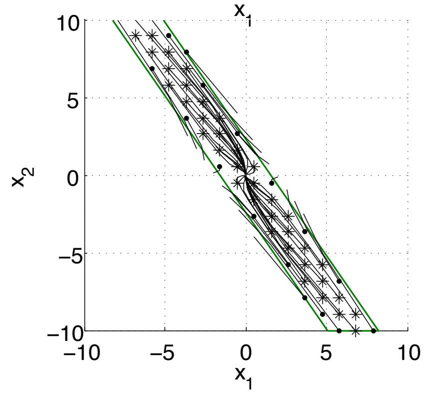
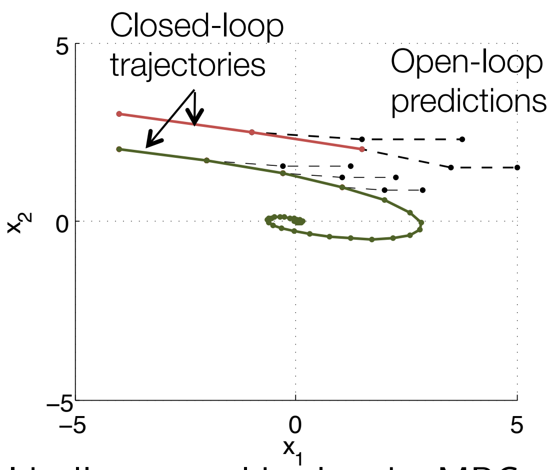
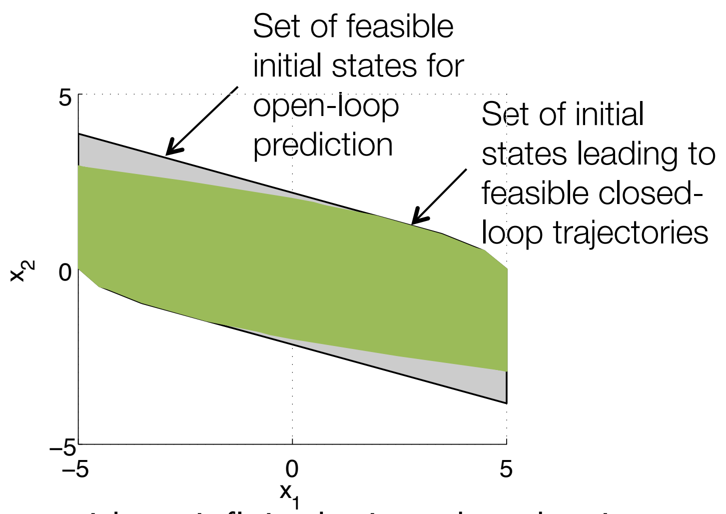


Green lines denote the set of all feasible initial points. They depend on the horizon N but not on the cost $R \Longrightarrow$ Parameters have complex effect and trajectories.

Summary: Feasibility and Stability

Problems originate from the use of a 'short sighted' strategy
⇒ Finite horizon causes deviation between the open-loop prediction and the closed-loop system:





Ideally we would solve the MPC problem with an infinite horizon, but that is computationally intractable
⇒ Design finite horizon problem such that it approximates the infinite horizon

Summary: Feasibility and Stability

- Infinite-Horizon
  If we solve the RHC problem for $N = \infty$ (as done for LQR), then the open loop trajectories are the same as the closed loop trajectories. Hence
  - If problem is feasible, the closed loop trajectories will be always feasible
  - If the cost is finite, then states and inputs will converge asymptotically to the origin

- Finite-Horizon
  RHC is "short-sighted" strategy approximating infinite horizon controller. But
  - **Feasibility**. After some steps the finite horizon optimal control problem may become infeasible. (Infeasibility occurs without disturbances and model mismatch!)
  - **Stability**. The generated control inputs may not lead to trajectories that converge to the origin.

Feasibility and stability in MPC - Solution

Main idea: Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

$$
\begin{aligned}
J_0^*(x_0) = \quad & \min_{U_0} \quad p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) && \text{Terminal Cost} \\
& \text{subj. to} \\
& \qquad x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
& \qquad x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& \qquad x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& \qquad x_0 = x(t)
\end{aligned}
$$

$p(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon.

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Feasibility and Stability of MPC: Proof

Main steps:

- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Two cases:

1. Terminal constraint at zero: $x_N = 0$
2. Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

General notation:

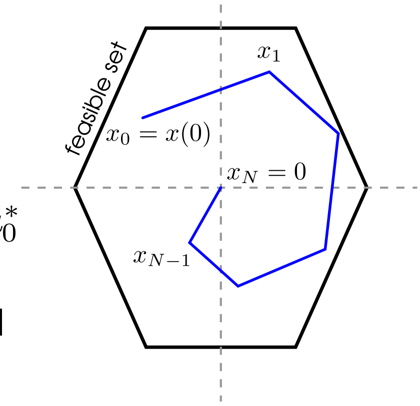
$$J_0^*(x_0) = \min_{U_0} \underbrace{p(x_N)}_{\text{terminal cost}} + \sum_{i=0}^{N-1} \underbrace{q(x_i, u_i)}_{\text{stage cost}}$$

Quadratic case: $q(x_i, u_i) = x_i^T Q x_i + u_i^T R u_i,\ p(x_N) = x_N^T P x_N$

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of x_0 and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ be the corresponding state trajectory
- Apply u_0^* and let system evolve to $x(1) = Ax_0 + Bu_0^*$
- At $x(1)$ the control sequence $\{u_1^*,\ u_2^*,\ \ldots,\ u_{N-1}^*,\ 0\}$ is feasible (apply 0 control input $\Rightarrow x_{N+1} = 0$)



$\Rightarrow$ *Recursive feasibility* ✔

$\Rightarrow$ *$J_0^*(x)$ is a Lyapunov function $\rightarrow$ (Lyapunov) Stability* ✔

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of x_0 and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ be the corresponding state trajectory
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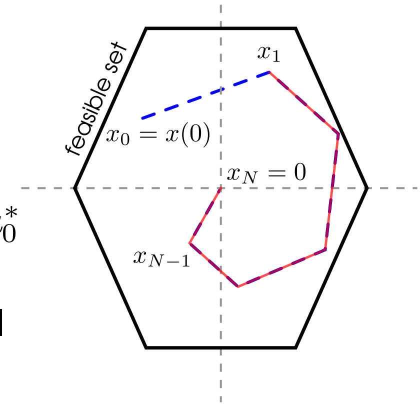


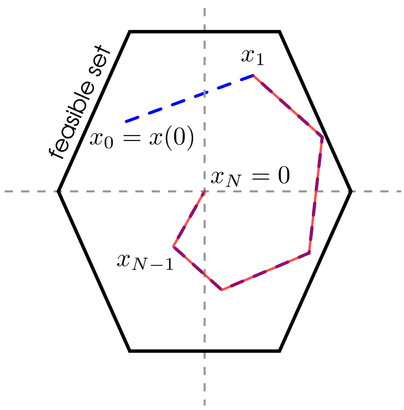
$\Rightarrow$ *Recursive feasibility* ✔

$\Rightarrow$ *$J_0^*(x)$ is a Lyapunov function $\rightarrow$ (Lyapunov) Stability* ✔

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

Goal: Show $J_0^*(x_1) < J_0^*(x_0) \quad \forall x_0 \neq 0$



$$J_0^*(x_0) = \underbrace{p(x_N)}_{=0} + \sum_{i=0}^{N-1} q(x_i, u_i^*)$$

$$\begin{aligned} J_0^*(x_1) \leq \tilde{J}_0(x_1) &= \sum_{i=1}^{N} q(x_i, u_i^*) \\ &= \sum_{i=0}^{N-1} q(x_i, u_i^*) - q(x_0, u_0^*) + q(x_N, u_N) \\ &= J_0^*(x_0) - \underbrace{q(x_0, u_0^*)}_{\substack{\text{Subtract cost} \\ \text{at stage } 0}} + \underbrace{q(0,0)}_{\substack{=0,\ \text{Add cost} \\ \text{for staying at } 0}} \end{aligned}$$

$\Rightarrow$ *$J_0^*(x)$ is a Lyapunov function $\rightarrow$ (Lyapunov) Stability* ✔

Example: Impact of Horizon with Zero Terminal Constraint

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

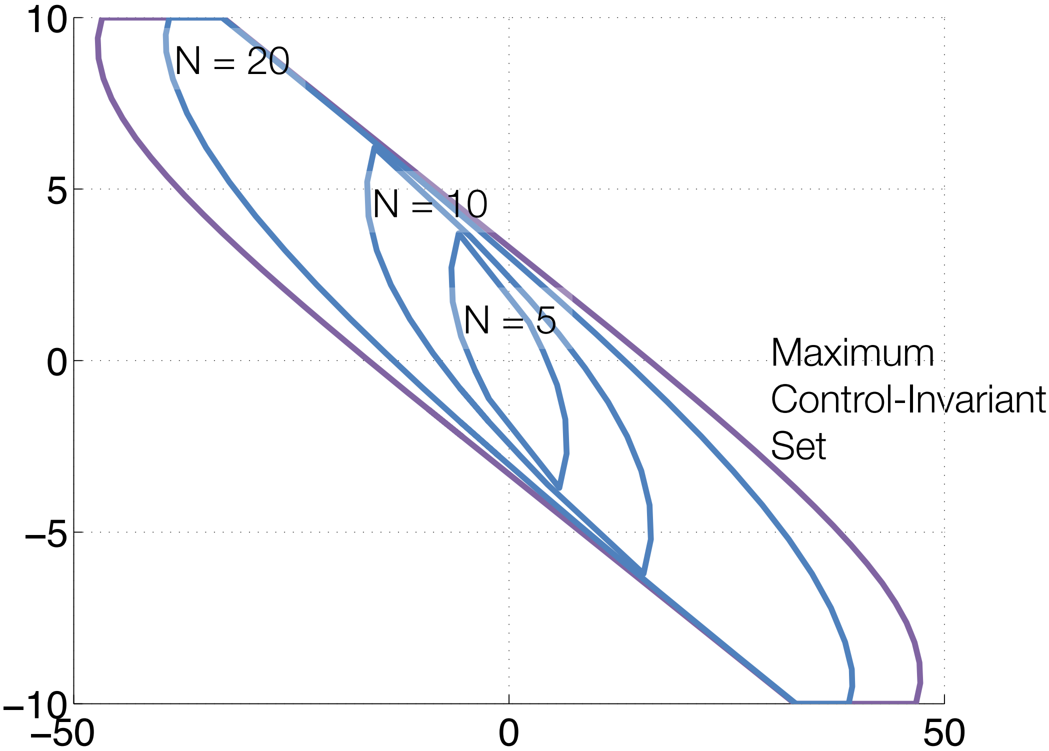
Constraints:

$$\mathcal{X} := \{x \mid -50 \leq x_1 \leq 50, \ -10 \leq x_2 \leq 10\} = \{x \mid A_x x \leq b_x\}$$
$$\mathcal{U} := \{u \mid \|u\|_\infty \leq 1\} = \{u \mid A_u u \leq b_u\}$$

Stage cost:

$$q(x, u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^T u$$

Example: Impact of Horizon with Zero Terminal Constraint



The horizon can have a strong impact on the region of attraction.

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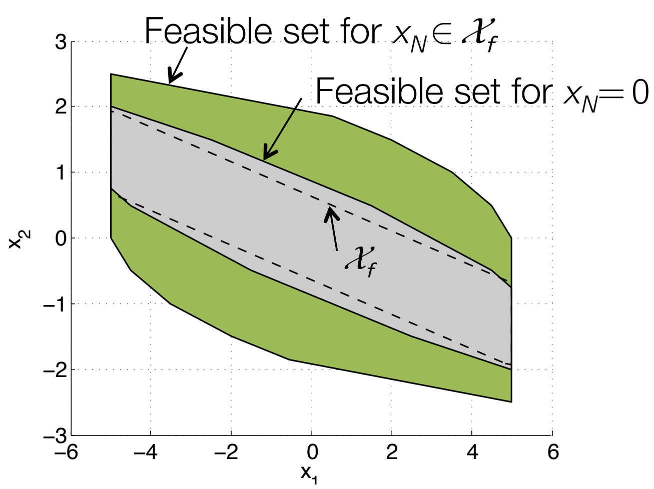
Extension to More General Terminal Sets

Problem: The terminal constraint $x_N = 0$ reduces the size of the feasible set
Goal: Use convex set $\mathcal{X}_f$ to increase the region of attraction



Double integrator

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \le x(t) \le \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$-0.5 \le u(t) \le 0.5$$

$$N = 5,\ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = 10$$

Goal: Generalize proof to the constraint $x_N \in \mathcal{X}_f$

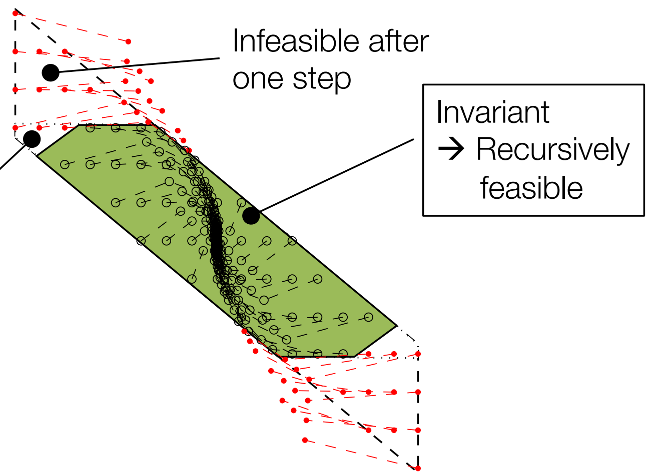
Invariant sets

Definition: Invariant set

A set $\mathcal{O}$ is called *positively invariant* for system $x(t+1) = f_{cl}(x(t))$, if

$$x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}_+$$

The positively invariant set that contains every closed positively invariant set is called the maximal positively invariant set $\mathcal{O}_\infty$.



Stability of MPC - Main Result

Assumptions

1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $v(x_k)$:

$$x_{k+1} = Ax_k + Bv(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

   All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\ v(x_k) \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$p(x_{k+1}) - p(x_k) \leq -q(x_k, v(x_k)), \quad \text{for all } x_k \in \mathcal{X}_f$$

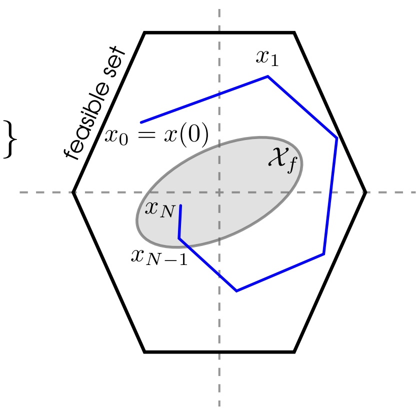
Under those 3 assumptions:

Theorem

The closed-loop system under the MPC control law $u_0^*(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for the system $x(k+1) = Ax + Bu_0^*(x)$.

Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ the corresponding state trajectory



- At $x(1)$, $\{u_1^*,\ u_2^*,\ \ldots,\ v(x_N)\}$ is feasible:

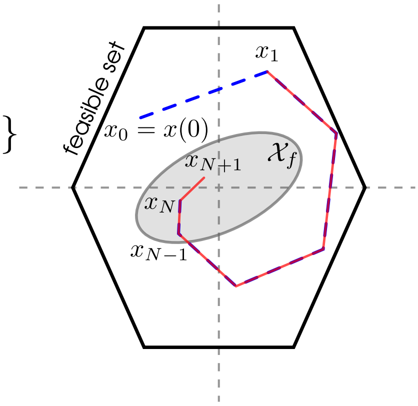
  x_N is in $\mathcal{X}_f \rightarrow v(x_N)$ is feasible

  and $x_{N+1} = Ax_N + Bv(x_N)$ in $\mathcal{X}_f$

$\Rightarrow$ *Terminal constraint provides recursive feasibility*

Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*,\ u_1^*,\ \dots,\ u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0),\ x_1,\ \dots,\ x_N\}$ the corresponding state trajectory



- At $x(1)$, $\{u_1^*,\ u_2^*,\ \dots,\ v(x_N)\}$ is feasible:

  x_N is in $\mathcal{X}_f \to v(x_N)$ is feasible

  and $x_{N+1} = Ax_N + Bv(x_N)$ in $\mathcal{X}_f$

$\Rightarrow$ *Terminal constraint provides recursive feasibility*

Asymptotic Stability of MPC - Outline of the Proof

$$J_0^*(x_0) = \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N)$$

Feasible, sub-optimal sequence for x_1 : $\{u_1^*,\ u_2^*,\ \dots,\ v(x_N)\}$

$$\begin{aligned}
J_0^*(x_1) &\le \sum_{i=1}^{N} q(x_i, u_i^*) + p(Ax_N + Bv(x_N)) \\
&= \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N) - q(x_0, u_0^*) + p(Ax_N + Bv(x_N)) \\
&\qquad - p(x_N) + q(x_N, v(x_N)) \\
&= J_0^*(x_0) - q(x_0, u_0^*) + \underbrace{p(Ax_N + Bv(x_N)) - p(x_N) + q(x_N, v(x_N))}_{p(x)\le 0}
\end{aligned}$$

$$\implies J_0^*(x_1) - J_0^*(x_0) \le -q(x_0, u_0^*), \quad q > 0$$

$J_0^*(x)$ is a Lyapunov function decreasing along the closed loop trajectories
$\Rightarrow$ The closed-loop system under the MPC control law is asymptotically stable

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

$$
\begin{aligned}
J_0^*(x_0) = \quad & \min_{U_0} && x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k && \text{Terminal Cost} \\
& \text{subj. to} && && \\
& && x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 && \\
& && x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 && \\
& && x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& && x_0 = x(t) &&
\end{aligned}
$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

- Design unconstrained LQR control law

$$F_\infty = -(B' P_\infty B + R)^{-1} B' P_\infty$$

where P_∞ is the solution to the discrete-time algebraic Riccati equation:

$$P_\infty = A' P_\infty A + Q - A' P_\infty B (B' P_\infty B + R)^{-1} B' P_\infty A$$

- Choose the terminal weight $P = P_\infty$
- Choose the terminal set $\mathcal{X}_f$ to be the maximum invariant set for the closed-loop system $x_{k+1} = (A + BF_\infty)x_k$:

$$x_{k+1} = Ax_k + BF\infty(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\ F_\infty x_k \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

1. The stage cost is a positive definite function
2. By construction the terminal set is **invariant** under the local control law $v = F_\infty x$
3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$\begin{aligned} x'_{k+1}Px_{k+1} - x'_kPx_k &= x'_k(-P_\infty + A'P_\infty A - A'P_\infty B(B'P_\infty B + R)^{-1}B'P_\infty A)x_k \\ &= -x'_kQx_k \end{aligned}$$

All the Assumptions of the Feasibility and Stability Theorem are verified.

Example: Unstable Linear System

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

Constraints:

$$\begin{aligned} &\mathcal{X} := \{x \mid -50 \le x_1 \le 50,\ -10 \le x_2 \le 10\} = \{x \mid A_x x \le b_x\} \\ &\mathcal{U} := \{u \mid \|u\|_\infty \le 1\} = \{u \mid A_u u \le b_u\} \end{aligned}$$

Stage cost:

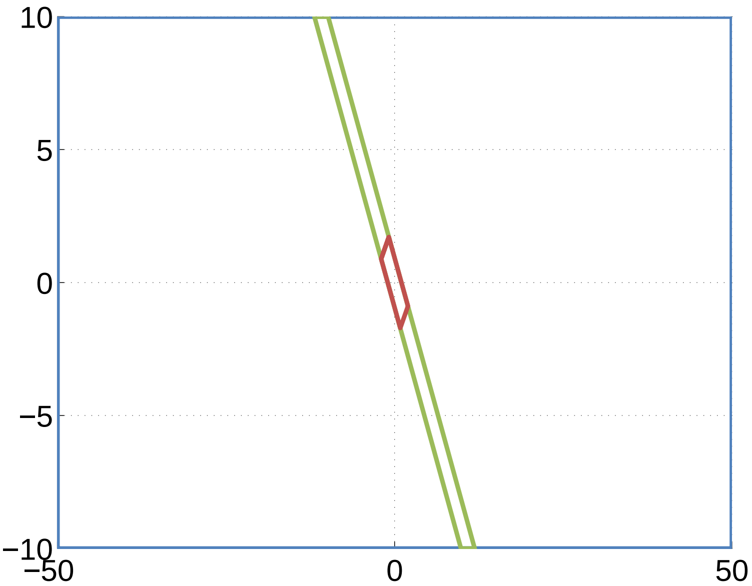
$$q(x,u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^T u$$

Horizon: $N = 10$

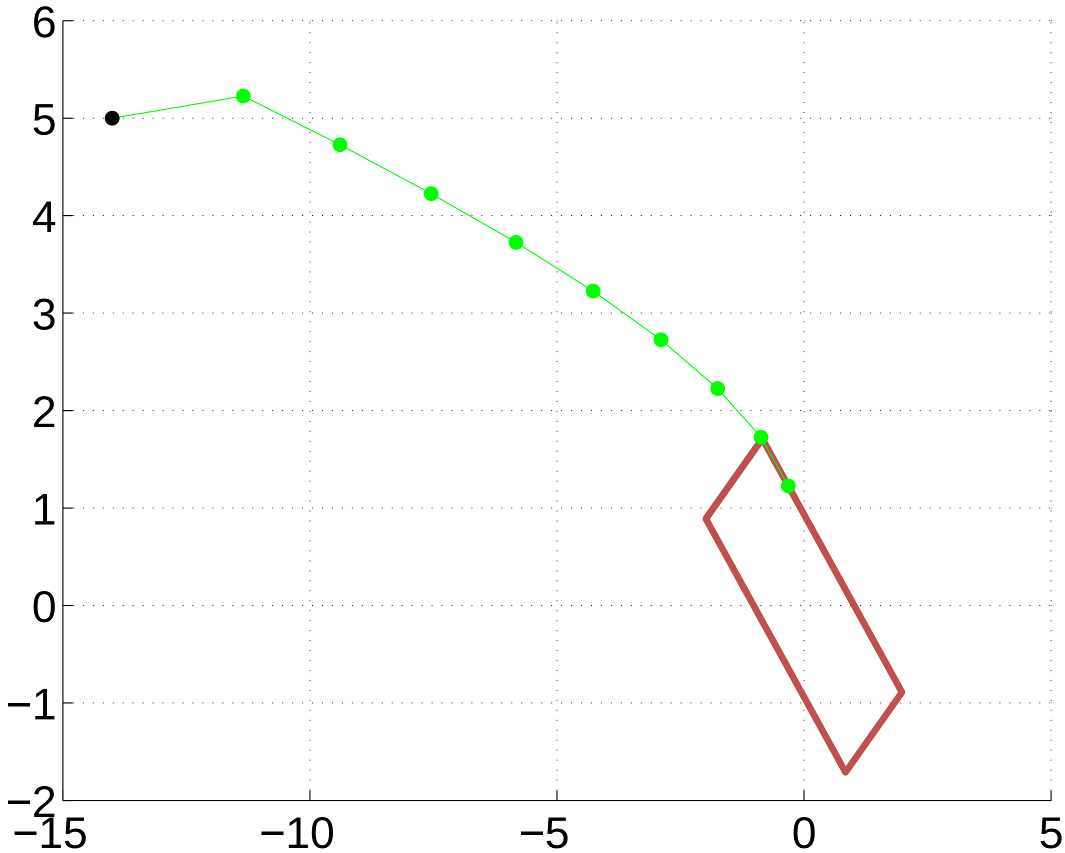
Example: Designing MPC Problem

1. Compute the optimal LQR controller and cost matrices: F_∞, P_∞
2. Compute the maximal invariant set $\mathcal{X}_f$ for the closed-loop linear system $x_{k+1} = (A + BF_\infty)x_k$ subject to the constraints

$$\mathcal{X}_{\text{cl}} := \left\{ x \;\middle|\; \begin{bmatrix} A_x \\ A_u F_\infty \end{bmatrix} x \le \begin{bmatrix} b_x \\ b_u \end{bmatrix} \right\}$$



Example: Closed-loop behaviour



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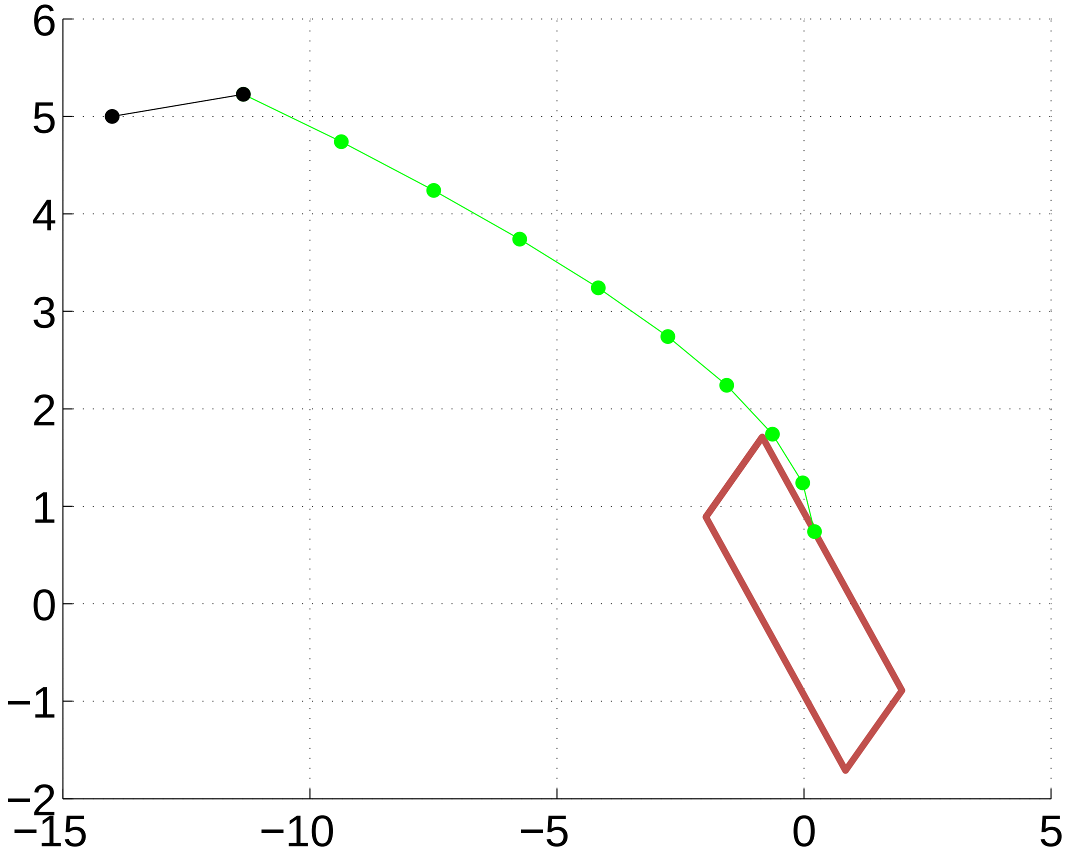

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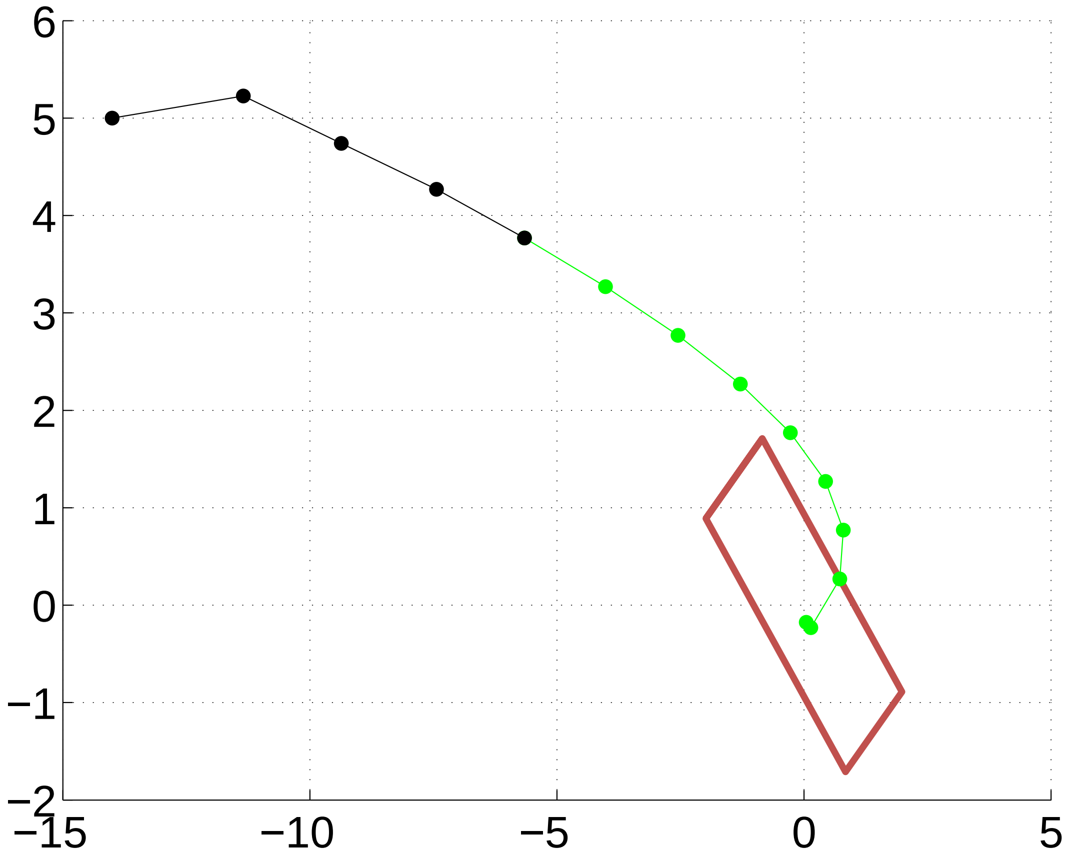

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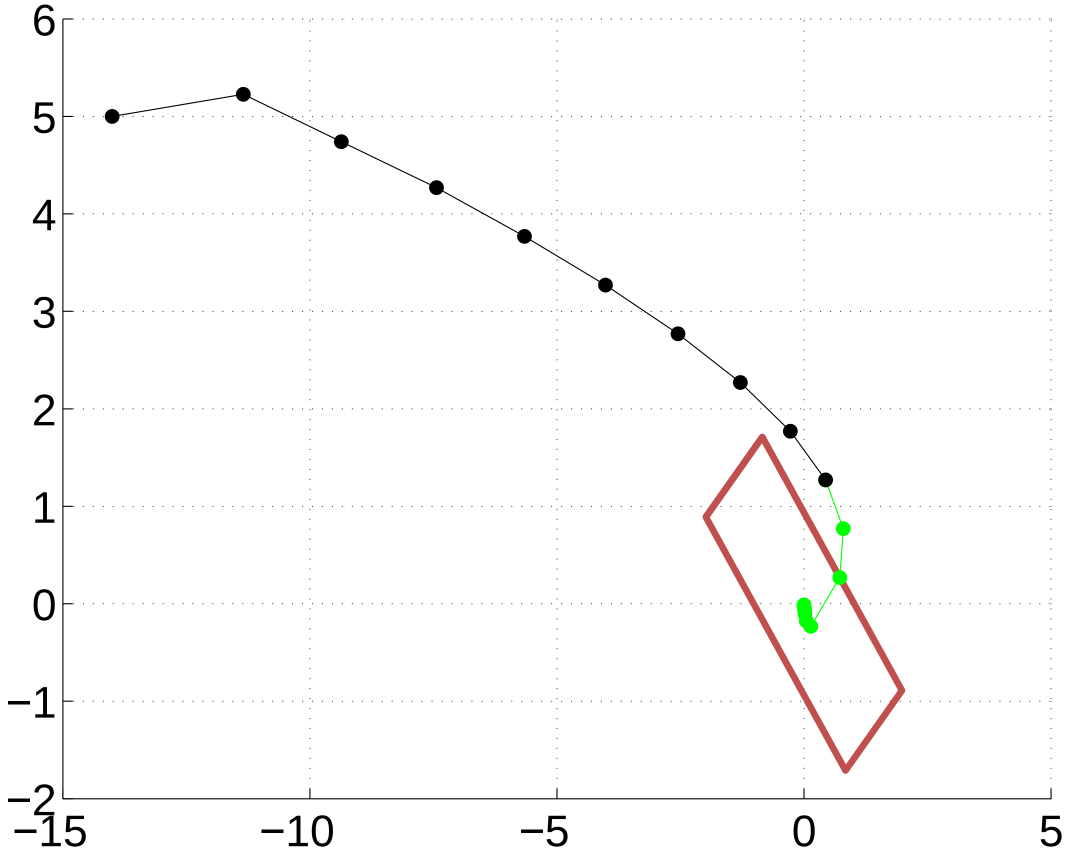
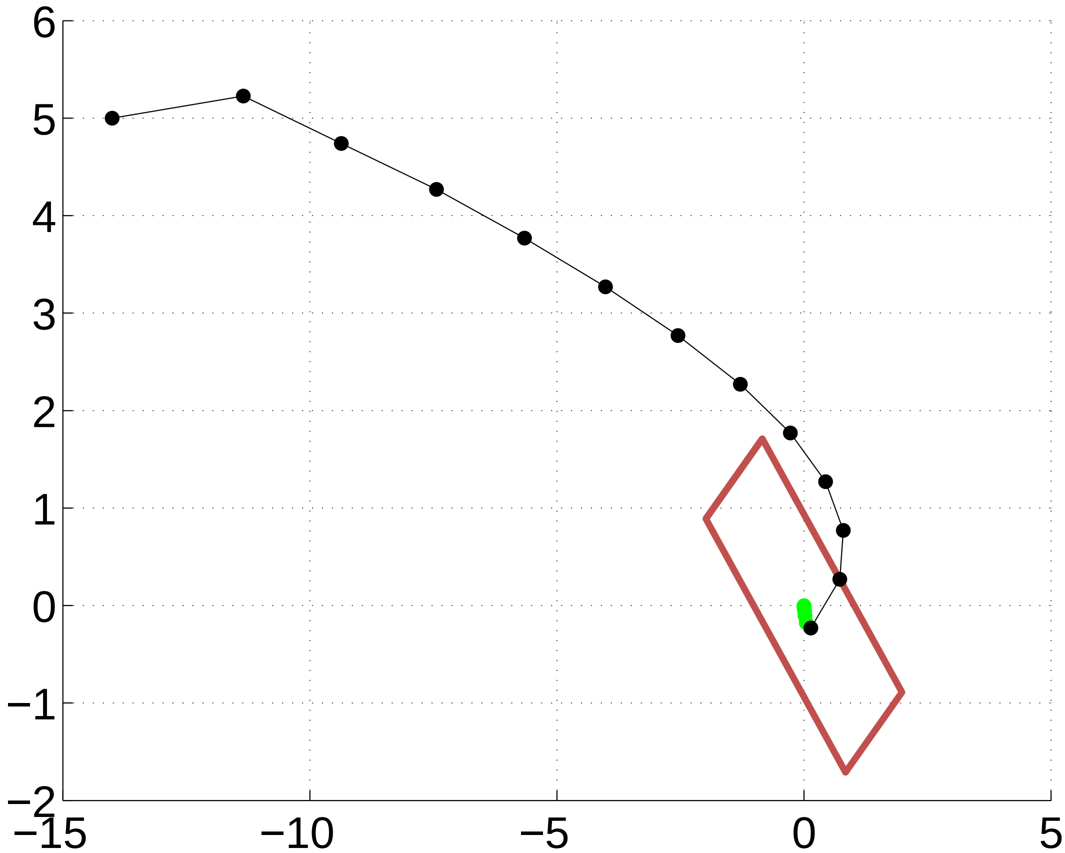
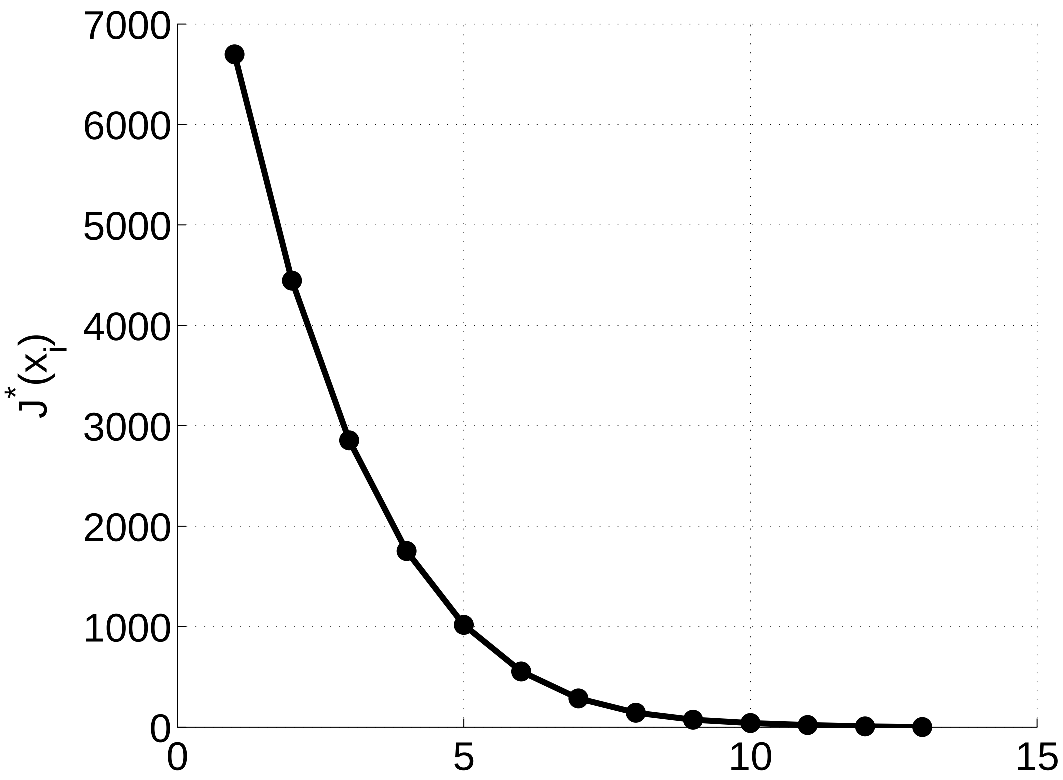


Example: Closed-loop behaviour



Example: Lyapunov Decrease of Optimal Cost



Stability of MPC - Remarks

- The terminal set $\mathcal{X}_f$ and the terminal cost ensure recursive feasibility and stability of the closed-loop system.
  But: the terminal constraint reduces the region of attraction.
  (Can extend the horizon to a sufficiently large value to increase the region)

Are terminal sets used in practice?

- Generally not...
  - Not well understood by practitioners
  - Requires advanced tools to compute (polyhedral computation or LMI)
- Reduces region of attraction
  - A 'real' controller must provide *some* input in *every* circumstance
- Often unnecessary
  - Stable system, long horizon $\rightarrow$ will be stable and feasible in a (large) neighbourhood of the origin

Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attaction for small N
- Solution for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N, region of attraction approaches maximum control invariant set

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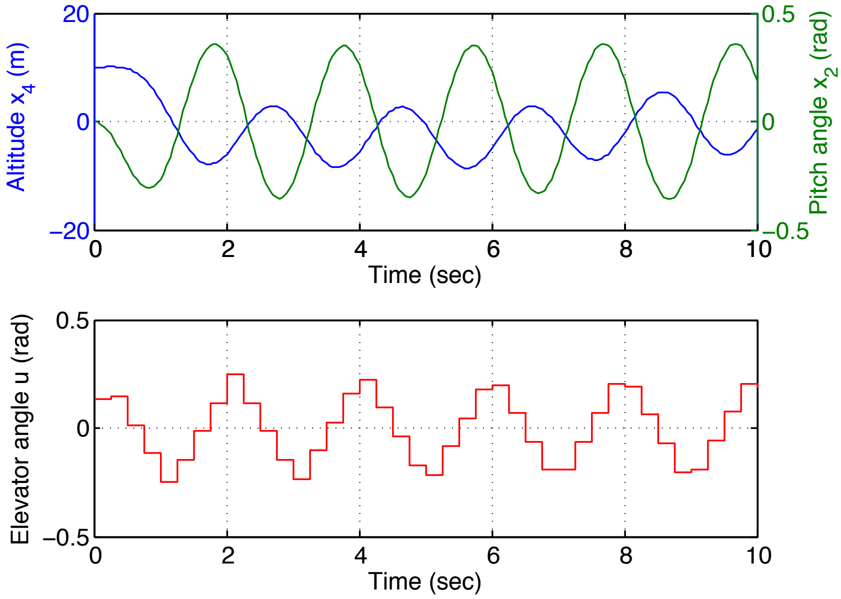
Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 4$

Decrease in the prediction horizon causes loss of the stability properties



6 6. Feasibility and Stability 6.3 Example

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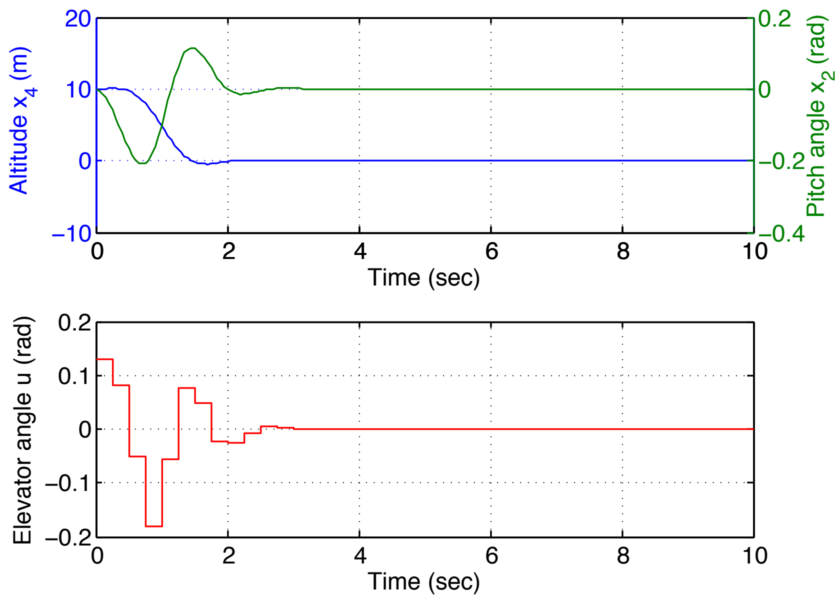
Inclusion of terminal cost and constraint provides stability



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Example: Short horizon

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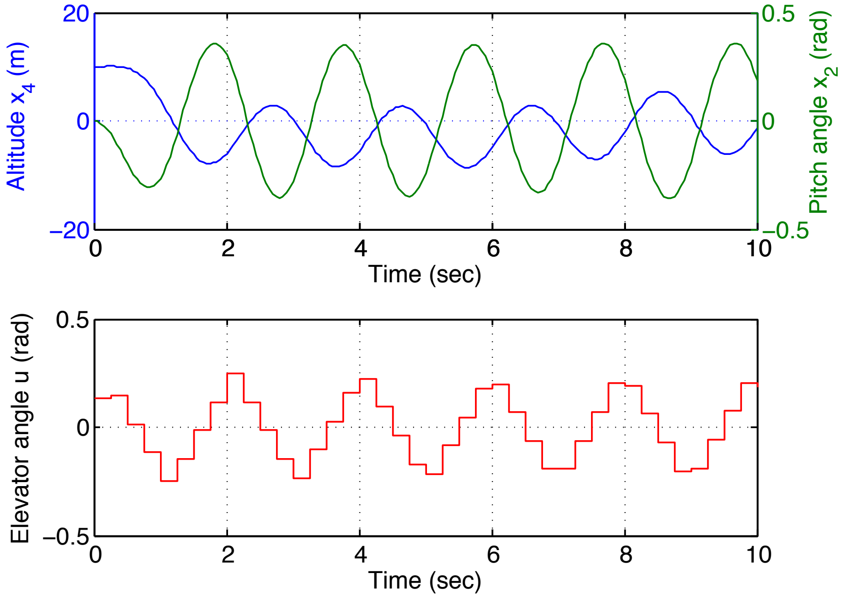

6 6. Feasibility and Stability 6.3 Example

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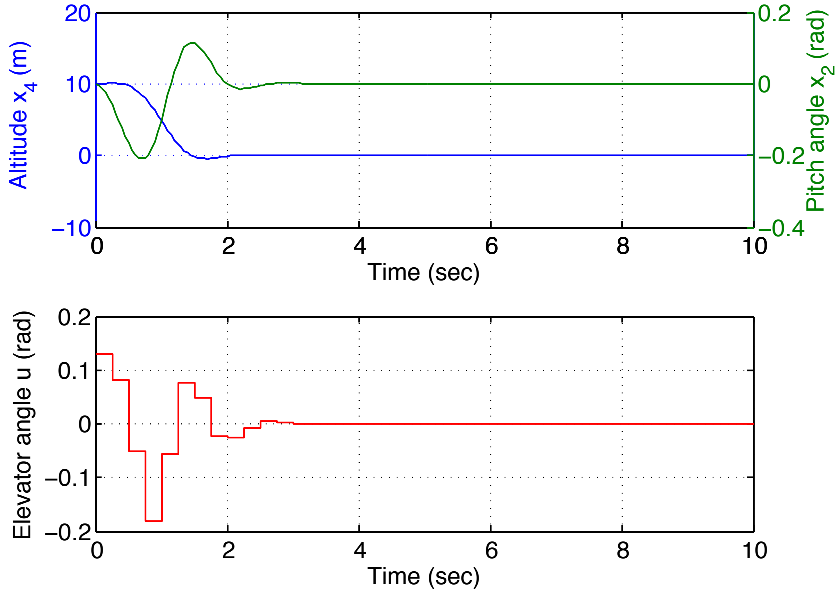

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Summary

Finite-horizon MPC may not be stable!

Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We 'fake' infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.

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Extension to Nonlinear MPC

Consider the nonlinear system dynamics: $x(t+1) = g(x(t), u(t))$

$$
\begin{array}{lll}
J_0^*(x(t)) = & \min\limits_{U_0} & p(x_N) + \sum\limits_{k=0}^{N-1} q(x_k, u_k) \\
& \text{subj. to} & x_{k+1} = g(x_k, u_k),\ k = 0, \ldots, N-1 \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& & x_N \in \mathcal{X}_f \\
& & x_0 = x(t)
\end{array}
$$

- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems

$\rightarrow$ Results can be directly extended to nonlinear systems.

However, computing the sets $\mathcal{X}_f$ and function p can be very difficult!